**Frobenius difference equations
and algebraic independence of zeta values
in positive equal characteristic**

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By analogy with the Riemann zeta function at positive integers, for each finite field $\mathbb{F}_{p^r}$ with fixed characteristic p, we consider Carlitz zeta values $\zeta_r(n)$ at positive integers n. Our theorem asserts that among the zeta values in the set $\bigcup_{r=1}^{\infty} \{\zeta_r(1), \zeta_r(2), \zeta_r(3), \dots\}$, all the algebraic relations are those relations within each individual family $\{\zeta_r(1), \zeta_r(2), \zeta_r(3), \dots\}$. These are the algebraic relations coming from the Euler–Carlitz and Frobenius relations. To prove this, a motivic method for extracting algebraic independence results from systems of Frobenius difference equations is developed.

1. Introduction

1.1. *Motivic transcendence theory.* Classically, Grothendieck's period conjecture for abelian varieties predicts that the dimension of the Mumford–Tate group of an abelian variety over $\overline{\mathbb{Q}}$ should be equal to the transcendence degree of the field generated by its period matrix over $\overline{\mathbb{Q}}$. Conjecturally, the Mumford–Tate group is the motivic Galois group from Tannakian duality, and therefore Grothendieck's conjecture provides an interpretation of the algebraic relations among periods in question by way of motivic Galois groups.

We are concerned with the algebraic independence of special zeta values over function fields with varying finite constant fields in positive equal characteristic. In the positive characteristic world, there is the concept of *t*-motives introduced by Anderson [1986], dual to the concept of *t*-modules. The third author showed that the structure of *t*-modules is key for proving many interesting linear independence results about special values in this setting [Yu 1997]. The breakthrough in passing from linear independence to algebraic independence by way of *t*-motives began

with Anderson, Brownawell and the second author, and in particular with the linear independence criterion of [Anderson et al. 2004] (the *ABP criterion*).

By introducing a Tannakian formalism for rigid analytically trivial pre-*t*-motives and relating it to the Galois theory of Frobenius difference equations, the second author [Papanikolas 2008] has shown that the Galois group of a rigid analytically trivial pre-*t*-motive is isomorphic to its difference Galois group. Furthermore, the second author has successfully used the ABP criterion to show that the transcendence degree of the field generated by the period matrix of an ABP motive (that is, the pre-*t*-motive comes from a uniformizable abelian *t*-module) is equal to the dimension of its Galois group. More generally, we say that a rigid analytically trivial pre-*t*-motive has the *GP (Grothendieck period) property* if the transcendence degree of the field generated by its period matrix is equal to the dimension of its Galois group (for more details of terminology, see Section 2).

Using a refined version of the ABP criterion proved by the first author [Chang 2009], we observe that there are many pre-*t*-motives that are not ABP motives but that have the GP property. This motivates us to introduce a method of uniformizing the Frobenius twisting operators with respect to different constant fields for those pre-*t*-motives that have the GP property. The pre-*t*-motive we obtain in this way is defined over a larger constant field, but still has the GP property (see Corollary 2.2.4). This technique is very useful when dealing with the problem of determining all the algebraic relations among various special values of arithmetic interest in a fixed positive characteristic. It is used in this paper to study special zeta values. For another application to special arithmetic gamma values, see [Chang et al. 2010].

1.2. *Carlitz zeta values.* Let p be a prime, and let $\mathbb{F}_{p^r}[\theta]$ be the polynomial ring in θ over the finite field $\mathbb{F}_{p^r}$ of p^r elements. Our aim is to determine all the algebraic relations among the zeta values

$$\zeta_r(n) := \sum_{\substack{a \in \mathbb{F}_{p^r}[\theta] \\ a \text{ monic}}} \frac{1}{a^n} \in \mathbb{F}_{p^r}((1/\theta)) \subseteq \overline{\mathbb{F}_p}((1/\theta)),$$

where r and n vary over all positive integers. Each $\zeta_r(n)$ lies in $\mathbb{F}_p((1/\theta))$, since it is fixed by the automorphism $\left(\sum a_i(1/\theta)^i \mapsto \sum a_i^p(1/\theta)^i\right): \overline{\mathbb{F}_p}((1/\theta)) \to \overline{\mathbb{F}_p}((1/\theta))$.

The study of these zeta values was begun by Carlitz [1935]. For a fixed positive integer r, he discovered that there is a constant $\tilde{\pi}_r$, algebraic over $\mathbb{F}_{p^r}((1/\theta))$, such that $\zeta_r(n)/\tilde{\pi}_r^n$ lies in $\mathbb{F}_p(\theta)$ if n is divisible by $p^r - 1$. The quantity $\tilde{\pi}_r$ arises as a fundamental period of the Carlitz $\mathbb{F}_{p^r}[t]$-module C_r, and Wade [1941] showed that $\tilde{\pi}_r$ is transcendental over $\mathbb{F}_p(\theta)$.

We say that a positive integer n is (p, r)-*even* if it is a multiple of $p^r - 1$. Thus the situation of Carlitz zeta values at (p, r)-even positive integers is completely

analogous to that of the Riemann zeta function at even positive integers. For these (p, r)-even n, we call the $\mathbb{F}_p(\theta)$-linear relations between $\zeta_r(n)$ and $\tilde{\pi}_r^n$ the *Euler–Carlitz* relations. Because the characteristic is positive, there are also *Frobenius p-th power* relations among these zeta values: for positive integers m, n,

$$\zeta_r(p^m n) = \zeta_r(n)^{p^m}.$$

Anderson and Thakur [1990] and Yu [1991; 1997] made several breakthroughs in understanding Carlitz zeta values. Using the t-module method, the transcendence of $\zeta_r(n)$ for all positive integers n, and in particular for *odd n* (that is, n not divisible by $p^r - 1$), was proved, and it was also proved that the Euler–Carlitz relations are the only $\overline{\mathbb{F}_p(\theta)}$-linear relations among $\{\zeta_r(n), \tilde{\pi}_r^m; \ m, n \in \mathbb{N}\}$. In [Chang and Yu 2007], the first and third authors used ABP motives instead of t-modules to show that for fixed r, the Euler–Carlitz relations and the Frobenius p-th power relations account for all the algebraic relations over $\overline{\mathbb{F}_p(\theta)}$ among the Carlitz zeta values

$$\tilde{\pi}_r, \zeta_r(1), \zeta_r(2), \zeta_r(3), \ldots.$$

To complete the story of Carlitz zeta values, the next natural question is what happens if r varies. Denis [1998] proved the algebraic independence of all fundamental periods $\{\tilde{\pi}_1, \tilde{\pi}_2, \tilde{\pi}_3, \ldots\}$ as the constant field varies. Thus, in view of [Chang and Yu 2007], one expects that for the bigger set of zeta values,

$$\bigcup_{r=1}^{\infty} \{\zeta_r(1), \zeta_r(2), \zeta_r(3), \ldots\},$$

the Euler–Carlitz relations and the Frobenius p-th power relations still account for all the algebraic relations. This is indeed the case, as we find from the following theorem (stated subsequently as Corollary 4.5.2).

Theorem 1.2.1. *Given any positive integers s and d, the transcendence degree of the field*

$$\overline{\mathbb{F}_p(\theta)}\Big(\bigcup_{r=1}^{d} \{\tilde{\pi}_r, \zeta_r(1), \ldots, \zeta_r(s)\}\Big)$$

over $\overline{\mathbb{F}_p(\theta)}$ is

$$\sum_{r=1}^{d} \Big(s - \Big\lfloor \frac{s}{p} \Big\rfloor - \Big\lfloor \frac{s}{p^r - 1} \Big\rfloor + \Big\lfloor \frac{s}{p(p^r - 1)} \Big\rfloor + 1\Big).$$

1.3. *Outline.* Our strategy is to construct a pre-t-motive that has the GP property and whose period matrix accounts for the Carlitz zeta values in question. In [Chang and Yu 2007], an ABP motive has already been constructed for Carlitz zeta values with respect to a fixed constant field. The problem here is one concerning varying the constant fields in a fixed characteristic, and one has to uniformize Frobenius powers in order to apply the method developed in [Papanikolas 2008].

This paper is organized as follows. In Section 2, we review Papanikolas' theory and investigate the pre-t-motives that have the GP property. Here we introduce the mechanism of uniformizing Frobenius twisting operators while taking direct sums. Section 3 includes discussions about rigid analytically trivial pre-t-motives of type **SV**, that is, whose Galois groups are extensions of split tori by vector groups. The heart of this section is Theorem 3.2.2, where we determine the dimensions of Galois groups of direct sums of pre-t-motives of type **SV**. The pre-t-motive for Theorem 1.2.1 is constructed in Section 4, and we prove that it satisfies the conditions of Theorem 3.2.2. Finally, we calculate its dimension explicitly in Theorem 4.5.1, which then has Theorem 1.2.1 as direct consequence.

2. t-motivic Galois groups

2.1. *Notation.*

2.1.1. *Table of symbols.*
$\mathbb{F}_p :=$ the finite field of p elements, p a prime number.
$k := \mathbb{F}_p(\theta) :=$ the rational function field in the variable θ over $\mathbb{F}_p$.
$k_\infty := \mathbb{F}_p((1/\theta))$, completion of k with respect to the infinite place.
$\overline{k_\infty} :=$ a fixed algebraic closure of k_∞.
$\overline{k} :=$ the algebraic closure of k in $\overline{k_\infty}$.
$\mathbb{C}_\infty :=$ completion of $\overline{k_\infty}$ with respect to the canonical extension of the infinite place.
$|\cdot|_\infty :=$ a fixed absolute value for the completed field $\mathbb{C}_\infty$ with $|\theta|_\infty = p$.
$\mathbb{C}_\infty[[t]] :=$ the ring of power series in the variable t over $\mathbb{C}_\infty$.
$\mathbb{C}_\infty((t)) :=$ the field of Laurent series in the variable t over $\mathbb{C}_\infty$.
$\mathbb{T} := \{f \in \mathbb{C}_\infty[[t]] \mid f$ converges on $|t|_\infty \le 1\}$. This is known as the Tate algebra.
$\mathbb{L} :=$ the fraction field of $\mathbb{T}$.
$\sigma := \sigma_p := \left(\sum_i a_i t^i \mapsto \sum_i a_i^{1/p} t^i\right) : \mathbb{C}_\infty((t)) \to \mathbb{C}_\infty((t))$.
$\mathbb{G}_a :=$ the additive group.
$\mathrm{GL}_r / F :=$ for a field F, the F-group scheme of invertible $r \times r$ square matrices.
$\mathbb{G}_m := \mathrm{GL}_1$, the multiplicative group.

2.1.2. *Block diagonal matrices.* Let $A_i \in \mathrm{Mat}_{m_i}(\mathbb{L})$ for $i = 1, \ldots, n$, and $m := m_1 + \cdots + m_n$. We define $\bigoplus_{i=1}^n A_i \in \mathrm{Mat}_m(\mathbb{L})$ to be the canonical block diagonal matrix, that is, the matrix with $A_1, \ldots, A_n$ down the diagonal and zeros elsewhere.

2.1.3. *n-fold twisting.* For $n \in \mathbb{Z}$ and a formal Laurent series $f = \sum_i a_i t^i \in \mathbb{C}_\infty((t))$, we define the n-fold twisting $f^{(n)} := \sigma^{-n}(f) := \sum_i a_i^{p^n} t^i$. The n-fold twisting operation is an automorphism of the Laurent series field $\mathbb{C}_\infty((t))$ that stabilizes several subrings, for example, $\overline{k}[[t]]$, $\overline{k}[t]$, and $\mathbb{T}$. More generally, for any matrix B with entries in $\mathbb{C}_\infty((t))$, we define $B^{(n)}$ by the rule $B^{(n)}{}_{ij} = B_{ij}{}^{(n)}$.

2.1.4. *Entire power series.* A power series $f = \sum_{i=0}^{\infty} a_i t^i \in \mathbb{C}_{\infty}[\![t]\!]$ that satisfies

$$\lim_{i \to \infty} \sqrt[i]{|a_i|_{\infty}} = 0$$

and

$$[k_{\infty}(a_0, a_1, a_2, \dots) : k_{\infty}] < \infty$$

is called an entire power series. As a function of t, such a power series f converges on all $\mathbb{C}_{\infty}$ and, when restricted to $\overline{k_{\infty}}$, f takes values in $\overline{k_{\infty}}$. The ring of the entire power series is denoted by $\mathbb{E}$.

2.2. *Pre-t-motives and the GP property.* For r a positive integer, let $\bar{k}(t)[\sigma^r, \sigma^{-r}]$ be the noncommutative ring of Laurent polynomials in σ^r with coefficients in $\bar{k}(t)$, subject to the relation

$$\sigma^r f := f^{(-r)} \sigma^r \text{ for all } f \in \bar{k}(t).$$

A pre-t-motive M of level r is a left $\bar{k}(t)[\sigma^r, \sigma^{-r}]$-module that is finite-dimensional over $\bar{k}(t)$. Letting $\mathbf{m} \in \mathrm{Mat}_{n \times 1}(M)$ comprise a $\bar{k}(t)$-basis of M, multiplication by σ^r on M is represented by

$$\sigma^r(\mathbf{m}) = \Phi \mathbf{m}$$

for some matrix $\Phi \in \mathrm{GL}_n(\bar{k}(t))$. Furthermore, M is called rigid analytically trivial if there exists $\Psi \in \mathrm{GL}_n(\mathbb{L})$ such that

$$\sigma^r(\Psi) := \Psi^{(-r)} = \Phi \Psi.$$

Such a matrix Ψ is called a rigid analytic trivialization of the matrix Φ. We also say that Ψ is a rigid analytic trivialization of M (with respect to $\mathbf{m}$). Note that if $\Psi' \in \mathrm{GL}_n(\mathbb{L})$ is also a rigid analytic trivialization of Φ, then by [Papanikolas 2008, §4.1.6] we have

$$\Psi'^{-1}\Psi \in \mathrm{GL}_n(\mathbb{F}_{p^r}(t)). \tag{1}$$

Moreover, if we put $\mathbf{m}' := B\mathbf{m}$ for any fixed $B \in \mathrm{GL}_n(\bar{k}(t))$, then $\Phi' := B^{(-1)}\Phi B^{-1}$ represents multiplication by σ^r on M with respect to the $\bar{k}(t)$-basis $\mathbf{m}'$ of M, and $\Psi' := B\Psi$ is a rigid analytic trivialization of Φ'.

Definition 2.2.1. Suppose we are given a rigid analytically trivial pre-t-motive M of level r that is of dimension n over $\bar{k}(t)$. If there exists a $\bar{k}(t)$-basis $\mathbf{m} \in \mathrm{Mat}_{n \times 1}(M)$ such that there exists $\Psi \in \mathrm{GL}_n(\mathbb{L}) \cap \mathrm{Mat}_n(\mathbb{E})$ that is a rigid analytic trivialization of M with respect to $\mathbf{m}$ and satisfies

$$\mathrm{tr. deg}_{\bar{k}(t)} \bar{k}(t)(\Psi) = \mathrm{tr. deg}_{\bar{k}} \bar{k}(\Psi(\theta)),$$

then we say that M has the GP property, where $\bar{k}(t)(\Psi)$ (resp. $\bar{k}(\Psi(\theta))$) is the field generated by all entries of Ψ (resp. $\Psi(\theta)$) over $\bar{k}(t)$ (resp. $\bar{k}$). The GP property is independent of the choices of Ψ for a fixed $\mathbf{m}$ because of (1).

Given a rigid analytically trivial pre-t-motive M of level r with $(\mathbf{m}, \Phi, \Psi)$ as above, for any $s \in \mathbb{N}$ we define its s-th derived pre-t-motive $M^{(s)}$, which is a pre-t-motive of level rs: the underlying space of $M^{(s)}$ is the same as M, but it is now regarded as a left $\bar{k}(t)[\sigma^{rs}, \sigma^{-rs}]$-module. Letting

$$\Phi' := \Phi^{(-(s-1)r)} \cdots \Phi^{(-r)} \Phi,$$

we have $\sigma^{rs}\mathbf{m} = \Phi'\mathbf{m}$ and $\sigma^{rs}\Psi := \Psi^{(-rs)} = \Phi'\Psi$, and hence Ψ is also a rigid analytic trivialization of $M^{(s)}$.

Proposition 2.2.2. *Let M be a rigid analytically trivial pre-t-motive of level r that has the GP property. For any positive integer s, the s-th derived pre-t-motive $M^{(s)}$ of M is also rigid analytically trivial and has the GP property.*

Theorem 2.2.3 [Chang 2009, Theorem 1.2; Papanikolas 2008, Theorem 5.2.2]. *Suppose $\Phi \in \mathrm{Mat}_n(\bar{k}[t])$ defines a rigid analytically trivial pre-t-motive M of level r with a rigid analytic trivialization $\Psi \in \mathrm{Mat}_n(\mathbb{T}) \cap \mathrm{GL}_n(\mathbb{L})$. If $\det \Phi(0) \neq 0$ and $\det \Phi(\theta^{1/p^{ri}}) \neq 0$ for all $i = 1, 2, 3, \ldots$, then M has the GP property.*

By [Anderson et al. 2004, Proposition 3.1.3], the condition $\det \Phi(0) \neq 0$ implies $\Psi \in \mathrm{Mat}_n(\mathbb{E})$. Combining Theorem 2.2.3 and Proposition 2.2.2, we have:

Corollary 2.2.4. *Given an integer $d \geq 2$, we let $\ell := \mathrm{lcm}(1, \ldots, d)$. For each $1 \leq r \leq d$, let $\ell_r := \ell/r$ and let $\Phi_r \in \mathrm{Mat}_{n_r}(\bar{k}[t]) \cap \mathrm{GL}_{n_r}(\bar{k}(t))$ define a pre-t-motive M_r of level r with a rigid analytic trivialization $\Psi_r \in \mathrm{Mat}_{n_r}(\mathbb{T}) \cap \mathrm{GL}_{n_r}(\mathbb{L})$. Suppose that each Φ_r satisfies the hypotheses of Theorem 2.2.3 for $r = 1, \ldots, d$. Then the direct sum*

$$M := \bigoplus_{r=1}^{d} M_r^{(\ell_r)}$$

is a rigid analytically trivial pre-t-motive of level ℓ that has the GP property.

Proof. For each $1 \leq r \leq d$, we define

$$\Phi'_r := \Phi_r^{(-(\ell_r - 1)r)} \cdots \Phi_r^{(-r)} \Phi_r.$$

Moreover, if we define

$$\Phi' := \oplus_{r=1}^{d} \Phi'_r, \quad \Psi' := \oplus_{r=1}^{d} \Psi_r,$$

then we have

$$\Psi'^{(-\ell)} = \Phi'\Psi'.$$

Note that the matrix representing multiplication by σ^ℓ on M with respect to the evident $\bar{k}(t)$-basis is given by Φ'.

Our task is to show that Φ' satisfies the hypotheses of Theorem 2.2.3 (with respect to the operator σ^ℓ), whence the result. It is obvious that $\det \Phi'(0) \neq 0$

since $\det \Phi_r(0) \neq 0$ for each $1 \leq r \leq d$. Suppose that $\det \Phi'(\theta^{1/p^{\ell j}}) = 0$ for some $j \in \mathbb{N}$. This implies that there exists $1 \leq r \leq d$ and $0 \leq m \leq \ell_r - 1$ such that

$$\det \Phi_r^{(-rm)}(\theta^{(-\ell j)}) = 0.$$

However, this is equivalent to

$$\det \Phi_r(\theta^{(-(\ell j - rm))}) = 0. \tag{2}$$

Since $0 \leq m \leq \ell_r - 1$ and $r | \ell$, we have that $(\ell j - rm) > 0$ and $r | (\ell j - rm)$. Thus, (2) contradicts the hypothesis that $\det \Phi_r(\theta^{(-rh)}) \neq 0$ for all $h = 1, 2, 3, \ldots$. $\square$

2.3. *Difference Galois groups and transcendence.* In this section, we review the related theory developed in [Papanikolas 2008]. Let r be a fixed positive integer. The category of pre-t-motives of level r forms a rigid abelian $\mathbb{F}_{p^r}(t)$-linear tensor category. Also, the category $\mathcal{R}$ of rigid analytically trivial pre-t-motives of level r forms a neutral Tannakian category over $\mathbb{F}_{p^r}(t)$. Given an object M in $\mathcal{R}$, we let $\mathcal{R}_M$ be the strictly full Tannakian subcategory of $\mathcal{R}$ generated by M. That is, $\mathcal{R}_M$ consists of all objects of $\mathcal{R}$ isomorphic to subquotients of finite direct sums of

$$M^{\otimes u} \otimes (M^\vee)^{\otimes v} \text{ for various } u, v,$$

where $M^\vee$ is the dual of M. By Tannakian duality, $\mathcal{R}_M$ is representable by an affine algebraic group scheme Γ_M over $\mathbb{F}_{p^r}(t)$. The group Γ_M is called the Galois group of M and it is described explicitly as follows.

Suppose that $\Phi \in \mathrm{GL}_n(\bar{k}(t))$ provides multiplication by σ^r on M with respect to a fixed basis $\mathbf{m} \in \mathrm{Mat}_{n \times 1}(M)$ over $\bar{k}(t)$. Let $\Psi \in \mathrm{GL}_n(\mathbb{L})$ be a rigid analytic trivialization for Φ. Let $X := (X_{ij})$ be an $n \times n$ matrix whose entries are independent variables X_{ij}, and define a $\bar{k}(t)$-algebra homomorphism $\nu : \bar{k}(t)[X, 1/\det X] \to \mathbb{L}$ such that $\nu(X_{ij}) = \Psi_{ij}$ for all $1 \leq i, j \leq n$. We let

$$\Sigma_\Psi := \mathrm{im}\, \nu = \bar{k}(t)[\Psi, 1/\det \Psi] \subseteq \mathbb{L},$$

$$Z_\Psi := \mathrm{Spec}\, \Sigma_\Psi.$$

Then Z_Ψ is a closed $\bar{k}(t)$-subscheme of $\mathrm{GL}_n/\bar{k}(t)$. Let $\Psi_1, \Psi_2 \in \mathrm{GL}_n(\mathbb{L} \otimes_{\bar{k}(t)} \mathbb{L})$ be the matrices satisfying $(\Psi_1)_{ij} = \Psi_{ij} \otimes 1$ and $(\Psi_2)_{ij} = 1 \otimes \Psi_{ij}$ for all $1 \leq i, j \leq n$. Let $\widetilde{\Psi} := \Psi_1^{-1} \Psi_2$. We have an $\mathbb{F}_{p^r}(t)$-algebra homomorphism $\mu : \mathbb{F}_{p^r}(t)[X, 1/\det X] \to \mathbb{L} \otimes_{\bar{k}(t)} \mathbb{L}$ such that $\mu(X_{ij}) = \widetilde{\Psi}_{ij}$ for all $1 \leq i, j \leq n$. Furthermore, we define

$$\Delta := \mathrm{im}\, \mu,$$
$$\Gamma_\Psi := \mathrm{Spec}\, \Delta. \tag{3}$$

Theorem 2.3.1 [Papanikolas 2008, Theorems 4.2.11, 4.3.1, 4.5.10]. *The scheme Γ_Ψ is a closed $\mathbb{F}_{p^r}(t)$-subgroup scheme of $\mathrm{GL}_n/\mathbb{F}_{p^r}(t)$, which is isomorphic to the Galois group Γ_M over $\mathbb{F}_{p^r}(t)$. Moreover, Γ_Ψ has the following properties*:

(a) *Γ_Ψ is smooth over $\overline{\mathbb{F}_p(t)}$ and is geometrically connected.*

(b) *$\dim \Gamma_\Psi = \mathrm{tr.deg}_{\bar{k}(t)} \bar{k}(t)(\Psi)$.*

(c) *Z_Ψ is a Γ_Ψ-torsor over $\bar{k}(t)$.*

In particular, if M has the GP property, then

(d) *$\dim \Gamma_\Psi = \mathrm{tr.deg}_{\bar{k}} \bar{k}(\Psi(\theta))$.*

We call Γ_Ψ the Galois group associated to the difference equation $\Psi^{(-r)} = \Phi\Psi$. This Γ_Ψ is independent of the analytic trivialization Ψ, up to isomorphism over $\mathbb{F}_{p^r}(t)$. Throughout this paper we always identify Γ_M with Γ_Ψ, and regard it as a linear algebraic group over $\mathbb{F}_{p^r}(t)$ because of Theorem 2.3.1(a).

Remark 2.3.2. Let n_1, n_2 be positive integers and $\mathbf{0} := 0_{n_1 \times n_2}$ be the zero matrix of size $n_1 \times n_2$. Suppose that the matrix

$$\Phi := \begin{bmatrix} \Phi_1 & \mathbf{0} \\ \Phi_3 & \Phi_2 \end{bmatrix} \in \mathrm{GL}_{n_1+n_2}(\bar{k}(t))$$

defines a rigid analytically trivial pre-t-motive M of level r. Then one can always find its rigid analytic trivialization of the form

$$\Psi := \begin{bmatrix} \Psi_1 & \mathbf{0} \\ \Psi_3 & \Psi_2 \end{bmatrix} \in \mathrm{GL}_{n_1+n_2}(\mathbb{L}).$$

By (3), we have that

$$\Gamma_\Psi \subseteq \left\{ \begin{bmatrix} * & \mathbf{0} \\ * & * \end{bmatrix} \right\} \subseteq \mathrm{GL}_{n_1+n_2}/\mathbb{F}_{p^r}(t).$$

Let N be the sub-pre-t-motive (of level r) of M defined by $\Phi_1 \in \mathrm{GL}_{n_1}(\bar{k}(t))$ with rigid analytic trivialization Ψ_1; then by the Tannakian theory we have a natural surjective morphism

$$\pi : \Gamma_\Psi(\overline{\mathbb{F}_p(t)}) \twoheadrightarrow \Gamma_{\Psi_1}(\overline{\mathbb{F}_p(t)}),$$
$$\gamma \mapsto \pi(\gamma). \tag{4}$$

In fact, $\pi(\gamma)$ comes from the restriction of the action of γ to the fiber functor of $\mathcal{R}_N$ (which is a full subcategory of $\mathcal{R}_M$). Precisely, $\pi(\gamma)$ is the matrix cut out from the upper left square of γ with size n_1 (for detailed arguments, see [Papanikolas 2008, §6.2.2]).

3. A dimension criterion

3.1. *Pre-t-motives of type* SV. Let r be a fixed positive integer and let $\{n_1, \ldots, n_h\}$ be h nonnegative integers. We say that a pre-t-motive M of level r is of type **SV** (its Galois group being an extension of a split torus by a vector group) if the multiplication by σ^r on M is represented by the matrix

$$\Phi := \bigoplus_{i=1}^{h} A_i, \quad A_i := \begin{bmatrix} a_i & 0 & \cdots & 0 \\ a_{i1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{in_i} & 0 & \cdots & 1 \end{bmatrix} \in \mathrm{GL}_{1+n_i}(\bar{k}(t)),$$

which has rigid analytic trivialization

$$\Psi := \bigoplus_{i=1}^{h} F_i, \quad F_i := \begin{bmatrix} f_i & 0 & \cdots & 0 \\ f_{i1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f_{in_i} & 0 & \cdots & 1 \end{bmatrix} \in \mathrm{GL}_{1+n_i}(\mathbb{L}).$$

Let T be the Galois group associated to the difference equation

$$\begin{bmatrix} f_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & f_h \end{bmatrix}^{(-r)} = \begin{bmatrix} a_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_h \end{bmatrix} \begin{bmatrix} f_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & f_h \end{bmatrix},$$

and note that by (3), T is a subtorus of the h-dimensional split torus in $\mathrm{GL}_h /\mathbb{F}_{p^r}(t)$. By the same reason as (4), we have the natural projection of Galois groups

$$\Gamma_\Psi \twoheadrightarrow T, \tag{5}$$

given in terms of coordinates by

$$\bigoplus_{i=1}^{h} \begin{bmatrix} x_i & 0 & \cdots & 0 \\ x_{i1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_{in_i} & 0 & \cdots & 1 \end{bmatrix} \mapsto [x_1] \oplus \cdots \oplus [x_h].$$

One has also the exact sequence of linear algebraic groups

$$1 \to V \to \Gamma_\Psi \twoheadrightarrow T \to 1, \tag{6}$$

where V is a vector group contained in the $\left(\sum_{i=1}^{h} n_i\right)$-dimensional "coordinate" vector group G, which is defined as the set of matrices

$$\bigoplus_{i=1}^{h} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ x_{i1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_{in_i} & 0 & \cdots & 1 \end{bmatrix}$$

with usual multiplication. This subgroup V is the unipotent radical of the (solvable) Galois group Γ_Ψ, and $\dim \Gamma_\Psi = \dim V + \dim T$.

Definition 3.1.1. Let M be a rigid analytically trivial pre-t-motive of level r that is of **SV** type given as above. We say that its Galois group Γ_Ψ is *full* if $\dim V = \sum_{i=1}^{h} n_i$, that is, $V = G$.

3.2. *Criterion for direct sum motives to have full Galois group.* We continue with the notation of Section 3.1. For each i, $1 \le i \le h$, the 1×1 matrix $[a_i]$ defines a sub-pre-t-motive (of level r) of M, which is one-dimensional over $\bar{k}(t)$. Its rigid analytic trivialization is given by f_i satisfying

$$[f_i]^{(-r)} = [a_i][f_i].$$

We call one such sub-pre-t-motive of level r a *diagonal* of the pre-t-motive M. By Theorem 2.3.1, the Galois group of a diagonal of M is $\mathbb{G}_m$ if and only if the corresponding trivialization f_i is transcendental over $\bar{k}(t)$. In this situation, the canonical projection $T \to \mathbb{G}_m$ on the i-th coordinate of T is surjective.

There is a canonical action of T on G given in terms of coordinates by

$$([x_1] \oplus \cdots \oplus [x_h]) * \left(\bigoplus_{i=1}^{h} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ x_{i1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_{in_i} & 0 & \cdots & 1 \end{bmatrix} \right) := \bigoplus_{i=1}^{h} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ x_i x_{i1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_i x_{in_i} & 0 & \cdots & 1 \end{bmatrix}.$$

This action induces an action of T on V compatible with the one coming from (6). Hence, we have that given any element $\gamma \in V$ whose x_{ij}-coordinate is nonzero, the orbit inside V given by the action of T on γ must be infinite if f_i is transcendental over $\bar{k}(t)$.

Definition 3.2.1. Let M_i be a rigid analytically trivial pre-t-motive of level r that is of type **SV** for $i = 1, \ldots, d$. We say that this set of pre-t-motives $\{M_i\}_{i=1}^{d}$ is diagonally independent if for any $1 \le i, j \le d$, $i \ne j$, the Galois group of $N_i \oplus N_j$ is a two-dimensional torus over $\mathbb{F}_{p^r}(t)$, where N_i is any diagonal of M_i and N_j is any diagonal of M_j.

Theorem 3.2.2. *Given any positive integer r, let $M_1, \ldots, M_d$ be rigid analytically trivial pre-t-motives of level r that are of type **SV** and have full Galois groups. Suppose that the set $\{M_i\}_{i=1}^d$ is diagonally independent. Then the Galois group of $M := \bigoplus_{i=1}^d M_i$ is also full.*

Proof. We first explain that without loss of generality, we may assume $d = 2$. We prove the result by induction on d. Consider $M = \left(\bigoplus_{i=1}^{d-1} M_i\right) \oplus M_d$. Then the induction hypothesis implies that the Galois group of $\left(\bigoplus_{i=1}^{d-1} M_i\right)$ is full. Since the following argument of the case $d = 2$ can be applied to the computation of the Galois group of the direct sum of $\left(\bigoplus_{i=1}^{d-1} M_i\right)$ and M_d, we may assume $d = 2$.

Now we set $d = 2$ and let the pre-t-motive M_i be defined by a matrix Φ_i, for $i = 1, 2$, with $\Psi = \Psi_1 \oplus \Psi_2$ a rigid analytic trivialization of M, and let Γ_Ψ, Γ_{Ψ_1}, Γ_{Ψ_2} be the Galois groups. The unipotent radicals of these groups are denoted by V, V_1, V_2 respectively. Let G, G_1, G_2 be the coordinate vector groups containing V, V_1, V_2, respectively. Suppose the matrix Φ_1 has h diagonal blocks, and let the coordinates of G_1 be denoted by x_{ij}, $i = 1, \ldots, h$, $j = 1, \ldots, n_i$. Similarly, let y_{ij}, $i = 1, \ldots, \ell$, $j = 1, \ldots, m_i$ denote the coordinates of G_2. Any subspace $W \subseteq G$ obtained by setting some of these coordinates to 0 is called a linear coordinate subspace. The hypothesis that the Galois group of M_i is full means exactly that $G_i = V_i$, for $i = 1, 2$. We are going to prove that $G = V$.

Suppose V has codimension s in the coordinate vector group G. We can find a linear coordinate subspace $W \subseteq G$ of dimension s such that $W \cap V$ is of dimension 0. Since $W \cap V$ is invariant under T, it must be equal to the neutral element of G, because the hypothesis that M_1 and M_2 are diagonally independent implies in particular that the Galois group of any diagonal of M is $\mathbb{G}_m$.

Let $W' \subseteq G$ be the linear coordinate space given by those coordinates disjoint from those of W. Then the natural projection from G to W' induces on V an isomorphism of vector groups. Composing the inverse of this isomorphism with the surjective morphism π_1 in the diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & V & \longrightarrow & \Gamma_\Psi & \longrightarrow & T & \longrightarrow & 1 \\
& & \pi_1 \downarrow & & \pi_1 \downarrow & & \downarrow & & \\
1 & \longrightarrow & G_1 & \longrightarrow & \Gamma_{\Psi_1} & \longrightarrow & T_1 & \longrightarrow & 1,
\end{array}
$$

we obtain a morphism π_1 from W' onto G_1 that is furthermore a T-morphism.

We contend that under the hypothesis that M_1 and M_2 are diagonally independent, π_1 maps $G_2 \cap W'$ to zero. This contention results from the following basic lemma by taking any diagonal N_1 (resp. N_2) of M_1 (resp. M_2) and considering the restriction of the above morphism π_1 to a single block. Now since $G = G_1 \times G_2$, it follows that $\pi_1(G_1 \cap W') = G_1$. Thus $G_1 \subseteq W'$. Similarly we also have $G_2 \subseteq W'$, and hence $G = W'$ and $G = V$. $\qquad\square$

Lemma 3.2.3. *Let G_1 (resp. G_2) be the vector group with coordinates*

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ x_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_n & 0 & \cdots & 1 \end{bmatrix}, \quad \left(resp. \begin{bmatrix} 1 & 0 & \cdots & 0 \\ y_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ y_m & 0 & \cdots & 1 \end{bmatrix} \right).$$

Let $\mathbb{G}_m^2$ act on G_1 (resp. G_2) by

$$\mathbb{G}_m^2 \ni (x, y) : x_i \mapsto xx_i, 1 \le i \le n, \quad (resp.\ y_j \mapsto yy_j, 1 \le j \le m).$$

If $\pi_1 : G_2 \to G_1$ is a $\mathbb{G}_m^2$-morphism, then $\pi_1 \equiv 0$.

4. Application to zeta values

Throughout this section, we use ri for two subscripts (for example, D_{ri} depends on the two parameters r and i), and use $*, rn$ for three subscripts (for example, $L_{\alpha,rn}$ depends on the three parameters α, r and n).

4.1. *Carlitz theory.* Throughout Section 4 we fix a positive integer r. Recall the Carlitz $\mathbb{F}_{p^r}[t]$-module, denoted by C_r, which is given by the $\mathbb{F}_{p^r}$-linear ring homomorphism

$$C_r = (t \mapsto (x \mapsto \theta x + x^{p^r})) : \mathbb{F}_{p^r}[t] \to \mathrm{End}_{\mathbb{F}_{p^r}}(\mathbb{G}_a).$$

Note that when we regard C_r as a Drinfeld $\mathbb{F}_p[t]$-module, it is of rank r [Goss 1996; Thakur 2004]. One has the Carlitz exponential associated to C_r:

$$\exp_{C_r}(z) := \sum_{i=0}^{\infty} \frac{z^{p^{ri}}}{D_{ri}}.$$

Here we set

$$D_{r0} := 1, \qquad D_{ri} := \prod_{j=0}^{i-1}(\theta^{p^{ri}} - \theta^{p^{rj}}), \quad i \ge 1.$$

Now $\exp_{C_r}(z)$ is an entire power series in z satisfying the functional equation

$$\exp_{C_r}(\theta z) = \theta \exp_{C_r}(z) + \exp_{C_r}(z)^{p^r}.$$

Moreover one has the product expansion

$$\exp_{C_r}(z) = z \prod_{0 \neq a \in \mathbb{F}_{p^r}[\theta]} \left(1 - \frac{z}{a\tilde{\pi}_r}\right),$$

where

$$\tilde{\pi}_r = \theta(-\theta)^{1/(p^r - 1)} \prod_{i=1}^{\infty} \left(1 - \theta^{1-p^{ri}}\right)^{-1}.$$

is a fundamental period of C_r. We fix once and for all a choice of $(-\theta)^{1/(p^r-1)}$ so that $\tilde{\pi}_r$ is a well-defined element in $\overline{\mathbb{F}_p((1/\theta))}$. We also choose these roots in a compatible way so that when r divides r', the number $(-\theta)^{1/(p^r-1)}$ is a power of $(-\theta)^{1/(p^{r'}-1)}$.

The formal inverse of Carlitz exponential is the Carlitz logarithm. It is the power series

$$\log_{C_r}(z) = \sum_{i=0}^{\infty} \frac{z^{p^{ri}}}{L_{ri}},$$

where

$$L_{r0} := 1, \qquad L_{ri} := \prod_{j=1}^{i} (\theta - \theta^{p^{rj}}).$$

As a function in z, $\log_{C_r}(z)$ converges for all $z \in \mathbb{C}_\infty$ with $|z|_\infty < |\theta|_\infty^{p^r/(p^r-1)}$. It satisfies the functional equation

$$\theta \log_{C_r}(z) = \log_{C_r}(\theta z) + \log_{C_r}(z^{p^r})$$

whenever the values in question are defined.

For a positive integer n, the n-th Carlitz polylogarithm associated to C_r is the series

$$\text{Plog}_{rn}(z) := \sum_{i=0}^{\infty} \frac{z^{p^{ri}}}{L_{ri}^n}, \tag{7}$$

which converges ∞-adically for all $z \in \mathbb{C}_\infty$ with $|z|_\infty < |\theta|_\infty^{np^r/(p^r-1)}$. Its value at a particular $z = \alpha \neq 0$ is called the n-th polylogarithm of α associated to C_r. In transcendence theory we are interested in those polylogarithms of $\alpha \in \bar{k}^\times$, as analogous to classical logarithms of algebraic numbers.

4.2. *Algebraic independence of special functions.* For any positive integer r, let

$$\Omega_r(t) := (-\theta)^{-p^r/(p^r-1)} \prod_{i=1}^{\infty} \left(1 - \frac{t}{\theta^{p^{ri}}}\right) \in \overline{k_\infty}[[t]] \subseteq \mathbb{C}_\infty((t)).$$

$\Omega_r \in \mathbb{E}$, since $|\theta^{p^{ri}}|_\infty \to \infty$. Furthermore, Ω_r satisfies the functional equation

$$\Omega_r^{(-r)}(t) = (t - \theta)\Omega_r(t), \tag{8}$$

and its specialization at $t = \theta$ gives $\Omega_r(\theta) = -1/\tilde{\pi}_r$.

By (8), the function Ω_r provides a rigid analytic trivialization of the Carlitz motive $\mathscr{C}_r$ of level r that has the GP property (see Theorem 2.2.3). This is the pre-t-motive with underlying space $\bar{k}(t)$ itself and σ^r acts by $\sigma^r f = (t - \theta) f^{(-r)}$ for $f \in \mathscr{C}_r$.

For any $d \in \mathbb{N}$, we let $\ell := \text{lcm}(1, \ldots, d)$ and $\ell_r := \ell/r$ for $r = 1, \ldots, d$. We let $\mathscr{C}_r^{(\ell_r)}$ be the ℓ_r-th derived pre-t-motive of $\mathscr{C}_r$ that is a rigid analytically

trivial pre-t-motive of level ℓ (see Section 2.2). Then we define the direct sum $M = M_d := \bigoplus_{r=1}^{d} \mathscr{C}_r^{(\ell_r)}$. By Corollary 2.2.4, M also has the GP property. We note that the canonical rigid analytical trivialization of M is the diagonal matrix $\Psi \in \mathrm{Mat}_d(\mathbb{E}) \cap \mathrm{GL}_d(\mathbb{L})$ with diagonal entries $\Omega_1, \ldots, \Omega_d$.

Lemma 4.2.1. *Given any positive integer $d \geq 2$, we let $\ell := \mathrm{lcm}(1, \ldots, d)$. Let $M = M_d$ be the rigid analytically trivial pre-t-motive of level ℓ with rigid analytic trivialization Ψ defined as above. Then we have $\dim \Gamma_\Psi = d$. In particular, the functions $\Omega_1, \ldots, \Omega_d$ are algebraically independent over $\bar{k}(t)$ and the values $\tilde{\pi}_1, \ldots, \tilde{\pi}_d$ are algebraically independent over $\bar{k}$.*

Proof. Suppose $\dim \Gamma_\Psi < d$. Since Ψ is a diagonal matrix with diagonal entries $\Omega_1, \ldots, \Omega_d$, by (3) we have that $\Gamma_\Psi \subseteq \mathrm{T}$, where T is the split torus of dimension d in $\mathrm{GL}_d /\mathbb{F}_{p^\ell}(t)$. We let $X_1, \ldots, X_d$ be the coordinates of T and χ_j the character of T that projects the j-th diagonal position to $\mathbb{G}_m$. Note that $\{\chi_j\}_{j=1}^d$ generates the character group of T. Hence Γ_Ψ is the kernel of some characters of T, that is, canonical generators of the defining ideal for Γ_Ψ can be of the form $X_1^{m_1} \cdots X_d^{m_d} - 1$ for some integers $m_1, \ldots, m_d$, not all zero. By (3) we have that

$$(\Omega_1^{-m_1} \ldots \Omega_d^{-m_d}) \otimes (\Omega_1^{m_1} \ldots \Omega_d^{m_d}) = 1 \in \mathbb{L} \otimes_{\bar{k}(t)} \mathbb{L},$$

and hence

$$\beta := \Omega_1^{m_1} \ldots \Omega_d^{m_d} \in \bar{k}(t)^\times. \tag{9}$$

We recall that Ω_r has zeros on $\{\theta^{p^{rj}}\}_{j=1}^\infty$. Since $\beta \in \bar{k}(t)^\times$, it has only finitely many zeros and poles, and hence $\mathrm{ord}_{t=\theta^{p^h}}(\beta) = 0$ for $h \gg 0$. Choose a prime number p' sufficiently large that

- $p' > d$, and
- the order of vanishing of β at $t = \theta^{p^{p'}}$ is zero.

These conditions imply that $m_1 = 0$ from (9). Iterating this argument, we conclude that $m_1 = \cdots = m_d = 0$, a contradiction. $\qquad\square$

4.3. *Algebraic independence of polylogarithms.* Given $n \in \mathbb{N}$ and $\alpha \in \bar{k}^\times$ with $|\alpha|_\infty < |\theta|_\infty^{np^r/(p^r-1)}$, we consider the power series

$$L_{\alpha, rn}(t) := \alpha + \sum_{i=1}^{\infty} \frac{\alpha^{p^{ri}}}{(t - \theta^{p^r})^n \cdots (t - \theta^{p^{ri}})^n},$$

which as a function on $\mathbb{C}_\infty$ converges on $|t|_\infty < |\theta|_\infty^{p^r}$. We note that $L_{\alpha, rn}(\theta)$ is exactly the n-th polylogarithm of α associated to C_r, that is,

$$L_{\alpha, rn}(\theta) = \mathrm{Plog}_{rn}(\alpha).$$

Given a collection of such numbers α, say $\alpha_1, \ldots, \alpha_m$, we define

$$\Phi_{rn}(\alpha_1, \ldots, \alpha_m) := \begin{bmatrix} (t-\theta)^n & 0 & \cdots & 0 \\ \alpha_1^{(-r)}(t-\theta)^n & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ \alpha_m^{(-r)}(t-\theta)^n & 0 & \cdots & 1 \end{bmatrix} \in \mathrm{GL}_{m+1}(\bar{k}(t)) \cap \mathrm{Mat}_{m+1}(\bar{k}[t])$$

and

$$\Psi_{rn}(\alpha_1, \ldots, \alpha_m) := \begin{bmatrix} \Omega_r^n & 0 & \cdots & 0 \\ \Omega_r^n L_{\alpha_1,rn} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Omega_r^n L_{\alpha_m,rn} & 0 & \cdots & 1 \end{bmatrix} \in \mathrm{GL}_{m+1}(\mathbb{L}) \cap \mathrm{Mat}_{m+1}(\mathbb{E}).$$

Then one has [Chang and Yu 2007, §3.1.2]

$$\Psi_{rn}(\alpha_1, \ldots, \alpha_m)^{(-r)} = \Phi_{rn}(\alpha_1, \ldots, \alpha_m) \Psi_{rn}(\alpha_1, \ldots, \alpha_m). \tag{10}$$

Hence, $\Phi_{rn}(\alpha_1, \ldots, \alpha_m)$ defines a rigid analytically trivial pre-t-motive of level r that has the GP property.

In [Chang and Yu 2007], we followed Papanikolas' methods to generalize the algebraic independence of Carlitz logarithms to algebraic independence of poly-logarithms. Precisely, by [Chang and Yu 2007, Theorem 3.1] and Theorem 2.2.3 we have:

Theorem 4.3.1. *Given any positive integers r and n, let $\alpha_1, \ldots, \alpha_m \in \bar{k}^\times$ satisfy $|\alpha_i|_\infty < |\theta|_\infty^{np^r/(p^r-1)}$ for $i = 1, \ldots, m$. Then*

$$\dim_{\mathbb{F}_{p^r}(\theta)} N_{rn} = \mathrm{tr.\,deg}_{\bar{k}}\, \bar{k}(\tilde{\pi}_r^n, L_{\alpha_1,rn}(\theta), \ldots, L_{\alpha_m,rn}(\theta))$$

$$= \mathrm{tr.\,deg}_{\bar{k}(t)}\, \bar{k}(t)(\Omega_r^n, L_{\alpha_1,rn}, \ldots, L_{\alpha_m,rn}),$$

where

$$N_{rn} := \mathbb{F}_{p^r}(\theta)\text{-}\mathrm{Span}\{\tilde{\pi}_r^n, L_{\alpha_1,rn}(\theta), \ldots, L_{\alpha_m,rn}(\theta)\}.$$

4.4. *Formulas for zeta values.*

4.4.1. *Euler–Carlitz relations.* Fix a positive integer r for this subsection. Carlitz [1935] introduced the power sum

$$\zeta_r(n) := \sum_{\substack{a \in \mathbb{F}_{p^r}[\theta] \\ a \text{ monic}}} \frac{1}{a^n} \in \mathbb{F}_p((1/\theta)) \quad (n \text{ a positive integer}),$$

which are the Carlitz zeta values associated to $\mathbb{F}_{p^r}[\theta]$.

Writing down a p^r-adic expansion $\sum_i n_{ri} p^{ri}$ of n, we let

$$\Gamma_{r(n+1)} := \prod_{i=0}^{\infty} D_{ri}^{n_{ri}}.$$

We call $\Gamma_{r(n+1)}$ the *Carlitz factorials* associated to $\mathbb{F}_{p^r}[\theta]$. The *Bernoulli–Carlitz numbers* B_{rn} in $\mathbb{F}_p(\theta)$ are given by the following expansions from the Carlitz exponential series

$$\frac{z}{\exp_{C_r}(z)} = \sum_{n=0}^{\infty} \frac{B_{rn}}{\Gamma_{r(n+1)}} z^n.$$

We state the *Euler–Carlitz relations*:

Theorem 4.4.2 [Carlitz 1935]. *For all positive integers n divisible by $p^r - 1$, one has*

$$\zeta_r(n) = \frac{B_{rn}}{\Gamma_{r(n+1)}} \tilde{\pi}_r^n. \tag{11}$$

We call a positive integer n (p, r)-*even* if it is divisible by $p^r - 1$; otherwise we call it (p, r)-*odd*. Thus, when $p = 2$ and $r = 1$, all positive integers are even.

4.4.3. *The Anderson–Thakur formula.* Anderson and Thakur [1990] introduced the n-th tensor power of the Carlitz $\mathbb{F}_{p^r}[t]$-module C_r, and they related $\zeta_r(n)$ to the last coordinate of the logarithm associated to the n-th tensor power of C_r for each positive integer n. More precisely, they interpreted $\zeta_r(n)$ as $\mathbb{F}_{p^r}(\theta)$-linear combinations of n-th Carlitz polylogarithms of algebraic numbers:

Theorem 4.4.4 [Anderson and Thakur 1990]. *Given any positive integers r and n, one can find a sequence $h_{0,rn}, \ldots, h_{l_{rn},rn} \in \mathbb{F}_{p^r}(\theta)$, $l_{rn} < np^r/(p^r - 1)$, such that*

$$\zeta_r(n) = \sum_{i=0}^{l_{rn}} h_{i,rn} \operatorname{Plog}_{rn}(\theta^i), \tag{12}$$

where $\operatorname{Plog}_{rn}(z)$ is defined as in (7). In the special case of $n \leq p^r - 1$,

$$\zeta_r(n) = \operatorname{Plog}_{rn}(1).$$

Definition 4.4.5. Given any positive integer r, for each $n \in \mathbb{N}$, $(p^r - 1) \nmid n$, with l_{rn} as given by (12), we fix a finite subset

$$\{\alpha_{0,rn}, \ldots, \alpha_{m_{rn},rn}\} \subseteq \{1, \theta, \ldots, \theta^{l_{rn}}\}$$

such that

$$\{\tilde{\pi}_r^n, \mathcal{L}_{0,rn}(\theta), \ldots, \mathcal{L}_{m_{rn},rn}(\theta)\} \quad \text{and} \quad \{\tilde{\pi}_r^n, \zeta_r(n), \mathcal{L}_{1,rn}(\theta), \ldots, \mathcal{L}_{m_{rn},rn}(\theta)\}$$

are $\mathbb{F}_{p^r}(\theta)$-bases for N_{rn}, where $\mathcal{L}_{j,rn}(t) := L_{\alpha_j,rn}(t)$ for $j = 0, \ldots, m_{rn}$. This can be done because of (12) (see [Chang and Yu 2007, § 4.1]), and note that $m_{rn} + 2$

is the dimension of N_{rn} over $\mathbb{F}_{p^r}(\theta)$. In the case of $p = 2$ and $r = 1$, the $\mathbb{F}_p(\theta)$-dimension of N_{11} is 1 and we set $m_{11} := -1$.

Definition 4.4.6. Given any positive integers s and d with $d \geq 2$, for each $1 \leq r \leq d$, we define

$$\begin{cases} U_r(s) = \{1\} & \text{if } p = 2 \text{ and } r = 1; \\ U_r(s) = \{1 \leq n \leq s; \, p \nmid n, \, (p^r - 1) \nmid n\} & \text{otherwise.} \end{cases}$$

For each $n \in U_r(s)$, we define that if $p = 2$ and $r = 1$,

$$\Phi_{rn} := (t - \theta) \in \mathrm{GL}_1(\bar{k}(t)),$$

$$\Psi_{rn} := \Omega_1 \in \mathrm{GL}_1(\mathbb{L}),$$

otherwise

$$\Phi_{rn} := \Phi_{rn}(\alpha_{0,rn}, \ldots, \alpha_{m_{rn},rn}) \in \mathrm{GL}_{(m_{rn}+2)}(\bar{k}(t)),$$

$$\Psi_{rn} := \Psi_{rn}(\alpha_{0,rn}, \ldots, \alpha_{m_{rn},rn}) \in \mathrm{GL}_{(m_{rn}+2)}(\mathbb{L}).$$

By Theorem 4.3.1, we have that

$$\dim \Gamma_{\Psi_{rn}} = \mathrm{tr.\,deg}_{\bar{k}(t)} \bar{k}(t)(\Psi_{rn}) = m_{rn} + 2.$$

Put $\Phi_r := \bigoplus_{n \in U_r(s)} \Phi_{rn}$; then Φ_r defines a rigid analytically trivial pre-t-motive M_r of level r with rigid analytic trivialization $\Psi_r := \bigoplus_{n \in U_r(s)} \Psi_{rn}$ (see (10)). Also, M_r is of type **SV**, and by Theorem 2.2.3, M_r has the GP property. The main theorem of [Chang and Yu 2007] is the following:

Theorem 4.4.7 [Chang and Yu 2007, Theorem 4.5]. *For any positive integers s and r, the Galois group Γ_{M_r} over $\mathbb{F}_{p^r}(t)$ is full, that is,*

$$\dim \Gamma_{M_r} = 1 + \sum_{n \in U_r(s)} (m_{rn} + 1).$$

4.5. *Proof of Theorem 1.2.1.* Given any integer $d \geq 2$, we put $\ell := \mathrm{lcm}(1, \ldots, d)$ and $\ell_r := \ell/r$ for $r = 1, \ldots, d$. For each $1 \leq r \leq d$, let $\mathbf{M}_r := M_r^{(\ell_r)}$ be the ℓ_r-th derived pre-t-motive of M_r defined as above. Note that $\mathbf{M}_r$ is a rigid analytically trivial pre-t-motive of level ℓ that is still of type **SV**. By Proposition 2.2.2, each $\mathbf{M}_r$ has the GP property, and by Theorem 4.4.7, its Galois group $\Gamma_{\mathbf{M}_r}$ is full. Further, for each $1 \leq r \leq d$ any diagonal of $\mathbf{M}_r$ has canonical rigid analytic trivialization given by Ω_r^n for some $n \in U_r(s)$, and hence its Galois group is $\mathbb{G}_m$ because Ω_r is transcendental over $\bar{k}(t)$.

Since by Lemma 4.2.1 the functions $\Omega_1, \ldots, \Omega_d$ are algebraically independent over $\bar{k}(t)$, particularly the Galois group of $N_i \oplus N_j$ is a two-dimensional torus over $\mathbb{F}_{p^\ell}(t)$ for any diagonal N_i (resp. N_j) of $\mathbf{M}_i$ (resp. $\mathbf{M}_j$) with $i \neq j$, $1 \leq i, j \leq d$. Put $\mathbf{M} := \bigoplus_{r=1}^d \mathbf{M}_r$, and note that $\mathbf{M}$ has the GP property by Corollary 2.2.4. Applying Theorem 3.2.2 to this situation, we obtain the explicit dimension of $\Gamma_{\mathbf{M}}$:

Theorem 4.5.1. *Given any positive integers s and d with $d \geq 2$, let* **M** *be defined as above. Then the Galois group $\Gamma_{\mathbf{M}}$ is full*, *that is*,

$$\dim \Gamma_{\mathbf{M}} = d + \sum_{r=1}^{d} (m_{rn} + 1).$$

As a consequence, we completely determine all the algebraic relations among the families of Carlitz zeta values:

Corollary 4.5.2. *Given any positive integers d and s*, *the transcendence degree of the field*

$$\bar{k} \Big(\bigcup_{r=1}^{d} \{\tilde{\pi}_r, \zeta_r(1), \ldots, \zeta_r(s)\} \Big)$$

over $\bar{k}$ is

$$\sum_{r=1}^{d} \Big(s - \Big\lfloor \frac{s}{p} \Big\rfloor - \Big\lfloor \frac{s}{p^r - 1} \Big\rfloor + \Big\lfloor \frac{s}{p(p^r - 1)} \Big\rfloor + 1 \Big).$$

Proof. We may assume $d \geq 2$, since the case $d = 1$ is already given in [Chang and Yu 2007, Corollary 4.6]. For $1 \leq r \leq d$, let

$$\begin{cases} V_1(s) := \varnothing & \text{if } p = 2; \\ V_r(s) := U_r(s) & \text{otherwise.} \end{cases}$$

Since **M** has the GP property, by Theorem 4.5.1 we see that the elements of the set

$$\{\Omega_1(\theta), \ldots, \Omega_d(\theta)\} \cup \Big(\bigcup_{r=1}^{d} \bigcup_{n \in V_r(s)} \{\mathscr{L}_{0,rn}(\theta), \ldots, \mathscr{L}_{m_{rn},rn}(\theta)\} \Big) \qquad (13)$$

are algebraically independent over $\bar{k}$. In particular, by Definition 4.4.5, we have that

$$\{\tilde{\pi}_1, \ldots, \tilde{\pi}_d\} \cup \Big(\bigcup_{r=1}^{d} \bigcup_{n \in V_r(s)} \{\zeta_r(n)\} \Big)$$

is an algebraically independent set over $\bar{k}$. Counting the cardinality of $V_r(s)$ for each $1 \leq r \leq d$, we complete the proof. $\qquad \square$

Acknowledgements

References

[Anderson 1986] G. W. Anderson, "*t*-motives", *Duke Math. J.* **53**:2 (1986), 457–502. MR 87j:11042 Zbl 0679.14001

[Anderson and Thakur 1990] G. W. Anderson and D. S. Thakur, "Tensor powers of the Carlitz module and zeta values", *Ann. of Math.* (2) **132**:1 (1990), 159–191. MR 91h:11046 Zbl 0713.11082

[Anderson et al. 2004] G. W. Anderson, W. D. Brownawell, and M. A. Papanikolas, "Determination of the algebraic relations among special Γ-values in positive characteristic", *Ann. of Math.* (2) **160**:1 (2004), 237–313. MR 2005m:11140 Zbl 1064.11055

[Carlitz 1935] L. Carlitz, "On certain functions connected with polynomials in a Galois field", *Duke Math. J.* **1**:2 (1935), 137–168. MR 1545872 Zbl 0012.04904

[Chang 2009] C.-Y. Chang, "A note on a refined version of Anderson–Brownawell–Papanikolas criterion", *J. Number Theory* **129**:3 (2009), 729–738. MR 2010a:11144 Zbl 1187.11022

[Chang and Yu 2007] C.-Y. Chang and J. Yu, "Determination of algebraic relations among special zeta values in positive characteristic", *Adv. Math.* **216**:1 (2007), 321–345. MR 2008j:11114 Zbl 1123.11025

[Chang et al. 2010] C.-Y. Chang, M. A. Papanikolas, D. S. Thakur, and J. Yu, "Algebraic independence of arithmetic gamma values and Carlitz zeta values", *Adv. Math.* **223**:4 (2010), 1137–1154. MR 2011d:11180 Zbl 05671646

[Denis 1998] L. Denis, "Indépendance algébrique de différents π", *C. R. Acad. Sci. Paris Sér. I Math.* **327**:8 (1998), 711–714. MR 99i:11059 Zbl 0922.11055

[Goss 1996] D. Goss, *Basic structures of function field arithmetic*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **35**, Springer, Berlin, 1996. MR 97i:11062 Zbl 0874.11004

[Papanikolas 2008] M. A. Papanikolas, "Tannakian duality for Anderson–Drinfeld motives and algebraic independence of Carlitz logarithms", *Invent. Math.* **171**:1 (2008), 123–174. MR 2009b:11127 Zbl 05236753

[Thakur 2004] D. S. Thakur, *Function field arithmetic*, World Scientific, River Edge, NJ, 2004. MR 2005h:11115 Zbl 1061.11001

[Wade 1941] L. I. Wade, "Certain quantities transcendental over $GF(p^n, x)$", *Duke Math. J.* **8** (1941), 701–720. MR 3,263f Zbl 0063.08101

[Yu 1991] J. Yu, "Transcendence and special zeta values in characteristic p", *Ann. of Math.* (2) **134**:1 (1991), 1–23. MR 92g:11075 Zbl 0734.11040

[Yu 1997] J. Yu, "Analytic homomorphisms into Drinfeld modules", *Ann. of Math.* (2) **145**:2 (1997), 215–233. MR 98c:11054 Zbl 0881.11055

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