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We consider diophantine subsets of function fields of curves and show, roughly speaking, that they are either very small or very large. In particular, this implies that the ring of polynomials $k[t]$ is not a diophantine subset of the field of rational functions $k(t)$ for many fields k.

Let R be a commutative ring. A subset $D \subset R$ is called *diophantine* if there are polynomials

$$F_i(x, y_1, \dots, y_n) \in R[x, y_1, \dots, y_n]$$

such that the system of equations

$$F_i(r, y_1, \dots, y_n) = 0 \quad \forall i$$

has a solution $(y_1, \dots, y_n) \in R^n$ if and only if $r \in D$.

Equivalently, if there is a (possibly reducible) algebraic variety X_R over R and a morphism $\pi : X_R \to \mathbb{A}^1{}_R$ such that $D = \pi(X_R(R))$. In this situation we call

$$\mathrm{dioph}(X_R, \pi) := \pi(X_R(R)) \subset R$$

the *diophantine set* corresponding to X_R and π.

A characterization of diophantine subsets of $\mathbb{Z}$ was completed in connection with Hilbert's 10th problem, but a description of diophantine subsets of $\mathbb{Q}$ is still not known. In particular, it is not known if $\mathbb{Z}$ is a diophantine subset of $\mathbb{Q}$ or not. (See [Poonen 2003] or the volume [Denef et al. 2000] for surveys and many recent results.)

In this paper we consider analogous questions where $R = k(t)$ is a function field of one variable and k is an uncountable *large field* of characteristic 0. That is, for any k-variety Y with a smooth k-point, $Y(k)$ is Zariski dense. Examples of uncountable large fields are

(1) $\mathbb{C}$ or any uncountable algebraically closed field,

(2) $\mathbb{R}$ or any uncountable real closed field, and

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(3) $\mathbb{Q}_p$, $\mathbb{Q}((x))$ or the quotient field of any uncountable local Henselian domain.

Roughly speaking, we show that for such fields, a diophantine subset of $k(t)$ is either very small or very large. The precise result is somewhat technical, but here are two easy-to-state consequences which serve as motivating examples.

Corollary 1. *Let k be an uncountable large field of characteristic 0. Then $k[t]$ is not a diophantine subset of $k(t)$.*

Corollary 2. *Let k be an uncountable large field of characteristic 0 and*

$$K_2 \supset K_1 \supset k(t)$$

finite field extensions. Then K_1 is a diophantine subset of K_2 if and only if $K_1 = K_2$.

The latter gives a partial answer to a question of Bogomolov: When is a subfield $K_1 \subset K_2$ diophantine in K_2?

It is possible that both of these corollaries hold for any field k. Unfortunately, my method says nothing about countable fields. The geometric parts of the proof for Propositions 12 and 13 (see 16 and 20) work for any uncountable field, but the last step 23 uses in an essential way that k is large.

We use two ways to measure how large a diophantine set is.

3 (Diophantine dimension and polar sets). Let B be a smooth, projective, irreducible curve over k. One can think of a rational function $f \in k(B)$ as a section of the first projection $\pi_1 : B \times \mathbb{P}^1 \to B$. This establishes a one-to-one correspondence

$$k(B) \cup \{\infty\} \quad \leftrightarrow \quad \{\text{sections of } \pi_1 : B \times \mathbb{P}^1 \to B\}.$$

Any section $\sigma : B \to B \times \mathbb{P}^1$ can be identified with its image, which gives a point in the Chow variety of curves of $B \times \mathbb{P}^1$. This gives an injection

$$k(B) \cup \{\infty\} \quad \hookrightarrow \quad \mathrm{Chow}_1(B \times \mathbb{P}^1).$$

Let U be a countable (disjoint) union of k-varieties and $D \subset U(k)$ a subset. Define the *diophantine dimension* of D over k as the smallest $n \in \{-1, 0, 1, \dots, \infty\}$ such that D is contained in a countable union of irreducible k-subvarieties of U of dimension $\le n$. It is denoted by $\text{d-dim}_k D$. Note that $\text{d-dim}_k D = -1$ if and only if $D = \emptyset$ and $\text{d-dim}_k D \le 0$ if and only if D is countable.

In particular, we can talk about the diophantine dimension of

$$\mathrm{dioph}(X, f) \subset k(B) \subset \mathrm{Chow}_1(B \times \mathbb{P}^1).$$

For $f \in k(B)$, let $\mathrm{pole}(f)$ denote its divisor of poles. For $D \subset k(B)$, set

$$\mathrm{Pole}_n(D) := \{\mathrm{pole}(f) : f \in D \text{ and } \deg \mathrm{pole}(f) = n\}.$$

I think of $\mathrm{Pole}_n(D)$ as a subset of the $\bar{k}$-points of the n-th symmetric power $S^n B$.

Taking each point with multiplicity $r \geq 1$ gives embeddings $S^m B \to S^{rm} B$, whose image I denote by $r \cdot S^m B$.

With these definitions, the main result is the following theorem illustrating the "very small or very large" dichotomy.

Theorem 4. *Let k be an uncountable large field of characteristic* 0 *and B a smooth, projective, irreducible curve over k. Let $X_{k(B)}$ be a (possibly reducible) algebraic variety of dimension n over $k(B)$ and $\pi_{k(B)} : X_{k(B)} \to \mathbb{A}^1{}_{k(B)}$ a morphism. Then either*

(1) d-dim$_k$ dioph$(X_{k(B)}, \pi_{k(B)}) \leq n$, *or*

(2) d-dim$_k$ dioph$(X_{k(B)}, \pi_{k(B)}) = \infty$ *and there is a* 0*-cycle $P_a \in S^a B$ and $r > 0$ such that for every $m > 0$ there is a smooth, irreducible k-variety D_m and a morphism $\rho_m : D_m \to S^{a+rm} B$ such that*

   (a) $D_m(k) \neq \varnothing$,

   (b) $\mathrm{Pole}_{a+rm}\big(\mathrm{dioph}(X_{k(B)}, \pi_{k(B)})\big) \supset \rho_m(D_m(k))$, *and*

   (c) *the Zariski closure of $\rho_m(D_m(k))$ contains $P_a + r \cdot S^m B \subset S^{a+rm} B$.*

5 (Proof of the Corollaries). In trying to write a subset $D \subset k(B)$ as

$$D = \mathrm{dioph}(X_{k(B)}, \pi_{k(B)}),$$

we do not have an a priori bound on $\dim X_{k(B)}$; thus the assertion

$$\text{d-dim}_k\, \mathrm{dioph}(X_{k(B)}, \pi_{k(B)}) = \infty$$

is hard to use. The Corollaries 1 and 2 both follow from the more precise results about the distribution of poles.

If $B = \mathbb{P}^1$, then a rational function with at least 2 poles on $\mathbb{P}^1$ is not a polynomial; thus Theorem 4 implies Corollary 1.

Next consider Corollary 2. Let $K_1 = k(B_1) \subsetneq K_2 = k(B_2)$ be a degree $d > 1$ extension of function fields of smooth, projective, irreducible k-curves. By Riemann–Roch, any zero cycle of degree $\geq 2g(B_1)$ defined over k is the polar set of some $f \in k(B_1)$. Pulling back gives a map $j : S^m B_1 \to S^{md} B_2$; thus

$$\mathrm{Pole}_n(K_1) = \begin{cases} j((S^m B_1)(k)) & \text{if } n = md \geq 2dg(B_1), \text{ and} \\ \varnothing \quad \text{if } d \nmid n. \end{cases}$$

If $b_1 \neq b_2 \in B_2$ map to the same point of B_1, then a 0-cycle in $j(S^m B_1)$ contains either both b_1 and b_2 or neither. Thus the Zariski closed set $j(S^m B_1)$ never contains a set of the form $P_a + r \cdot S^m B_2$. By (2.c) of Theorem 4, this shows that K_1 is not diophantine in K_2, proving Corollary 2.

Example 6. (1) The bound n in (1) of Theorem 4 is actually sharp, as shown by the following.

Note first that any $k(t)$-solution of $x^3+y^3=1$ is constant. Set

$$X_n := (x_1^3+y_1^3=\cdots=x_n^3+y_n^3=1)\subset\mathbb{A}^{2n}$$

and

$$\pi:(x_1,y_1,\ldots,x_n,y_n)\mapsto x_1+x_2t+\cdots+x_nt^{n-1}.$$

Then $\dim X=n$ and for $k=\mathbb{C}$ or $k=\mathbb{R}$, $\operatorname{dioph}(X_n,\pi)$ is the set of all degree $\le n-1$ polynomials.

Using similar constructions one can see that any (finite dimensional) k-algebraic subset of $k(t)$ is diophantine when k is algebraically closed or real closed. These are the "small" diophantine subsets of $k(t)$.

(2) The somewhat unusual looking condition about the Zariski closure of D_m in (2.c) of Theorem 4 is also close to being optimal. For $g\in k(t)$ and $r>0$ consider the diophantine set

$$L_{g,r} := \{f\in k(t):\exists\, h \text{ such that } f=gh^r\}.$$

Then, up to some lower dimensional contribution coming from possible cancellations between poles and zeros of g and h^r,

$$\operatorname{Pole}_n(L_{g,r})=\begin{cases}\operatorname{pole}(g)+r\cdot(S^mB)(k) & \text{if } n=\deg\operatorname{pole}(g)+rm, \text{ and}\\ \varnothing \quad \text{otherwise.}\end{cases}$$

7. If $k=\mathbb{C}$ then our proof shows that in case (2) of Theorem 4 there is a finite set $P\subset B(\mathbb{C})$ such that for every $p\in B(\mathbb{C})\setminus P$ there is an $f_p\in\operatorname{dioph}(X,\pi)$ with a pole at p.

If $k=\mathbb{R}$, then we guarantee many poles, but one may not get any real poles. To get examples, note that $h\in\mathbb{R}(t)$ is everywhere nonnegative on $\mathbb{R}$ if and only if h is a sum of two squares. Thus for any $g\in\mathbb{R}(t)$, the set

$$L_1(g) := \{f\in\mathbb{R}(t): f(t)\le g(t)\ \forall t\in\mathbb{R}\}$$

is diophantine. $L_1(g)$ is infinite dimensional but if $g\in\mathbb{R}[t]$ then no element of $L_1(g)$ has a real pole.

From the point of view of our proof a more interesting example is the diophantine set

$$L_2(g) := \{f\in\mathbb{R}(t):\exists\, c\in\mathbb{R},\ f^2(t)\le c^2\cdot g^2(t)\ \forall t\in\mathbb{R}\}.$$

The elements of $L_2(g)$ are unbounded everywhere yet no element of $L_2(g)$ has a pole in $\mathbb{R}$ if g is a polynomial.

This leads to the following question.

Question 8. Is $\mathbb{R}[t]_{\mathbb{R}}$, the set of all rational functions without poles in $\mathbb{R}$, diophantine?

There should be some even stronger variants of the "very small or very large" dichotomy, especially over $\mathbb{C}$. As a representative case, I propose the following.

Conjecture 9. *Let $D \subset \mathbb{C}(t)$ be a diophantine subset which contains a Zariski open subset of $\mathbb{C}[t]$. (Meaning, for instance, that D contains a Zariski open subset of the space of degree $\leq n$ polynomials for infinitely many n.) Then $\mathbb{C}(t) \setminus D$ is finite.*

In connection with Bogomolov's question, I would hazard the following:

Conjecture 10. *Let k be a large field and $K_1 \subset K_2$ function fields of k-varieties. Then K_1 is diophantine in K_2 if and only if K_1 is algebraically closed in K_2.*

11. The proof of Theorem 4 relies on the theory of rational curves on algebraic varieties. A standard reference is [Kollár 1996], but nonexperts may prefer the more introductory [Araujo and Kollár 2003].

The proof is divided into three steps.

First we show that if $\text{d-dim}_k \operatorname{dioph}(X_{k(B)}, \pi) \geq n+1$ then there is a rationally connected (see 18) subvariety $Z_{k(B)} \subset X_{k(B)}$ such that $\pi|_Z$ is nonconstant and $Z_{k(B)}$ has a smooth $k(B)$-point. This relies on the *bend and break* method of [Mori 1979]. In a similar context it was first used in [Graber et al. 2005].

Then we show, using the *deformation of combs* technique developed in [Kollár et al. 1992; Kollár 1996; 2004; Graber et al. 2003], that for any such $Z_{k(B)}$, there are infinitely many k-varieties S_m and maps $S_m \times B \dashrightarrow Z_{k(B)}$ which give injections $S_m(k) \hookrightarrow Z_{k(B)}(k(B))$.

Both of these steps are geometric, but the statements are formulated to work over an arbitrary field L.

Finally, if k is a large field, then each $S_m(k)$ is "large", which shows that $Z_{k(B)}(k(B))$ is "very large".

For all three steps it is better to replace $\pi : X_{k(B)} \to \mathbb{A}^1{}_{k(B)}$ with a morphism of k-varieties $f : X \to B \times \mathbb{P}^1$.

Proposition 12. *Let L be any field and B a smooth, projective, irreducible curve over L. Let $f : X \to B \times \mathbb{P}^1$ be an L-variety of dimension $n+1$ and consider the corresponding diophantine set* $\operatorname{dioph}(X_{L(B)}, f) \subset L(B)$. *Then either*

(1) $\text{d-dim}_L \operatorname{dioph}(X_{L(B)}, f) \leq n$, *or*

(2) *there is a subvariety $Z \subset X$ such that*
   (a) *$Z \to B \times \mathbb{P}^1$ is dominant,*
   (b) *the generic fiber of $Z \to B$ is rationally connected, and*
   (c) *there is a rational section $\sigma : B \dashrightarrow Z$ whose image is not contained in* $\operatorname{Sing} Z$.

Proposition 13. *Let L be an infinite field and B a smooth, projective, irreducible curve over L. Let $f : Z \to B \times \mathbb{P}^1$ be a smooth, projective L-variety such that*

(1) $Z \to B \times \mathbb{P}^1$ *is dominant,*

(2) *the generic fiber of $Z \to B$ is separably rationally connected, and*

(3) *there is a section $\sigma : B \to Z$.*

Then, for some $r > 0$ and for all $m > 0$ in an arithmetic progression, there are

(4) *a smooth, irreducible L-variety S_m with an L-point, and*

(5) *a dominant rational map $\sigma_m : S_m \times B \dashrightarrow Z$ which commutes with projection to B,*

such that the Zariski closure of the image of $f \circ \sigma_m : S_m \dashrightarrow \mathrm{Chow}_1(B \times \mathbb{P}^1)$ contains

$$[f \circ \sigma(B)] + r[\{b_1\} \times \mathbb{P}^1] + \cdots + r[\{b_m\} \times \mathbb{P}^1] \quad \textit{for every} \quad b_i \in B(\bar{L}).$$

14 (Spaces of sections). Let L be any field, B a smooth, projective, irreducible curve over L and $f : X \to B$ a projective morphism. A section of f (defined over some $L' \supset L$) can be identified with the corresponding L'-point in the Chow variety of 1-cycles $\mathrm{Chow}_1(X)$. All sections $\Sigma(X/B)$ defined over $\bar{L}$ form an open set of $\mathrm{Chow}_1(X)$. Indeed, if H is an ample line bundle on B of degree d then a 1-cycle C is a section if and only if C is irreducible (an open condition) and $(C \cdot f^*H) = d$ (an open and closed condition). This procedure realizes $X_{k(B)}(k(B))$ as the set of k-points of a countable union of algebraic k-varieties $\Sigma(X/B) = \bigcup_i \Sigma_i$.

The choice of the Σ_i is not canonical. Given $X \to B$, we get "natural" irreducible components, but for fixed generic fiber $X_{k(B)}$, these components depend on the choice of X. Any representation gives, however, the same constructible sets. We usually make a further decomposition. Since every variety is a finite set-theoretic union of locally closed smooth subvarieties, we may choose the Σ_i such that each one is smooth and irreducible.

As an explicit example, consider $B = \mathbb{P}^1$. Then $k(B) \cong k(t)$ and every $f \in k(t)$ can be uniquely written (up to scalars) as

$$f = \frac{a_0 + a_1 t + \cdots + a_n t^n}{b_0 + b_1 t + \cdots + b_n t^n},$$

where the nominator and the denominator are relatively prime and at least one of a_n or b_n is nonzero. For any n, all such f form an open subset

$$\Sigma_n \subset \mathbb{P}(a_0 : a_1 : \cdots : a_n : b_0 : b_1 : \cdots : b_n) \cong \mathbb{P}^{2n+1}.$$

15 (Very dense subsets). Let U be an irreducible variety over a field L. We say that a subset $D \subset U(\bar{L})$ is *Zariski very dense* if D is not contained in a countable union of L-subvarieties $V_i \subsetneq U$.

It is easy to see that for any D, there are countably many closed, irreducible L-subvarieties $W_i \subset U$ such that $D \subset \bigcup_i W_i(\bar{L})$ and $D \cap W_i(\bar{L})$ is Zariski very dense in W_i for every i. There is a unique irredundant choice of these W_i.

16 (Proof of Proposition 12). Write $X = \bigcup X_i$ as a finite set-theoretic union of locally closed, smooth, connected varieties. If Proposition 12 holds for each X_i then it also holds for X; thus we may assume that X is smooth and irreducible. Let $X' \supset X$ be a smooth compactification such that f extends to $f' : X' \to B \times \mathbb{P}^1$.

As before, there are countably many disjoint, irreducible, smooth L-varieties $\bigcup_i \Sigma_i = \Sigma(X'/B)$ and morphisms $u_i : B \times \Sigma_i \to X'$ commuting with projection to B giving all $\bar{L}$-sections of f'. As in 15, there are countably many disjoint, irreducible, smooth L-varieties $S_i \subset \Sigma(X'/B)$ such that each $S_i(L)$ is Zariski very dense in S_i and the L-sections of $X' \to B$ are exactly given by $\bigcup_i S_i(L)$.

Composing u_i with f', we obtain maps

$$f'_* : S_i \to \Sigma \subset \mathrm{Chow}_1(B \times \mathbb{P}^1).$$

There are 2 distinct possibilities. Either

(1) $\dim_L f'_*(S_i) \leq n$ whenever $u_i(B \times S_i) \cap X \neq \varnothing$, or

(2) there is an i_0 such that $\dim_L f'_*(S_{i_0}) \geq n+1$ and $u_{i_0}(B \times S_{i_0}) \cap X \neq \varnothing$.

In the first case $\mathrm{dioph}(X', f')$ is contained in the union of the constructible sets $f'_*(S_i)$, thus we have (1) of Proposition 12. This is always the case if L is countable.

In the second case we construct Z as required by (2) of Proposition 12 using only the existence of $u_{i_0} : B \times S_{i_0} \to X$. Set $S := S_{i_0}$ and $u := u_{i_0}$. We can replace X' by a desingularization of the closure of the image $u(B \times S)$. By shrinking S we may assume that u lifts to $u : B \times S \to X'$.

For a point $x \in X'$ let $S_x \subset S$ be the subvariety parametrizing those sections that pass through x.

Let us now fix $b \in B(\bar{L})$ such that $u(\{b\} \times S)$ is dense in X'_b and let x run through X'_b, the fiber of X' over b. Since every section intersects X'_b,

$$S = \bigcup_{x \in X'_b} S_x \quad \text{and so} \quad f'_*(S) = \bigcup_{x \in X'_b} f'_*(S_x).$$

By assumption $\dim_L f'_*(S) \geq n+1$ and $\dim_L X'_b = n$; hence $\dim_L f'_*(S_x) \geq 1$ for general $x \in X'_b(\bar{L})$. In particular, there is a 1-parameter family of sections $C_x \subset S_x$ such that

$$f' \circ u : B \times C \to X \to B \times \mathbb{P}^1$$

is a nonconstant family of sections passing through the point $f'(x)$.

By 17, this leads to a limit 1-cycle of the form

$$A + \{b\} \times \mathbb{P}^1 + (\text{other fibers of } \pi_1)$$

where A is a section of $\pi_1 : B \times \mathbb{P}^1 \to B$.

Correspondingly, we get a limit 1-cycle in X' of the form

$$A_x + R_x + (\text{other rational curves})$$

where A_x is a section of $X' \to B$ which dominates A and R_x is a connected union of rational curves which dominates $\{b\} \times \mathbb{P}^1$. Note also that $x \in R_x$.

Thus we conclude that for general $x \in X'_b(\bar{L})$, there is a connected union of rational curves $x \in R_x \subset X'_b$ which dominates $\{b\} \times \mathbb{P}^1$.

As in 19, let us take the relative MRC-fibration $f' : X' \overset{w}{\dashrightarrow} W' \dashrightarrow B$.

For very general $x \in X'(\bar{L})$ let X'_x be the fiber of w containing x. By 19, X'_x is closed in X' and every rational curve in X' that intersects X'_x is contained in X'_x. In particular, $R_x \subset X'_x$ and hence X'_x dominates $\{b\} \times \mathbb{P}^1$.

Let now $p \in S(L)$ be a general point and $C \subset X'$ the corresponding section. By assumption $S(L)$ is Zariski dense in S; hence we may assume that w is smooth at the generic point of C. Let $Z' \subset w^{-1}(w(C))$ be the unique irreducible component that dominates C and $Z = Z' \cap X$. It satisfies all the required properties. □

17 (Bend-and-break for sections [Mori 1979; Kollár 1996, § II.5; Graber et al. 2005, Lemma 3.2]). Let $h : Y \to B$ be a proper morphism onto a smooth projective curve B. Let C be a smooth curve and $u : B \times C \to Y$ a nonconstant family of sections passing through a fixed point $y \in Y$.

Then C can not be a proper curve and for a suitable point $c \in \overline{C} \setminus C$ the corresponding limit 1-cycle is of the form

$$\Sigma_y = A_y + R_y,$$

where A_y is a section of h (which need not pass through y) and R_y is a nonempty union of rational curves contained in finitely many fibers of h. Furthermore, $A_y + R_y$ is connected and $y \in R_y$.

This holds whether we take the limit in the Chow variety of 1-cycles, in the Hilbert scheme or in the space of stable maps.

18 (Rationally connected varieties [Kollár et al. 1992; Kollár 1996, Chapter IV; Araujo and Kollár 2003, Section 7]). Let k be a field and $K \supset k$ an uncountable algebraically closed field. A smooth projective k-variety X is called *rationally connected* or *RC* if for every pair of points $x_1, x_2 \in X(K)$ there is a K-morphism $f : \mathbb{P}^1 \to X$ such that $f(0) = x_1$ and $f(\infty) = x_2$. X is called *separably rationally connected* or *SRC* if for every point $x \in X(K)$ there is a K-morphism $f : \mathbb{P}^1 \to X$

such that $f(0) = x$ and f^*T_X is an ample vector bundle. (That is, a sum of positive degree line bundles.) Furthermore, $f : \mathbb{P}^1 \to X$ can be taken to be an embedding if $\dim X \geq 3$. It is known that SRC implies RC and the two notions are equivalent in characteristic 0.

We may not have any rational curves over k, but we can work with the universal family of these maps $f : \mathbb{P}^1 \to X$. Thus, if $\dim X \geq 3$ and $p \in X$ is a k-point, then there is an irreducible, smooth k-variety U and a k-morphism $G : U \times \mathbb{P}^1 \to X$ such that

(1) $G(U \times \{0\}) = p$,

(2) $G_u : \{u\} \times \mathbb{P}^1 \to X_{\bar{k}}$ is an embedding for every $u \in U(\bar{k})$, and

(3) ${G^*}_u T_X$ is ample for every $u \in U(\bar{k})$.

By [Kollár 1999, Theorem 1.4], if k is large then we can choose U such that $U(k) \neq \emptyset$.

19 (MRC fibrations [Kollár et al. 1992; Kollár 1996, § IV.5]). Let $K \supset k$ be as above. Let X be a smooth projective k-variety and $g : X \to S$ a k-morphism. There is a unique (up to birational maps) factorization

$$g : X \overset{w}{\dashrightarrow} W \overset{h}{\dashrightarrow} S$$

such that

(1) for general $p \in W(K)$, the fiber $w^{-1}(p)$ is closed in X and rationally connected, and

(2) for very general $p \in W(K)$ (that is, for p in a countable intersection of dense open subsets) every rational curve in $X(K)$ which intersects $w^{-1}(p)$ and maps to a point in S is contained in $w^{-1}(p)$.

The map $w : X \dashrightarrow W$ is called the (relative) *maximal rationally connected fibration* or *MRC fibration* of $X \to S$. Note that if X contains very few rational curves (for example, if X is an Abelian variety or a K3 surface) then $X = W$.

20 (Proof of Proposition 13). Here we essentially reverse the procedure of the first part. Instead of degenerating a 1-parameter family of sections to get a 1-cycle consisting of a section plus rational curves, we start with a section, add to it suitably chosen rational curves and prove that this 1-cycle can be written as the limit of sections in many different ways.

We assume that Z is smooth, projective. If necessary, we take its product with $\mathbb{P}^3$ to achieve that $\dim Z \geq 4$. This changes the space of sections $\Sigma(Z/B)$ but it does not change the image of $\Sigma(Z/B)$ in $L(B)$.

Apply 18 to $X = Z_{L(B)}$ and the point $p = \sigma(B)$ to get

$$G : U_{L(B)} \times \mathbb{P}^1 \to Z_{L(B)}.$$

Next replace $U_{L(B)}$ by an L-variety $\tau : U \to B$ such that G extends to

$$g : U \times \mathbb{P}^1 \to Z.$$

By shrinking U if necessary, we may assume that for general $b \in B(\bar{L})$, the corresponding

$$g_b : U_b \times \mathbb{P}^1 \to Z_b$$

is a family of smooth rational curves passing through $\sigma(b)$ and ${g^*}_{b,u} T_{Z_b}$ is ample for every $u \in U_b$ where $g_{b,u}$ is the restriction of g_b to $\{u\} \times \mathbb{P}^1$.

Given distinct points $b_1, \ldots, b_m \in B(\bar{L})$, let $B(b_1, \ldots, b_m)$ be the comb assembled from B and m copies of $\mathbb{P}^1$ where we attach $\mathbb{P}^1_i$ to B at b_i (see 21).

By [Kollár 2004, Theorem 16], there are $b_1, \ldots, b_{m_0} \in B(\bar{L})$ and an embedding

$$\sigma(g_1, \ldots, g_{m_0}) : B(b_1, \ldots, b_{m_0}) \to Z$$

given by σ on B and by $g_i := g_{b_i,u_i}$ on $\mathbb{P}^1_i$ for some $u_i \in U_{b_i}$ such that the image, denoted by $B(g_1, \ldots, g_{m_0}) \subset Z$ is defined over L and its normal bundle is as positive as one wants. In particular, by 22, $B(g_1, \ldots, g_{m_0})$ gives a smooth point of the Hilbert scheme of Z. Furthermore, for any further distinct points $b_{m_0+1}, \ldots, b_m$ and g_i for $i = m_0 + 1, \ldots, m$, the resulting

$$\sigma(g_1, \ldots, g_m) : B(b_1, \ldots, b_m) \to Z$$

also gives a smooth point of the Hilbert scheme of Z.

Let S_m denote the smooth locus of the corresponding L-irreducible component of the Hilbert scheme of Z. $B(g_1, \ldots, g_m)$ gives an L-point of S_m; hence S_m is geometrically irreducible. By 22 the general point of S_m corresponds to a section of f, the universal family is a product over a dense open subset of S_m and we have a dominant rational map $\sigma_m : S_m \times B \dashrightarrow Z$.

For a given m, it is not always possible to choose $b_{m_0+1}, \ldots, b_m$ such that the set $b_1, \ldots, b_m$ is defined over L. To achieve this, choose a generically finite and dominant map $\rho : U \dashrightarrow {\mathbb{A}_L}^{\dim U}$. For general $c \in \mathbb{A}^{\dim U}(L)$, its preimage $\rho^{-1}(c)$ gives $\deg \rho$ general points in U which are defined over L. Thus we can choose $b_1, \ldots, b_m$ to be defined over L whenever $m - m_0$ is a multiple of $\deg \rho$.

Let us now consider $f_*(S_m) \subset \mathrm{Chow}_1(B \times \mathbb{P}^1)$. By construction, it contains

$$f_*(B(g_1, \ldots, g_m)) = A + r(\{b_1\} \times \mathbb{P}^1) + \cdots + r(\{b_m\} \times \mathbb{P}^1)$$

where $A = f \circ \sigma(B)$ is a section of $B \times \mathbb{P}^1$ and $r \geq 1$ is the common (geometric) degree of the maps

$$f \circ g_{b,u} : \{u\} \times \mathbb{P}^1 \to \{b\} \times \mathbb{P}^1 \subset B \times \mathbb{P}^1.$$

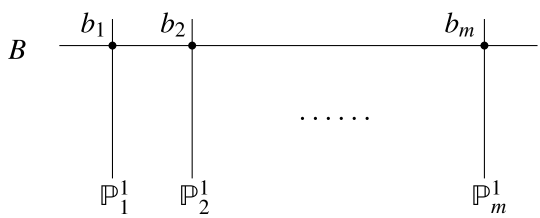
The combs $B(g_1, \dots, g_m) \subset Z$ have some obvious deformations where we keep B fixed and vary the points b_i and the maps g_i with them. By construction, the points b_i can vary independently. The images of these 1-cycles in $B \times \mathbb{P}^1$ are of the form

$$A + r(\{b_1'\} \times \mathbb{P}^1) + \cdots + r(\{b_m'\} \times \mathbb{P}^1),$$

where the b_i' vary independently. $\square$

21 (Combs). A *comb* assembled from B and m copies of $\mathbb{P}^1$ attached at the points b_i is a curve $B(b_1, \dots, b_m)$ obtained from B and $\{b_1, \dots, b_m\} \times \mathbb{P}^1$ by identifying the points $b_i \in B$ and $(b_i, 0) \in \{b_i\} \times \mathbb{P}^1$.

A comb (with m teeth) can be pictured as below:



22 (Deformation of reducible curves). Let X be a smooth projective variety and $C \subset X$ a connected curve with ideal sheaf I_C. Assume that C has only nodes as singularities. By the smoothness criterion of the Hilbert scheme [Grothendieck 1962, exposé 221, p. 21], if $H^1(C, (I_C/I_C{}^2)^*) = 0$ then $[C]$ is a smooth point of the Hilbert scheme $\mathrm{Hilb}(X)$ and there is a unique irreducible component $H_C \subset \mathrm{Hilb}(X)$ containing $[C]$. Let $U_C \to H_C$ be the universal family.

If, in addition, $(I_C/I_C{}^2)^*$ is generated by global sections, then a general point of H_C corresponds to a smooth curve and the natural map $U_C \to X$ is dominant.

23 (Proof of Theorem 4). Let us start with the $k(B)$-variety $X_{k(B)}$. We can write it as the generic fiber of a quasiprojective k-variety $X \to B$ and extend $\pi_{k(B)}$ to $f : X \to B \times \mathbb{P}^1$. If (1) of Theorem 4 fails then using Proposition 12 we obtain $Z \subset X$. Take a compactification $\bar{Z}$ and a resolution $Z_1 \to \bar{Z}$ such that the composite map $Z_1 \dashrightarrow B \times \mathbb{P}^1$ is a morphism. Next apply Proposition 13 to $Z_1 \to B \times \mathbb{P}^1$. We obtain, for every m in an arithmetic progression, a dominant family of sections $\sigma_m : S_m \times B \to Z_1$.

There is a dense open subset $D_m \subset S_m$ such that for every $q \in D_m(\bar{k})$,

(1) the corresponding section $\sigma_m(\{q\} \times B) \subset Z_1$ intersects Z, and

(2) the corresponding rational function $f \circ \sigma_m : \{q\} \times B \to \mathbb{P}^1$ has exactly $a + rm$ poles where a is the number of poles of $f \circ \sigma(B)$.

Thus the composite map

$$\rho_m := \mathrm{pole} \circ f \circ \sigma_m : D_m \to S^{a+rm} B$$

is defined. The condition (2.b) of Theorem 4 holds by construction and the Zariski closure of $\rho_m(D_m)$ contains $P_a + r \cdot S^m B \subset S^{a+rm} B$ by (5) of Proposition 13, where P_a denotes the polar divisor of the section σ, that is, the 0-cycle

$$f \circ \sigma(B) \cap (B \times \{\infty\}).$$

S_m has a smooth k-point by Proposition 13 and k-points are Zariski dense since k is a large field. Thus $D_m(k)$ is Zariski dense in D_m. This implies both (2.a) and (2.c) of Theorem 4.

Finally, $D_m(k)$ is Zariski very dense in D_m by Lemma 24 and

$$\text{d-dim}_k \operatorname{dioph}(X, \pi) \geq \text{d-dim}_k \rho_m(D_m(k)) = \dim_k \rho_m(D_m) \geq \dim_k S^m B = m,$$

where the middle equality holds by Lemma 24. Thus $\text{d-dim}_k \operatorname{dioph}(X, \pi) = \infty$.

The only remaining issue is that our m runs through an arithmetic progression and is not arbitrary. If the progression is $m = b + m'c$ then

$$a + r(b + m'c) = (a + rb) + (rc)m',$$

so by changing $a \mapsto a + rb$, $r \mapsto rc$ we get Theorem 4. □

Lemma 24. *Let X be a smooth and irreducible variety over a large field k such that $X(k) \neq \varnothing$. Then $X(k)$ is not contained in the union of fewer than $|k|$ subvarieties $X_\lambda \subsetneq X$. In particular, if k is uncountable then $X(k)$ is Zariski very dense in X.*

Proof. Assume to the contrary that $X(k) \neq \varnothing$ but $X(k) \subset \bigcup_{\lambda \in \Lambda} X_\lambda$ where $|\Lambda| < |k|$ and $X_\lambda \neq X$.

If $\dim X \geq 2$, then pick $p \in X(k)$ and let $\{H_t : t \in \mathbb{P}^1_k\}$ be a general pencil of hypersurface sections of X passing through p. Since $|\Lambda| < |k|$, there is an H_t such that H_t is smooth at p and $H_t \not\subset X_\lambda$ for every λ. Thus $H_t(k) \subset \bigcup_{\lambda \in \Lambda} (H_t \cap X_\lambda)$ is a lower dimensional counterexample. Thus it is enough to prove Lemma 24 when X is a curve. Then lower dimensional k-subvarieties are just points, thus we need to show that $|X(k)| = |k|$.

Let $\{H_t : t \in \mathbb{P}^1\}$ be a linear system on $X \times X$ which has (p, p) as a base point and whose general member is smooth at (p, p). Since k is large, each H_t contains a k-point different from (p, p). Thus $|X(k)| = |X(k) \times X(k)| \geq |k|$. □

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