# Algebra & Number Theory

Volume 5

2011

No. 6


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mathematical sciences publishers

ALGEBRA AND NUMBER THEORY 5:6(2011)

msp

# Higher direct images of the structure sheaf in positive characteristic

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We prove vanishing of the higher direct images of the structure (and the canonical) sheaf for a proper birational morphism with source a smooth variety and target the quotient of a smooth variety by a finite group of order prime to the characteristic of the ground field. We also show that for smooth projective varieties the cohomology of the structure sheaf is a birational invariant. These results are well known in characteristic zero.



## Introduction

In characteristic zero it is a well-known and frequently used fact that the higher direct images $R^i f_* \mathcal{O}_X$ of a projective birational morphism $f : X \to Y$ between smooth schemes vanish for $i > 0$. This statement was proved as a corollary of Hironaka's resolution of singularities by resolving the indeterminacies of $f^{-1}$ by successively blowing up smooth subvarieties of $Y$. In this article we consider the situation over an arbitrary field $k$ and prove this and related results.

In the following all schemes are assumed to be separated and of finite type over $k$, and all morphisms are assumed to be $k$-morphisms. The two main results of this paper are as follows.

This work has been supported by the SFB/TR 45 "Periods, moduli spaces and arithmetic of algebraic varieties".

*MSC2000:* primary 14E05; secondary 14C17.

*Keywords:* birational geometry, rational singularities.

**Theorem 1.** *Assume $k$ is perfect. Let $S$ be an arbitrary scheme and let $X$ and $Y$ be integral $S$-schemes. Assume that $X$ and $Y$ are smooth over $k$ and properly birational over $S$, that is, there exists an integral scheme $Z$ and a commutative diagram*

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow^{\tau_X} & & \searrow^{\tau_Y} & \\ X & & & & Y \\ & \searrow_{f} & & \swarrow_{g} & \\ & & S & & \end{array}$$

*such that $\tau_X$ and $\tau_Y$ are proper and birational ($f$ and $g$ being the fixed morphisms to $S$). Then for all $i$, there are isomorphisms of $\mathcal{O}_S$-modules*

$$R^i f_*\mathcal{O}_X \cong R^i g_*\mathcal{O}_Y \quad \text{and} \quad R^i f_*\omega_X \cong R^i g_*\omega_Y.$$

**Theorem 2.** *Consider a diagram*

$$\begin{array}{ccc} & & Y \\ & & \downarrow{\scriptstyle f} \\ \tilde{X} & \xrightarrow{\pi} & X, \end{array}$$

*where $Y$ and $\tilde{X}$ are connected smooth schemes, $X$ is integral and normal, $f$ is surjective and finite such that $\deg(f) \in k^*$, and $\pi$ is birational and proper. Then $X$ is Cohen–Macaulay and*

$$R\pi_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X \quad \text{and} \quad R\pi_*\omega_{\tilde{X}} = \omega_X,$$

*where $\omega_X$ is the dualizing sheaf of $X$.*

By duality, the two identities in Theorem 2 imply each other.

Both theorems are known in characteristic zero: Theorem 1 follows from Hironaka's resolution of singularities; for Theorem 2, see [Viehweg 1977] (which also uses resolution of singularities). If resolution of singularities is available in positive characteristic then it easily yields Theorem 1.

Recall from [Kempf et al. 1973, I, Section 3, page 50] that a rational resolution of an integral normal scheme $X$ is a resolution (that is, a proper birational morphism $g : \tilde{X} \to X$ with $\tilde{X}$ smooth) that satisfies $R^i g_*(\mathcal{O}_{\tilde{X}}) = 0 = R^i g_*(\omega_{\tilde{X}})$ for all $i > 0$. Thus Theorem 1 implies that if an integral normal scheme over a perfect field has a rational resolution, then any resolution of $X$ is a rational resolution. For a smooth scheme $X$ we obtain $R^i g_*(\mathcal{O}_{\tilde{X}}) = 0 = R^i g_*(\omega_{\tilde{X}})$ for $i > 0$ and any resolution $g$ (Corollary 3.2.10). Theorem 2 asserts that $\pi : \tilde{X} \to X$ is a rational resolution; this includes the important special case where $X$ is the quotient of $Y$ by a finite group of order prime to the characteristic of $k$.

Since resolution of singularities is not yet available in positive characteristic, we develop a different approach based on algebraic correspondences. To get an idea of the methods involved, let us sketch the proof of Theorem 1 for $S = \operatorname{Spec} k$ and $X$ and $Y$ projective (see Section 3 for the details and a rigorous proof).

For a scheme $X$ we write $H(X) = \bigoplus_{i,j} H^i(X, \Omega^j_X)$ and $\mathrm{CH}(X) = \bigoplus_i \mathrm{CH}^i(X)$, with $\mathrm{CH}^i(X)$ the Chow group of codimension $i$ cycles on $X$. Given smooth projective schemes $X$, $Y$, $Z$, there is a composition of correspondences:

$$\mathrm{CH}(X \times Y) \otimes_{\mathbb{Z}} \mathrm{CH}(Y \times Z) \to \mathrm{CH}(X \times Z),$$
$$H(X \times Y) \otimes_k H(Y \times Z) \to H(X \times Z).$$

Moreover $a \in \mathrm{CH}(X \times Y)$ and $\in H(X \times Y)$ define maps on the cohomologies $\mathrm{CH}(X) \to \mathrm{CH}(Y)$ and $H(X) \to H(Y)$, respectively, by $c \mapsto \mathrm{pr}_{2*}(\mathrm{pr}_1^*(c) \cup a)$, and composition of correspondences corresponds to the composition of maps. Furthermore, there is a cycle map $cl : \mathrm{CH} \to H$, which is compatible with composition.

Now, the proof proceeds as follows. By assumption there exists a closed integral subscheme $Z \subset X \times Y$ projecting birational to $X$ and $Y$. Let $Z' \subset Y \times X$ be its transpose. By using the refined intersection product of Fulton we will see that

$$[Z'] \circ [Z] = \mathrm{id}_X + E \quad \text{in } \ \mathrm{CH}\big(\mathrm{pr}_{13}((Z \times X) \cap (X \times Z'))\big),$$

with $E$ a cycle on $X \times X$ that projects on both sides to subsets of codimension at least one in $X$. We will show that the map defined by $cl(E) \in H(X \times X)$ acts as zero on $H^*(X, \mathcal{O}_X) \oplus H^*(X, \omega_X)$. A similar argument applies for $[Z] \circ [Z']$. Thus the maps defined by $cl([Z])$ and $cl([Z'])$ are inverse to each other (when restricted to $H^*(X, \mathcal{O}_X) \oplus H^*(X, \omega_X)$). This proves Theorem 1 in the case $S = \operatorname{Spec} k$ and $X$ and $Y$ projective.

It is not hard to deduce the general statement of Theorem 1 once we know it in the case $S = \operatorname{Spec} k$. Therefore we have to generalize the argument above to the case of smooth but not necessarily proper $k$-schemes. The problem is that in general a push-forward on CH or $H$ does not exist. However, the variety $Z \subset X \times Y$ is proper over $X$ and $Y$, and by working with cohomology (or Chow groups) with support we can conclude as outlined above.

One of the main points in this paper is the construction of a cycle map, or natural transformation between cohomology theories with support, $\mathrm{CH} \to H$. For this, we first give a definition for (weak) cohomology theories with support. We introduce two categories $V^*$ and $V_*$. The objects in both categories are $(X, \Phi)$, where $X$ is smooth and $\Phi$ is a family of supports on $X$ (see Definition 1.1.1 for the definition of a family with supports). A morphism $f : X \to Y$ induces a morphism $(X, \Phi) \to (Y, \Psi)$ in $V_*$ if and only if $f|_\Phi$ is proper and $f(\Phi) \subset \Psi$; $f$ induces a morphism in $V^*$ if and only if $f^{-1}(\Psi) \subset \Phi$.

Then we consider the data $(F_*, F^*, T, e)$, where

$$F_* : V_* \to \mathbf{GrAb} \quad \text{and} \quad F^* : (V^*)^{\mathrm{op}} \to \mathbf{GrAb},$$

are functors to graded abelian groups with $F_*(X, \Phi) = F^*(X, \Phi) =: F(X, \Phi)$ as abelian groups, $T$ gives for all $(X, \Phi)$ and $(Y, \Psi)$ a morphism of abelian groups

$$T_{(X,\Phi),(Y,\Psi)} : F(X, \Phi) \otimes_{\mathbb{Z}} F(Y, \Psi) \to F(X \times Y, \Phi \times \Psi),$$

and $e : \mathbb{Z} \to F(\operatorname{Spec} k)$ is a morphism of abelian groups.

These data define a weak cohomology theory with support if the following conditions are satisfied.

(1) The map $T$ is functorial for $F_*$ and $F^*$.

(2) For all diagrams

$$\begin{array}{ccc} (X', \Phi') & \xrightarrow{f'} & (Y', \Psi') \\ \downarrow{\scriptstyle g_X} & & \downarrow{\scriptstyle g_Y} \\ (X, \Phi) & \xrightarrow{f} & (Y, \Psi), \end{array}$$

in which the underlying diagram of schemes is cartesian and *transversal*, with $g_X, g_Y \in V^*$ and $f, f' \in V_*$, we have

$$F^*(g_Y) \circ F_*(f) = F_*(f') \circ F^*(g_X).$$

(3) Some more (very natural) conditions.

The conditions allow us to obtain a calculus with correspondences. One key example of a weak cohomology theory with supports is the Chow group

$$\mathrm{CH}(X, \Phi) := \varinjlim_{W \in \Phi} \mathrm{CH}(W),$$

with $\mathrm{CH}_*$ the (proper) push forward for cycles and $\mathrm{CH}^*$ the refined Gysin homomorphism. Another example is the Hodge cohomology

$$H(X, \Phi) := \bigoplus_{i,j} H^i_\Phi(X, \Omega^j_X).$$

Here, the definition of $H^*$ is straightforward, but for $H_*$ we use Grothendieck duality for singular schemes since smooth compactifications are not available in characteristic $p$. That the Hodge cohomology defines a (weak) cohomology theory with supports is a nontrivial fact, the proof of which occupies Section 2.

In Theorem 1.2.3 we give necessary and sufficient conditions for a (weak) cohomology theory with supports $F$ to be target of a morphism from CH. Unfortunately, we can do this only with an additional semipurity assumption on $F$ (see Definition 1.2.1). As an application we prove the existence of a cycle map $\mathrm{CH} \to H$. We hope that Theorem 1.2.3 will turn out to be useful for proving similar results for the Witt vector cohomology.

Let us give a short overview of the content of each section.

In Section 1 we define weak cohomology theories with supports and prove basic properties. We show that CH is an example and prove Theorem 1.2.3. Moreover, we explain the calculus of correspondences attached to a cohomology theory.

In Section 2 we show that the Hodge cohomology is another example for a cohomology theory with supports. The hard part is the definition of push-forward maps. We use Grothendieck's duality theory for singular schemes as developed in [Hartshorne 1966; Conrad 2000], and make extensive use of the results given in these references. There are also other approaches to duality theory that are more elegant (see for example [Lipman and Hashimoto 2009]). But since we use at several places the explicit description of duality theory as developed by Grothendieck and since it is not clear to the authors how this classical approach compares to the one, for example, in [Lipman and Hashimoto 2009], we will solely stick to the references [Hartshorne 1966; Conrad 2000].

In Section 3 we show the existence of a cycle map $\mathrm{CH} \to H$. We also prove a vanishing statement Proposition 3.2.2, enabling us to prove Theorem 1 and 2.

In Section 4 we generalize Theorem 1 to the case where $X$ and $Y$ are tame quotients (see Theorem 4.3.1). This theorem also implies Theorem 2.

We finish with some open questions.

In the appendix we describe the trace morphism for closed embeddings between smooth schemes and for finite and surjective morphisms between smooth schemes; this is well known but needed in Section 2.

## 1. Chow groups with support

**1.1.** ***Cohomology theories with support.*** Let $k$ be a field. We assume all schemes are of finite type and separated over $k$. We begin by recalling basic definitions and notation concerning families of supports.

**Definition 1.1.1.** A family of supports $\Phi$ on $X$ is a nonempty set of closed subsets of $X$ such that the following holds:

(i) The union of two elements in $\Phi$ is contained in $\Phi$.

(ii) Every closed subset of an element in $\Phi$ is contained in $\Phi$.

Let $A$ be any set of closed subsets of $X$. The smallest family of supports $\Phi_A$ that contains $A$ is given by

$$\Phi_A := \Big\{ \bigcup_{i=1}^{n} Z_i' \,;\; Z_i' \underset{\text{closed}}{\subset} Z_i \in A \Big\}. \tag{1.1.2}$$

For a closed subset $Z \subset X$ we write $\Phi_Z$ for $\Phi_{\{Z\}}$.

**Notation 1.1.3.** Let $f : X \to Y$ be a morphism of schemes and let $\Phi$ and $\Psi$ be families of supports of $X$ and $Y$, respectively.

(1) We denote by $f^{-1}(\Psi)$ the smallest family of supports on $X$ that contains $\{f^{-1}(Z) : Z \in \Psi\}$.

(2) We say that $f|_\Phi$ is proper if $f|_Z$ is proper for every $Z \in \Phi$. If $f|_\Phi$ is proper, then $f(\Phi)$ is a family of supports on $Y$.

(3) If $\Phi_1$ and $\Phi_2$ are two families of supports, then $\Phi_1 \cap \Phi_2$ is a family of supports.

(4) If $\Phi$ and $\Psi$ are families of supports of $X$ and $Y$, respectively, then we denote by $\Phi \times \Psi$ the smallest family of supports on $X \times_k Y$ that contains

$$\{Z_1 \times Z_2;\ Z_1 \in \Phi,\ Z_2 \in \Psi\}.$$

When working with cohomology theories with support, it is convenient to define the following two categories $V_*$ and $V^*$, where for the morphisms in $V_*$ a "push-forward" map can be expected and for the morphisms in $V^*$ a "pull-back" map can be expected.

**Definition 1.1.4.** We denote by $V_*$ the category with objects $(X, \Phi)$, where $X$ is a smooth scheme and $\Phi$ is a family of supports of $X$, and morphisms

$$\mathrm{Hom}_{V_*}((X, \Phi), (Y, \Psi)) = \{f \in \mathrm{Hom}_k(X, Y);\ f|_\Phi \text{ is proper and } f(\Phi) \subset \Psi\}.$$

We denote by $V^*$ the category with objects $(X, \Phi)$, where $X$ is a smooth scheme and $\Phi$ is a family of supports of $X$ ($\mathrm{ob}(V_*) = \mathrm{ob}(V^*)$), and morphisms

$$\mathrm{Hom}_{V^*}((X, \Phi), (Y, \Psi)) = \{f \in \mathrm{Hom}_k(X, Y);\ f^{-1}(\Psi) \subset \Phi\}.$$

The composition and the identity comes in both cases from the category of schemes (over $k$).

**1.1.5.** Let $X$ be a smooth scheme. For a closed subscheme $W \subset X$ we write $(X, W) := (X, \Phi_W)$ in $V^*$ and $V_*$, respectively. We simply write $X$ for $(X, X)$.

We respectively have forgetful functors $V_* \to \mathbf{Sch}_k$ and $V^* \to \mathbf{Sch}_k$ to the category of schemes, and we often denote the morphism of schemes induced by a morphism in $V_*$ and $V^*$, respectively, by the same letter.

For a morphism $f$ in $V_*$ we will say that $f$ is an immersion, flat, ..., if the corresponding morphism of schemes has this property, and similarly for morphisms in $V^*$. We say that a diagram

$$\begin{array}{ccc} (X', \Phi') & \xrightarrow{f'} & (Y', \Psi') \\ \downarrow{\scriptstyle g_X} & & \downarrow{\scriptstyle g_Y} \\ (X, \Phi) & \xrightarrow{f} & (Y, \Psi) \end{array} \tag{1.1.6}$$

is cartesian if the diagram of the corresponding schemes is cartesian.

**1.1.7.** *Coproducts and "products".* For both categories $V^*$ and $V_*$, finite coproducts exist:

$$(X, \Phi) \coprod (Y, \Psi) = (X \coprod Y, \Phi \cup \Psi).$$

For $(X, \Phi)$, let $X = \coprod_i X_i$ be the decomposition into connected components; then

$$(X, \Phi) = \coprod_i (X_i, \Phi \cap \Phi_{X_i}).$$

In general products don't exist, and we define

$$(X, \Phi) \otimes (Y, \Psi) := (X \times Y, \Phi \times \Psi),$$

which together with the unit object $\mathbf{1} = \mathrm{Spec}(k)$ and the obvious isomorphism $(X, \Phi) \otimes (Y, \Psi) \to (Y, \Psi) \otimes (X, \Phi)$ makes $V_*$ and $V^*$ into a symmetric monoidal category; see [Mac Lane 1998, VII.1].

**1.1.8.** Consider the following data $(F_*, F^*, T, e)$:

(1) Two functors $F_* : V_* \to \mathbf{GrAb}$ and $F^* : (V^*)^{\mathrm{op}} \to \mathbf{GrAb}$ to the symmetric monoidal category of graded abelian groups, such that $F_*(X) = F^*(X)$ as ungraded groups for every object $X \in \mathrm{ob}(V_*) = \mathrm{ob}(V^*)$. We will simply write $F(X) := F_*(X) = F^*(X)$. We use lower indexes for the grading on $F_*(X)$, that is, $F_*(X) = \oplus_i F_i(X)$, and upper indexes for $F^*(X)$.

(2) For every two objects $X, Y \in \mathrm{ob}(V_*) = \mathrm{ob}(V^*)$, a morphism of graded abelian groups (for both gradings):

$$T_{X,Y} : F(X) \otimes_{\mathbb{Z}} F(Y) \to F(X \otimes Y).$$

(3) A morphism of abelian groups $e : \mathbb{Z} \to F(\mathrm{Spec}(k))$. For all smooth schemes $\pi : X \to \mathrm{Spec}(k)$ we denote by $1_X$ the image of $1 \in \mathbb{Z}$ via the map

$$\mathbb{Z} \xrightarrow{e} F^*(\mathrm{Spec}(k)) \xrightarrow{F^*(\pi)} F^*(X).$$

**1.1.9.** The data $(F_*, F^*, T, e)$ is called a *weak cohomology theory with supports* if the following conditions are satisfied:

(1) The functor $F_*$ preserves coproducts and $F^*$ maps coproducts to products, and for $(X, \Phi_1), (X, \Phi_2) \in \mathrm{ob}(V_*)$ with $\Phi_1 \cap \Phi_2 = \{\varnothing\}$, the map

$$F^*(\jmath_1) + F^*(\jmath_2) : F^*(X, \Phi_1) \oplus F^*(X, \Phi_2) \to F^*(X, \Phi_1 \cup \Phi_2),$$

with $\jmath_1 : (X, \Phi_1 \cup \Phi_2) \to (X, \Phi_1)$ and $\jmath_2 : (X, \Phi_1 \cup \Phi_2) \to (X, \Phi_2)$ in $V^*$, is an isomorphism.

(2) The data $(F_*, T, e)$ and $(F^*, T, e)$ respectively define a (right-lax) symmetric monoidal functor (see below).

(3) Grading: For $(X, \Phi)$ such that $X$ is connected, we have

$$F_i(X, \Phi) = F^{2\dim X - i}(X, \Phi) \quad \text{for all } i.$$

(4) For all cartesian diagrams (1.1.6) with $g_X, g_Y \in V^*$ and $f, f' \in V_*$ such that either $g_Y$ is smooth or $g_Y$ is a closed immersion and $f$ is transversal to $g_Y$, we have

$$F^*(g_Y) \circ F_*(f) = F_*(f') \circ F^*(g_X).$$

Recall that $f$ is transversal to $g_Y$ if $(f')^* N_{Y'/Y} = N_{X'/X}$, where $N$ denotes the normal bundle. The case $X' = \varnothing$ is also admissible; in this case the equality 1.1.9(4) reads $F^*(g_Y) \circ F_*(f) = 0$. The condition 1.1.9(4) implies the projection formula (see Proposition 1.1.16) and will be needed for a calculus with correspondences.

Recall that $(F_*, T, e)$ is called a right-lax symmetric monoidal functor if

- $T$ is associative, that is, for $X, Y, Z \in \mathrm{ob}(V_*)$, the diagram

$$\begin{array}{ccc} F(X) \otimes F(Y) \otimes F(Z) & \xrightarrow{\mathrm{id} \otimes T} & F(X) \otimes F(Y \otimes Z) \\ \downarrow{\scriptstyle T \otimes \mathrm{id}} & & \downarrow{\scriptstyle T} \\ F(X \otimes Y) \otimes F(Z) & \xrightarrow{T} & F(X \otimes Y \otimes Z) \end{array}$$

  is commutative;

- $T$ is commutative, that is, for $X, Y \in \mathrm{ob}(V_*)$, the diagram

$$\begin{array}{ccc} F(X) \otimes F(Y) & \xrightarrow{T} & F(X \otimes Y) \\ \downarrow & & \downarrow \\ F(Y) \otimes F(X) & \xrightarrow{T} & F(Y \otimes X) \end{array}$$

  is commutative. Here, for two graded abelian groups $A$ and $B$, the morphism $A \otimes B \to B \otimes A$ maps $a \otimes b \mapsto (-1)^{\deg(a)\deg(b)} b \otimes a$;

- the map $e : \mathbb{Z} \to F(\mathrm{Spec}(k))$ renders commutative the diagrams

$$\begin{array}{ccccc} F(X) \otimes_{\mathbb{Z}} \mathbb{Z} & \xrightarrow{\mathrm{id} \otimes e} & F(X) \otimes F(\mathrm{Spec}(k)) & \xrightarrow{T} & F(X \otimes \mathrm{Spec}(k)) \\ & {\scriptstyle =}\searrow & & \swarrow{\scriptstyle =} & \\ & & F(X), & & \end{array}$$

$$\begin{array}{ccccc} \mathbb{Z} \otimes_{\mathbb{Z}} F(X) & \xrightarrow{e \otimes \mathrm{id}} & F(\mathrm{Spec}(k)) \otimes F(X) & \xrightarrow{T} & F(\mathrm{Spec}(k) \otimes X) \\ & {\scriptstyle =}\searrow & & \swarrow{\scriptstyle =} & \\ & & F(X); & & \end{array}$$

- $T$ is a natural transformation

$$T : (V_* \times V_* \xrightarrow{F_* \times F_*} \mathbf{GrAb} \times \mathbf{GrAb} \xrightarrow{\otimes} \mathbf{GrAb}) \to (V_* \times V_* \xrightarrow{\otimes} V_* \xrightarrow{F_*} \mathbf{GrAb}).$$

**Example 1.1.10.** The Chow group

$$(X, \Phi) \mapsto \varinjlim_{W \in \Phi} \mathrm{CH}_*(W)$$

satisfies these conditions (see Proposition 1.1.34). The push-forward $V_* \to \mathbf{GrAb}$ is defined in the usual way. To define the pull-back $(V^*)^{\mathrm{op}} \to \mathbf{GrAb}$, we use Fulton's refined Gysin homomorphism. However, in order to get a symmetric functor we have to put the Chow group $\mathrm{CH}_d(W)$ in degree $= 2d$.

It will be shown in Section 2 that the Hodge cohomology with support

$$(X, \Phi) \mapsto \bigoplus_{i,j} H^i_\Phi(X, \Omega^j_X)$$

is another example. The push-forward is an application of Grothendieck's duality theory.

An example not further considered in this paper is the étale cohomology with support

$$(X, \Phi) \mapsto H^*_\Phi(X \times \bar{k}, \mathbb{Q}_\ell).$$

**Definition 1.1.11.** Let $(F_*, F^*, T, e)$ and $(G_*, G^*, U, \epsilon)$ be as in 1.1.8 and satisfy the conditions 1.1.9. By a morphism

$$(F_*, F^*, T, e) \to (G_*, G^*, U, \epsilon), \tag{1.1.12}$$

we mean a morphism of graded abelian groups (for both gradings)

$$\phi : F(X) \to G(X) \quad \text{for every } X \in \mathrm{ob}(V_*) = \mathrm{ob}(V^*),$$

such that $\phi$ induces a natural transformation of (right-lax) symmetric monoidal functors

$$\phi : (F_*, T, e) \to (G_*, U, \epsilon) \quad \text{and} \quad \phi : (F^*, T, e) \to (G^*, U, \epsilon),$$

that is, $\phi$ induces natural transformations $F_* \to G_*$, $F^* \to G^*$, and

$$\phi \circ T = U \circ (\phi \otimes \phi), \quad \phi \circ e = \epsilon. \tag{1.1.13}$$

We denote by $\mathbf{T}$ the category of weak cohomology theories with supports, that is, it is the category consisting of objects $(F_*, F^*, T, e)$ as in 1.1.8, and satisfying the properties 1.1.9, together with morphisms (1.1.12).

**1.1.14.** *Cup product.* Let $(F_*, F^*, T, e) \in \mathbf{T}$. For all $(X, \Phi) \in \mathrm{ob}(V^*)$ we obtain a cup product

$$\cup : F(X, \Phi_1) \otimes F(X, \Phi_2) \xrightarrow{T} F(X \times X, \Phi_1 \times \Phi_2) \xrightarrow{F^*(\Delta_X)} F(X, \Phi_1 \cap \Phi_2),$$

where $\Delta_X : (X, \Phi_1 \cap \Phi_2) \to (X \times X, \Phi_1 \times \Phi_2)$ is induced by the diagonal immersion. The cup product is associative and graded commutative.

By functoriality we obtain

$$F^*(f_1)(a) \cup F^*(f_2)(b) = F^*(f_3)(a \cup b) \tag{1.1.15}$$

for all morphisms $f_1 : (X', \Phi'_1) \to (X, \Phi_1)$, $f_2 : (X', \Phi'_2) \to (X, \Phi_2)$ in $V^*$ with $f_1 = f_2 := f$ as morphisms of schemes; $f_3 : (X', \Phi'_1 \cap \Phi'_2) \to (X, \Phi_1 \cap \Phi_2)$ in $V^*$ is induced by $f$.

**Proposition 1.1.16** (projection formula). *Let $(F_*, F^*, T, e) \in \mathbf{T}$ and let $f : X \to Y$ be a morphism between smooth schemes, inducing morphisms*

$$f_1 : (X, \Phi_1) \to (Y, \Phi_2) \quad \textit{in } V_*,$$
$$f_2 : (X, f^{-1}(\Psi)) \to (Y, \Psi) \quad \textit{in } V^*.$$

*Then $f$ also induces a morphism*

$$f_3 : (X, \Phi_1 \cap f^{-1}(\Psi)) \to (Y, \Phi_2 \cap \Psi) \quad \textit{in } V_*$$

*and for all $a \in F(X, \Phi_1)$ and $b \in F(Y, \Psi)$ we have in $F(Y, \Phi_2 \cap \Psi)$*

$$F_*(f_3)(a \cup F^*(f_2)(b)) = F_*(f_1)(a) \cup b,$$
$$F_*(f_3)(F^*(f_2)(b) \cup a) = b \cup F_*(f_1)(a).$$

*Proof.* We prove the first equality of the statement; the second is proved in the same way. The diagram

$$\begin{array}{ccc}
(X, \Phi_1 \cap f^{-1}(\Psi)) & \xrightarrow{f_3} & (Y, \Phi_2 \cap \Psi) \\
\downarrow{\scriptstyle \Delta_X} & & \\
(X \times X, \Phi_1 \times f^{-1}(\Psi)) & & \downarrow{\scriptstyle \Delta_Y} \\
\downarrow{\scriptstyle \mathrm{id} \times f_2} & & \\
(X \times Y, \Phi_1 \times \Psi) & \xrightarrow{f_1 \times \mathrm{id}} & (Y \times Y, \Phi_2 \times \Psi)
\end{array}$$

is cartesian and $f \times \mathrm{id}$ is transversal to $\Delta_Y$. Thus by 1.1.9(4) we get

$$\begin{aligned}
F_*(f_3)(a \cup F^*(f_2)(b)) &= F_*(f_3) F^*(\Delta_X) F^*(\mathrm{id} \times f_2)(T(a \otimes b)) \\
&= F^*(\Delta_Y) F_*(f_1 \times \mathrm{id})(T(a \otimes b)) = F_*(f_1)(a) \cup b. \quad \square
\end{aligned}$$

The proof of the following lemma is straightforward.

**Lemma 1.1.17.** (1) *For all* $(X, \Phi)$ *and* $a \in F(X, \Phi)$ *the equality*

$$1_X \cup a = a = a \cup 1_X$$

*holds. In particular* $F^*(X)$ *is a* (*graded*) *ring.*

(2) *For smooth schemes* $X$ *and* $Y$, *we have*

$$T(1_X \otimes 1_Y) = 1_{X\times Y}.$$

**1.1.18.** *Definition of Chow groups with support.* In the following we define a first example of an object $(\mathrm{CH}_*, \mathrm{CH}^*, \times, e) \in \mathbf{T}$.

**Definition 1.1.19** (Chow groups with support). Let $\Phi$ be a family of supports on $X$. We define

$Z_\Phi(X) :=$ abelian group freely generated by irreducible closed subsets $Z \in \Phi$;

$\mathrm{Rat}_\Phi(X) :=$ subgroup of $Z_\Phi(X)$ generated by $\mathrm{div}(f)$, where $f \in k(W)^*$ is a nonzero rational function and $W \in \Phi$ is irreducible;

$\mathrm{CH}(X, \Phi) :=$ $Z_\Phi(X)/\operatorname{Rat}\Phi(X)$.

For $(X, \Phi)$ and $(Y, \Psi)$ we obtain

$$\mathrm{CH}((X \coprod Y, \Phi \cup \Psi)) = \mathrm{CH}(X, \Phi) \oplus \mathrm{CH}(Y, \Psi). \tag{1.1.20}$$

**1.1.21.** *Grading.* The groups $Z_\Phi(X)$ and $\operatorname{Rat}\Phi(X)$ can be graded by dimension:

$$\mathrm{CH}_*(X, \Phi) = \bigoplus_{d\geq 0} \mathrm{CH}_d(X, \Phi)[2d],$$

where the bracket $[2d]$ means that $\mathrm{CH}_d(X, \Phi)$ is considered to be in degree $2d$.

There is also a grading by codimension. Let $X = \coprod_i X_i$ be the decomposition into connected components. Then $\mathrm{CH}^*(X, \Phi) = \bigoplus_i \mathrm{CH}^*(X_i, \Phi \cap \Phi_{X_i})$ and

$$\mathrm{CH}^*(X_i, \Phi \cap \Phi_{X_i}) = \bigoplus_{d\geq 0} \mathrm{CH}^d(X_i, \Phi \cap \Phi_{X_i})[2d],$$

where $\mathrm{CH}^d(X_i, \Phi)$ is generated by cycles $[Z]$ with $Z \in \Phi \cap \Phi_X$, $Z$ irreducible, and $\mathrm{codim}_{X_i}(Z) = d$.

**1.1.22.** *Examples.* If $W \subset X$ is a closed subset, then we get

$$\mathrm{CH}(X, \Phi_W) = \mathrm{CH}(X, W) = \mathrm{CH}(W),$$

the usual Chow group of $W$.

If $X$ is proper, $U$ is affine, and $\Phi := \{Z' ;\ Z' \subset U\}$, then

$$\mathrm{CH}(X, \Phi) = Z_\Phi(X) = \text{freely generated by closed points of } U.$$

**1.1.23.** *Push forward for Chow groups.* Let $\Phi$ be a family of supports on $X$ as in Definition 1.1.1. If $W \subset X$ is a closed subscheme with $W \in \Phi$, then $Z(W) = Z_{\Phi_W}(X) \subset Z_\Phi(X)$, $\operatorname{Rat}(W) = \operatorname{Rat}\Phi_W(X) \subset \operatorname{Rat}\Phi(X)$ (with $\Phi_W$ as defined in (1.1.2)), and we obtain a map

$$\mathrm{CH}(W) = \mathrm{CH}(X, W) \to \mathrm{CH}(X, \Phi). \tag{1.1.24}$$

Obviously, $\mathrm{CH}(X, \Phi)$ is the largest quotient of $Z_\Phi(X)$ such that there are push-forward maps (1.1.24) for every $W \in \Phi$.

**1.1.25.** In general, let $f : (X, \Phi) \to (Y, \Psi)$ be a morphism in $V_*$. There is a push forward of cycles

$$f_* : Z_\Phi(X) \to Z_\Psi(Y) \quad \text{and} \quad f_*([Z]) = \deg(Z/f(Z)) \cdot [f(Z)],$$

for $Z \in \Phi$ irreducible ($\deg(Z/f(Z)) = 0$ if $\dim(f(Z)) < \dim(Z)$). Push forward is functorial [Fulton 1998, Section 1.4].

**Lemma 1.1.26.** *With the assumption of 1.1.25, we get* $f_*(\operatorname{Rat}\Phi(X)) \subset \operatorname{Rat}\Psi(Y)$.

*Proof.* Indeed, $\operatorname{Rat}\Phi(X)$ is generated by the images of $\operatorname{Rat}(W)$ where $W \in \Phi$. The restriction $f|_W$ is proper and [Fulton 1998, Proposition 1.4] yields

$$f_*(\operatorname{Rat}(W)) \subset \operatorname{Rat}(f(W)).$$ □

Thus we get an induced map

$$f_* : \mathrm{CH}(X, \Phi) \to \mathrm{CH}(Y, \Psi) \tag{1.1.27}$$

and a functor

$$\mathrm{CH}_* : V_* \to \mathbf{GrAb}, \quad \mathrm{CH}_*(X, \Phi) := \mathrm{CH}(X, \Phi), \quad \mathrm{CH}_*(f) := f_*. \tag{1.1.28}$$

**Proposition 1.1.29.** *Let $\Phi$ be a family of supports of $X$. The map*

$$\varinjlim_{W \in \Phi} \mathrm{CH}(X, W) \to \mathrm{CH}(X, \Phi)$$

*is an isomorphism.*

*Proof.* This is straightforward. □

**1.1.30.** *Pull-back for Chow groups.* To define a functor

$$\mathrm{CH}^* : (V^*)^{\mathrm{op}} \to \mathbf{GrAb}$$

we recall Fulton's work on refined Gysin morphisms [Fulton 1998, Section 6.6].

Let $f : X \to Y$ be a morphism between smooth schemes and let $V \subset Y$ be a closed subscheme. There is a morphism

$$f^! : \mathrm{CH}(V) \to \mathrm{CH}(f^{-1}(V))$$

of abelian groups (where $f^{-1}(V) = X \times_Y V$) with the following properties:

(1) For a closed subscheme $V' \subset Y$ with $V \subset V'$ (denote the immersion by $\imath$ and the immersion $f^{-1}(V) \subset f^{-1}(V')$ by $\jmath$), we have the equality

$$f^! \imath_* = \jmath_* f^!$$

(as maps $\mathrm{CH}(V) \to \mathrm{CH}(f^{-1}(V'))$).

(2) If $g : Y \to Z$ is another morphism between smooth schemes and $S \subset Z$ a closed subscheme, then

$$f^! \circ g^! = (g \circ f)^!$$

as maps $\mathrm{CH}(S) \to \mathrm{CH}((g \circ f)^{-1}(S))$.

(3) If $f : X \to Y$ is flat, then $f^! = f^*$ where $f^*$ is the usual pull-back map for flat morphisms.

(4) Let

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle g_X} & & \downarrow{\scriptstyle g_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

be a cartesian diagram of smooth schemes and $W \subset X$ a closed subscheme such that $f|_W$ is proper. Assume that either $g_Y$ is flat or $g_Y$ is a closed immersion and $f$ is transversal to $g_Y$. Then

$$g_Y^! f_* = f'_* g_X^!$$

as maps $\mathrm{CH}(W) \to \mathrm{CH}(f'(g_X^{-1} W)) = \mathrm{CH}(g_Y^{-1} f(W))$. This statement is proved in [Fulton 1998, Proposition 1.7] for flat morphisms and in [Fulton 1998, Theorem 6.2(a),(c)] for the case of a closed immersion.

**Remark 1.1.31.** Note that $\mathrm{CH}(W) = \mathrm{CH}(W_{\mathrm{red}})$ for every scheme $W$.

**1.1.32.** *Definition of the pull-back map.* Let $f : X \to Y$ be a morphism between smooth schemes and let $V \subset Y$ be a closed subscheme; thus $f : (X, f^{-1}(V)) \to (Y, V)$ is a morphism in $V^*$. We define

$$\mathrm{CH}^*(f) := f^! : \mathrm{CH}(Y, V) = \mathrm{CH}(V) \to \mathrm{CH}(f^{-1}(V)) = \mathrm{CH}(X, f^{-1}(V)).$$

For the general case, let $f : (X, \Phi) \to (Y, \Psi)$ be any morphism in $V^*$. For every $V \in \Psi$ the map $f$ induces $(X, f^{-1}(V)) \to (Y, V)$ in $V^*$. Because of 1.1.30(1) and Proposition 1.1.29, we obtain

$$\mathrm{CH}^*(f) : \mathrm{CH}(Y, \Psi) = \varinjlim_{V \in \Psi} \mathrm{CH}(Y, V) \to \varinjlim_{W \in \Phi} \mathrm{CH}(X, W) = \mathrm{CH}(X, \Phi).$$

The assignment

$$\mathrm{CH}^*: (V^*)^{\mathrm{op}} \to \mathbf{GrAb}, \quad \mathrm{CH}^*(X, \Phi) = \mathrm{CH}(X, \Phi), \quad f \mapsto \mathrm{CH}^*(f), \tag{1.1.33}$$

defines a functor by 1.1.30(1) and (2).

**Proposition 1.1.34.** *With the exterior product* $\times$ (see [Fulton 1998, Section 1.10]) *and the obvious unit* $1 : \mathbb{Z} \to \mathrm{CH}(\mathrm{Spec}(k))$, *we obtain an object*

$$(\mathrm{CH}_*, \mathrm{CH}^*, \times, e) \in \mathbf{T}.$$

*Proof.* The formula 1.1.9(4) follows from 1.1.30(1) and (4). □

**1.2. *Chow groups with support as initial object.*** Given $(\mathrm{CH}_*, \mathrm{CH}^*, \times, 1)$ we are interested in objects $(F_*, F^*, T, e) \in \mathbf{T}$ that admit a morphism $(\mathrm{CH}_*, \mathrm{CH}^*, \times, 1) \to (F_*, F^*, T, e)$. Such morphisms should be viewed as a kind of cycle map, which is compatible with push-forward and pull-back. Unfortunately, we can only give a satisfactory answer under an additional hypothesis on $(F_*, F^*, T, e)$, which we call semipurity.

**Definition 1.2.1** (semipurity). We say that $(F_*, F^*, T, e)$ satisfies the semipurity condition if the following holds:

- For all smooth schemes $X$ and irreducible closed subsets $W \subset X$, the groups $F_i(X, W)$ vanish if $i > 2 \dim W$.
- For all smooth schemes $X$, closed subsets $W \subset X$, and open sets $U \subset X$ such that $U$ contains the generic point of every irreducible component of $W$, we require the map

  $$F^*(\jmath) : F_{2 \dim W}(X, W) \to F_{2 \dim W}(U, W \cap U),$$

  induced by $\jmath : (U, W \cap U) \to (X, W)$ in $V^*$, to be *injective*.

**Remark 1.2.2.** For $(\mathrm{CH}_*, \mathrm{CH}^*, \times, 1)$ the condition is satisfied since

$$\mathrm{CH}_{2 \dim W}(X, W) = \mathbb{Z} \cdot [W] \quad \text{and} \quad \mathrm{CH}_i(X, W) = 0 \quad \text{for } i > 2 \dim W.$$

Let $c$ be the codimension of $W$ in $X$, so that $F_{2 \dim W} = F^{2c}$. Whenever there are exact sequences

$$F^{2c}(X, W \setminus U) \to F^{2c}(X, W) \to F^{2c}(U, U \cap W),$$

the conditions in 1.2.1 follow from $F^i(X, W) = 0$ for $i < 2c$ (and all pairs $(X, W)$); this is known as semipurity in the literature.

**Theorem 1.2.3.** *Suppose $k$ is a perfect field and assume* $(F_*, F^*, T, e) \in \mathbf{T}$ *satisfies the semipurity condition 1.2.1. Then* $\mathrm{Hom}_{\mathbf{T}}((\mathrm{CH}_*, \mathrm{CH}^*, \times, 1), (F_*, F^*, T, e))$ *is either empty or contains only one element; it is nonempty if and only if the following conditions hold:*

(1) *If $f : X \to Y$ is a finite morphism between smooth connected schemes of equal dimension, then*

$$F_*(f)(1_X) = \deg(f) \cdot 1_Y.$$

(2) *For the* 0*-point* $\imath_0 : \mathrm{Spec}(k) \to \mathbb{P}^1$ *and the* $\infty$*-point* $\imath_\infty : \mathrm{Spec}(k) \to \mathbb{P}^1$ *the following equality holds*:

$$F_*(\imath_0) \circ e = F_*(\imath_\infty) \circ e.$$

(3) *For a closed immersion* $\imath : X \to Y$ *between smooth schemes and an effective smooth divisor* $D \subset Y$ *such that*

- $D$ *meets* $X$ *properly and thus* $D \cap X := D \times_Y X$ *is a divisor on* $X$,
- $D' := (D \cap X)_{\mathrm{red}}$ *is smooth and connected and thus* $D \cap X = n \cdot D'$ *as divisors* (*for some* $n \in \mathbb{Z}$, $n \geq 1$),

*we denote by* $\imath_X : X \to (Y, X)$ *and* $\imath_{D'} : D' \to (D, D')$ *the morphisms in* $V_*$ *induced by* $\imath$, *and we define* $g : (D, D') \to (Y, X)$ *in* $V^*$ *by the inclusion* $D \subset Y$. *Then the following equality is required to hold*:

$$F^*(g)(F_*(\imath_X)(1_X)) = n \cdot F_*(\imath_{D'})(1_{D'}).$$

(4) *If* $W \subset X$ *is an irreducible closed subset, then there is an element* $cl_{(X,W)} \in F_{2\dim W}(X, W)$ *with*

$$F^*(\jmath)(cl_{(X,W)}) = F_*(\imath)(1_{U \cap W})$$

*for all open sets* $U \subset X$ *such that* $U \cap W \neq \varnothing$ *is smooth, and where* $\jmath : (U, W \cap U) \to (X, W)$ *in* $V^*$ *and* $\imath : W \cap U \to (U, W \cap U)$ *in* $V_*$.

We will give the proof after the proof of the following proposition.

**Proposition 1.2.4.** *Let $k$ be a perfect field and let* $F := (F_*, F^*, T, e) \in \mathbf{T}$ *satisfy the semipurity condition 1.2.1. We also assume that the conditions (1)–(4) of Theorem 1.2.3 hold for $F$. Then there is a unique natural transformation of* (*right-lax*) *symmetric monoidal functors*

$$\phi : (\mathrm{CH}_*, \times, 1) \to (F_*, T, e)$$

*such that* $\phi(1_X) = 1_X$ *for every smooth scheme $X$.*

*Proof. Uniqueness*: In view of the semipurity condition 1.2.1,

$$\phi([W]) = cl_{(X,W)} \tag{1.2.5}$$

is the only choice for an irreducible closed subset $W$ of $X$, $[W] \in \mathrm{CH}_*(X, W)$. For a general family of supports $\Phi$ of $X$, the group $\mathrm{CH}_*(X, \Phi)$ is generated by the images of $[W]$ via $\mathrm{CH}_*(X, W) \to \mathrm{CH}_*(X, \Phi)$, where $W$ runs through all irreducible closed subsets $W \in \Phi$.

*Existence*: For every smooth scheme $X$ and a family of supports $\Phi$ of $X$, we define a homomorphism of abelian groups

$$\phi'_{(X,\Phi)} : Z_\Phi(X) \to F(X, \Phi) \tag{1.2.6}$$

by $\phi'_{(X,\Phi)}([W]) = F_*(\imath_W)(cl_{(X,W)})$ for every irreducible closed subset $W \in \Phi$ and $\imath_W : (X, W) \to (X, \Phi)$ in $V_*$ induced by $\mathrm{id}_X$.

*1st step*: For every morphism $f : (X, \Phi) \to (Y, \Psi)$ in $V_*$ the push-forward $f_* : Z_\Phi(X) \to Z_\Psi(Y)$ is well defined by 1.1.25. We claim that

$$\phi'_{(Y,\Psi)} \circ f_* = F_*(f) \circ \phi'_{(X,\Phi)} \tag{1.2.7}$$

for every $f : (X, \Phi) \to (Y, \Psi)$ in $V_*$.

Let $W \in \Phi$ be irreducible. If $\dim(f(W)) < \dim(W)$, then $F_{2\dim W}(Y, f(W)) = 0$ by semipurity 1.2.1; thus (1.2.7) holds in this case.

In the case $\dim f(W) = \dim W =: d$, the map $W \to f(W)$ is generically finite, so that we may find an open $U \subset Y$ such that $U \cap f(W) \neq \varnothing$, $U \cap f(W)$ is smooth, $f^{-1}(U) \cap W$ is smooth, and $f' : f^{-1}(U) \cap W \to U \cap f(W)$ induced by $f$ is finite. Consider the commutative diagram

$$\begin{array}{ccccc}
F_{2d}(X, W) & \xrightarrow{F^*(\jmath')} & F_{2d}(f^{-1}(U), W \cap f^{-1}(U)) & \xleftarrow{F_*(\imath')} & F_{2d}(W \cap f^{-1}(U)) \\
\downarrow{\scriptstyle F_*(f)} & & \downarrow{\scriptstyle F_*(f)} & & \downarrow{\scriptstyle F_*(f')} \\
F_{2d}(Y, f(W)) & \xrightarrow{F^*(\jmath)} & F_{2d}(U, f(W) \cap U) & \xleftarrow{F_*(\imath)} & F_{2d}(f(W) \cap U),
\end{array}$$

where $\jmath : (U, f(W) \cap U) \to (Y, f(W))$ and $\jmath' : (f^{-1}(U), W \cap f^{-1}(U)) \to (X, W)$ in $V^*$ are induced by the obvious open immersions; the obvious closed immersions respectively induce $\imath : f(W) \cap U \to (U, f(W) \cap U)$ and $\imath' : W \cap f^{-1}(U) \to (f^{-1}(U), W \cap f^{-1}(U))$ in $V_*$. From the diagram and condition 1.2.3(1), we obtain

$$F^*(\jmath)F_*(f)(cl_{(X,W)}) = \deg(W/f(W)) \cdot F_*(\imath)(1_{f(W)\cap U}).$$

Now, semipurity 1.2.1 implies

$$F_*(f)(cl_{(X,W)}) = \deg(W/f(W)) \cdot cl_{(Y,f(W))},$$

which proves the claim (1.2.7).

*2nd step*: Let $X$ be a smooth scheme, $W \subset X$ an irreducible closed subset, and $D \subset X$ a smooth divisor intersecting $W$ properly, so that $W \cap D := W \times_X D$ is an effective Cartier divisor on $W$. We denote by $[W \cap D]$ the associated Weil divisor and claim that

$$F^*(\imath_D)(\phi'_{(X,W)}([W])) = \phi'_{(D,W\cap D)}([W \cap D]), \tag{1.2.8}$$

where $\imath_D : (D, W \cap D) \to (X, W)$ is induced by $D \subset X$.

Note that by semipurity we may replace $X$ by an open subset that contains the generic points of $(W \cap D)_{\mathrm{red}}$. In particular, we may assume that the irreducible components of $(W \cap D)_{\mathrm{red}}$ are disjoint. Letting $V_1, \ldots, V_r$ be the irreducible components of $(W \cap D)_{\mathrm{red}}$, we obtain

$$\bigoplus_{i=1}^{r} F(D, V_i) \xrightarrow{\cong} F(D, W \cap D)$$

from 1.1.9(1); thus we may assume that $r = 1$. If $W$ is regular (that is, smooth) in codimension one (for example, $W$ is normal), then we can find an open $U \subset X$ such that $W \cap U$ and $V_1 \cap U \neq \varnothing$ is smooth; thus (1.2.8) follows from 1.2.3(3).

Now, let $W$ be not necessarily normal. Since we may assume that $X$ is affine we can find a closed immersion $\tilde{W} \to W \times \mathbb{P}^n$ (over $W$) of the normalization $\tilde{W}$ of $W$. Setting

$$\tilde{X} := X \times \mathbb{P}^n, \quad \tilde{D} := D \times \mathbb{P}^n, \quad \tilde{\imath} : (\tilde{D}, \tilde{V} \cap \tilde{D}) \to (\tilde{X}, \tilde{V}),$$

we obtain

$$\begin{aligned}
F^*(\imath)(\phi'_{(X,W)}[W]) &= F^*(\imath)F_*(\mathrm{pr}_1)(\phi'_{(\tilde{X},\tilde{W})}([\tilde{W}])) && \text{by (1.2.7)}\\
&= F_*(\mathrm{pr}_1|_{\tilde{D}})F^*(\tilde{\imath})(\phi'_{(\tilde{X},\tilde{W})}([\tilde{W}])) && \text{by 1.1.9(4)}\\
&= F_*(\mathrm{pr}_1|_{\tilde{D}})(\phi'_{(\tilde{D},\tilde{W}\cap\tilde{D})}([\tilde{W} \cap \tilde{D}]))\\
&= \phi'_{(D,W\cap D)}(\mathrm{pr}_{1*}([\tilde{W} \cap \tilde{D}])) && \text{by (1.2.7)}\\
&= \phi'_{(D,W\cap D)}([W \cap D]).
\end{aligned}$$

*3rd step*: For all $(X, \Phi)$ we claim that the map $\phi'_{(X,\Phi)}$ satisfies

$$\phi'_{(X,\Phi)}(\mathrm{Rat}_\Phi(X)) = 0; \tag{1.2.9}$$

and thus induces a natural transformation $\phi : \mathrm{CH}_* \to F_*$.

Let $W \subset X \times \mathbb{P}^1$ be irreducible such that $\mathrm{pr}_1(W) \in \Phi$ and $W \to \mathbb{P}^1$ is dominant. By using the 2nd step's (1.2.8), we obtain

$$F^*(\imath_\epsilon)(\phi'_{(X\times\mathbb{P}^1,W)}([W])) = \phi'_{(X,\mathrm{pr}_1(W))}([W \cap (X \times \{\epsilon\})])$$

for $\epsilon \in \{0, \infty\}$, $\imath_\epsilon : (X \times \{\epsilon\},\ \mathrm{pr}_1(W)) \to (X \times \mathbb{P}^1, W)$.

Thus $F^*(\imath_0) = F^*(\imath_\infty)$ will prove the claim (1.2.9). It is not difficult to see that this follows from the projection formula and

$$F_*(\imath'_0)(1_X) = F_*(\imath'_\infty)(1_X) \tag{1.2.10}$$

in $F(X \times \mathbb{P}^1)$, where $\imath'_\epsilon : X \times \{\epsilon\} \xrightarrow{\subset} X \times \mathbb{P}^1$.

In view of 1.1.9(4) the equality (1.2.10) is implied by 1.2.3(2).

*4th step*: The only assertion left to prove is

$$\phi \circ \times = T \circ (\phi \otimes \phi), \quad \phi \circ 1 = e.$$

The second equality holds by definition. For the first it suffices to show

$$\phi'_{(X \times Y, W \times V)}([W] \times [V]) = T(\phi'_{(X,W)}([W]) \otimes \phi'_{(Y,V)}([V]))$$

for smooth schemes $X$, $Y$ and irreducible closed subsets $W \subset X$, $V \subset Y$. Again by semipurity we may assume that $W$ and $V$ are smooth, in which case the statement follows from Lemma 1.1.17. □

*Proof of Theorem 1.2.3.* Set $\mathrm{CH} := (\mathrm{CH}_*, \mathrm{CH}^*, \times, 1)$ and $F := (F_*, F^*, T, e)$. For $\phi \in \mathrm{Hom}_{\mathbf{T}}(\mathrm{CH}, F)$, we get

$$\phi(1_X) = 1_X$$

for all smooth schemes $X$; thus Proposition 1.2.4 implies that $\mathrm{Hom}_{\mathbf{T}}(\mathrm{CH}, F)$ is either empty or contains only one element.

Obviously the conditions (1)–(4) of 1.2.3 are necessary for $\mathrm{Hom}_{\mathbf{T}}(\mathrm{CH}, F)$ to be nonempty. So let us assume that the conditions are satisfied. Proposition 1.2.4 yields a natural transformation of right-lax symmetric monoidal functors

$$\phi : (\mathrm{CH}_*, \times, 1) \to (F_*, T, e).$$

We need to prove that $\phi$ induces a natural transformation $\phi : \mathrm{CH}^* \to F^*$.

*1st step*: Assume that $f : (X, \Phi) \to (Y, \Psi)$ in $V^*$ is smooth. We claim diagram

$$\begin{array}{ccc} \mathrm{CH}(Y, \Psi) & \xrightarrow{\mathrm{CH}^*(f)} & \mathrm{CH}(X, \Phi) \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ F(Y, \Psi) & \xrightarrow{F^*(f)} & F(X, \Phi) \end{array}$$

commutes. It suffices to prove

$$F^*(f)(\phi_{(Y,V)}([V])) = \phi_{(X, f^{-1}(V))}(f^*[V])$$

for all irreducible closed subsets $V \subset Y$. By using semipurity we may replace $Y$ by an open set and thus assume that $V$ is smooth. We obtain

$$\begin{aligned} F^*(f)(\phi_{(Y,V)}([V])) &= F^*(f) F_*(\imath_V)(1_V) \\ &= F_*(\imath_{f^{-1}(V)}) F^*(f|_{f^{-1}(V)})(1_V) \qquad \text{by 1.1.9(4)} \\ &= F_*(\imath_{f^{-1}(V)})(1_{f^{-1}(V)}) \\ &= \phi_{(X, f^{-1}(V))}([f^{-1}(V)]), \end{aligned}$$

where $\imath_V : V \to (Y, V)$ and $\imath_{f^{-1}(V)} : f^{-1}(V) \to (X, f^{-1}(V))$.

*2nd step*: Let $p : E \to X$ be a vector bundle and let $s : X \to E$ be the zero section. We claim that for every closed subscheme $W \subset X$ the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{CH}(E, p^{-1}(W)) & \xrightarrow{\mathrm{CH}^*(s)} & \mathrm{CH}(X, W) \\ \downarrow \phi & & \downarrow \phi \\ F(E, p^{-1}(W)) & \xrightarrow{F^*(s)} & F(X, W). \end{array}$$

Indeed, by homotopy invariance we may write any $a \in \mathrm{CH}(p^{-1}(W))$ as $a = \mathrm{CH}^*(p)(b)$ with $b \in \mathrm{CH}(W)$. Thus by the 1st step,

$$F^*(s)(\phi(a)) = F^*(s)F^*(p)(\phi(b)) = \phi(b) = \phi(\mathrm{CH}^*(s)(a)).$$

*3rd step*: For every closed subscheme $W \subset X$, denote by

$$\imath_0 : (X, W) \to (X \times \mathbb{P}^1, W \times \mathbb{P}^1) \quad \text{and} \quad \imath_\infty : (X, W) \to (X \times \mathbb{P}^1, W \times \mathbb{P}^1),$$

the morphisms in $V^*$ and $V_*$ induced by the inclusions $X \times \{0\} \subset X \times \mathbb{P}^1$ and $X \times \{\infty\} \subset X \times \mathbb{P}^1$, respectively. We claim that

$$F^*(\imath_0) = F^*(\imath_\infty).$$

Indeed, if $p : (X \times \mathbb{P}^1, W \times \mathbb{P}^1) \to (X, W)$ is the first projection, then

$$F^*(\imath_\epsilon)(a) = F_*(p)F_*(\imath_\epsilon)(\phi([X]) \cup F^*(\imath_\epsilon)(a)) = F_*(p)(F_*(\imath_\epsilon)(\phi([X])) \cup a)$$

for $\epsilon \in \{0, \infty\}$. Since $F_*(\imath_\epsilon)(\phi([X])) = \phi([X \times \{\epsilon\}])$ the claim follows from $[X \times \{0\}] = [X \times \{\infty\}]$ in $\mathrm{CH}^1(X \times \mathbb{P}^1)$.

*4th step*: Let $f : X \to Y$ be a closed immersion and $V \subset Y$ a closed subscheme; set $W := f^{-1}(V) = V \times_Y X$. Then $f$ induces $f : (X, W) \to (Y, V)$ in $V^*$ and we claim that

$$\begin{array}{ccc} \mathrm{CH}(Y, V) & \xrightarrow{\mathrm{CH}^*(f)} & \mathrm{CH}(X, W) \\ \downarrow \phi & & \downarrow \phi \\ F(Y, V) & \xrightarrow{F^*(f)} & F(X, W) \end{array}$$

is a commutative diagram.

Again, it is sufficient to prove

$$F^*(f)(\phi([V])) = \phi(\mathrm{CH}^*(f))$$

for $V$ integral.

For the proof we use deformation to the normal cone [Fulton 1998, Section 5]. Let

$$M^0 := \mathrm{Bl}_{X\times\{\infty\}}(Y\times\mathbb{P}^1)\setminus \mathrm{Bl}_{X\times\{\infty\}}(Y\times\{\infty\}),$$
$$\tilde{M}^0 := \mathrm{Bl}_{W\times\{\infty\}}(V\times\mathbb{P}^1)\setminus \mathrm{Bl}_{W\times\{\infty\}}(V\times\{\infty\});$$

then $\tilde{M}^0 \subset M^0$ is closed, and $M^0$ and $\tilde{M}^0$ are flat over $\mathbb{P}^1$. We have closed immersions $\imath_X : X\times\mathbb{P}^1 \to M^0$ and $\imath_W : W\times\mathbb{P}^1 \to \tilde{M}^0$ that deform the immersions $X\subset Y$ and $W\subset V$, respectively, over $\mathbb{P}^1\setminus\{\infty\}$ to the zero section of the normal cone over $\infty$.

Since $W\times\mathbb{P}^1 = \tilde{M}^0\cap(X\times\mathbb{P}^1)$ we obtain morphisms

$$\imath_\epsilon : (X\times\{\epsilon\}, W\times\{\epsilon\}) \to (M^0, \tilde{M}^0)$$

in $V^*$ for $\epsilon\in\{0,\infty\}$. By the 3rd step we know that $F^*(\imath_0) = F^*(\imath_\infty)$.

Consider the projection $p : (Y\times(\mathbb{P}^1\setminus\{\infty\}), V\times(\mathbb{P}^1\setminus\{\infty\})) \to (Y, V)$ in $V^*$. Note that $\tilde{M}^0$ is the closure of $V\times(\mathbb{P}^1\setminus\{\infty\})$ in $M^0$, and thus

$$\mathrm{CH}^*(p)([V]) = \mathrm{CH}^*(\jmath)([\tilde{M}^0])$$

with $\jmath : (Y\times(\mathbb{P}^1\setminus\{\infty\}), V\times(\mathbb{P}^1\setminus\{\infty\})) \to (M^0, \tilde{M}^0)$ the open immersion. By using the 1st step we get $F^*(p)(\phi([V])) = F^*(\jmath)(\phi([\tilde{M}^0]))$ and thus

$$F^*(f)(\phi([V])) = F^*(\imath_0)(\phi([\tilde{M}^0])) = F^*(\imath_\infty)(\phi([\tilde{M}^0])).$$

Now, let us compute $F^*(\imath_\infty)$. The morphism $\imath_\infty$ has a factorization

$$\imath_\infty : (X, W) \xrightarrow{s} (N_{Y/X}, C_{V/W}) \xrightarrow{t} (M^0, \tilde{M}^0),$$

where $N_{Y/X}$ is the normal bundle and $C_{V/W}$ is the normal cone. Note that $N_{Y/X}$ is a smooth divisor in $M^0$, which intersects $\tilde{M}^0$ properly (being the fiber of $M^0\to\mathbb{P}^1$ over $\infty$), so that we may apply (1.2.8) to $t$. Moreover $s$ is the zero section of the normal bundle. The zero section also induces a morphism

$$s' : (X, W) \to (N_{Y/X}, N_{Y/X}\times_X W) \quad \text{in } V^*;$$

denote by $\tau : (N_{Y/X}, C_{V/W}) \to (N_{Y/X}, N_{Y/X}\times_X W)$ the morphism in $V_*$ induced by the identity map. Then 1.1.9(4) yields

$$F^*(s) = F^*(s')\circ F_*(\tau).$$

Thus we get

$$\begin{aligned} F^*(\imath_\infty)(\phi([\tilde{M}^0])) &= F^*(s')F_*(\tau)F^*(t)(\phi([\tilde{M}^0])) \\ &= F^*(s')F_*(\tau)(\phi(\mathrm{CH}^*(t)([\tilde{M}^0]))) && \text{by (1.2.8)} \\ &= \phi(\mathrm{CH}^*(\imath_\infty)([\tilde{M}^0])) && \text{by the 2nd step} \\ &= \phi(\mathrm{CH}^*(\imath_0)([\tilde{M}^0])) = \phi(\mathrm{CH}^*(f)([V])). \end{aligned}$$

*5th step*: Let $f : (X, \Phi) \to (Y, \Psi)$ be any morphism in $V^*$. We have to prove that

$$\phi \circ \mathrm{CH}^*(f) = F^*(f) \circ \phi.$$

Indeed, $f$ factors through

$$f : (X, \Phi) \xrightarrow{(\mathrm{id}, f)} (X \times Y, \mathrm{pr}_2^{-1}(\Psi)) \xrightarrow{\mathrm{pr}_2} (Y, \Psi).$$

By the 1st step we may reduce to the case of the closed immersion (id, $f$), and by using Proposition 1.1.29 the statement follows from the 4th step. □

**1.3.** ***Correspondences.*** Let $(F_*, F^*, T, e) \in \mathbf{T}$. Let $X_i$ for $i = 1, 2, 3$ be smooth varieties and $\Phi_{ij}$ for $ij = 12, 23, 13$ be families of supports on $X_i \times X_j$. Denote by $p_{ij} : X_1 \times X_2 \times X_3 \to X_i \times X_j$ the projection. Suppose that

$$\begin{cases} p_{13}|_{p_{12}^{-1}(\Phi_{12}) \cap p_{23}^{-1}(\Phi_{23})} \text{ is proper,} \\ p_{13}(p_{12}^{-1}(\Phi_{12}) \cap p_{23}^{-1}(\Phi_{23})) \subset \Phi_{13}. \end{cases} \tag{1.3.1}$$

Then we define

$$F(X_1 \times X_2, \Phi_{12}) \otimes F(X_2 \times X_3, \Phi_{23}) \to F(X_1 \times X_3, \Phi_{13}), \quad a \otimes b \mapsto b \circ a$$

to be the composition

$$\begin{aligned} &F(X_1 \times X_2, \Phi_{12}) \otimes F(X_2 \times X_3, \Phi_{23}) \\ &\xrightarrow{F^*(p_{12}) \otimes F^*(p_{23})} F(X_1 \times X_2 \times X_3, p_{12}^{-1}(\Phi_{12})) \otimes F(X_1 \times X_2 \times X_3, p_{23}^{-1}(\Phi_{23})) \\ &\xrightarrow{\cup} F(X_1 \times X_2 \times X_3, p_{12}^{-1}(\Phi_{12}) \cap p_{23}^{-1}(\Phi_{23})) \\ &\xrightarrow{F_*(p_{13})} F(X_1 \times X_3, \Phi_{13}). \end{aligned} \tag{1.3.2}$$

**1.3.3.** Let $\Phi'_{ij}$ for $ij = 12, 23, 13$ be families of supports on $X_i \times X_j$. Suppose that

$$\begin{cases} p_{13}|_{p_{12}^{-1}(\Phi'_{12}) \cap p_{23}^{-1}(\Phi'_{23})} \text{ is proper,} \\ p_{13}(p_{12}^{-1}(\Phi'_{12}) \cap p_{23}^{-1}(\Phi'_{23})) \subset \Phi'_{13}, \end{cases}$$

and $\Phi'_{ij} \subset \Phi_{ij}$ for $ij = 12, 23, 13$. Obviously, the diagram

$$\begin{array}{ccc} F(X_1 \times X_2, \Phi'_{12}) \otimes F(X_2 \times X_3, \Phi'_{23}) & \xrightarrow{\circ} & F(X_1 \times X_3, \Phi'_{13}) \\ \downarrow & & \downarrow \\ F(X_1 \times X_2, \Phi_{12}) \otimes F(X_2 \times X_3, \Phi_{23}) & \xrightarrow{\circ} & F(X_1 \times X_3, \Phi_{13}) \end{array}$$

is commutative.

The most important case for us will be $(\mathrm{CH}_*, \mathrm{CH}^*, \times, 1)$. For later use we record the following particular case of the discussion above.

**Lemma 1.3.4.** *Let $X_i$ for $i=1,2,3$ be smooth schemes and $\Phi_{ij}$ for $ij=12,23,13$ be families of supports on $X_i \times X_j$ that satisfy (1.3.1). For $a \in Z_{\Phi_{12}}(X_1 \times X_2)$ and $b \in Z_{\Phi_{23}}(X_2 \times X_3)$, we define*

$$\operatorname{supp}(a,b) := p_{13}(p_{12}^{-1}(\operatorname{supp}(a)) \cap p_{23}^{-1}(\operatorname{supp}(b))), \tag{1.3.5}$$

*which is a closed subset contained in $\Phi_{13}$. The families of supports*

$$\Phi'_{12} = \Phi_{\operatorname{supp}(a)}, \quad \Phi'_{23} = \Phi_{\operatorname{supp}(b)}, \quad \Phi'_{13} = \Phi_{\operatorname{supp}(a,b)}$$

*satisfy (1.3.1). The cycles $a$ and $b$ define in the obvious way classes*

$$\begin{aligned} &\tilde{a} \in \operatorname{CH}(\operatorname{supp}(a)), && \tilde{b} \in \operatorname{CH}(\operatorname{supp}(b)), \\ &a \in \operatorname{CH}(X_1 \times X_2, \Phi_{12}), && b \in \operatorname{CH}(X_2 \times X_3, \Phi_{23}). \end{aligned}$$

*Then $b \circ a$ is the image of $\tilde{b} \circ \tilde{a}$ via the map $\operatorname{CH}(\operatorname{supp}(a,b)) \to \operatorname{CH}(X_1 \times X_3, \Phi_{13})$.*

Lemma 1.3.4 helps to understand the composition of two cycles $a$ and $b$ via the purely set-theoretic computation of $\operatorname{supp}(a,b)$. Frequently we are able to compute the composition over suitable good open subsets; this is the motivation for the next lemma.

**Lemma 1.3.6.** *Let $X_i$ for $i=1,2,3$ be smooth schemes. Let $a \in Z(X_1 \times X_2)$ and $b \in Z(X_2 \times X_3)$ be algebraic cycles such that*

$$p_{13}|_{p_{12}^{-1}\operatorname{supp}(a) \cap p_{23}^{-1}\operatorname{supp}(b)} \quad \text{is proper.}$$

*Let $X'_1 \subset X_1$, $X'_3 \subset X_3$ be open subsets; define $a' \in Z(X'_1 \times X_2)$, $b' \in Z(X_2 \times X'_3)$ as the restrictions of $a$, $b$. We denote by $p'_{ij}$ the projections from $X'_1 \times X_2 \times X'_3$.*

(1) *The restriction of $p'_{13}$ to $p'^{-1}_{12}\operatorname{supp}(a') \cap p'^{-1}_{23}\operatorname{supp}(b')$ is proper.*

(2) *The equality*

$$\operatorname{supp}(a',b') = \operatorname{supp}(a,b) \cap (X'_1 \times X'_3)$$

*holds, where $\operatorname{supp}(a,b)$ is defined in (1.3.5).*

(3) *The composition $b' \circ a'$ is the image of $b \circ a$ via the localization map*

$$\operatorname{CH}(\operatorname{supp}(a,b)) \to \operatorname{CH}(\operatorname{supp}(a',b')).$$

*(Here $\operatorname{supp}(a',b') \subset \operatorname{supp}(a,b)$ is an open subset by (2)).*

*Proof.* By definition we obtain

$$\operatorname{supp}(a') = \operatorname{supp}(a) \cap (X'_1 \times X_2), \quad \operatorname{supp}(b') = \operatorname{supp}(b) \cap (X_2 \times X'_3).$$

For (1): Let $Z_{12} \subset X_1 \times X_2$, $Z_{23} \subset X_2 \times X_3$ be closed subsets such that

$$p_{13}|_{p_{12}^{-1}Z_{12} \cap p_{23}^{-1}Z_{23}} \quad \text{is proper.}$$

Set $Z'_{12} = Z_{12} \cap (X'_1 \times X_2)$ and $Z'_{23} = Z_{23} \cap (X_2 \times X'_3)$. Obviously,

$$p'^{-1}_{12} Z'_{12} \cap p'^{-1}_{23} Z'_{23} = (p^{-1}_{12} Z_{12} \cap p^{-1}_{23} Z_{23}) \cap (X'_1 \times X_2 \times X'_3).$$

Thus, if $p^{-1}_{12} Z_{12} \cap p^{-1}_{23} Z_{23}$ is proper over $X_1 \times X_3$, then $p'^{-1}_{12} Z'_{12} \cap p'^{-1}_{23} Z'_{23}$ is proper over $X'_1 \times X'_3$.

Statement (2) is a straightforward computation. For (3): By using the definition of $\circ$ in (1.3.2) it is straightforward to show that the diagram

$$\begin{array}{ccc} \mathrm{CH}(X_1 \times X_2, \mathrm{supp}(a)) \otimes \mathrm{CH}(X_2 \times X_3, \mathrm{supp}(b)) & \xrightarrow{\circ} & \mathrm{CH}(X_1 \times X_3, \mathrm{supp}(a, b)) \\ \downarrow & & \downarrow \\ \mathrm{CH}(X'_1 \times X_2, \mathrm{supp}(a')) \otimes \mathrm{CH}(X_2 \times X'_3, \mathrm{supp}(b')) & \xrightarrow{\circ} & \mathrm{CH}(X'_1 \times X'_3, \mathrm{supp}(a', b')) \end{array}$$

is commutative. □

**1.3.7.** For two smooth schemes $X$ and $Y$ and families $\Phi$ and $\Psi$ of supports of $X$ and $Y$, respectively, we define a family of supports $P(\Phi, \Psi)$ on the product by

$$P(\Phi, \Psi) := \{Z \subset X \times Y;\ Z \text{ is closed, } \mathrm{pr}_2|_Z \text{ is proper, } Z \cap \mathrm{pr}_1^{-1}(W) \in \mathrm{pr}_2^{-1}(\Psi) \text{ for every } W \in \Phi\}. \quad (1.3.8)$$

Let $X_i$ for $i = 1, 2, 3$ be smooth schemes and let $\Phi_i$ be a family of supports on $X_i$ for $i = 1, 2, 3$. It is easy to see that $\Phi_{ij} := P(\Phi_i, \Phi_j)$ satisfy the condition (1.3.1) and therefore

$$F(X_1 \times X_2, P(\Phi_1, \Phi_2)) \otimes F(X_2 \times X_3, P(\Phi_2, \Phi_3)) \to F(X_1 \times X_3, P(\Phi_1, \Phi_3)), \quad (1.3.9)$$

where $a \otimes b \mapsto b \circ a$, is well defined.

**Proposition 1.3.10.** (1) *Let $X_i$ for $i = 1, \ldots, 4$ be a smooth scheme and let $\Phi_i$ for $i = 1, \ldots, 4$ be a family of supports of $X_i$. We have*

$$a_{34} \circ (a_{23} \circ a_{12}) = (a_{34} \circ a_{23}) \circ a_{12} \quad \textit{for all } a_{ij} \in F(X_i \times X_j, P(\Phi_i, \Phi_j)).$$

(2) *For any $(X, \Phi)$, the diagonal immersion induces a morphism*

$$\imath : X \to (X \times X, P(\Phi, \Phi)) \quad \textit{in } V_*.$$

*We set $\Delta_{(X,\Phi)} := F_*(\imath)(1_X)$. The equality $\Delta_{(X,\Phi)} \circ g = g$ holds for all $(Y, \Psi)$ and $g \in F(Y \times X, P(\Psi, \Phi))$, and $g \circ \Delta_{(X,\Phi)} = g$ holds for all $(Y, \Psi)$ and $g \in F(X \times Y, P(\Phi, \Psi))$.*

*Proof.* The proof of the first statement is as in [Fulton 1998, Proposition 16.1.1] but one has to keep track of the supports, which is straightforward.

The second statement is an easy computation. □

**1.3.11.** *Grading.* For $(X, \Phi)$ and $(Y, \Psi)$, $F_*$ and $F^*$ give rise to two different gradings on $F(X \times Y, P(\Phi, \Psi))$. Unfortunately, neither are compatible with the $\circ$ from (1.3.9). We define a new grading by

$$F(X \times Y, P(\Phi, \Psi))^i = \bigoplus_{X'} F^{2\dim(X')+i}(X' \times Y, P(\Phi, \Psi)),$$

where $X'$ runs through the connected components of $X$. With this grading, $\circ$ becomes a morphism of graded abelian groups.

By the definition of the grading there are choices. We could also define a grading

$$F(X \times Y, P(\Phi, \Psi))_i = \bigoplus_{X'} F_{2\dim(X')+i}(X' \times Y, P(\Phi, \Psi)).$$

**Definition 1.3.12.** To an object $F = (F_*, F^*, T, e) \in \mathbf{T}$, we attach the graded additive symmetric monoidal category $\mathrm{Cor}_F$ with objects $\mathrm{ob}(\mathrm{Cor}_F) = \mathrm{ob}(V_*) = \mathrm{ob}(V^*)$ and morphisms

$$\mathrm{Hom}_{\mathrm{Cor}_F}((X, \Phi), (Y, \Psi)) = F(X \times Y, P(\Phi, \Psi))$$

with composition law $a \otimes b \mapsto b \circ a$ (1.3.9). The identity is $\Delta_{(X,\Phi)}$.

The product $\otimes$ on $\mathrm{Cor}_F$ is defined by

$$(X, \Phi) \otimes (Y, \Psi) := (X \times Y, \Phi \times \Psi),$$

and for two morphisms $f \in F(X \times X', P(\Phi, \Phi'))$ and $g \in F(Y \times Y', P(\Psi, \Psi'))$, we define

$$f \otimes g \in \mathrm{Hom}_{\mathrm{Cor}_F}((X, \Phi) \otimes (Y, \Psi), (X', \Phi') \otimes (Y', \Psi')),$$
$$f \otimes g := F_*(\mathrm{id}_X \times \mu_{X',Y} \times \mathrm{id}_{Y'})(T(f \otimes g)),$$

where $\mu_{X',Y}$ is the permutation of the factors $(X', \Phi')$ and $(Y, \Psi)$.

**1.3.13.** Given two objects $F, G \in \mathbf{T}$ and a morphism $\phi : F \to G$ in $\mathbf{T}$, we obtain a functor of graded additive symmetric monoidal categories

$$\mathrm{Cor}(\phi) : \mathrm{Cor}_F \to \mathrm{Cor}_G$$

that is given by

$$\phi : F(X \times Y, P(\Phi, \Psi)) \to G(X \times Y, P(\Phi, \Psi))$$

for all $(X, \Phi)$ and $(Y, \Psi)$. This provides a functor

$$\mathrm{Cor} : \mathbf{T} \to \mathbf{Cat}_{\mathbf{GrAb},\otimes}, \quad F \mapsto \mathrm{Cor}_F, \quad \phi \mapsto \mathrm{Cor}(\phi).$$

Here, $\mathbf{Cat}_{\mathbf{GrAb},\otimes}$ is the category of graded additive symmetric monoidal categories.

**1.3.14.** In order to state the properties of Cor, it is convenient to introduce the category $V$ with objects $\mathrm{ob}(V) = \mathrm{ob}(V_*) = \mathrm{ob}(V^*)$ and only morphisms the identity $\mathrm{id}_X$ (for every $X \in \mathrm{ob}(V)$). There are obvious functors $V \to V_*$, $V \to V^*$, and $V \to \mathrm{Cor}_F$ for all $F \in \mathbf{T}$. We define $\mathbf{Cat}_{V/\mathbf{GrAb},\otimes}$ to be the category with functors $V \to X$ as objects ($X \in \mathbf{Cat}_{\mathbf{GrAb},\otimes}$) and commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y, \\ & \nwarrow \quad \nearrow & \\ & V & \end{array} \tag{1.3.15}$$

with $f \in \mathrm{Hom}_{\mathbf{Cat}_{\mathbf{GrAb},\otimes}}(X, Y)$, as morphisms. In general a functor $f$ is said to be *under* $V$ if the diagram (1.3.15) is commutative.

**Proposition 1.3.16.** *The functor* $\mathrm{Cor} : \mathbf{T} \to \mathbf{Cat}_{V/\boldsymbol{GrAb},\otimes}$ *is fully faithful.*

*Proof.* Given $F, G \in \mathbf{T}$ and $\phi : F \to G$, we can recover $\phi : F(X) \to G(X)$ for $X \in \mathrm{ob}(V)$ from the map $\mathrm{Cor}(\phi)$:

$$\mathrm{Hom}_{\mathrm{Cor}_F}(\mathrm{Spec}(k), X) \to \mathrm{Hom}_{\mathrm{Cor}_G}(\mathrm{Spec}(k), X). \tag{1.3.17}$$

On the other hand, given $\psi : \mathrm{Cor}_F \to \mathrm{Cor}_G$ in $\mathbf{Cat}_{V/\mathbf{GrAb},\otimes}$, the map (1.3.17) defines a morphism $F \to G$ in $\mathbf{T}$. □

**1.3.18.** For all $F \in \mathbf{T}$, there is a functor

$$\rho_F : \mathrm{Cor}_F \to \mathbf{GrAb}$$

defined by

$$\rho_F(X, \Phi) = F(X, \Phi)$$
$$\rho_F(\gamma) = (a \mapsto F_*(\mathrm{pr}_2)(F^*(\mathrm{pr}_1)(a) \cup \gamma)) \quad \text{for } \gamma \in F(X \times Y, P(\Phi, \Psi)).$$

The map $\rho_F(\gamma) : F(X, \Phi) \to F(Y, \Psi)$ is well defined since $\mathrm{pr}_2$ restricted to $\mathrm{pr}_1^{-1}(\Phi) \cap P(\Phi, \Psi)$ is proper and $\mathrm{pr}_1^{-1}(\Phi) \cap P(\Phi, \Psi) \subset \mathrm{pr}_2^{-1}(\Psi)$ by definition of $P(\Phi, \Psi)$. Functoriality is again a straightforward computation.

Moreover, there are functors

$$\tau_*^F : V_* \to \mathrm{Cor}_F \quad \text{and} \quad \tau_F^* : (V^*)^{\mathrm{op}} \to \mathrm{Cor}_F,$$

(under $V$) such that

$$\rho_F \circ \tau_*^F = F_* \quad \text{and} \quad \rho_F \circ \tau_F^* = F^*.$$

The functor $\tau_*^F : V_* \to \mathrm{Cor}_F$ is defined by mapping a morphism $f : (X, \Phi) \to (Y, \Psi)$ to $F_*(\mathrm{id}, f)(1_X)$, where $(\mathrm{id}, f) : X \to (X \times Y, P(\Phi, \Psi))$ is in $V_*$. Similarly, the functor $\tau_F^* : V^* \to \mathrm{Cor}_F$ is defined by mapping a morphism $f : (X, \Phi) \to (Y, \Psi)$

to $F_*(f, \mathrm{id})(1_X)$ with $(f, \mathrm{id}) : X \to (Y \times X, P(\Psi, \Phi))$ in $V_*$. Then the equalities $\rho_F \circ \tau_*^F = F_*$ and $\rho_F \circ \tau_F^* = F^*$ follow easily from the projection formula.

**Lemma 1.3.19.** *If $\phi : F \to G$ is a morphism in* $\mathbf{T}$, *then*

$$\mathrm{Cor}(\phi) \circ \tau_*^F = \tau_*^G \quad \textit{and} \quad \mathrm{Cor}(\phi) \circ \tau_F^* = \tau_G^*.$$

*Proof.* For the first equality, let $f : (X, \Phi) \to (Y, \Psi)$ be a morphism in $V_*$. We get

$$\begin{aligned}\mathrm{Cor}(\phi)(\tau_*^F(f)) &= \mathrm{Cor}(\phi)(F_*(\mathrm{id}, f)(1_X)) = \phi(F_*(\mathrm{id}, f)(1_X)) \\ &= G_*(\mathrm{id}, f)(\phi(1_X)) = G_*(\mathrm{id}, f)(1_X) = \tau_*^G(f).\end{aligned}$$

The second equality is proved in the same way. □

## 2. Hodge cohomology with support

For a smooth scheme $X$ and a family of supports $\Phi$ of $X$, we define

$$H(X, \Phi) := \bigoplus_{i,j} H^i_\Phi(X, \Omega^j_X),$$

and call this $k$-vector space the Hodge cohomology of $X$ with support in $\Phi$. We denote by $H^*(X, \Phi)$ the graded abelian group, which in degree $n$ equals

$$H^n(X, \Phi) = \bigoplus_{i+j=n} H^i_\Phi(X, \Omega^j_X). \tag{2.0.1}$$

We denote by $H_*(X, \Phi)$ the graded abelian group, which in degree $n$ equals

$$H_n(X, \Phi) = \bigoplus_r H^{2\dim X_r - n}(X_r, \Phi), \tag{2.0.2}$$

where $X = \coprod_r X_r$ is the decomposition into connected components. We define

$$e : \mathbb{Z} \to H(\mathrm{Spec}\, k) = k \tag{2.0.3}$$

to be the natural map sending 1 to 1.

The goal of this section is to provide the object functions $H_*$ and $H^*$ with the structure of functors

$$H_* : V_* \to \mathbf{GrAb} \quad \text{and} \quad H^* : (V^*)^{\mathrm{op}} \to \mathbf{GrAb}$$

and to define for each $(X, \Phi), (Y, \Psi) \in \mathrm{ob}(V_*) = \mathrm{ob}(V^*)$ a morphism

$$T_{(X,\Phi),(Y,\Psi)} : H(X, \Phi) \otimes H(Y, \Psi) \to H(X \times Y, \Phi \times \Psi)$$

of graded abelian groups (for both gradings) such that $(H_*, H^*, T, e)$ is an object in $\mathbf{T}$, that is, it is a datum as in 1.1.8 and satisfies the properties 1.1.9.

### 2.1. *Pullback.*

**2.1.1.** We work in the bounded derived category of quasicoherent sheaves $D^b(X)$ on a scheme $X$. (The bounded derived category of coherent sheaves will be denoted by $D^b_c(X)$.) Let $f : X \to Y$ be a morphism of schemes; let $\Phi$ and $\Psi$ be families of supports of $X$ and $Y$, respectively. There is an isomorphism of functors

$$R\Gamma_{f^{-1}(\Psi)} \xrightarrow{\cong} R\Gamma_{\Psi} Rf_*. \tag{2.1.2}$$

If $\Psi \subset \Psi'$ for another family of supports $\Psi'$, then the diagram

$$\begin{array}{ccc} R\Gamma_{f^{-1}(\Psi')} & \xrightarrow{(2.1.2)} & R\Gamma_{\Psi'} Rf_* \\ \uparrow & & \uparrow \\ R\Gamma_{f^{-1}(\Psi)} & \xrightarrow{(2.1.2)} & R\Gamma_{\Psi} Rf_* \end{array} \tag{2.1.3}$$

is commutative. Moreover, if $g : Z \to X$ is another morphism of schemes then the following diagram is commutative:

$$\begin{array}{ccc} R\Gamma_{(f\circ g)^{-1}(\Psi)} & \xrightarrow{(2.1.2) \text{ for } f\circ g} & R\Gamma_{\Psi} R(f\circ g)_* \\ \Big\downarrow{\scriptstyle (2.1.2) \text{ for } g} & \nearrow{\scriptstyle (2.1.2) \text{ for } f} & \\ R\Gamma_{f^{-1}(\Psi)} Rg_*. & & \end{array} \tag{2.1.4}$$

**2.1.5.** For a morphism $f : X \to Y$ of schemes, we have

$$\mathrm{id} \to Rf_* Lf^*,$$

and thus we obtain a morphism of functors

$$R\Gamma_{\Psi} \to R\Gamma_{f^{-1}(\Psi)} Lf^*; \tag{2.1.6}$$

it easily follows from (2.1.3) that the diagram

$$\begin{array}{ccc} R\Gamma_{\Psi'} & \xrightarrow{(2.1.6)} & R\Gamma_{f^{-1}(\Psi')} Lf^* \\ \uparrow & & \uparrow \\ R\Gamma_{\Psi} & \xrightarrow{(2.1.6)} & R\Gamma_{f^{-1}(\Psi)} Lf^* \end{array} \tag{2.1.7}$$

commutes for $\Psi \subset \Psi'$. From (2.1.7) and (2.1.4) it follows that for another morphism $g : Z \to X$ of schemes the following diagram is commutative:

$$\begin{array}{ccc} R\Gamma_{\Psi} & \xrightarrow{(2.1.6) \text{ for } f} & R\Gamma_{f^{-1}(\Psi)} Lf^* \\ \Big\downarrow{\scriptstyle (2.1.6) \text{ for } f\circ g} & \swarrow{\scriptstyle (2.1.6) \text{ for } g} & \\ R\Gamma_{(f\circ g)^{-1}(\Psi)} L(f\circ g)^*. & & \end{array} \tag{2.1.8}$$

For a morphism $f : (X, \Phi) \to (Y, \Psi)$ in $V^*$ (that is, $f^{-1}(\Psi) \subset \Phi$), the morphism

$$R\Gamma_\Psi \Omega^d_Y \to R\Gamma_{f^{-1}(\Psi)} Lf^* \Omega^d_Y = R\Gamma_{f^{-1}(\Psi)} f^* \Omega^d_Y \to R\Gamma_{f^{-1}(\Psi)} \Omega^d_X \to R\Gamma_\Phi \Omega^d_X$$

(for $d \geq 0$) gives a morphism

$$H^*(f) : H(Y, \Psi) \to H(X, \Phi). \tag{2.1.9}$$

By a straightforward computation, $f \mapsto H^*(f)$ defines a functor $(V^*)^{\mathrm{op}} \to \mathbf{GrAb}$.

**2.2.** ***Push-forward in the derived category.*** We recall the following notations from duality theory [Hartshorne 1966; Conrad 2000]: Let $X$ be a separated $k$-scheme of finite type with structure map $\pi : X \to \operatorname{Spec} k$. We have $\pi^! k \in D^b_c(X)$. (In fact if $X$ has dimension $d$, then $\pi^! k$ has nonzero cohomology only in the interval $[-d, 0]$. This follows from [Hartshorne 1966, Chapter V, Proposition 7.3 and its proof] and [Conrad 2000, (3.1.25)].) We denote

$$D_X := R\,\mathcal{H}om_X(\,\cdot\,, \pi^! k) : D^b_c(X) \to D^b_c(X). \tag{2.2.1}$$

If $f : X \to Y$ is a *proper* morphism between $k$-schemes, we have the trace map

$$\mathrm{Tr}_f : Rf_* f^! \to \mathrm{id}, \tag{2.2.2}$$

which is a natural transformation of functors on $D^+_c(Y)$. For maps $f : X \to Y$ and $g : Y \to Z$, we have the canonical isomorphisms

$$c_{f,g} : (gf)^! \xrightarrow{\simeq} f^! g^! \quad \text{in } D^+_c(X). \tag{2.2.3}$$

**Notation 2.2.4.** Given a bounded complex $C$ in $D(X)$ and a morphism of complexes $\varphi : A \to B$ in $D(X)$, we will often denote the morphism $R\,\mathcal{H}om_X(C, \varphi) : R\,\mathcal{H}om_X(C, A) \to R\,\mathcal{H}om_X(C, B)$ simply by $\varphi$ and the morphism $R\,\mathcal{H}om_X(\varphi, C) : R\,\mathcal{H}om_X(B, C) \to R\,\mathcal{H}om_X(A, C)$ by $\varphi^\vee$. It will always be clear from the context what $C$ is in the particular situation.

**Definition 2.2.5.** Let $f : X \to Y$ be a proper $k$-morphism. Let $\pi_X$ and $\pi_Y$ denote the structure maps of $X$ and $Y$ respectively. Then we define

$$f_* : Rf_* D_X(\Omega^q_X) \to D_Y(\Omega^q_Y) \quad \text{for } q \geq 0,$$

to be the composition

$$\begin{aligned} Rf_* R\,\mathcal{H}om_X(\Omega^q_X, \pi^!_X k) &\xrightarrow{c_{f,\pi_Y}} Rf_* R\,\mathcal{H}om_X(\Omega^q_X, f^! \pi^!_Y k) \\ &\xrightarrow{\text{nat.}} R\,\mathcal{H}om_Y(Rf_* \Omega^q_X, Rf_* f^! \pi^!_Y k) \xrightarrow{\mathrm{Tr}_f} R\,\mathcal{H}om_Y(Rf_* \Omega^q_X, \pi^!_Y k) \\ &\xrightarrow{(f^*)^\vee} R\,\mathcal{H}om_Y(\Omega^q_Y, \pi^!_Y k). \end{aligned}$$

**Remark 2.2.6.** (1) Notice that the composition of the middle two arrows in the composition above is just the standard Grothendieck duality isomorphism (see for example [Conrad 2000, (3.4.10)])

$$Rf_* R\,\mathcal{H}\mathrm{om}_X(\,\cdot\,, f^!(\,\cdot\,)) \xrightarrow{\simeq} R\,\mathcal{H}\mathrm{om}_Y(Rf_*(\,\cdot\,), \,\cdot\,).$$

(2) It is straightforward to check that the push-forward above also equals the composition

$$Rf_* D_X(\Omega_X^q) \xrightarrow{({}^a f^*)^\vee} Rf_* D_X(Lf^*\Omega_Y^q) \xrightarrow{c_{f,\pi_Y}} Rf_* R\,\mathcal{H}\mathrm{om}_X(Lf^*\Omega_Y^q, f^!\pi_Y^! k)$$
$$\xrightarrow{\text{adj.}} R\,\mathcal{H}\mathrm{om}_Y(\Omega_Y^q, Rf_* f^!\pi_Y^! k) \xrightarrow{\mathrm{Tr}_f} D_Y(\Omega_Y^q).$$

Here adj. denotes the isomorphism

$$Rf_* R\,\mathcal{H}\mathrm{om}_X(Lf^*(\,\cdot\,), \,\cdot\,) \cong R\,\mathcal{H}\mathrm{om}_Y(\,\cdot\,, Rf_*(\,\cdot\,)) \quad \text{on } D_{\mathrm{c}}^-(Y) \times D^+(X)$$

(see [Hartshorne 1966, Chapter II, Proposition 5.10]) and ${}^a f^* : Lf^*\Omega_Y^q \to \Omega_X^q$ is the morphism corresponding to $\Omega_Y^q \to Rf_*\Omega_X^q$ under $H^0(Y, \,\cdot\,)$ applied to the isomorphism above.

**Proposition 2.2.7.** (1) $\mathrm{id}_* = \mathrm{id}$.

(2) *Let $f : X \to Y$ and $g : Y \to Z$ be two proper maps with $X$ and $Y$ of pure dimension $d_X$ and $d_Y$, respectively. Then*

$$(g \circ f)_* = g_* \circ Rg_*(f_*) : Rg_* Rf_* D_X(\Omega_X^q) \to D_Z(\Omega_Z^q).$$

(3) *Let*

$$\begin{array}{ccc} X' & \xrightarrow{u'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{u} & Y \end{array}$$

*be a cartesian diagram with $f$ proper, $u$ étale and $X$ of pure dimension $d$. Then the diagram*

$$\begin{array}{ccc} u^* Rf_* D_X(\Omega_X^q) & \xrightarrow{u^*(f_*)} & u^* D_Y(\Omega_Y^q) \\ \simeq\downarrow & & \downarrow\simeq \\ Rf'_* D_X(\Omega_{X'}^q) & \xrightarrow{f'_*} & D_{Y'}(\Omega_{Y'}^q) \end{array} \tag{2.2.8}$$

*commutes, where the vertical maps are the natural isomorphisms (in the proof we will make these isomorphisms precise).*

*Proof.* (1) is clear. By [Conrad 2000, Lemma 3.4.3, (TRA1) and p. 139, (VAR1)], we have

$$\mathrm{Tr}_{g\circ f} = \mathrm{Tr}_g \circ Rg_*(\mathrm{Tr}_f)\circ R(g\circ f)_*(c_{f,g}) : Rg_*Rf_*(g\circ f)^! \xrightarrow{\simeq} \mathrm{id}\,. \tag{2.2.9}$$

and

$$c_{f,g}\circ c_{g\circ f,h} = f^!(c_{g,h})\circ c_{f,h\circ g} : (h\circ g\circ f)^! \to f^!g^!h^!, \tag{2.2.10}$$

where $h : Z \to W$ is a third map. This implies (2).

Now to make the vertical maps in (3) precise we need some further notation: Let

$$\alpha : u^*Rf_* \xrightarrow{\simeq} Rf_*u'^*, \qquad e_u : u^* \xrightarrow{\simeq} u^!,$$

$$\beta_u : u^*R\,\mathscr{H}\mathrm{om}(\,\cdot\,,\,\cdot\,) \xrightarrow{\simeq} R\,\mathscr{H}\mathrm{om}(u^*(\,\cdot\,), u^*(\,\cdot\,))$$

be the natural isomorphisms. Then the vertical map on the left of (2.2.8) is given by $c^{-1}_{u',\pi_X}\circ e_{u'}\circ\beta_{u'}\circ\alpha$ and the vertical map on the right of (2.2.8) is given by $c^{-1}_{u,\pi_Y}\circ e_u\circ\beta_u$. Thus we have to prove

$$c^{-1}_{u,\pi_Y}\circ e_u\circ\beta_u\circ u^*(f_*) = f'_*\circ c^{-1}_{u',\pi_X}\circ e_{u'}\circ\beta_{u'}\circ\alpha. \tag{2.2.11}$$

Denote by $b_{u,f} : u'^*f^! \xrightarrow{\simeq} f'^!u^*$ the isomorphism of [Hartshorne 1966, Chapter VII, Corollary 3.4(a)(5)]; see also [Conrad 2000, (3.3.24)]. Then it is easy (but tedious) to check that (2.2.11) follows from

$$u^*(\mathrm{Tr}_f) = \mathrm{Tr}_{f'}\circ Rf'_*(b_{u,f})\circ\alpha : u^*Rf_*f^! \to Rf'_*f'^!u^*$$

(see [Conrad 2000, Lemma 3.4.3, (TRA4)]) and the following lemma. □

**Lemma 2.2.12.** *Let*

$$\begin{array}{ccc} X' & \xrightarrow{u'} & X \\ {\scriptstyle f'}\downarrow & \searrow^{h} & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{u} & Y \end{array}$$

*be a cartesian diagram with u étale. Then the following diagram commutes (with notation as above):*

$$\begin{array}{ccccc} u'^*f^! & & \xrightarrow{\quad b_{u,f}\quad} & & f'^!u^* \\ {\scriptstyle e_{u'}}\downarrow & & & & \downarrow{\scriptstyle f'^!(e_u)} \\ u'^!f^! & \xleftarrow{c_{u',f}} & h^! & \xrightarrow{c_{f',u}} & f'^!u^!. \end{array}$$

*Proof.* We extensively use the notation of [Hartshorne 1966; Conrad 2000]. All maps and functors involved in the statement are defined, for example, in [Conrad 2000, (3.3.6), (3.3.15), (3.3.21), (3.3.25)]. Using these definitions for the residual complex $K = \pi_Y^{\Delta} k$ on $Y$ together with the identity $u^* K = u^{\Delta} K$ and the commutativity of

$$\begin{array}{ccc} u^* D_K D_{u^*K} D_{u^*K} & \xrightarrow{\eta} & u^* D_K \\ \downarrow{\scriptstyle \beta_u} & & \downarrow{\scriptstyle \beta_u} \\ D_{u^*K} u^* D_K D_K & \xrightarrow{\eta} & D_{u^*K} u^*, \end{array}$$

one checks that one is reduced to proving the commutativity of the diagram

$$\begin{array}{ccccc} u'^* f^{\Delta} K & & \xleftarrow{d_{u,f}} & & f'^{\Delta} u^* K \\ \downarrow{\scriptstyle \varphi_{u'}^{-1}} & & & & \downarrow{\scriptstyle f'^{!}(\varphi_u^{-1})} \\ u'^{\Delta} f^{\Delta} K & \xleftarrow{c_{u',f}} & h^{\Delta} K & \xrightarrow{c_{f',u}} & f'^{\Delta} u^{\Delta} K. \end{array} \tag{2.2.13}$$

Here the maps are the analogs in the category of residual complexes of the maps in the lemma; see [Hartshorne 1966, Chapter IV, Theorems 3.1 and 5.5]. Since we work with actual complexes now, the commutativity of the diagram above is a local question. Thus take $U \subset X$ open such that $f|_U$ factors as $U \xrightarrow{i} P \xrightarrow{p} Y$ with $i$ a closed immersion and $p$ smooth. Then $f'|_{U'}$ also factors as $U' \xrightarrow{i'} P' \xrightarrow{p'} Y'$. By the construction of $f^{\Delta}$ in the proof of [Hartshorne 1966, Chapter VI, Theorem 3.1(a)] we have $f^{\Delta} = i^y p^z$ and also $f'^{\Delta} = i'^y p'^z$. Now by [Hartshorne 1966, Chapter VI, Theorem 5.5(2)], we have

$$c_{p,i} \circ d_{u,f} = d_{u_P,i} \circ i'^{\Delta}(d_{u,p}) \circ c_{p',i'},$$

with $u_P$ being the base change of $u$ by $p : P \to Y$. This equality implies that it is sufficient to prove the commutativity of (2.2.13) for $i$ and $p$ separately. Thus we are reduced to consider the two cases $f$ is finite or $f$ is smooth. The latter case smooth is immediate, while the first follows from [Conrad 2000, Theorems 3.3.1, 2.(VAR4)]. □

**Remark 2.2.14.** (1) Let $\pi : X \to \operatorname{Spec} k$ be smooth of pure dimension $d$. Then there is a canonical isomorphism $\pi^! k \cong \Omega_X^d[d] =: \omega_X[d]$. More generally, for any $j \geq 0$ and $n \in \mathbb{Z}$ we have the isomorphism

$$\Omega_X^j[n] \xrightarrow{\simeq} D_X(\Omega_X^{d-j})[n-d], \tag{2.2.15}$$

which is defined to be the composition of

$$\Omega_X^j[n] \xrightarrow{\simeq} \mathcal{H}om_X(\Omega_X^{d-j}, \Omega_X^d)[n], \quad \alpha \mapsto (\beta \mapsto \alpha \wedge \beta)$$

(notice that we make the choice of a sign here) with

$$\mathcal{H}om_X(\Omega_X^{d-j}, \Omega_X^d)[n] = \mathcal{H}om^{\bullet}(\Omega_X^{d-j}, \Omega_X^d[d])[n-d] \cong R\,\mathcal{H}om(\Omega_X^{d-j}, \pi^! k)[n-d].$$

(2) Let $X$ be a $k$-scheme of pure dimension $d$ and $U \subset X$ a smooth open subscheme; then we have for all $j \geq 0$

$$\Omega_U^j \simeq \mathcal{H}om_U^\bullet(\Omega_U^{d-j}, \Omega_U^d[d])[-d] \simeq D_X(\Omega_X^{d-j})|_U[-d],$$

where the first isomorphism is as in (1) and the second is given by restriction (or to be more precise, first use the isomorphism $\Omega_U^d[d] \simeq \pi_U^! k$ and then the vertical isomorphism on the right in (2.2.8) with $U \hookrightarrow X$ instead of $u : Y' \to Y$).

**Lemma 2.2.16.** *Let $\pi_X : X \to \operatorname{Spec} k$ be proper of pure dimension $d_X$ and let $\pi_Y : Y \to \operatorname{Spec} k$ be smooth of pure dimension $d_Y$. We denote by $\mathrm{pr}_2 : X \times Y \to Y$ the projection (it is proper) and set $d := \dim(X \times Y)$. Then for all $j \geq 0$, there is a morphism in $D_c^+(X \times Y)$*

$$\gamma : \mathrm{pr}_2^!(\mathcal{O}_Y) \otimes (\mathrm{pr}_2^* \Omega_Y^{j-d_X}[d_Y]) \to D_{X\times Y}(\mathrm{pr}_2^* \Omega_Y^{d-j})$$

*satisfying the following conditions:*

(1) *For $U \subset X$ open and smooth denote by $p_2 : U \times X \to Y$ the restriction of $\mathrm{pr}_2$. Then $\gamma|_{U\times Y}$ is the composition*

$$(\mathrm{pr}_2^!(\mathcal{O}_Y) \otimes \mathrm{pr}_2^* \Omega_Y^{j-d_X}[d_Y])|_{U\times Y} \xrightarrow{\simeq} \Omega^{d_X}_{U\times Y/Y}[d_X] \otimes p_2^*\Omega_Y^{j-d_X}[d_Y]$$
$$\xrightarrow{\simeq} \Omega^{d_X}_{U\times Y/Y}[d_X] \otimes p_2^* R\,\mathcal{H}om_Y(\Omega_Y^{d-j}, \Omega_Y^{d_Y}[d_Y]) \xrightarrow{\simeq\ nat.} D_{U\times Y}(p_2^*\Omega_Y^{d-j}).$$

*Here the last isomorphism is induced by the composition of the canonical isomorphisms*

$$\Omega^{d_X}_{U\times Y/Y}[d_X] \otimes p_2^*\Omega_Y^{d_Y}[d_Y] \cong \Omega^d_{U\times Y}[d] \cong \pi^!_{U\times Y}(k).$$

(2) *The following diagram commutes:*

$$\begin{array}{ccc}
R\,\mathrm{pr}_{2*}(\mathrm{pr}_2^!(\mathcal{O}_Y) \otimes \mathrm{pr}_2^* \Omega_Y^{j-d_X}[d_Y]) & \xrightarrow{\gamma} & R\,\mathrm{pr}_{2*}\, D_{X\times Y}(\mathrm{pr}_2^* \Omega_Y^{d-j}) \\
\downarrow{\scriptstyle \textit{proj. formula}} & & \downarrow \\
R\,\mathrm{pr}_{2*}(\mathrm{pr}_2^!(\mathcal{O}_Y)) \otimes \Omega_Y^{j-d_X}[d_Y] & & R\,\mathcal{H}om_Y(\Omega_Y^{d-j}, \pi_Y^! k) \\
& \searrow{\scriptstyle \mathrm{Tr}_{\mathrm{pr}_2} \otimes \mathrm{id}} \quad \nearrow{\scriptstyle \simeq\ (2.2.15)} & \\
& \Omega_Y^{j-d_X}[d_Y], &
\end{array}$$

*where the vertical map on the right is $\mathrm{Tr}_{\mathrm{pr}_2} \circ \mathrm{adjunction} \circ c_{\mathrm{pr}_2,\pi_Y}$.*

*Proof.* Conrad [2000, (4.3.12)] defines a map

$$e_{\mathrm{pr}_2} : \mathrm{pr}_2^!(\mathcal{O}_Y) \otimes^L \mathrm{pr}_2^* \pi_Y^! k \longrightarrow \mathrm{pr}_2^! \pi_Y^! k$$

such that

$$\begin{array}{ccc} (\mathrm{pr}_2^!(\mathcal{O}_Y)\otimes^L \mathrm{pr}_2^*\,\pi_Y^!k)|_{U\times Y} & \xrightarrow{e_{\mathrm{pr}_2}|_{U\times Y}} & (\mathrm{pr}_2^!\,\pi_Y^!k)|_{U\times Y} \\ \uparrow\simeq & & \simeq\uparrow \\ \Omega^{d_X}_{U\times Y/Y}[d_X]\otimes p_2^*\Omega_Y^{d_Y}[d_Y] & \xrightarrow{\simeq} & \Omega^d_{U\times Y}[d] \end{array} \tag{2.2.17}$$

commutes, where the vertical map on the left is the composition of the canonical isomorphism $\Omega^d_{U\times Y}[d]\cong \pi^!_{U\times Y}(k)$ with $c_{p_2,\pi_Y}:\pi^!_{U\times Y}\cong p_2^!\pi_Y^!$. Furthermore by [Conrad 2000, Theorem 4.4.1] the following diagram commutes:

$$\begin{array}{ccc} R\,\mathrm{pr}_{2*}(\mathrm{pr}_2^!\,\mathcal{O}_Y\otimes^L \mathrm{pr}_2^*\,\pi_Y^!k) & \xrightarrow{e_{\mathrm{pr}_2}} & R\,\mathrm{pr}_{2*}\,\mathrm{pr}_2^!(\pi_Y^!k) \\ {\scriptstyle\text{proj. formula}}\downarrow & & \downarrow{\scriptstyle\mathrm{Tr}_{\mathrm{pr}_2}} \\ R\,\mathrm{pr}_{2*}(\mathrm{pr}_2^!\,\mathcal{O}_Y)\otimes^L \pi_Y^!k & \xrightarrow[\mathrm{Tr}_{\mathrm{pr}_2}\otimes\mathrm{id}]{} & \pi_Y^!k. \end{array} \tag{2.2.18}$$

We define $\gamma$ to be the composition

$$\begin{aligned} \mathrm{pr}_2^!(\mathcal{O}_Y)\otimes \mathrm{pr}_2^*\,\Omega_Y^{j-d_X}[d_Y] &\xrightarrow{\mathrm{id}\otimes(2.2.15)} \mathrm{pr}_2^!(\mathcal{O}_Y)\otimes \mathrm{pr}_2^*\,R\,\mathcal{H}om_Y(\Omega_Y^{d-j},\pi_Y^!k) \\ &\xrightarrow{\text{nat.}} R\,\mathcal{H}om(\mathrm{pr}_2^*\,\Omega_Y^{d-j},\mathrm{pr}_2^!(\mathcal{O}_Y)\otimes^L \mathrm{pr}_2^*\,\pi_Y^!k) \\ &\xrightarrow{c^{-1}_{\mathrm{pr}_2,\pi_Y}\circ e_{\mathrm{pr}_2}} D_{X\times Y}(\mathrm{pr}_2^*\,\Omega_Y^{d-j}). \end{aligned}$$

It follows from (2.2.17) and (2.2.18) that $\gamma$ satisfies (1) and (2). □

**Proposition 2.2.19.** *Let $i:X\hookrightarrow Y$ be a closed immersion of pure codimension $c$ between smooth $k$-schemes of pure dimension $d_X$ and $d_Y$, respectively. Then*

$$R\underline{\Gamma}_X\Omega_Y^q[c]\cong \mathcal{H}^c_X(\Omega_Y^q)\quad \text{in } D^b_{\mathrm{qc}}(\mathcal{O}_Y)\quad \text{for all } q\geq 0.$$

*Suppose further the ideal sheaf of $X$ in $\mathcal{O}_Y$ is generated by a sequence $t=t_1,\dots,t_c$ of global sections of $\mathcal{O}_Y$. Define a morphism $\imath_X^q$ by*

$$\imath_X^q: i_*\Omega_X^q\to \mathcal{H}^c_X(\Omega_Y^{c+q}),\quad \alpha\mapsto (-1)^c\begin{bmatrix} dt\tilde{\alpha} \\ t\end{bmatrix},$$

*where $\tilde{\alpha} \in \Omega^q_Y$ is any lift of $\alpha$ and $dt = dt_1 \wedge \cdots \wedge dt_c$. (Here we use the notation of Section A.1.) Then the following diagram commutes in $D^b_{\mathrm{qc}}(\mathcal{O}_Y)$:*

$$\begin{array}{ccc} i_* D_X(\Omega^{d_X-q}_X)[-d_X] & \xrightarrow{i_*} & D_Y(\Omega^{d_X-q}_Y)[-d_X] \\ \uparrow{\scriptstyle (2.2.15)} & & \downarrow{\scriptstyle (2.2.15)} \\ i_*\Omega^q_X & & \Omega^{c+q}_Y[c] \\ {\scriptstyle \iota^q_X}\downarrow & & \uparrow \\ \mathcal{H}^c_X(\Omega^{c+q}_Y) & \xrightarrow{\simeq} & R\underline{\Gamma}_X(\Omega^{c+q}_Y)[c]. \end{array} \tag{2.2.20}$$

*Proof.* The first statement is well known; see also Lemma A.2.5. It remains to prove the commutativity of (2.2.20). Let $\pi_X : X \to \operatorname{Spec} k$ and $\pi_Y : Y \to \operatorname{Spec} k$ be the structure maps. By Definition 2.2.5, the top row in (2.2.20) is given by the following composition in $D^b_{\mathrm{qc}}(\mathcal{O}_Y)$:

$$\begin{aligned} i_*\Omega^q_X &\xrightarrow{(2.2.15)} i_* R\,\mathcal{H}om(\Omega^{d_X-q}_X, \pi^!_X k)[-d_X] \\ &\xrightarrow{c_{i,\pi_Y}} i_* R\,\mathcal{H}om(\Omega^{d_X-q}_X, i^!\pi^!_Y k)[-d_X] \\ &\xrightarrow{\text{nat.}} R\,\mathcal{H}om(i_*\Omega^{d_X-q}_X, i_*i^!\pi^!_Y k)[-d_X] \\ &\xrightarrow{\mathrm{Tr}_i} R\,\mathcal{H}om(i_*\Omega^{d_X-q}_X, \pi^!_Y k)[-d_X] \\ &\xrightarrow{(i^*)^\vee} R\,\mathcal{H}om(\Omega^{d_X-q}_Y, \pi^!_Y k)[-d_X] \xrightarrow{(2.2.15)^{-1}} \Omega^{c+q}_Y[c]. \end{aligned}$$

We set $\iota_X := \iota^{d_X}_X$. Then it follows from Lemma A.2.12 and the definition of (2.2.15) that the composition above equals

$$\begin{aligned} i_*\Omega^q_X &\xrightarrow{\text{multipl.}} i_*\,\mathcal{H}om(\Omega^{d_X-q}_X, \Omega^{d_X}_X) \\ &\xrightarrow{\text{nat.}} \mathcal{H}om(i_*\Omega^{d_X-q}_X, i_*\Omega^{d_X}_X) \\ &\xrightarrow{\iota_X} \mathcal{H}om(i_*\Omega^{d_X-q}_X, \mathcal{H}^c_X(\Omega^{d_Y}_Y)) \\ &\xrightarrow{(i^*)^\vee} \mathcal{H}om(\Omega^{d_X-q}_Y, \mathcal{H}^c_X(\Omega^{d_Y}_Y)) \\ &\xrightarrow{(*)} \mathcal{H}om^\bullet(\Omega^{d_X-q}_Y, \Omega^{d_Y}_Y[c]) \xrightarrow{\text{multipl.}^{-1}} \Omega^{c+q}_Y[c], \end{aligned} \tag{2.2.21}$$

where $(*)$ is induced by $\mathcal{H}^c_X(\Omega^{d_Y}_Y) \cong R\underline{\Gamma}_X(\Omega^{d_Y}_Y)[c] \to \Omega^{d_Y}_Y[c]$. There is a natural isomorphism

$$\varphi : \mathcal{H}^c_X(\Omega^{c+q}_Y) \xrightarrow{\simeq} \mathcal{H}om(\Omega^{d_X-q}_Y, \mathcal{H}^c_X(\Omega^{d_Y}_Y))$$

coming from the isomorphisms

$$\begin{aligned}\mathcal{H}om(\Omega_Y^{d_X-q}, \mathcal{H}_X^c(\Omega_Y^{d_Y})) &\cong R\,\mathcal{H}om(\Omega_Y^{d_X-q}, R\underline{\Gamma}_X(\Omega_Y^{d_Y})[c]) \\ &\cong R\underline{\Gamma}_X(R\,\mathcal{H}om(\Omega_Y^{d_X-q}, \Omega_Y^{d_Y}))[c] \cong R\underline{\Gamma}_X(\Omega_Y^{c+q})[c] \\ &\cong \mathcal{H}_X^c(\Omega_Y^{c+q}).\end{aligned}$$

This isomorphism is explicitly given by

$$\varphi : \mathcal{H}_X^c(\Omega_Y^{c+q}) \xrightarrow{\simeq} \mathcal{H}om(\Omega_Y^{d_X-q}, \mathcal{H}_X^c(\Omega_Y^{d_Y})), \qquad \begin{bmatrix}\alpha \\ t^n\end{bmatrix} \mapsto \left(\beta \mapsto \begin{bmatrix}\alpha\beta \\ t^n\end{bmatrix}\right).$$

The composition (2.2.21) equals

$$i_*\Omega_X^q \xrightarrow{\varphi^{-1}\circ(i^*)^\vee\circ\iota_X\circ(\text{nat.})\circ(\text{multipl.})} \mathcal{H}_X^c(\Omega_Y^{c+q}) \cong R\underline{\Gamma}_X(\Omega_Y^{c+q})[c] \to \Omega_Y^{c+q}[c].$$

It is straightforward to check that $\iota_X^q = \varphi^{-1}\circ(i^*)^\vee\circ\iota_X\circ(\text{nat.})\circ(\text{multipl.})$, and this implies the commutativity of (2.2.20). □

**Corollary 2.2.22.** *Assume we have a cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{i'} & Y' \\ \downarrow{\scriptstyle g_X} & & \downarrow{\scriptstyle g_Y} \\ X & \xrightarrow{i} & Y, \end{array}$$

*in which $X$, $X'$, $Y$, $Y'$ are smooth of pure dimension $d_X$, $d_{X'}$, $d_Y$, $d_{Y'}$, $i$ is a closed immersion, and $c := d_Y - d_X = d_{Y'} - d_{X'}$. Then for all $q \geq 0$ the following diagram commutes in $D^b_{\mathrm{qc}}(Y)$:*

$$\begin{array}{ccc} i_*Rg_{X*}\Omega_{X'}^q = Rg_{Y*}i'_*\Omega_{X'}^q & \longrightarrow & Rg_{Y*}\Omega_{Y'}^{c+q}[c] \\ \uparrow{\scriptstyle g_X^*} & & \uparrow{\scriptstyle g_Y^*} \\ i_*\Omega_X^q & \longrightarrow & \Omega_Y^{c+q}[c], \end{array}$$

*where the lower horizontal morphism is given by the composition*

$$i_*\Omega_X^q \xrightarrow{(2.2.15)} i_*D_X(\Omega_X^{d_X-q})[-d_X] \xrightarrow{i_*} D_Y(\Omega_Y^{d_X-q})[-d_X] \xrightarrow{(2.2.15)} \Omega_Y^{c+q}[c]$$

*and the upper horizontal morphism by $Rg_{Y*}$ applied to the analogous map for $i'$.*

*Proof.* Since $R\underline{\Gamma}_X Rg_{Y*} = Rg_{Y*}R\underline{\Gamma}_{X'}$, we naturally have a commutative diagram

$$\begin{array}{ccccc} Rg_{Y*}\mathcal{H}_{X'}^c(\Omega_{Y'}^{c+q}) & \xrightarrow{\simeq} & Rg_{Y*}R\underline{\Gamma}_{X'}(\Omega_{Y'}^{c+q})[c] & \longrightarrow & Rg_{Y*}(\Omega_{Y'}^{c+q})[c] \\ \uparrow{\scriptstyle g_Y^*} & & \uparrow{\scriptstyle g_Y^*} & & \uparrow{\scriptstyle g_Y^*} \\ \mathcal{H}_X^c(\Omega_Y^{c+q}) & \xrightarrow{\simeq} & R\underline{\Gamma}_X(\Omega_Y^{c+q})[c] & \longrightarrow & \Omega_Y^{c+q}[c], \end{array}$$

where the $g_Y^*$ on the very left is defined in such a way that the left square commutes. By Proposition 2.2.19 it thus suffices to prove the commutativity of

$$\begin{array}{ccc} i_*g_{X*}\Omega^q_{X'} = g_{Y*}i'_*\Omega^q_{X'} & \xrightarrow{\iota^q_{X'}} & g_{Y*}\mathcal{H}^c_{X'}(\Omega^{c+q}_{Y'}) \\ {\scriptstyle g_X^*}\uparrow & & \uparrow{\scriptstyle g_Y^*} \\ i_*\Omega^q_X & \xrightarrow{\iota^q_X} & \mathcal{H}^c_X(\Omega^{c+q}_Y). \end{array}$$

This is a local question, we may therefore assume that the ideal of $X$ in $Y$ is generated by a sequence $t_1, \dots, t_c$ of global sections of $\mathcal{O}_Y$. Then $g_Y^*t_1, \dots, g_Y^*t_c$ is a sequence of global sections of $\mathcal{O}_{Y'}$, which generate the ideal sheaf of $X'$ in $Y'$. Hence the assumption follows from the explicit description of $\iota^q_X$ and $\iota^q_{X'}$ in Proposition 2.2.19. □

**Proposition 2.2.23.** *Let $f : X \to Y$ be a finite and surjective morphism between smooth schemes, which are both of pure dimension $n$. We denote by*

$$\tau_f : \bigoplus_q f_*\Omega^q_X \to \bigoplus_q \Omega^q_Y$$

*the composition*

$$\bigoplus_q f_*\Omega^q_X \xrightarrow{(2.2.15)} \bigoplus_q f_*D_X(\Omega^{n-q}_X) \xrightarrow{f_*} \bigoplus_q D_Y(\Omega^{n-q}_Y) \xrightarrow{(2.2.15)} \bigoplus_q \Omega^q_Y.$$

*Then we have the following:*

(1) *In degree* 0, *the map $\tau_f$ equals the usual trace on the finite and locally free $\mathcal{O}_Y$-module $f_*\mathcal{O}_X$, that is,* $\mathrm{Tr}_{X/Y} : f_*\mathcal{O}_X \to \mathcal{O}_Y$.

(2) *For $\alpha \in f_*\Omega^a_X$ and $\beta \in \Omega^b_Y$, we have*

$$\tau_f(\alpha f^*\beta) = \tau_f(\alpha)\beta.$$

(3) *The composition $\tau_f \circ f^* : \bigoplus_q \Omega^q_Y \to \bigoplus_q \Omega^q_Y$ equals multiplication with the degree of $f$.*

*Proof.* All statements are local in $Y$. We may therefore assume that $f$ factors as $X \xrightarrow{i} P \xrightarrow{\pi} Y$, where $i$ is a regular closed immersion of pure codimension $d$ and $\pi$ is smooth of relative dimension $d$; further we may assume that the ideal sheaf of $X$ in $P$ is generated by $d$ global sections $t_1, \dots, t_d$ of $\mathcal{O}_P$. Then in degree $n$ the map $\tau_f$ equals the trace map $\tau^n_f : f_*\omega_X \to \omega_Y$ from Section 1.3.1 and in degree $q$ the map $\tau_f$ thus equals the composition

$$f_*\Omega^q_X \cong f_*\,\mathcal{H}om_X(\Omega^{n-q}_X, \omega_X) \xrightarrow{\text{nat.}} \mathcal{H}om_Y(f_*\Omega^{n-q}_X, f_*\omega_X)$$
$$\xrightarrow{\tau^n_f} \mathcal{H}om(f_*\Omega^{n-q}_X, \omega_Y) \xrightarrow{\circ f^*} \mathcal{H}om(\Omega^{n-q}_Y, \omega_Y) \cong \Omega^q_Y.$$

Thus for $\alpha \in f_*\Omega^q_X$, Lemma A.3.3 gives the following formula for $\tau_f(\alpha)$: In

$$i^*\Omega^{d+q}_P = \bigoplus_{r+s=d+q} i^*(\Omega^r_{P/Y}) \otimes f^*\Omega^s_Y,$$

write

$$i^*(dt_d \wedge \cdots \wedge dt_1 \wedge \tilde{\alpha}) = \sum_{r+s=d+q} \sum_j i^*\gamma_{j,r} \otimes f^*\beta_{j,s} \quad \text{for } \gamma_{j,r} \in \Omega^r_{P/Y},\ \beta_{j,s} \in \Omega^s_Y,$$

where $\tilde{\alpha} \in \Omega^q_P$ is a lift of $\alpha$. Then

$$\tau_f(\alpha) = (-1)^{d(d-1)/2} \sum_j \operatorname{Res}_{P/Y} \begin{bmatrix} \gamma_{j,d} \\ t_1, \dots, t_d \end{bmatrix} \beta_{j,q} \in \Omega^q_Y. \tag{2.2.24}$$

This formula immediately implies (2). For any $a \in \mathcal{O}_X$, we have

$$\tau_f(a) = (-1)^{d(d-1)/2} \operatorname{Res}_{P/Y} \begin{bmatrix} \tilde{a}\, dt_d \wedge \cdots \wedge dt_1 \\ t_1, \dots, t_d \end{bmatrix} = \operatorname{Res}_{P/Y} \begin{bmatrix} \tilde{a}\, dt_1 \wedge \cdots \wedge dt_d \\ t_1, \dots, t_d \end{bmatrix},$$

which equals $\operatorname{Tr}_{X/Y}(a)$ by [Conrad 2000, page 240, (R6)]; hence (1). Finally (3) is a direct consequence of (1) and (2). □

**Remark 2.2.25.** The trace map from Proposition 2.2.23 and its properties are well known; see for example [Kunz 1986, §16], where the trace is considered in much greater generality. There the construction is done via an ad hoc method not using the duality formalism. Therefore the connection to the trace map above is not a priori clear.

### 2.3. *Push-forward for Hodge cohomology with support.*

**Definition 2.3.1.** Let $f : (X, \Phi) \to (Y, \Psi)$ be a morphism in $V_*$ with $X$ equidimensional. We define a compactification of $f$ to be a factorization

$$f = \bar{f} \circ j : (X, \Phi) \hookrightarrow (\overline{X}, \Phi) \to (Y, \Psi),$$

where $\overline{X}$ is equidimensional (but possibly singular), $j$ is an open immersion and $\bar{f}$ is proper. Notice that since $f|_\Phi$ is proper, $\Phi$ is also a family of supports on $\overline{X}$. The compactification will be denoted by $(j, \bar{f})$.

By Nagata's compactification theorem (see, for example [Conrad 2007]) any $f$ in $V_*$ admits a compactification.

**Definition 2.3.2** (push-forward). Let $f : (X, \Phi) \to (Y, \Psi)$ be a morphism in $V_*$ and assume that $X$ and $Y$ are of pure dimension $d_X$ and $d_Y$, respectively, and set $r := d_X - d_Y$. Let

$$(X, \Phi) \xrightarrow{j} (\overline{X}, \Phi) \xrightarrow{\bar{f}} (Y, \Psi)$$

be a compactification of $f$. We define the push-forward

$$H_*(f) : H(X, \Phi) \to H(Y, \Psi)$$

as the composition

$$H(X,\Phi) \simeq \bigoplus_{i,j} H^i_\Phi(\overline{X}, D_{\overline{X}}(\Omega^{d_X-j}_{\overline{X}})[-d_X]) \xrightarrow{\text{nat.}} \bigoplus_{i,j} H^{i-d_X}_{\bar{f}^{-1}(\Psi)}(\overline{X}, D_{\overline{X}}(\Omega^{d_X-j}_{\overline{X}}))$$
$$\xrightarrow{\oplus \bar{f}^j_*} \bigoplus_{i,j} H^{i-d_X}_\Psi(Y, D_Y(\Omega^{d_X-j}_Y)) \xrightarrow{\simeq\,(2.2.15)} \bigoplus_{i,j} H^{i-r}_\Psi(Y, \Omega^{j-r}_Y) = H(Y,\Psi),$$

where the first isomorphism is the composition of (2.2.15) for $n = 0$ with the excision isomorphism. Notice that we obtain a morphism of graded abelian groups $H_*(f) : H_*(X, \Phi) \to H_*(Y, \Psi)$; see (2.0.2).

This definition is independent of the chosen compactification.

We extend the definition to the case of nonequidimensional $X$ and $Y$ additively.

*Proof.* We have to prove the independence of $H_*(f)$ from the chosen compactification. Let

$$\begin{array}{ccc} X & \xrightarrow{j_2} & X_2 \\ {\scriptstyle j_1}\downarrow & \overset{g}{\nearrow} & \downarrow{\scriptstyle f_2} \\ X_1 & \xrightarrow[f_1]{} & Y \end{array}$$

be a commutative diagram with $d := \dim X_1 = \dim X_2 = d_X$, $j_1$ and $j_2$ open and $f_1$ and $f_2$ proper. Notice that $g$ is automatically proper. Then the diagram

$$\begin{array}{ccccccc} & \overset{\simeq}{\nearrow} & H^{i-d}_\Phi(D_{X_2}(\Omega^{d-j}_{X_2})) & \to & H^{i-d}_{f_2^{-1}(\Psi)}(D_{X_2}(\Omega^{d-j}_{X_2})) & \overset{f_{2*}}{\searrow} & \\ H^i_\Phi(\Omega^j_X) & & \uparrow{\scriptstyle g_*} & & \uparrow{\scriptstyle g_*} & & H^{i-d}_\Psi(D_Y(\Omega^{d-j}_Y)). \\ & \underset{\simeq}{\searrow} & H^{i-d}_\Phi(D_{X_1}(\Omega^{d-j}_{X_1})) & \to & H^{i-d}_{f_1^{-1}(\Psi)}(D_{X_1}(\Omega^{d-j}_{X_1})) & \underset{f_{1*}}{\nearrow} & \end{array}$$

commutes. The left triangle commutes since $g_*|_X = \mathrm{id}_*$ by Proposition 2.2.7(3), the square in the middle obviously commutes, and the triangle on the right commutes by Proposition 2.2.7(2).

Two arbitrary compactifications of $f$ always receive a map from a third one and thus the general case follows from the case above. □

**Proposition 2.3.3.** (1) $H_*(\mathrm{id}) = \mathrm{id}$.

(2) *Let $f : (X, \Phi) \to (Y, \Psi)$ and $g : (Y, \Psi) \to (Z, \Xi)$ be two morphisms in $V_*$. Then $H_*(g \circ f) = H_*(g) \circ H_*(f) : H(X, \Phi) \to H(Z, \Xi)$.*

(3) *If* $f:(X,\Phi)\to(Y,\Psi)$ *in* $V_*$ *is finite, then* $H_*(f)$ *is induced by the trace map* $\tau_f$ *from Proposition 2.2.23.*

*Proof.* (1) follows from Proposition 2.2.7(1). Now for (2) we may assume that $X$, $Y$ and $Z$ are connected. Let $(j_X, f_1)$ and $(j_Y, g_1)$ be compactifications of $f$ and $g$, respectively. Let $(j_{X_1}, f_2)$ be a compactification of $j_Y\circ f_1$. Thus we have a commutative diagram

$$\begin{array}{ccccc} X_2 & & & & \\ {\scriptstyle j_{X_1}}\uparrow & \overset{f_2}{\searrow} & & & \\ X_1 & & Y_1 & & \\ {\scriptstyle j_X}\uparrow & \overset{f_1}{\searrow} & \uparrow{\scriptstyle j_Y} & \overset{g_1}{\searrow} & \\ X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z, \end{array}$$

with vertical arrows open immersions and diagonal arrows proper. By replacing $X_1$ by $f_2^{-1}(Y)$, we may assume that the parallelogram is cartesian. Then by Proposition 2.2.7(3) the diagram

$$\begin{array}{ccc} H^i_{f_2^{-1}(\Psi)}(D_{X_2}(\Omega^j_{X_2})) & \xrightarrow{f_{2*}} & H^i_{\Psi}(D_{Y_1}(\Omega^j_{Y_1})) \\ \downarrow\simeq & & \downarrow\simeq \\ H^i_{f_1^{-1}(\Psi)}(D_{X_1}(\Omega^j_{X_1})) & \xrightarrow{f_{1*}} & H^i_{\Psi}(D_Y(\Omega^j_Y)) \end{array}$$

commutes. Thus (2) follows from Proposition 2.2.7(2). Finally (3) follows immediately from the definitions. □

**Lemma 2.3.4.** *Consider a cartesian diagram*

$$\begin{array}{ccc} (X\times Y', \Phi') & \xrightarrow{f'} & (Y', \Psi') \\ {\scriptstyle g_{X\times Y}}\downarrow & & \downarrow{\scriptstyle g_Y} \\ (X\times Y, \Phi) & \xrightarrow{f} & (Y, \Psi), \end{array}$$

*such that* $f$ *is induced by the projection to* $Y$, *with* $f, f'\in V_*$ *and* $g_{X\times Y}, g_Y\in V^*$. *Then* $H^*(g_Y)\circ H_*(f) = H_*(f')\circ H^*(g_{X\times Y})$.

*Furthermore,* $H_*(f): H(X\times Y,\Phi)\to H(Y,\Psi)$ *factors over the projection*

$$H(X\times Y,\Phi)\to\bigoplus_{i,j} H^i_\Phi(X\times Y, \mathrm{pr}_1^*\,\Omega_X^{d_X}\otimes\mathrm{pr}_2^*\,\Omega_Y^j).$$

*Proof.* We may assume $X$ and $Y$ are of pure dimension $d_X$ and $d_Y$, respectively, and we set $d := d_X + d_Y$. We embed $X$ as an open in a proper $k$-scheme $\overline{X}$ of pure dimension $d_X$. Then

$$(X\times Y,\Phi)\xrightarrow{j}(\overline{X}\times Y,\Phi)\xrightarrow{\mathrm{pr}_2}(Y,\Psi)$$

is a compactification of $f$, where $j$ is the open embedding and $\mathrm{pr}_2$ is induced by the projection to $Y$. Similarly we obtain a compactification for $f'$, in which case we write $\mathrm{pr}_2'$ for the projection to $Y'$. The second statement of the lemma follows from Definition 2.3.2, Remark 2.2.6(2) and the commutative diagram

$$\begin{array}{ccc} \Omega^j_{X\times Y} & \xrightarrow{\ \simeq\ } & D_{X\times Y}(\Omega^{d-j}_{X\times Y})[-d] \\ {\scriptstyle \text{projection}}\big\downarrow & & \big\downarrow{\scriptstyle (\mathrm{pr}_2^*)^\vee} \\ \mathrm{pr}_1^*\,\Omega^{d_X}_X\otimes \mathrm{pr}_2^*\,\Omega^{j-d_X}_Y & \xrightarrow{\ \simeq\ } & D_{X\times Y}(\mathrm{pr}_2^*\,\Omega^{d-j}_Y)[-d]. \end{array}$$

Now we come to the first statement of the lemma. Consider the diagram (we use a shortened notation)

$$\begin{array}{ccccc} & & H^{i-d}_{\mathrm{pr}_2^{-1}(\Psi)}(D_{\overline{X}\times Y}(\Omega^{j-d}_{\overline{X}\times Y})) & & \\ & \nearrow & \big\downarrow{\scriptstyle (\mathrm{pr}_2^*)^\vee} & \overset{\mathrm{pr}_{2*}}{\searrow} & \\ H^i_\Phi(\Omega^j_{X\times Y}) & \longrightarrow & H^{i-d}_{\mathrm{pr}_2^{-1}(\Psi)}(D_{\overline{X}\times Y}(\mathrm{pr}_2^*\,\Omega^{j-d}_Y)) & \longrightarrow & H^{i-d_X}_\Psi(\Omega^{j-d_X}_Y), \\ & \searrow & {\scriptstyle \gamma}\big\uparrow & \nearrow & \\ & & H^{i-d}_{\mathrm{pr}_2^{-1}(\Psi)}(\mathrm{pr}_2^!(\mathcal{O}_Y)\otimes \mathrm{pr}_2^*\,\Omega^{j-d_X}_Y[d_Y]) & & \end{array}$$

where we use the notation of Lemma 2.2.16, the upper map on the left is induced by excision, the middle and the lower map on the left are induced by projection and excision, and the middle and the lower map on the right are induced by the corresponding maps from Lemma 2.2.16(2). It follows from Lemma 2.2.16 and Remark 2.2.6(2) that all the triangles in this diagram commute. Replacing $Y$ by $Y'$ and $\mathrm{pr}_2$ by $\mathrm{pr}_2'$, we obtain a similar commutative diagram. Thus it remains to show that the diagram

$$\begin{array}{ccccc} H^i_\Phi(\Omega^j_{X\times Y}) & \xrightarrow{\text{proj.}} & H^{i-d}_{\mathrm{pr}_2^{-1}(\Psi)}(\mathrm{pr}_2^!(\mathcal{O}_Y)\otimes \mathrm{pr}_2^*\,\Omega^{j-d_X}_Y[d_Y]) & \xrightarrow{\mathrm{Tr}_{\mathrm{pr}_2}\otimes\, \mathrm{id}} & H^{i-d_X}_\Psi(\Omega^{j-d_X}_Y) \\ \big\downarrow{\scriptstyle H^*(g_{X\times Y})} & & & & \big\downarrow{\scriptstyle H^*(g_Y)} \\ H^i_{\Phi'}(\Omega^j_{X\times Y'}) & \xrightarrow{\text{proj.}} & H^{i-d'}_{\mathrm{pr}_2'^{-1}(\Psi')}(\mathrm{pr}_2'^{!}(\mathcal{O}_{Y'})\otimes \mathrm{pr}_2'^{*}\Omega^{j-d_X}_{Y'}[d_{Y'}]) & \xrightarrow{\mathrm{Tr}_{\mathrm{pr}_2'}\otimes\, \mathrm{id}} & H^{i-d_X}_{\Psi'}(\Omega^{j-d_X}_{Y'}) \end{array} \tag{2.3.5}$$

is commutative, where $d' = d_X + d_{Y'}$. To this end we define the map

$$\tau_f : Rf_*R\underline{\Gamma}_\Phi(\omega_{X\times Y/Y}[d_X]) \to R\underline{\Gamma}_\Psi\mathcal{O}_Y,$$

to be the composition

$$Rf_*R\underline{\Gamma}_\Phi(\omega_{X\times Y/Y}[d_X]) \xrightarrow{\text{excision} \simeq} R\,\mathrm{pr}_{2*}\,R\underline{\Gamma}_\Phi(\mathrm{pr}_2^!\,\mathcal{O}_Y)$$
$$\xrightarrow{\text{nat.}} R\,\mathrm{pr}_{2*}\,R\underline{\Gamma}_{\mathrm{pr}_2^{-1}(\Psi)}(\mathrm{pr}_2^!\,\mathcal{O}_Y) \xrightarrow{\simeq} R\underline{\Gamma}_\Psi R\,\mathrm{pr}_{2*}\,\mathrm{pr}_2^!\,\mathcal{O}_Y \xrightarrow{\mathrm{Tr}_{\mathrm{pr}_2}} R\underline{\Gamma}_\Psi\mathcal{O}_Y.$$

Then the upper horizontal line in diagram (2.3.5) equals $H^{i-d}(Y,\,\cdot)$ applied to the composition

$$Rf_*R\underline{\Gamma}_\Phi\Omega^j_{X\times Y}[d] \xrightarrow{\text{projection}} Rf_*R\underline{\Gamma}_\Phi(\omega_{X\times Y/Y}[d_X]\otimes f^*\Omega_Y^{j-d_X}[d_Y])$$
$$\xrightarrow{\simeq} Rf_*R\underline{\Gamma}_\Phi(\omega_{X\times Y/Y}[d_X])\otimes\Omega_Y^{j-d_X}[d_Y] \xrightarrow{\tau_f\otimes\mathrm{id}} R\underline{\Gamma}_\Psi(\Omega_Y^{j-d_X})[d_Y].$$

(That there is no intervention of signs in the definition of the projection map is compatible with the fact that the isomorphism $\omega_{X\times Y}[d]\cong\omega_{X\times Y/Y}[d_X]\otimes f^*\omega_Y[d_Y]$ is defined without a sign; see [Conrad 2000, (2.2.6)].) The lower horizontal line in the diagram (2.3.5) equals $H^{i-d'}(Y',\,\cdot)$ applied to the analog composition for $f'$. Then it is straightforward to check that the commutativity of diagram (2.3.5) is implied by the commutativity of

$$\begin{array}{ccc} Rf_*R\underline{\Gamma}_\Phi(\omega_{X\times Y/Y}[d_X]) & \xrightarrow{\tau_f} & R\underline{\Gamma}_\Psi\mathcal{O}_Y \\ \downarrow{\scriptstyle g^*_{X\times Y}} & & \downarrow{\scriptstyle g^*_Y} \\ Rg_{Y*}Rf'_*R\underline{\Gamma}_{\Phi'}(\omega_{X\times Y'/Y'}[d_X]) & \xrightarrow{\tau_{f'}} & Rg_{Y*}R\underline{\Gamma}_{\Psi'}\mathcal{O}_{Y'}. \end{array} \tag{2.3.6}$$

To prove the commutativity of this last diagram, we can clearly assume (by definition of the pull-back and $\tau_f$) that $\Phi'=g^{-1}_{X\times Y}(\Phi)$ and $\Psi'=g_Y^{-1}(\Psi)$. We define the map

$$\alpha : R\,\mathrm{pr}_{2*}\,\mathrm{pr}_2^!\,\mathcal{O}_Y \to Rg_{Y*}R\,\mathrm{pr}'_{2*}(\mathrm{pr}'_2)^!\mathcal{O}_{Y'}$$

to be the composition

$$R\,\mathrm{pr}_{2*}\,\mathrm{pr}_2^!(\pi_Y^*k) \xrightarrow{b^{-1}_{\pi_Y,\pi_{\overline{X}}}} R\,\mathrm{pr}_{2*}\,\mathrm{pr}_1^*(\pi^!_{\overline{X}}k)$$
$$\to Rg_{Y*}R\,\mathrm{pr}'_{2*}(\mathrm{pr}'_1)^*(\pi^!_{\overline{X}}k)$$
$$\xrightarrow{b_{\pi_{Y'},\pi_{\overline{X}}}} Rg_{Y*}R\,\mathrm{pr}'_{2*}(\mathrm{pr}'_2)^!(\pi^*_{Y'}k) = Rg_{Y*}R\,\mathrm{pr}'_{2*}(\mathrm{pr}'_2)^!\mathcal{O}_{Y'},$$

where $b_{\pi_Y,\pi_{\overline{X}}}:\mathrm{pr}_1^*\,\pi^!_{\overline{X}}\simeq\mathrm{pr}_2^!\,\pi_Y^*$ is the isomorphism from [Hartshorne 1966, Chapter VII, Corollaries 3.4(a)(5)] and the middle map is the composition of the natural maps

$$R\,\mathrm{pr}_{2*}\,\mathrm{pr}_1^* \to Rg_{Y*}Lg_Y^*R\,\mathrm{pr}_{2*}\,\mathrm{pr}_1^* \to Rg_{Y*}R\,\mathrm{pr}'_{2*}\,Lg^*_{\overline{X}\times Y}\,\mathrm{pr}_1^* \cong Rg_{Y*}R\,\mathrm{pr}'_{2*}(\mathrm{pr}'_1)^*.$$

Now the commutativity of diagram (2.3.6) follows from the commutativity of

$$\begin{array}{ccc} Rf_*R\underline{\Gamma}_\Phi(\omega_{X\times Y/Y}[d_X]) & \longrightarrow & R\underline{\Gamma}_\Psi R\,\mathrm{pr}_{2*}\,\mathrm{pr}_2^!\,\mathcal{O}_Y \\ {\scriptstyle g^*_{X\times Y}}\downarrow & & \downarrow{\scriptstyle \alpha} \\ Rg_{Y*}Rf'_*R\underline{\Gamma}_{\Phi'}(\omega_{X\times Y'/Y'}[d_X]) & \longrightarrow & R\underline{\Gamma}_\Psi Rg_{Y*}R\,\mathrm{pr}'_{2*}(\mathrm{pr}'_2)^!\mathcal{O}_{Y'}, \end{array}$$

which is clear by the explicit description of the isomorphisms $b_{(\cdot,\cdot)}$ in the smooth case (see [Hartshorne 1966, Chapter VII, Corollary 3.4(a), Var 6]), and from the commutativity of the diagram

$$\begin{array}{ccc} R\underline{\Gamma}_\Psi R\,\mathrm{pr}_{2*}\,\mathrm{pr}_2^!\,\mathcal{O}_Y & \xrightarrow{\mathrm{Tr}_{\mathrm{pr}_2}} & R\underline{\Gamma}_\Psi(\mathcal{O}_Y) \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle g_Y^*} \\ R\underline{\Gamma}_\Psi Rg_{Y*}R\,\mathrm{pr}'_{2*}(\mathrm{pr}'_2)^!\mathcal{O}_{Y'} & \xrightarrow{\mathrm{Tr}_{\mathrm{pr}'_2}} & R\underline{\Gamma}_\Psi Rg_{Y*}(\mathcal{O}_{Y'}), \end{array}$$

which follows from [Hartshorne 1966, Chapter VII, Corollary 3.4(b), TRA 4]. Hence the statement. □

**Proposition 2.3.7.** *Let*

$$\begin{array}{ccc} (X',\Phi') & \xrightarrow{f'} & (Y',\Psi') \\ {\scriptstyle g_X}\downarrow & & \downarrow{\scriptstyle g_Y} \\ (X,\Phi) & \xrightarrow{f} & (Y,\Psi), \end{array}$$

*be a cartesian square with $f, f'\in V_*$ and $g_X, g_Y\in V^*$. Assume either that $g_Y$ is flat or $g_Y$ is a closed immersion and $f$ is transversal to $Y'$. Then*

$$H^*(g_Y)\circ H_*(f) = H_*(f')\circ H^*(g_X).$$

*Proof.* After embedding $X$ in $X\times Y$ via the graph morphism, the diagram above splits as

$$\begin{array}{ccccc} (X',\Phi') & \hookrightarrow & (X\times Y',\Phi') & \xrightarrow{\mathrm{pr}_2} & (Y',\Psi') \\ \downarrow{\scriptstyle g_X} & & \downarrow{\scriptstyle \mathrm{id}\times g_Y} & & \downarrow{\scriptstyle g_Y} \\ (X,\Phi) & \hookrightarrow & (X\times Y,\Phi) & \xrightarrow{\mathrm{pr}_2} & (Y,\Psi). \end{array}$$

Both squares are cartesian, the projections $\mathrm{pr}_2$ are smooth and the inclusions are closed. If $g_Y$ is a closed immersion and $f$ is transversal to $Y'$, then $\mathrm{id}\times g_Y : X\times Y'\to X\times Y$ is transversal to $X\hookrightarrow X\times Y$. Thus the statement follows from Proposition 2.3.3(2), Corollary 2.2.22 and Lemma 2.3.4. □

**Lemma 2.3.8.** *Let $X$ be smooth and $\iota : D\hookrightarrow X$ the inclusion of a smooth divisor. Let $\Phi$ be a family of supports on $D$ and denote by $\iota_1 : (D,\Phi)\to(X,\Phi)$ the map in*

*$V_*$ induced by $\iota$. Then $H_*(\iota_1) : H^i_\Phi(D, \Omega^j_D) \to H^{i+1}_\Phi(X, \Omega^{j+1}_X)$ is the connecting homomorphism of the long exact cohomology sequence associated to the exact sequence*

$$0 \to \Omega^{j+1}_X \to \Omega^{j+1}_X(\log D) \xrightarrow{\mathrm{Res}} \iota_*\Omega^j_D \to 0, \tag{2.3.9}$$

*where $\mathrm{Res}(\frac{dt}{t}\alpha) = \iota^*(\alpha)$ for $t \in \mathcal{O}_X$ a regular element defining $D$ and $\alpha \in \Omega^j_X$. In particular, if $\Phi \subset X$ is supported in codimension $\geq i+1$ in $X$, then $H_*(\iota_1)$ is injective on $H^i_\Phi$.*

*Proof.* By Remark 2.2.14(1), the map $\iota_*$ from Definition 2.2.5 induces a map (also denoted by $\iota_*$)

$$\iota_* : \iota_*\Omega^j_D \to \Omega^{j+1}_X[1].$$

It suffices to show that this map coincides with the edge homomorphism coming from the distinguished triangle (2.3.9), which we denote by $\partial_{\mathrm{Res}}$. The diagram

$$\begin{array}{ccc}
\mathcal{H}om(\Omega^{n-(j+1)}_X, \iota_*\omega_D) & \longrightarrow & \mathcal{H}om(\Omega^{n-(j+1)}_X, \omega_X[1]) \\
\uparrow & & \uparrow \simeq \\
\iota_*\Omega^{j-1}_D & \longrightarrow & \Omega^j_X[1],
\end{array}$$

where $n = \dim X$ and the vertical maps are induced by multiplication from the left, is commutative for both $\iota_*$ and $\partial_{\mathrm{Res}}$. Thus we only need to consider the case $j = n-1$.

Let $K^\bullet$ be the complex $\mathcal{O}_X(-D) \to \mathcal{O}_X$ in degree $[-1, 0]$. Then $K^\bullet \to \iota_*\mathcal{O}_D$ is a locally free resolution. We denote by $\mathrm{Tr}'_\iota$ the composition

$$\iota_*\omega_{D/X}[-1] \xrightarrow{\eta_\iota} \iota_*\iota^!\mathcal{O}_X \xrightarrow{\mathrm{Tr}_\iota} \mathcal{O}_X,$$

where $\eta_\iota$ is the fundamental local isomorphism (see (A.2.1)). Then $\mathrm{Tr}'_\iota$ is given by

$$\mathrm{Tr}'_\iota : \iota_*\omega_{D/X}[-1] \xleftarrow{\simeq} \mathcal{H}om^\bullet(K^\bullet, \mathcal{O}_X) \to \mathcal{O}_X.$$

Here the first map is in degree 1 given by (see (A.2.2))

$$\mathcal{H}om(\mathcal{O}_X(-D), \mathcal{O}_X) = \mathcal{O}_X(D) \to \iota_*\omega_{D/X}, \quad 1/t \mapsto -t^\vee,$$

where $t$ is a regular parameter defining $D$, and the second map (in degree 0) by $\mathcal{H}om(\mathcal{O}_X, \mathcal{O}_X) = \mathcal{O}_X$. (See the proof of Lemma A.2.5 and in particular (A.2.8).)

It thus follows from the commutative diagram (A.2.14) that $\iota_* : \iota_*\omega_D \to \omega_X$ equals the composition

$$\iota_*\omega_D \to \iota_*\omega_{D/X} \otimes \omega_X \xrightarrow{\mathrm{Tr}'_\iota[1]\otimes\mathrm{id}} \omega_X[1],$$

where the first map is given by $\alpha \mapsto t^\vee \otimes (dt \wedge \tilde{\alpha})$, with $\tilde{\alpha}$ a lift. Obviously the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \omega_X & \longrightarrow & \omega_X(\log D) & \xrightarrow{-\mathrm{Res}} & \iota_*\omega_D & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{H}om(\mathcal{O}_X, \mathcal{O}_X) \otimes \omega_X & \longrightarrow & \mathcal{O}_X(D) \otimes \omega_X & \xrightarrow{-1} & \iota_*\omega_{D/X} \otimes \omega_X & \longrightarrow & 0. \end{array}$$

Now, by the above (and the sign conventions from [Conrad 2000, 1.3]), the map $\iota_*\omega_{D/X} \otimes \omega_X \to \mathcal{H}om(\mathcal{O}_X, \mathcal{O}_X)[1] \otimes \omega_X$ induced by the lower exact sequence equals $-(\mathrm{Tr}'_\iota \otimes \mathrm{id})$. (Here we need that $\mathcal{H}om^\bullet(K^\bullet, \mathcal{O}_X)[1] = \mathcal{H}om(K^{-(\bullet+1)}, \mathcal{O}_X)$.) Thus the commutativity of the diagram above yields $i_* = -\partial_{-\mathrm{Res}} = \partial_{\mathrm{Res}}$. □

**2.4. *The Künneth morphism.*** For $(X, \Phi)$ and $(Y, \Psi) \in \mathrm{ob}(V_*) = \mathrm{ob}(V^*)$, the Künneth morphism

$$\times : H^i_\Phi(X, \Omega^p_X) \times H^j_\Psi(Y, \Omega^q_Y) \to H^{i+j}_{\Phi\times\Psi}(X \times Y, \Omega^{p+q}_{X\times Y}) \tag{2.4.1}$$

is defined as the composition of the cartesian product with multiplication. Choose flasque resolutions $\Omega^p_X \to I^\bullet$ and $\Omega^q_Y \to J^\bullet$ and write $K^i_\Phi = \mathrm{Ker}(\Gamma_\Phi I^i \to \Gamma_\Phi I^{i+1})$ and $K^j_\Psi = \mathrm{Ker}(\Gamma_\Psi J^j \to \Gamma_\Psi J^{j+1})$. Then $\mathrm{pr}_1^{-1} I^\bullet \otimes_k \mathrm{pr}_2^{-1} J^\bullet$ is a resolution of $\mathrm{pr}_1^{-1}\Omega^p_X \otimes_k \mathrm{pr}_2^{-1}\Omega^q_Y$ and (2.4.1) is induced by the composition of the natural maps

$$\begin{aligned} K^i_\Phi \otimes_k K^j_\Psi &\to H^{i+j}_{\Phi\times\Psi}(\mathrm{pr}_1^{-1} I^\bullet \otimes_k \mathrm{pr}_2^{-1} J^\bullet) \\ &\to H^{i+j}_{\Phi\times\Psi}(X \times Y, \mathrm{pr}_1^{-1}\Omega^p_X \otimes_k \mathrm{pr}_2^{-1}\Omega^q_Y) \to H^{i+j}_{\Phi\times\Psi}(X \times Y, \Omega^{p+q}_{X\times Y}). \end{aligned}$$

We define

$$T : H(X, \Phi) \otimes H(Y, \Psi) \to H(X \times Y, \Phi \times \Psi) \tag{2.4.2}$$

by the formula

$$T(\alpha_{i,p} \otimes \beta_{j,q}) = (-1)^{(i+p)\cdot j}(\alpha_{i,p} \times \beta_{j,q}),$$

where $\alpha_{i,p} \in H^i_\Phi(X, \Omega^p_X)$, $\beta_{j,q} \in H^j_\Psi(Y, \Omega^q_Y)$, and $\times$ is the map in (2.4.1).

**Proposition 2.4.3.** *The triples $(H_*, T, e)$ and $(H^*, T, e)$ define right-lax symmetric monoidal functors (see Section 1.1.9).*

**Lemma 2.4.4.** *Let $f : X \to Y$ be a morphism. Assume $Y$ to be smooth and $X$ of pure dimension $d$. Then for any $p, q \geq 0$, there is a morphism*

$$\mu : D_X(\Omega^{d-p}_X) \otimes f^*\Omega^q_Y \longrightarrow D_X(\Omega^{d-(p+q)}_X),$$

*such that*

(1) *if $U \subset X$ is a smooth open subset, then the diagram*

$$\begin{array}{ccc} D_U(\Omega_U^{d-p})[-d] \otimes f|_U{}^*\Omega_Y^q & \xrightarrow{\mu|_U[-d]} & D_U(\Omega_U^{d-(q+p)})[-d] \\ {\scriptstyle (2.2.15)}\downarrow\simeq & & {\scriptstyle (2.2.15)}\downarrow\simeq \\ \Omega_U^p \otimes f|_U{}^*\Omega_Y^q & \longrightarrow & \Omega_U^{p+q} \end{array}$$

*commutes, where the lower horizontal map is given by $\alpha \otimes \beta \mapsto \alpha \wedge f^*(\beta)$;*

(2) *if $f$ is proper, then the diagram*

$$\begin{array}{ccccc} Rf_*D_X(\Omega_X^{d-p}) \otimes \Omega_Y^q & \xrightarrow{\simeq} & Rf_*(D_X(\Omega_X^{d-p}) \otimes f^*\Omega_Y^q) & \xrightarrow{\mu} & Rf_*D_X(\Omega_X^{d-(p+q)}) \\ & \searrow^{f_*\otimes \mathrm{id}} & & & \downarrow f_* \\ & & D_Y(\Omega_Y^{d-p}) \otimes \Omega_Y^q & \longrightarrow & D_Y(\Omega_Y^{d-(p+q)}) \end{array}$$

*commutes, where the lower horizontal map is induced by*

$$\mathcal{H}om(\Omega_Y^{d-p}, \Omega_Y^{d_Y}) \otimes \Omega_Y^q \to \mathcal{H}om(\Omega_Y^{d-(q+p)}, \Omega_Y^{d_Y}), \quad \varphi \otimes \alpha \mapsto \varphi(\alpha \wedge (\cdot)).$$

*Proof.* We denote by $\pi_X$ and $\pi_Y$ the structure maps of $X$ and $Y$, respectively. Since $\pi_X^! k$ and $\pi_Y^! k$ are dualizing complexes, they are represented by bounded complexes of injectives $I_X^\bullet$ and $I_Y^\bullet$, and $\mathrm{Tr}_f : f_*\pi_X^! k \cong f_* f^! \pi_Y^! k \to \pi_Y^! k$ is thus represented by a morphism of complexes $\mathrm{Tr}_f : f_* I_X^\bullet \to I_Y^\bullet$. Now the map

$$\mu : \mathcal{H}om_X(\Omega_X^{d-p}, I_X^\bullet) \otimes f^*\Omega_Y^q \to \mathcal{H}om_X(\Omega_X^{d-(p+q)}, I_Y^\bullet)$$

is in degree $n$ given by

$$\mathcal{H}om_X(\Omega_X^{d-p}, I_X^n) \otimes f^*\Omega_Y^q \to \mathcal{H}om_X(\Omega_X^{d-(p+q)}, I_Y^n), \quad \theta \otimes \alpha \mapsto \theta(f^*(\alpha) \wedge \cdot).$$

It is immediate that this defines a map of complexes that satisfies (1). For (2) we observe that it suffices to check the commutativity of

$$\begin{array}{ccc} \mathcal{H}om_X(f_*\Omega_X^{d-p}, f_*I_X^\bullet) \otimes \Omega_Y^q & \xrightarrow{\mu} & \mathcal{H}om_X(f_*\Omega_X^{d-(p+q)}, f_*I_X^\bullet) \\ {\scriptstyle \mathrm{Tr}_f \circ(\cdot)\circ f^*}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}_f \circ(\cdot)\circ f^*} \\ \mathcal{H}om_Y(\Omega_Y^{d-p}, I_Y^\bullet) \otimes \Omega_Y^q & \longrightarrow & \mathcal{H}om(\Omega_Y^{d-(p+q)}, I_Y^\bullet), \end{array}$$

which is straightforward. □

*Proof of Proposition 2.4.3.* Recall that $H^*(X, \Phi)$ is graded by (2.0.1) and $H_*(X, \Phi)$ is graded by (2.0.2). The morphism $T$ respects the grading for both gradings. In the following, we will work with the upper grading $H^*$, but all arguments will also work for the lower grading $H_*$ because the difference between lower and upper grading is an even integer.

By using the associativity of $\times$ (defined in (2.4.1)) it is straightforward to prove the associativity of $T$. Let us prove the commutativity of $T$, that is, that the diagram

$$\begin{array}{ccc} H(X,\Phi)\otimes H(Y,\Psi) & \xrightarrow{T} & H(X\times Y,\Phi\times\Psi) \\ \downarrow & & \downarrow \\ H(Y,\Psi)\otimes H(X,\Phi) & \xrightarrow{T} & H(Y\times X,\Psi\times\Phi) \end{array} \tag{2.4.5}$$

is commutative. The left vertical map is defined by $a\otimes b\mapsto(-1)^{\deg(a)\deg(b)}b\otimes a$, and the right vertical map is given by $H^*(\epsilon_1)$ and $H_*(\epsilon_2)$, respectively, with

$$\epsilon_1 : (Y\times X,\Psi\times\Phi)\to(X\times Y,\Phi\times\Psi),$$
$$\epsilon_2 : (X\times Y,\Phi\times\Psi)\to(Y\times X,\Psi\times\Phi)$$

the obvious morphisms $\epsilon_1\in V^*$ and $\epsilon_2\in V_*$. Obviously, $H^*(\epsilon_1)=H_*(\epsilon_2)$; thus we may work with $H^*(\epsilon_1)$ in the following. Note that the diagram

$$\begin{array}{ccc} H^i_\Phi(X,\Omega^p_X)\times H^j_\Psi(Y,\Omega^q_Y) & \xrightarrow{(2.4.1)} & H^{i+j}_{\Phi\times\Psi}(X\times Y,\Omega^{p+q}_{X\times Y}) \\ \downarrow & & \downarrow{\scriptstyle (-1)^{p\cdot q}H^*(\epsilon_1)} \\ H^j_\Psi(Y,\Omega^q_Y)\times H^i_\Phi(X,\Omega^p_X) & \xrightarrow{(2.4.1)} & H^{i+j}_{\Psi\times\Phi}(Y\times X,\Omega^{p+q}_{Y\times X}) \end{array}$$

is commutative, where the left vertical arrow is defined by $a\times b\mapsto(-1)^{i\cdot j}(b\times a)$. By using this diagram it is a straightforward calculation to prove the commutativity of (2.4.5).

We still need to prove the functoriality of $T$ for $H^*$ and $H_*$. For $H^*$ this follows immediately from the definitions. Let us prove the functoriality for $H_*$. We will write $H^i_\Phi(\,\cdot\,)$ instead of $H^i_\Phi(X,\,\cdot\,)$. By using the commutativity of $T$ (that is, (2.4.5)) it is enough to prove that the diagram

$$\begin{array}{ccc} H^i_\Phi(\Omega^p_X)\times H^j_\Psi(\Omega^q_Y) & \xrightarrow{T} & H^{i+j}_{\Phi\times\Psi}(\Omega^{p+q}_{X\times Y}) \\ {\scriptstyle H_*(h)\times\mathrm{id}}\downarrow & & \downarrow{\scriptstyle H_*(h\times\mathrm{id})} \\ H^{i-r}_{\Phi'}(\Omega^{p-r}_{X'})\times H^j_\Psi(\Omega^q_Y) & \xrightarrow{T} & H^{i+j-r}_{\Phi'\times\Psi}(\Omega^{p+q-r}_{X'\times Y}) \end{array} \tag{2.4.6}$$

commutes for any $(Y,\Psi)\in V_*$, $h:(X,\Phi)\to(X',\Phi')$ in $V_*$ and $r=\dim X-\dim X'$ ($X$ and $X'$ are assumed to be equidimensional). Equivalently, the diagram (2.4.6), but with $\times$ instead of $T$ as horizontal arrows, commutes. Observe that $\times$ can be factored as

$$\times : H^i_\Phi(\Omega^p_X) \times H^j_\Psi(\Omega^q_Y) \xrightarrow{H^*(\mathrm{pr}_1)\times \mathrm{id}} H^i_{\Phi\times Y}(\Omega^p_{X\times Y}) \times H^j_\Psi(\Omega^q_Y) \to H^{i+j}_{\Phi\times\Psi}(\Omega^{p+q}_{X\times Y}),$$

where the map at right is the composition of the cartesian product with the multiplication map $\Omega^p_{X\times Y} \otimes_k \mathrm{pr}_2^{-1}\Omega^q_Y \to \Omega^{p+q}_{X\times Y}$, $\alpha\otimes\beta \mapsto \alpha\wedge \mathrm{pr}_2^*\beta$. By Proposition 2.3.7 the diagram

$$\begin{array}{ccc} H^i_\Phi(\Omega^p_X) & \xrightarrow{H^*(\mathrm{pr}_1)} & H^i_{\Phi\times Y}(\Omega^p_{X\times Y}) \\ {\scriptstyle H(h_*)}\downarrow & & \downarrow{\scriptstyle H_*(h\times \mathrm{id})} \\ H^{i-r}_{\Phi'}(\Omega^{p-r}_{X'}) & \xrightarrow{H^*(\mathrm{pr}_1)} & H^{i-r}_{\Phi'\times Y}(\Omega^{p-r}_{X'\times Y}) \end{array} \tag{2.4.7}$$

commutes. Thus it suffices to prove that

$$\begin{array}{ccc} H^i_{\Phi\times Y}(\Omega^p_{X\times Y}) \times H^j_\Psi(\Omega^q_Y) & \longrightarrow & H^{i+j}_{\Phi\times\Psi}(\Omega^{p+q}_{X\times Y}) \\ {\scriptstyle H_*(h\times \mathrm{id})\times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle H_*(h\times \mathrm{id})} \\ H^{i-r}_{\Phi'\times Y}(\Omega^{p-r}_{X'\times Y}) \times H^j_\Psi(\Omega^q_Y) & \longrightarrow & H^{i+j-r}_{\Phi'\times\Psi}(\Omega^{p+q-r}_{X'\times Y}) \end{array}$$

commutes.

Now let $\bar{h} : \overline{X} \to X'$ be a compactification of $h$ and set $d = \dim X + \dim Y$. We write

$$\omega^p_{\overline{X}\times Y} := D_{\overline{X}\times Y}(\Omega^{d-p}_{\overline{X}\times Y}) \quad \text{and} \quad \omega^p_{X'\times Y} := D_{X'\times Y}(\Omega^{d-p}_{X'\times Y}).$$

Notice that $\omega^p_{\overline{X}\times Y}|_{X\times Y} \cong \Omega^p_{X\times Y}[d]$ and $\omega^p_{X'\times Y} \cong \Omega^{p-r}_{X'\times Y}[d-r]$. With this notation the push-forward is a morphism

$$(\bar{h}\times \mathrm{id})_* : R(\bar{h}\times\mathrm{id})_*\omega^p_{\overline{X}\times Y} \to \omega^p_{X'\times Y}$$

and we have to show that the following diagram commutes:

$$\begin{array}{ccc} H^i_{\Phi\times Y}(\omega^p_{\overline{X}\times Y}) \times H^j_\Psi(\Omega^p_Y) & \longrightarrow & H^{i+j}_{\Phi\times\Psi}(\omega^{p+q}_{\overline{X}\times Y}) \\ {\scriptstyle (h\times \mathrm{id})_*\times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle (h\times \mathrm{id})_*} \\ H^i_{\Phi'\times Y}(\omega^p_{X'\times Y}) \times H^j_\Psi(\Omega^p_Y) & \longrightarrow & H^{i+j}_{\Phi'\times\Psi}(\omega^{p+q}_{X'\times Y}), \end{array}$$

where the upper map is given by the cartesian product composed with the $\mu$ from Lemma 2.4.4. Clearly we may assume $\Phi = \bar{h}^{-1}(\Phi')$; thus

$$H^i_{\Phi\times Y}(\omega^p_{\overline{X}\times Y}) = H^i_{\Phi'\times Y}(R(\bar{h}\times\mathrm{id})_*\omega^p_{\overline{X}\times Y}). \tag{2.4.8}$$

Now it follows from Lemma 2.4.4(2) that it is enough to prove the commutativity of the diagram

$$\begin{array}{ccc} H^i_{\Phi\times Y}(\omega^p_{\bar{X}\times Y})\times H^j_\Psi(\Omega^q_Y) & \longrightarrow & H^{i+j}_{\Phi'\times\Psi}(R(\bar{h}\times\mathrm{id})_*(\omega^p_{\bar{X}\times Y})\otimes_k \mathrm{pr}_2^{-1}\Omega^q_Y) \\ \downarrow{\scriptstyle (\bar{h}\times\mathrm{id})_*\times\mathrm{id}} & & \downarrow{\scriptstyle (\bar{h}\times\mathrm{id})_*\otimes\mathrm{id}} \\ H^i_{\Phi'\times Y}(\omega^p_{X'\times Y})\times H^j_\Psi(\Omega^q_Y) & \longrightarrow & H^{i+j}_{\Phi'\times\Psi}(\omega^p_{X'\times Y}\otimes_k \mathrm{pr}_2^{-1}\Omega^q_Y), \end{array}$$

where the upper horizontal map is the composition of (2.4.8) with the cartesian product, and the diagram

$$\begin{array}{ccc} H^{i+j}_{\Phi'\times\Psi}(R(\bar{h}\times\mathrm{id})_*(\omega^p_{\bar{X}\times Y})\otimes_k\mathrm{pr}_2^{-1}\Omega^q_Y) & \longrightarrow & H^{i+j}_{\Phi'\times\Psi}(R(\bar{h}\times\mathrm{id})_*(\omega^p_{\bar{X}\times Y})\otimes_{\mathcal{O}}\mathrm{pr}_2^*\Omega^q_Y) \\ \downarrow{\scriptstyle (\bar{h}\times\mathrm{id})_*\otimes\mathrm{id}} & & \downarrow{\scriptstyle (\bar{h}\times\mathrm{id})_*\otimes\mathrm{id}} \\ H^{i+j}_{\Phi'\times\Psi}(\omega^p_{X'\times Y}\otimes_k\mathrm{pr}_2^{-1}\Omega^q_Y) & \longrightarrow & H^{i+j}_{\Phi'\times\Psi}(\omega^p_{X'\times Y}\otimes_{\mathcal{O}}\mathrm{pr}_2^*\Omega^q_Y). \end{array}$$

For this take injective resolutions $\omega^p_{\bar{X}\times Y}\to I^\bullet$ and $\omega^p_{X'\times Y}\to J^\bullet$; then the push-forward is given by an actual morphism $(\bar{h}\times\mathrm{id})_*I^\bullet\to J^\bullet$. Now the commutativity of the first diagram is easily checked by taking an injective resolution of $\Omega^q_Y$. For the commutativity of the second we observe that

$$(\bar{h}\times\mathrm{id})_*I^\bullet\otimes_k\mathrm{pr}_2^{-1}\Omega^q_Y \quad\text{and}\quad (\bar{h}\times\mathrm{id})_*I^\bullet\otimes_{\mathcal{O}}\mathrm{pr}_2^*\Omega^q_Y$$

still represent

$$R(\bar{h}\times\mathrm{id})_*\omega^p_{\bar{X}\times Y}\otimes_k\mathrm{pr}_2^{-1}\Omega^q_Y \quad\text{and}\quad R(\bar{h}\times\mathrm{id})_*\omega^p_{\bar{X}\times Y}\otimes_{\mathcal{O}}\mathrm{pr}_2^*\Omega^q_Y,$$

respectively, and similarly with $(\bar{h}\times\mathrm{id})_*I^\bullet$ replaced by $J^\bullet$ and $(\bar{h}\times\mathrm{id})_*\omega^p_{\bar{X}\times Y}$ by $\omega^p_{X'\times Y}$. Thus it is enough to check the commutativity using these complexes, which is obvious. □

**2.5. *Summary.*** Let $(H_*, H^*, T, e)$ be the datum defined above, that is, we define $H_*: V_*\to\mathbf{GrAb}$ on objects by (2.0.2) and on morphisms by Definition 2.3.2, we define $H^*: (V^*)^{\mathrm{op}}\to\mathbf{GrAb}$ on objects by (2.0.1) and on morphisms by (2.1.9), and we define $T$ by (2.4.2) and $e$ by (2.0.3).

**Theorem 2.5.1.** *The datum $(H_*, H^*, T, e)$ is an object in* $\mathbf{T}$, *that is, it is a datum as in* 1.1.8 *and satisfies the properties* 1.1.9.

We denote by $HP$ the pure part of $H$, that is,

$$HP(X,\Phi) := \bigoplus_{n\geq 0} H^n_\Phi(X,\Omega^n_X)\quad\text{for } (X,\Phi)\in V_*, \tag{2.5.2}$$

and let $HP^*(X, \Phi)$ be the graded abelian group, which in degree $2n$ equals

$$HP^{2n}(X, \Phi) = H^n_\Phi(X, \Omega^n_X)$$

and is zero in odd degrees. The graded abelian group $HP_*(X, \Phi)$ is defined as in (2.0.2). Then the functors $H_*$ and $H^*$ induce functors $HP_*$ and $HP^*$, and $T$ and $e$ restrict to $HP$.

**Corollary 2.5.3.** *The datum* $(HP_*, HP^*, T, e)$ *is an object in* $\mathbf{T}$. *Furthermore* $HP$ *satisfies the semipurity condition from Definition 1.2.1 and the natural map* $(HP_*, HP^*, T, e) \to (H_*, H^*, T, e)$ *is a morphism in* $\mathbf{T}$.

*Proof.* The semipurity condition follows from [Grothendieck 1968, Exposé III, Proposition 3.3]. □

## 3. Cycle class map to Hodge cohomology and applications

*In this section k is assumed to be a perfect field (unless stated otherwise).*

### 3.1. *Cycle class.*

**Proposition 3.1.1** (cycle class). *Let* $X$ *be a smooth scheme and let* $W \subset X$ *be an irreducible closed subset of codimension* $c$. *There is a class* $cl_{(X,W)} = cl(W) \in H^c_W(X, \Omega^c_X)$ *with the property that*

$$H^*(\jmath)(cl(W)) = H_*(\imath_{U\cap W})(1)$$

*for every open subset* $U \subset X$ *such that* $U \cap W$ *is smooth* (*and nonempty*), *where* $\jmath : (U, W \cap U) \to (X, W)$ *and* $\imath : W \cap U \to (U, W \cap U)$ *are induced by the open and closed immersion*, *respectively*, *and* $1$ *is the identity element of the ring* $H^0(X, \mathcal{O}_X)$.

**Remark 3.1.2.** The cycle class in the proposition is Grothendieck's "fundamental class"; see for example [Lipman 1984, page 39, (ii)]. For the convenience of the reader and to be sure about the compatibility with the push-forward constructed in the previous section, we give a proof of the proposition, which is standard.

*Proof. 1st step*: Let $\eta$ be the generic point of $W$. We define

$$H^c_\eta(X, \Omega^c_X) := \varinjlim_{\eta\in U} H^c_{U\cap W}(U, \Omega^c_U),$$

where the inductive limit runs over all open sets $U \subset X$ with $\eta \in U$. Choose $U$ such that $U \cap W \neq \varnothing$ is smooth. The image of $H_*(\imath_{U\cap W})(1) \in H^c_{U\cap W}(U, \Omega^c_U)$ in $H^c_\eta(X, \Omega^c_X)$ doesn't depend on the choice of $U$ by Section 1.1.9(4). We denote this class by $cl(W)_\eta$.

*2nd step*: A class $a \in H^c_\eta(X, \Omega^c_X)$ is in the image of

$$H^c_W(X, \Omega^c_X) \to H^c_\eta(X, \Omega^c_X)$$

(that is, extends to a global class) if and only if for all 1-codimensional points $x$ in $W$ there is an open subset $U \subset X$ containing $x$, so that $a$ lies in the image of

$$H^c_{W\cap U}(U, \Omega^c_U) \to H^c_\eta(X, \Omega^c_X).$$

Indeed, the Cousin resolution yields an exact sequence

$$0 \to H^c_W(X, \Omega^c_X) \to H^c_\eta(X, \Omega^c_X) \to \bigoplus_{x\in W, cd(x)=1} H^{c+1}_x(X, \Omega^c_X), \tag{3.1.3}$$

and $H^c_{U\cap W}(U, \Omega^c_U) \to H^c_\eta(X, \Omega^c_X) \to H^{c+1}_x(X, \Omega^c_X)$ vanishes for all $x$ and $U$ as above.

*3rd step*: If $W$ is normal then $cl(W)_\eta$ extends (uniquely) to a class in $H^c_W(X, \Omega^c_X)$. Indeed, since $W$ is regular in codimension one and we assume that $k$ is perfect, we may choose an open $U \subset X$ such that $U \cap W$ is smooth and $U \cap W$ contains all points of codimension 1 of $W$. So that the class extends by the 2nd step. Note that the extension is unique because of the exact sequence (3.1.3).

*4th step*: We claim that the class $cl(W)_\eta$ extends to a class in $H^c_W(X, \Omega^c_X)$. In view of the 2nd step it is sufficient to extend the class at all points $x \in W$ of codimension 1. Thus we may assume that $X$ (and therefore $W$) is affine. The normalization $\tilde{W} \to W$ is a finite morphism and thus projective. Choose an embedding $\tilde{W} \to W \times_k \mathbb{P}^n_k$ over $W$. The previous step yields a class $cl(\tilde{W}) \in H^{n+c}_{\tilde{W}}(X \times \mathbb{P}^n, \Omega^{n+c}_{X\times\mathbb{P}^n})$. Consider $H_*(\mathrm{pr}_1)(cl(\tilde{W})) \in H^c_W(X, \Omega^c_X)$; for an open $U \subset X$ such that $W \cap U \neq \varnothing$ and $U \cap W$ is smooth, we obtain

$$\begin{aligned} H^*(\jmath)H_*(\mathrm{pr}_1)(cl(\tilde{W})) &= H_*(\mathrm{pr}_{1|U\times\mathbb{P}^n})H^*(\jmath')(cl(\tilde{W})) \\ &= H_*(\mathrm{pr}_{1|U\times\mathbb{P}^n})H_*(\imath_{(U\times\mathbb{P}^n)\cap\tilde{W}})(1) = H_*(\imath_{U\cap W})(1), \end{aligned}$$

with $\jmath' : (U \times \mathbb{P}^n, (U \times \mathbb{P}^n) \cap \tilde{W}) \to (X \times \mathbb{P}^n, \tilde{W})$. Thus $H_*(\mathrm{pr}_1)(cl(\tilde{W}))$ is the desired lift. □

**3.1.4.** *Explicit description of the cycle class.* Let $X$ be a smooth scheme and let $W \subset X$ be an irreducible closed subset of codimension $c$ with generic point $\eta \in X$. Denote $A = \mathcal{O}_{X,\eta}$. Then

$$H^c_\eta(X, \Omega^c_X) = \varinjlim_{f\subset\mathfrak{m}_\eta} \frac{\Omega^c_A}{(f)},$$

where the limit is over all $A$-sequences $f = (f_1, \dots, f_c)$ of length $c$ that are contained in $\mathfrak{m}_\eta$ (in particular $\sqrt{(f_1, \dots, f_c)} = \mathfrak{m}_\eta$). The class of $\omega \in \Omega^c_A$ under the composition in $\Omega^c_A \to \Omega^c_A/(f) \to H^c_\eta(X, \Omega^c_X)$ is denoted by $\left[\begin{smallmatrix}\omega\\ f\end{smallmatrix}\right]$. See Section A.1 for details.

Now let $U$ be an affine open subset of $X$ such that $U \cap W$ is smooth and the ideal of $W \cap U$ in $\mathcal{O}_U$ is generated by global sections $t_1, \ldots, t_c$ on $U$. Then by Proposition 2.2.19

$$cl(W)_\eta = (-1)^c \begin{bmatrix} dt_1 \cdots dt_c \\ t_1, \ldots, t_c \end{bmatrix}.$$

**Lemma 3.1.5.** *For a closed immersion $\imath : X \to Y$ between smooth schemes and an effective smooth divisor $D \subset Y$ such that*

- *$D$ meets $X$ properly, and thus $D \cap X := D \times_Y X$ is a divisor on $X$,*
- *$D' := (D \cap X)_{\mathrm{red}}$ is smooth and connected, and thus $D \cap X = n \cdot D'$ as divisors (for some $n \in \mathbb{Z}$ with $n \geq 1$),*

*we denote by $\imath_X : X \to (Y, X)$ and $\imath_{D'} : D' \to (D, D')$ the morphisms in $V_*$ induced by $\imath$, and we define $g_2 : (D, D') \to (Y, X)$ in $V^*$ by the inclusion $g : D \hookrightarrow Y$. Then the following equality holds:*

$$H^*(g_2)(H_*(\imath_X)(1_X)) = n \cdot H_*(\imath_{D'})(1_{D'}).$$

*Proof.* Let $c$ be the codimension of $X$ in $Y$ and $g_3 : (D, D') \to (Y, D')$ be induced by the inclusion $D \subset Y$. Then

$$H_*(g_3) : H^c_{D'}(D, \Omega^c_D) \to H^{c+1}_{D'}(Y, \Omega^{c+1}_Y)$$

is injective (by Lemma 2.3.8), and thus we need to prove

$$H_*(g_3) H^*(g_2) H_*(\imath_X)(1_X) = n \cdot H_*(g_3) H_*(\imath_{D'})(1_{D'}).$$

Let $g_1 : D \to (Y, D)$ be induced by $g$; then projection formula 1.1.16 gives

$$H_*(g_3) H^*(g_2)(H_*(\imath_X)(1_X)) = H_*(g_1)(1_D) \cup H_*(\imath_X)(1_X).$$

Therefore it suffices to prove

$$H_*(g_1)(1_D) \cup H_*(\imath_X)(1_X) = n \cdot H_*(g_3 \circ \imath_{D'})(1_{D'}). \tag{3.1.6}$$

Let $\eta$ be the generic point of $D'$. Since $H^{c+1}_Z(Y, \Omega^{c+1}_Y) = 0$ for all closed subsets $Z \subset Y$ of codimension $\geq c+2$, by [Grothendieck 1968, Exposé III, Proposition 3.3], the restriction map

$$H^{c+1}_{D'}(Y, \Omega^{c+1}_Y) \to H^{c+1}_\eta(Y, \Omega^{c+1}_Y) \tag{3.1.7}$$

is injective. Thus it is sufficient to prove the equality (3.1.6) in $H^{c+1}_\eta(Y, \Omega^{c+1}_Y)$.

Since $X$ is smooth we may find a regular sequence $t_1, \dots, t_c \in \mathcal{O}_{Y,\eta}$ that generates the ideal of $X$. If $D = \mathrm{div}(f)$ around $\eta$, then

$$(-1)\begin{bmatrix} df \\ f \end{bmatrix} \cup (-1)^c \begin{bmatrix} dt_1 \wedge \cdots \wedge dt_c \\ t_1, \dots, t_c \end{bmatrix} = (-1)^{c+1} \begin{bmatrix} df \wedge dt_1 \wedge \cdots \wedge dt_c \\ f, t_1, \dots, t_c \end{bmatrix}$$

is the image of $H_*(g_1)(1_D) \cup H_*(\imath_X)(1_X)$ in $H^{c+1}_\eta(Y, \Omega^{c+1}_Y)$.

Let $\pi \in \mathcal{O}_{Y,\eta}$ be a lift of a generator of the maximal ideal in $\mathcal{O}_{X,\eta}$. By the explicit description of the cycle class in Section 3.1.4 we get

$$H_*(g_3 \circ \imath_{D'})(1_{D'}) = (-1)^{c+1} \begin{bmatrix} d\pi \wedge dt_1 \wedge \cdots \wedge dt_c \\ \pi, t_1, \dots, t_c \end{bmatrix}.$$

Obviously $f = a\pi^n$ in $\mathcal{O}_{X,\eta}$ for a unit $a \in \mathcal{O}^*_{X,\eta}$. Choose a lift $\tilde{a} \in \mathcal{O}^*_{Y,\eta}$ of $a$; thus $f = \tilde{a}\pi^n$ modulo $(t_1, \dots, t_c)$, and we obtain

$$\begin{aligned} \begin{bmatrix} df \wedge dt_1 \wedge \cdots \wedge dt_c \\ f, t_1, \dots, t_c \end{bmatrix} &= \begin{bmatrix} n\tilde{a}\pi^{n-1} \cdot d\pi \wedge dt_1 \wedge \cdots \wedge dt_c \\ \tilde{a}\pi^n, t_1, \dots, t_c \end{bmatrix} \\ &= n \cdot \begin{bmatrix} d\pi \wedge dt_1 \wedge \cdots \wedge dt_c \\ \pi, t_1, \dots, t_c \end{bmatrix}, \end{aligned}$$

which proves (3.1.6). □

**Theorem 3.1.8.** *There exists a morphism* $cl : \mathrm{CH} \to H = (H_*, H^*, T, e)$ *in* **T**.

*Proof.* Since there is a morphism $HP = (HP_*, HP^*, T, e) \to H$ in **T**, it suffices to prove the existence of $cl : \mathrm{CH} \to HP$.

This follows from Theorem 1.2.3, since $HP$ satisfies all the conditions listed there: $HP$ is in **T** and satisfies the semipurity condition 1.2.1 by Corollary 2.5.3. It satisfies 1.2.3(1) by Proposition 2.3.3(3) and 1.2.3(3) by Lemma 3.1.5. Finally the element $cl_{(X,W)}$ from 1.2.3(4) is the cycle class constructed in Proposition 3.1.1 and 1.2.3(2) is obvious. □

### 3.2. *Main theorems.*

**3.2.1.** Let $f : (X, \Phi) \to (Y, \Psi)$ be a morphism in $V_*$ or $V^*$. By Theorem 3.1.8, Section 1.3.18 and Lemma 1.3.19, the morphisms

$$H_*(f) : H_*(X, \Phi) \to H_*(Y, \Psi) \quad \text{and} \quad H^*(f) : H^*(Y, \Psi) \to H^*(X, \Phi)$$

are respectively given by (we write $cl$ instead of $\mathrm{Cor}(cl)$)

$$H_*(f) = \rho_H \circ cl \circ \tau^{\mathrm{CH}}_*(f) \quad \text{and} \quad H^*(f) = \rho_H \circ cl \circ \tau^*_{\mathrm{CH}}(f).$$

Thus we may use composition of correspondences in $\mathrm{Cor}_{\mathrm{CH}}$ to compute $H^*(f) \circ H_*(f)$, $H_*(f) \circ H^*(f)$, etc.

**Proposition 3.2.2.** *Let $X$, $Y$ be smooth and connected, and let*

$$\alpha \in \mathrm{Hom}_{\mathrm{Cor}_{\mathrm{CH}}}(X, Y)^0 = CH^{\dim X}(X \times Y, P(\Phi_X, \Phi_Y)).$$

(1) *If the support of $\alpha$ projects to an $r$-codimensional subset in $Y$, then the restriction of $\rho_H \circ cl(\alpha)$ to $\bigoplus_{j<r,i} H^i(X, \Omega_X^j)$ vanishes.*

(2) *If the support of $\alpha$ projects to an $r$-codimensional subscheme in $X$, then the restriction of $\rho_H \circ cl(\alpha)$ to $\bigoplus_{j\geq \dim X - r+1,i} H^i(X, \Omega_X^j)$ vanishes.*

*Proof.* (1) We may assume $\alpha = [V]$ for $V \subset X \times Y$ a closed irreducible subset of dimension $\dim(Y) =: d_Y$, with $p_Y(V) \subset Y$ of codimension $r$. We set $d_X = \dim X$.

By definition of $\rho_H$ (see Section 1.3.18) and Lemma 2.3.4 it is sufficient to prove that for all $0 \leq q \leq r-1$ the image of the class $cl(V)$ vanishes via the map

$$H_V^{d_X}(\Omega_{X\times Y}^{d_X}) \xrightarrow{\text{proj.}} H_V^{d_X}(\mathrm{pr}_1^* \,\Omega_X^{d_X-q} \otimes \mathrm{pr}_2^* \,\Omega_Y^q).$$

To prove this we may also localize at the generic point $\eta$ of $V$ [Grothendieck 1968, Exposé III, Proposition 3.3].

We write $B = \mathbb{O}_{X\times Y,\eta}$ and $A = \mathbb{O}_{Y,p_Y(\eta)}$. Now $A$ is a regular local ring of dimension $r$ and $B$ is formally smooth over $A$. Let $t_1, \dots, t_r \in A$ be a regular system of parameters of $A$. Since $B/(1\otimes t_1, \dots, 1\otimes t_r)$ is a local regular ring there exist elements $s_{r+1}, \dots, s_{d_X} \in B$ such that $1\otimes t_1, \dots, 1\otimes t_r, s_{r+1}, \dots, s_{d_X}$ is a system of regular parameters for $B$. Thus by the explicit description of the cycle class in Section 3.1.4 we obtain

$$cl(V)_\eta = (-1)^{d_X} \begin{bmatrix} d(1\otimes t_1)\wedge\cdots\wedge d(1\otimes t_r)\wedge ds_{r+1}\wedge\cdots\wedge ds_{d_X} \\ 1\otimes t_1, \dots, 1\otimes t_r, s_{r+1}, \dots, s_{d_X} \end{bmatrix}.$$

This clearly implies the claim.

(2) Let $\alpha = [V]$ be as in (1) and suppose $p_X(V)$ has codimension $r$ in $X$. As above it suffices to prove that for all $0 \leq q \leq r-1$ the image of the class $cl(V)$ vanishes under the projection map

$$H_V^{d_X}(\Omega_{X\times Y}^{d_X}) \xrightarrow{\text{proj.}} H_V^{d_X}(\mathrm{pr}_1^* \,\Omega_X^{q} \otimes \mathrm{pr}_2^* \,\Omega_Y^{d_X-q}).$$

Write $C = \mathbb{O}_{X,p_X(\eta)}$. Then as in (1) we find $\tau_1, \dots, \tau_r \in C$ and $\sigma_{r+1}, \dots, \sigma_{d_X} \in B$, such that $\tau_1\otimes 1, \dots, \tau_r\otimes 1, \sigma_{r+1}, \dots, \sigma_{d_X}$ is a system of regular parameters for $B$. Thus

$$cl(V)_\eta = (-1)^{d_X} \begin{bmatrix} d(\tau_1\otimes 1)\wedge\cdots\wedge d(\tau_r\otimes 1)\wedge d\sigma_{r+1}\wedge\cdots\wedge d\sigma_{d_X} \\ \tau_1\otimes 1, \dots, \tau_r\otimes 1, \sigma_{r+1}, \dots, \sigma_{d_X} \end{bmatrix},$$

which implies the claim. □

**3.2.3.** Let $S$ be a $k$-scheme and let $f : X \to S$ and $g : Y \to S$ be two integral $S$-schemes that are smooth over $k$. Let $Z \subset X \times_S Y$ be a closed integral subscheme of dimension equal to the dimension of $Y$ and such that $\mathrm{pr}_2|_Z : Z \to Y$ is proper. For an open subset $U \subset S$, we denote by $Z_U \subset f^{-1}(U) \times_U g^{-1}(U)$ the pullback of $Z$ over $U$. This gives a correspondence $[Z_U] \in \mathrm{Hom}_{\mathrm{Cor}_{\mathrm{CH}}}(f^{-1}(U), g^{-1}(U))^0$, which induces a morphism of $k$-vector spaces

$$\rho_H \circ cl([Z_U]) : H^i(f^{-1}(U), \Omega^j_{f^{-1}(U)}) \to H^i(g^{-1}(U), \Omega^j_{g^{-1}(U)}) \quad \text{for all } i, j.$$

**Proposition 3.2.4.** *In the situation above, the set* $\{\rho_H \circ cl([Z_U]) \mid U \subset Z \text{ open}\}$ *induces a morphism of quasicoherent* $\mathcal{O}_S$*-modules*

$$\rho_H(Z/S) : R^i f_*\Omega^j_X \to R^i g_*\Omega^j_Y \quad \textit{for all } i, j.$$

*Proof.* We need to show the following statements:

(1) The maps $\rho_H \circ cl([Z_U])$ are compatible with restriction to open sets.

(2) The maps $\rho_H \circ cl([Z_U])$ are $\mathcal{O}(U)$-linear.

To show (1), let us denote by

$$\mathrm{pr}_{1,U} : f^{-1}(U) \times g^{-1}(U) \to f^{-1}(U) \quad \text{for } \mathrm{pr}_{1,U} \in V^*,$$
$$\mathrm{pr}_{2,U} : (f^{-1}(U) \times g^{-1}(U), P(\Phi_{f^{-1}(U)}, \Phi_{g^{-1}(U)}) \to g^{-1}(U) \quad \text{for } \mathrm{pr}_{2,U} \in V_*,$$

the morphism induced by the projections (see (1.1.2) and (1.3.8) for the definition of $P(\Phi_{f^{-1}(U)}, \Phi_{g^{-1}(U)})$). Let $j : V \hookrightarrow U$ be an open immersion and denote by

$$j_f : f^{-1}(V) \to f^{-1}(U) \quad \text{and} \qquad j_g : g^{-1}(V) \to g^{-1}(U)$$

the morphisms in $V^*$ induced by $j$.

We have to show that for all $a \in H^i(f^{-1}(U), \Omega^j_{f^{-1}(U)})$

$$H^*(j_g)H_*(\mathrm{pr}_{2,U})(H^*(\mathrm{pr}_{1,U})(a) \cup cl([Z_U])) \\ = H_*(\mathrm{pr}_{2,V})(H^*(\mathrm{pr}_{1,V})(H^*(j_f)(a) \cup cl([Z_V])). \quad (3.2.5)$$

As a first step from the left side to the right side we observe that

$$H^*(j_g)H_*(\mathrm{pr}_{2,U}) = H_*(\mathrm{pr}'_{2,V})H^*(\mathrm{id}_{f^{-1}(U)} \times j_g),$$

where $\mathrm{pr}'_{2,V} : (f^{-1}(U) \times g^{-1}(V), \Phi) \to g^{-1}(V)$ in $V_*$ is induced by the projection and $\Phi := (\mathrm{id} \times j_g)^{-1} P(\Phi_{f^{-1}(U)}, \Phi_{g^{-1}(U)})$. Denoting $\mathrm{pr}'_{1,U} : f^{-1}(U) \times g^{-1}(V) \to f^{-1}(U)$ as a morphism in $V^*$, we obtain the equality

$$H^*(\mathrm{id}_{f^{-1}(U)} \times j_g)(H^*(\mathrm{pr}_{1,U})(a) \cup cl([Z_U])) = H^*(\mathrm{pr}'_{1,U})(a) \cup cl([Z_V])$$

in $H(f^{-1}(U)\times g^{-1}(V),\Phi)$; here we consider $Z_V\in\Phi$ as a closed subset of $f^{-1}(U)\times g^{-1}(V)$. Next, consider the morphisms

$$\begin{aligned} j_f\times \mathrm{id}_{g^{-1}(V)} &: f^{-1}(V)\times g^{-1}(V)\to f^{-1}(U)\times g^{-1}(V),\\ \tau &: (f^{-1}(V)\times g^{-1}(V), Z_V)\to (f^{-1}(U)\times g^{-1}(V),\Phi),\\ \mathrm{id}' &: (f^{-1}(V)\times g^{-1}(V), Z_V)\to (f^{-1}(V)\times g^{-1}(V), P(\Phi_{f^{-1}(V)},\Phi_{g^{-1}(V)})),\end{aligned}$$

with

$$j_f\times \mathrm{id}_{g^{-1}(V)}\in V^*,\quad \tau\in V_*,\quad \mathrm{id}'\in V_*,$$

and where id′ is induced by the identity. The projection formula yields

$$\begin{aligned} H^*(\mathrm{pr}'_{1,U})(a)\cup cl([Z_V]) &= H^*(\mathrm{pr}'_{1,U})(a)\cup cl(\mathrm{CH}_*(\tau)([Z_V]))\\ &= H_*(\tau)(H^*(j_f\times\mathrm{id}_{g^{-1}(V)})H^*(\mathrm{pr}'_{1,U})(a)\cup cl([Z_V])).\end{aligned}$$

Now the equalities

$$\begin{aligned} H_*(\mathrm{pr}'_{2,V})H_*(\tau) &= H_*(\mathrm{pr}_{2,V})H_*(\mathrm{id}'),\\ H^*(j_f\times\mathrm{id}_{g^{-1}(V)})H^*(\mathrm{pr}'_{1,U}) &= H^*(\mathrm{pr}_{1,V})H^*(j_f),\end{aligned}$$

imply the claim (3.2.5).

For (2), it suffices to consider the case $U=S=\operatorname{Spec} R$. The ring homomorphisms $g^*:R\to H^0(X,\mathcal{O}_X)$ and $f^*:R\to H^0(Y,\mathcal{O}_Y)$ induce $R$-module structures on $H(X)$ and $H(Y)$, respectively.

We have to prove the following equality for all $r\in R$ and $a\in H^i(X,\Omega^j_X)$:

$$g^*(r)\cup H_*(\mathrm{pr}_2)(H^*(\mathrm{pr}_1)(a)\cup cl([Z])) = H_*(\mathrm{pr}_2)(H^*(\mathrm{pr}_1)(f^*(r)\cup a)\cup cl([Z])).$$

For this, it is enough to show that

$$\begin{aligned} &H^*(\mathrm{pr}_2)(g^*(r))\cup cl([Z])\\ &\qquad = H^*(\mathrm{pr}_1)(f^*(r))\cup cl([Z])\quad \text{in } H^d_Z(X\times Y,\Omega^d_{X\times Y}). \qquad (3.2.6)\end{aligned}$$

Choose an open set $U\subset X\times Y$ such that $Z\cap U$ is nonempty and smooth. Since the natural map $H^d_Z(X\times Y,\Omega^d_{X\times Y})\to H^d_{Z\cap U}(U,\Omega^d_U)$ is injective, it suffices to check (3.2.6) on $H^d_{Z\cap U}(U,\Omega^d_U)$. We write $\imath_1:Z\cap U\to(U,Z\cap U)$ in $V_*$ and $\imath_2:Z\cap U\to U$ in $V^*$ for the obvious morphisms. By using the projection formula and $cl([Z\cap U])=H_*(\imath_1)(1)$ we reduce to the statement

$$H^*(\imath_2)H^*(\mathrm{pr}_2)(g^*(r)) = H^*(\imath_2)H^*(\mathrm{pr}_1)(f^*(r)).$$

This follows from $g\circ\mathrm{pr}_2\circ\imath_2 = f\circ\mathrm{pr}_1\circ\imath_2$. □

**Definition 3.2.7.** Two integral schemes $X$ and $Y$ over a base $S$ are called properly birational over $S$ if there is an integral scheme $Z$ over $S$ and morphisms over $S$:

$$\begin{array}{ccc} & Z & \\ {}^{\text{proper, birational}}\swarrow & & \searrow^{\text{proper, birational}} \\ X & & Y. \end{array}$$

**Theorem 3.2.8.** *Let $S$ be a scheme over a perfect field $k$. Let $f : X \to S$ and $g : Y \to S$ be integral $S$-schemes, that are smooth over $k$ and properly birational over $S$. Let $Z$ be an integral scheme together with proper birational morphisms $Z \to X$ and $Z \to Y$ such that*

$$\begin{array}{ccc} & Z & \\ \swarrow & & \searrow \\ X & & Y \\ {}_{f}\searrow & & \swarrow_{g} \\ & S & \end{array}$$

*is commutative. We denote by $Z_0$ be the image of $Z$ in $X \times_S Y$. Then, for all $i$, $\rho_H(Z_0/S)$ induces isomorphisms of $\mathcal{O}_S$-modules* ($d = \dim X = \dim Y$)

$$R^i f_* \mathcal{O}_X \xrightarrow{\simeq} R^i g_* \mathcal{O}_Y \quad \textit{and} \quad R^i f_* \Omega^d_X \xrightarrow{\simeq} R^i g_* \Omega^d_Y.$$

*Proof.* Recall that $\rho_H(Z_0/S)$ is defined in Proposition 3.2.4 as the sheafication of the maps

$$\rho_H \circ cl([(Z_{0,U}]) : H^i(f^{-1}(U), \Omega^j_{f^{-1}(U)}) \to H^i(g^{-1}(U), \Omega^j_{g^{-1}(U)}), \qquad (3.2.9)$$

where $U$ runs over all open sets of $S$ and $Z_{0,U}$ denotes the restriction of $Z_0$ to $f^{-1}(U) \times_U g^{-1}(U)$. By Proposition 3.2.4, $\rho_H(Z_0/S)$ is a morphism of $\mathcal{O}_S$-modules.

Obviously, it is sufficient to prove that (3.2.9), for $j = 0$ and $j = d$, is an isomorphism for every open $U$. Thus we may suppose that $U = S$, $f^{-1}(U) = X$ and $g^{-1}(U) = Y$, $Z_{0,U} = Z$, and we need to prove that

$$\rho_H \circ cl([(Z_0]) : H^i(X, \mathcal{O}_X) \to H^i(Y \mathcal{O}_Y),$$
$$\rho_H \circ cl([(Z_0]) : H^i(X, \Omega^d_X) \to H^i(Y, \Omega^d_Y),$$

are isomorphisms for all $i$. In other words, we reduced to the case $S = \mathrm{Spec}(k)$.

Obviously, we may assume that $Z \subset X \times Y$. Let $Z' \subset Z$, $X' \subset X$ and $Y' \subset Y$ be nonempty open subsets such that $\mathrm{pr}_1^{-1}(X') = Z'$ and $\mathrm{pr}_2^{-1}(Y') = Z'$, and such that $\mathrm{pr}_1 : Z' \to X'$ and $\mathrm{pr}_2 : Z' \to Y'$ are isomorphisms.

We obtain a correspondence $[Z] \in \mathrm{Hom}_{\mathrm{Cor}_{\mathrm{CH}}}(X, Y)^0$, and we denote by $[Z^t] \in \mathrm{Hom}_{\mathrm{Cor}_{\mathrm{CH}}}(Y, X)^0$ the correspondence defined by $Z$ considered as subset of $Y \times X$.

We claim that

$$[Z]\circ[Z^t] = \Delta_Y + E_1 \quad \text{and} \quad [Z^t]\circ[Z] = \Delta_X + E_2,$$

with cycles $E_1$ and $E_2$, supported in $(Y \setminus Y') \times (Y \setminus Y')$ and $(X \setminus X') \times (X \setminus X')$, respectively. Indeed, in view of Lemma 1.3.4, $[Z^t]\circ[Z]$ is naturally supported in

$$\mathrm{supp}(Z, Z^t) = \{(x_1, x_2) \in X \times X \mid (x_1, y) \in Z,\ (y, x_2) \in Z^t \text{ for some } y \in Y\}.$$

By using Lemma 1.3.6 for the open $X' \subset X$, we conclude that $[Z^t]\circ[Z]$ maps to $[\Delta_{X'}]$ via the localization map

$$\mathrm{CH}(\mathrm{supp}(Z, Z^t)) \to \mathrm{CH}(\mathrm{supp}(Z, Z^t) \cap (X' \times X')).$$

Thus

$$[Z^t]\circ[Z] = \Delta_X + E_2$$

with $E_2$ supported in $\mathrm{supp}(Z, Z^t) \setminus (X' \times X')$. Finally, we observe that

$$\mathrm{supp}(Z, Z^t) \cap ((X' \times X) \cup (X \times X')) = \Delta_{X'} = \mathrm{supp}(Z, Z^t) \cap (X' \times X'),$$

and thus $E_2$ has support in $(X \times X) \setminus ((X' \times X) \cup (X \times X')) = (X \setminus X') \times (X \setminus X')$. The same argument works for $[Z]\circ[Z^t]$.

Now, Proposition 3.2.2 implies that $\rho_H \circ cl([Z])$ induce isomorphisms

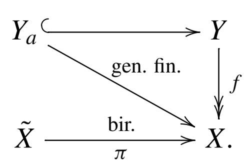
$$H^*(X, \mathcal{O}_X) \xrightarrow{\simeq} H^*(Y, \mathcal{O}_Y) \quad \text{and} \quad H^*(X, \Omega^d_X) \xrightarrow{\simeq} H^*(Y, \Omega^d_Y). \qquad \square$$

**Corollary 3.2.10.** *Let $k$ be an arbitrary field and let $f : X \to Y$ be a proper birational morphism between smooth schemes $X$ and $Y$. Then*

$$Rf_*(\mathcal{O}_X) = \mathcal{O}_Y \quad \textit{and} \quad Rf_*(\omega_X) = \omega_Y.$$

*Proof.* By base change we may assume that $k$ is algebraically closed. The claim follows from Theorem 3.2.8 for $S = Y$ and $X = Z$. $\square$

**3.2.11.** Consider a commutative diagram




Here, all morphisms are proper and all schemes are integral, $\tilde{X}$ and $Y$ are smooth of dimension $d_X$ and $d_Y$, $\pi$ is birational, $f$ is surjective, $Y_a \to X$ is generically finite and surjective, and $Y_a \hookrightarrow Y$ is a closed immersion. Let $\eta$ be the generic point of $X$; then $Y_a \times_X \eta$ is finite over $\mathrm{Spec}\, k(\eta)$ of degree $\deg(Y_a/X)$.

Choose a nonempty open set $U \subset X$ with $\pi : \pi^{-1}(U) \xrightarrow{\cong} U$ and such that $Y_a' := Y_a \cap f^{-1}(U) \subset f^{-1}(U) \xrightarrow{f} U$ is a finite morphism. Set

$$Z_a = \overline{Y_a' \times_U \pi^{-1}(U)} \subset Y_a \times_X \tilde{X},$$

which gives a morphism $[Z_a] : Y \to \tilde{X}$ in $\mathrm{Cor}_{\mathrm{CH}}$. Furthermore, set

$$\Gamma = \overline{\pi^{-1}(U) \times_U f^{-1}(U)} \subset \tilde{X} \times_X Y,$$

which defines an element $[\Gamma] : \tilde{X} \to Y$ in $\mathrm{Cor}_{\mathrm{CH}}$. By using Lemmas 1.3.4 and 1.3.6, we obtain

$$[Z_a] \circ [\Gamma] = \deg(Y_a/X) \cdot \mathrm{id}_{\tilde{X}} + E_1,$$

where $E_1$ has support in $\pi^{-1}(X \backslash U) \times \pi^{-1}(X \backslash U)$; thus $\rho_H \circ cl(E_1)$ acts trivially on $H^*(\tilde{X}, \mathcal{O}_{\tilde{X}}) \oplus H^*(\tilde{X}, \Omega_{\tilde{X}}^{d_X})$ by Proposition 3.2.2. On the other hand, Lemmas 1.3.4 and 1.3.6 imply

$$[\Gamma] \circ [Z_a] = \overline{Y_a' \times_U f^{-1}(U)} + E_2,$$

where $E_2$ has support in $f^{-1}(X \setminus U) \times f^{-1}(X \setminus U)$; thus $\rho_H \circ cl(E_2) = 0$ on $H^*(Y, \mathcal{O}_Y) \oplus H^*(Y, \Omega_Y^{d_Y})$. Moreover, by using Lemmas 1.3.4 and 1.3.6 again,

$$[\Gamma] \circ [Z_a] \circ [\Gamma] \circ [Z_a] = \deg(Y_a/X) \cdot \overline{Y_a' \times_U f^{-1}(U)} + E_3,$$

with a cycle $E_3$ supported in $f^{-1}(X \backslash U) \times f^{-1}(X \backslash U)$, and therefore $\rho_H \circ cl(E_3) = 0$ on $H^*(Y, \mathcal{O}_Y) \oplus H^*(Y, \Omega_Y^{d_Y})$.

We obtain an endomorphism

$$P(Y_a) := \rho_H \circ cl([\Gamma] \circ [Z_a])|_{H^*(Y, \mathcal{O}_Y) \oplus H^*(Y, \Omega_Y^{d_Y})}$$

of $H^*(Y, \mathcal{O}_Y) \oplus H^*(Y, \Omega_Y^{d_Y})$ such that $P(Y_a)^2 = \deg(Y_a/X) \cdot P(Y_a)$. Note that $P(Y_a)$ does not depend on $\tilde{X}$, because it is given by

$$P(Y_a) = \rho_H(cl([\overline{Y_a' \times_U f^{-1} U}])).$$

**Proposition 3.2.12.** *If* $\deg(Y_a/X)$ *is invertible in* $k$ *then*

$$\rho_H \circ cl(\Gamma) : H^*(\tilde{X}, \mathcal{O}_{\tilde{X}}) \oplus H^*(\tilde{X}, \Omega_{\tilde{X}}^{d_X}) \to image\big(P(Y_a)|_{H^*(Y, \mathcal{O}_Y) \oplus H^*(Y, \Omega_Y^{d_Y})}\big)$$

*is a well-defined isomorphism.*

*Proof.* Indeed

$$((\rho_H \circ cl(Z_a)) \circ (\rho_H \circ cl(\Gamma)))|_{H^*(\tilde{X}, \mathcal{O}_{\tilde{X}}) \oplus H^*(\tilde{X}, \Omega_{\tilde{X}}^{d_X})}$$

is multiplication by $\deg(Y_a/X)$. It follows that $\rho_H \circ cl(\Gamma)$ is injective and the image is contained in the image of $P(Y_a)$. The opposite inclusion is obvious. □

**Corollary 3.2.13.** *Let $Y$, $\tilde{X}$ and $X$ be as in Section 3.2.11. Let $a \in Y$ be a closed point of the generic fiber $f^{-1}(\eta)$ with $\deg_{k(\eta)}(a) \in k^*$; we denote the corresponding closed subvariety by $Y_a$. For $i \geq 0$, the following are equivalent*:

(1) $R^i\pi_*(\mathcal{O}_{\tilde{X}}) \oplus R^i\pi_*(\Omega^{d_X}_{\tilde{X}}) = 0.$

(2) $P(Y_a \cap f^{-1}(X'))$ *vanishes on*

$$H^i(f^{-1}(X'), \mathcal{O}_{f^{-1}(X')}) \oplus H^i(f^{-1}(X'), \Omega^{d_X}_{f^{-1}(X')})$$

*for every affine open subset $X' \subset X$.*

*Proof.* In view of Proposition 3.2.12 we get

$$H^i(\pi^{-1}(X'), \mathcal{O}_{\pi^{-1}(X')}) = \operatorname{image}\big(P(Y_a \cap f^{-1}(X'))|_{H^i(f^{-1}(X'),\mathcal{O}_{f^{-1}(X')})}\big)$$

and

$$H^i(\pi^{-1}(X'), \Omega^{d_X}_{\pi^{-1}(X')}) = \operatorname{image}\big(P(Y_a \cap f^{-1}(X'))|_{H^i(f^{-1}(X'),\Omega^{d_Y}_{f^{-1}(X')})}\big)$$

for every open subset $X' \subset X$. $\square$

**Theorem 3.2.14.** *Let $k$ be an arbitrary field. Consider*

$$\begin{array}{ccc} & & Y \\ & & \downarrow{\scriptstyle f} \\ \tilde{X} & \xrightarrow{\pi} & X, \end{array}$$

*where $Y$ and $\tilde{X}$ are smooth and connected, $X$ is integral and normal, $f$ is surjective and finite with $\deg(f) \in k^*$, and finally $\pi$ is birational and proper. Then $X$ is Cohen–Macaulay and*

$$R\pi_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X \quad \text{and} \quad R\pi_*\omega_{\tilde{X}} = \omega_X,$$

*where $\omega_X$ is the dualizing sheaf of $X$.*

*Proof.* Choose an algebraic closure $\bar{k}$ of $k$. We claim that $X$ is geometrically normal, that is,

$$X \times_k \bar{k} = \coprod_{i=1}^{r} X_i \quad \text{(disjoint union)} \tag{3.2.15}$$

with $X_i$ integral and normal for all $i$. Indeed, since $X$ is normal we obtain

$$\mathcal{O}_X \xrightarrow{=} \pi_*\mathcal{O}_{\tilde{X}},$$

and this isomorphism is stable under the base change to $\bar{k}$. Because $\tilde{X}$ is smooth, $\tilde{X} \times_k \bar{k}$ is a disjoint union of smooth schemes

$$\tilde{X} \times_k \bar{k} = \coprod_{i=1}^{r} \tilde{X}_i.$$

From $\mathcal{O}_{X\times_k\bar{k}} \xrightarrow{=} \pi_*\mathcal{O}_{\tilde{X}\times_k\bar{k}}$ we conclude that $\pi \times_k \bar{k}$ has connected fibres; thus we obtain the equality (3.2.15) with $X_i := (\pi \times_k \bar{k})(\tilde{X}_i)$. Of course, $\tilde{X}_i \to X_i$ is birational, and $\mathcal{O}_{X_i} \xrightarrow{=} (\pi \times_k \bar{k})_*\mathcal{O}_{\tilde{X}_i}$ implies that $X_i$ is normal.

We denote by $Y_{\bar{k}}$, $X_{\bar{k}}$, $\tilde{X}_{\bar{k}}$ the base change to $\bar{k}$, and by $\sigma_X : X_{\bar{k}} \to X$ the obvious morphism. Since

$$\sigma_X^* R^i\pi_*\mathcal{O}_{\tilde{X}} = R^i\pi_{\bar{k}*}\mathcal{O}_{\tilde{X}_{\bar{k}}}, \quad \text{and} \quad \sigma_X^* R^i\pi_*\omega_{\tilde{X}} = R^i\pi_{\bar{k}*}\omega_{\tilde{X}_{\bar{k}}},$$

and $\sigma_X$ is faithfully flat, it is sufficient to prove

$$R^i\pi_{\bar{k}*}\mathcal{O}_{\tilde{X}_{\bar{k}}} = 0 = R^i\pi_{\bar{k}*}\omega_{\tilde{X}_{\bar{k}}} \quad \text{for all } i > 0.$$

Now $\tilde{X} \times_k \bar{k} = \coprod_{i=1}^{r} \tilde{X}_i$ with $\tilde{X}_i$ smooth and connected such that $\pi_{\bar{k}}|_{\tilde{X}_i} : \tilde{X}_i \to X_i$ is birational. We define $Y_i := Y_{\bar{k}} \cap f_{\bar{k}}^{-1}(X_i)$ and let $Y_i = \coprod_j Y_{i,j}$ be the decomposition into connected (smooth) components. Since $\deg(Y_i/X_i) \in k^*$ is invertible, there exists $j$ such that $\deg(Y_{i,j}/X_i) \in k^*$. Thus we are reduced to proving the claim for an algebraically closed field $k$.

Since $f$ is affine, the statement $R\pi_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$ follows from Corollary 3.2.13 and $X$ normal. Applying $D_X$ and shifting by $[-d_X]$ (with $d_X = \dim X$), we obtain $R\pi_*\Omega^{d_X}_{\tilde{X}} = \pi^!_X k[-d_X]$ (with $\pi_X : X \to \operatorname{Spec} k$ the structure map). Now again by Corollary 3.2.13, we obtain $R^i\pi_*\Omega^{d_X}_{\tilde{X}} = 0$ for all $i \neq 0$. Thus $\pi^!_X k[-d_X] = \omega_X$ is a sheaf and hence $X$ is Cohen–Macaulay. □

## 4. Generalization to tame quotients

The goal of this section is to generalize Theorem 3.2.8 by replacing the assumption on the smoothness of $X$ and $Y$ with the weaker assumption that $X$ and $Y$ are tame quotients (see Definition 4.2.6). We already proved in Theorem 3.2.14 that the cohomology of the structure sheaf and the dualizing sheaf of a tame quotient behaves like for smooth schemes. Therefore it is a natural question to extend Theorem 3.2.8 in order to include tame quotients.

**4.1.** ***The action of finite correspondences.*** Let $X$ be a smooth scheme and $Z$ a closed integral subscheme of pure codimension $c$. Then $\mathcal{H}^j_Z(\Omega^c_X) = 0$ for $j < c$. Consequently, there is a natural morphism

$$\mathcal{H}^c_Z(\Omega^c_X)[-c] \to R\underline{\Gamma}_Z\Omega^c_X, \tag{4.1.1}$$

which induces an isomorphism $H^c_Z(X, \Omega^c_X) = H^0(X, \mathcal{H}^c_Z(\Omega^c_X))$.

**Definition 4.1.2.** Let $f : X \to Y$ be a morphism between smooth $k$-schemes of pure dimension $d_X$ and $d_Y$, respectively. Let $Z \subset X$ be a $c := d_X - d_Y$ codimensional integral subscheme such that the restriction of $f$ to $Z$ is *finite*. Then we define for $q \geq 0$ the local push-forward

$$f_{Z*} : f_* \mathcal{H}^c_Z(\Omega^q_X) \to \Omega^{q-c}_Y$$

in the following way: Choose a compactification of $f$, that is, a proper morphism $\bar{f} : \bar{X} \to Y$ and an open immersion $j : X \hookrightarrow \bar{X}$ such that $f = \bar{f} \circ j$, and then define $f_{Z*}$ as the composition in $D^+_{\mathrm{qc}}(Y)$ of the natural map

$$f_* \mathcal{H}^c_Z(\Omega^q_X) \xrightarrow{(4.1.1)} Rf_*(R\underline{\Gamma}_Z(\Omega^q_X)[c])$$

with

$$\begin{aligned} Rf_*(R\underline{\Gamma}_Z(\Omega^q_X)[c]) &\xrightarrow[\text{+excision}]{(2.2.15)} R\bar{f}_* R\underline{\Gamma}_Z(D_{\bar{X}}(\Omega^{d_X-q}_{\bar{X}})[c - d_X]) \\ &\xrightarrow{\text{forget support}} R\bar{f}_* D_{\bar{X}}(\Omega^{d_X-q}_{\bar{X}})[-d_Y] \\ &\xrightarrow{\bar{f}_*} D_Y(\Omega^{d_X-q}_Y)[-d_Y] \xrightarrow{(2.2.15)} \Omega^{q-c}_Y, \end{aligned} \tag{4.1.3}$$

where $\bar{f}_*$ is the morphism from Definition 2.2.5.

Applying $H^0(X, \cdot)$ gives a morphism

$$H^0(Y, f_*\mathcal{H}^c_Z(\Omega^q_X)) = H^0(X, \mathcal{H}^c_Z(\Omega_X)) = H^c_Z(X, \Omega^q_X) \to H^0(Y, \Omega^{q-c}_Y),$$

which by the definition coincides with the cohomological degree zero part of the pushforward for $f : (X, Z) \to Y$; see Definition 2.3.2. This also implies that $f_{Z*}$ is independent of the chosen compactification.

**Definition 4.1.4.** Let $S$ be a $k$-scheme and let $f : X \to S$ and $g : Y \to S$ be two integral $S$-schemes, which are smooth over $k$. Let $Z \subset X \times_S Y$ be a closed integral subscheme such that $\mathrm{pr}_2|_Z : Z \to Y$ is *finite and surjective*. In particular the codimension of $Z$ in $X \times Y$ equals $\dim X := c$. The projections from $X \times Y$ to $X$ and $Y$ are denoted by $\mathrm{pr}_1$ and $\mathrm{pr}_2$, respectively. For all $q \geq 0$ we define a morphism

$$\varphi^q_Z : f_*\Omega^q_X \to g_*\Omega^q_Y$$

as follows: Let

$$cl(Z) \in H^c_Z(X \times Y, \Omega^c_{X\times Y}) = H^0(X \times Y, \mathcal{H}^c_Z(\Omega^c_{X\times Y}))$$

be the cycle class of $Z$. The cup product with $cl(Z)$ yields a morphism

$$\cup\, cl(Z) : \Omega^q_{X\times Y} \to \mathcal{H}^c_Z(\Omega^{c+q}_{X\times Y}) \tag{4.1.5}$$

and hence a morphism $f_* \operatorname{pr}_{1*} \Omega^q_{X\times Y} \to f_* \operatorname{pr}_{1*} \mathcal{H}^c_Z(\Omega^{c+q}_{X\times Y})$. We claim that it also induces a morphism of $\mathcal{O}_S$-modules

$$f_* \operatorname{pr}_{1*} \Omega^q_{X\times Y} \to g_* \operatorname{pr}_{2*} \mathcal{H}^c_Z(\Omega^{q+c}_{X\times Y}). \tag{4.1.6}$$

Indeed since $\mathcal{H}^c_Z(\Omega^{q+c}_{X\times Y})$ has support in $Z \subset X \times_S Y$, the two abelian sheaves

$$g_* \operatorname{pr}_{2*} \mathcal{H}^c_Z(\Omega^{q+c}_{X\times Y}) \quad \text{and} \quad f_* \operatorname{pr}_{1*} \mathcal{H}^c_Z(\Omega^{q+c}_{X\times Y})$$

are equal; we denote this abelian sheaf on $S$ by $\mathcal{A}$. Now there are two $\mathcal{O}_S$-module structures on $\mathcal{A}$: They are induced by

$$\mathcal{O}_S \xrightarrow{g^*} g_*\mathcal{O}_Y \xrightarrow{\operatorname{pr}_2^*} g_* \operatorname{pr}_{2*} \mathcal{O}_{X\times Y} \quad \text{and} \quad \mathcal{O}_S \xrightarrow{f^*} f_*\mathcal{O}_X \xrightarrow{\operatorname{pr}_1^*} f_* \operatorname{pr}_{1*} \mathcal{O}_{X\times Y}.$$

The claim (4.1.6) is now a consequence of the following equality in $\mathcal{A}$:

$$\operatorname{pr}_2^* g^*(a)\cdot(\beta \cup cl(Z)) = \operatorname{pr}_1^* f^*(a)\cdot(\beta\cup cl(Z))$$

for all $a \in \mathcal{O}_S$, $\beta \in f_* \operatorname{pr}_{1*} \Omega^q_{X\times cl(Z))Y}$, which holds by (3.2.6). We can then define the morphism $\varphi^q_Z$ as the composition

$$f_*\Omega^q_X \xrightarrow{\operatorname{pr}_1^*} f_* \operatorname{pr}_{1*} \Omega^q_{X\times Y} \xrightarrow{(4.1.6)} g_* \operatorname{pr}_{2*} \mathcal{H}^c_Z(\Omega^{c+q}_{X\times Y}) \xrightarrow{\operatorname{pr}_{2,Z*}} g_*\Omega^q_Y. \tag{4.1.7}$$

We write $\varphi_Z = \bigoplus_q \varphi^q_Z$.

Let $\alpha = \sum_i n_i[Z_i]$ be a formal sum of integral closed subschemes $Z_i$ of $X\times_S Y$, which are finite and surjective over $Y$, with coefficients $n_i$ in $\mathbb{Z}$. Then we define

$$\varphi_\alpha := \sum_i n_i \varphi_{Z_i} : \bigoplus_q f_*\Omega^q_X \to \bigoplus_q g_*\Omega^q_Y. \tag{4.1.8}$$

**Lemma 4.1.9.** *In the situation above, assume additionally that $g$ is affine. Then for any cycle $\alpha = \sum_i n_i[Z_i]$, with $Z_i \subset X\times_S Y$ integral closed subschemes, which are finite and surjective over $Y$, and $n_i \in \mathbb{Z}$, we have the equality*

$$\oplus_i H^i(S,\varphi_\alpha) = \rho_H(cl(\bar\alpha)) : \bigoplus_{i,j} H^i(X,\Omega^j_X) \to \bigoplus_{i,j} H^i(Y,\Omega^j_Y),$$

*where $\bar\alpha$ is the image of $\alpha$ in $\mathrm{CH}_{\dim Y}(X\times Y, P(\Phi_X,\Phi_Y))$ with $P(\Phi_X,\Phi_X)$ as in (1.3.8), $\rho_H$ is defined in 1.3.18 and $cl$ is a shorthand notation for* Cor$(cl)$ *with $cl : CH \to H$ the morphism from 3.1.8.*

*Proof.* Let $\pi : S \to \operatorname{Spec} k$ be the structure map. We may assume $\alpha = [Z]$ with $Z \subset X\times_S Y$ an integral closed subscheme, which is finite and surjective over $Y$. It is easy to see that $\rho_H(cl(\bar{\alpha}))$ is induced by taking the cohomology of the composition

$$
\begin{aligned}
R\pi_* f_*\Omega^q_X &\xrightarrow{\mathrm{pr}_1^*} R\pi_* R(f\,\mathrm{pr}_1)_*\Omega^q_{X\times Y} &\quad (4.1.10)\\
&\xrightarrow{\cup cl(Z)} R\pi_* R(f\,\mathrm{pr}_1)_*\mathcal{H}^c_Z(\Omega^{q+c}_{X\times Y})\\
&\xrightarrow{Z\subset X\times_S Y} R\pi_* R(g\,\mathrm{pr}_2)_*\mathcal{H}^c_Z(\Omega^{q+c}_{X\times Y})\\
&\xrightarrow{\simeq} R\pi_*(g\,\mathrm{pr}_2)_*\mathcal{H}^c_Z(\Omega^{q+c}_{X\times Y}) &\quad (4.1.11)\\
&\xrightarrow{(4.1.1)} R\pi_* R(g\,\mathrm{pr}_2)_* R\underline{\Gamma}_Z(\Omega^{q+c}_{X\times Y})[c]\\
&\xrightarrow{(4.1.3)} R\pi_* g_*\Omega^q_Y. &\quad (4.1.12)
\end{aligned}
$$

We used for the fourth arrow the isomorphism

$$(g\,\mathrm{pr}_2)_*\mathcal{H}^c_Z(\Omega^{q+c}_{X\times Y}) \xrightarrow{\simeq} R(g\,\mathrm{pr}_2)_*\mathcal{H}^c_Z(\Omega^{q+c}_{X\times Y}),$$

because $\mathcal{H}^c_Z(\Omega^{q+c}_{X\times Y})$ is a quasicoherent $\mathcal{O}_{X\times Y}$-module with support in $Z$, the map $Z\to Y$ is finite, and $g: Y\to S$ is affine. For the third arrow, notice that there is no map $R(f\,\mathrm{pr}_1)_*\mathcal{H}^c_Z \to R(g\,\mathrm{pr}_2)_*\mathcal{H}^c_Z$ in the derived category of $\mathcal{O}_S$-modules, but in the derived category of sheaves of $k$-vector spaces on $S$ these two complexes are isomorphic and that's all we need to define the third arrow.

We have to compare the morphism from (4.1.10) to (4.1.12) with $R\pi_*\varphi_Z$, where $\varphi_Z$ is defined in (4.1.7). Obviously, the morphism from (4.1.10) to (4.1.11) is equal to $R\pi_*((4.1.6)\circ\mathrm{pr}_1^*)$. The morphism from (4.1.11) to (4.1.12) equals $R\pi_*(\mathrm{pr}_{2,Z*})$, which proves the claim. □

## 4.2. *Tame quotients.*

**4.2.1.** Let $X$ be a $k$-scheme that is normal, Cohen–Macaulay (CM) and equidimensional of pure dimension $n$ and denote by $\pi: X\to \operatorname{Spec} k$ its structure map.

Then $H^i(\pi^! k)=0$ for all $i\neq -n$; see [Conrad 2000, Theorem 3.5.1]. The *dualizing sheaf of* $X$ is then by definition

$$\omega_X := H^{-n}(\pi^! k).$$

We list some well-known properties:

(1) $\omega_X[n]$ is canonically isomorphic to $\pi^! k$ in $D^+_c(X)$.

(2) $\omega_X$ is a dualizing complex on $X$, that is, $\omega_X$ is coherent, has finite injective dimension and the natural map $\mathcal{O}_X \to R\,\mathcal{H}\mathrm{om}(\omega_X,\omega_X)$ is an isomorphism. (Indeed by [Hartshorne 1966, Chapter V, 10.1 and 10.2], $\pi^! k$ is a dualizing complex.)

(3) $\omega_X$ is CM with respect to the codimension filtration on $X$, that is,

$$\operatorname{depth}_{\mathcal{O}_{X,x}} \omega_{X,x} = \dim \mathcal{O}_{X,x} \quad \text{for all } x\in X,$$

(By [Hartshorne 1966, Chapter V, Proposition 7.3] $\omega_X$ is Gorenstein, in particular CM, with respect to its associated filtration. Therefore we have to show that the associated codimension function to $\omega_X$ [Hartshorne 1966, V.7] is the usual codimension function. By [Hartshorne 1966, Chapter V, Proposition 7.1] it suffices to show that $\mathrm{Ext}^0_{\mathcal{O}_{X,\eta}}(k(\eta), \omega_{X,\eta}) \neq 0$ for all generic points $\eta \in X$. But $X$ is normal and thus $\omega_{X,\eta} \cong k(\eta)$.

(4) In case $X$ is smooth, $\omega_X$ is canonically isomorphic to $\Omega^n_X$, via the isomorphism

$$\omega_X \cong \pi^! k[-n] \xrightarrow{e_\pi, \simeq} \pi^\# k[-n] = \Omega^n_X,$$

where $e_\pi : \pi^! \cong \pi^\#$ is the isomorphism from [Conrad 2000, (3.3.21)].

(5) If $u : U \to X$ is étale, then $u^*\omega_X$ is canonically isomorphic to $\omega_U$ via the isomorphism

$$u^*\omega_X = u^\#\omega_X \xrightarrow{e_u, \simeq} u^!\omega_X \cong u^!\pi^! k \xrightarrow{c^{-1}_{u,\pi}, \simeq} (\pi \circ u)^! k \cong \omega_U,$$

where $c_{u,\pi} : (\pi \circ u)^! \cong u^! \circ \pi^!$ is the isomorphism from [Conrad 2000, 3.3.14].

(6) Let $U$ be an open subscheme of $X$ that is smooth over $k$ and contains all 1-codimensional points and denote by $j : U \hookrightarrow X$ the corresponding open immersion. Then adjunction induces an isomorphism

$$\omega_X \xrightarrow{\simeq} j_* j^* \omega_X \cong j_*\omega_U \cong j_*\Omega^n_U,$$

where the last two isomorphisms are induced by (4) and (5). (This follows from (3). Indeed, let $V \subset X$ be open; then $\Gamma(V, \omega_X) \to \Gamma(V, j_* j^*\omega_X) = \Gamma(V \cap U, \omega_X)$ is the restriction. Since all points in the complement of $U$ have codimension $\geq 2$, we obtain from (3) that $\mathrm{depth}(\omega_{X,x}) \geq 2$ for all $x \in X \setminus U$. Therefore $\Gamma(V, \omega_X) \to \Gamma(V \cap U, \omega_X)$ is bijective by [Grothendieck 1968, Exposé III, Corollary 3.5].)

**4.2.2.** Let $X$ be smooth and $Y$ a normal CM scheme both of pure dimension $n$, and let $f : X \to Y$ be a finite and surjective morphism. Then we have the usual pull-back on the structure sheaves $f^* : \mathcal{O}_Y \to f_*\mathcal{O}_X$ as well as a trace map $\tau^0_f : f_*\mathcal{O}_X \to \mathcal{O}_Y$, which extends the usual trace over the smooth locus of $Y$ (over which $f$ is flat). We define a pull-back and a trace between the dualizing sheaves as follows.

**Definition 4.2.3.** Let $X$ be smooth and let $Y$ be a normal CM scheme, both of pure dimension $n$, and let $f : X \to Y$ be a finite and surjective morphism.

(1) We define a pullback morphism

$$f^* : \omega_Y \to f_*\omega_X$$

as follows: Choose $j : U \hookrightarrow Y$ open and smooth over $k$ such that it contains all 1-codimensional points of $Y$; let $j' : U' = X \times_Y U \to X$ and $f' : U' \to U$ be the base changes of $j$ and $f$. Then we define $f^*$ as the composition

$$\omega_Y \simeq j_*\Omega^n_U \xrightarrow{f'^*} j_* f'_* \Omega^n_{U'} = f_* j'_* \omega_{U'} \cong f_*\omega_X;$$

for the last isomorphism observe that $U'$ contains all 1-codimensional points of $X$. It is straightforward to check that this morphism is independent of the choice of $U$. (One only needs the compatibility statements (VAR1) and (VAR3) of [Hartshorne 1966, Chapter VII, Corollary 3.4(a)].)

(2) We define the trace

$$\tau^n_f : f_*\omega_X \to \omega_Y$$

as the composition in $D^+_c(Y)$

$$f_*\omega_X \cong f_*\pi^!_X k[-n] \xrightarrow{c_{f,\pi_Y}} f_* f^! \pi^!_Y k[-n] \xrightarrow{\mathrm{Tr}_f} \pi^!_Y k[-n] \cong \omega_Y,$$

where $\pi_X$ and $\pi_Y$ are the structure maps of $X$ and $Y$ and $\mathrm{Tr}_f$ is the trace morphism [Conrad 2000, (3.3.2)].

We write $f^* : \mathcal{O}_Y \oplus \omega_Y \to f_*(\mathcal{O}_X \oplus \omega_X)$ for the sum of the usual pull-back with the pull-back defined in (1), and write $\tau_f := \tau^0_f \oplus \tau^n_f : f_*(\mathcal{O}_X \oplus \omega_X) \to \mathcal{O}_Y \oplus \omega_Y$.

**Remark 4.2.4.** By its definition, the $\tau_f$ constructed above equals, when restricted to the smooth locus of $Y$, the $\tau_f$ from Proposition 2.2.23.

**Corollary 4.2.5.** *Let $X$, $Y$ and $f$ be as in Definition 4.2.3. Suppose that $X$ is connected. Then the composition*

$$\tau_f \circ f^* : \mathcal{O}_Y \oplus \omega_Y \to \mathcal{O}_Y \oplus \omega_Y$$

*is equal to multiplication with the degree of $f$.*

*Proof.* We have to check that the section $s = \tau_f \circ f^* - \deg f$ of

$$H^0(Y, \mathcal{H}om_Y(\mathcal{O}_Y, \mathcal{O}_Y)) \oplus H^0(Y, \mathcal{H}om_Y(\omega_Y, \omega_Y))$$

is zero. But $H^0(Y, \mathcal{H}om_Y(\mathcal{O}_Y, \mathcal{O}_Y)) = H^0(Y, \mathcal{O}_Y) = H^0(Y, \mathcal{H}om_Y(\omega_Y, \omega_Y))$ (for the last equality we need that $\omega_Y$ is a dualizing complex). Therefore it is enough to check that $s$ is zero over an open and dense subset $U$ of $Y$. We may choose $U$ such that it is smooth and contains all 1-codimensional points of $Y$. Thus the statement follows from Proposition 2.2.23(3). □

**Definition 4.2.6.** Let $X$ be a $k$-scheme. We say that $X$ is a *tame quotient* if $X$ is integral and normal and there exists a smooth and integral scheme $X'$ with a finite and surjective morphism $f : X' \to X$ whose degree is invertible in $k$.

**Remark 4.2.7.** Assume $X$ is a tame quotient. Then $X$ is CM; see [Kollár and Mori 1998, Proposition 5.7(1)].

We may describe the cohomology of the structure sheaf and of the dualizing sheaf of a tame quotient as a direct summand of the corresponding cohomology of a smooth scheme as follows.

**Proposition 4.2.8.** *Let $f : X \to Y$ be a finite and surjective morphism between integral schemes. Assume $X$ is smooth and $Y$ is normal. Furthermore, we assume that* $\deg f$ *is invertible in $k$. Set*

$$\alpha := [X \times_Y X] \quad \textit{in } \mathrm{CH}_{\dim X}(X \times X, P(\Phi_X, \Phi_X)) =: \mathrm{Hom}_{\mathrm{Cor}_{\mathrm{CH}}}(X, X)^0,$$

(*see (1.3.8) for the definition of $P(\Phi_X, \Phi_X)$*). *Then, for all $i$, the pull-back morphism*

$$f^* : (H^i(Y, \mathcal{O}_Y) \oplus H^i(Y, \omega_Y)) \to (H^i(X, \mathcal{O}_X) \oplus H^i(X, \omega_X))$$

*induces an isomorphism*

$$(H^i(Y, \mathcal{O}_Y) \oplus H^i(Y, \omega_Y)) \cong \rho_H(cl(\alpha))(H^i(X, \mathcal{O}_X) \oplus H^i(X, \omega_X)).$$

(*The functor $\rho_H$ is defined in 1.3.18 and $cl$ is a shorthand notation for* $\mathrm{Cor}(cl)$ *with $cl : CH \to H$ the morphism from Theorem 3.1.8.*)

*Proof.* Write $\alpha = [X \times_Y X] = \sum_T n_T[T]$, where the sum is over all irreducible components $T$ of $X \times_Y X$. Notice that all the $T$ have dimension equal to $\dim X$ and project (via both projections) finitely and surjectively to $X$. Therefore

$$\varphi_\alpha : f_*(\mathcal{O}_X \oplus \omega_X) \to f_*(\mathcal{O}_X \oplus \omega_X)$$

is defined, where $\varphi_\alpha$ is the morphism from Definition 4.1.4. By Lemma 4.1.9 we have, for all $i$,

$$H^i(Y, \varphi_\alpha) = \rho_H \circ cl(\alpha) : H^i(X, \mathcal{O}_X \oplus \omega_X) \to H^i(X, \mathcal{O}_X \oplus \omega_X). \tag{4.2.9}$$

We claim

$$\varphi_\alpha = f^* \circ \tau_f : f_*(\mathcal{O}_X \oplus \omega_X) \to f_*(\mathcal{O}_X \oplus \omega_X). \tag{4.2.10}$$

Let $U \subset Y$ be a nonempty smooth open subscheme that contains all 1-codimensional points of $Y$. Then $f^{-1}(U)$ is smooth and contains all 1-codimensional points of $X$. Hence for any open $V \subset Y$, the restriction map

$$H^0(f^{-1}(V), \mathcal{O}_X \oplus \omega_X) \to H^0(f^{-1}(V) \cap f^{-1}(U), \mathcal{O}_X \oplus \omega_X)$$

is an isomorphism; see [Grothendieck 1968, Exposé III, Corollary 3.5]. Since both maps in (4.2.10) are compatible with restriction to open subsets of $Y$, we may therefore assume that $Y$ is smooth. In particular $f$ is flat and thus $\alpha$ equals $[\Gamma_f^t] \circ [\Gamma_f]$, where $\Gamma_f$ is the graph of $f$ and $\Gamma_f^t$ its transposed. Now the identity

(4.2.10) follows from (4.2.9) (in the case $i = 0$), Proposition 2.3.3(3) and 3.2.1. Thus applying again (4.2.9), we obtain

$$\rho_H(cl(\alpha))(H^i(X, \mathcal{O}_X) \oplus H^i(X, \omega_X)) \\ = \mathrm{Image}(f^* \circ \tau_f : H^i(X, \mathcal{O}_X \oplus \omega_X)) \to H^i(X, \mathcal{O}_X \oplus \omega_X).$$

Since $\tau_f \circ f^* : (H^i(Y, \mathcal{O}_Y) \oplus H^i(Y, \omega_Y)) \to (H^i(Y, \mathcal{O}_Y) \oplus H^i(Y, \omega_Y))$ is multiplication with the degree of $f$ (by Corollary 4.2.5) the proposition follows. □

**4.3. *Main theorem for tame quotients.***

**Theorem 4.3.1.** *Let $S$ be a scheme over a perfect field $k$. Let $\pi_X : X \to S$ and $\pi_Y : Y \to S$ be two integral $S$-schemes, which are tame quotients (see Definition 4.2.6). Furthermore, we assume that $X$ and $Y$ are properly birational equivalent. Then any $Z$ as in Definition 3.2.7 induces isomorphisms of $\mathcal{O}_S$-modules*

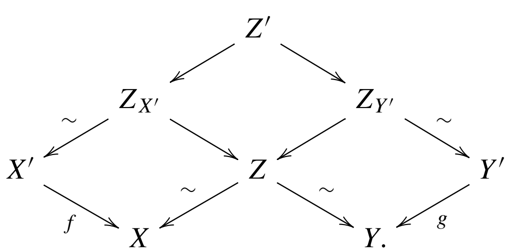
$$R^i\pi_{X*}\mathcal{O}_X \cong R^i\pi_{Y*}\mathcal{O}_Y \quad \textit{and} \quad R^i\pi_{X*}\omega_X \cong R^i\pi_{Y*}\omega_Y \quad \textit{for all } i \geq 0.$$

*These isomorphisms depend only on the $\mathcal{O}_{S,\eta}$-isomorphism $k(X) \cong k(Y)$ induced by $Z$, where $\eta = \pi_X$(generic point of $X$) $= \pi_Y$(generic point of $Y$).*

*Proof.*

**Claim 1.** *There are isomorphisms as in the statement in the case $S = \operatorname{Spec} k$.*

Choose integral and smooth schemes $X'$ and $Y'$ with finite and surjective morphisms $f : X' \to X$ and $g : Y' \to Y$ whose degree is invertible in $k$. Choose $Z$ as in Definition 3.2.7. We may assume that $Z \subset X \times Y$ is a closed integral subscheme. We define $Z_{X'}$, $Z_{Y'}$ and $Z'$ by the cartesian diagram



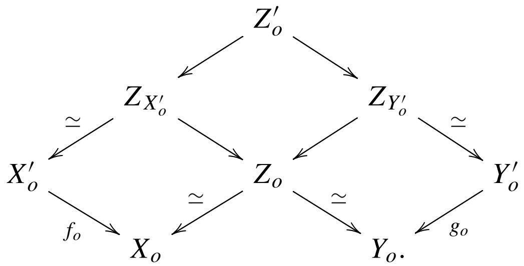

Here the arrows with an $\sim$ are proper and birational morphisms between integral schemes, and all other morphisms are finite and surjective. We may identify $Z'$ with a closed subscheme of $X' \times Y'$ whose irreducible components are proper and surjective over both $X'$ and $Y'$, and all irreducible components have the same dimension equal to $d := \dim X = \dim X' = \dim Y = \dim Y'$ (since $f$ and $g$ are finite and universally equidimensional). Therefore $Z'$ and its transpose define cycles

$$[Z'] \in \mathrm{CH}^d(X' \times Y', P(\Phi_{X'}, \Phi_{Y'})) \quad \text{and} \quad [Z']^t \in \mathrm{CH}^d(Y' \times X', P(\Phi_{Y'}, \Phi_{X'})).$$

Now choose nonempty smooth open subschemes $X_o$, $Y_o$ of $X, Y$ such that the morphisms $Z \to X$, $Z \to Y$ induce isomorphisms $Z_o \xrightarrow{\simeq} X_o$, $Z_o \xrightarrow{\simeq} Y_o$ with

$$Z_o := X_o \times_X Z = Z \times_Y Y_o.$$

Set $X'_o = f^{-1}(X_o)$ and $Y'_o = g^{-1}(Y_o)$ and denote by $f_o$ and $g_o$ the restrictions of $f$ and $g$ to $X'_o$ and $Y'_o$, respectively. We define $Z_{X'_o}$, $Z_{Y'_o}$ and $Z'_o$ by the cartesian diagram




Here the arrows with an $\simeq$ are isomorphisms, and all other arrows are finite and surjective. We set $X'_c = X' \setminus X'_o$ and $Y'_c = Y' \setminus Y'_o$; these are closed subsets of codimension $\geq 1$. Now we define

$$\alpha := [X' \times_X X'] \in \mathrm{CH}^d(X' \times X', P(\Phi_{X'}, \Phi_{X'})),$$
$$\beta := [Y' \times_Y Y'] \in \mathrm{CH}^d(Y' \times Y', P(\Phi_{Y'}, \Phi_{Y'})).$$

We claim

$$\deg g \cdot ([Z'] \circ \alpha) - \deg f \cdot (\beta \circ [Z']) \in \mathrm{image}(\mathrm{CH}_*(X'_c \times Y'_c)), \tag{4.3.2}$$
$$\deg f \cdot ([Z']^t \circ \beta) - \deg g \cdot (\alpha \circ [Z']^t) \in \mathrm{image}(\mathrm{CH}_*(Y'_c \times X'_c)), \tag{4.3.3}$$
$$([Z']^t \circ [Z'] \circ \alpha) - \deg f \deg g \cdot \alpha \in \mathrm{image}(\mathrm{CH}_*(X'_c \times X'_c)), \tag{4.3.4}$$
$$([Z'] \circ [Z']^t \circ \beta) - \deg f \deg g \cdot \beta \in \mathrm{image}(\mathrm{CH}_*(Y'_c \times Y'_c)). \tag{4.3.5}$$

By symmetry, it suffices to prove (4.3.2) and (4.3.4). Let us prove (4.3.2). By using Lemma 1.3.4 we can consider

$$\alpha \in \mathrm{CH}(X' \times_X X'), \quad \beta \in \mathrm{CH}(Y' \times_Y Y'), \quad [Z'] \in \mathrm{CH}(Z'),$$

and see that $[Z'] \circ \alpha$ and $\beta \circ [Z']$ are naturally supported in $\mathrm{CH}(Z')$.

Since $Z' \cap ((X'_o \times Y') \cup (X' \times Y'_o)) = Z' \cap (X'_o \times Y'_o)$, Lemma 1.3.6 and the localization sequence for Chow groups implies the claim if the equality

$$\deg(g) \cdot [Z']|_{X' \times Y'_o} \circ \alpha|_{X'_o \times X'} = \deg(f) \cdot \beta|_{Y' \times Y'_o} \circ [Z']|_{X'_o \times Y'} \tag{4.3.6}$$

holds in $\mathrm{CH}(Z' \cap (X'_o \times Y'_o))$. Here we have

$$\begin{aligned} \alpha|_{X'_o \times X'} &\in \mathrm{CH}(X'_o \times_X X') = \mathrm{CH}(X'_o \times_{X_o} X'_o), \\ \beta|_{Y' \times Y'_o} &\in \mathrm{CH}(Y' \times_Y Y'_o) = \mathrm{CH}(Y'_o \times_{Y_o} Y'_o), \\ [Z']|_{X'_o \times Y'} &\in \mathrm{CH}(Z' \cap (X'_o \times Y')) = \mathrm{CH}(Z' \cap (X'_o \times Y'_o)), \\ [Z']|_{X' \times Y'_o} &\in \mathrm{CH}(Z' \cap (X' \times Y'_o)) = \mathrm{CH}(Z' \cap (X'_o \times Y'_o)). \end{aligned}$$

Obviously,

$$\begin{aligned} \alpha|_{X'_o \times X'} &= [X'_o \times_{X_o} X'_o] = [\Gamma^t_{f_o}] \circ [\Gamma_{f_o}], \\ \beta|_{Y' \times Y'_o} &= [Y'_o \times_{Y_o} Y'_o] = [\Gamma^t_{g_o}] \circ [\Gamma_{g_o}], \\ [Z']|_{X'_o \times Y'} &= [Z'_o] = [\Gamma^t_{g_o}] \circ [Z_o] \circ [\Gamma_{f_o}], \\ [Z']|_{X' \times Y'_o} &= [Z'_o]. \end{aligned} \tag{4.3.7}$$

Thus (4.3.6) follows from

$$\begin{aligned} [\Gamma_{f_o}] \circ [\Gamma^t_{f_o}] &= \deg(f)[\Delta_{X_o}], \\ [\Gamma_{g_o}] \circ [\Gamma^t_{g_o}] &= \deg(g)[\Delta_{Y_o}]. \end{aligned} \tag{4.3.8}$$

This finishes the proof of (4.3.2). The proof of (4.3.4) is similar. The cycles $[Z']^t \circ [Z'] \circ \alpha$ and $\alpha$ are supported in

$$B = \{(x'_1, x'_2) \in X' \times X' \mid (f(x'_1), y) \in Z, (f(x'_2), y) \in Z \text{ for some } y \in Y\}.$$

We see that $B \cap ((X'_o \times X') \cup (X' \times X'_o)) = B \cap (X'_o \times X'_o)$, and by using Lemma 1.3.6 it is sufficient to prove

$$[Z'_o]^t \circ [Z'_o] \circ [X'_o \times_{X_o} X'_o] = \deg f \deg g \cdot [X'_o \times_{X_o} X'_o].$$

In view of (4.3.7), this follows immediately from (4.3.8).

Since $\deg f$ and $\deg g$ are invertible in $k$, it follows from Proposition 3.2.2 and (4.3.2) that $(\deg f)^{-1} \rho_H \circ cl([Z'])$ induces a morphism

$$(\rho_H \circ cl(\alpha)) H^*(X', \mathbb{O}_{X'} \oplus \omega_{X'}) \to (\rho_H \circ cl(\beta)) H^*(Y', \mathbb{O}_{Y'} \oplus \omega_{Y'})$$

and by (4.3.3), $(\deg g)^{-1} \rho_H \circ cl([Z']^t)$ induces a morphism

$$(\rho_H \circ cl(\beta)) H^*(Y', \mathbb{O}_{Y'} \oplus \omega_{Y'}) \to (\rho_H \circ cl(\alpha)) H^*(X', \mathbb{O}_{X'} \oplus \omega_{X'}).$$

By (4.3.4) and (4.3.5) these two morphisms mutually inverse. So Proposition 4.2.8 yields isomorphisms

$$H^i(X, \mathbb{O}_X) \cong H^i(Y, \mathbb{O}_Y) \quad \text{and} \quad H^i(X, \omega_X) \cong H^i(Y, \omega_Y) \quad \text{for all } i \geq 0.$$

This proves Claim 1.

**Claim 2.** *The isomorphisms constructed in Claim* 1 *depends only on the isomorphism $k(X) \cong k(Y)$ induced by a $Z$.*

We use the shorthand notation $H^i(X) = H^i(X, \mathcal{O}_X \oplus \omega_X)$. Choose $Z$ as in Definition 3.2.7. Denote by $Z_0$ the image of $Z$ in $X \times Y$. Choose $f_1 : X_1 \to X$ and $g_1 : Y_1 \to Y$ finite and surjective, with $X_1$, $Y_1$ smooth and integral and $\deg f_1$, $\deg g_1 \in k^*$. Define

$$\underline{\alpha}_1 := \rho_H \circ cl([X_1 \times_X X_1]), \quad \underline{\beta}_1 := \rho_H \circ cl([Y_1 \times_Y Y_1]),$$
$$\underline{\gamma}_1(Z) := (\deg f_1)^{-1} \rho_H \circ cl([X_1 \times_X Z_0 \times_Y Y_1]).$$

Then, as seen in the proof of Claim 1 above, we obtain isomorphisms

$$H^i(X) \xrightarrow{f_1^*, \simeq} \underline{\alpha}_1 H^i(X_1) \xrightarrow{\underline{\gamma}_1(Z), \simeq} \underline{\beta}_1 H^i(Y_1) \xleftarrow{g_1^*, \simeq} H^i(Y).$$

Now choose two different $Z$ as in Definition 3.2.7, say $Z_1$ and $Z_2$, which induce the same isomorphism $k(X) \cong k(Y)$. Then we can find smooth open subschemes $X_o$, $Y_o$, $Z_{1,o}$, $Z_{2,o}$ of $X$, $Y$, $Z_1$, $Z_2$ such that for $i = 1, 2$, we have

$$Z_{i,o} = Z_i \times_X X_o = Z_i \times_Y Y_o,$$

the projections $Z_{i,o} \xrightarrow{\simeq} X_o$, $Z_{i,o} \xrightarrow{\simeq} Y_o$ are isomorphisms, and the induced isomorphisms $h_i : X_o \xrightarrow{\simeq} Z_{o,i} \xrightarrow{\simeq} Y_o$ for $i = 1, 2$ are *equal*. Proposition 3.2.2 implies $\underline{\gamma}_1(Z_1) = \underline{\gamma}_1(Z_2)$ on $H^i(X_1)$. Therefore $\underline{\gamma}_1(Z)$ depends only on the isomorphism $k(X) \cong k(Y)$, which $Z$ induces. From now on we fix such an isomorphism and simply write $\underline{\gamma}_1$.

Now choose $f_2 : X_2 \to X$ and $g_2 : Y_2 \to Y$ finite and surjective, with $X_2$, $Y_2$ smooth and integral and $\deg f_2$, $\deg g_2 \in k^*$. Define $\underline{\alpha}_2$, $\underline{\beta}_2$ and $\underline{\gamma}_2$ as above (in the above formulas replace 1 by 2) and set

$$\underline{\alpha}_{12} := (\deg f_1)^{-1} \rho_H \circ cl([X_1 \times_X X_2]), \quad \underline{\alpha}_{21} := (\deg f_2)^{-1} \rho_H \circ cl([X_2 \times_X X_1])$$
$$\underline{\beta}_{12} := (\deg g_1)^{-1} \rho_H \circ cl([Y_1 \times_Y Y_2]), \quad \underline{\beta}_{21} := (\deg g_2)^{-1} \rho_H \circ cl([Y_2 \times_Y Y_1]).$$

Then one checks as in the proof of Claim 1 that $\underline{\alpha}_{12} : H^i(X_1) \to H^i(X_2)$ induces an isomorphism $\underline{\alpha}_1 H^i(X_1) \xrightarrow{\simeq} \underline{\alpha}_2 H^i(X_2)$ with inverse $\underline{\alpha}_{21}$ and $\underline{\beta}_{12} : H^i(Y_1) \to H^i(Y_2)$ induces an isomorphism $\underline{\beta}_1 H^i(Y_1) \xrightarrow{\simeq} \underline{\beta}_2 H^i(Y_2)$ with inverse $\underline{\beta}_{21}$. Further one checks that $\underline{\beta}_{12} \circ \underline{\gamma_1} \circ \underline{\alpha}_1 = \underline{\gamma}_2 \circ \underline{\alpha}_{12} \circ \underline{\alpha}_1$. Thus we obtain the commutative diagram

$$\begin{array}{ccccccc} & & \underline{\alpha}_1 H^i(X_1) & \xrightarrow{\underline{\gamma}_1, \simeq} & \underline{\beta}_1 H^i(Y_1) & & \\ & \nearrow^{f_1^*, \simeq} & & & & \nwarrow^{g_1^*, \simeq} & \\ H^i(X) & & \simeq \downarrow \underline{\alpha}_{12} & & \simeq \downarrow \underline{\beta}_{12} & & H^i(Y). \\ & \searrow_{f_2^*, \simeq} & & & & \swarrow_{g_2^*, \simeq} & \\ & & \underline{\alpha}_2 H^i(X_2) & \xrightarrow{\underline{\gamma}_2, \simeq} & \underline{\beta}_2 H^i(Y_2) & & \end{array} \tag{4.3.9}$$

Therefore the isomorphisms of Claim 1 do not depend on the choice of $f_1$ and $g_1$. This proves Claim 2 and also the theorem in the case $S = \operatorname{Spec} k$.

*Finally, consider the case of a general basis S.* Choose $Z$ as in Definition 3.2.7 and choose integral, smooth schemes $X'$ and $Y'$ with finite, surjective morphisms $f : X' \to X$ and $g : Y' \to Y$ whose degree is invertible in $k$. For $U \subset S$ open denote by $X_U$, $f_U$, etc. the pull-backs over $U$. By Proposition 4.2.8, the pull-back $f_U^*$ realizes $H^i(X_U, \mathbb{O}_{X_U} \oplus \omega_{X_U})$ as a direct summand of $H^i(X'_U, \mathbb{O}_{X'_U} \oplus \omega_{X'_U})$. This is clearly compatible with restrictions along opens $V \subset U \subset S$ and thus the pull-back $f^*$ realizes $R^i\pi_{X*}(\mathbb{O}_X \oplus \omega_X)$ as a direct summand of the $\mathbb{O}_S$-module $R^i\pi_{X*}f_*(\mathbb{O}_{X'} \oplus \omega_{X'})$. In the same way, $g^*$ realizes $R^i\pi_{Y*}(\mathbb{O}_Y \oplus \omega_Y)$ as a direct summand of the $\mathbb{O}_S$-module $R^i\pi_{Y*}g_*(\mathbb{O}_{Y'} \oplus \omega_{Y'})$. Further, by the case $S = \operatorname{Spec} k$ considered above, the map

$$(\deg f)^{-1}\rho_H \circ cl([Z'_U]) : H^*(X'_U, \mathbb{O}_{X'_U} \oplus \omega_{X'_U}) \to H^*(Y'_U, \mathbb{O}_{Y'_U} \oplus \omega_{Y'_U})$$

induces an isomorphism between $H^*(X_U, \mathbb{O}_{X_U} \oplus \omega_{X_U})$ and $H^*(Y_U, \mathbb{O}_{Y_U} \oplus \omega_{Y_U})$. Write $[Z'] = \sum_T n_T[T]$, where the sum is over the irreducible components of $Z'$. Then the collection $\{(\deg f)^{-1}\rho_H \circ cl([Z'_U]) \mid U \subset S\}$ induces a morphism of $\mathbb{O}_S$-modules (by Proposition 3.2.4)

$$\rho_H(Z'/S) := \sum_T n_T\, \rho_H(T/S) : R^i\pi_{X*}f_*(\mathbb{O}_{X'} \oplus \omega_{X'}) \to R^i\pi_{Y*}g_*(\mathbb{O}_{Y'} \oplus \omega_{Y'}),$$

which by the above induces an isomorphism

$$R^i\pi_{X*}(\mathbb{O}_X \oplus \omega_X) \xrightarrow{\simeq} R^i\pi_{Y*}(\mathbb{O}_Y \oplus \omega_Y). \tag{4.3.10}$$

Claim 2 implies that (4.3.10) depends only on the $\mathbb{O}_{S,\eta}$-isomorphism $k(X) \cong k(Y)$ induced by $Z$. □

**Remark 4.3.11.** Theorem 4.3.1 implies Theorem 3.2.8 and Theorem 3.2.14.

**Corollary 4.3.12.** *Let $\pi : X \to Y$ be a birational and proper morphism between integral schemes over a perfect field $k$. Assume $X$ and $Y$ are tame quotients. Then $\pi^*$ induces isomorphisms*

$$R\pi_*\mathbb{O}_X \cong \mathbb{O}_Y \quad \textit{and} \quad R\pi_*\omega_X \cong \omega_Y.$$

*Proof.* In Theorem 4.3.1 take $S = Y$, $\pi_X = \pi$ and $\pi_Y = \mathrm{id}_Y$. □

**4.4. *Open questions.*** Questions in $\operatorname{char}(k) = p$.

**4.4.1.** Do the statements in Theorem 3.2.8 and Theorem 3.2.14 hold when $k$ is not perfect and smooth is replaced by regular?

**4.4.2.** Let $f : Y \to X$ be a surjective projective morphism with connected fibres between smooth varieties $Y$ and $X$. Is $R^{\dim(Y)-\dim(X)} f_*\omega_Y = \omega_X$? In $\operatorname{char}(k) = 0$ this holds by [Kollár 1986, Proposition 7.6].

**4.4.3.** Let $f : Y \to X$ be a surjective projective morphism with connected fibres between smooth varieties $Y, X$. Is $R^e f_*\omega_Y = 0$ for $e > \dim(Y) - \dim(X)$? In $\mathrm{char}(k) = 0$ this holds by [Kollár 1986, Theorem 2.1(ii)].

## Appendix

*All schemes in this appendix are assumed to be finite-dimensional and noetherian.*

**A.1.** ***Local Cohomology.*** Let $Y = \operatorname{Spec} B$ be an affine scheme and $X \subset Y$ a closed subscheme of pure codimension $c$, defined by the ideal $I \subset B$. We assume that there exists a $B$-regular sequence $t = t_1, \dots, t_c \in I$ with $\sqrt{(t)} = \sqrt{I}$, where $(t)$ denotes the ideal $(t_1, \dots, t_c) \subset B$. We denote by $K^\bullet(t)$ the Koszul complex of the sequence $t$, that is, $K^{-q}(t) = K_q(t) = \bigwedge^q B^c$ for $q = 0, \dots, c$, and if $\{e_1, \dots, e_c\}$ is the standard basis of $B^c$ and $e_{i_1,\dots,i_q} := e_{i_1} \wedge \cdots \wedge e_{i_q}$, then the differential is given by

$$d^{-q}_{K^\bullet}(e_{i_1,\dots,i_q}) = d^{K_\bullet}_q(e_{i_1,\dots,i_q}) = \sum_{j=1}^{q} (-1)^{j+1} t_{i_j} e_{i_1,\dots,\widehat{i_j},\dots i_q}.$$

For any $B$-module $M$ we define the complex

$$K^\bullet(t, M) := \operatorname{Hom}_B(K^{-\bullet}(t), M),$$

and denote its $n$-th cohomology by $H^n(t, M)$. The map

$$\operatorname{Hom}_B\Big(\bigwedge^c B^c, M\Big) \to M/(t)M, \quad \varphi \mapsto \varphi(e_{1,\dots,c})$$

induces a canonical isomorphism $H^c(t, M) \simeq M/(t)M$.

If $t$ and $t'$ are two sequences as above with $(t') \subset (t)$, then there exists a $c \times c$-matrix $T$ with coefficients in $B$ such that $t' = Tt$ and $T$ induces a morphism of complexes $K^\bullet(t') \to K^\bullet(t)$, which is the unique (up to homotopy) morphism lifting the natural map $B/(t') \to B/(t)$. Furthermore we observe that, for any pair of sequences $t$ and $t'$ as above, there exists an $N \geq 0$ such that $(t^N) \subset (t')$, where $t^N$ denotes the sequence $t_1^N, \dots, t_c^N$. Thus the sequences $t$ form a directed set and $H^c(t, M) \to H^c(t', M)$ for $(t') \subset (t)$ becomes a direct system. It follows from [Grothendieck 1968, Exposé II, Proposition 5] that we have an isomorphism

$$\varinjlim_t M/(t)M = \varinjlim_t H^c(t, M) \cong H^c_X(Y, \widetilde{M}),$$

where the limit is over all $B$-regular sequences $t = t_1, \dots, t_c$ in $B$ with $V((t)) = X$ and $\widetilde{M}$ is the sheaf associated to $M$. We denote by $\left[\begin{smallmatrix} m \\ t \end{smallmatrix}\right]$ the image of $m \in M$ under the composition

$$M \to M/(t)M \to H^c(t, M) \to H^c_X(Y, \widetilde{M}).$$

It is a consequence of the explanations above that we have the following properties:

(1) Let $t$ and $t'$ be two sequences as above with $(t') \subset (t)$. Let $T$ be a $c \times c$-matrix with $t' = Tt$; then
$$\begin{bmatrix} \det(T)\, m \\ t' \end{bmatrix} = \begin{bmatrix} m \\ t \end{bmatrix}.$$

(2) $\begin{bmatrix} m+m' \\ t \end{bmatrix} = \begin{bmatrix} m \\ t \end{bmatrix} + \begin{bmatrix} m' \\ t \end{bmatrix}$ and $\begin{bmatrix} t_i m \\ t \end{bmatrix} = 0$ for all $i$.

(3) If $M$ is any $B$-module of finite rank, then
$$H^c_X(Y, \mathcal{O}_Y) \otimes_B M \xrightarrow{\simeq} H^c_X(Y, \widetilde{M}), \qquad \begin{bmatrix} b \\ t \end{bmatrix} \otimes m \mapsto \begin{bmatrix} bm \\ t \end{bmatrix}$$
is an isomorphism.

**Remark A.1.1.** Since for a $B$-regular sequence $t$ as above $K^\bullet(t) \to B/(t)$ is a free resolution, we have an isomorphism for all $n$, given by
$$\mathrm{Ext}^n(B/(t), M) \simeq H^n(\mathrm{Hom}^\bullet_B(K^\bullet(t), M)).$$
We also have an isomorphism
$$\mathrm{Hom}^\bullet_B(K^\bullet(t), M) \simeq K^\bullet(t, M),$$
which is given by multiplication with $(-1)^{n(n+1)/2}$ in degree $n$. We obtain an isomorphism
$$\psi_{t,M} : \mathrm{Ext}^c(B/(t), M) \xrightarrow{\simeq} H^c(t, M) = M/(t)M,$$
which has the sign $(-1)^{c(c+1)/2}$ in it. In particular, under the composition
$$\mathrm{Ext}^c(B/(t), M) \xrightarrow{\psi_{t,M}} M/(t)M \to H^c_X(Y, \widetilde{M}),$$
the class of a map $\varphi \in \mathrm{Hom}_B(\bigwedge^c B^c, M)$ is sent to
$$(-1)^{c(c+1)/2} \begin{bmatrix} \varphi(e_{1,\dots,c}) \\ t \end{bmatrix}.$$

**Lemma A.1.2.** *Let $Y = \mathrm{Spec}\, B$ be as above, let $\mathcal{M}$ be a quasicoherent sheaf on $Y$, let $c \geq 0$ and let $t_1, \dots, t_{c+1}$ be a $B$-regular sequence. Set $X' := V(t_1, \dots, t_{c+1}) \subset X := V(t_1, \dots, t_c)$. Let $\partial : H^c_{X \setminus X'}(Y \setminus X', \mathcal{M}) \to H^{c+1}_{X'}(Y, \mathcal{M})$ be the boundary map of the localization long exact sequence. Then*
$$\partial \begin{bmatrix} m/t_{c+1} \\ t_1, \dots, t_c \end{bmatrix} = \begin{bmatrix} m \\ t_1, \dots, t_c, t_{c+1} \end{bmatrix}.$$

*Proof.* Let $M$ be the $B$-module of global sections of $\mathcal{M}$. By [Grothendieck 1968, Exposé II, Corollary 4] and Čech computations, we may identify

$$H^c_{X\setminus X'}(Y\setminus X',\mathcal{M})=\frac{M_{t_1\cdots t_c t_{c+1}}}{\sum_{i=1}^{c} M_{t_1\cdots\widehat{t_i}\cdots t_c t_{c+1}}},\qquad H^{c+1}_{X'}(Y,\mathcal{M})=\frac{M_{t_1\cdots t_{c+1}}}{\sum_{i=1}^{c+1} M_{t_1\cdots\widehat{t_i}\cdots t_{c+1}}},$$

and $\partial$ is the natural map from left to right. Under these identifications, the map $M_{t_{c+1}}/(t_1,\ldots,t_c)=H^c(K^\bullet(t,M))\to H^c_{X\setminus X'}(Y,\mathcal{M})$ sends the class of $m/t_{c+1}$, for $m\in M$, to the class of $(m/t_{c+1})/(t_1\cdots t_c)$ and similarly for $M/(t_1,\ldots,t_{c+1})\to H^{c+1}_{X'}(Y,\mathcal{M})$. □

**A.2.** ***The trace for a regular closed embedding.*** We now explicitly describe the trace morphism for a regular closed embedding. This is well known and appears in various incarnations in the literature; see for example [Lipman 1984; Hübl and Seibert 1997, Section 4]. However in all the articles we are aware of, more elementary versions of duality theory are used (for example no derived categories appear). Since the compatibility of these theories with the one we are working with — namely the one developed in [Hartshorne 1966; Conrad 2000] — is not evident to us, and also to be sure about the signs, we recall the description of the trace in this situation.

Let $i: X\hookrightarrow Y$ be a closed immersion of pure codimension $c$ between two Gorenstein schemes and assume that the ideal sheaf $\mathcal{I}$ of $X$ is generated by a sequence $t=(t_1,\ldots,t_c)$ of global sections of $\mathcal{O}_Y$. Then the image of $t$ in any local ring of $Y$ is automatically a regular sequence. We denote by $K^\bullet(t)$ the sheafified Koszul complex of $t$ and set

$$\omega_{X/Y}:=\bigwedge^c \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}/\mathcal{I}^2,\mathcal{O}_X).$$

The fundamental local isomorphism (see for example [Conrad 2000, 2.5]) gives an isomorphism in $D^b_c(Y)$

$$\eta_i: i_*\omega_{X/Y}[-c]\xleftarrow{\simeq}\mathcal{H}om^\bullet_Y(K^\bullet(t),\mathcal{O}_Y)\cong R\,\mathcal{H}om_Y(i_*\mathcal{O}_X,\mathcal{O}_Y)\cong i_*i^!\mathcal{O}_Y. \quad (A.2.1)$$

The first map is induced by

$$\begin{gathered}\mathcal{H}om(\bigwedge^c\mathcal{O}_Y^c,\mathcal{O}_Y)=\mathcal{H}om^c(K^\bullet(t),\mathcal{O}_Y)\to i_*\omega_{X/Y},\\ \varphi\mapsto(-1)^{c(c+1)/2}\varphi(e_{1,\ldots,c})t_1^\vee\wedge\cdots\wedge t_c^\vee.\end{gathered}\quad (A.2.2)$$

(The reason for the sign is Remark A.1.1.) Composing the morphism $\eta_i$ with the trace $\mathrm{Tr}_i: i_*i^!\mathcal{O}_Y\to\mathcal{O}_Y$ (see for example [Conrad 2000, 3.4]), we obtain a morphism in $D^b_c(Y)$

$$i_*\omega_{X/Y}[-c]\xrightarrow{\eta_i} i_*i^!\mathcal{O}_Y\xrightarrow{\mathrm{Tr}_i}\mathcal{O}_Y, \quad (A.2.3)$$

which factors in $D^b_{\mathrm{qc}}(Y)$ as

$$i_*\omega_{X/Y}[-c] \xrightarrow{\eta_i} i_* i^! \mathcal{O}_Y \xrightarrow{\mathrm{Tr}_i} R\underline{\Gamma}_X \mathcal{O}_Y. \tag{A.2.4}$$

**Lemma A.2.5.** *In the situation above there is a natural isomorphism*

$$R\underline{\Gamma}_X \mathcal{O}_Y \cong \mathcal{H}^c_X(\mathcal{O}_Y)[-c] \quad \textit{in } D^b_{\mathrm{qc}}(Y) \tag{A.2.6}$$

*and* $\mathcal{H}^c$((A.2.4)) *is given by*

$$i_*\omega_{X/Y} \longrightarrow \mathcal{H}^c_X(\mathcal{O}_Y), \quad a t_1^\vee \wedge \cdots \wedge t_c^\vee \mapsto (-1)^{c(c+1)/2} \begin{bmatrix} \tilde{a} \\ t_1, \ldots, t_c \end{bmatrix}, \tag{A.2.7}$$

*where* $\tilde{a} \in \mathcal{O}_Y$ *is any lift of* $a \in \mathcal{O}_X$.

*Proof.* The first statement is equivalent to $\mathcal{H}^i_X(\mathcal{O}_Y) = 0$, for $i \neq c$, and hence we may assume that $Y$ is affine. We have the vanishing for $i < c$ since $Y$ is CM by [Grothendieck 1968, Exposé III, Proposition 3.3] and for $i > c$ since the ideal of $X$ in $Y$ is generated by $c$ elements, which by a Čech argument implies that $H^i(Y \setminus X, \mathcal{O}_Y) = 0$ for $i > c$.

We denote by $E^\bullet = E^\bullet(\mathcal{O}_Y)$ the Cousin complex of $\mathcal{O}_Y$; see e.g., [Hartshorne 1966, Chapter IV, Section 2]. In particular $E^\bullet$ is an injective resolution of $\mathcal{O}_Y$ (since $Y$ is Gorenstein) and if $Y^{(c)}$ denotes the set of points of codimension $c$ in $Y$, then

$$E^q = \bigoplus_{y \in Y^{(c)}} i_{y*} H^q_y(Y, \mathcal{O}_Y),$$

where $i_y : y \to Y$ is the inclusion and $H^q_y(Y, \mathcal{O}_Y) = \mathrm{colim}_{y \in U} H^q_{\bar{y} \cap U}(Y, \mathcal{O}_Y)$, the limit being over all open subsets $U \subset Y$ that contain $y$. We write $K^\bullet := K^\bullet(t)$.

The trace $\mathrm{Tr}_i : i_* i^! \mathcal{O}_Y \to \mathcal{O}_Y$ is now induced by the "evaluation at 1" morphism $\mathcal{H}om^\bullet(i_*\mathcal{O}_X, E^\bullet) \to E^\bullet$. Furthermore the augmentation morphisms $K^\bullet \to i_*\mathcal{O}_X$ and $\mathcal{O}_Y \to E^\bullet$ induce quasiisomorphisms

$$\mathcal{H}om^\bullet(K^\bullet, \mathcal{O}_Y) \xrightarrow{\sim} \mathcal{H}om^\bullet(K^\bullet, E^\bullet) \xleftarrow{\sim} \mathcal{H}om^\bullet(i_*\mathcal{O}_X, E^\bullet). \tag{A.2.8}$$

To prove the second statement, we may assume $a = 1 \in \mathcal{O}_X$. We define $\alpha \in \mathcal{H}om^c(K^\bullet, \mathcal{O}_Y) = \mathcal{H}om(\wedge^c \mathcal{O}^c_Y, \mathcal{O}_Y)$ by $\alpha(e_{1,\ldots,c}) = 1$ and $\beta \in \mathcal{H}om^c(i_*\mathcal{O}_X, E^\bullet) = \mathcal{H}om(i_*\mathcal{O}_X, E^c)$ by

$$\beta(1) = (\beta_y) \in E^c = \bigoplus_{y \in Y^{(c)}} i_{y*} H^c_y(Y, \mathcal{O}_Y),$$

with

$$\beta_y = \begin{cases} \begin{bmatrix} 1 \\ t_1, \ldots, t_c \end{bmatrix} & \text{if } y \text{ is a generic point of } X, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\mathrm{Tr}_i(\bar{\beta}) = \begin{bmatrix} 1 \\ t_1, \ldots, t_c \end{bmatrix} \in \mathcal{H}^c_X(\mathcal{O}_Y),$$

where $\bar{\beta}$ is the residue class of $\beta$ in $\mathcal{H}^c(\mathcal{H}om^\bullet(i_*\mathcal{O}_X, E^\bullet))$ and

$$\eta_i(\alpha) = (-1)^{c(c+1)/2} t_1^\vee \wedge \cdots \wedge t_c^\vee \in i_*\omega_{X/Y}.$$

Thus the second statement of the lemma follows if we can show that the images of $\alpha$ and $\beta$ in $\mathcal{H}om^c(K^\bullet, E^\bullet)$ differ by an element in $d^{c-1}_{\mathcal{H}om^\bullet}(\mathcal{H}om^{c-1}(K^\bullet, E^\bullet))$.

For $j = 0, \ldots, c-1$, we define

$$\gamma^{c-1-j} = (\gamma^{c-1-j}_y) \in E^{c-1-j} = \bigoplus_{y \in Y^{(c-1-j)}} i_{y*} H^{c-1-j}_y(Y, \mathcal{O}_Y)$$

by

$$\gamma^{c-1-j}_y := \begin{cases} \left[\begin{smallmatrix} 1/t_{c-j} \\ t_1, \ldots, t_{c-j-1} \end{smallmatrix}\right] & \text{if } y \in Y^{(c-j-1)} \cap V(t_1, \ldots, t_{c-1-j}), \\ 0 & \text{otherwise.} \end{cases}$$

In particular $\gamma^0 = 1/t_1 \in \bigoplus_i H^0_{\eta_i}(Y, \mathcal{O}_Y) = \oplus_i k(\eta_i)$, with $\eta_i$ the generic points of $Y$. Notice that (by Lemma A.1.2)

$$d_E \gamma^{c-1-j} = t_{c-j+1} \gamma^{c-j} \qquad \text{for all } j \geq 1, \tag{A.2.9}$$

$$t_i \gamma^{c-1-j} = 0 \qquad \text{for all } j \geq 0,\ i \in \{1, \ldots, c-1-j\}. \tag{A.2.10}$$

Further define

$$\psi = (\psi_0, \ldots, \psi_{c-1}) \in \mathcal{H}om^{c-1}(K^\bullet, E^\bullet) = \bigoplus_{j=0}^{c-1} \mathcal{H}om(K_j, E^{c-1-j})$$

by

$$\psi_j(e_{i_1,\ldots,i_j}) = \begin{cases} (-1)^{j(c+j)} \gamma^{c-1-j} & \text{if } (i_1, \ldots, i_j) = (c+1-j, \ldots, c), \\ 0 & \text{otherwise.} \end{cases}$$

By definition and (A.2.10) we have

$$t_{i_q} \psi_{j-1}(e_{i_1,\ldots,\widehat{i_q},\ldots,i_j}) \neq 0 \text{ if } q \neq 1 \text{ or } (i_1, \ldots, i_j) \neq (c+1-j, \ldots, c). \tag{A.2.11}$$

Now we calculate the boundary of $\psi$,

$$d^{c-1}_{\mathcal{H}om}\psi = (d^{c-1}_E \circ \psi_0, \ldots, d^{c-1-j}_E \circ \psi_j + (-1)^c \psi_{j-1} \circ d^K_j, \ldots, (-1)^c \psi_{c-1} \circ d^K_c).$$

*1st Case*: $j = 0$.

$$d^{c-1}_E \circ \psi_0(1) = d^{c-1}_E \gamma^{c-1} = \beta(1) \quad \text{by (A.2.9).}$$

*2nd Case*: $1 \leq j \leq c-2$. By (A.2.11) and the definition of $\psi$, we have

$$(d^{c-1-j}_E \circ \psi_j + (-1)^c \psi_{j-1} \circ d^K_j)(e_{i_1,\ldots,i_j}) = 0$$

if $(i_1,\dots,i_j)\neq(c+1-j,\dots,c)$ and otherwise

$$\begin{aligned}(d_E^{c-1-j}\circ\psi_j+(-1)^c\psi_{j-1}\circ d_j^K)(e_{c+1-j,\dots,c})\\ &=(-1)^{j(c+j)}d_E^{c-1-j}\gamma^{c-1-j}+(-1)^c t_{c+1-j}\psi_{j-1}(e_{c+2-j,\dots,c})\\ &=(-1)^{j(c+j)}(d_E^{c-1-j}\gamma^{c-1-j}-t_{c-j+1}\gamma^{c-j})=0\quad\text{by (A.2.9).}\end{aligned}$$

*3rd Case*: $j=c-1$. By (A.2.11), we have

$$(-1)^c\psi_{c-1}(d_c^K(e_{1,\dots,c}))=(-1)^c t_1\psi_{c-1}(e_{2,\dots,c})=-1=-\alpha(e_{1,\dots,c}).$$

All in all we obtain

$$d^{c-1}_{\mathcal{H}om}(\psi)=(\beta,0,\dots,0,-\alpha).$$ □

**Lemma A.2.12.** *Let $S$ be a Gorenstein scheme and $i:X\hookrightarrow Y$ a closed immersion between smooth, separated and equidimensional $S$-schemes with structure maps $\pi_X:X\to S$ and $\pi_Y:Y\to S$ and denote by $d_{X/S}$ and $d_{Y/S}$ their relative dimensions. We set $\omega_{X/S}:=\Omega^{d_{X/S}}_{X/S}$, $\omega_{Y/S}:=\Omega^{d_{Y/S}}_{Y/S}$ and $c=d_{Y/S}-d_{X/S}$. Assume that the ideal sheaf of $X$ in $Y$ is generated by a sequence $t=t_1,\dots,t_c$ of global sections of $\mathcal{O}_Y$. Define a morphism $\iota_X$ by*

$$\iota_X:i_*\omega_{X/S}\to\mathcal{H}^c_X(\omega_{Y/S}),\quad \alpha\mapsto(-1)^c\begin{bmatrix}dt\tilde{\alpha}\\ t\end{bmatrix},$$

*with $\tilde{\alpha}\in\Omega^{d_{X/S}}_{Y/S}$ any lift of $\alpha$ and $dt=dt_1\wedge\cdots\wedge dt_c$. Then the following diagram in $D^b_{\mathrm{qc}}(\mathcal{O}_Y)$ is commutative*:

$$\begin{array}{ccccc} i_*\pi_X^!\mathcal{O}_S & \xrightarrow{c_{i,\pi_Y}} & i_*i^!\pi_Y^!\mathcal{O}_S & \xrightarrow{\mathrm{Tr}_i} & \pi_Y^!\mathcal{O}_S \\ \simeq\uparrow & & & & \uparrow \\ i_*\omega_{X/S}[d_{X/S}] & \xrightarrow{\iota_X} & \mathcal{H}^c_X(\omega_{Y/S})[d_{X/S}] & \xrightarrow{\simeq} & R\underline{\Gamma}_X(\omega_{Y/S})[d_{Y/S}], \end{array} \tag{A.2.13}$$

*where the vertical map on the left is the well-known canonical isomorphism (see* [Conrad 2000, (3.3.21)]), *the vertical map on the right is the composition of the forget supports map $R\underline{\Gamma}_X(\omega_{Y/S})[d_{Y/S}]\to\omega_{Y/S}[d_{Y/S}]$ with the canonical isomorphism $\omega_{Y/S}[d_{Y/S}]\cong\pi_Y^!\mathcal{O}_S$ and $c_{i,\pi_Y}:\pi_X^!\cong i^!\pi_Y^!$ is the canonical isomorphism* [Conrad 2000, (3.3.14)].

*Proof.* Let $\mathcal{I}\subset\mathcal{O}_Y$ be the ideal sheaf of $X$. As above we write

$$\omega_{X/Y}:=\bigwedge^c\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}/\mathcal{I}^2,\mathcal{O}_X).$$

Further let $\tau_Y:\pi_Y^!\mathcal{O}_S\cong\omega_{Y/S}[d_{Y/S}]$ and $\tau_X:\pi_X^!\mathcal{O}_S\cong\omega_{X/S}[d_{X/S}]$ be the canonical isomorphisms and $\eta_i:\omega_{X/Y}[-c]\cong i^!\mathcal{O}_Y$ the fundamental local isomorphism

(A.2.1). Consider the following diagram in $D^b_{\mathrm{c}}(\mathcal{O}_Y)$:

$$\begin{array}{ccccc}
i_*\pi_X^!\mathcal{O}_S & \xrightarrow{c_{i,\pi_Y},\,\simeq} & i_*i^!\pi_Y^!\mathcal{O}_S & \xrightarrow{\mathrm{Tr}_i} & \pi_Y^!\mathcal{O}_S \\
\| & & \downarrow{\scriptstyle\simeq} & & \| \\
i_*\pi_X^!\mathcal{O}_S & \xrightarrow{\simeq} & i_*i^!\mathcal{O}_Y \otimes^L \pi_Y^!\mathcal{O}_S & \xrightarrow{\mathrm{Tr}_i\otimes\mathrm{id}} & \pi_Y^!\mathcal{O}_S \\
{\scriptstyle\simeq}\downarrow{\scriptstyle\tau_X} & & {\scriptstyle\simeq}\downarrow{\scriptstyle\eta_i^{-1}\otimes\tau_Y} & & {\scriptstyle\simeq}\downarrow{\scriptstyle\tau_Y} \\
i_*\omega_{X/S}[d_{X/S}] & \xrightarrow{\simeq} & i_*\omega_{X/Y}[-c]\otimes\omega_{Y/S}[d_{Y/S}] & \xrightarrow{\mathrm{Tr}'_i\otimes\mathrm{id}} & \omega_{Y/S}[d_{Y/S}].
\end{array} \tag{A.2.14}$$

Here some explanations: The middle horizontal arrow on the left is defined such that the upper left square commutes. We have a canonical identification $i_*i^!(\cdot) = R\,\mathcal{H}om_{\mathcal{O}_Y}(i_*\mathcal{O}_X, (\cdot))$ and since $\pi_Y^!\mathcal{O}_S$ is isomorphic to a shifted locally free $\mathcal{O}_Y$-module, we have $R\,\mathcal{H}om_{\mathcal{O}_Y}(i_*\mathcal{O}_X, \pi_Y^!\mathcal{O}_S) = R\,\mathcal{H}om_{\mathcal{O}_Y}(i_*\mathcal{O}_X, \mathcal{O}_Y)\otimes^L \pi_Y^!\mathcal{O}_S$; this defines the upper vertical arrow in the middle. Furthermore $\mathrm{Tr}_i : i_*i^!(\cdot)\to(\cdot)$ may be identified with $R\,\mathcal{H}om_{\mathcal{O}_Y}(i_*\mathcal{O}_X,\cdot)\to(\cdot)$ given by the evaluation at 1. This shows, that in the diagram above the upper square on the right commutes. The map $\mathrm{Tr}'_i : i_*\omega_{X/Y}[-c]\to\mathcal{O}_Y$ on the right bottom is the composition (A.2.3) and thus the lower square on the right commutes by definition. The horizontal isomorphism on the lower left is given by (see [Conrad 2000, page 29, 30(c) and (2.2.6)])

$$\begin{aligned} i_*\omega_{X/S}[d_{X/S}] &\to i_*\omega_{X/Y}[-c]\otimes\omega_{Y/S}[d_{Y/S}],\\ \alpha &\mapsto (t_1^\vee\wedge\cdots\wedge t_c^\vee)\otimes dt_c\wedge\cdots\wedge dt_1\wedge\tilde\alpha, \end{aligned} \tag{A.2.15}$$

with $\tilde\alpha\in\Omega^{d_{X/S}}_{Y/S}$ any lift of $\alpha$. That the square on the lower left commutes follows from [Conrad 2000, Theorem 3.3.1, (3.3.27)] and [Conrad 2000, Lemma 3.5.3]. (Notice that by [Conrad 2001, p. 5, pp. 160–164] the statement of [Conrad 2000, Lemma 3.5.3] should be "(3.5.8) is equal to (3.5.7)" instead of "(3.5.8) is equal to $(-1)^{n(N-n)}$ times (3.5.7)".) Thus the whole diagram commutes. The upper line equals the upper line in (A.2.13) and the lower line factors as the composition

$$\begin{aligned} &i_*\omega_{X/S}[d_{X/S}]\to i_*\omega_{X/Y}[-c]\otimes\omega_{Y/S}[d_{Y/S}]\\ &\qquad\xrightarrow{\mathrm{tr}_i\otimes\mathrm{id}}\mathcal{H}^c_X(\mathcal{O}_Y)[-c]\otimes\omega_{Y/S}[d_{Y/S}]\simeq\mathcal{H}^c_X(\omega_{Y/S})[d_{X/S}] \end{aligned} \tag{A.2.16}$$

with the natural map

$$\mathcal{H}^c_X(\omega_{Y/S})[d_{X/S}]\simeq R\underline{\Gamma}_X(\omega_{Y/S})[d_{Y/S}]\to\omega_{Y/S}[d_{Y/S}].$$

Here $\mathrm{tr}_i$ denotes the composition of $R\underline{\Gamma}_X(\mathrm{Tr}'_i)$ with the isomorphism $R\underline{\Gamma}_X(\mathcal{O}_Y)\cong\mathcal{H}^c_X(\mathcal{O}_Y)[-c]$. Thus the lemma is proved once we know that (A.2.16) equals $\iota_X$.

But by Lemma A.2.5 the map $\mathrm{tr}_i$ is given by

$$i_*\omega_{X/Y} \to \mathcal{H}^c_X(\mathcal{O}_Y), \quad t_1^\vee \wedge \cdots \wedge t_c^\vee \mapsto (-1)^{c(c+1)/2}\begin{bmatrix} 1 \\ t \end{bmatrix}.$$

Together with (A.2.15) we obtain that (A.2.16) is given by

$$\begin{aligned} \alpha &\mapsto (t_1^\vee \wedge \cdots \wedge t_c^\vee) \otimes dt_c \wedge \cdots \wedge dt_1 \wedge \tilde{\alpha} \\ &\mapsto (-1)^{c(c+1)/2}\begin{bmatrix} 1 \\ t \end{bmatrix} \otimes dt_c \wedge \cdots \wedge dt_1 \wedge \tilde{\alpha} = (-1)^c \begin{bmatrix} dt\tilde{\alpha} \\ t \end{bmatrix}, \end{aligned}$$

which by definition equals $\iota_X$. This proves the lemma. □

## A.3. *The trace for a finite and surjective morphism.*

**1.3.1.** Let $S$ be a Gorenstein scheme and $f : X \to Y$ a finite and surjective morphism of smooth, separated and equidimensional $S$-schemes, both of which have relative dimension $n$. We denote by $\pi_X : X \to S$ and $\pi_Y : Y \to S$ the respective structure maps. Then we define the trace map

$$\tau_f^n : f_*\omega_{X/S} \to \omega_{Y/S} \tag{A.3.2}$$

to be the composition

$$f_*\omega_{X/S} \cong Rf_*\pi_X^!\mathcal{O}_S[-n] \xrightarrow{\simeq c_{f,\pi_Y}} Rf_*f^!\pi_Y^!\mathcal{O}_S[-n] \xrightarrow{\mathrm{Tr}_f} \pi_Y^!\mathcal{O}_S[-n] \cong \omega_{Y/S}.$$

In the lemma below we give a well-known explicit description of this trace map, for which we could not find an appropriate reference. There are well-studied ad hoc definitions of this trace map not using the machinery of duality theory (see for example [Kunz 1986, §16]), but it is a priori not clear that these construction coincide with the one above.

**Lemma A.3.3.** *Let $f : X \to Y$ be as above and assume it factors as*

$$\begin{array}{ccc} X & \overset{i}{\hookrightarrow} & P \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle \pi} & \\ Y, & & \end{array}$$

*where $\pi$ is smooth and separated of pure relative dimension $d$ and $i$ is a closed immersion whose ideal sheaf $\mathcal{I} \subset \mathcal{O}_P$ is generated by global sections $t_1, \ldots, t_d \in \Gamma(P, \mathcal{O}_P)$. Then for any local section $\alpha \in f_*\omega_{X/S}$, we have a formula for $\tau_f^n(\alpha)$: Let $\tilde{\alpha} \in \Omega^n_{P/S}$ be any lift of $\alpha$ and write*

$$i^*(dt_d \wedge \cdots \wedge dt_1 \wedge \tilde{\alpha}) = \sum_j i^*\gamma_j \otimes f^*\beta_j, \quad \textit{where } \gamma_j \in \omega_{P/Y},\ \beta_j \in \omega_{Y/S}$$

*in* $i^*\Omega^{n+d}_{P/S} = i^*\omega_{P/S} = i^*(\omega_{P/Y}) \otimes f^*\omega_{Y/S}$. *Then*

$$\tau^n_f(\alpha) = (-1)^{d(d-1)/2} \sum_j \mathrm{Res}_{P/Y}\begin{bmatrix} \gamma_j \\ t_1, \dots, t_d \end{bmatrix} \beta_j \in \omega_{Y/S},$$

*where* $\mathrm{Res}_{P/Y}\left[\begin{smallmatrix}\gamma_j\\ t_1,\dots,t_d\end{smallmatrix}\right] \in \mathcal{O}_Y$ *is the residue symbol defined in* [Conrad 2000, (A.1.4)].

*Proof.* The proof is a collection of compatibility statements from [Conrad 2000]. First we collect some notation.

(1) $\zeta'_{i,\pi_P} : \omega_{X/S} \to \omega_{X/P} \otimes i^*\omega_{P/S}$ is defined in [Conrad 2000, pp. 29–30, (c)] and sends $\alpha$ to $(t_1^\vee \wedge \cdots \wedge t_d^\vee) \otimes i^*(dt_d \wedge \cdots \wedge dt_1 \wedge \tilde{\alpha})$, where we identify $\omega_{X/P} = \bigwedge^d (\mathcal{I}/\mathcal{I}^2)^\vee$.

(2) $\eta_i : \mathcal{E}xt^d_P(i_*\mathcal{O}_X, \cdot\,) \xrightarrow{\simeq} \omega_{X/P} \otimes i^*(\cdot\,)$ is the fundamental local isomorphism [Conrad 2000, (2.5.1)].

(3) For a smooth and separated morphism of pure relative dimension $n$ between two schemes $g : V \to W$, $e_g : g^! \xrightarrow{\simeq} g^\# = \omega_{V/W}[n] \otimes^L (\cdot\,)$ denotes the natural transformation [Conrad 2000, (3.3.21)].

(4) In case $g$ as above factors as $g = h \circ i$ with $i : V \to Z$ finite and $h : Z \to W$ smooth, $\psi_{i,h} : g^\# \xrightarrow{\simeq} i^\flat h^\#$ is the isomorphism defined in [Conrad 2000, (2.7.5)], where $i^\flat(\cdot\,) = i^{-1} R\,\mathcal{H}om_Z(i_*\mathcal{O}_V, \cdot\,) \otimes_{i^{-1}i_*\mathcal{O}_V} \mathcal{O}_V$ is defined in [Conrad 2000, (2.2.8)].

(5) $d_f : f^! \xrightarrow{\simeq} f^\flat$ is the isomorphism defined in [Conrad 2000, (3.3.19)].

(6) $\mathrm{Tr}_f : f_*f^! \to \mathrm{id}$ is the trace morphism defined in [Conrad 2000, 3.4], and $\mathrm{Trf}_f : f_*f^\flat \to \mathrm{id}$ is the finite trace morphism defined in [Conrad 2000, (2.2.9)] and which is induced by evaluation at 1.

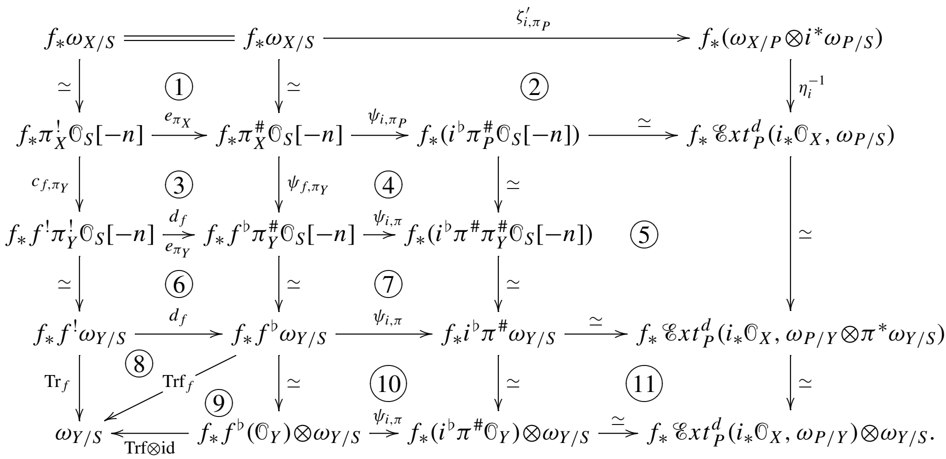
Consider the diagram on page 773. Let us describe the different squares and triangles in this diagram:

(1) The vertical isomorphism on the right in square 1 is immediate from the definition of $\pi_Y^\#$; the left vertical isomorphism is defined such that the square commutes.

(2) See [Conrad 2000, Theorem 3.5.1, Corollary 3.5.2] for the isomorphism in the lower right of square 2. The square commutes by [Conrad 2000, Lem. 3.5.3]. (By [Conrad 2001, comment to pp. 160–164] the last statement of [Conrad 2000, Lemma 3.5.3] should be "…, *then* (3.5.8) *is equal to* (3.5.7)".)

(3) Square 3 commutes by [Conrad 2000, (3.3.27)].

(4) The vertical isomorphism on the right of square 4 is induced by the natural isomorphism $(\pi\pi_Y)^\# \cong \pi_Y^\# \pi^\#$. For the commutativity of the square, see the discussion in [Conrad 2000, p. 83–84] (our case is point three).




(5) The isomorphism on the right of square 5 is induced by the natural isomorphism $\omega_{P/S}\cong\omega_{P/Y}\otimes\pi^*\omega_{Y/S}$. The square commutes by the functoriality of the horizontal isomorphisms, which are just induced by taking the 0-th cohomology (the other cohomology groups being zero).

(6) The vertical isomorphism on the right of square 6 is induced by $\omega_{Y/S}\cong\pi_Y^!\mathcal{O}_S[-n]\cong\pi_Y^\#\mathcal{O}_S[-n]$. Thus the square commutes by the functoriality of $d_f$.

(7) The vertical isomorphism on the right of square 7 is defined as above. Thus the square commutes by the functoriality of $\psi_{i,\pi}$.

(8) Triangle 8 commutes by [Conrad 2000, Lemma 3.4.3, (TRA2)].

(9) By [Hartshorne 1966, proof of Chapter III, Proposition 6.5], we may identify $f_*f^\flat\omega_{Y/S}$ with the sheaf $\mathcal{H}om_Y(f_*\mathcal{O}_X,\omega_{Y/S})$ (since $\omega_{Y/S}$ is locally free) and $\mathrm{Trf}_f$ is given by evaluation at 1. The vertical map on the right of triangle 9 is defined by the isomorphism $\mathcal{H}om_Y(f_*\mathcal{O}_X,\omega_{Y/S})\cong\mathcal{H}om_Y(f_*\mathcal{O}_X,\mathcal{O}_Y)\otimes\omega_{Y/S}$. The triangle thus obviously commutes.

(10) By [Conrad 2000, (2.8.3) and the paragraph after this, pp. 100–101] we have a commutative square

$$\begin{array}{ccc} f^\flat\omega_{Y/S} & \xrightarrow{\psi_{i,\pi}} & i^\flat\pi^\#\omega_{Y/S} \\ \downarrow\simeq & & \downarrow\simeq \\ f^\flat\mathcal{O}_Y\otimes f^*\omega_{Y/S} & \xrightarrow{\psi_{i,\pi}} & i^\flat\pi^\#\mathcal{O}_Y\otimes f^*\omega_{Y/S}. \end{array}$$

Applying $f_*$ to this diagram and using projection formula defines the commutative square 10.

(11) The horizontal maps in square 11 are induced by taking the 0-th cohomology, the vertical maps are the natural isomorphisms ($\omega_{Y/S}$ is locally free and projection formula). The commutativity of the diagram is clear.

(12) The isomorphism in the upper right of triangle 12 is induced by the isomorphism $\omega_{P/S} \cong \omega_{P/Y} \otimes \pi^*\omega_{Y/S}$ and the projection formula. The triangle commutes by [Conrad 2000, Theorem 2.5.2, 1].

Thus diagram on page 773 is commutative. The composition of the vertical maps along the left outer edge of the diagram equals $\tau_f^n$ by definition. The composition of the vertical maps along the right outer edge of the diagram is by [Conrad 2000, Theorem 2.5.2, 1] equal to the composition

$$f_*(\omega_{X/P} \otimes i^*\omega_{P/S}) \xrightarrow{\simeq} f_*(\omega_{X/P} \otimes i^*\omega_{P/Y}) \otimes \omega_{Y/S} \xrightarrow{\eta_i^{-1}\otimes \mathrm{id}} f_* \mathcal{E}xt_P^d(i_*\mathcal{O}_X, \omega_{P/Y}) \otimes \omega_{Y/S}.$$

All together we see that $\tau_f^n$ equals the composition

$$f_*\omega_{X/S} \xrightarrow{\zeta'_{i,\pi_P}} f_*(\omega_{X/P} \otimes i^*\omega_P) \cong f_*(\omega_{X/P} \otimes i^*\omega_{P/Y}) \otimes \omega_{Y/S} \xrightarrow{\eta_i^{-1}\otimes \mathrm{id}} f_*(i^\flat \pi^\# \mathcal{O}_Y) \otimes \omega_{Y/S} \xrightarrow{\psi_{i,\pi}^{-1}} f_* f^\flat \mathcal{O}_Y \otimes \omega_Y \cong \mathcal{H}om_Y(f_*\mathcal{O}_X, \mathcal{O}_Y) \otimes \omega_{Y/S} \xrightarrow{\text{eval. at } 1} \omega_{Y/S}.$$

Hence the claim follows from the definition of $\zeta'_{i,\pi_P}$ (see (1) above) and the definition of the residue symbol in [Conrad 2000, (A.1.4)]. □

## Acknowledgments

We are deeply grateful to Hélène Esnault for her encouragement and patience. We thank Manuel Blickle, Georg Hein and Eckart Viehweg for useful discussions. Finally, we thank the referee for a careful reading and for suggesting many improvements in the exposition.

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Communicated by János Kollár
Received 2009-12-02 Revised 2011-01-17 Accepted 2011-03-01

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Algebra & Number Theory (ISSN 1937-0652) at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY
mathematical sciences publishers
http://msp.org/
A NON-PROFIT CORPORATION
Typeset in LaTeX

Algebra & Number Theory

Volume 5 No. 6 2011