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## Log minimal models according to Shokurov

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# Log minimal models according to Shokurov

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Following Shokurov's ideas, we give a short proof of the following klt version of his result: termination of terminal log flips in dimension $d$ implies that any klt pair of dimension $d$ has a log minimal model or a Mori fibre space. Thus, in particular, any klt pair of dimension 4 has a log minimal model or a Mori fibre space.

## 1. Introduction

All the varieties in this paper are assumed to be over an algebraically closed field $k$ of characteristic zero. We refer the reader to Section 2 for notation and terminology.

The following conjecture is perhaps the most important problem in birational geometry.

**Conjecture 1.1** (Minimal model). Let $(X/Z, B)$ be a Kawamata log terminal (klt) pair. Then it has a log minimal model or a Mori fibre space.

The 2-dimensional case of this conjecture is considered to be classical. The 3-dimensional case was settled in the 80's and 90's by the efforts of many mathematicians, in particular Mori, Shokurov and Kawamata. The higher-dimensional case has seen considerable progress in recent years, thanks primarily to Shokurov's work on the existence of log flips, which paved the way for further progress. The conjecture is also settled for pairs of general type [Birkar et al. 2006], and inductive arguments have been proposed for pairs of nonnegative Kodaira dimension [Birkar 2007]. For a more detailed account of the known cases of this conjecture, see the introduction to [Birkar 2007].

Shokurov [2006] proved that the log minimal model program (LMMP) in dimension $d-1$ and termination of terminal log flips in dimension $d$ imply Conjecture 1.1 in dimension $d$ even for log canonical (lc) pairs. (In this paper, by termination of *terminal log flips in dimension $d$*, we will mean termination of any sequence $X_i \dashrightarrow X_{i+1}/Z_i$ of log flips$/Z$ starting with a $d$-dimensional klt pair $(X/Z, B)$ which is terminal in codimension $\geq 2$; see Section 2 for a more precise formulation.)

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*Keywords:* minimal models, Mori fibre spaces.

Following Shokurov's method and using results of [Birkar et al. 2006], we give a short proof of:

**Theorem 1.2.** *Termination of terminal log flips in dimension $d$ implies Conjecture 1.1 in dimension $d$; more precisely, for a klt pair $(X/Z, B)$ of dimension $d$ one constructs a log minimal model or a Mori fibre space by running the LMMP$/Z$ on $K_X + B$ with scaling of a suitable big$/Z$ $\mathbb{R}$-divisor and proving that it terminates.*

As in [Shokurov 2006], one immediately derives the following:

**Corollary 1.3.** *Conjecture 1.1 holds in dimension* 4.

Note that when $(X/Z, B)$ is effective (for example if it is of nonnegative Kodaira dimension), log minimal models are constructed in [Birkar 2007], using different methods, in dimension $\leq 5$.

## 2. Basics

Let $k$ be an algebraically closed field of characteristic zero. For an $\mathbb{R}$-divisor $D$ on a variety $X$ over $k$, we use $D^\sim$ to denote the birational transform of $D$ on a specified birational model of $X$.

**Definition 2.1.** A *pair* $(X/Z, B)$ consists of normal quasiprojective varieties $X$, $Z$ over $k$, an $\mathbb{R}$-divisor $B$ on $X$ with coefficients in $[0, 1]$ such that $K_X + B$ is $\mathbb{R}$-Cartier, and a projective morphism $X \to Z$. $(X/Z, B)$ is called *log smooth* if $X$ is smooth and Supp $B$ has simple normal crossing singularities.

For a prime divisor $D$ on some birational model of $X$ with a nonempty centre on $X$, $a(D, X, B)$ denotes the log discrepancy. $(X/Z, B)$ is *terminal in codimension* $\geq 2$ if $a(D, X, B) > 1$ whenever $D$ is exceptional$/X$. Log flips preserve this condition but divisorial contractions may not.

Let $(X/Z, B)$ be a klt pair. By a *log flip$/Z$* we mean the flip of a $K_X + B$-negative extremal flipping contraction$/Z$. A *sequence of log flips$/Z$ starting with* $(X/Z, B)$ is a sequence $X_i \dashrightarrow X_{i+1}/Z_i$ in which $X_i \to Z_i \leftarrow X_{i+1}$ is a $K_{X_i} + B_i$-flip$/Z$ and $B_i$ is the birational transform of $B_1$ on $X_1$, and $(X_1/Z, B_1) = (X/Z, B)$. By termination of terminal log flips in dimension $d$ we mean termination of such a sequence in which $(X_1/Z, B_1)$ is a $d$-dimensional klt pair which is terminal in codimension $\geq 2$. Now assume that $G$ is an $\mathbb{R}$-Cartier divisor on $X$. A *sequence of $G$-flops$/Z$ with respect to* $(X/Z, B)$ is a sequence $X_i \dashrightarrow X_{i+1}/Z_i$ in which $X_i \to Z_i \leftarrow X_{i+1}$ is a $G_i$-flip$/Z$ such that $K_{X_i} + B_i \equiv 0/Z_i$ where $G_i$ is the birational transform of $G$ on $X = X_1$.

**Definition 2.2** ([Birkar 2007, §2]). Let $(X/Z, B)$ be a klt pair, $(Y/Z, B_Y)$ a $\mathbb{Q}$-factorial klt pair, $\phi\colon X \dashrightarrow Y/Z$ a birational map such that $\phi^{-1}$ does not contract divisors, and $B_Y$ be the birational transform of $B$ (Note that since $X \to Z$ and

$Y \to Z$ are both projective, by the definition of a pair, $X$ and $Y$ have the same image on $Z$). Moreover, assume that

$$a(D, X, B) \leq a(D, Y, B_Y)$$

for any prime divisor $D$ on birational models of $X$ and assume that the strict inequality holds for any prime divisor $D$ on $X$ which is exceptional/$Y$.

We say that $(Y/Z, B_Y)$ is a *log minimal model* of $(X/Z, B)$ if $K_Y+B_Y$ is nef/$Z$. On the other hand, we say that $(Y/Z, B_Y)$ is a *Mori fibre space* of $(X/Z, B)$ if there is a $K_Y + B_Y$-negative extremal contraction $Y \to Y'/Z$ such that $\dim Y' < \dim Y$.

Typically, one obtains a log minimal model or a Mori fibre space by a finite sequence of divisorial contractions and log flips.

**Remark 2.3.** Let $(X/Z, B)$ be a klt pair and $W \to X$ a log resolution. Let $B_W = B^{\sim} + (1-\epsilon)\sum E_i$ where $0 < \epsilon \ll 1$ and $E_i$ are the exceptional/$X$ divisors on $W$. Remember that $B^{\sim}$ is the birational transform of $B$. If $(Y/X, B_Y)$ is a log minimal model of $(W/X, B_W)$, which exists by [Birkar et al. 2006], then by the negativity lemma $Y \to X$ is a small $\mathbb{Q}$-factorialisation of $X$. To find a log minimal model or a Mori fibre space of $(X/Z, B)$, it is enough to find one for $(Y/Z, B_Y)$. So, one could assume that $X$ is $\mathbb{Q}$-factorial by replacing it with $Y$.

We recall a variant of the LMMP with scaling which we use in this paper. Let $(X/Z, B+C)$ be a $\mathbb{Q}$-factorial klt pair such that $K_X+B+C$ is nef/$Z$ and $B, C \geq 0$. By [Birkar 2007, Lemma 2.7], either $K_X + B$ is nef/$Z$ or there is an extremal ray $R/Z$ such that

$$(K_X + B)\cdot R < 0 \ \text{ and } \ (K_X + B + \lambda_1 C)\cdot R = 0,$$

where

$$\lambda_1 := \inf\{t \geq 0 \mid K_X + B + tC \text{ is nef}/Z\}$$

and $K_X + B + \lambda_1 C$ is nef/$Z$. Now assume that $R$ defines a divisorial contraction or a log flip $X \dashrightarrow X'$. We can consider $(X'/Z, B' + \lambda_1 C')$, where $B' + \lambda_1 C'$ is the birational transform of $B + \lambda_1 C$ and continue the argument. That is, either $K_{X'} + B'$ is nef/$Z$ or there is an extremal ray $R'/Z$ such that $(K_{X'} + B')\cdot R' < 0$ and $(K_{X'} + B' + \lambda_2 C')\cdot R' = 0$, where

$$\lambda_2 := \inf\{t \geq 0 \mid K_{X'} + B' + tC' \text{ is nef}/Z\}$$

and $K_{X'} + B' + \lambda_2 C'$ is nef/$Z$. By continuing this process, we obtain a special kind of LMMP on $K_X + B$ which we refer to as the *LMMP with scaling of* $C$. If it terminates, then we obviously get a log minimal model or a Mori fibre space for $(X/Z, B)$. Note that the required log flips exist by [Birkar et al. 2006].

## 3. Extremal rays

In this section, for convenience of the reader, we give the proofs of some results about extremal rays [Shokurov 2006, Corollary 9, Addendum 4]. The norm $\|G\|$ of an $\mathbb{R}$-divisor $G$ denotes the maximum of the absolute value of its coeffecients.

Let $X \to Z$ be a projective morphism of normal quasiprojective varieties. A curve $\Gamma$ on $X$ is called *extremal*$/Z$ if it generates an extremal ray $R/Z$ which defines a contraction $X \to S/Z$, and if for some ample$/Z$ divisor $H$ we have $H \cdot \Gamma = \min\{H \cdot \Sigma\}$, where $\Sigma$ ranges over curves generating $R$. If $(X/Z, B)$ is divisorial log terminal (dlt) and $(K_X + B) \cdot R < 0$, then by [Kawamata 1991] there is a curve $\Sigma$ generating $R$ such that $(K_X + B) \cdot \Sigma \geq -2 \dim X$. On the other hand, since $\Gamma$ and $\Sigma$ both generate $R$ we have

$$\frac{(K_X + B) \cdot \Gamma}{H \cdot \Gamma} = \frac{(K_X + B) \cdot \Sigma}{H \cdot \Sigma},$$

hence

$$(K_X + B) \cdot \Gamma = (K_X + B) \cdot \Sigma \frac{H \cdot \Gamma}{H \cdot \Sigma} \geq -2 \dim X.$$

**Remark 3.1.** Let $(X/Z, B)$ be a $\mathbb{Q}$-factorial klt pair, $F$ be a reduced divisor on $X$ whose support contains that of $B$, and $V$ be the $\mathbb{R}$-vector space of divisors generated by the components of $F$.

(i) By [Shokurov 1992, Property 1.3.2; 1996, First Main Theorem 6.2 and Remark 6.4], the sets

$$\mathscr{L} = \{\Delta \in V \mid (X/Z, \Delta) \text{ is lc}\} \quad \text{and} \quad \mathscr{N} = \{\Delta \in \mathscr{L} \mid K_X + \Delta \text{ is nef}/Z\}$$

are rational polytopes in $V$. Since $B \in \mathscr{L}$, there are rational boundaries $B^1, \dots, B^r \in \mathscr{L}$ and nonnegative real numbers $a_1, \dots, a_r$ such that $B = \sum a_j B^j$, $\sum a_j = 1$, and each $(X/Z, B^j)$ is klt. In particular, there is $m \in \mathbb{N}$ such that $m(K_X + B^j)$ are Cartier, and for any curve $\Gamma$ on $X$ the intersection number $(K_X + B) \cdot \Gamma$ can be written as $\sum a_j \frac{n_j}{m}$ for certain $n_1, \dots, n_r \in \mathbb{Z}$. Moreover, if $\Gamma$ is extremal$/Z$, then the $n_j$ satisfy $n_j \geq -2m \dim X$.

(ii) If $K_X + B$ is nef$/Z$, then $B \in \mathscr{N}$ and so one can choose the $B^j$ so that $K_X + B^j$ are nef$/Z$.

**Lemma 3.2.** *Let $(X/Z, B)$ be a $\mathbb{Q}$-factorial klt pair. There is a real number $\alpha > 0$ such that*:

(i) *If $\Gamma$ is any extremal curve$/Z$ and if $(K_X + B) \cdot \Gamma > 0$, then $(K_X + B) \cdot \Gamma > \alpha$.*

(ii) *If $K_X + B$ is nef$/Z$, then for any $\mathbb{R}$-divisor $G$, any sequence $X_i \dashrightarrow X_{i+1}/Z_i$ of $G$-flops$/Z$ with respect to $(X/Z, B)$, and any extremal curve $\Gamma$ on $X_i$, if $(K_{X_i} + B_i) \cdot \Gamma > 0$, then $(K_{X_i} + B_i) \cdot \Gamma > \alpha$ where $B_i$ is the birational transform of $B$.*

*Proof.* (i) If $B$ is a $\mathbb{Q}$-divisor, then the statement is trivially true. Let $B^1, \dots, B^r$, $a_1, \dots, a_r$, and $m$ be as in Remark 3.1(i). Let $\Gamma$ be an extremal curve$/Z$. Then, $(K_X + B) \cdot \Gamma = \sum a_j (K_X + B^j) \cdot \Gamma$ and since for each $j$ we have $(K_X + B^j) \cdot \Gamma \geq -2 \dim X$, the existence of $\alpha$ is clear for (i).

(ii) By Remark 3.1(ii) we may also assume that $K_X + B^j$ are nef$/Z$. Then, the sequence $X_i \dashrightarrow X_{i+1}/Z_i$ is also a sequence of flops with respect to each $(X/Z, B^j)$. In particular, $(X_i/Z, B_i^j)$ is klt and $m(K_{X_i} + B_i^j)$ is Cartier for any $j, i$ where $B_i^j$ is the birational transform of $B^j$. The rest is as in (i). □

**Proposition 3.3.** *Let $(X/Z, B)$ be a $\mathbb{Q}$-factorial klt pair, $F$ a reduced divisor on $X$ whose support contains that of $B$, and $\mathcal{L}$ as in Remark 3.1. There is a rational polytope $\mathcal{H} \subset \mathcal{L}$ of klt boundaries and of maximal dimension containing an open neighborhood of $B$ in $\mathcal{L}$ (with respect to the topology on $\mathcal{L}$ induced by the norm $\|\cdot\|$) such that*

(i) *if $\Delta \in \mathcal{H}$ and $(K_X + \Delta) \cdot R < 0$ for an extremal ray $R/Z$, then $(K_X + B) \cdot R \leq 0$, and*

(ii) *if $K_X + B$ is nef$/Z$, $\Delta \in \mathcal{H}$, we have a sequence $X_i \dashrightarrow X_{i+1}/Z_i$ of $K_X + \Delta$-flops$/Z$ with respect to $(X/Z, B)$, and $(K_{X_i} + \Delta_i) \cdot R < 0$ for an extremal ray $R/Z$ on some $X_i$, then $(K_{X_i} + B_i) \cdot R = 0$, where $\Delta_i, B_i$ are the birational transforms of $\Delta$, $B$ respectively.*

*Proof.* (i) Let $\mathcal{M} \subset \mathcal{L}$ be a rational polytope of klt boundaries and of maximal dimension containing an open neighborhood of $B$ in $\mathcal{L}$. If the statement is not true then there is an infinite sequence of $\Delta_l \in \mathcal{M}$ and extremal rays $R_l/Z$ such that for each $l$ we have

$$(K_X + \Delta_l) \cdot R_l < 0, \quad (K_X + B) \cdot R_l > 0,$$

and $\|\Delta_l - B\|$ converges to 0. Let $\Omega_l$ be the point on the boundary of $\mathcal{M}$ such that $\Omega_l - \Delta_l = b_l(\Delta_l - B)$ for some real number $b_l \geq 0$ and such that $\|\Omega_l - B\|$ is maximal. So, $\Omega_l$ is the most far away point in $\mathcal{M}$ which is on the line determined by $B$ and $\Delta_l$, in the direction of $\Delta_l$. Since $\|\Delta_l - B\|$ converges to 0, $b_l$ converges to $+\infty$.

By assumptions, $(X/Z, \Omega_l)$ is klt and if $\Gamma_l$ is an extremal curve$/Z$ generating $R_l$, then

$$(\Omega_l - \Delta_l) \cdot \Gamma_l = (K_X + \Omega_l) \cdot \Gamma_l - (K_X + \Delta_l) \cdot \Gamma_l \geq -2 \dim X.$$

This is not possible because by Lemma 3.2,

$$(K_X + \Delta_l) \cdot \Gamma_l + (B - \Delta_l) \cdot \Gamma_l = (K_X + B) \cdot \Gamma_l > \alpha,$$

and by the same arguments $(B - \Delta_l) \cdot \Gamma_l$ approaches 0.

By definition, the sequence $X_i \dashrightarrow X_{i+1}/Z_i$ is a sequence of $K_X+\Delta$-flips which are numerically trivial with respect to $K_X+B$. Let $\mathcal{H}$ be as in (i). Assume that $R$ is an extremal ray$/Z$ on $X_i$ such that $(K_{X_i}+\Delta_i)\cdot R<0$ but $(K_{X_i}+B_i)\cdot R>0$. Let $\Gamma$ be an extremal curve$/Z$ generating $R$. Let $\Omega$ be the point on the boundary of $\mathcal{H}$ which is chosen for $\Delta$ similarly as in (i). By assumptions, $(X_i/Z, \Delta_i)$ and $(X_i/Z, \Omega_i)$ are klt where $\Omega_i$ is the birational transform of $\Omega$. So,

$$(\Omega_i-\Delta_i)\cdot\Gamma=(K_{X_i}+\Omega_i)\cdot\Gamma-(K_{X_i}+\Delta_i)\cdot\Gamma\geq -2\dim X.$$

On the other hand, $(K_{X_i}+B_i)\cdot\Gamma>\alpha$ where $\alpha$ is as in Lemma 3.2. By construction, there is some $b\geq 0$ such that $b(\Delta_i-B_i)=\Omega_i-\Delta_i$. Therefore,

$$(K_{X_i}+\Delta_i)\cdot\Gamma=(K_{X_i}+B_i)\cdot\Gamma+(\Delta_i-B_i)\cdot\Gamma>\alpha-\frac{2\dim X}{b}.$$

which is not possible if $b\alpha>2\dim X$. In other words, if $\Delta$ is close enough to $B$ then the statement of $(ii)$ also holds, that is, we only need to shrink $\mathcal{H}$ appropriately. □

## 4. Proof of the main results

*Proof of Theorem 1.2.* Let $(X/Z, B)$ be a klt pair of dimension $d$. By Remark 2.3, we can assume that $X$ is $\mathbb{Q}$-factorial. Let $H\geq 0$ be an $\mathbb{R}$-divisor which is big$/Z$ so that $K_X+B+H$ is klt and nef$/Z$. Run the LMMP$/Z$ on $K_X+B$ with scaling of $H$. If the LMMP terminates, then we get a log minimal model or a Mori fibre space. Suppose that we get an infinite sequence $X_i \dashrightarrow X_{i+1}/Z_i$ of log flips$/Z$, where we may also assume that $(X_1/Z, B_1)=(X/Z, B)$.

Let $\lambda_i$ be the threshold on $X_i$ determined by the LMMP with scaling as explained in Section 2. So, $K_{X_i}+B_i+\lambda_i H_i$ is nef$/Z$,

$$(K_{X_i}+B_i)\cdot R_i<0 \quad\text{and}\quad (K_{X_i}+B_i+\lambda_i H_i)\cdot R_i=0,$$

where $B_i$ and $H_i$ are the birational transforms of $B$ and $H$ and, $R_i$ is the extremal ray which defines the flipping contraction $X_i\to Z_i$. Obviously, $\lambda_i\geq\lambda_{i+1}$.

Put $\lambda=\lim_{i\to\infty}\lambda_i$. If the limit is attained, that is, $\lambda=\lambda_i$ for some $i$, then the sequence terminates by Corollary 1.4.2 of [Birkar et al. 2006]. So, we assume that the limit is not attained. Actually, if $\lambda>0$, again [Birkar et al. 2006] implies that the sequence terminates. However, we do not need to use [Birkar et al. 2006] in this case. In fact, by replacing $B_i$ with $B_i+\lambda H_i$, we can assume that $\lambda=0$ hence $\lim_{i\to\infty}\lambda_i=0$.

Put $\Lambda_i:=B_i+\lambda_i H_i$. Since we are assuming that terminal log flips terminate, or, alternatively, by Corollary 1.4.3 of [Birkar et al. 2006], we can construct a terminal (in codimension $\geq 2$) crepant model $(Y_i/Z, \Theta_i)$ of $(X_i/Z, \Lambda_i)$. A slight modification of the argument in Remark 2.3 would do this. Note that we can assume that all the $Y_i$ are isomorphic to $Y_1$ in codimension one, perhaps after truncating the sequence. Let $\Delta_1 = \lim_{i\to\infty} \Theta_i^{\sim}$ on $Y_1$ and let $\Delta_i$ be its birational transform on $Y_i$. The limit is obtained componentwise.

Since $H_i$ is big$/Z$ and $K_{X_i} + \Lambda_i$ is klt and nef$/Z$, $K_{X_i} + \Lambda_i$ and $K_{Y_i} + \Theta_i$ are semiample$/Z$ by the base point freeness theorem for $\mathbb{R}$-divisors. Thus, $K_{Y_i} + \Delta_i$ is a limit of movable$/Z$ divisors which in particular means that it is pseudo-effective$/Z$. Note that if $K_{Y_i} + \Delta_i$ is not pseudo-effective$/Z$, we get a contradiction by Corollary 1.3.2 of [Birkar et al. 2006].

Now run the LMMP$/Z$ on $K_{Y_1} + \Delta_1$. No divisor will be contracted again because $K_{Y_1} + \Delta_1$ is a limit of movable$/Z$ divisors. Since $K_{Y_1} + \Delta_1$ is terminal in codimension $\geq 2$, by assumptions, the LMMP terminates with a log minimal model $(W/Z, \Delta)$. By construction, $\Delta$ on $W$ is the birational transform of $\Delta_1$ on $Y_1$, and $G_i := \Theta_i^{\sim} - \Delta$ on $W$ satisfies $\lim_{i\to\infty} G_i = 0$.

By Proposition 3.3, for each $G_i$ with $i \gg 0$, we can run the LMMP$/Z$ on $K_W + \Delta + G_i$ which will be a sequence of $G_i$-flops, that is, $K + \Delta$ would be numerically zero on all the extremal rays contracted in the process. No divisor will be contracted because $K_W + \Delta + G_i$ is movable$/Z$. The LMMP ends up with a log minimal model $(W_i/Z, \Omega_i)$. Here, $\Omega_i$ is the birational transform of $\Delta + G_i$ and so of $\Theta_i$. Let $S_i$ be the lc model of $(W_i/Z, \Omega_i)$ which is the same as the lc model of $(Y_i/Z, \Theta_i)$ and that of $(X_i/Z, \Lambda_i)$ because $K_{W_i} + \Omega_i$ and $K_{Y_i} + \Theta_i$ are nef$/Z$ with $W_i$ and $Y_i$ being isomorphic in codimension one, and $K_{Y_i} + \Theta_i$ is the pullback of $K_{X_i} + \Lambda_i$. Also note that since $K_{X_i} + B_i$ is pseudoeffective$/Z$, $K_{X_i} + \Lambda_i$ is big$/Z$; hence $S_i$ is birational to $X_i$.

By construction, $K_{W_i} + \Delta^{\sim}$ is nef$/Z$ and it turns out that $K_{W_i} + \Delta^{\sim} \sim_{\mathbb{R}} 0/S_i$. Suppose that this is not the case. Then, $K_{W_i} + \Delta^{\sim}$ is not numerically zero$/S_i$ hence there is some curve $C/S_i$ such that $(K_{W_i} + \Delta^{\sim} + G_i^{\sim}) \cdot C = 0$ but $(K_{W_i} + \Delta^{\sim}) \cdot C > 0$ which implies that $G_i^{\sim} \cdot C < 0$. Hence, there is a $K_{W_i} + \Delta^{\sim} + (1+\tau)G_i^{\sim}$-negative extremal ray $R/S_i$ for any $\tau > 0$. This contradicts Proposition 3.3 because we must have

$$(K_{W_i} + \Delta^{\sim} + G_i^{\sim}) \cdot R = (K_{W_i} + \Delta^{\sim}) \cdot R = 0.$$

Therefore, $K_{W_i} + \Delta^{\sim} \sim_{\mathbb{R}} 0/S_i$. Now $K_{X_i} + \Lambda_i \sim_{\mathbb{R}} 0/Z_i$ implies that $Z_i$ is over $S_i$ and so $K_{Y_i} + \Delta_i \sim_{\mathbb{R}} 0/S_i$. On the other hand, $K_{X_i} + B_i$ is the pushdown of $K_{Y_i} + \Delta_i$; hence $K_{X_i} + B_i \sim_{\mathbb{R}} 0/S_i$. Thus, $K_{X_i} + B_i \sim_{\mathbb{R}} 0/Z_i$ and this contradicts the fact that $X_i \to Z_i$ is a $K_{X_i} + B_i$-flipping contraction. So, the sequence of flips terminates and this completes the proof. □

*Proof of Corollary 1.3.* Since terminal log flips terminate in dimension 4 by [Fujino 2004; Shokurov 2004] (see also [Alexeev et al. 2007]), the result follows from Theorem 1.2. □

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