Positive motivic measures are counting measures

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A motivic measure is a ring homomorphism from the Grothendieck group of a field $K$ (with multiplication coming from the fiber product over Spec $K$) to some field. We show that if a real-valued motivic measure $\mu$ satisfies $\mu([V]) \geq 0$ for all $K$-varieties $V$, then $\mu$ is a counting measure; that is, there exists a finite field $L$ containing $K$ such that $\mu([V]) = |V(L)|$ for all $K$-varieties $V$.

Let $K$ be a field. By a $K$-variety, we mean a geometrically reduced, separated scheme of finite type over $K$. Let $K_0(\text{Var}_K)$ denote the Grothendieck group of $K$, that is, the free abelian group generated by isomorphism classes $[V]$ of $K$-varieties, with the scissors relations $[V] = [W] - [V \setminus W]$ whenever $W$ is a closed $K$-subvariety of $V$. There is a unique product on $K_0(\text{Var}_K)$ characterized by the relation

$$[V] \cdot [W] = [V \times W],$$

where $\times$ denotes the fiber product over Spec $K$. This product gives $K_0(\text{Var}_K)$ a commutative ring structure with identity $[\text{Spec } K]$. For every extension $L$ of $K$, extension of scalars gives a natural ring homomorphism $K_0(\text{Var}_K) \to K_0(\text{Var}_L)$. The map $K \mapsto K_0(\text{Var}_K)$ can be regarded as a functor from fields to commutative rings. Throughout the paper, we follow the usual convention of writing $\mathbb{L}$ for $[\mathbb{A}_K^1]$.

Following the terminology of [Larsen and Lunts 2003], we call a ring homomorphism from $K_0(\text{Var}_K)$ to a field $F$ a motivic measure. Note that the original meaning of this term [Hales 2005; Looijenga 2002] is different (though related). If $K$ is a finite field, the map $[V] \mapsto |V(K)|$ extends to a homomorphism $\mu_K : K_0(\text{Var}_K) \to \mathbb{Z}$, and therefore to an $F$-valued measure for any field $F$. More generally, if $L$ is an extension of $K$ which is also a finite field, the composition of $\mu_L$ with the natural map $K_0(\text{Var}_K) \to K_0(\text{Var}_L)$ gives for each $F$ a motivic measure. We will call all such measures counting measures.

In this paper, we consider positive motivic measures, by which we mean $\mathbb{R}$-valued measures $\mu$ such that $\mu([V]) \geq 0$ for all $K$-varieties $V$. We now state our main result.

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Theorem 1. Every positive motivic measure is a counting measure. In other words, if $K$ is any field and $\mu : K_0(\text{Var}_K) \to \mathbb{R}$ is positive, there exists a finite field $L$ containing $K$ such that $\mu([V]) = |V(L)|$ for all $K$-varieties $V$.

Of course, for other choices of $F$ there may still be motivic measures such that $\mu([V])$ lies in some interesting semiring of $F$ for all $K$-varieties $V$. For example, if $F$ is $\mathbb{C}(u, v)$ and $K = \mathbb{C}$, the measure sending $V$ to its Hodge–Deligne polynomial takes values in the semiring of polynomials in $u, v$ whose term of highest total degree is a positive multiple of a power of $uv$.

We begin with a direct proof of the following obvious corollary of Theorem 1.

Proposition 2. If $K$ is infinite, there are no positive motivic measures on $K_0(\text{Var}_K)$.

Proof. Let $\mu$ be such a measure. For any finite subset $S$ of $K$, which we regard as a zero-dimensional subvariety of $\mathbb{A}^1$,

$$0 \leq \mu(\mathbb{A}^1 \setminus S) = \mu(\mathbb{L}) - |S|.$$ 

Thus, $\mu(\mathbb{L}) \geq |S|$ for all subsets $S$ of $K$, which proves the proposition. \qed

For the remainder of the paper we may and do assume that $K$ is finite, of cardinality $q$. We write $\mathbb{F}_q^n$ for the degree $n$ extension of $K$.

Proposition 3. Let $\Omega^n$ denote the variety obtained from $\mathbb{A}^n$ by removing all proper affine-linear subspaces defined over $\mathbb{F}_q$. Then

$$[\Omega^n] = (\mathbb{L} - q)(\mathbb{L} - q^2) \cdots (\mathbb{L} - q^n).$$

Proof. For any $\mathbb{F}_q$-rational affine-linear subspace $A$ of $\mathbb{A}^n$, let $A^\circ$ denote the open subvariety of $A$ which is the complement of all proper $\mathbb{F}_q$-rational affine-linear subspaces of $A$. Then $[A^\circ] = [\Omega^{\dim A}]$, and one can write recursively

$$[\Omega^n] = \mathbb{L}^n - \sum_{i=1}^{n-1} a_{n,i} [\Omega^i],$$

where $a_{n,i}$ is the number of $\mathbb{F}_q$-rational $i$-dimensional affine linear subspaces of $\mathbb{A}^n$. Thus, $[\Omega^i]$ can be expressed as $P_n(\mathbb{L})$, where $P_n \in \mathbb{Z}[x]$ is monic and of degree $n$. It suffices to prove that $q^d$ is a root of $P_n(x)$ for all integers $d \in \{1, 2, \ldots, n\}$.

For any $d$ in this range $\Omega^n(\mathbb{F}_{q^d})$ is empty. Indeed, if $x \in \mathbb{A}^n(\mathbb{F}_{q^d})$, then the $n$ coordinates of $x$ together with $1$ cannot be linearly independent over $\mathbb{F}_q$, which implies that $x$ lies in a proper $\mathbb{F}_q$-rational affine-linear subspace of $\mathbb{A}^n$. Thus,

$$0 = \mu_{\mathbb{F}_{q^d}}(\Omega^n) = P_n(q^d).$$

\qed

Corollary 4. If $\mu$ is a positive measure on $K_0(\text{Var}_{\mathbb{F}_q})$, there exists a positive integer $n$ such that $\mu(\mathbb{L}) = q^n$. 

Proof. If \( q^{n-1} < \mu(L) < q^n \) for some integer \( n \), then \( \mu(\Omega^n) < 0 \), contrary to positivity.

Our goal is then to prove that \( \mu(L) = q^n \) implies \( \mu = q^n \). We prove first that these measures coincide for varieties of the form \( \text{Spec} \mathbb{F}_q^d \) and deduce that they coincide for all affine varieties. As \( K_0(\text{Var}_F) \) is generated by the classes of affine varieties, this implies Theorem 1.

Lemma 5. Let \( \mu \) be a real-valued motivic measure of \( K_0(\text{Var}_F) \) and \( m \) a positive integer. Then

\[
\mu(\text{Spec} \mathbb{F}_q^m) \in \{0, m\}.
\]

If \( \text{Spec} \mathbb{F}_q^m \) has measure \( m \), then \( \text{Spec} \mathbb{F}_q^d \) has measure \( d \) whenever \( d \) divides \( m \).

Proof. As \( \mathbb{F}_q^m \otimes \mathbb{F}_q \mathbb{F}_q^m = \mathbb{F}_q^m \), the class \( x \) of \( \text{Spec} \mathbb{F}_q^m \) satisfies \( x^2 = mx \). If \( d \) divides \( m \),

\[
\mathbb{F}_q^d \otimes \mathbb{F}_q \mathbb{F}_q^m = \mathbb{F}_q^d,
\]

so \( \mu(\text{Spec} \mathbb{F}_q^m) = m \) implies \( \mu(\text{Spec} \mathbb{F}_q^d) = d \). Of course,

\[
\mu(\text{Spec} \mathbb{F}_q^m) = \begin{cases} m & \text{if } m \mid n, \\ 0 & \text{otherwise}. \end{cases}
\]

We will prove the same thing for the values of \( \mu(\text{Spec} \mathbb{F}_q^m) \). We begin with:

Proposition 6. If \( \mu(L) = q^n \) and \( \mu(\text{Spec}(\mathbb{F}_q^d)) = k \) for some \( k \geq n \), then

\[
\mu(\text{Spec} \mathbb{F}_q^m) = \begin{cases} m & \text{if } m \mid n, \\ 0 & \text{otherwise.} \end{cases}
\]  \( \text{(1)} \)

For any integer \( k \), we denote by \( X_k \) the complement in \( \mathbb{A}^1 \) of the set of all points with residue field contained in \( \mathbb{F}_q^k \).

Proof. By Lemma 5, \( \mu(\text{Spec} \mathbb{F}_q^d) = d \) when \( d \) divides \( k \). Choose an \( m \) not dividing \( k \), and let \( Y_{k,m} \) denote the complement in \( X_k \) of the set of points with residue field \( \mathbb{F}_q^m \). Then

\[
\mu([Y_{k,m}]) = \mu(L) - \sum_{d \mid k} c_d d - c_m \mu(\text{Spec} \mathbb{F}_q^m),
\]

where \( c_i \) is the number of points in \( \mathbb{A}^1 \) with residue field \( \mathbb{F}_q^i \). From the positivity of \( \mu([Y_{k,m}]) \) and the fact that

\[
0 = \mu(\mathbb{F}_q^k([Y_{k,m}])) = q^k - \sum_{d \mid k} c_d d,
\]

we see that \( \mu(L) - q^k = q^n - q^k \) must be nonnegative, which is to say \( k = n \), and that \( \mu(\text{Spec} \mathbb{F}_q^m) = 0 \). \( \square \)
Proposition 7. If \( \mu(L) = q^n \), then \( \mu(\text{Spec } \mathbb{F}_{q^n}) = n \).

Proof. The assertion is clear for \( n = 1 \), so we assume \( n > 1 \). Let \( c_i \) denote the number of points in \( \mathbb{A}^1 \) with residue field \( \mathbb{F}_{q^i} \). Thus \( i c_i \leq q^i - 1 \) for all \( i > 1 \). If \( \mu(\text{Spec } \mathbb{F}_{q^n}) = 0 \), then \( \mu(\text{Spec } \mathbb{F}_{q^i}) = 0 \) for all \( i \geq n \), so for all \( k > 0 \) we have

\[
\mu([X_k]) \geq q^n - q - \sum_{i=2}^{n-1} (q^i - 1) \geq 2.
\]

Now we consider all curves in \( \mathbb{A}^2 \) of the form \( y = P(x) \) where \( P(x) \in \mathbb{F}_q[x] \) has degree \( \leq 2n \). The total number of such curves is greater than \( q^{2n} \), and for any intersection point \((\alpha, \beta)\) of any two distinct curves of this family, \( \alpha \) satisfies a polynomial equation of degree \( \leq 2n \) over \( \mathbb{F}_q \). Therefore, the open curves \( C_P := \{ (x, P(x)) | x \notin \mathbb{F}_{q^{2n}} \} \), indexed by polynomials \( P \) of degree \( \leq 2n \), are mutually disjoint. If \( C \) denotes the closure of the union of the \( C_P \) in \( \mathbb{A}^2 \), it follows that

\[
\mu([C]) > q^{2n} \mu([X(2n)]) > q^{2n},
\]

so \( \mu([\mathbb{A}^2 \setminus C]) < 0 \), which is absurd. \( \square \)

Together, the two preceding propositions imply (1).

We can now prove Theorem 1. We assume \( \mu(L) = q^n \). It suffices to check that \( \mu([V]) = |V(\mathbb{F}_{q^n})| \) for all affine \( \mathbb{F}_q \)-varieties \( V \).

Each closed point of \( V \) with residue field \( \mathbb{F}_{q^d} \) corresponds to a \( d \)-element Galois orbit in \( V(\mathbb{F}_{q^d}) \). If \( d \) divides \( n \), it gives a \( d \)-element subset of \( V(\mathbb{F}_{q^n}) \) and the subsets arising from different closed points are mutually disjoint. Since \( V(\mathbb{F}_{q^n}) \) is the union of all these subsets, and \( \mu(\text{Spec } \mathbb{F}_{q^n}) = d \), we have

\[
\mu([V]) \geq |V(\mathbb{F}_{q^n})| \tag{2}
\]

for each \( \mathbb{F}_q \)-variety \( V \). However, embedding \( V \) as a closed subvariety of \( \mathbb{A}^m \) for some \( m \), the complement \( W = \mathbb{A}^m \setminus V \) is again a variety, so

\[
\mu([W]) \geq |W(\mathbb{F}_{q^n})| \tag{3}
\]

Since

\[
q^{mn} = \mu([\mathbb{A}^m]) = \mu([V]) + \mu([W]) \geq |V(\mathbb{F}_{q^n})| + |W(\mathbb{F}_{q^n})| = |\mathbb{A}^m(\mathbb{F}_{q^n})| = q^{mn},
\]

we must have equality in (2) and (3).
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References


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