Haglund–Haiman–Loehr type formulas for Hall–Littlewood polynomials of type $B$ and $C$

Cristian Lenart
Haglund–Haiman–Loehr type formulas for Hall–Littlewood polynomials of type $B$ and $C$

Cristian Lenart

In previous work we showed that two apparently unrelated formulas for the Hall–Littlewood polynomials of type $A$ are, in fact, closely related. The first is the tableau formula obtained by specializing $q = 0$ in the Haglund–Haiman–Loehr formula for Macdonald polynomials. The second is the type $A$ instance of Schwer’s formula (rephrased and rederived by Ram) for Hall–Littlewood polynomials of arbitrary finite type; Schwer’s formula is in terms of so-called alcove walks, which originate in the work of Gaussent and Littelmann and of the author with Postnikov on discrete counterparts to the Littelmann path model. We showed that the tableau formula follows by “compressing” Ram’s version of Schwer’s formula. In this paper, we derive new tableau formulas for the Hall–Littlewood polynomials of type $B$ and $C$ by compressing the corresponding instances of Schwer’s formula.

1. Introduction

Hall–Littlewood polynomials are at the center of many recent developments in representation theory and algebraic combinatorics. They were originally defined in type $A$, as a basis for the algebra of symmetric functions depending on a parameter $t$. This basis interpolates between two fundamental bases: the one of Schur functions, at $t = 0$, and the one of monomial functions, at $t = 1$. Besides the original motivation for defining Hall–Littlewood polynomials, which comes from the Hall algebra [Littlewood 1961], there are many other applications; see for example [Lenart 2011] and the references therein.

Macdonald [1971] showed that there is a formula for the spherical functions corresponding to a Chevalley group over a $p$-adic field that generalizes the formula for

Cristian Lenart was partially supported by the National Science Foundation grant DMS-0701044. 

MSC2000: primary 05E05; secondary 33D52.

Keywords: Hall–Littlewood polynomials, Macdonald polynomials, alcove walks, Schwer’s formula, the Haglund–Haiman–Loehr formula.
the Hall–Littlewood polynomials. Thus, the Macdonald spherical functions gener-
alize the Hall–Littlewood polynomials to all root systems, and the two names are
used interchangeably in the literature. There are two families of Hall–Littlewood
functions of arbitrary type, called \( P \) and \( Q \), which form dual bases for the Weyl
group invariants. The \( P \)-polynomials specialize to the Weyl characters at \( t = 0 \).
The transition matrix between Weyl characters and \( P \)-polynomials is given by Lusztig’s
\( t \)-analog of weight multiplicities (Kostka–Foulkes polynomials of arbitrary type),
which are certain affine Kazhdan–Lusztig polynomials [Kato 1982; Lusztig 1983].
On the combinatorial side, we have the Lascoux–Schützenberger formula [1979]
for the Kostka–Foulkes polynomials in type \( A \), but no generalization of this for-
mula to other types is known. Other applications of the type \( A \) Hall–Littlewood
polynomials that extend to arbitrary type are those related to fermionic multiplicity
formulas [Ardonne and Kedem 2007] and affine crystals [Lecouvey and Shimozono
2007]. We refer to [Nelsen and Ram 2003; Stembridge 2005] for surveys on Hall–
Littlewood polynomials of arbitrary type.

Macdonald [1992; 2000] defined a remarkable family of orthogonal polyno-
mials depending on parameters \( q, t \), which bear his name. These polynomials
generalize the spherical functions for a \( p \)-adic group, the Jack polynomials, and
the zonal polynomials. At \( q = 0 \), the Macdonald polynomials specialize to the
Hall–Littlewood polynomials, and thus they further specialize to the Weyl charac-
ters (upon setting \( t = 0 \) as well). There has been considerable interest recently in
the combinatorics of Macdonald polynomials. This stems in part from a combi-
natorial formula for the ones corresponding to type \( A \), which is due to Haglund,
Haiman, and Loehr [Haglund et al. 2005]. This formula is in terms of fillings of
Young diagrams, and uses two statistics, called inv and maj, on such fillings. The
Haglund–Haiman–Loehr formula has already found important applications, such
as new proofs of the positivity theorem for Macdonald polynomials, which states
that the two-parameter Kostka–Foulkes polynomials have nonnegative integer co-
efficients. One of these proofs is based on Hecke algebras [Grojnowski and Haiman
2007], while the other is purely combinatorial and leads to a positive formula for
the two-parameter Kostka–Foulkes polynomials [Assaf 2010]. Moreover, in the
one-parameter case (that is, when \( q = 0 \)), the Haglund–Haiman–Loehr formula
was used to give a concise derivation of the Lascoux–Schützenberger formula for
the Kostka–Foulkes polynomials of type \( A \) [Haglund et al. 2005, Section 7].

An apparently unrelated development, at the level of arbitrary finite root systems,
led to Schwer’s formula [2006], rephrased and rederived by Ram [2006], for the
Hall–Littlewood polynomials of arbitrary type. The latter formulas are in terms of
so-called alcove walks, which originate in the work of Gaussent and Littelmann
[2005] and of the author with Postnikov [Lenart and Postnikov 2007; 2008] on dis-
crete counterparts to the Littelmann path model [Littelmann 1994; 1995]. Schwer’s
formula was recently generalized by Ram and Yip [2011] to a similar formula for
the Macdonald polynomials. The generalization consists in the fact that the latter
formula is in terms of alcove walks with both “positive” and “negative” foldings,
whereas in the former only “positive” foldings appear.

In [Lenart 2011], we related Schwer’s formula to the Haglund–Haiman–Loehr
formula. More precisely, we showed that we can group the terms in the type A
instance of Schwer’s formula (in fact, we used Ram’s version of it) for \( P_\lambda(x; t) \)
into equivalence classes, such that the sum in each equivalence class is a term
in the Haglund–Haiman–Loehr formula for \( q = 0 \). An equivalence class consists
of all the terms corresponding to alcove walks that produce the same filling of
a Young diagram \( \lambda \) (indexing the Hall–Littlewood polynomial) via a simple con-
struction. In fact, we first considered the case when the partition \( \lambda \) has no two
parts identical (that is, it is a regular weight); the general case, which displays
additional complexity, was considered in the Appendix to the same paper, written
with Lubovsky. The work referring to a regular weight \( \lambda \) was then extended in
[Lenart 2009], by showing that the type A instance of the Ram–Yip formula for
Macdonald polynomials compresses, in a similar way, to a formula analogous to
the Haglund–Haiman–Loehr one, but with fewer terms.

In this paper we extend the results in [Lenart 2011] to types \( B \) and \( C \). More
precisely, we derive new formulas for the Hall–Littlewood polynomials of type
\( B \) and \( C \) indexed by regular weights in terms of fillings of Young diagrams; we
do this by compressing the corresponding instances of Schwer’s formula (in fact,
we again use Ram’s version of it). Note that no tableau formula for the Hall–
Littlewood or Macdonald polynomials exists beyond type \( A \) so far. Our approach
provides a natural way to obtain such formulas, and suggests that this method
could be further extended to type \( D \) (this case is slightly more complex than types
\( B \) and \( C \), as seen below), as well as to Macdonald polynomials; these problems
are currently explored, as is the compression in the case of a Hall–Littlewood
polynomial indexed by a nonregular weight (by extending the type A result in the
Appendix of [Lenart 2011]). Our formula is more complex than the corresponding
one in type \( A \) (that is, the Haglund–Haiman–Loehr formula at \( q = 0 \)). However,
the statistic we use is, in the case of some special fillings, completely similar to the
Haglund–Haiman–Loehr inversion statistic (which is the more intricate of their
two statistics). The naturality of our formula is also supported by the fact that
the Kashiwara–Nakashima tableaux [1994] of type \( B \) and \( C \) are, essentially, the
surviving fillings in this formula when we set \( t = 0 \). We also note that the passage
from (Ram’s version of) Schwer’s formula to ours results in a considerably larger
reduction in the number of terms in type \( B \) and \( C \) compared to type \( A \). In terms of
applications, it would be very interesting to see whether our formula could be used
to derive, in the spirit of [Haglund et al. 2005, Section 7], a positive combinatorial
formula for Lusztig’s \( t \)-analog of weight multiplicities in type \( B \) and \( C \), which has been long sought.

2. The tableau formula in type \( C \)

Let us start by recalling the Weyl group of type \( B/C \), viewed as the group of signed permutations \( B_n \). Such permutations are bijections \( w \) from

\[ [\bar{n}] := \{1 < 2 < \cdots < n < \bar{n} < \bar{n}-1 < \cdots < \bar{1}\} \]

to \([\bar{n}]\) satisfying \( w(\bar{i}) = \bar{w}(i) \). Here \( \bar{r} \) is viewed as \(-i\), so \( \bar{i} = i \). We use the window notation \( w = w(1) \ldots w(n) \). Given \( 1 \leq i < j \leq \bar{n} \), we denote by \((i, j)\) the reflection that transposes the entries in positions \( i \) and \( j \) (upon right multiplication). Similarly, we denote by \((i, \bar{j})\), again for \( i < j \), the transposition of entries in positions \( i \) and \( j \) followed by the sign change of those entries. Finally, we denote by \((i, \bar{i})\) the sign change in position \( i \). Given \( w \) in \( B_n \), we define

\[
\begin{align*}
\ell_+(w) &:= |\{(k, l) : 1 \leq k < l \leq n, \ w(k) > w(l)\}|, \\
\ell_-(w) &:= |\{(k, l) : 1 \leq k \leq l \leq n, \ w(k) > \bar{w}(l)\}|.
\end{align*}
\]

(2-1)

Then the length of \( w \) is given by \( \ell(w) := \ell_+(w) + \ell_-(w) \).

Let \( \lambda \) be a partition corresponding to a regular weight in type \( C_n \) for \( n \geq 2 \), that is, \( \lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0) \) with \( \lambda_i \in \mathbb{Z} \). We identify \( \lambda \) with its Young (or Ferrers) diagram, as usual, but we draw this diagram in “Japanese style” (as opposed to the more common English or French styles), that is, we embed it in the third quadrant, where \( n = 3 \):

\[ \lambda = (4, 3, 2) = \begin{array}{ccc}
\begin{array}{c}
\hline
\hline
\hline
\end{array}
\end{array} \]

Consider the shape \( \hat{\lambda} \) obtained from \( \lambda \) by replacing each column of height \( k \) with \( k \) or \( 2k-1 \) (adjacent) copies of it, depending on the given column being the rightmost one or not. In this example, we have

\[ \hat{\lambda} = (12, 11, 8) = \begin{array}{ccc}
\begin{array}{c}
\hline
\hline
\hline
\end{array}
\end{array} \]

Here \( \hat{\lambda} \) is shown divided into rectangular blocks, each of which corresponds to a column of \( \lambda \); the heights of the blocks (from right to left) are given by the conjugate partition \( \lambda' = (3, 3, 2, 1) \).

We are representing a filling \( \sigma \) of \( \hat{\lambda} \) as a concatenation of columns \( C_{ij} \) and \( C_{ik}' \), where \( i = 1, \ldots, \lambda_1 \), while for a given \( i \) we have \( j = 1, \ldots, \lambda_i' \) if \( i > 1 \), \( j = 1 \) if
Hall–Littlewood polynomials of type $B$ and $C$

$i = 1$, and $k = 2, \ldots, \lambda_i’$; the columns $C_{ij}$ and $C’_{ik}$ have height $\lambda_i’$. More precisely, we let

$$\sigma = \epsilon_i^{\lambda_1} \ldots \epsilon_1^1,$$

(2-2)

where

$$\epsilon_i^i := \begin{cases} C’_{i2} \ldots C’_{i,\lambda’_i} C_{i1} \ldots C_{i,\lambda’_i} & \text{if } i > 1, \\ C’_{i2} \ldots C’_{i,\lambda’_i} C_{i1} & \text{if } i = 1. \end{cases}$$

Note that the leftmost column is $C_{\lambda_1,1}$ and the rightmost column is $C_{11}$.

**Example 2.1.** The following is a filling for the partition considered above, where we use the same division into blocks as above:

$$\sigma = \begin{array}{cccccccc}
C_{41} & C’_{32} & C_{31} & C’_{32} & C’_{22} & C_{23} & C_{21} & C_{23} & C’_{12} & C’_{13} & C_{11} \\
\bar{1} & \bar{1} & \bar{1} & 3 & 2 & 2 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 3 & 3 & 2 & 2 & 2 & 3 & 3 & 3 \\
\bar{1} & \bar{1} & \bar{3} & \bar{3} & \bar{3} & \bar{3} & \bar{3} & \bar{3} & \bar{3} & \bar{3} & \bar{3} \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
\end{array}.$$

Essentially, the description (2-2) of a filling of $\hat{\lambda}$ says that the column to the right of $C_{ij}$ is $C_{i,j+1}$, whereas the column to the right of $C_{ik}$ is $C’_{i,k+1}$. Here we are assuming that the mentioned columns exist, up to the conventions

$$C_{i,\lambda’_i+1} = \begin{cases} C’_{i-1,2} & \text{if } i > 1 \text{ and } \lambda’_{i-1} > 1, \\ C_{i-1,1} & \text{if } i > 1 \text{ and } \lambda’_{i-1} = 1, \end{cases} \quad C’_{i,\lambda’_i+1} = C_{i1}.$$  

(2-3)

The entry in position $i$, counted from the top, in some column $C$ is denoted by $C(i)$. We also write $C[i,j]$ for the portion of column $C$ consisting of the entries in positions $i, i+1, \ldots, j$; this is empty if $i > j$.

We consider the set $\mathcal{F}(\lambda)$ of fillings of $\hat{\lambda}$ with entries in $[\bar{n}]$ that satisfy the following conditions:

(1) The rows are weakly decreasing from left to right.

(2) No column contains two entries $a, b$ with $a = \pm b$.

(3) Each column after the first is related to its left neighbor as indicated in the next paragraph. (Essentially, consecutive columns differ by a signed cycle, that is, a composition $(r_1, j) \ldots (r_p, j)$, where $1 \leq r_1 < \cdots < r_p < j$; furthermore, $j$ varies from 1 to the length of the column in question, as we consider the columns from left to right.)

Here we let the reflections in $B_n$ act on columns $C$ like they do on signed permutations; for instance, $C(a, b)$ is the column obtained from $C$ by transposing the entries in positions $a, b$ and by changing their signs. Let us first explain the passage from some column $C_{ij}$ to $C_{i,j+1}$. There exist positions $1 \leq r_1 < \cdots < r_p < j$ (possibly $p = 0$) such that $C_{i,j+1}$ differs from $D = C_{ij}(r_1, j) \ldots (r_p, j)$ only in
position \( j \), while \( C_{i,j+1}(j) \leq D(j) \). To include the case \( j = \lambda'_i \) in this description, just replace \( C_{i,j+1} \) everywhere by \( C_{i,j+1}[1, \lambda'_i] \) and use the conventions (2-3). Let us now explain the passage from some column \( C'_{ik} \) to \( C'_{i,k+1} \). There exist positions \( 1 \leq r_1 < \cdots < r_p < k \) (possibly \( p = 0 \)) such that \( C'_{i,k+1} = C'_{ik}(r_1, \bar{k}) \cdots (r_p, \bar{k}) \). This description includes the case \( k = \lambda'_i \), based on the conventions (2-3).

Note that the filling \( \sigma \) in Example 2.1 satisfies the above conditions. Indeed, conditions (1) and (2) are clearly verified. Then compare, for instance,

\[
C_{33}[1, 2] = C'_{22}[1, 2] = \frac{2}{3} \quad \text{and} \quad D = C_{32}(1, \bar{2}) = \frac{3}{2} (1, \bar{2}) = \frac{2}{3} ;
\]

they only differ in position 2, while \( C'_{22}(2) = 3 < D(2) = \bar{3} \). Similarly, we have

\[
C'_{24} = C_{21} = C'_{23}(1, \bar{3})(2, \bar{3}) .
\]

Also note that, while the rows are weakly decreasing (from left to right), the columns need not be always increasing or always decreasing (compare \( C'_{32} = C_{31} \) with the other columns).

Let us now define the content of a filling. For this purpose, we first associate with a filling \( \sigma \) a compressed version of it, namely the filling \( \bar{\sigma} \) of the partition \( 2\lambda \). This is defined as follows:

\[
\bar{\sigma} = \bar{c}^{\lambda_1} \cdots \bar{c}^{1} , \quad \text{where} \quad \bar{c}^i := C'_{i2} C_{i1} ,
\]

(2-4)

where the conventions (2-3) are used again. Now define \( \text{ct} \sigma = (c_1, \ldots, c_n) \), where \( c_i \) is half the difference between the number of occurrences of the entries \( i \) and \( \bar{i} \) in \( \bar{\sigma} \). Sometimes, this vector is written in terms of the coordinate vectors \( \varepsilon_i \):

\[
\text{ct} \sigma = c_1 \varepsilon_1 + \cdots + c_n \varepsilon_n = \frac{1}{2} \sum_{b \in \bar{\sigma}} \varepsilon_{\bar{\sigma}(b)} ;
\]

(2-5)

here the last sum is over all boxes \( b \) of \( \bar{\sigma} \), and we set \( \varepsilon_{\bar{i}} := -\varepsilon_i \). In our running example, we have

\[
\bar{\sigma} = \begin{array}{cccccccc}
\text{T} & \text{T} & \text{T} & \text{T} & 2 & 1 & 1 & 1 \\
2 & 2 & 3 & 2 & 2 & 2 & 2 & \text{T} \\
\text{T} & 3 & 3 & 3 & \text{T} & \text{T} & \text{T} & \text{T}
\end{array}
\]

so \( \text{ct} \sigma = (-1, 1, 1) \).

We now define two statistics on fillings in \( \mathcal{F}(\lambda) \) that will be used in our compressed formula for Hall–Littlewood polynomials. Intervals refer to the totally ordered set \( [\bar{n}] \). Let

\[
\sigma_{ab} := \begin{cases} 
1 & \text{if } a, b \geq \bar{n}, \\
0 & \text{otherwise}.
\end{cases}
\]

(2-6)

Given a word \( w \), we use the notation \( N_{ab}(w) \) for the number of entries in \( w \) contained in the interval \( (a, b) \).
Given two columns $D, C$ of the same height $d$ such that $D \geq C$ componentwise, we will define two statistics $N(D, C)$ and des$(D, C)$ in some special cases, as specified below.

**Case 0.** If $D = C$, then $N(D, C) := 0$ and des$(D, C) := 0$.

**Case 1.** Assume that $C = D(r, j)$ with $r < j$. Let $a := D(r)$ and $b := D(j)$. In this case, we set

$$N(D, C) := N_{ba}(D[r+1, j-1]) + \left| (\bar{b}, a) \setminus \{ \pm D(i) : i = 1, \ldots, j \}\right| + \sigma_{ab},$$

and des$(D, C) := 1$.

**Case 2.** Assume that $C = D(r_1, j) \ldots (r_p, j)$, where $1 \leq r_1 < \cdots < r_p < j$. Let $D_i := D(r_1, j) \ldots (r_i, j)$ for $i = 0, \ldots, p$, so that $D_0 = D$ and $D_p = C$. We define

$$N(D, C) := \sum_{i=1}^{p} N(D_{i-1}, D_i), \quad \text{des}(D, C) := p.$$

For instance, in the example above, we have

$$N(C'_{23}C_{21}) = N\left(\begin{array}{cc} 2 & 1 \\ 3 & 2 \\ 1 & 3 \end{array}\right) = N\left(\begin{array}{cc} 2 & 1 \\ 3 & 3 \\ 1 & 2 \end{array}\right) + N\left(\begin{array}{cc} 1 & 1 \\ 3 & 2 \\ 2 & 3 \end{array}\right) = N_{12}(3) + N_{23}(\emptyset) = 0,$$

and des$(C'_{23}C_{21}) = 2$.

**Case 3.** Assume that $C$ differs from $D' := D(r_1, j) \ldots (r_p, j)$ with $1 \leq r_1 < \cdots < r_p < j$ (possibly $p = 0$) only in position $j$, while $C(j) < D'(j)$. We define

$$N(D, C) := N(D, D') + N_{C(j), D'(j)}(D[j+1, d]), \quad \text{des}(D, C) := p + 1.$$

For instance, in our running example, we have

$$N(C_{31}C_{32}) = N\left(\begin{array}{c} 1 \\ 3 \end{array}\right) = N_{3T}(\frac{2}{2}) = 1,$$

and des$(C_{31}C_{32}) = 1$.

If the height of $C$ is larger than the height $d$ of $D$ (necessarily by 1), and $N(D, C[1, d])$ can be computed as above, we let $N(D, C) := N(D, C[1, d])$ and des$(D, C) := \text{des}(D, C[1, d])$. For instance, we have

$$N(C_{32}C'_{22}) = N\left(\begin{array}{c} 3 \\ \frac{2}{2} \end{array}\right) = N\left(\begin{array}{c} \frac{3}{2} \\ \frac{3}{2} \end{array}\right) + N_{33}(\emptyset) = N_{23}(\emptyset) = 0,$$

and des$(C_{32}C'_{22}) = 2$. 
Given a filling $\sigma$ in $\mathcal{F}(\lambda)$ with columns $C_m, \ldots, C_1$, we set

$$N(\sigma) := \sum_{i=1}^{m-1} N(C_{i+1}, C_i) + \ell_+(C_1);$$

here $\ell_+(C_1)$ is defined as in (2-1). We also set

$$\text{des} \sigma := \sum_{i=1}^{m-1} \text{des}(C_{i+1}, C_i).$$

Note that $\text{des} \sigma$ essentially counts the descents in the rows of $\sigma$. In our running example, we have $N(\sigma) = 1$ and $\text{des} \sigma = 6$.

We can now state our new formula for the Hall–Littlewood polynomials of type $C$, which follows as a corollary of our main result, Theorem 4.6. A completely similar formula in type $B$ is discussed in Section 5. We refer to Proposition 2.4 and Remarks 4.7 for more insight into our formula. In particular, the Kashiwara–Nakashima tableaux of type $C$ are, essentially, the surviving fillings in this formula when we set $t = 0$. Furthermore, in some special cases, the statistic $N(\sigma)$ is completely similar to the Haglund–Haiman–Loehr inversion statistic (the more intricate of their two statistics); more precisely, this happens when the related chains in Bruhat order contain no reflections of type $B$, that is $(i, j)$, where $i$ and $j$ are less than the height of the corresponding column of the filling (see Proposition 2.4).

**Theorem 2.2.** Given a regular weight $\lambda$, we have

$$P_\lambda(X; t) = \sum_{\sigma \in \mathcal{F}(\lambda)} t^{N(\sigma)} (1 - t)^{\text{des} \sigma} x^{\text{ct} \sigma},$$

where $x^{(c_1, \ldots, c_n)} := x_1^{c_1} \cdots x_n^{c_n}$.

**Example 2.3.** Consider the simplest case, namely $n = 2$ and $\lambda = (2, 1)$. This leads to considering fillings of the shape $(3, 2)$ with elements in $[2]$, namely

$$\begin{array}{|c|c|}
  e & c \\
  \hline
  a & d \\
  b &
\end{array}$$

The fillings need to satisfy the following conditions:

- $a \leq c \leq e$, $b \leq d$.
- $a \neq \pm b$.
- either $c = a$ and $d = b$, or $c = \bar{b}$ and $d = \bar{a}$.

For $i \in \{1, 2\}$, let $n_i$ be half the difference between the number of $i$’s and $\bar{i}$’s in the multiset $\{a, b, c, d, e, e\}$. Given a proposition $A$, we let $\chi(A)$ be 1 or 0, depending
on the logical value of \( A \) being true or false. Then
\[
P_{(2,1)}(x_1, x_2; t) = \sum_{(a,b,c,d,e)} t^\chi(a>b) + \chi(a,b \leq 2, a\neq c)(1-t)^\chi(c\neq e) x_1^{n_1} x_2^{n_2}.
\]

It turns out that there are 27 terms in this sum, versus 70 terms in (Ram’s version of) Schwer’s formula. For instance, the terms contributing to the coefficient of \( x_2 \) correspond to the fillings
\[
\begin{array}{cccc}
1 & 1 & 1 \\
2 & 2 \\
\end{array}, \quad \begin{array}{cccc}
1 & 2 & 2 \\
1 & 1 \\
\end{array}, \quad \begin{array}{cccc}
2 & 2 & 1 \\
1 & 2 \\
\end{array},
\]
the associated polynomials in \( t \) are
\[
1 - t, \quad t(1-t), \quad 1 - t,
\]
respectively. Note that these polynomials are obtained by compressing 3, 2, and 2 terms in Schwer’s formula, respectively. By symmetry, the coefficients of \( x_1, x_2, x_1^{-1}, \) and \( x_2^{-1} \) in \( P_{(2,1)}(x_1, x_2; t) \) are all \((t+2)(1-t)\). Other fillings have an even larger number of terms in Schwer’s formula corresponding to them, such as
\[
\begin{array}{cccc}
1 & 2 & 2 \\
2 & 1 & 1 \\
\end{array},
\]
which has 7; in other words, the associated polynomial in \( t \), namely \( 1 - t \), which contributes to the coefficient of \( x_1^{-2}x_2^{-1} \), is the sum of 7 polynomials of the form \( t^r(1-t)^s \) in Schwer’s formula. In conclusion, we have
\[
P_{(2,1)}(x_1, x_2; t) = x_1^2x_2 + x_1x_2^2 + x_1^2x_2^{-1} + x_1x_2^{-2} + x_1^{-1}x_2^2 + x_1^{-2}x_2
\]
\[
+ x_1^{-1}x_2^{-2} + x_1^{-2}x_2^{-1} + (t+2)(1-t)(x_1 + x_2 + x_1^{-1} + x_2^{-1}).
\]

In order to relate our statistic \( N(\sigma) \) to the Haglund–Haiman–Loehr inversion statistic and to compare our formula to its type \( A \) counterpart (see [Haglund et al. 2005, Proposition 8.1] or [Lenart 2011, Theorem 2.10]), let us recall some definitions from [Haglund et al. 2005; Lenart 2011]. We start by considering fillings \( \tau \) of the shape \( \lambda \) with entries in \([\tilde{n}]\), which are again displayed in Japanese style, as a sequence of columns \( \tau = C_{\lambda_1} \ldots C_1 \); here \( C_i \) is a sequence \( (C_i(1), \ldots, C_i(\lambda_i')) \), so the entry in cell \( u = (i, j) \) is \( \tau(u) = C_j(i) \). Two cells \( u, v \in \lambda \) are said to \textit{attack} each other if they are in one of the following two relative positions:
\[
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \\
\end{array}, \quad \begin{array}{cc}
\bullet & \\
\bullet & \bullet \\
\end{array}.
\]
An inversion of \( \tau \) is a pair of attacking cells \((u, v)\) that have one of the following two relative positions, where \( a := \tau(u) < b := \tau(v) \):

\[
\begin{array}{ccc}
  a & & b \\
  b & & a
\end{array}
\]

The Haglund–Haiman–Loehr statistic \( \text{inv}\ \tau \) is defined as the number of inversions of \( \tau \). The descent statistic, denoted \( \text{des}\ \tau \) (which is similar to \( \text{des} \) for fillings of \( \hat{\lambda} \) defined above, as seen below), is the number of cells \( u = (i, j) \) with \( j \neq 1 \) and \( \tau(u) > \tau(v) \), where \( v = (i, j-1) \). As usual, let

\[
n(\lambda) := \sum_i (i - 1)\lambda_i,
\]

and assume that \( \tau \) has the following two properties: (i) \( \tau(u) \neq \tau(v) \) whenever \( u \) and \( v \) attack each other; and (ii) \( \tau \) is weakly decreasing in rows. Then it was shown in [Lenart 2011, Proposition 2.12] that the so-called complementary inversion statistic \( \text{cin}v\ \tau := n(\lambda) - \text{inv}\ \tau \) counts the triples of cells filled with \( a < b < c \) that have the following relative position (here the cell supposed to contain \( c \) might be outside the shape \( \lambda \), in which case we only require \( a < b \)):

\[
\begin{array}{ccc}
  b & & c \\
  a & &
\end{array}
\]

**Proposition 2.4.** Let \( \sigma \) in \( \mathcal{F}(\lambda) \) be a filling satisfying the properties that \( C_{i,j+1}' = C_{i,j} \) for all \( i \) and \( j = 2, \ldots, \lambda_i' \); and that \( C_{i,j+1} \) differs from \( C_{ij} \) at most in position \( j \), for all \( i \) and \( j = 1, \ldots, \lambda_i' \). Let \( \tilde{\sigma} \) be the filling of \( \lambda \) given by

\[
\tilde{\sigma} := C_{\lambda_1,1} C_{\lambda_1-1,1} \ldots C_{11}.
\]

Then \( N(\sigma) = \text{cin}v\ \tilde{\sigma} \) and \( \text{des}\ \sigma = \text{des}\ \tilde{\sigma} \).

Before presenting the proof, let us exhibit an example.

**Example 2.5.** For the partition \( \lambda = (4, 3, 2) \) considered above, a filling satisfying the conditions in Proposition 2.4 is

\[
\sigma = \begin{array}{cccccccc}
1 & & & & & & & \\
2 & 2 & 2 & 1 & & & & \\
3 & 3 & 3 & & & & & \\
\end{array}
\]

\[
\begin{array}{cccccccc}
\begin{array}{cccc}
C_{41} & C_{32}' & C_{31} & C_{32} \\
1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 \\
\end{array} & & & & & & & \\
\begin{array}{cccc}
C_{22}' & C_{23}' & C_{21} & C_{22} \\
1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 \\
\end{array} & & & & & & & \\
\begin{array}{cccc}
C_{12}' & C_{13}' & C_{11} \\
1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 \\
\end{array}
\end{array}
\]
Proof of Proposition 2.4. The equality $\text{des} \sigma = \text{des} \tilde{\sigma}$ is clear, so we concentrate on the other equality. Let $m := \lambda_1$ be the number of columns of $\lambda$, and let $C_m = C_m, \ldots, C_1 = C_{11}$ be the columns of $\tilde{\sigma}$, of lengths $c_m := \lambda'_m, \ldots, c_1 := \lambda'_1$; let $C_k := C_k[1, c_k+1]$, for $k = 1, \ldots, m-1$. We refer to a pair $(i, j)$ with $1 \leq i < j \leq c_k$ and $C_k(i) > C_k(j)$ as a (type $A$) inversion in $C_k$. It is easy to see that $\tilde{\sigma}$ satisfies the properties considered above: (i) $\tilde{\sigma}(u) \neq \tilde{\sigma}(v)$ whenever $u$ and $v$ attack each other; (ii) $\tilde{\sigma}$ is weakly decreasing in rows. We start by evaluating $N(\ell^k C_{k-1,1})$, with $\ell^k$ as in (2.2). By definition, $N(\ell^k C_{k-1,1}) = \sum_{i=1}^{\ell_k} N_{C_{k-1}(i), C_k(i)}(C_k[1+i, c_k])$. This is the number of inversions $(i, j)$ in $C_k$ for which $C_{k-1}(i) < C_k(j)$. If $(i, j)$ is an inversion in $C_k$ not satisfying the previous condition, then $C_{k-1}(i) > C_k(j)$ (by property (i) of $\tilde{\sigma}$), and thus $(i, j)$ is an inversion in $C'_{k-1}$ (by property (ii) of $\tilde{\sigma}$). Moreover, the only inversions of $C'_{k-1}$ that do not arise in this way are those counted by the statistic $\text{cinv}(C_k C'_{k-1})$, so $N(\ell^k C_{k-1,1}) = \ell_+(C_k) - (\ell_+(C'_{k-1}) - \text{cinv}(C_k C'_{k-1}))$.

We conclude that $N(\sigma) - \ell_+(C_1) = \sum_{k=2}^{m} \ell_+(C_k) - \ell_+(C'_{k-1}) + \text{cinv}(C_k C'_{k-1})$.

Now recall that $\lambda$ has no two parts identical. We clearly have $c_m = 1$, so $\ell_+(C_m) = 0$. Therefore, $N(\sigma) = \sum_{k=2}^{m} \ell_+(C_k-1) - \ell_+(C'_{k-1}) + \text{cinv}(C_k C'_{k-1}) = \sum_{k=2}^{m} \text{cinv}(C_k C_{k-1}) = \text{cinv} \tilde{\sigma}$. \hfill \Box

3. Background on Ram’s version of Schwer’s formula

We recall some background information on finite root systems and affine Weyl groups.

3.1. Root systems. Let $g$ be a complex semisimple Lie algebra, and $\mathfrak{h}$ a Cartan subalgebra, whose rank is $r$. Let $\Phi \subset \mathfrak{h}^*$ be the corresponding irreducible root system, $\mathfrak{h}_{\mathbb{R}}^* \subset \mathfrak{h}^*$ the real span of the roots, and $\Phi^+ \subset \Phi$ the set of positive roots. Let $\alpha_1, \ldots, \alpha_r \in \Phi^+$ be the corresponding simple roots. We denote by $\langle \cdot, \cdot \rangle$ the
nondegenerate scalar product on $h^*_R$ induced by the Killing form. Given a root $\alpha$, we consider the corresponding coroot $\alpha^\vee := 2\alpha/\langle \alpha, \alpha \rangle$ and reflection $s_{\alpha}$.

Let $W$ be the corresponding Weyl group, whose Coxeter generators are denoted, as usual, by $s_i := s_{\alpha_i}$. The length function on $W$ is denoted by $\ell(\cdot)$. The Bruhat graph on $W$ is the directed graph with edges $u \to w$, where $w = us_\beta$ for some $\beta \in \Phi^+$, and $\ell(w) > \ell(u)$; we usually label such an edge by $\beta$ and write $u \xrightarrow{\beta} w$. The reverse Bruhat graph is obtained by reversing the directed edges above. The Bruhat order on $W$ is the transitive closure of the relation corresponding to the Bruhat graph.

The weight lattice $\Lambda$ is given by

$$\Lambda := \{ \lambda \in h^*_R : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for any } \alpha \in \Phi \}. \quad (3-1)$$

The weight lattice $\Lambda$ is generated by the fundamental weights $\omega_1, \ldots, \omega_r$, which form the dual basis to the basis of simple coroots, that is, $\langle \omega_i, \alpha^\vee_j \rangle = \delta_{ij}$. The set $\Lambda^+$ of dominant weights is given by

$$\Lambda^+ := \{ \lambda \in \Lambda : \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for any } \alpha \in \Phi^+ \}.$$ The subgroup of $W$ stabilizing a weight $\lambda$ is denoted by $W_\lambda$, and the set of minimum coset representatives in $W/W_\lambda$ by $W_\lambda$. Let $\mathbb{Z}[\Lambda]$ be the group algebra of the weight lattice $\Lambda$, which has a $\mathbb{Z}$-basis of formal exponents $\{x^\lambda : \lambda \in \Lambda \}$ with multiplication $x^\lambda \cdot x^\mu := x^{\lambda + \mu}$.

Given $\alpha \in \Phi$ and $k \in \mathbb{Z}$, we denote by $s_{\alpha,k}$ the reflection in the affine hyperplane

$$H_{\alpha,k} := \{ \lambda \in h^*_R : \langle \lambda, \alpha^\vee \rangle = k \}. \quad (3-2)$$

These reflections generate the affine Weyl group $W_{aff}$ for the dual root system $\Phi^\vee := \{ \alpha^\vee : \alpha \in \Phi \}$. The hyperplanes $H_{\alpha,k}$ divide the real vector space $h^*_R$ into open regions, called alcoves. The fundamental alcove $A_0$ is given by

$$A_0 := \{ \lambda \in h^*_R : 0 < \langle \lambda, \alpha^\vee \rangle < 1 \text{ for all } \alpha \in \Phi^+ \}.$$ 3.2. Alcove walks. We say that two alcoves $A$ and $B$ are adjacent if they are distinct and have a common wall. Given two such alcoves, we write $A \xrightarrow{\beta} B$ if the common wall is of the form $H_{\beta,k}$ and the root $\beta \in \Phi$ points in the direction from $A$ to $B$.

**Definition 3.1.** An alcove path is a sequence of alcoves such that any two consecutive ones are adjacent. We say that an alcove path $(A_0, A_1, \ldots, A_m)$ is reduced if $m$ is the minimal length of all alcove paths from $A_0$ to $A_m$.

We need the following generalization of alcove paths.
Definition 3.2. An alcove walk is a sequence
\[ \Omega = (A_0, F_1, A_1, F_2, \ldots, F_m, A_m, F_\infty) \]
such that \( A_0, \ldots, A_m \) are alcoves; \( F_i \) is a codimension-one common face of the alcoves \( A_{i-1} \) and \( A_i \), for \( i = 1, \ldots, m \); and \( F_\infty \) is a vertex of the last alcove \( A_m \). The weight \( F_\infty \) is called the weight of the alcove walk, and is denoted by \( \mu(\Omega) \).

The folding operator \( \phi_i \) is the operator that acts on an alcove walk by leaving its initial segment from \( A_0 \) to \( A_{i-1} \) intact and by reflecting the remaining tail in the affine hyperplane containing the face \( F_i \). In other words, we define
\[ \phi_i(\Omega) := (A_0, F_1, A_1, \ldots, A_{i-1}, F'_i = F_i, A'_i, F'_{i+1}, A'_{i+1}, \ldots, A'_m, F'_\infty); \]
here \( A'_j := \rho_i(A_j) \) for \( j \in \{i, \ldots, m\} \), \( F'_j := \rho_i(F_j) \) for \( j \in \{i, \ldots, m\} \cup \{\infty\} \), and \( \rho_i \) is the affine reflection in the hyperplane containing \( F_i \). Note that any two folding operators commute. An index \( j \) such that \( A_{j-1} = A_j \) is called a folding position of \( \Omega \). Let \( \text{fp}(\Omega) := \{j_1 < \cdots < j_s\} \) be the set of folding positions of \( \Omega \). If this set is empty, \( \Omega \) is called unfolded. We define the operator “unfold”, producing an unfolded alcove walk, by
\[ \text{unfold}(\Omega) = \phi_{j_1} \cdots \phi_{j_s}(\Omega). \]

Definition 3.3. An alcove walk \( \Omega = (A_0, F_1, A_1, F_2, \ldots, F_m, A_m, F_\infty) \) is called positively folded if, for any folding position \( j \), the alcove \( A_{j-1} = A_j \) lies on the positive side of the affine hyperplane containing the face \( F_j \).

We now fix a dominant weight \( \lambda \) and a reduced alcove path
\[ \Pi := (A_0, A_1, \ldots, A_m) \]
from \( A_0 = A_0 \) to its translate \( A_0 + \lambda = A_m \). Assume that
\[ A_0 \xrightarrow{\beta_1} A_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_m} A_m, \]
where \( \Gamma := (\beta_1, \ldots, \beta_m) \) is a sequence of positive roots. This sequence, which determines the alcove path, is called a \( \lambda \)-chain (of roots). Two equivalent definitions of \( \lambda \)-chains (in terms of reduced words in affine Weyl groups, and an interlacing condition) can be found in [Lenart and Postnikov 2007, Definition 5.4] and [Lenart and Postnikov 2008, Definition 4.1 and Proposition 4.4]; note that the \( \lambda \)-chains considered in the these papers are obtained by reversing the ones in this paper. We also let \( r_i := s_{\beta_i} \), and let \( \hat{r}_i \) be the affine reflection in the common wall of \( A_{i-1} \) and \( A_i \), for \( i = 1, \ldots, m \); in other words, \( \hat{r}_i := s_{\beta_i, l_i} \), where \( l_i := |\{j \leq i : \beta_j = \beta_i\}| \) is the cardinality of the corresponding set. Given
\[ J = \{j_1 < \cdots < j_s\} \subseteq [m] := \{1, \ldots, m\}, \]
we define the Weyl group element $\phi(J)$ and the weight $\mu(J)$ by
\begin{equation}
\phi(J) := r_{j_1} \ldots r_{j_s}, \quad \mu(J) := \hat{r}_{j_1} \ldots \hat{r}_{j_s}(\lambda).
\end{equation}

Given $w \in W$, we define the alcove path $w(\Pi) := (w(A_0), w(A_1), \ldots, w(A_m))$. Consider the set of alcove paths
\[ \mathcal{P}(\Gamma) := \{ w(\Pi) : w \in W^{\lambda} \}. \]
We identify any $w(\Pi)$ with the obvious unfolded alcove walk of weight $\mu(w(\Pi)) := w(\lambda)$.

Let us now consider the set of alcove walks
\[ \mathcal{F}_+(\Gamma) := \{ \text{positively folded alcove walks } \Omega : \text{unfold}(\Omega) \in \mathcal{P}(\Gamma) \}. \]
We can encode an alcove walk $\Omega$ in $\mathcal{F}_+(\Gamma)$ by the pair $(w, J)$ in $W^{\lambda} \times 2^{[m]}$, where $\text{fp}(\Omega) = J$ and $\text{unfold}(\Omega) = w(\Pi)$.

Clearly, we can recover $\Omega$ from $(w, J)$ with $J = \{ j_1 < \ldots < j_s \}$ by
\[ \Omega = \phi_{j_1} \ldots \phi_{j_s}(w(\Pi)). \]
Let $\mathcal{A}(\Gamma)$ be the image of $\mathcal{F}_+(\Gamma)$ under the map $\Omega \mapsto (w, J)$. We call a pair $(w, J)$ in $\mathcal{A}(\Gamma)$ an admissible pair, and the subset $J \subseteq [m]$ in this pair a $w$-admissible subset.

**Proposition 3.4 [Lenart 2011].** If $\Omega \mapsto (w, J)$, then $\mu(\Omega) = w(\mu(J))$. Moreover,
\[ \mathcal{A}(\Gamma) = \{(w, J) \in W^{\lambda} \times 2^{[m]} : J = \{ j_1 < \ldots < j_s \}, \quad w > wr_{j_1} > \ldots > wr_{j_1} \ldots r_{j_s} = w\phi(J) \}; \quad (3-4) \]
where the decreasing chain is in the Bruhat order on the Weyl group, its steps not being covers necessarily.

The formula for the Hall–Littlewood $P$-polynomials in [Schwer 2006] was re-derived in [Ram 2006] in a slightly different version, based on positively folded alcove walks. Based on Proposition 3.4, we now restate the latter formula in terms of admissible pairs.

**Theorem 3.5 [Ram 2006; Schwer 2006].** Given a dominant weight $\lambda$, we have
\begin{equation}
P_\lambda(X; t) = \sum_{(w, J) \in \mathcal{A}(\Gamma)} t^{(1/2)(\ell(w)+\ell(w\phi(J))-|J|)} (1-t)^{|J|} \chi^w(\mu(J)). \quad (3-5)\end{equation}
4. Specializing Ram’s version of Schwer’s formula to type $C$

We now restrict ourselves to the root system of type $C_n$. We can identify the space $\mathfrak{h}_R^*$ with $V := \mathbb{R}^n$, the coordinate vectors being $\varepsilon_1, \ldots, \varepsilon_n$. The root system $\Phi$ can be represented as $\Phi = \{\pm\varepsilon_i \pm \varepsilon_j : 1 < i < j \leq n\} \cup \{\pm2\varepsilon_i : 1 \leq i \leq n\}$. The simple roots are $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, for $i = 1, \ldots, n-1$ and $\alpha_n = 2\varepsilon_n$. The fundamental weights are $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$, for $i = 1, \ldots, n$. The weight lattice is $\Lambda = \mathbb{Z}^n$. A dominant weight $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_{n-1} + \lambda_n\varepsilon_n$ is identified with the partition $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n-1 \geq \lambda_n \geq 0)$ of length at most $n$. A dominant weight is regular if all these inequalities are strict: that is, the corresponding partition has all parts distinct and nonzero. We fix such a partition $\lambda$ for the remainder of this paper.

The corresponding Weyl group $W$ is the group of signed permutations $B_n$. For simplicity, we use the same notation for roots and the corresponding reflections (see Section 2). For instance, given $1 \leq i < j \leq n$, we denote by $(i, j)$ the positive root $\varepsilon_i - \varepsilon_j$, by $(i, \bar{j})$ the positive root $\varepsilon_i + \varepsilon_j$, and by $(i, \bar{i})$ the positive root $2\varepsilon_i$.

Let

$$\Gamma(k) := \Gamma'_2 \cdots \Gamma'_k \Gamma_1(k) \cdots \Gamma_k(k),$$

where

$$\Gamma'_j := (1, \bar{j}), \quad (2, \bar{j}), \quad \ldots, \quad (j-1, \bar{j});$$

$$\Gamma_j(k) := \begin{cases} (1, \bar{j}), & (2, \bar{j}), & \ldots, & (j-1, \bar{j}), \\ (j, k+1), & (j, k+2), & \ldots, & (j, n), & (j, \bar{j}), \\ (j, n), & (j, n-1), & \ldots, & (j, k+1) \end{cases}.$$ 

**Lemma 4.1.** $\Gamma(k)$ is an $\omega_k$-chain.

**Proof.** We use the criterion for $\lambda$-chains given in [Lenart and Postnikov 2008, Definition 4.1 and Proposition 4.4] (see also Proposition 10.2 of the same reference). This criterion says that a chain of roots $\Gamma$ is a $\lambda$-chain if and only if it satisfies the following conditions:

(R1) The number of occurrences of any positive root $\alpha$ in $\Gamma$ is $\langle \lambda, \alpha^\vee \rangle$.

(R2) For each triple of positive roots $(\alpha, \beta, \gamma)$ with $\gamma^\vee = \alpha^\vee + \beta^\vee$, the subsequence of $\Gamma$ consisting of $\alpha$, $\beta$, $\gamma$ is a concatenation of pairs $(\gamma, \alpha)$ and $(\gamma, \beta)$ (in any order).

Letting $\lambda = \omega_k = \varepsilon_1 + \cdots + \varepsilon_k$, the first condition is easily checked; for instance, a root $(a, \bar{b})$ appears twice in $\Gamma(k)$ if $a < b \leq k$, once if $a \leq k < b$, and zero times otherwise. For the second condition, we use a case by case analysis, as follows, where $a < b < c$:

1. $\alpha = (a, b)$, $\beta = (b, c)$, $\gamma = (a, c)$;
2. $\alpha = (a, b)$, $\beta = (b, \bar{c})$, $\gamma = (a, \bar{c})$;
3. $\alpha = (a, c)$, $\beta = (b, \bar{c})$, $\gamma = (a, \bar{b})$;
4. $\alpha = (b, c)$, $\beta = (a, \bar{c})$, $\gamma = (a, \bar{b})$;
5. $\alpha = (a, b)$, $\beta = (b, \bar{b})$, $\gamma = (a, \bar{a})$;
6. $\alpha = (a, \bar{a})$, $\beta = (b, \bar{b})$, $\gamma = (a, \bar{b})$. 


Case (1) is the same as in type $A$. Cases (2)–(4) each have the three subcases $k \geq c$, \( b \leq k < c \), and \( a \leq k < b \); while cases (5) and (6) each have the two subcases $k \geq b$ and $a \leq k < b$. For instance, if $b \leq k < c$ in case (3), the subsequence of $\Gamma(k)$ consisting of $\alpha$, $\beta$, $\gamma$ is $((a, \bar{b}), (a, c), (a, \bar{b}), (b, \bar{c}))$. \( \square \)

Hence, we can construct a $\lambda$-chain as a concatenation $\Gamma := \Gamma^{\lambda_1} \ldots \Gamma^1$, where

$$
\Gamma^i = \Gamma(\lambda'_i) = \Gamma'_{i_2} \ldots \Gamma'_{i_{\lambda'_i}} \Gamma_{i_1} \ldots \Gamma_{i_{\lambda'_i}} \quad \text{and} \quad \Gamma_{ij} = \Gamma_j(\lambda'_i), \quad \Gamma'_{ij} = \Gamma'_j. \quad (4-1)
$$

This $\lambda$-chain is fixed for the remainder of this paper. Thus, we can replace the notation $\mathcal{A}(\Gamma)$ with $\mathcal{A}(\lambda)$.

**Example 4.2.** Consider $n = 3$ and $\lambda = (3, 2, 1)$, for which we have the $\lambda$-chain below. The factorization of $\Gamma$ into subchains is indicated with vertical bars, while the double vertical bars separate the subchains corresponding to different columns. The underlined pairs are only relevant in Example 4.3 below.

$$
\begin{align*}
\Gamma &= \Gamma_{31} \parallel \Gamma'_{21} \Gamma_{22} \parallel \Gamma'_{12} \Gamma_{13} \Gamma_{11} \Gamma_{12} \\
&= ((1, \bar{2}), (1, \bar{3}), (1, \bar{1}), (1, 3), (1, 2) \parallel (1, \bar{2}) | (1, \bar{3}), (1, \bar{1}), (1, 3) \parallel (1, \bar{2}), (2, \bar{3}), (2, \bar{2}), (2, 3) \parallel (1, \bar{2}) | (1, \bar{3}), (2, \bar{3}) | (1, \bar{1}) | (1, \bar{2}), (2, \bar{2}) | (1, \bar{3}), (2, \bar{3}), (3, \bar{3})).
\end{align*}
$$

(4-2)

We represent the Young diagram of $\lambda$ inside a broken $3 \times 2$ rectangle, as below. In this way, a reflection in $\Gamma$ can be viewed as swapping entries and/or changing signs in the two parts of each column, or only the top part.

\[
\begin{array}{ccc}
1 & 1 & 1 \\
2 & 2 \\
3 & &
\end{array}
\]

Given the $\lambda$-chain $\Gamma$ above, in Section 3.2 we considered subsets

$$
J = \{j_1 < \cdots < j_s\}
$$

of $[m]$, where $m$ is the length of $\Gamma$. Instead of $J$, it is now convenient to use the subsequence $T$ of the roots in $\Gamma$ whose positions are in $J$. This is viewed as a concatenation with distinguished factors $T_{ij}$ and $T'_{ik}$ induced by the factorization (4-1) of $\Gamma$.

All the notions defined in terms of $J$ are now redefined in terms of $T$. As such, from now on we will write $\phi(T)$, $\mu(T)$, and $|T|$, the latter being the size of $T$; see (3-3). If $J$ is a $w$-admissible subset for some $w$ in $B_n$, we will also call the corresponding $T$ a $w$-admissible sequence, and $(w, T)$ an admissible pair. We
Hall–Littlewood polynomials of type $B$ and $C$ will use the notation $\mathcal{A}(\Gamma)$ and $\mathcal{A}(\lambda)$ accordingly. We denote by $wT_{\lambda_1,1} \ldots T_{ij}$ and $wT_{\lambda_1,1} \ldots T'_{ik}$ the permutations obtained from $w$ via right multiplication by the transpositions in $T_{\lambda_1,1}, \ldots, T_{ij}$ and $T_{\lambda_1,1}, \ldots, T'_{ik}$, considered from left to right. This agrees with the convention of using pairs to denote both roots and the corresponding reflections. As such, $\phi(T)$ can now be written simply as $T$.

**Example 4.3.** We continue Example 4.2, by picking the admissible pair $(w, J)$ with $w = \overline{123} \in B_3$ and $J = \{2, 6, 12, 13\}$ (see the underlined positions in (4-2)). Thus, we have

$$T = T_{31} \parallel T'_{22}T_{21}T_{22} \parallel T'_{12}T'_{13}T_{11}T_{12}T_{13} = ((1, \overline{3}) \parallel (1, \overline{2}) \parallel (2, \overline{3}), (2, 3) \parallel 1)$$

The corresponding decreasing chain in Bruhat order is the following, where the swapped entries are shown in bold (we represent permutations as broken columns starting with $w$, as discussed in Example 4.2, and we display the splitting of the chain into subchains induced by the splitting of $T$ just given):

$$\begin{array}{cccccccc}
\bar{1} & 3 & 2 & 2 & 2 & 2 & 2 & 2 \\
\bar{2} & 1 & 1 & 1 & 1 & 1 & 1 & 3
\end{array}$$

Given a (not necessarily admissible) pair $(w, T)$, with $T$ split into factors $T_{ij}$ and $T'_{ik}$ as above, we consider the permutations

$$\pi_{ij} = \pi_{ij}(w, T) := wT_{\lambda_1,1} \ldots T_{i,j-1}, \quad \pi'_{ik} = \pi'_{ik}(w, T) := wT_{\lambda_1,1} \ldots T'_{i,k-1};$$

when undefined, $T_{i,j-1}$ and $T'_{i,k-1}$ are given by conventions similar to (2-3), based on the corresponding factorization (4-1) of the $\lambda$-chain $\Gamma$. In particular, $\pi_{\lambda_1,1} = w$.

**Definition 4.4.** The *filling map* is the map $f$ from pairs $(w, T)$, not necessarily admissible, to fillings $\sigma = f(w, T)$ of the shape $\hat{\lambda}$, defined (based on the notation (2-2)) by

$$C_{ij} = \pi_{ij}[1, \lambda'_i], \quad C'_{ik} = \pi'_{ik}[1, \lambda'_i]. \quad (4-3)$$

**Example 4.5.** Given $(w, T)$ as in Example 4.3, we have

$$f(w, T) = \begin{array}{cccc}
\bar{1} & 3 & 2 & 2 & 2 \\
\bar{2} & 3 & 2 & 1 & 1 \\
3 & 3 & 3 
\end{array}$$

The following theorem describes the way in which our tableau formula (2-7) is obtained by compressing Ram’s version of Schwer’s formula (3-5). Recall that $\lambda$ is a regular weight, so $B^\lambda_n = B_n$, and thus the pairs $(w, J)$ in $\mathcal{A}(\lambda)$ are only subject to the decreasing chain condition in (3-4); this fact is implicitly used in the proof of the theorem.
Theorem 4.6. (i) We have \( f(\mathcal{A}(\lambda)) = \mathcal{F}(\lambda) \).

(ii) Given any \( \sigma \in \mathcal{F}(\lambda) \) and \( (w, T) \in f^{-1}(\sigma) \), we have \( ct f(w, T) = w(\mu(T)) \).

(iii) The following compression formula holds:

\[
\sum_{(w, T) \in f^{-1}(\sigma)} t^{\frac{1}{2}(\ell(w) + \ell(wT) - |T|)} (1 - t)^{|T|} = t^{N(\sigma)} (1 - t)^{\text{des} \sigma}.
\] (4-4)

Proof. We start with part (i). That we have \( f(\mathcal{A}(\lambda)) \subseteq \mathcal{F}(\lambda) \) is clear from the definition of the set of fillings \( \mathcal{F}(\lambda) \) in Section 2 and the construction (4-1) of the fixed \( \lambda \)-chain \( \Gamma \). Vice versa, given a filling \( \sigma \) in \( \mathcal{F}(\lambda) \), it is not hard to construct an admissible pair \( (w, T) \) in \( f^{-1}(\sigma) \). We will assign to the columns \( C_{ij} \) and \( C'_{ij} \) signed permutations \( \rho_{ij} \) and \( \rho'_{ij} \) in \( B_n \) recursively, starting with \( \rho_{11} := C_{11} \); in parallel, we construct the reverse \( \text{rev} \ T \) of the mentioned chain of roots \( T \), and conclude by letting \( w := \rho_{21,1} \). Each time we pass to the left neighbor \( C'_{ik} \) of a column \( C'_{i,k+1} = C_{ik}(r_1, \bar{k}) \ldots (r_p, \bar{k}) \), we append to the part of \( \text{rev} \ T \) already constructed the roots \( (r_p, \bar{k}), \ldots, (r_1, \bar{k}) \) and let \( \rho'_{ik} := \rho'_{i,k+1}(r_p, \bar{k}) \ldots (r_1, \bar{k}) \). We proceed similarly when passing to the left neighbor \( C_{ij} \) of a column \( C_{i,j+1} \), where \( C_{i,j+1} \) differs from \( D = C_{ij}(r_1, \bar{j}) \ldots (r_p, \bar{j}) \) only in position \( j \); the only difference is that, in this case, we start by applying to \( \rho_{i,j+1} \) and appending to \( \text{rev} \ T \) the reflection that exchanges the entry \( C_{i,j+1}(j) \) with \( D(j) \), and then we proceed as above.

Parts (ii) and (iii) of the theorem are proved in Sections 6 and 7. \( \square \)

Remarks 4.7. (i) The Kashiwara–Nakashima tableaux [1994] of shape \( \lambda \) index the basis elements of the irreducible representation of \( \mathfrak{sp}_{2n} \) of highest weight \( \lambda \). It is shown in Proposition 4.8 below that these tableaux correspond precisely to the surviving fillings in our formula (2-7) when we set \( t = 0 \).

(ii) In (4-4), in general, we cannot replace the filling map \( f \) with the map \( \bar{f} \), sending \( (w, T) \) to the compressed version \( f(w, T) \) of \( f(w, T) \). Indeed, consider \( n = 2, \lambda = (3, 2) \), and the following filling of \( 2\lambda = (6, 4) \), which happens to be the “doubled” version of a Kashiwara–Nakashima tableau:

\[
\tilde{\sigma} = \begin{array}{cccccc}
2 & 2 & 2 & 1 & 1 & \\
\hline
T & T & 2 & 2 & \\
\end{array}
\]

If \( (w, T) \in f^{-1}(\tilde{\sigma}) \), we must have \( w = 2\bar{1} \) and

\[
T \subseteq \Gamma_{21}\Gamma_{22} = ((1, \bar{1}),(1, \bar{2}),(2, \bar{2})) ,
\]

where the full \( \lambda \)-chain factors as follows:

\[
\Gamma = \Gamma_{31} \parallel \Gamma'_{22} \Gamma_{21} \parallel \Gamma'_{12} \Gamma_{11} \Gamma_{12}.
\]
There are two elements \((w, T^1)\) and \((w, T^2)\) in \(\bar{f}^{-1}(\bar{\sigma})\), namely
\[
T^1 = (1, \bar{2}) \quad \text{and} \quad T^2 = ((1, \bar{1}), (1, \bar{2}), (2, \bar{2})).
\]
But we have
\[
\sum_{(w, T) \in \bar{f}^{-1}(\bar{\sigma})} t^{(\ell(w) + \ell(wT) - |T|)}(1-t)^{|T|} = t(1-t) + (1-t)^3 = (1-t)(1-t + t^2).
\]
In general, this sum has several factors not of the form \(t\) or \((1-t)\), which are hard to control.

(iii) To measure the compression phenomenon, we define the compression factor \(c(\lambda)\) as in [Lenart 2011], as the ratio of the number of terms in Ram’s version of Schwer’s formula for \(\lambda\) and the number of terms in the tableau formula. The compression factor is considerably larger in type \(C\). For instance, for \(\lambda = (3, 2, 1, 0)\) in \(C_4\) we have 23,495 terms in the compressed formula, while \(c(\lambda) = 44.9\).

Proposition 4.8. The map \(\sigma \mapsto \bar{\sigma}\) defined by (2-4) is a bijection between the fillings \(\sigma\) in \(\mathcal{F}(\lambda)\) with \(N(\sigma) = 0\) and the “doubled” versions of the type \(C\) Kashiwara–Nakashima tableaux of shape \(\lambda\).

Proof. Adamczak and the author [2009] proved that for each type \(C\) Kashiwara–Nakashima tableau of shape \(\lambda\) there is a unique admissible pair \((w, T)\) whose associated chain in Bruhat order is saturated and ends at the identity, such that the compressed version \(\bar{\sigma}\) of \(\sigma = f(w, T)\) is the “doubled” version of the given tableau. It follows that the term associated to \((w, T)\) in (3-5) is \(t^0(1-t)^{|T|}x^{w(\mu(T))}\), and therefore \(N(\sigma) = 0\), by (4-4). On the other hand, since \(P_\lambda(x; 0)\) is the irreducible character indexed by \(\lambda\), which is expressed in terms of Kashiwara–Nakashima tableaux, we conclude that all \(\sigma\) in \(\mathcal{F}(\lambda)\) with \(N(\sigma) = 0\) arise in this way. □

5. The tableau formula in type \(B\)

We now restrict ourselves to the root system of type \(B_n\). This can be represented as \(\Phi = \{\pm \varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq n\} \cup \{\pm \varepsilon_i : 1 \leq i \leq n\}\). The simple roots are \(\alpha_i = \varepsilon_i - \varepsilon_{i+1}\), for \(i = 1, \ldots, n-1\) and \(\alpha_n = \varepsilon_n\). The fundamental weights are \(\omega_i = \varepsilon_1 + \cdots + \varepsilon_i\), for \(i = 1, \ldots, n-1\) and \(\omega_n = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n)\). A dominant weight \(\lambda = \alpha_1 \omega_1 + \cdots + \alpha_n \omega_n\), where \(\alpha_i \in \mathbb{Z}_{\geq 0}\), is identified with the partition \(\mu = (n^{\alpha_n}, \ldots, 1^{\alpha_1})\); we let \(\ell(\mu) := \alpha_1 + \cdots + \alpha_n\), and write \(\mu = (\mu_1, \ldots, \mu_{\ell(\mu)})\). A dominant weight is regular if \(\alpha_i > 0\) for all \(i\). Let us now fix such a weight \(\lambda\).

The corresponding Weyl group \(W\) is the same group of signed permutations \(B_n\) considered above. For simplicity, we again use the same notation for roots and the corresponding reflections; see Section 2. The pairs \((i, j)\) and \((i, \bar{i})\) have the same meaning as in type \(C\), whereas \((i)\) denotes the positive root \(\varepsilon_i\). Note that, as a reflection in \(B_n\), \((i)\) is the same as \((i, \bar{i})\) in type \(C\).
The canonical $\omega_k$-chains and $\lambda$-chains are very similar to those in type $C$. If $k < n$, let
\[
\Gamma(k) := \Gamma'_1 \ldots \Gamma'_k \Gamma_1(k) \ldots \Gamma_k(k),
\]
where
\[
\Gamma'_j := ((1, j), (2, j), \ldots, (j-1, j), (j));
\]
\[
\Gamma_j(k) := ((1, j), (2, j), \ldots, (j-1, j), (j, k+1), (j, k+2), \ldots, (j, \bar{n}), (j), (j, n), \ldots, (j, k+1)).
\]
On the other hand, we let
\[
\Gamma(n) := \Gamma'_1 \ldots \Gamma'_n = \Gamma_1(n) \ldots \Gamma_n(n).
\]
As in the type $C$ case, we can prove that $\Gamma(k)$ is an $\omega_k$-chain for any $k$. Hence, we can construct a $\lambda$-chain as a concatenation $\Gamma := \Gamma^{\ell(\mu)} \ldots \Gamma^1$, where $\Gamma^i = \Gamma(\mu_i)$.

The filling map is defined as in Definition 4.4. This gives rise to fillings
\[
\sigma = \mathcal{C}^{\ell(\mu)} \ldots \mathcal{C}^1,
\]
where each $\mathcal{C}^i$ is a concatenation of columns of height $\mu_i$:
\[
\mathcal{C}^i := \begin{cases} 
C_{i1} \ldots C_{i,\mu_i} C_{1i} \ldots C_{i,\mu_i} & \text{if } \mu_i < n, \\
C_{i1} \ldots C_{i,\mu_i} & \text{if } i \neq 1 \text{ and } \mu_i = n, \\
C_{11} & \text{if } i = 1.
\end{cases}
\]
The fillings are subject to the same conditions (1)–(3) as in type $C$ in Section 2, where condition (3) is made more precise below. In fact, the $\lambda$-chain $\Gamma$ above specifies the way in which each column is related to its left neighbor. Essentially, everything is similar to type $C$, except for a small difference in the passage from some column $C'_{ik}$ to $C'_{i,k+1}$. Namely, there exist positions $1 \leq r_1 < \cdots < r_p < k$ (possibly $p = 0$) such that $C'_{i,k+1} = C'_{ik}(r_1, \bar{k}) \ldots (r_p, \bar{k})$, like in type $C$, or $C'_{i,k+1} = C'_{ik}(r_1, \bar{k}) \ldots (r_p, \bar{k})(k)$, in which case we also require $C'_{i,k+1}(k) \leq n$.

The weight of a filling and the statistics $N(\sigma)$ and $\des(\sigma)$ are defined completely similarly to type $C$. The only minor addition is the definition of $N(D, C)$ and $\des(D, C)$ when $C = D(r_1, \bar{k}) \ldots (r_p, \bar{k})(k)$. With the notation in Case 2 of the definition of $N(D, C)$, we set
\[
N(D, C) := N(D, D_p) + N(D_p, C), \quad \des(D, C) := p + 1.
\]
Here $N(D, D_p)$ is defined in Case 2, whereas $N(D_p, C)$ is given by the second formula in (7-1); more precisely,
\[
N(D_p, C) := \frac{1}{2} \left| (\bar{a}, a) \setminus \{ \pm D_p(i) : i = 1, \ldots, k \} \right|,
\]
where $a := D_p(k)$. 

Given these constructions, the proof of the following theorem is completely similar to its counterparts in type $C$, since no new situations arise.

**Theorem 5.1.** Theorems 2.2 and 4.6 hold in type $B$, with the appropriate constructions explained above.

**Remark 5.2.** The situation in type $D$ is slightly more complex. In this case, applying the preceding ideas leads to an analog of the compression formula (4-4) that contains factors of the form $1 - t^k$ with $k > 1$ in the right side. However, these factors are not hard to control, while no extra factors appear.

### 6. Proof of Theorem 4.6(ii)

Recall the $\lambda$-chain $\Gamma$ in Section 4. Let us write $\Gamma = (\beta_1, \ldots, \beta_m)$, as in Section 3.2. As such, we recall the hyperplanes $H_{\beta_k, l_k}$ and the corresponding affine reflections $\hat{r}_k = s_{\beta_k, l_k} = s_{\beta_k} + l_k \beta_k$.

Now fix a signed permutation $w$ in $B_n$ and a subset $J = \{j_1 < \cdots < j_s\}$ of $[m]$ (not necessarily $w$-admissible). Let $\Pi$ be the alcove path corresponding to $\Gamma$, and define the alcove walk $\Omega$ as in Section 3.2, by

$$\Omega := \phi_{j_1} \cdots \phi_{j_s}(w(\Pi)).$$

Given $k$ in $[m]$, let $i = i(k)$ be the largest index in $[s]$ for which $j_i < k$, and let $\gamma_k := w r_{j_1} \cdots r_{j_i} (\beta_k)$. Then the hyperplane containing the face $F_k$ of $\Omega$ (see Definition 3.2) is of the form $H_{\gamma_k, m_k}$; in other words,

$$H_{\gamma_k, m_k} = w \hat{r}_{j_1} \cdots \hat{r}_{j_i} (H_{\beta_k, l_k}).$$

Our first goal is to describe $m_k$ purely in terms of the filling associated to $(w, J)$.

Let $\hat{t}_k$ be the affine reflection in the hyperplane $H_{\gamma_k, m_k}$. Note that

$$\hat{t}_k = w \hat{r}_{j_1} \cdots \hat{r}_{j_i} \hat{r}_k \hat{r}_{j_i} \cdots \hat{r}_{j_1} w^{-1}.$$

Thus, we can see that

$$w \hat{r}_{j_1} \cdots \hat{r}_{j_i} = \hat{t}_{j_i} \cdots \hat{t}_{j_1} w.$$

Let $T = ((a_1, b_1), \ldots, (a_s, b_s))$ be the subsequence of $\Gamma$ indexed by the positions in $J$; see Section 4. Let $T^i$ be the initial segment of $T$ with length $i$, let $w_i := w T^i$, and let $\sigma_i := f(w, T^i)$; see (2-4). In particular, $\sigma_0$ is the filling with all entries in row $i$ equal to $w(i)$, and $\sigma := \sigma_s = f(w, T)$. The columns of a filling of $2\lambda$ are numbered left to right by $2\lambda_1$ to 1. We split each segment $\Gamma_k^i$ of $\Gamma$ into two parts: the head, which is a concatenation of $\Gamma_k^i$, and the tail, which is a concatenation of $\Gamma_k^t$; see (4-1). We say that the head corresponds to column $2k$ of the Young diagram $2\lambda$, whereas the tail corresponds to column $2k - 1$ (see the construction of $f(w, T)$ in Section 4 and (2-4)). If $\beta_{j_i+1} = (a_{i+1}, b_{i+1}) = (a, b)$ falls in the
segment of \( \Gamma \) corresponding to column \( p \) of \( 2\lambda \), then \( \sigma_{i+1} \) is obtained from \( \sigma_i \) by replacing the entry \( w_i(a) \) with \( w_i(b) \) in the columns \( p-1, \ldots, 1 \) of \( \sigma_i \), as well as, possibly, the entry \( w_i(\tilde{b}) \) with \( w_i(\tilde{a}) \) in the same columns.

Now fix a position \( k \), and consider \( i = i(k) \) and the roots \( \beta_k, \gamma := \gamma_k \), as above, where \( \gamma_k \) might be negative. Assume that \( \beta_k \) falls in the segment of \( \Gamma \) corresponding to column \( q \) of \( 2\lambda \). Given a filling \( \phi \), we denote by \( \phi[\pi] \) the part of \( \phi \) consisting of columns \( 2\lambda_1, 2\lambda_1-1, \ldots, \pi \), and by \( \phi^{(p,q)} \) the part consisting of columns \( p-1, p-2, \ldots, q \). We also recall the definition (2-5) and conventions related to the content of a filling; this definition now applies to any filling of some Young diagram.

**Proposition 6.1.** With the same notation, we have

\[
m_k = (\text{ct} \sigma^{[q]}_i, \gamma').
\]

**Proof.** We apply induction on \( i \), which starts at \( i = 0 \), when the verification is straightforward. We now proceed from \( j_1 < \cdots < j_i < k \), where \( i = s \) or \( k \leq j_{i+1} \), to \( j_1 < \cdots < j_{i+1} < k \), and we freely use the notation above.

Assume that \( \beta_{j_{i+1}} \) falls in the segment of \( \Gamma \) corresponding to column \( p \) of \( 2\lambda \), where \( p \geq q \).

We need to compute

\[
w_{\hat{i}_j_1} \cdots \hat{i}_{j_{i+1}}(H_{\beta_k}, \lambda_k) = \hat{i}_{j_{i+1}} \cdots \hat{i}_j w(H_{\beta_k}, \lambda_k) = \hat{i}_{j_{i+1}}(H_{\gamma', \lambda}).
\]

where \( m = (\text{ct} \sigma^{[q]}_i, \gamma') \), by induction. Let \( \gamma' := \gamma_{j_{i+1}} \), and \( \hat{i}_{j_{i+1}} = s_{\gamma', \lambda'} \), where \( m' = (\text{ct} \sigma^{[\pi]}_i, (\gamma')') \), by induction. We use the formula

\[
s_{\gamma', \lambda'}(H_{\gamma, \lambda}) = H_{s_{\gamma', \lambda}(\gamma), \lambda-m'(\gamma', \lambda')}.\]

Thus, the proof is reduced to showing that

\[
m - m'(\gamma', \gamma') = (\text{ct} \sigma^{[q]}_{i+1}, s_{\gamma'}(\gamma')).\]

An easy calculation, based on the information above, shows that the latter equality is nontrivial only if \( p > q \), in which case it is equivalent to

\[
(\text{ct} \sigma^{(p,q)}_{i+1} - \text{ct} \sigma^{(p,q)}_i, \gamma') = (\gamma', \gamma') (\text{ct} \sigma^{(p,q)}_{i+1}, (\gamma')').\]

This equality is a consequence of

\[
\text{ct} \sigma^{(p,q)}_{i+1} = s_{\gamma'}(\text{ct} \sigma^{(p,q)}_i),
\]

which follows from the construction of \( \sigma_{i+1} \) from \( \sigma_i \) explained above. \( \square \)

**Proof of Theorem 4.6(ii).** We apply induction on the size of \( T \), using freely the notation above. We prove the statement for \( T = (\beta_{j_1}, \ldots, \beta_{j_{i+1}}) \), assuming it holds
for $T^s = (\beta_{j_1}, \ldots, \beta_{j_s})$. We have

$$w(\mu(T)) = w\hat{r}_{j_1} \cdots \hat{r}_{j_{s+1}}(\lambda) = \hat{t}_{j_{s+1}} \cdots \hat{t}_{j_1} w(\lambda) = \hat{t}_{j_{s+1}}(ct\sigma_s),$$

by induction. We need to show that

$$\hat{t}_{j_{s+1}}(ct\sigma_s) = ct\sigma_{s+1}. \quad (6-1)$$

Let $\gamma := \gamma_{j_{s+1}}$ and assume that $\beta_{j_{s+1}}$ falls in the segment of $\Gamma$ corresponding to column $p$ of $2\lambda$. Based on Proposition 6.1, (6-1) is rewritten as

$$s_\gamma (ct\sigma_s) + \langle ct\sigma_s[p], \gamma^\vee \rangle \gamma = ct\sigma_{s+1}. \quad (6-2)$$

Decomposing $ct\sigma_s$ as $ct\sigma_s[p] + ct\sigma_s[p,1]$ (using the notation above), and $ct\sigma_{s+1}$ in a similar way, (6-2) would follow from

$$s_\gamma (ct\sigma_s[p]) + \langle ct\sigma_s[p], \gamma^\vee \rangle \gamma = ct\sigma_{s+1}^{[p]},$$

$$s_\gamma (ct\sigma_s^{[p,1]}) = ct\sigma_{s+1}^{[p,1]}.$$

The first equality is clear since $\sigma_s[p] = \sigma_s[p]$, while the second one follows from the construction of $\sigma_{s+1}$ from $\sigma_s$ explained above. $\square$

7. Proof of Theorem 4.6(iii)

We start by recalling some basic facts about the group $B_n$ and some notation from Section 2. We will use the following notation related to a word $w = w_1 \cdots w_l$ with integer letters, which is sometimes identified with its set of letters

$$w[i, j] := w_i \cdots w_j, \quad N_{ab}(w) := |(a, b) \cap w|, \quad N_{ab}(\pm w) := N_{ab}(w) + N_{ab}(-w),$$

where $-w := \overline{w_1} \cdots \overline{w_l}$. Given words $w^1, \ldots, w^p$, we let

$$N_{ab}(w^1, \ldots, w^p) := N_{ab}(w^1) + \cdots + N_{ab}(w^p).$$

We also let

$$\tau_{ab} := \begin{cases} 1 & \text{if } a, b \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

With this notation, given a signed permutation $w$ in $B_n$ and $1 \leq i < j \leq n$, $a := w(i)$, $b := w(j)$, we have

$$\frac{\ell(w(i, j)) - \ell(w) - 1}{2} = N_{ab}(w[i, j]),$$

$$\frac{\ell(w(i, j)) - \ell(w) - 1}{2} = N_{aa}(w[i, j]),$$

$$\frac{\ell(w(i, j)) - \ell(w) - 1}{2} = N_{ab}(w[i, j-1], \pm w[j+1, n]) + \tau_{ab}. \quad (7-1)$$
assuming that the left side is nonnegative (that is, that we go up in Bruhat order upon applying the indicated reflection); these facts are used implicitly throughout.

Given a chain of roots \( \Delta \), we define \( \mathcal{A}'(\Delta) \) as in (3-4) except that we impose an increasing chain condition and \( w \in W \). The following simple lemma will be useful throughout, for splitting chains into subchains.

**Lemma 7.1.** Consider \((w, T)\) with \( T \) written as a concatenation \( S_1 \ldots S_p \). Let \( w_i := wS_1 \ldots S_i, \) so \( w_0 = w \). Then

\[
\frac{1}{2}(\ell(w)+\ell(wT)-|T|) = \frac{1}{2}(\ell(w_{p-1})+\ell(w_p)-|S_p|) + \sum_{i=1}^{p-1} \frac{1}{2}(\ell(w_{i-1})-\ell(w_i)-|S_i|)
\]

Let \( \Delta \) be the chain

\[
\Delta := \big( (1, p+1), (1, p+2), \ldots, (1, n), (1, \overline{1}), (1, \overline{n}), (1, n-1), \ldots, (1, p+1) \big).
\]

**Proposition 7.2.** Consider a signed permutation \( w \) in \( B_n \) with \( a := w(1) \), a position \( 1 \leq p \leq n \), and a value \( b \in \{ \pm a \} \cup (\pm w[p+1, n]) \) such that \( b \geq a \). Then

\[
\sum_{T: (w, T) \in \mathcal{A}'(\Delta)} t^{\frac{1}{2}(\ell(wT)-\ell(w)-|T|)} (1-t)^{|T|} = t^{N_{ab}(w[2, p])} (1-t)^{1-\delta_{ab}},
\]

where \( \delta_{ab} \) is the Kronecker delta.

**Proof.** Given \( s \in \{ \overline{1}, \pm(p+1), \ldots, \pm n \} \), we let \( \Delta_s \) be the subchain of \( \Delta \) starting with \((1, s)\). We also let

\[
S(w, s) := \sum_{T: (w, T) \in \mathcal{A}'(\Delta_s)} t^{\frac{1}{2}(\ell(wT)-\ell(w)-|T|)} (1-t)^{|T|}.
\]

We consider three cases: \( b = w(q), b = \overline{w(q)}, \) and \( b = \overline{a} \). The proof in the first case is identical to that of the analogous result for type \( A \), namely [Lenart 2011, Proposition 5.3].

**Case (ii):** \( b = \overline{w(q)} \). Let \( c := w(q) = \overline{b} \) and \( p < q \leq s \). We start by showing that

\[
S(w, \overline{s}) = t^{N_{ac}(w[2, q-1], w[q+1, s], \pm w[s+1, n]+\tau_{ac})} (1-t).
\]

We use induction on \( s \), which starts at \( s = q \). For \( s > q \), let \( w^1 := w[2, q-1], w^2 := w[q+1, s-1], w^3 := w[s+1, n], \) and \( d := w(s) \). The sum \( S(w, \overline{s}) \) splits into two sums: over \( s \) such that \( (1, \overline{s}) \notin T \) and such that \( (1, \overline{s}) \in T \). By induction, the first sum is

\[
S(w, s-1) = t^{N_{ac}(w^1, w^2, \pm dw^3)+\tau_{ac}} (1-t).
\]
Again by induction, if \( a < \bar{d} < \bar{c} \), then the second sum is

\[
t^{N_{ad}(w^1 c w^2, \pm w^3) + \tau_{ad}} (1 - t) S(w(1, \bar{s}), s - 1) = t^{N_{ad}(w^1 c w^2, \pm w^3) + N_{\bar{c}e}(w^1, w^2, \pm \bar{a} w^3) + \tau_{ad} + \tau_{\bar{c}e}} (1 - t)^2;
\]

otherwise, it is empty. Adding up the two sums into which \( S(w, \bar{s}) \) splits, we obtain

\[
t^{N_{ae}(w^1, w^2 d, \pm w^3) + \tau_{ac}} (1 - t).
\]

This last claim rests on the easily verified facts that if \( a < \bar{d} < \bar{c} \), then

\[
\tau_{ad} + \tau_{\bar{c}e} = \tau_{ac}, \quad N_{ad}(c) + N_{\bar{c}e}(\bar{a}) = N_{ae}(d).
\]

Still assuming that \( c = w(q) = \bar{b} \) and \( p < q \), we now show that

\[
S(w, \bar{1}) = t^{N_{ae}(w[2.q-1], w[q+1,n]) + \tau_{ac}'} (1 - t), \tag{7-4}
\]

where

\[
\tau_{ac}' := \begin{cases} 1 & \text{if } a < c \leq n, \\ 0 & \text{otherwise}. \end{cases}
\]

Let \( w^1 := w[2, q-1] \), as before, and let \( w^2 := w[q+1, n] \). The sum \( S(w, \bar{1}) \) splits into two sums, depending on whether \( (1, \bar{1}) \notin T \) or and \( (1, \bar{1}) \in T \). By (7-3), the first sum is

\[
S(w, \bar{n}) = t^{N_{ae}(w^1, w^2) + \tau_{ac} (1 - t)}.
\]

Again by (7-3), if \( c < a \leq n \), then the second sum is

\[
t^{N_{ae}(w^1 c w^2)} (1 - t) S(w(1, \bar{1}), \bar{n}) = t^{N_{ae}(w^1 c w^2) + N_{\bar{c}e}(w^1, w^2) + \tau_{\bar{c}e} (1 - t)^2};
\]

otherwise, it is empty. Adding up the two sums into which \( S(w, \bar{s}) \) splits, we obtain

\[
t^{N_{ae}(w^1, w^2) + \tau_{ac}'} (1 - t).
\]

Assuming that \( c = w(q) = \bar{b} \) and \( p < q < s \), we now show that

\[
S(w, s) = t^{N_{ae}(w[2,q-1], w[q+1,s-1]) + \tau_{ac}'} (1 - t). \tag{7-5}
\]

We use decreasing induction on \( s \). As before, we let \( w^1 := w[2, q-1] \), \( w^2 := w[q+1, s-1] \), and \( d := w(s) \). The sum \( S(w, s) \) splits into two sums, depending on whether \( (1, s) \notin T \) or \( (1, s) \in T \). By induction, the first sum is

\[
S(w, s + 1) = t^{N_{ae}(w^1, w^2 d) + \tau_{ac}'} (1 - t).
\]

Again by induction, if \( a < d < \bar{c} \), then the second sum is

\[
t^{N_{ad}(w^1 c w^2)} (1 - t) S(w(1, s), s + 1) = t^{N_{ad}(w^1 c w^2) + N_{\bar{e}}(w^1, w^2 a) + \tau_{ac}'(1 - t)^2};
\]
otherwise, it is empty. (In both calculations, induction works by substituting $\bar{\tau}$ for $n + 1$ when $s = n$, and by using (7-4) in this case.) Adding up the two sums into which $S(w, s)$ splits, we obtain

$$t^{N_{ac}(w^1, w^2)} + t'_{ac}(1 - t).$$

This last claim rests on the easily verified fact that if $a < d < \bar{c}$, then

$$N_{ad}(c) + t'_{dc} = t'_{ac}.$$

**Case (iii):** $b = \bar{a}$. We need to show that

$$S(w, p + 1) = t^{N_{a\overline{a}}(w^2, p)}(1 - t).$$

We use decreasing induction on $p$, which starts at $p = n$; in this case $\Delta$ only contains the pair $(1, \bar{1})$, so the convention of substituting $\bar{\tau}$ for $n + 1$ works well here too. For $p < n$, we let $d := w(p + 1)$. The sum $S(w, p + 1)$ splits into two sums, depending on whether $(1, p + 1) \not\in T$ or $(1, p + 1) \in T$. By induction, the first sum is

$$S(w, p + 2) = t^{N_{a\bar{a}}(w^2, p)[d]}(1 - t).$$

If $a < d < \bar{a}$, then by (7-5) of case (ii), the second sum is

$$t^{N_{ad}(w^2, p)}(1 - t)S(w(1, p + 1), p + 2) = t^{N_{ad}(w^2, p) + N_{a\overline{a}}(w^2, p) + \tau'_{da}}(1 - t)^2;$$

otherwise, it is empty. Adding up the two sums into which $S(w, p + 1)$ splits, we obtain the desired result.

**Case (ii) (continued).** Assuming that $c = w(q) = \bar{b}$ and $p < q$, we now show that

$$S(w, q) = t^{N_{a\overline{a}}(w^2, q - 1)}(1 - t).$$

The sum $S(w, q)$ splits into two sums, depending on whether $(1, q) \not\in T$ or $(1, q) \in T$. By (7-5) of case (ii), the first sum is

$$S(w, q + 1) = t^{N_{a\overline{a}}(w^2, q - 1) + \tau'_{a}}(1 - t).$$

If $a < c \leq n$, then by (7-6) of case (iii), the second sum is

$$t^{N_{ac}(w^2, q - 1)}(1 - t)S(w(1, q), q + 1) = t^{N_{ac}(w^2, q - 1) + N_{c\overline{c}}(w^2, q)}(1 - t)^2;$$

otherwise, it is empty. Adding up the two sums into which $S(w, q)$ splits, we obtain the desired result.

The final step in case (ii) is to prove that

$$S(w, p + 1) = t^{N_{ac}(w^2, p)}(1 - t).$$
This can be done by decreasing induction on \( p \), starting with \( p = q - 1 \), which is the case proved in (7-7). The procedure is completely similar to the ones above, and, in fact, to the one for type \( A \) in [Lenart 2011, Proposition 5.3]. \( \square \)

Consider the chain
\[
\Phi := \Gamma_1(n) \ldots \Gamma_n(n) = (1, \bar{1}), \quad (1, 2), (2, \bar{2}), \quad \ldots \quad (1, \bar{n}), (2, \bar{n}), \ldots, (n-1, \bar{n}) \, .
\]

(7-9)

We denote by \( \Phi_{ij} \) the subchain of \( \Phi \) starting with \((i, \bar{j})\). Given a signed permutation \( w \), recall the definition (2-1) of \( \ell_+(w) \) and \( \ell_-(w) \). Given \((i, j)\) with \( 1 \leq i \leq j \leq n \), we also define
\[
\ell^i_j(w) := |\{(k, l) : (k, \bar{l}) \in \Phi \setminus \Phi_{ij}, w(k) > \overline{w(l)}\}|, \\
\ell^i_j := \ell_-(w) - \ell^i_j(w).
\]

(7-10)

**Proposition 7.3.** Fix \((i, j)\) with \( 1 \leq i \leq j \leq n \) and a signed permutation \( w \) in \( B_n \). We have
\[
\sum_{T : (w, T) \in \mathcal{E}(\Phi_{ij})} t^{\frac{1}{2}(\ell(w)+\ell(wT)-|T|)} (1 - t)^{|T|} = t^{\ell_+(w)+\ell^i_j(w)} .
\]

(7-11)

In particular, if this sum is over \((w, T) \in \mathcal{E}(\Phi)\), then the right side is \( t^{\ell_+(w)} \).

**Proof.** Let us denote the sum in the left side of (7-11) by \( S(w, i, j) \), and the corresponding sum over \((w, T) \in \mathcal{E}(\Phi_{ij} \setminus \{(i, \bar{j})\})\) by \( S'(w, i, j) \). We view the chain \( \Phi \) as a total order on the pairs \((i, \bar{j})\), with \((1, \bar{1})\) being the smallest pair. With this in mind, we use decreasing induction on pairs \((i, \bar{j})\). Given such a pair, if \( w(i) < \overline{w(j)} \), then the induction step is clear, so assume the contrary. We can now split \( S(w, i, j) \) into two sums, depending on whether \((i, \bar{j}) \notin T \) or \((i, \bar{j}) \in T \). By induction, the first sum is
\[
S'(w, i, j) = t^{1+\ell_+(w)+\ell^i_j(w)} .
\]

By induction and Lemma 7.1, the second sum is
\[
(1 - t)t^{\frac{1}{2}(\ell(w)-\ell(w(i, \bar{j}))-1)} S'(w(i, \bar{j}), i, j) = (1 - t)t^{\frac{1}{2}(\ell(w)-\ell(w(i, \bar{j}))-1)+\ell_+(w(i, \bar{j}))+\ell^i_j(w(i, \bar{j}))} .
\]

The induction step is completed once we show that
\[
\ell_+(w) + \ell^i_j(w) = \frac{1}{2} (\ell(w) - \ell(w(i, \bar{j})) - 1) + \ell_+(w(i, \bar{j})) + \ell^i_j(w(i, \bar{j})) .
\]
This equality can be rewritten as
\[ \Delta \ell_+(w) + \Delta \ell_{ij}^+(w) - 1 = \Delta \ell_{ij}^-(w), \]
where \( \Delta \ell_+(w) := \ell_+(w) - \ell_+(w(i, j)) \), and similarly for the other two variations. To prove this, we first fix \( k \) between \( i \) and \( j \), and analyze the contribution to the three variations of the pairs \((i, k)\) and \((k, j)\); see (2-1) and (7-10). For simplicity, let \( a := w(i) \), \( b := w(k) \), and \( c := w(j) \), where \( a > c \). The nonzero contributions are as follows:

- the pair \((i, k)\) contributes 1 to \( \Delta \ell_+(w) \) if \( a > b > c \);
- the pair \((k, j)\) contributes -1 to \( \Delta \ell_+(w) \) if \( a < b < c \), which is equivalent to \( a > b > c \);
- the pair \((i, k)\) contributes 1 to \( \Delta \ell_{ij}^+(w) \) if \( a > b > c \);
- the pair \((k, j)\) contributes 1 to \( \Delta \ell_{ij}^-(w) \) if \( a > b > c \).

The second and third contributions cancel out, whereas the first and fourth are equal. The analysis is completely similar if \( k < i \) or \( k > j \). The pair \((i, j)\) only contributes 1 to \( \Delta \ell_{ij}^+(w) \). As far as the pairs \((i, j)\) and \((j, k)\) are concerned, the contribution of the first one to \( \Delta \ell_{ij}^+(w) \) and of the second one to \( \Delta \ell_{ij}^-(w) \) are both equal to \( \sigma_{ac} \); see (2-6).

**Proof of Theorem 4.6(iii).** Fix a filling \( \sigma \) in \( \bar{F}(\lambda) \) with columns \( C_{ij} \) and \( C_{ij}' \), as explained in Section 2. Recall the chain \( \Phi := \Gamma_1(n) \ldots \Gamma_n(n) = \Gamma_{11} \ldots \Gamma_{nn} \) in (7-9). By splitting the \( \lambda \)-chain \( \Gamma \) into the tail \( \Phi \) and its complement, and by using Lemma 7.1, the sum in the left side of (4-4) can be rewritten as

\[
\sum_{(w, T) \in f^{-1}(\sigma)} t^{\frac{1}{2}( \ell(w) + \ell(wT) - |T| )} (1 - t)^{|T|} \left( \sum_{(w, T) \in f^{-1}(\sigma)} t^{(1/2)( \ell(w) - \ell(wT) - |T| )} (1 - t)^{|T|} \right) \times \left( \sum_{T: (C_{11}, T) \in \mathcal{A}(\Phi)} t^{\frac{1}{2}( \ell(C_{11}) + \ell(C_{11}T) - |T| )} (1 - t)^{|T|} \right). \tag{7-12}
\]

Here the column \( C_{11} \), which has height \( n \), is viewed as a signed permutation in \( B_n \). By Proposition 7.3, the second bracket is \( t^{\ell_+ (C_{11})} \).

To evaluate the first bracket, we reverse all chains. Let us start by recalling the construction (4-1) of the \( \lambda \)-chain \( \Gamma \), and in particular the order in which the subchains \( \Gamma_{ij} \) and \( \Gamma_{ij}' \) are concatenated (including the conventions in Section 2 related to \( \Gamma_{i,j+1} \) and \( \Gamma_{i,j+1}' \)). We denote by \( \Gamma_{ij}'^r \) and \( (\Gamma_{ij})^r \) the corresponding reverse chains. Also recall that we defined \( \mathcal{A}'(\cdot) \) as in (3-4) except that we imposed an
increasing chain condition and $w \in W$. We consider pairs $(w_{ij}, S_{ij})$ in $\mathcal{A}^r(\Gamma_{ij}^r)$ and $(w'_{ij}, S'_{ij})$ in $\mathcal{A}^r((\Gamma_{ij}^r)^c)$, where $w_{ij}$ and $w'_{ij}$ are defined by

$$w_{ij} := C_{11}S'_{1,\lambda'_1} \ldots S_{i,j+1}, \quad w'_{ij} := C_{11}S'_{1,\lambda'_1} \ldots S_{i,j+1},$$

where the concatenation order for $S_{ij}$ and $S'_{ij}$ comes from that for $\Gamma_{ij}$ and $\Gamma_{ij}^r$; in particular, $w'_{1,\lambda'_1} = C_{11}$. Given this notation, we define the sum

$$\Sigma_{ij} := \sum_{S_{ij}, (w_{ij}, S_{ij}) \in \mathcal{A}^r(\Gamma_{ij}^r)} \sum_{w_{ij}S_{ij}[1, \lambda'_1] = C_{ij}} t^\frac{1}{2}(\ell(w_{ij}S_{ij}) - \ell(w_{ij}) - |S_{ij}|)(1 - t)|S_{ij}|,$$

and the similar sum $\Sigma'_{ij}$. We can now evaluate the first bracket in the right side of (7-12):

$$\sum_{(w, T) \in f^{-1}(\sigma)} \sum_{T_{11} = \ldots = T_{1n} = \emptyset} t^\frac{1}{2}(\ell(w) - \ell(wT) - |T|)(1 - t)^{|T|} = \Sigma_{i,1,1} \ldots \Sigma_{i,j} \ldots \Sigma'_{1,1,1}.$$}

In fact, we first write the sum in the left side as an iterated sum, which factors in the way shown above because $\Sigma_{ij}$ only depends on $w_{ij}[1, \lambda'_i] = C_{i,j+1}[1, \lambda'_i]$ (rather than the whole $w_{ij}$), by Proposition 7.2.

We conclude the proof by calculating the sum $\Sigma_{ij}$, the calculation for $\Sigma'_{ij}$ being similar but simpler. For simplicity, let $d := \lambda'_i$, $w = w_{ij}$, $C := C_{i,j+1}[1, \lambda'_i]$, and $D := C_{ij}$. Assume that $C$ differs from $D' := D(r_1, \bar{j}) \ldots (r_p, \bar{j})$ with $1 \leq r_1 < \cdots < r_p < j$ (possibly $p = 0$) only in position $j$. Let $\Gamma_{ij}^* = \Delta \Delta'$, where

$$\Delta := ((j, d+1), (j, d+2), \ldots, (j, n),
  (j, \bar{j}),
  (j, n), (j, n-1), \ldots, (j, d+1)),$$

$$\Delta' := ((j-1, \bar{j}), \ldots, (2, \bar{j}), (1, \bar{j})).$$

Correspondingly, the chains $S_{ij}$ split into a head $S$, which can vary, and a fixed tail $S' := ((r_p, \bar{j}), \ldots, (r_1, \bar{j})).$

We have

$$\Sigma_{ij} = t^e(1 - t)^p \sum_{S_{[(w, S) \in \mathcal{A}^r(\Delta)]} \omega_{S(j)} = D'(j)} t^\frac{1}{2}(\ell(wS') - \ell(wS) - p)(1 - t)^{|S'|},$$

where $e := \frac{1}{2}(\ell(wSS') - \ell(wS) - p)$. By Proposition 7.2, the sum in the right side is

$$t^{N_{C(j), D'(j)}(\ell[j+1, d])}(1 - t);$$

note that this sum is missing when $D' = C$, which is another possibility. The exponent $e$ is calculated based on (7-1).
References


Hall–Littlewood polynomials of type $B$ and $C$


Communicated by Georgia Benkart

Received 2009-07-16 Revised 2010-07-11 Accepted 2010-10-13

lenart@albany.edu

Department of Mathematics and Statistics, State University of New York at Albany, 1400 Washington Avenue, Albany, NY 12222, United States

http://www.albany.edu/~lenart
Hochschild cohomology and homology of quantum complete intersections 821  
**Steffen Oppermann**

Meromorphic continuation for the zeta function of a Dwork hypersurface 839  
**Thomas Barnet-Lamb**

Equations for Chow and Hilbert quotients 855  
**Angela Gibney** and **Diane Maclagan**

Haglund–Haiman–Loehr type formulas for Hall–Littlewood polynomials of type $B$ and $C$ 887  
**Cristian Lenart**

On exponentials of exponential generating series 919  
**Roland Bacher**

On families of $\varphi$, $\Gamma$-modules 943  
**Kiran Kedlaya** and **Ruochuan Liu**