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In a recent paper, Gopal Prasad and Jiu-Kang Yu introduced the notion of a quasireductive group scheme $\mathcal{G}$ over a discrete valuation ring R, in the context of Langlands duality. They showed that such a group scheme $\mathcal{G}$ is necessarily of finite type over R, with geometrically connected fibres, and its geometric generic fibre is a reductive algebraic group; however, they found examples where the special fibre is nonreduced, and the corresponding reduced subscheme is a reductive group of a different type. In this paper, the formalism of vanishing cycles in étale cohomology is used to show that the generic fibre of a quasireductive group scheme cannot be a restriction of scalars of a group scheme in a nontrivial way; this answers a question of Prasad, and implies that nonreductive quasireductive group schemes are essentially those found by Prasad and Yu.

Gopal Prasad and Jiu-Kang Yu [2006] introduced the notion of a *quasireductive group scheme* over a discrete valuation ring R in a recent paper: this is an affine, flat group scheme $\pi : \mathcal{G} \to \operatorname{Spec} R$, such that

(i) the generic fibre $\mathcal{G}_K$ is a smooth, connected group scheme over the quotient field K of R;

(ii) the reduced geometric special fibre $(\mathcal{G}_{\overline{k}})_{\mathrm{red}}$ is of finite type over the algebraic closure of the residue field k of R, and its identity component is a reductive affine algebraic group;

(iii) $\dim \mathcal{G}_K = \dim \mathcal{G}_k$.

They showed that $\mathcal{G}$ is necessarily of finite type over R, with geometrically connected fibres, and its geometric generic fibre is a reductive algebraic group. Further, $\mathcal{G}$ is a reductive group scheme over $\operatorname{Spec} R$, except possibly when R has residue characteristic 2 and the geometric generic fibre $\mathcal{G}_{\overline{K}}$ has a nontrivial normal subgroup of type SO_{2n+1}, for some $n \geq 1$. They gave examples to show that in case $\mathcal{G}_{\overline{K}} = \mathrm{SO}_{2n+1}$, reductivity can fail to hold, with a nonreduced geometric special fibre, and they gave a classification of such $\mathcal{G}$. Their work arose in response to

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a question of Vilonen to Prasad, in connection with a Tannakian construction of Langlands dual groups; see [Mirković and Vilonen 2004].

In this context, it is natural to ask if there are any other possibilities for nonreductive, quasireductive group schemes $\mathscr{G}$, except the examples found by Prasad and Yu, and the others obtained from these by simple modifications (like products and so forth). From their results, this boils down to the following specific question:

> Does there exist a quasireductive group scheme $\pi : \mathscr{G} \to \operatorname{Spec} R$, where R is a complete DVR with algebraically closed residue field, such that for some finite, separable (totally ramified) extension field L of K, of degree > 1, the generic fibre $\mathscr{G}_K$ is isomorphic to $R_{L/K}(\mathrm{SO}_{2n+1})_L$, the Weil restriction of scalars of $(\mathrm{SO}_{2n+1})_L$?

One aim of this paper is to show that *there do not exist any such quasireductive group schemes*; see Corollary 2 below. Gopal Prasad has obtained a stronger conclusion, combining Corollary 2 with the arguments based on [Prasad and Yu 2006]; at his urging, this is included below (Theorem 11).

The nonexistence proof is based on a topological result, Theorem 1, on the ℓ-adic cohomology of a quasireductive group scheme; it says roughly that, though a quasireductive group scheme may not be smooth over the base, it is almost so from the point of view of ℓ-adic cohomology. This property of quasireductive group schemes (including the nonsmooth ones) may also be of interest in potential applications of such group schemes. This topological result was motivated by the well known Serre–Tate criterion [1968] for good reduction of abelian varieties, which relies ultimately on the theory of Néron models. In a sense, [Prasad and Yu 2006] also relies on some aspects of this theory.

Theorem 1. *Let R be a complete DVR with quotient field K and algebraically closed residue field k. Let*

$$\pi : \mathscr{G} \to \operatorname{Spec} R$$

be a quasireductive group scheme. Let $G \to \operatorname{Spec} K$ be the generic fibre. Let ℓ be a prime number, invertible in R. Then the action of the inertia group $\mathrm{Gal}(\overline{K}/K)$ on the étale cohomology group $H^i_{\mathrm{et}}(G_{\overline{K}}, \mathbb{Z}/\ell^n\mathbb{Z})$ is trivial, for any $i, n \geq 0$. Thus, the inertia action on the ℓ-adic cohomology $H^i_{\mathrm{et}}(G_{\overline{K}}, \mathbb{Q}_\ell)$ is trivial, for all $i \geq 0$.

For a more technical assertion, which implies the above result, and may be viewed as the key new observation in this paper, see Proposition 8 in Section 2 below.

Corollary 2. *Let R be a complete DVR with quotient field K, and algebraically closed residue field k. Let L be a finite extension field of K, and let*

$$\pi : \mathscr{G} \to \operatorname{Spec} R$$

be a quasireductive group scheme, whose generic fibre $\mathcal{G}_K$ is isomorphic to the restriction of scalars of a positive dimensional reductive affine algebraic group G over L. Then we must have $L = K$.

Proof. We first note that since $G_{\overline{K}}$ is a positive dimensional reductive algebraic group over an algebraically closed field, it has a nonzero ℓ-adic Betti number in some positive degree; for example, this is a simple consequence of the classification of reductive groups over algebraically closed fields. Let $i > 0$ be the smallest such degree.

Next, since the generic fibre of $\mathcal{G} \to \operatorname{Spec} R$ is a reductive group and is obtained by the restriction of scalars from L to K, the extension field L/K is necessarily separable. (If L/K is a purely inseparable finite extension and G is an algebraic group over L, then the kernel of the natural homomorphism $R_{L/K}(G)_L \to G$ is unipotent; see [Oesterlé 1984, A.3.5], for example.)

Now, if L/K is a separable extension of degree $n > 1$, then the geometric generic fibre $\mathcal{G}_{\overline{K}}$ is isomorphic to a product of n copies of $G_{\overline{K}}$, and the inertia group $\operatorname{Gal}(\overline{K}/K)$ permutes the n factors transitively. From the Kunneth formula, it follows that for the chosen $i > 0$, the étale cohomlogy group $H^i_{\text{et}}(\mathcal{G}_{\overline{K}}, \mathbb{Q}_\ell)$ is a direct sum of (a positive number of) copies of a nontrivial permutation Galois module. This contradicts Theorem 1. □

1. Some preliminaries

Before proving Theorem 1, we discuss some preliminaries.

Recall that, if k is an algebraically closed field, a *unipotent isogeny* between connected reductive algebraic k-groups is a homomorphism, which is a finite surjective morphism, whose kernel does not contain any nontrivial subgroup scheme of multiplicative type (that is, isomorphic to a subgroup scheme of $\mathbb{G}_m^e$ for some $e \geq 1$).

The following lemma sheds more light on unipotent isogenies (see Corollary 4). We thank Conrad for explaining this argument to us; the reader might compare this with [Prasad and Yu 2006, Lemma 2.2].

Lemma 3. *Let H be a reduced group scheme over a perfect field k, let G be a closed normal subgroup scheme of H and let G^{red} be the reduced subscheme of G. Then G^{red} is also a normal subgroup scheme of H. If H is connected and G is finite then G^{red} is in the center of H.*

Proof. We first recall that since k is perfect, the product of reduced k-schemes is reduced, so the morphism $G^{\text{red}} \times G^{\text{red}} \to G$ induced by the product morphism of G factors through G^{red} and similarly for the inverse morphism. Hence G^{red} is a subgroup scheme of G. Since H is reduced, so is $H \times G^{\text{red}}$, and hence $(H \times G)^{\text{red}} = H \times G^{\text{red}}$.

Let $c : H \times G \to G$ be the morphism giving the conjugation action of H on G and let $i : G^{\mathrm{red}} \to G$ be the inclusion. Then there is a unique morphism $c^{\mathrm{red}} : H \times G^{\mathrm{red}} \to G^{\mathrm{red}}$ making the diagram below commute,

$$\begin{tikzcd} H \times G^{\mathrm{red}} \arrow[r, "c^{\mathrm{red}}"] \arrow[d, "\mathrm{Id}\times i"'] & G^{\mathrm{red}} \arrow[d, "i"] \\ H \times G \arrow[r, "c"] & G. \end{tikzcd}$$

Thus G^{red} is normal.

Now suppose G is finite and H is connected. Since $H(k)$ is nonempty, H is geometrically connected over k [EGA 6, 4.5.13]. We may assume that k is algebraically closed and so G^{red} is a disjoint union of copies of $\operatorname{Spec} k$. Then the inclusion $e : \operatorname{Spec} k \to H$ given by the identity induces a bijection of connected components of G^{red} with those of $H \times G^{\mathrm{red}}$. Since c^{red} is continuous, it follows that

$$c^{\mathrm{red}} = p_{G^{\mathrm{red}}},$$

the projection onto G^{red}. Thus G^{red} is central. □

Corollary 4. *The kernel of a unipotent isogeny between connected reductive algebraic groups over an algebraically closed field k is infinitesimal, so that such an isogeny must be purely inseparable.*

Proof. If H is a connected reductive algebraic group over k, and G is the kernel of a unipotent isogeny with domain H, then G is a finite, normal subgroup scheme of H. By Lemma 3, G^{red} is a central subgroup scheme, hence contained in a maximal torus. Since G, and hence G^{red}, has no nontrivial subgroup scheme of multiplicative type, this means G^{red} is trivial, that is, G is infinitesimal. □

Lemma 5. *Let k be an algebraically closed field of characteristic $p > 0$, and ℓ a prime distinct from p. Let $f : G_1 \to G_2$ be either*

(i) *a unipotent isogeny between connected reductive algebraic groups over k, or*

(ii) *a closed immersion of k-schemes of finite type, which induces an isomorphism on the underlying reduced schemes.*

Then

$$\mathcal{F} \mapsto f_*\mathcal{F}, \quad \mathcal{F}' \mapsto f^*\mathcal{F}'$$

determine an equivalence of categories between étale sheaves on G_1 and G_2, and there are natural isomorphisms $H^i_{\mathrm{et}}(G_2, f_*\mathcal{F}) \cong H^i_{\mathrm{et}}(G_1, \mathcal{F})$ for all i.

Proof. A finite, surjective, radicial morphism induces an equivalence of categories on étale sheaves, and hence isomorphisms on étale cohomology — see [SGA 4 II, Exposé VIII, Théorème 1.1, Cor. 1.2]. □

The main input in the proof of Theorem 1 is the formalism of vanishing cycles, and in particular, the notion of the *complex of nearby cycles*, as explained in [SGA 7 II, Exposé XIII]. We briefly review what we need.

Suppose given a morphism of schemes $\pi : X \to T$, where T is the spectrum of a complete discrete valuation ring with algebraically closed residue field. Denote the generic point of T by η, and fix an algebraic closure of the quotient field of the DVR, giving a geometric generic point $\overline{\eta}$ of T. Let X_0 be the closed fibre, and let $X_{\overline{\eta}}$ be the geometric generic fibre.

If $\mathcal{F}$ is any étale sheaf of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules on X, then one defines the complex of nearby cycles $R\psi_T(\mathcal{F})$ on the closed fibre X_0 as follows: if $i : X_0 \to X$ is the inclusion, and $j : X_{\overline{\eta}} \to X$ the evident morphism, then

$$R\psi_T(\mathcal{F}) = i^* Rj_* j^* \mathcal{F}.$$

The adjunction map $\mathrm{id} \to i_* i^*$ gives a map $Rj_* j* \to i_* i^* Rj_* j*$ and the adjunction map $\mathrm{id} \to Rj_* j^*$ gives a map $i^* \to i^* Rj_* j^*$. These give rise to maps on cohomology:

$$\begin{aligned} H^i_{\mathrm{et}}(X_{\overline{\eta}}, j^*\mathcal{F}) &\to H^i_{\mathrm{et}}(X_0, R\psi_T(\mathcal{F})), \\ H^i_{\mathrm{et}}(X_0, \mathcal{F}_0) &\to H^i_{\mathrm{et}}(X_0, R\psi_T(\mathcal{F})). \end{aligned} \tag{1-1}$$

Further, $H^i_{\mathrm{et}}(X_0, R\psi_T(\mathcal{F}))$ carries an action of the inertia group $\mathrm{Gal}(k(\overline{\eta})/k(\eta))$, such that the above two maps on cohomology are equivariant (where the inertia action on $H^i_{\mathrm{et}}(X_0, \mathcal{F}_0)$ is taken to be trivial). We may of course replace the closed fibre X_0 by its reduced subscheme in the above, since the categories of étale sheaves on X_0 and $(X_0)_{\mathrm{red}}$ are equivalent. If $T = \mathrm{Spec}\, R$, we may write ψ_R instead of ψ_T.

The adjunctions above fit into a square

$$\begin{tikzcd} \mathrm{id} \arrow[r] \arrow[d] & Rj_* j^* \arrow[d] \\ i_* i^* \arrow[r] & i_* i^* Rj_* j^* \end{tikzcd}$$

which gives a commutative diagram

$$\begin{tikzcd} H^i_{\mathrm{et}}(X, \mathcal{F}) \arrow[r] \arrow[d] & H^i_{\mathrm{et}}(X_{\overline{\eta}}, j^*\mathcal{F}) \arrow[d] \\ H^i_{\mathrm{et}}(X_0, \mathcal{F}_0) \arrow[r] & H^i_{\mathrm{et}}(X_0, R\psi_T(\mathcal{F})). \end{tikzcd} \tag{1-2}$$

Here the left vertical arrow is an isomorphism if f is proper [SGA 5, proper base change theorem, Exposé XII].

Lemma 6. *If in the above situation, $f : X \to T$ is smooth, and $\mathcal{F}$ is a locally constant constructible sheaf of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules, with ℓ invertible in $\mathcal{O}_T$, then the natural map*

$$\mathcal{F}_0 \to R\psi_T(\mathcal{F})$$

is an isomorphism, and so induces isomorphisms on étale cohomology.

Proof. This follows from the definition of $R\psi_T$, and the smooth base change theorem [SGA 7 II, Exposé XIII, Reformulation 2.1.5 and above]. □

Lemma 7. *Let X be a noetherian scheme, $i : Y \to X$ a closed embedding, $\beta : X' \to X$ a finite morphism, $Y' = X' \times_X Y$ with induced embedding $i' : Y' \to X'$ and finite morphism $\alpha : Y' \to Y$. For all $\mathcal{I} \in D^+(X_{\text{et}})$ and $r \in \mathbb{Z}$ the restriction map*

$$H^r_{\text{et}}(X', \mathcal{I}) \to H^r_{\text{et}}(Y', i'^*\mathcal{I})$$

is equal to the composite

$$H^r_{\text{et}}(X', \mathcal{I}) \to H^r_{\text{et}}(X, R\beta_*\mathcal{I}) \to H^r_{\text{et}}(Y, i^*R\beta_*\mathcal{I}) \to H^r_{\text{et}}(Y, R\alpha_*i'^*\mathcal{I}) \to H^r_{\text{et}}(Y', i'^*\mathcal{I})$$

where the first and the last map are the natural isomorphisms, the second is the restriction map and the third is induced by the base change map.

Proof. If $\mathcal{I}$ is represented by a single sheaf $\mathcal{F}$ (in degree 0) and $r = 0$ then the equality follows from the very definition of the base change map [SGA 5, Exposé XII, §4].

We now assume that $\mathcal{I}$ is (represented by) a bounded below complex of injective sheaves

$$\cdots \mathcal{I}^{j-1} \to \mathcal{I}^j \to \mathcal{I}^{j+1} \to \cdots .$$

Then $i'^*\mathcal{I}$ is (represented by) the complex

$$\cdots i'^*\mathcal{I}^{j-1} \to i'^*\mathcal{I}^j \to i'^*\mathcal{I}^{j+1} \to \cdots .$$

Let $\mathcal{J}$ be a complex of injective sheaves on Y' with a quasiisomorphism $q' : i'^*\mathcal{I} \to \mathcal{J}$. Then the map $H^r_{\text{et}}(X', \mathcal{I}) \to H^r_{\text{et}}(Y', i'^*\mathcal{I})$ is induced by the map of complexes of abelian groups which in cohomological degree r is the composite

$$\Gamma(X', \mathcal{I}^r) \to \Gamma(Y', i'^*\mathcal{I}^r) \to \Gamma(Y', \mathcal{J}^r) .$$

Since the pushforward by a finite morphism is exact on the category of étale sheaves,

$$\alpha_*(q) : \alpha_* i'^*\mathcal{I} \to \alpha_*\mathcal{J}$$

is a quasiisomorphism and $\alpha_*\mathcal{J}$ is also a complex of injective sheaves on Y. Using the base change isomorphism we may view $q' := \alpha_*(q)$ as a quasiisomorphism

$i^*\beta_*\mathcal{I} \to \alpha_*\mathcal{J}$, and hence we may use it to compute the second map in the sequence of the lemma. Since the diagram

$$\begin{tikzcd}
\Gamma(Y', i'^*\mathcal{J}^j) \arrow[r, "q"] \arrow[d] & \Gamma(Y', \mathcal{J}^j) \arrow[d] \\
\Gamma(Y, \alpha_* i'^*\mathcal{J}^j) \arrow[r, "q'"] & \Gamma(Y, \alpha_*\mathcal{J}^j),
\end{tikzcd}$$

where the vertical maps are the canonical isomorphisms, commutes for all j, the lemma then follows from the first step of the proof. □

2. Proof of the theorem

We now give the proof of Theorem 1.

If we apply the formalism of nearby cycles to our quasireductive group scheme

$$\pi : \mathcal{G} \to \operatorname{Spec} R$$

(which is of course not proper), with geometric generic fibre $\mathcal{G}_{\overline{K}}$, special fibre $\mathcal{G}_0$, and

$$\mathcal{F} = (\mathbb{Z}/\ell^n\mathbb{Z})_{\mathcal{G}},$$

where ℓ is invertible in R, then from (1-1) we obtain homomorphisms

$$H^i_{\mathrm{et}}(\mathcal{G}_{\overline{K}}, \mathbb{Z}/\ell^n\mathbb{Z}) \to H^i_{\mathrm{et}}(\mathcal{G}_0, R\psi_R(\mathbb{Z}/\ell^n\mathbb{Z})), \tag{2-1}$$

$$H^i_{\mathrm{et}}(\mathcal{G}_0, \mathbb{Z}/\ell^n\mathbb{Z}) \to H^i_{\mathrm{et}}(\mathcal{G}_0, R\psi_R(\mathbb{Z}/\ell^n\mathbb{Z})). \tag{2-2}$$

These are equivariant for the action of the inertia $\mathrm{Gal}(\overline{K}/K)$, where the action on $H^i_{\mathrm{et}}(\mathcal{G}_0, Z/\ell^n\mathbb{Z})$ is trivial.

Thus, Theorem 1 follows from:

Proposition 8. *With the above notation*, *the maps in* (2-1), (2-2) *are* isomorphisms, *for any n.*

We first consider the situation of a smooth reductive group scheme.

Lemma 9. *Let R be a complete DVR with algebraically closed residue field. Let*

$$\varphi : \mathcal{H} \to \operatorname{Spec} R$$

be a smooth, *reductive group scheme. Let H_0 be the closed fibre of φ, and $H_{\overline{\eta}}$ the geometric generic fibre. Then for any prime ℓ which is invertible in R*, *we have the following.*

(i) *The canonical map*

$$(\mathbb{Z}/\ell^n\mathbb{Z})_{H_0} \to R\psi_R(\mathbb{Z}/\ell^n\mathbb{Z})_{\mathcal{H}}$$

is an isomorphism.

(ii) *The canonical maps*

$$H^i_{\mathrm{et}}(H_{\bar\eta}, \mathbb{Z}/\ell^n Z) \to H^i_{\mathrm{et}}(H_0, R\psi_R(\mathbb{Z}/\ell^n\mathbb{Z})),$$
$$H^i_{\mathrm{et}}(H_0, \mathbb{Z}/\ell^n\mathbb{Z}) \to H^i_{\mathrm{et}}(H_0, R\psi_R(\mathbb{Z}/\ell^n\mathbb{Z}))$$

are isomorphisms.

Proof. Since φ is a smooth morphism, the isomorphism in (i) holds by Lemma 6.

To get the isomorphisms in (ii), consider (1-2) constructed with $X = \mathcal{H}$, $f = \varphi$, $\mathcal{F} = \mathbb{Z}/\ell^n\mathbb{Z}$:

$$\begin{tikzcd}
H^i_{\mathrm{et}}(\mathcal{H}, \mathbb{Z}/\ell^n\mathbb{Z}) \arrow[r] \arrow[d] & H^i_{\mathrm{et}}(H_{\bar\eta}, \mathbb{Z}/\ell^n\mathbb{Z}) \arrow[d] \\
H^i_{\mathrm{et}}(H_0, \mathbb{Z}/\ell^n\mathbb{Z}) \arrow[r] & H^i_{\mathrm{et}}(H_0, R\psi_R(\mathbb{Z}/\ell^n\mathbb{Z})).
\end{tikzcd}$$

We claim that the left vertical and top horizontal arrows are isomorphisms; this follow at once from [Raynaud 1968, Théorème 3.7] or [SGA 5, Exposé VII, Proposition 6.2, p. 315]. From (i), the bottom horizontal arrow is also an isomorphism, and so the right vertical arrow must be one as well. □

We now return to the case of a "general" quasireductive group scheme.

Lemma 10. *Let $\pi : \mathcal{G} \to \operatorname{Spec} R$ be a quasireductive group scheme, where R is a complete DVR with algebraically closed residue field, and ℓ a prime invertible in R. Let $\mathcal{G}_0$ be the closed fibre of π. Then the canonical map*

$$(\mathbb{Z}/\ell^n\mathbb{Z})_{\mathcal{G}_0} \to R\psi_R(\mathbb{Z}/\ell^n\mathbb{Z})_{\mathcal{G}}$$

is an isomorphism.

Proof. Combining Propositions 3.4 and 4.3 of [Prasad and Yu 2006], we see that there is a finite extension field K' of K (contained in our chosen algebraic closure $\bar K$) with the following property. Let R' be the integral closure of R in K', and set

$$\tilde{\mathcal{G}} = \text{ normalization of } \mathcal{G} \times_{\operatorname{Spec} R} \operatorname{Spec} R'.$$

Then

(i) R' is a complete DVR (with the same residue field as R),

(ii) $\widetilde{\mathcal{G}} \to \operatorname{Spec} R'$ is a smooth, reductive group scheme with connected fibres, and

(iii) the induced morphism on reduced, geometric special fibres $\widetilde{\mathcal{G}}_0 \to (\mathcal{G}_0)_{\mathrm{red}}$ is a unipotent isogeny between connected, reductive groups of the same dimension.

The propositions cited rely on a result due independently to Raynaud and Faltings, whose proof is given in [Conrad 2006].

We note that there is a commutative diagram

$$\begin{tikzcd} \widetilde{\mathscr{G}} \arrow[r] \arrow[d, "\pi'"'] & \mathscr{G} \arrow[d, "\pi"] \\ \operatorname{Spec} R' \arrow[r] & \operatorname{Spec} R. \end{tikzcd}$$

By choice, the geometric point $\overline{\eta}$ of Spec R is also a geometric point of Spec R'.

Let $\widetilde{\mathscr{G}}_1 = \widetilde{\mathscr{G}} \times_{\mathscr{G}} \mathscr{G}_0$. We may regard the special fibre $\widetilde{\mathscr{G}}_0$ of $\pi' : \widetilde{\mathscr{G}} \to \operatorname{Spec} R'$ as a closed subscheme of $\widetilde{\mathscr{G}}_1$, and in fact it is just the underlying reduced subscheme. Thus the inclusion

$$i_0 : \widetilde{\mathscr{G}}_0 \hookrightarrow \widetilde{\mathscr{G}}_1$$

induces an equivalence of categories between étale sheaves on the two schemes; under this equivalence, the constant sheaves $\mathbb{Z}/\ell^n\mathbb{Z}$ on the two schemes correspond. There is a commutative diagram

$$\begin{tikzcd} & & \mathscr{G}_{\overline{K}} \arrow[d, "j'"] \\ \widetilde{\mathscr{G}}_0 \arrow[r, "i_0"] \arrow[dr] & \widetilde{\mathscr{G}}_1 \arrow[r, "i_1"] \arrow[d, "\alpha"'] & \widetilde{\mathscr{G}} \arrow[d, "\beta"] \\ & \mathscr{G}_0 \arrow[r, "i"] & \mathscr{G} \end{tikzcd}$$

where the square is a pullback, and the inclusion $i' : \widetilde{\mathscr{G}}_0 \to \widetilde{\mathscr{G}}$ is the composition $i' = i_1 \circ i_0$.

From the definitions, we have that

$$R\psi_{R'}(\mathbb{Z}/\ell^n\mathbb{Z})_{\widetilde{\mathscr{G}}} = i'^* Rj'_*(\mathbb{Z}/\ell^n\mathbb{Z})_{\mathscr{G}_{\overline{K}}} \ .$$

From Lemma 9, this is isomorphic to the constant sheaf $\mathbb{Z}/\ell^n\mathbb{Z}$ on $\widetilde{\mathscr{G}}_0$. Hence we obtain isomorphisms

$$(\mathbb{Z}/\ell^n\mathbb{Z})_{\widetilde{\mathscr{G}}_0} \cong i'^* Rj'_*(\mathbb{Z}/\ell^n\mathbb{Z})_{\mathscr{G}_{\overline{K}}} \cong i_0^* i_1^* Rj'_*(\mathbb{Z}/\ell^n\mathbb{Z})_{\mathscr{G}_{\overline{K}}} \ .$$

This implies that

$$(\mathbb{Z}/\ell^n\mathbb{Z})_{\widetilde{\mathscr{G}}_1} \cong i_1^* Rj'_*(\mathbb{Z}/\ell^n\mathbb{Z})_{\mathscr{G}_{\overline{K}}} \ .$$

Applying $R\beta_* = \beta_*$ (since β is a finite morphism), and using the proper base-change theorem, we get isomorphisms

$$\alpha_*(\mathbb{Z}/\ell^n\mathbb{Z})_{\widetilde{\mathscr{G}}_1} \cong i^* R(\beta \circ j')_*(\mathbb{Z}/\ell^n\mathbb{Z})_{\mathscr{G}_{\overline{K}}} = R\psi_R(\mathbb{Z}/\ell^n\mathbb{Z})_{\mathscr{G}} \ .$$

Now the three arrows

$$i_0 : \widetilde{\mathcal{G}}_0 \to \widetilde{\mathcal{G}}_1, \quad \widetilde{\mathcal{G}}_0 \to (\mathcal{G}_0)_{\mathrm{red}}, \quad (\mathcal{G}_0)_{\mathrm{red}} \hookrightarrow \mathcal{G}_0$$

induce equivalences of categories between the respective categories of étale sheaves (the middle arrow is a unipotent isogeny, the other two are inclusions of underlying reduced schemes). Hence α_* also induces such an equivalence of categories, so $R\psi_R(\mathbb{Z}/\ell^n\mathbb{Z})_{\mathcal{G}}$ is isomorphic to $(\mathbb{Z}/\ell^n\mathbb{Z})_{\mathcal{G}_0}$. It follows from the description of the stalks of $R^i\psi_R(\mathbb{Z}/\ell^n\mathbb{Z})_{\mathcal{G}}$ [SGA 7 II, Exposé 12, Proposition 2.1.4] that the canonical map $(\mathbb{Z}/\ell^n\mathbb{Z})_{\mathcal{G}_0} \to R^0\psi_R(\mathbb{Z}/\ell^n\mathbb{Z})_{\mathcal{G}}$ is an injection of sheaves. Since any injection from $(\mathbb{Z}/\ell^n\mathbb{Z})_{\mathcal{G}_0}$ to itself must be an isomorphism, the lemma follows. □

In particular, we see that the map (2-2) is an isomorphism. It remains to consider the map (2-1)

$$H^i_{\mathrm{et}}(\mathcal{G}_{\overline{K}}, \mathbb{Z}/\ell^n\mathbb{Z}) \to H^i_{\mathrm{et}}(\mathcal{G}_0, R\psi_R\mathbb{Z}/\ell^n\mathbb{Z}).$$

This map is constructed as the composition

$$H^i_{\mathrm{et}}(\mathcal{G}_{\overline{K}}, \mathbb{Z}/\ell^n\mathbb{Z}) \cong H^i_{\mathrm{et}}(\mathcal{G}, Rj_*\mathbb{Z}/\ell^n\mathbb{Z}) \to H^i_{\mathrm{et}}(\mathcal{G}_0, i^*Rj_*\mathbb{Z}/\ell^n\mathbb{Z}) = H^i_{\mathrm{et}}(\mathcal{G}_0, R\psi_R\mathbb{Z}/\ell^n\mathbb{Z}).$$

Thus it suffices to show that the restriction map

$$H^i_{\mathrm{et}}(\mathcal{G}, Rj_*\mathbb{Z}/\ell^n\mathbb{Z}) \to H^i_{\mathrm{et}}(\mathcal{G}_0, i^*Rj_*\mathbb{Z}/\ell^n\mathbb{Z}) \tag{2-3}$$

is an isomorphism.

The analogous map for the group scheme $\pi' : \widetilde{\mathcal{G}} \to \operatorname{Spec} R'$ is similarly expressed as a composition

$$H^i_{\mathrm{et}}(\mathcal{G}_{\overline{K}}, \mathbb{Z}/\ell^n\mathbb{Z}) \cong H^i_{\mathrm{et}}(\widetilde{\mathcal{G}}, Rj'_*\mathbb{Z}/\ell^n\mathbb{Z}) \to H^i_{\mathrm{et}}(\widetilde{\mathcal{G}}_0, i'^*Rj'_*\mathbb{Z}/\ell^n\mathbb{Z}) = H^i_{\mathrm{et}}(\widetilde{\mathcal{G}}_0, R\psi_{R'}\mathbb{Z}/\ell^n\mathbb{Z}).$$

As seen in Lemma 9, this composition is an isomorphism.

Since $i' = i_0 \circ i_1$, where i_0 is finite, surjective and radicial, we see that the restriction map

$$H^i_{\mathrm{et}}(\widetilde{\mathcal{G}}, Rj'_*\mathbb{Z}/\ell^n\mathbb{Z}) \to H^i_{\mathrm{et}}(\widetilde{\mathcal{G}}_1, i_1^*Rj'_*\mathbb{Z}/\ell^n\mathbb{Z}) \tag{2-4}$$

is also an isomorphism. The formula $j = \beta \circ j'$, Lemma 7, the proper base change theorem and (2-4) imply that the map in (2-3) is indeed an isomorphism, thus completing the proof of the theorem.

3. An application

Gopal Prasad has given the following application of Corollary 2; we include his proof here.

Theorem 11. *Let R be a strictly Henselian DVR with algebraically closed residue field, and K be its field of fractions. Then the generic fibre $G = \mathscr{G}_K$ of any quasireductive R-scheme $\mathscr{G}$ splits over K.*

Proof. In view of [Prasad and Yu 2006, Proposition 4.4 (i)], we can assume that G is either a torus or a semisimple K-simple group. Now if G is a torus, then it follows from [SGA 3 II, Exposé X, Théorème 8.8], that $\mathscr{G}$ is a R-torus, which implies that it splits over R [SGA 3 III, Exposé XXII, Proposition 2.1], and hence the generic fibre G is a K-split torus. We assume now that G is a semisimple K-simple group. If G does not contain a normal subgroup, defined and isomorphic over the algebraic closure $\overline{K}$ of K, to SO_{2n+1} for some $n \geqslant 1$, then according to [Prasad and Yu 2006, Theorem 1.2], $\mathscr{G}$ is smooth and reductive, so again by [SGA 3 III, Exposé XXII, Proposition 2.1], $\mathscr{G}$ is split, and so its generic fibre G is K-split. On the other hand, if G contains a normal subgroup defined and isomorphic over $\overline{K}$ to SO_{2n+1} for some $n \geqslant 1$, then as SO_{2n+1} is a group of adjoint type, and G is K-simple, there exists a finite separable extension $L \subset \overline{K}$ of K, and an absolutely simple L-group H such that

(i) H is $\overline{K}$-isomorphic to SO_{2n+1}, and

(ii) $G \cong R_{L/K}(H)$; see [Borel and Tits 1965, 6.21(ii) and 6.17].

Now Corollary 2 implies that since $\mathscr{G}$ is quasireductive, $L = K$. Thus G is a K-group which is isomorphic to SO_{2n+1} over $\overline{K}$. But as K is a field of cohomological dimension $\leqslant 1$, according to a well known theorem of Steinberg [1965] (if K is imperfect, see also [Borel and Springer 1968, 8.6]), G is quasisplit over K. But as G is an absolutely simple K-group of type B_n, if it is quasisplit over K, then it is K-split. This completes the proof of the above theorem. $\square$

Remark 12. Let R and K be as in the theorem above. According to [Prasad and Yu 2006, 8.2], a quasireductive group scheme $\mathscr{G}$ is by definition a *good* quasireductive model of its generic fibre G if $\mathscr{G}(R)$ is a hyperspecial parahoric subgroup of $G(K) = \mathscr{G}(K)$. If G admits a good quasireductive model, then it is K-split, by Lemma 8.1 of the same references. Theorems 9.3–9.5 of Prasad and Yu classify all good quasireductive models of G. It is an interesting problem to determine all quasireductive models of a connected K-split reductive group G. For $G = \mathrm{SO}_{2n+1}$, all such models have been determined in Section 10 of the same article.

4. Further remarks

We briefly discuss an analogue of quasireductive group schemes wherein we replace reductive algebraic groups by abelian varieties.

Definition 13. For a scheme S, we call a group scheme $\pi : \mathscr{A} \to S$ *quasiabelian* if it is proper and flat over S and if it is an abelian scheme when restricted to an open dense subset of S.

If all residue fields of S are of characteristic zero then a quasiabelian scheme is necessarily an abelian scheme by Cartier's theorem [SGA 3 I, Exposé $\mathrm{VI_B}$, Corollaire 1.6.1].

Now suppose S is the spectrum of a DVR R with residue characteristic $p > 0$ and $\pi : \mathscr{A} \to S$ a quasiabelian scheme. The following statements are in contrast with the quasireductive case.

1. If $\mathscr{A}$ is normal then it is an abelian scheme. This follows from (i) the existence of Néron models and (ii) the fact that for any commutative group scheme $\mathscr{G}$, flat and of finite type over S, the morphism

$$[n] : \mathscr{G} \to \mathscr{G}$$

of multiplication by n, where $n \in \mathbb{Z}$ and $(n, p) = 1$, is étale. (One can use [SGA 3 I, Exposé $\mathrm{VI_A}$, p. 316, Proposition] to prove that $[n]$ is flat; it is unramified because n is a unit in R).

Since $\mathscr{A}$ is proper and its geometric special fibre contains no rational curves, it follows that the rational map from $\mathscr{A}'$, the Néron model of $\mathscr{A}$, to $\mathscr{A}$ extending the identity morphism on the generic fibres, is actually a morphism. By examining prime to p torsion (using (ii)) we deduce that the induced morphism on special fibres is dominant, which implies that $\mathscr{A}'$ is an abelian scheme. We then use Zariski's main theorem to conclude.

2. For any prime number p there exists S as above and a quasiabelian scheme over S which is *not* an abelian scheme. Such schemes can be constructed as follows: Let $\mathscr{B}'$, $\mathscr{B}$ be abelian schemes over S and $\phi : \mathscr{B}' \to \mathscr{B}$ a flat isogeny with kernel $\mathscr{K}'$. Suppose there is an abelian subscheme $\mathscr{A}'$ of $\mathscr{B}'$ such that $\mathscr{K}' \cap \mathscr{A}'$ is *not* flat over S. Then $\mathscr{A} := \phi(\mathscr{A}')$ is quasiabelian but not abelian. For any p one may easily find such data with $\mathscr{B}'$ the product of a one dimensional abelian scheme with itself.

One could generalize the definition of quasiabelian schemes by considering group schemes $\pi : \mathscr{A} \to S$ which are flat and of finite type over S, abelian over a dense open subset and such that all reduced geometric fibres are semiabelian. In this generality, we do not know if the analogue of item 1 above continues to hold (though it does if the relative dimension is one since there exist canonical regular compacifications in this case).

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