On some crystalline representations of $\text{GL}_2(\mathbb{Q}_p)$

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We show that the universal unitary completion of certain locally algebraic representation of \( G := \text{GL}_2(\mathbb{Q}_p) \) with \( p > 2 \) is nonzero, topologically irreducible, admissible and corresponds to a 2-dimensional crystalline representation with nonsemisimple Frobenius via the \( p \)-adic Langlands correspondence for \( G \).

1. Introduction

Let \( G := \text{GL}_2(\mathbb{Q}_p) \) and \( B \) be the subgroup of upper-triangular matrices in \( G \). Let \( L \) be a finite extension of \( \mathbb{Q}_p \).

**Theorem 1.1.** Assume that \( p > 2 \), let \( k \geq 2 \) be an integer and let \( \chi : \mathbb{Q}_p^\times \to L^\times \) be a smooth character with \( \chi(p^2) p^{k-1} \in \mathfrak{o}_L^\times \). Assume there exists a \( G \)-invariant norm \( \| \cdot \| \) on \( \text{Ind}_G^B \chi \otimes \chi | \cdot |^{-1} \otimes \text{Sym}^{k-2} L^2 \). Then the completion \( E \) is a topologically irreducible, admissible Banach space representation of \( G \). If we let \( E^0 \) be the unit ball in \( E \), then

\[
V_{k,\chi(p)^{-1}} \otimes (\chi | \cdot |) \cong L \otimes_{\mathfrak{o}_L} \lim_{\leftarrow} V(E^0/\sigma_L^r E^0),
\]

where \( V \) is Colmez’s Montreal functor and \( V_{k,\chi(p)^{-1}} \) is a 2-dimensional irreducible crystalline representation of \( \mathfrak{g}_{\mathbb{Q}_p} \), the absolute Galois group of \( \mathbb{Q}_p \), with Hodge–Tate weights \( (0, k-1) \) and the trace of crystalline Frobenius equal to \( 2 \chi(p)^{-1} \).

As we explain in Section 5, the existence of such \( G \)-invariant norm follows from [Colmez 2008]. Our result addresses [Berger and Breuil 2007, remarque 5.3.5]. In other words, the completion \( E \) fits into the \( p \)-adic Langlands correspondence for \( \text{GL}_2(\mathbb{Q}_p) \).

The idea is to approximate \( (\text{Ind}_G^B \chi \otimes \chi | \cdot |^{-1}) \otimes \text{Sym}^{k-2} L^2 \) with representations \( (\text{Ind}_B^G \chi \delta_x \otimes \chi \delta_x^{-1} | \cdot |^{-1}) \otimes \text{Sym}^{k-2} L^2 \), where \( \delta_x : \mathbb{Q}_p^\times \to L^\times \) is an unramified character with \( \delta_x(p) = x \in 1 + p_L \). If \( x \neq 1 \), then \( \chi \delta_x \neq \chi \delta_x^{-1} \) and the analogue of Theorem 1.1 is a result of Berger and Breuil [2007]. This allows to deduce admissibility. This approximation process relies on the results of [Vigneras 2008].

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Using Colmez’s functor $V$, we may then transfer the question of irreducibility to the Galois side. Here, we use the fact that for $p > 2$ the representation $V_{k, \pm 2p^{(k-1)/2}}$ sits in the $p$-adic family studied by Berger, Li and Zhu [2004].

2. Notation

We fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. We let val be the valuation on $\overline{\mathbb{Q}}_p$ such that $\text{val}(p) = 1$, and we set $|x| := p^{-\text{val}(x)}$. Let $L$ be a finite extension of $\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}}_p$, let $\sigma_L$ be the ring of integers of $L$, let $\sigma_L$ be a uniformizer, and let $p_L$ be the maximal ideal of $\sigma_L$. Given a character $\chi : Q_p^\times \to L^\times$, we consider $\chi$ as a character of the absolute Galois group $\mathcal{G}_{Q_p} \cong \mathbb{Q}_p$ via the local class field theory by sending the geometric Frobenius to $p$.

Let $G := \text{GL}_2(\mathbb{Q}_p)$, and let $B$ be the subgroup of upper-triangular matrices. Given two characters $\chi_1, \chi_2 : Q_p^\times \to L^\times$, we consider $\chi_1 \otimes \chi_2$ as a character of $B$ sending a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $\chi_1(a) \chi_2(d)$. Let $Z$ be the centre of $G$. Define

$$K := \text{GL}_2(\mathbb{Z}_p), \quad K_m := \begin{pmatrix} 1 + p^m \mathbb{Z}_p & p^m \mathbb{Z}_p \\ p^m \mathbb{Z}_p & 1 + p^m \mathbb{Z}_p \end{pmatrix} \quad \text{for } m \geq 1,$$

$$I := \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p \mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}, \quad I_m := \begin{pmatrix} 1 + p^m \mathbb{Z}_p & p^{m-1} \mathbb{Z}_p \\ p^m \mathbb{Z}_p & 1 + p^m \mathbb{Z}_p \end{pmatrix} \quad \text{for } m \geq 1.$$

Let $\mathcal{R}_0$ be the $G$-normalizer of $K$, so that $\mathcal{R}_0 = KZ$, and let $\mathcal{R}_1$ be the $G$-normalizer of $I$, so that $\mathcal{R}_1$ is generated as a group by $I$ and $\Pi := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We note that if $m \geq 1$, then $K_m$ is normal in $\mathcal{R}_0$ and $I_m$ is normal in $\mathcal{R}_1$. We denote $s := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

3. Diagrams

Let $R$ be a commutative ring, (typically $R = L$, $\sigma_L$ or $\sigma_L/p_L^\alpha$). By a diagram $D$ of $R$-modules, we mean the data $(D_0, D_1, r)$, where $D_0$ is an $R[\mathcal{R}_0]$-module, $D_1$ is an $R[\mathcal{R}_1]$-module and $r : D_1 \to D_0$ is a $\mathcal{R}_0 \cap \mathcal{R}_1 = IZ$-equivariant homomorphism of $R$-modules. A morphism $\alpha$ between two diagrams $D$ and $D'$ is given by $(\alpha_0, \alpha_1)$, where $\alpha_0 : D_0 \to D'_0$ is a morphism of $R[\mathcal{R}_0]$-modules, $\alpha_1 : D_1 \to D'_1$ is a morphism of $R[\mathcal{R}_1]$-modules, and the diagram

$$D_0 \xrightarrow{\alpha_0} D'_0$$

$$r \uparrow \quad \uparrow r'$$

$$D_1 \xrightarrow{\alpha_1} D'_1$$

commutes in the category of $R[IZ]$-modules. The condition (1) is important, since one can have two diagrams of $R$-modules $D$ and $D'$, such that $D_0 \cong D'_0$ as $R[\mathcal{R}_0]$-modules and $D_1 \cong D'_1$ as $R[\mathcal{R}_1]$-modules, but $D \not\cong D'$ as diagrams.
of $R$-modules with the above morphisms form an abelian category. To a diagram $D$ one may associate a complex

$$c\text{-Ind}_{\mathfrak{g}_i}^G D_1 \otimes \delta \xrightarrow{\hat{\partial}} c\text{-Ind}_{\mathfrak{g}_0}^G D_0$$

(2)

of $G$-representations, where $\delta : \mathfrak{g}_1 \to R^\times$ is the character $\delta(g) := (-1)^{\text{val}(\det(g))}$; $c\text{-Ind}_{\mathfrak{g}_i}^G D_1$ denotes the space of functions $f : G \to D_1$ such that $f(kg) = kf(g)$ for $k \in \mathfrak{g}_i$ and $g \in G$, and $f$ is supported only on finitely many cosets $\mathfrak{g}_i \cdot g$. To describe $\hat{\partial}$, we note that Frobenius reciprocity gives

$$\text{Hom}_G(c\text{-Ind}_{\mathfrak{g}_i}^G D_1 \otimes \delta, c\text{-Ind}_{\mathfrak{g}_0}^G D_0) \cong \text{Hom}_{\mathfrak{g}_i}(D_1 \otimes \delta, c\text{-Ind}_{\mathfrak{g}_0}^G D_0);$$

now $\text{Ind}_{IZ}^G D_0$ is a direct summand of the restriction of $c\text{-Ind}_{\mathfrak{g}_0}^G D_0$ to $\mathfrak{g}_i$, and

$$\text{Hom}_{\mathfrak{g}_i}(D_1 \otimes \delta, \text{Ind}_{IZ}^G D_0) \cong \text{Hom}_{IZ}(D_1, D_0),$$

since $\delta$ is trivial on $IZ$. Composition of the maps above yields a map

$$\text{Hom}_{IZ}(D_1, D_0) \to \text{Hom}_G(c\text{-Ind}_{\mathfrak{g}_i}^G D_1 \otimes \delta, c\text{-Ind}_{\mathfrak{g}_0}^G D_0).$$

We let $\hat{\partial}$ be the image of $r$. We define $H_0(D)$ to be the cokernel of $\hat{\partial}$ and $H_1(D)$ to be the kernel of $\hat{\partial}$. So we have this exact sequence of $G$-representations:

$$0 \to H_1(D) \to c\text{-Ind}_{\mathfrak{g}_i}^G D_1 \otimes \delta \xrightarrow{\hat{\partial}} c\text{-Ind}_{\mathfrak{g}_0}^G D_0 \to H_0(D) \to 0 \tag{3}$$

Further, if $r$ is injective then one may show that $H_1(D) = 0$; see [Vignéras 2008, Proposition 0.1]. To a diagram $D$ one may associate a $G$-equivariant coefficient system $\mathcal{V}$ of $R$-modules on the Bruhat–Tits tree; see [Paškūnas 2004, Section 5]. Then $H_0(D)$ and $H_1(D)$ compute the homology of the coefficient system $\mathcal{V}$, and the map $\hat{\partial}$ has a natural interpretation. Assume that $R = L$ (or any field of characteristic 0), and let $\pi$ be a smooth irreducible representation of $G$ on an $L$-vector space, so that for all $\nu \in \pi$ the subgroup $\{g \in G : gv = \nu\}$ is open in $G$. Since the action of $G$ is smooth, there exists an $m \geq 0$ such that $\pi^{m+1} \neq 0$. To $\pi$ we may associate a diagram $D := (\pi^m \hookrightarrow \pi^K)$. As a very special case of a result by Schneider and Stuhler [1997, Theorem V.1; 1993, Section 3], we obtain that $H_0(D) \cong \pi$.

We are going to compute such diagrams $D$, attached to smooth principal series representations of $G$ on $L$-vector spaces. Given smooth characters $\theta_1, \theta_2 : \mathbb{Z}_p^\times \to L^\times$ and $\lambda_1, \lambda_2 \in L^\times$, we define a diagram $D(\lambda_1, \lambda_2, \theta_1, \theta_2)$ as follows. Let $c \geq 1$ be an integer such that $\theta_1$ and $\theta_2$ are trivial on $1 + p^c \mathbb{Z}_p$. Set $J_c := (K \cap B) K_c = (I \cap B) K_c$, so that $J_c$ is a subgroup of $I$. Let $\theta : J_c \to L^\times$ be the character $\theta(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) := \theta_1(a) \theta_2(d)$. Let $D_0 := \text{Ind}_F^K \theta$, and let $p \in \mathbb{Z}$ act on $D_0$ by a scalar $\lambda_1 \lambda_2$, so that $D_0$ is a representation of $\mathfrak{g}_0$. Set $D_1 := D_0^{\lambda_1 \lambda_2}$, so that $D_1$ is naturally a representation of $IZ$. 

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We are going to put an action of $\Pi$ on $D_1$, so that $D_1$ is a representation of $\mathcal{R}_1$. Let

$$V_1 := \{ f \in D_1 : \text{Supp} f \subseteq I \}, \quad V_s := \{ f \in D_1 : \text{Supp} f \subseteq J_s I \}. \quad (4)$$

Since $I$ contains $K_1$, we have $J_{s1} I = (B \cap K) s I = Is I$; hence $D_1 = V_1 \oplus V_s$. For all $f_1 \in V_1$ and $f_s \in V_s$, we define $\Pi \cdot f_1 \in V_s$ and $\Pi \cdot f_s \in V_1$ such that

$$[\Pi \cdot f_1](sg) := \lambda_1 f_1(\Pi^{-1} g \Pi), \quad [\Pi \cdot f_s](g) := \lambda_2 f_s(s \Pi g \Pi^{-1}) \quad \text{for all } g \in I. \quad (5)$$

Every $f \in D_1$ can be written uniquely as $f = f_1 + f_s$, with $f_1 \in V_1$ and $f_s \in V_s$, and we define $\Pi \cdot f := \Pi \cdot f_1 + \Pi \cdot f_s$.

**Lemma 3.1.** Equation (5) defines an action of $\mathcal{R}_1$ on $D_1$. We denote the diagram $D_1 \hookrightarrow D_0$ by $D(\lambda_1, \lambda_2, \theta_1, \theta_2)$. Let $\pi := \text{Ind}_{K}^{G} \chi_1 \otimes \chi_2$ be a smooth principal series representation of $G$, with

$$\chi_1(p) = \lambda_1, \quad \chi_2(p) = \lambda_2, \quad \chi_1(z_p) = \theta_1, \quad \chi_2(z_p) = \theta_2.$$  

There exists an isomorphism of diagrams $D(\lambda_1, \lambda_2, \theta_1, \theta_2) \cong (\pi^I \hookrightarrow \pi^K)$. In particular, we have a $G$-equivariant isomorphism $H_0(D(\lambda_1, \lambda_2, \theta_1, \theta_2)) \cong \pi$.

**Proof.** We note that $p \in Z$ acts on $\pi$ by a scalar $\lambda_1 \lambda_2$. Since $G = BK$, we have $\pi|_K \cong \text{Ind}_{B \cap K}^{K} \theta$, and so the map $f \mapsto [g \mapsto f(g)]$ induces an isomorphism $t_0 : \pi^K \cong \text{Ind}_{B}^{G} \theta = D_0$. Let

$$\mathcal{F}_1 := \{ f \in \pi : \text{Supp} f \subseteq BI \} \quad \text{and} \quad \mathcal{F}_s := \{ f \in \pi : \text{Supp} f \subseteq Bs I \}.$$  

Iwasawa decomposition gives $G = BI \cup Bs I$; hence $\pi = \mathcal{F}_1 \oplus \mathcal{F}_s$. If $f_1 \in \mathcal{F}_1$, then $\text{Supp}(\Pi f_1) = (\text{Supp} f_1) \Pi^{-1} \subseteq BI \Pi^{-1} = Bs I$. Moreover,

$$[\Pi f_1](sg) = f_1(s g \Pi) = f_1(s \Pi (\Pi^{-1} g \Pi)) = \chi_1(p) f_1(\Pi^{-1} g \Pi) \quad \text{for all } g \in I. \quad (6)$$

Similarly, if $f_s \in \mathcal{F}_s$, then $\text{Supp}(\Pi f_s) = (\text{Supp} f_s) \Pi^{-1} \subseteq Bs I \Pi^{-1} = BI$, and

$$[\Pi f_s](g) = f_s(g \Pi) = f_1((\Pi s) s \Pi^{-1} g \Pi)) = \chi_2(p) f_s(\Pi^{-1} g \Pi) \quad \text{for all } g \in I. \quad (7)$$

Now $\pi^I = \mathcal{F}_1^I \oplus \mathcal{F}_s^I \subseteq \pi^K$. Let $t_1$ be the restriction of $t_0$ to $\pi^I$. Then it is immediate that $t_1(\mathcal{F}_1^I) = V_1$ and $t_1(\mathcal{F}_s^I) = V_s$, where $V_1$ and $V_s$ are as above. Moreover, if $f \in D_1$ and $\Pi \cdot f$ is given by (5), then $\Pi \cdot f = t_1(\Pi r_1^{-1}(f))$. Since $\mathcal{R}_1$ acts on $\pi^I$, Equation (5) defines an action of $\mathcal{R}_1$ on $D_1$ such that $t_1$ is $\mathcal{R}_1$-equivariant. Hence, $(t_0, t_1)$ is an isomorphism of diagrams $(\pi^I \hookrightarrow \pi^K) \cong (D_1 \hookrightarrow D_0). \quad \square$
4. The main result

**Lemma 4.1.** Let $U$ be a finite dimensional $L$-vector space with subspaces $U_1, U_2$ such that $U = U_1 \oplus U_2$. For $x \in L$ define a map $\phi_x : U \to U$ by $\phi_x(v_1 + v_2) = xv_1 + v_2$ for all $v_1 \in U_1$ and $v_2 \in U_2$. Let $M$ be an $\omega_L$-lattice in $V$. Then there exists an integer $a \geq 1$ such that $\phi_x(M) = M$ for $x \in 1 + p^a L$.

**Proof.** Let $N$ denote the image of $M$ in $U/U_2$. Then $N$ contains $(M \cap U_1) + U_2$, and both are lattices in $U/U_2$. Define $a \geq 1$ to be the smallest integer such that $p^a(M \cap U_1) + U_2$ contains $N$. Suppose that $x \in 1 + p^a$ and $v \in M$. We may write $v = \lambda v_1 + v_2$, with $v_1 \in M \cap U_1$, $v_2 \in U_2$ and $\lambda \in p^{-a}$. Now $\phi_x(v) = v + \lambda(v_1 - 1)v_1 \in M$. Hence we get $\phi_x(M) \subseteq M$ and $\phi_x^{-1}(M) \subseteq M$. Applying $\phi_x^{-1}$ to the first inclusion gives $M \subseteq \phi_x^{-1}(M)$. \hfill $\square$

We fix an integer $k \geq 2$ and set $W := \text{Sym}^{k-2} L^2$, an algebraic representation of $G$. Let $\pi := \pi(\chi_1, \chi_2) := \text{Ind}_H^G \chi_1 \otimes \chi_2$ be a smooth principal series $L$-representation of $G$. We say that $\pi \otimes W$ admits a $G$-invariant norm if there exists a norm $\| \cdot \|$ on $\pi \otimes W$ with respect to which $\pi \otimes W$ is a normed $L$-vector space such that $\|gv\| = \|v\|$ for all $v \in \pi \otimes W$ and $g \in G$.

Let $c \geq 1$ be an integer such that both $\chi_1$ and $\chi_2$ are trivial on $1 + pcZ_p$. Let $D$ be the diagram $\pi^{L_1} \otimes W \hookrightarrow \pi^{K_1} \otimes W$. Since $H_0(\pi^{L_1} \hookrightarrow \pi^{K_1}) \cong \pi$, by tensoring (2) with $W$ we obtain $H_0(D) \cong \pi \otimes W$. Assume that $\pi \otimes W$ admits a $G$-invariant norm $\| \cdot \|$, and set $(\pi \otimes W)^0 := \{v \in \pi \otimes W : \|v\| \leq 1\}$. Then we may define a diagram $\mathcal{D} = (\mathcal{D}_1 \hookrightarrow \mathcal{D}_0)$ of $\sigma_L$-modules by

$$\mathcal{D}_0 := ((\pi^{L_1} \otimes W) \cap (\pi \otimes W)^0 \hookrightarrow (\pi^{K_1} \otimes W) \cap (\pi \otimes W)^0).$$

In this case Vignéras [2008] has shown that the inclusion $\mathcal{D} \hookrightarrow D$ induces a $G$-equivariant injection $H_0(\mathcal{D}) \hookrightarrow H_0(D)$ such that $H_0(\mathcal{D}) \otimes_{\sigma_L} L = H_0(D)$ and $H_1(\mathcal{D}) = 0$. Moreover, $H_0(\mathcal{D})$ does not contain an $\omega_L$-submodule isomorphic to $L$; see [Vignéras 2008, Proposition 0.1]. Since $H_0(D)$ is an $L$-vector space of countable dimension, this implies that $H_0(\mathcal{D})$ is a free $\omega_L$-module. By tensoring (2) with $\sigma_L/p^a_L$, we obtain

$$H_0(\mathcal{D}) \otimes_{\omega_L} \sigma_L/p^a_L \cong H_0(\mathcal{D} \otimes_{\omega_L} \sigma_L/p^a_L). \quad (8)$$

**Proposition 4.2.** Let $\pi := \pi(\chi_1, \chi_2)$ be a smooth principal series representation, assume that $\pi \otimes W$ admits a $G$-invariant norm, and let $\mathcal{D}$ be as above. Then there exists an integer $a \geq 1$ such that for all $x \in 1 + p^a$, with $b \geq a$, there exists both a finitely generated $\omega_L(G)$-module $M$ in $\pi(\chi_1 \delta_{x^{-1}}, \chi_2 \delta_x) \otimes W$ that is free as an $\omega_L$-module, and a $G$-equivariant isomorphism

$$M \otimes_{\omega_L} \sigma_L/p^b_L \cong H_0(\mathcal{D}) \otimes_{\omega_L} \sigma_L/p^b_L,$$

where $\delta_x : \mathbb{Q}_p^\times \to L^\times$ is an unramified character with $\delta_x(p) = x$. 

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Proof. Apply Lemma 4.1 to $U = D_1$, $U = V_1 \otimes W$, $U_2 = V_1 \otimes W$ and $M = \mathcal{D}_1$, where $V_1$ and $V_2$ are given by (4). We get an integer $a \geq 1$ such that $\phi(x_{\mathcal{D}_1}) = \mathcal{D}_1$ for all $x \in 1 + p_L^a$. It is immediate that $\phi_x$ is $IZ$-equivariant. We define a new action $\ast$ of $\Pi$ on $D_1$ by $\Pi \ast v := \phi_x(\Pi \phi_x^{-1}(v))$. This gives us a new diagram $D(x)$, so that $D(x) = D_0$ as a representation of $\mathfrak{h}_0$. $D(x)_1 = D_1$ as a representation of $IZ$, the $IZ$-equivariant injection $D(x)_1 \hookrightarrow D(x)_0$ is equal to the $IZ$-equivariant injection $D_1 \hookrightarrow D_0$, but the action of $\Pi$ on $D_1$ is given by $\ast$, (here by $= \mathcal{D}_1$). Hence, we have an isomorphism of diagrams $D(x) \cong D(x)$, $D(x)_1 \cong D(x)_1$, which $\Pi$ acts on $D(x)_1$ by $\ast$, then the diagram $D(x) := (D(x)_1 \hookrightarrow D(x)_0)$ is an integral structure in $D(x)$ in the sense of [Vignéras 2008]. The results of Vignéras cited above imply that $M := H_0(D(x))$ is a finitely generated $\mathfrak{o}_L[G]$-submodule of $\pi(\chi_1, \cdot, \chi_2, \cdot) \otimes W$, which is free as an $\mathfrak{o}_L$-module, and $M \otimes \mathfrak{o}_L \cong \pi(\chi_1, \cdot, \chi_2, \cdot) \otimes W$. Moreover, since $\phi_x$ is the identity modulo $p_L^a$, we have $\Pi \ast v = \Pi \cdot v \pmod{\mathfrak{p}^b \mathcal{D}_1}$ for all $v \in \mathcal{D}_1$, and so the identity map $\mathcal{D}(x)_0 \hookrightarrow \mathcal{D}_0$ induces an isomorphism of diagrams $\mathcal{D}(x) \otimes \mathfrak{o}_L / \mathfrak{p}_L^b \cong \mathcal{D} \otimes \mathfrak{o}_L / \mathfrak{p}_L^b$ (9) gives $H_0(\mathcal{D}) \otimes \mathfrak{o}_L / \mathfrak{p}_L^b \cong M \otimes \mathfrak{o}_L / \mathfrak{p}_L^b$.

Let $k \geq 2$ be an integer and $a_p \in \mathfrak{p}_L$. Following [Breuil 2003] we define a filtered $\mathfrak{p}$-module $D_{k,a_p}$ as the following data: a 2-dimensional $L$-vector space $D$ with basis $\{e_1, e_2\}$, an $L$-linear automorphism $\varphi : D \rightarrow D$ given by

$$
\varphi(e_1) = p^{k-1}e_2 \quad \text{and} \quad \varphi(e_2) = -e_1 + a_pe_2,
$$

and a decreasing filtration $(\text{Fil}_i D)_{i \in \mathbb{Z}}$ by $L$-subspaces such that if $i \leq 0$ then $\text{Fil}_i D = D$, if $1 \leq i \leq k - 1$ then $\text{Fil}_i D = L e_1$, and if $i \geq k$ then $\text{Fil}_i D = 0$. We set $V_{k,a_p} := \text{Hom}_{\varphi, \text{Fil}}(D_{k,a_p}, B_{cris})$. Then $V_{k,a_p}$ is a 2-dimensional $L$-linear absolutely irreducible crystalline representation of $\mathfrak{g}_{\text{Q}_p} := \text{Gal}(\overline{\mathbb{Q}}_p / \mathbb{Q}_p)$ with Hodge–Tate weights 0 and $k - 1$. We denote by $\chi_{k,a_p}$ the trace character of $V_{k,a_p}$. Since $\mathfrak{g}_{\text{Q}_p}$ is
compact and the action is continuous, $\mathcal{G}_{\mathbb{Q}_p}$ stabilizes some $\mathfrak{o}_L$-lattice in $V_{k,a_p}$, and so $\chi_{k,a_p}$ takes values in $\mathfrak{o}_L$.

**Proposition 4.3.** Let $m$ be the largest integer such that $m \leq (k - 2)/(p - 1)$. Let $a_p, a'_p \in p_L$, and assume that $\text{val}(a_p) > m$ and $\text{val}(a'_p) > m$. Let $n \geq em$ be an integer, where $e := e(L/\mathbb{Q}_p)$ is the ramification index. Suppose $a_p \equiv a'_p \pmod{p^n_L}$. Then $\chi_{k,a_p}(g) \equiv \chi_{k,a'_p}(g) \pmod{p^{n-em}_L}$ for all $g \in \mathcal{G}_{\mathbb{Q}_p}$.

**Proof.** This a consequence of a result of Berger, Li and Zhu [Berger et al. 2004], where the authors construct $\mathcal{G}_{\mathbb{Q}_p}$-invariant lattices $T_{k,a_p}$ in $V_{k,a_p}$. The assumption $a_p \equiv a'_p \pmod{p^n_L}$ implies $T_{k,a_p} \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/p^{n-em}_L \cong T_{k,a'_p} \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/p^{n-em}_L$, see their [Remark 4.1.2(2)]. This implies the congruences of characters.

Let $k \geq 2$ be an integer and choose $\lambda_1, \lambda_2 \in L$ such that $\lambda_1 + \lambda_2 = a_p$ and $\lambda_1 \lambda_2 = p^{k-1}$ (enlarge $L$ if necessary). Assume $\text{val}(\lambda_1) \geq \text{val}(\lambda_2) > 0$. Let $\chi_1, \chi_2 : \mathbb{Q}_p^\times \to L^\times$ be unramified characters, with $\chi_1(p) = \lambda_1^{-1}$ and $\chi_2(p) = \lambda_2^{-1}$. Let $M$ be a finitely generated $\mathfrak{o}_L[G]$-module in $π(\chi_1, \chi_2 \cdot | - |^{-1}) \otimes W$, where $W := \text{Sym}^{k-2} L^2$. In the case $\lambda_1 \neq \lambda_2$, Berger and Breuil have shown that the unitary $L$-Banach space representation

$$E_{k,a_p} := L \otimes_{\mathfrak{o}_L} \underset{\leftarrow}{\lim} M/\mathfrak{m}^n_L M$$

of $G$ is nonzero, topologically irreducible, admissible in the sense of [Schneider and Teitelbaum 2002], and contains $π(\chi_1, \chi_2 \cdot | - |^{-1}) \otimes W$ as a dense $G$-invariant subspace [Berger and Breuil 2007, Section 5.3]. Moreover, the dual of $E_{k,a_p}$ is isomorphic to the representation of Borel subgroup $B$ constructed from the $(\varphi, \Gamma)$-module of $V_{k,a_p}$.

Let $\text{Rep}_{\varphi} G$ be the category of finite length $\mathfrak{o}_L[G]$-modules with a central character such that the action of $G$ is smooth (that is, the stabilizer of a vector is an open subgroup of $G$). Let $\text{Rep}_{\varphi, \mathcal{G}_{\mathbb{Q}_p}}$ be the category of continuous representations of $\mathcal{G}_{\mathbb{Q}_p}$ on $\mathfrak{o}_L$-modules of finite length. Colmez [2008, IV.2.14] has defined an exact covariant functor $V : \text{Rep}_{\varphi} G \to \text{Rep}_{\varphi, \mathcal{G}_{\mathbb{Q}_p}}$. The constructions in [Berger and Breuil 2007] and [Colmez 2008] are mutually inverse to one another. This means if we assume $\lambda_1 \neq \lambda_2$ and let $M$ be as above, then

$$V_{k,a_p} \cong L \otimes_{\mathfrak{o}_L} \underset{\leftarrow}{\lim} V(M/\mathfrak{m}^n_L M).$$

That $M/\mathfrak{m}^n_L M$ is an $\mathfrak{o}_L[G]$-module of finite length follows from [Berger 2005, Theorem A].

**Theorem 4.4.** Assume that $p > 2$. Let $\lambda = \pm p^{(k-1)/2}$, and let $\chi : \mathbb{Q}_p^\times \to L^\times$ be a smooth character with $\chi(p) = \lambda^{-1}$. Assume there exists a $G$-invariant norm $\| \cdot \|$ on $π(\chi, \chi \cdot | - |^{-1}) \otimes W$, where $W := \text{Sym}^{k-2} L^2$. Let $E$ be the completion of $π(\chi, \chi \cdot | - |^{-1}) \otimes W$ with respect to $\| \cdot \|$. Then $E$ is a nonzero, topologically
irreducible, admissible Banach space representation of $G$. If we let $E^0$ be the unit ball in $E$, then $V_{k,2}(\chi|\chi|) \cong \mathbb{L} \otimes_{\mathcal{O}_E} \varprojlim V(E^0/\varpi^n E^0)$.

**Proof.** Since the character $\chi|\chi|$ is integral, by twisting we may assume that $\chi$ is unramified. We denote the diagram

$$\pi(\chi, \chi|\chi|^{-1}) \otimes W \leftrightarrow \pi(\chi, \chi|\chi|^{-1})^{G_1} \otimes W$$

by $D = (D_1 \hookrightarrow D_0)$. Let $\mathcal{D} = (\mathcal{D}_1 \hookrightarrow \mathcal{D}_0)$ be the diagram of $\mathcal{O}_L$-modules with $\mathcal{D}_1 = D_1 \cap E^0$ and $\mathcal{D}_0 = D_0 \cap E^0$. Let $a \geq 1$ be the integer Proposition 4.2 gives. For each $j \geq 0$, we fix $x_j \in 1 + p_L^{a+j}$ with $x_j \neq 1$ and a finitely generated $\mathcal{O}_L[G]$-submodule $M_j$ in $\pi(\delta_{x_j^{-1}}, \delta_{x_j}|\cdot|^{-1}) \otimes W$ (which is then a free $\mathcal{O}_L$-module) such that

$$H_0(\mathcal{D}) \otimes_{\mathcal{O}_L} \mathcal{O}_L/p_L^{a+j} \cong M_j \otimes_{\mathcal{O}_L} \mathcal{O}_L/p_L^{a+j}.$$ 

This is possible by Proposition 4.2. To ease the notation we set $M := H_0(\mathcal{D})$. Let $a_p(j) := \lambda x_j^{-1} + \lambda x_j$, let $a_p := 2\lambda$, and let $m$ be the largest integer such that $m \leq (k-2)/(p-1)$. Since $p > 2$, $x_j + x_j^{-1}$ is a unit in $\mathcal{O}_L$, we have $\text{val}(a_p(j)) = \text{val}(a_p) = (k-1)/2 > m$. (Here we really need $p > 2$.) Moreover, we have $a_p \equiv a_p(j) \pmod{p_L^{j+a+em}}$, where $e := e(L/\mathbb{Q}_p)$ is the ramification index. Now since $x_j \neq 1$ we get that $\lambda x_j \neq \lambda x_j^{-1}$, and hence we may apply the results of Berger and Breuil to $\pi(\delta_{x_j^{-1}}, \delta_{x_j}|\cdot|^{-1}) \otimes W$. By (11),

$$T_{k,a_p(j)} := \varprojlim V(M_j/\varpi^n M_j)$$

is a $\mathcal{O}_{\mathbb{Q}_p}$-invariant lattice in $V_{k,a_p(j)}$. Since $M \otimes_{\mathcal{O}_L} \mathcal{O}_L/p_L^{a+j} \cong M_j \otimes_{\mathcal{O}_L} \mathcal{O}_L/p_L^{a+j}$ we get

$$V(M/\varpi^n M) \cong V(M_j/\varpi^n M_j) \cong T_{k,a_p(j)} \otimes_{\mathcal{O}_L} \mathcal{O}_L/p_L^{a+j}.$$ (12)

Set $V := L \otimes_{\mathcal{O}_L} \varprojlim V(M/\varpi^n M)$. Then (12) implies that $V$ is a 2-dimensional $L$-vector space. Let $\chi_V$ be the trace character of $V$. Then it follows from (12) that $\chi_V \equiv \chi_{k,a_p(j)} \pmod{p_L^{a+j}}$. Since $a_p \equiv a_p(j) \pmod{p_L^{a+j+em}}$, Proposition 4.3 says that $\chi_{k,a_p} \equiv \chi_{k,a_p(j)} \pmod{p_L^{a+j}}$. We obtain $\chi_V \equiv \chi_{k,a_p} \pmod{p_L^{a+j}}$ for all $j \geq 0$. This gives us $\chi_V = \chi_{k,a_p}$. Since $V_{k,a_p}$ is irreducible, the equality of characters implies $V \cong V_{k,a_p}$.

Set $\tilde{M} := \varprojlim M/\varpi^n M$, and $E' := \tilde{M} \otimes_{\mathcal{O}_L} L$. Since $M$ is a free $\mathcal{O}_L$-module, we get an injection $M \hookrightarrow \tilde{M}$. In particular, $E'$ contains $\pi(\chi, \chi|\cdot|^{-1}) \otimes W$ as a dense $G$-invariant subspace. We claim that $E'$ is a topologically irreducible and admissible $G$-representation. Now Theorem 4.1.1 and Proposition 4.1.4 of [Berger et al. 2004] say that the semisimplification of $T_{k,a_p(j)} \otimes_{\mathcal{O}_L} k_L$ is irreducible if $p+1 \nmid k-1$ and is otherwise isomorphic to
\[
\left(\mu_{\sqrt{-1}}, 0 \right) \otimes \omega^{(k-1)/(p+1)},
\]
where \(\mu_{\pm \sqrt{-1}}\) is the unramified character sending arithmetic Frobenius to \(\pm \sqrt{-1}\), and \(\omega\) is the cyclotomic character. Then [Berger 2005, Theorem A] implies that if \(p+1 \mid k-1\), then \(M_j \otimes_{\omega_L} k_L\) is an irreducible supersingular representation of \(G\), and if \(p+1 \nmid k-1\), then the semisimplification of \(M_j \otimes_{\omega_L} k_L\) is a direct sum of two irreducible principal series. The irreducibility of principal series follows from [Barthel and Livné 1994, Theorem 33], since \(\sqrt{-1} \neq \pm 1\), as \(p > 2\). Since \(M \otimes_{\omega_L} k_L \cong M_j \otimes_{\omega_L} k_L\), we get that \(M \otimes_{\omega_L} k_L\) is an admissible representation of \(G\) (so that for every open subgroup \(U\) of \(G\), the space of \(U\)-invariants is finite dimensional). This implies that \(E'\) is admissible.

Suppose that \(E_1\) is a closed \(G\)-invariant subspace of \(E'\) with \(E' \neq E_1\). Let \(E_1^0 := E_1 \cap \hat{M}\). We obtain a \(G\)-equivariant injection \(E_1^0 \otimes_{\omega_L} k_L \hookrightarrow M \otimes_{\omega_L} k_L\). If \(E_1^0 \otimes_{\omega_L} k_L = 0\) or \(M \otimes_{\omega_L} k_L\), then Nakayama’s lemma gives \(E_1^0 = 0\) or \(E_1 = \hat{M}\), respectively. If \(p+1 \mid k-1\), then \(M \otimes_{\omega_L} k_L\) is irreducible and we are done. If \(p+1 \nmid k-1\), then \(E_1^0 \otimes_{\omega_L} k_L\) is an irreducible principal series, and so \(V(E_1^0 \otimes_{\omega_L} k_L)\) is one-dimensional [Colmez 2008, IV.4.17]. But then \(V_1 := L \otimes_{\omega_L} \lim \to V(E_1^0 / \sigma_j^\infty E_1^0)\) is a 1-dimensional subspace of \(V_{k,a,p}\) stable under the action of \(\omega_{\hat{M},p}\). Since \(V_{k,a,p}\) is irreducible we obtain a contradiction.

Since \(E'\) is a completion of \(\pi(\chi, \chi | \cdot |^{-1}) \otimes W\) with respect to a finitely generated \(\omega_L[G]\)-submodule, \(E'\) is in fact the universal completion; see for example [Emerton 2005, Proposition 1.17]. In particular, we obtain a nonzero \(G\)-equivariant map of \(L\)-Banach space representations \(E' \to E\), but since \(E'\) is irreducible and \(\pi(\chi, \chi | \cdot |^{-1}) \otimes W\) is dense in \(E\), this map is an isomorphism. \(\square\)

**Corollary 4.5.** Assume that \(p > 2\), and let \(\chi : \mathbb{Q}_p^\times \to L^\times\) be a smooth character such that \(\chi(p)^2 p^{k-1} = 1\). Assume that there is a \(G\)-invariant norm \(\| \cdot \|\) on \(\pi(\chi, \chi | \cdot |^{-1}) \otimes W\), where \(W := \text{Sym}^{k-2} L^2\). Then every bounded \(G\)-invariant \(\omega_L\)-lattice in \(\pi(\chi, \chi | \cdot |^{-1}) \otimes W\) is finitely generated as an \(\omega_L[G]\)-module.

**Proof.** The existence of a \(G\)-invariant norm implies that the universal completion is nonzero. It follows from Theorem 4.4 that the universal completion is topologically irreducible and admissible. The assertion follows from the proof of [Berger and Breuil 2007, Corollary 5.3.4]. \(\square\)

For the purposes of [Paškūnas 2008] we record the following corollary to the proof of Theorem 4.4.

**Corollary 4.6.** Assume \(p > 2\), and let \(\chi : \mathbb{Q}_p^\times \to L^\times\) be a smooth character such that \(\chi^2(p)^k = 1\) is a unit in \(\omega_L\). Assume there exists a unitary \(L\)-Banach space representation \((E, \| \cdot \|)\) of \(G\) containing \((\text{Ind}_E^G \chi \otimes \chi | \cdot |^{-1}) \otimes \text{Sym}^{k-2} L^2\) as a dense \(G\)-invariant subspace and satisfying \(\|E\| \subseteq |L|\). Then there exists \(x \in 1 + p_L\)
with \( x^2 \neq 1 \) and a unitary completion \( E_x \) of \((\text{Ind}_B^G \chi \otimes \chi \cdot |^{-1}) \otimes \text{Sym}^{k-2} L^2 \) such that \( E^0 \otimes_{oL} k_L \cong E_x^0 \otimes_{oL} k_L \), where \( E^0_x \) is the unit ball in \( E_x \) and \( E^0 \) is the unit ball in \( E \).

**Proof.** Let \( \pi := \text{Ind}_B^G \chi \otimes \chi \cdot |^{-1} \) and \( M := (\pi \otimes W) \cap E^0 \). Now \( M \cap \omega_L E^0 = (\pi \otimes W) \cap \omega_L E^0 = \omega_L M \). So \( \iota : M/\omega_L M \hookrightarrow E^0/\omega_L E^0 \) is a \( G \)-equivariant injection. We claim that \( \iota \) is a surjection. Let \( v \in E^0 \). Since \( \pi \otimes W \) is dense in \( E \), there exists a sequence \( \{v_n\}_{n \geq 1} \) in \( \pi \otimes W \) such that \( \lim v_n = v \). We also have \( \lim \| v_n \| = \| v \| \). Since \( \| E \| \subseteq |L| \cong \mathbb{Z} \), there exists an \( n \geq 0 \) such that \( v_n \in M \) for all \( n \geq m \). This implies the surjectivity of \( \iota \). So we get \( M \otimes_{oL} k_L \cong E^0 \otimes_{oL} k_L \).

By Corollary 4.5 we may find \( u_1, \ldots, u_n \in M \) that generate \( M \) as an \( o_L[G] \)-module. Further, \( u_i = \sum_{j=1}^{m_i} v_{ij} \otimes w_{ij} \) with \( v_{ij} \in \pi \) and \( w_{ij} \in W \). Since \( \pi \) is a smooth representation of \( G \), there exists an integer \( c \geq 1 \) such that \( v_{ij} \) is fixed by \( K \) for all \( 1 \leq i \leq n \) and \( 1 \leq j \leq m_i \). Set

\[
\varnothing := ((\pi^c \otimes W) \cap \omega_L M) \cap M, \quad D := (\pi^c \otimes W) \cap M
\]
and let \( M' \) be the image of \( H_0(\varnothing) \hookrightarrow H_0(D) \cong \pi \otimes W \). It follows from (3) that \( M' \) is generated by \((\pi^c \otimes W) \cap M \) as an \( o_L[G] \)-module. Hence, \( M' \subseteq M \). By construction \( (\pi^c \otimes W) \cap M \) contains \( u_1, \ldots, u_n \), and so \( M \subseteq M' \). In particular, \( H_0(\varnothing) \otimes_{oL} k_L \cong M \otimes_{oL} k_L \). The claim follows from the proof of Theorem 4.4. \( \Box \)

5. Existence

Recent results of Colmez, which appeared after the first version of this note, imply the existence of a \( G \)-invariant norm on \((\text{Ind}_B^G \chi \otimes \chi \cdot |^{-1}) \otimes \text{Sym}^{k-2} L^2 \) for \( \chi^2(p)p^{k-1} \in o_L \chi \), thus making our results unconditional. We briefly explain this.

We continue to assume that \( p > 2 \), that \( k \geq 2 \) is an integer and that \( a_p = 2p^{(k-1)/2} \). The representation \( V_{k,a_p} \) of \( \mathcal{O}_p \) sits in the \( p \)-adic family of Berger, Li and Zhu, [2004, 3.2.5]. Moreover, all the other points in the family correspond to the crystalline representations with distinct Frobenius eigenvalues, to which the theory of [Berger and Breuil 2007] applies. Hence [Colmez 2008, II.3.1 and IV.4.11] imply that there exists an irreducible unitary \( L \)-Banach space representation \( \Pi \) of \( \text{GL}_2(\mathbb{Q}_p) \) such that \( V(\Pi) \cong V_{k,a_p} \). If \( p \geq 5 \) or \( p = 3 \) and \( k \neq (\text{mod} \ 8) \) and \( k \neq 7 \) (mod 8), the existence of such \( \Pi \) also follows from [Kisin 2008]. It follows from [Colmez 2008, VI.6.46] that the set of locally algebraic vectors \( \Pi^\text{alg} \) of \( \Pi \) is isomorphic to

\[
(\text{Ind}_B^G \chi \otimes \chi \cdot |^{-1}) \otimes \text{Sym}^{k-2} L^2,
\]
where \( \chi : \mathbb{Q}_p^\times \to L^\times \) is an unramified character with \( \chi(p) = p^{-(k-1)/2} \). The restriction of the \( G \)-invariant norm of \( \Pi \) to \( \Pi^\text{alg} \) solves the problem. Also, if \( \delta : \mathbb{Q}_p^\times \to L^\times \) is a unitary character, then we also obtain a \( G \)-invariant norm on \( \Pi^\text{alg} \otimes \delta \circ \det \).
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References


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