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The behavior of Hecke L-functions of real quadratic fields at $s = 0$

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For a family of real quadratic fields $\{K_n = \mathbb{Q}(\sqrt{f(n)})\}_{n\in\mathbb{N}}$, a Dirichlet character χ modulo q, and prescribed ideals $\{\mathfrak{b}_n \subset K_n\}$, we investigate the linear behavior of the special value of the partial Hecke L-function $L_{K_n}(s, \chi_n := \chi \circ N_{K_n}, \mathfrak{b}_n)$ at $s = 0$. We show that for $n = qk + r$, $L_{K_n}(0, \chi_n, \mathfrak{b}_n)$ can be written as

$$\frac{1}{12q^2}(A_\chi(r) + kB_\chi(r)),$$

where $A_\chi(r), B_\chi(r) \in \mathbb{Z}[\chi(1), \chi(2), \dots, \chi(q)]$ if a certain condition on $\mathfrak{b}_n$ in terms of its continued fraction is satisfied. Furthermore, we write $A_\chi(r)$ and $B_\chi(r)$ explicitly using values of the Bernoulli polynomials. We describe how the linearity is used in solving the class number one problem for some families and recover the proofs in some cases.

1. Introduction

In this paper, we are mainly concerned with linear behavior of the special values of the Hecke L-function at $s = 0$ for families of real quadratic fields.

Let $\{K_n = \mathbb{Q}(\sqrt{f(n)})\}_{n\in\mathbb{N}}$ be a family of real quadratic fields where $f(n)$ is a positive square free integer for each n. For example $f(x)$ can be a polynomial with integer coefficients.

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For a Dirichlet character χ modulo q, we have a ray class character $\chi_n := \chi \circ N_{K_n}$ for each n. Fixing an ideal $\mathfrak{b}_n$ in K_n for each n, one obtains an indexed family of partial Hecke L-functions $\{L_{K_n}(s, \chi_n, \mathfrak{b}_n)\}$, where the partial Hecke L-function for $(K, \chi, \mathfrak{b})$ is defined as

$$L_K(s, \chi, \mathfrak{b}) := \sum_{\substack{\mathfrak{a}\sim\mathfrak{b} \text{ integral} \\ (q,\mathfrak{a})=1}} \chi(\mathfrak{a}) N(\mathfrak{a})^{-s}.$$

and $\mathfrak{a} \sim \mathfrak{b}$ means that $\mathfrak{a} = \alpha\mathfrak{b}$ for totally positive $\alpha \in K$.

Roughly speaking, if $L_{K_n}(0, \chi_n, \mathfrak{b}_n)$ can be written as linear polynomial in k with coefficients depending only on r for $n = qk + r$, we say that $L_{K_n}(0, \chi_n, \mathfrak{b}_n)$ is linear.

Definition 1.1 (linearity). When the special values of $L_{K_n}(s, \chi_n, \mathfrak{b}_n)$ at $s = 0$ are expressed as

$$L_{K_n}(0, \chi_n, \mathfrak{b}_n) = \frac{1}{12q^2}(A_\chi(r) + kB_\chi(r))$$

for $n = qk+r$, $A_\chi(r), B_\chi(r) \in \mathbb{Z}[\chi(1), \chi(2), \dots, \chi(q)]$, we say that $L_{K_n}(0, \chi_n, \mathfrak{b}_n)$ is *linear*.

Linearity was originally observed by Biró in his proof of Yokoi's conjecture.

Theorem 1.2 [Biró 2003b]. *If the class number of* $\mathbb{Q}(\sqrt{n^2+4})$ *is* 1, *then* $n \leq 17$.

In Yokoi's conjecture, we take $K_n = \mathbb{Q}(\sqrt{n^2+4})$ and $\mathfrak{b}_n = O_{K_n}$. Biró [2003b, pp. 88, 89] expressed the special value of the Hecke L-function for (K_n, χ_n, O_{K_n}) at $s = 0$ for $n = qk + r$

$$L_{K_n}(0, \chi_n, \mathfrak{b}_n) = \frac{1}{q}(A_\chi(r) + kB_\chi(r)), \tag{1-1}$$

where

$$A_\chi(r) = \sum_{0\leq C,D\leq q-1} \chi(D^2 - C^2 - rCD)\left\lceil \frac{rC - D}{q} \right\rceil (C - q),$$

$$B_\chi(r) = \sum_{0\leq C,D\leq q-1} \chi(D^2 - C^2 - rCD)C(C - q).$$

When K_n is of class number 1, the unique ideal class can be represented by any ideal $\mathfrak{b}_n$. *A priori* the partial Hecke L-function equals the total Hecke L-function up to multiplication by 2 (that is,

$$L_{K_n}(0, \chi_n) = cL_{K_n}(0, \chi_n, O_{K_n})$$

where c is the number of narrow ideal classes).

From this identification, one can find the residue of n by sufficiently many primes p for which the class number of $\mathbb{Q}(\sqrt{n^2+4})$ is one. Moreover, by the linearity, this residue depends only on r. Consequently, one can tell whether or not p is inert in $\mathbb{Q}(\sqrt{n^2+4})$. As we have a bound for a smaller prime to inert depending on n, finally we have enough conditions to list all K_n of class number 1.

Other families $(K_n, \chi_n, \mathfrak{b}_n)$ that have linearity were discovered in [Biró 2003a; Byeon et al. 2007; Byeon and Lee 2008; Lee 2009a; 2009b]. Similarly, developing Biro's method, one can solve the associated class number one problems.

In this paper, we give a criterion on $(K_n, \chi_n, \mathfrak{b}_n)$ for $L_{K_n}(0, \chi_n, \mathfrak{b}_n)$ to be linear. The criterion is in terms of the continued fraction expression of $\delta(n)$, where $\mathfrak{b}_n^{-1} = [1, \delta(n)] := \mathbb{Z} + \delta(n)\mathbb{Z}$. Let $[\![a_0, a_1, \dots, a_n]\!]$ be the purely periodic continued fraction

$$[a_0, a_1, a_2, \dots, a_n, a_0, a_1, \dots],$$

where

$$[a_0, a_1, a_2, \dots] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}.$$

Our main theorem is as follows:

Theorem 1.3 (linearity criterion). *Let $\{K_n = \mathbb{Q}(\sqrt{f(n)})\}_{n\in\mathbb{N}}$ be a family of real quadratic fields where $f(n)$ is a positive square free integer for each n. Let χ be a Dirichlet character modulo q for a positive integer q and χ_n be a ray class character modulo q defined by $\chi \circ N_{K_n}$. Suppose $\mathfrak{b}_n$ is an integral ideal relatively prime to q such that $\mathfrak{b}_n^{-1} = [1, \delta(n)]$. Assume the continued fraction expansion of $\delta(n) - 1$*

$$\delta(n) - 1 = [\![a_0(n), a_1(n), \dots, a_{s-1}(n)]\!]$$

is purely periodic and of a fixed length s independent of n and $a_i(n) = \alpha_i n + \beta_i$ for some fixed $\alpha_i, \beta_i \in \mathbb{Z}$.

If $N_{K_n}(\mathfrak{b}_n(C + D\delta(n)))$ modulo q is a function only depending on C, D and r for $n = qk + r$, then $L_{K_n}(0, \chi_n, \mathfrak{b}_n)$ is linear.

Furthermore, we give a precise description of $A_\chi(r)$ and $B_\chi(r)$ using values of the Bernoulli polynomials (Proposition 3.8). From this description, for n with $h(K_n) = 1$, as in Biró's case, one can compute the residue of n modulo p depending on the mod-q residue r of n. There are possibly many (q, p) pairs. The more pairs of (q, p) we have, the more we can restrict possible n. There are many known families for which the class number one problem can be solved in this way. Many known results can be recovered by using the continued fraction expansion to show linearity and finding enough (q, p).

There are still other families of real quadratic fields with linearity whose class number one problems are not yet answered. Morally, once we obtain a reasonable class number one criterion, finding sufficiently many (q, p)-pairs should solve it.

This paper is composed as follows. In Section 2, we describe the special value at $s = 0$ of the partial Hecke L-function in terms of values of the Bernoulli polynomials. Section 3 is devoted to the proof of our main theorem. In Section 4, Biró's method is sketched as a prototype to apply the linearity. Section 5 concludes the paper with a possible generalization of the linearity criterion to a polynomial of higher order.

Notation and conventions. Throughout this article, we keep the following general notation and conventions. If necessary, we rewrite the notation at the place where it is used.

(1) K is a real quadratic field.

(2) For a real quadratic field K, we fix an embedding $\iota : K \to \mathbb{R}$. If there is no danger of confusion, we denote $\iota(\alpha)$ by an element $\alpha \in K$. α' denotes the conjugate of α as well as $\iota(\alpha')$.

(3) For $\alpha \in K$, $N_K(\alpha)$ denotes the norm of α over $\mathbb{Q}$. If there is no danger of confusion, we simply write $N(\alpha)$ to denote $N_K(\alpha)$. For an integral ideal $\mathfrak{a}$ of K, we let $N(\mathfrak{a}) := [\mathfrak{o}_K, \mathfrak{a}]$ denote the norm of $\mathfrak{a}$.

(4) For two linearly independent elements $\alpha, \beta \in K$ viewed as a vector space over $\mathbb{Q}$, $[\alpha, \beta]$ denotes the lattice (ie. free abelian group) generated by α and β. The lattice defined by a fractional ideal $\mathfrak{a}$ of K is denoted by $[\alpha, \beta]$ if $\{\alpha, \beta\}$ is a free basis of $\mathfrak{a}$.

(5) For a subset A of K, we denote by A^+ the set of totally positive elements in A.

(6) χ is a fixed Dirichlet character of modulus q.

(7) For a real number x,

$$\langle x \rangle := \begin{cases} x - [x] & \text{for } x \notin \mathbb{Z}, \\ 1 & \text{for } x \in \mathbb{Z}. \end{cases}$$

Equivalently, $\langle - \rangle$ is the composition $\mathbb{R} \xrightarrow{\text{mod } \mathbb{Z}} \mathbb{R}/\mathbb{Z} \to \mathbb{R}$, where $\mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is the unique map so that the composition is the identity on $(0, 1]$.

(8) For a real x, $[x]_1 := x - \langle x \rangle$.

(9) For an integer m, $\langle m \rangle_q$ denotes the residue of m by q taken in $[1, q]$ (i.e., $m = qk + \langle m \rangle_q$ for $k \in \mathbb{Z}$, $\langle m \rangle_q \in [1, q] \cap \mathbb{Z}$).

(10) For positive integers a_i, $[a_0, a_1, a_2, \dots]$ denotes the usual continued fraction:

$$[a_0, a_1, a_2, \dots] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

$[a_0, a_1, \dots, a_{i-1}, \overline{a_i, a_{i+1}, \dots, a_{i+j}}]$ denotes the continued fraction with periodic part $(a_i, a_{i+1}, \dots, a_{i+j})$.

$[\![a_0, a_1, \dots, a_n]\!]$ is the purely periodic continued fraction

$$[a_0, a_1, \dots, a_n, a_0, a_1, \dots].$$

(11) $(a_0, a_1, a_2, \dots)$ denotes the minus continued fraction:

$$(a_0, a_1, a_2, \dots) := a_0 - \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cdots}}$$

$((a_0, a_1, \dots, a_n))$ is the purely periodic minus continued fraction:

$$(a_0, a_1, a_2, \dots, a_n, a_0, a_1, \dots)$$

(12) For an integer s, $\mu(s) = 1$ if s is odd and $\mu(s) = \frac{1}{2}$ if s is even.

2. Partial Hecke L-function

Throughout this section, K denotes a real quadratic field and $\mathfrak{b}$ is a fixed integral ideal of K relatively prime to q such that $\mathfrak{b}^{-1} = [1, \delta]$ for $\delta \in K$ satisfying $0 < \delta' < 1$ and $\delta > 2$.

A *ray class character* modulo q is a homomorphism

$$\chi : I_K(q)/P_K(q) \to \mathbb{C}^*$$

where $I_K(q)$ is a group of fractional ideals of K which is relatively prime to q and $P_K(q)$ is a subgroup of principal ideals (α) for totally positive $\alpha \equiv 1 \pmod q$.

Define

$$F := \{(C, D) \in \mathbb{Z}^2 \mid 0 \le C, D \le q - 1, ((C + D\delta)\mathfrak{b}, q) = 1\}.$$

Let E^+ be the set of totally positive units in K, and E_q^+ the set of totally positive units congruent to 1 mod q. Then $\epsilon \in E^+$ acts on the set F by the rule

$$\epsilon * (C + D\delta) = C' + D'\delta,$$

where C' and D' are given by

$$\epsilon \cdot (C + D\delta) + q\mathfrak{b}^{-1} = C' + D'\delta + q\mathfrak{b}^{-1} \quad \text{for } \epsilon \in E^+.$$

Lemma 2.1. *(C, D) in F is fixed by the action of ϵ if and only if ϵ is in E_q^+.*

Proof. (C, D) is fixed by $\epsilon \in E^+$ if and only if $(C + D\delta)(\epsilon - 1) \in q\mathfrak{b}^{-1}$. Since $(\mathfrak{b}(C + D\delta), q) = 1$, the condition $(C + D\delta)(\epsilon - 1) \in q\mathfrak{b}^{-1}$ is equivalent to

$$\epsilon \equiv 1 \pmod q. \qquad \square$$

Lemma 2.2. *Suppose $0 \le C, D \le q - 1$. Then the following are equivalent:*

(1) *(C, D) is in F.*

(2) *For every $\alpha \in (C + D\delta)/q + \mathfrak{b}^{-1}$, the ideal $q\alpha\mathfrak{b}$ is relatively prime to q.*

(3) *For a $\alpha \in (C + D\delta)/q + \mathfrak{b}^{-1}$, the ideal $q\alpha\mathfrak{b}$ is relatively prime to q.*

Proof. Suppose that $(q, (C + D\delta)\mathfrak{b}) = 1$.

We have

$$\frac{q\alpha}{C + D\delta} \in 1 + \frac{q}{C + D\delta}\mathfrak{b}^{-1}$$

for $\alpha \in (C + D\delta)/q + \mathfrak{b}^{-1}$. Thus $(q, \mathfrak{b}(C + D\delta)) = 1$ implies that

$$\frac{q\alpha}{C + D\delta} \equiv 1 \pmod q.$$

Since

$$q\mathfrak{b}\alpha = \mathfrak{b}(C + D\delta)\frac{q\alpha}{C + D\delta},$$

we have

$$(q\mathfrak{b}\alpha, q) = 1.$$

If $(q, (C + D\delta)\mathfrak{b}) \ne 1$, then $(q, q\mathfrak{b}\alpha) \ne 1$ for $\alpha \in (C + D\delta)/q + \mathfrak{b}^{-1}$, since for $\alpha \in (C + D\delta)/q + \mathfrak{b}^{-1}$, we have

$$q\mathfrak{b}\alpha \subset (C + D\delta)\mathfrak{b} + qO_K. \qquad \square$$

Let $F' = F/E^+$ be the orbit space of the action of E^+ on F. Let $\tilde{F}'$ be a fundamental set of F'. Let ϵ be the totally positive fundamental unit. The order of the action of ϵ is $\lambda := [E^+ : E_q^+]$ by Lemma 2.1. Then we can decompose F as follows:

$$F = \bigsqcup_{i=0}^{\lambda-1} \epsilon^i \tilde{F}'. \tag{2-1}$$

According to this decomposition of F, we can further decompose the partial Hecke L-function:

Proposition 2.3. *Let q be a positive integer. Given an ideal $\mathfrak{b} \subset K$ as specified at the beginning of this section and a ray class character χ modulo q, we have*

$$\begin{aligned} L_K(s, \chi, \mathfrak{b}) &= \sum_{\substack{\mathfrak{a}\sim\mathfrak{b} \text{ integral} \\ (q,\mathfrak{a})=1}} \chi(\mathfrak{a}) N(\mathfrak{a})^{-s} \\ &= \sum_{(C,D)\in\tilde{F}'} \chi((C + D\delta)\mathfrak{b}) \sum_{\alpha\in(\frac{C+D\delta}{q}+\mathfrak{b}^{-1})^+/E_q^+} N(q\mathfrak{b}\alpha)^{-s}. \end{aligned}$$

Proof. For $\alpha_1, \alpha_2 \in (q^{-1}\mathfrak{b}^{-1})^+$, $q\alpha_1\mathfrak{b} = q\alpha_2\mathfrak{b}$ if and only if $\alpha_1/\alpha_2 \in E^+$.

So we have

$$\sum_{\substack{\mathfrak{a}\sim\mathfrak{b} \text{ integral} \\ (q,\mathfrak{a})=1}} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s} = \sum_{\substack{\mathfrak{a}\sim q\mathfrak{b} \text{ integral} \\ (q,\mathfrak{a})=1}} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s} = \sum_{\substack{\alpha\in(q^{-1}\mathfrak{b}^{-1})^+/E^+ \\ (q,q\alpha\mathfrak{b})=1}} \frac{\chi(q\alpha\mathfrak{b})}{N(q\alpha\mathfrak{b})^s}$$

For a totally positive fundamental unit $\epsilon > 1$, we also have

$$\begin{aligned} \sum_{\substack{\alpha\in(q^{-1}\mathfrak{b}^{-1})^+/E_q^+ \\ (q,q\mathfrak{b}\alpha)=1}} \frac{\chi(q\mathfrak{b}\alpha)}{N(q\mathfrak{b}\alpha)^s} &= \sum_{\substack{\alpha\in(q^{-1}\mathfrak{b}^{-1})^+/E^+ \\ (q,q\mathfrak{b}\alpha)=1}} \sum_{i=0}^{\lambda-1} \frac{\chi(q\mathfrak{b}\alpha\epsilon^i)}{N(q\mathfrak{b}\alpha\epsilon^i)^s} \\ &= \lambda \cdot \sum_{\substack{\alpha\in(q^{-1}\mathfrak{b}^{-1})^+/E^+ \\ (q,q\mathfrak{b}\alpha)=1}} \frac{\chi(q\mathfrak{b}\alpha)}{N(q\mathfrak{b}\alpha)^s}. \end{aligned}$$

And from Lemma 2.2, we have

$$\begin{aligned} \sum_{\substack{\alpha\in(q^{-1}\mathfrak{b}^{-1})^+/E_q^+ \\ (q,q\mathfrak{b}\alpha)=1}} \frac{\chi(q\mathfrak{b}\alpha)}{N(q\mathfrak{b}\alpha)^s} &= \sum_{(C,D)\in F} \sum_{\substack{\alpha\in(\frac{C+D\delta}{q}+\mathfrak{b}^{-1})^+/E_q^+ \\ (q,q\mathfrak{b}\alpha)=1}} \frac{\chi(q\mathfrak{b}\alpha)}{N(q\mathfrak{b}\alpha)^s} \\ &= \sum_{(C,D)\in F} \sum_{\alpha\in(\frac{C+D\delta}{q}+\mathfrak{b}^{-1})^+/E_q^+} \frac{\chi(q\mathfrak{b}\alpha)}{N(q\mathfrak{b}\alpha)^s}. \end{aligned}$$

By equation (2), the above is equal to

$$\sum_{(C,D)\in\tilde{F}'} \sum_{i=0}^{\lambda-1} \sum_{\alpha\in(\frac{(C+D\delta)\epsilon^i}{q}+\mathfrak{b}^{-1})^+/E_q^+} \frac{\chi(q\mathfrak{b}\alpha)}{N(q\mathfrak{b}\alpha)^s}.$$

Since

$$\sum_{\alpha\in(\frac{(C+D\delta)\epsilon^i}{q}+\mathfrak{b}^{-1})^+/E_q^+} \frac{\chi(q\mathfrak{b}\alpha)}{N(q\mathfrak{b}\alpha)^s} = \sum_{\alpha\in(\frac{(C+D\delta)}{q}+\mathfrak{b}^{-1})^+/E_q^+} \frac{\chi(q\mathfrak{b}\alpha\epsilon^i)}{N(q\mathfrak{b}\alpha\epsilon^i)^s},$$

the above also equal to

$$\lambda \cdot \sum_{(C,D)\in\tilde{F}'} \sum_{\alpha\in(\frac{C+D\delta}{q}+\mathfrak{b}^{-1})^+/E_q^+} \frac{\chi(q\mathfrak{b}\alpha)}{N(q\mathfrak{b}\alpha)^s}.$$

Note that for $\alpha \in (\frac{C+D\delta}{q}+\mathfrak{b}^{-1})^+$, $q\mathfrak{b}\alpha$ and $(C+D\delta)\mathfrak{b}$ are in the same ray class modulo q. Thus $\chi(q\mathfrak{b}\alpha) = \chi((C+D\delta)\mathfrak{b})$. This completes the proof. □

Shintani–Zagier cone decomposition. We review briefly the decomposition of $(\mathbb{R}^2)^+$ into cones due to Shintani [1976] and Zagier [1975] (see also [van der Geer 1988]). This depends on a real quadratic field K and a fixed ideal $\mathfrak{a}$ inside. Here for the sake of computation, we fix $\mathfrak{a} = \mathfrak{b}^{-1}$ where $\mathfrak{b}$ is set as in the beginning of this section.

K is embedded into $\mathbb{R}^2$ by $\iota = (\tau_1, \tau_2)$, where τ_1, τ_2 are two real embeddings of K. In particular, the totally positive elements of K land on $(\mathbb{R}^2)^+$. We are going to describe the fundamental domain of $(\frac{C+D\delta}{q}+\mathfrak{b}^{-1})^+/E_q^+$ embedded into $(\mathbb{R}^2)^+$.

The multiplicative action of ${E_q}^+$ on K^+ induces an action on $(\mathbb{R}^2)^+$ by coordinatewise multiplication:

$$\epsilon \circ (x, y) = (\tau_1(\epsilon)x, \tau_2(\epsilon)y).$$

A fundamental domain $\mathfrak{D}_{\mathbb{R}}$ of $(\mathbb{R}^2)^+/E_q^+$ is given by

$$\mathfrak{D}_{\mathbb{R}} := \{x\iota(1) + y\iota(\epsilon^{-\lambda}) \mid x > 0,\ y \ge 0\} \subset (\mathbb{R}^2)^+ \tag{2-2}$$

where $E_q^+ = \langle \epsilon^\lambda \rangle$ for an integer λ and $\epsilon > 1$ is the unique totally positive fundamental unit.

If we take the convex hull of $\iota(\mathfrak{b}^{-1}) \cap (\mathbb{R}^2)^+$ in $(\mathbb{R}^2)^+$, the vertices on the boundary are $\{P_i\}_{i\in\mathbb{Z}}$ for $P_i \in \iota(\mathfrak{b}^{-1})$, and determined by the conditions $P_0 = \iota(1)$, $P_{-1} = \iota(\delta)$ and $x(P_i) < x(P_{i-1})$ where $x(P_k)$ denotes the first coordinate of P_k for $k \in \mathbb{Z}$. Since any two consecutive boundary points make a basis of $\iota(\mathfrak{b}^{-1})$, we find that

$$\begin{pmatrix} 0 & 1 \\ -1 & b_i \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \end{pmatrix} = \begin{pmatrix} P_i \\ P_{i+1} \end{pmatrix},$$

for an integer b_i. It is easy to see that $b_i \ge 2$ from the convexity. Thus we obtain

$$x(P_{i-1}) + x(P_{i+1}) = b_i x(P_i). \tag{2-3}$$

Put $\delta_i := \dfrac{x(P_{i-1})}{x(P_i)} > 1$. Note that $\delta_0 = \delta$. δ_i satisfies a recursion relation:

$$\delta_i = b_i - \frac{1}{\delta_{i+1}} \quad \text{for } i \in \mathbb{Z}.$$

Therefore

$$\delta_i = b_i - \cfrac{1}{b_{i+1} - \cfrac{1}{b_{i+2} - \cdots}} = (b_i, b_{i+1}, b_{i+2}, \ldots).$$

Let $\epsilon > 1$ be the totally positive fundamental unit. Then ϵ moves a boundary point to another boundary point, preserving the order. Thus, there exists a positive integer m so that for all $i \in \mathbb{Z}$

$$\epsilon \circ P_i = P_{i-m}. \tag{2-4}$$

Therefore we obtain the following proposition.

Proposition 2.4. (1) $\delta_{i+m} = \delta_i$ *for all* $i \in \mathbb{Z}$.

(2) $\delta_i = ((b_i, b_{i+1}, \ldots, b_{i+m-1})) = b_i - \cfrac{1}{b_{i+1} - \cdots \cfrac{1}{b_{i+m-1} - \cfrac{1}{b_i - \cdots}}}$.

(3) $\iota(\epsilon^{-1}) = P_m$.

(4) $\epsilon^{-1} \circ P_i = P_{i+m}$.

(5) $\iota(\epsilon^{-\gamma}) = P_{\gamma m}$.

Proof. (1) $\delta_{i+m} = \dfrac{x(P_{i+m-1})}{x(P_{i+m})} = \dfrac{\epsilon x(P_{i-1})}{\epsilon x(P_i)} = \delta_i$.

(2) This is an immediate consequence of (1).

(3) From (2-4),

$$P_m = \epsilon^{-1} \circ P_0,$$

since $P_0 = \iota(1)$ and $\epsilon^{-1} \circ \iota(1) = \iota(\epsilon^{-1})$.

(4) This is immediate from (2-4).

(5) This follows trivially from (3) and (4). □

Using (2-2) and Proposition 2.4(4), the fundamental domain $\mathfrak{D}_{\mathbb{R}}$ of $(\mathbb{R}^2)^+/E_q^+$ is further decomposed into the disjoint union of λm smaller cones:

$$\mathfrak{D}_{\mathbb{R}} = \bigsqcup_{i=1}^{\lambda m} \{x P_{i-1} + y P_i \mid x > 0,\ \ y \geq 0\}.$$

Clearly, the fundamental set of the quotient $\big(\iota((C + D\delta)/q + \mathfrak{b}^{-1}) \cap (\mathbb{R}^2)^+\big)/E_q^+$ inside $\mathfrak{D}_{\mathbb{R}}$, which we denote by $\mathfrak{D}$, is given by a disjoint union:

$$\mathfrak{D} := \bigsqcup_{i=1}^{\lambda m} \Big(\iota\Big(\frac{C+D\delta}{q} + \mathfrak{b}^{-1}\Big) \cap \{x P_{i-1} + y P_i \mid x > 0,\ y \geq 0\}\Big).$$

Since $\{P_{i-1}, P_i\}$ is a $\mathbb{Z}$-basis of $\iota(\mathfrak{b}^{-1})$, there is a unique $(x^i_{C+D\delta}, y^i_{C+D\delta}) \in (0,1] \times [0,1)$ such that

$$x^i_{C+D\delta} P_{i-1} + y^i_{C+D\delta} P_i \in \iota\Big(\frac{C+D\delta}{q} + \mathfrak{b}^{-1}\Big),$$

for each $i, C, D \in \mathbb{Z}$. Thus

$$\begin{aligned}\iota\Big(\frac{C+D\delta}{q} + \mathfrak{b}^{-1}\Big) &\cap \{xP_{i-1} + yP_i \mid x > 0,\ \ y \geq 0\} \\ &= \{(x^i_{C+D\delta} + n_1)P_{i-1} + (y^i_{C+D\delta} + n_2)P_i \mid n_1, n_2 \in \mathbb{Z}_{\geq 0}\}. \quad (2\text{-}5)\end{aligned}$$

Yamamoto [2008, (2.1.3)] found that $(x^i_{C+D\delta}, y^i_{C+D\delta})$ satisfy the following recurrence relations:

$$\begin{aligned} x^{i+1}_{C+D\delta} &= \langle b_i x^i_{C+D\delta} + y^i_{C+D\delta}\rangle, \\ y^{i+1}_{C+D\delta} &= 1 - x^i_{C+D\delta}. \end{aligned} \tag{2-6}$$

Let $A_i := x(P_i)$ for all $i \in \mathbb{Z}$. Then from (2-5), we obtain the following:

$$\begin{aligned}\sum_{\alpha \in (\frac{C+D\delta}{q} + \mathfrak{b}^{-1})^+/E^+_q} \frac{1}{N(\alpha)^s} &= \sum_{i=1}^{\lambda m} \sum_{n_1, n_2 \geq 0} N\big((x^i_{C+D\delta} + n_1)A_{i-1} + (y^i_{C+D\delta} + n_2)A_i\big)^{-s} \\ &= \sum_{i=1}^{\lambda m} \sum_{n_1, n_2 \geq 0} N\big((x^i_{C+D\delta} + n_1)\delta_i + (y^i_{C+D\delta} + n_2)\big)^{-s} A_i^{-s}. \quad (2\text{-}7)\end{aligned}$$

Shintani [1976] evaluated $\sum_{n_1,n_2 \geq 0} N\big((x+n_1)\delta + (y+n_2)\big)^{-s}$ for nonpositive integers s. In particular, the value at $s = 0$ is expressed by first and second Bernoulli polynomials as follows:

Lemma 2.5 (Shintani).

$$\begin{aligned}\sum_{n_1,n_2 \geq 0} N\big((x+n_1)\delta + (y+n_2)\big)^{-s}\Big|_{s=0} \\ = \frac{\delta+\delta'}{4} B_2(x) + B_1(x)B_1(y) + \frac{1}{4}\Big(\frac{1}{\delta} + \frac{1}{\delta'}\Big) B_2(y).\end{aligned}$$

Using this, we have

$$\sum_{\alpha\in(\frac{C+D\delta}{q}+\mathfrak{b}^{-1})^+/E_q^+} \frac{1}{N(\alpha)^s}\Big|_{s=0}$$
$$=\sum_{i=1}^{\lambda m}\frac{\delta_i+\delta_i'}{4}B_2(x^i_{C+D\delta})+B_1(x^i_{C+D\delta})B_1(y^i_{C+D\delta})+\frac{1}{4}\Big(\frac{1}{\delta_i}+\frac{1}{\delta_i'}\Big)B_2(y^i_{C+D\delta}). \tag{2-8}$$

This simplifies further:

Lemma 2.6 [Yamamoto 2008, proof of Theorem 4.1.1].

$$\sum_{i=1}^{\lambda m}\frac{\delta_i+\delta_i'}{4}B_2(x^i_{C+D\delta})+\frac{1}{4}(\frac{1}{\delta_i}+\frac{1}{\delta_i'})B_2(y^i_{C+D\delta})=\sum_{i=1}^{\lambda m}\frac{b_i}{2}B_2(x^i_{C+D\delta}).$$

Finally, we have

$$\sum_{\alpha\in(\frac{C+D\delta}{q}+\mathfrak{b}^{-1})^+/E_q^+} \frac{1}{N(\alpha)^s}\Big|_{s=0}=\sum_{i=1}^{\lambda m}B_1(x^i_{C+D\delta})B_1(y^i_{C+D\delta})+\frac{b_i}{2}B_2(x^i_{C+D\delta}).$$

Lemma 2.7. *Let ϵ be the totally positive fundamental unit of K. Then*

$$x^{mi+j}_{C+D\delta}=x^{j}_{\epsilon^i*(C+D\delta)}\quad \textit{and}\quad y^{mi+j}_{C+D\delta}=y^{j}_{\epsilon^i*(C+D\delta)}$$

for $j=0,1,2,\dots,m-1$.

Proof. From (4) of Proposition 2.4, we have $A_{mi+j}=\epsilon^{-i}A_j$, for any integer i. Thus

$$x^{mi+j}_{C+D\delta}A_{mi+j-1}+y^{mi+j}_{C+D\delta}A_{mi+j}=x^{mi+j}_{C+D\delta}\epsilon^{-i}A_{j-1}+y^{mi+j}_{C+D\delta}\epsilon^{-i}A_j\in\frac{C+D\delta}{q}+\mathfrak{b}^{-1}.$$

Therefore,

$$x^{mi+j}_{C+D\delta}A_{j-1}+y^{mi+j}_{C+D\delta}A_j\in\frac{\epsilon^i\cdot(C+D\delta)}{q}+\mathfrak{b}^{-1}.\qquad\square$$

From Lemma 2.7 and the periodicity of b_i, we have:

Lemma 2.8.
$$\sum_{\alpha\in(\frac{C+D\delta}{q}+\mathfrak{b}^{-1})^+/E_q^+} \frac{1}{N(\alpha)^s}\Big|_{s=0}$$
$$=\sum_{i=1}^{m}\sum_{j=0}^{\lambda-1}B_1(x^i_{\epsilon^j*(C+D\delta)})B_1(y^i_{\epsilon^j*(C+D\delta)})+\frac{b_i}{2}B_2(x^i_{\epsilon^j*(C+D\delta)}).$$

Finally, we have:

Proposition 2.9. *For a ray class character χ modulo q and an ideal $\mathfrak{b}$ of K such that*

$$\mathfrak{b}^{-1} = [1, \delta]$$

for $\delta \in K$ with $\delta > 2$ and $0 < \delta' < 1$, we have

$$L_K(0, \chi, \mathfrak{b}) = \sum_{1 \le C, D \le q} \chi((C + D\delta)\mathfrak{b}) \sum_{i=1}^{m} B_1(x^i_{(C+D\delta)}) B_1(y^i_{C+D\delta}) + \frac{b_i}{2} B_2(x^i_{C+D\delta}).$$

Proof. From Proposition 2.3, we obtain

$$L_K(0, \chi, \mathfrak{b}) = \sum_{(C,D) \in \tilde{F}'} \chi((C + D\delta)\mathfrak{b}) \sum_{\alpha \in (\frac{C+D\delta}{q} + \mathfrak{b}^{-1})^+ / E_q^+} N(q\mathfrak{b}\alpha)^{-s}|_{s=0}.$$

Lemma 2.8 implies that this is equal to

$$\sum_{(C,D) \in \tilde{F}'} \chi\big((C+D\delta)\mathfrak{b}\big) \sum_{j=0}^{\lambda-1} \sum_{i=1}^{m} B_1(x^i_{\epsilon^j * (C+D\delta)}) B_1(y^i_{\epsilon^j * (C+D\delta)}) + \frac{b_i}{2} B_2(x^i_{\epsilon^j * (C+D\delta)}).$$

Since $(C + D\delta)\epsilon\mathfrak{b} = (C + D\delta)\mathfrak{b}$, this expression can be rewritten as

$$\sum_{(C,D) \in \tilde{F}'} \sum_{j=0}^{\lambda-1} \Bigg(\chi((C + D\delta)\epsilon^j \mathfrak{b}) \\ \times \sum_{i=1}^{m} B_1(x^i_{\epsilon^j * (C+D\delta)}) B_1(y^i_{\epsilon^j * (C+D\delta)}) + \frac{b_i}{2} B_2(x^i_{\epsilon^j * (C+D\delta)}) \Bigg).$$

In view of the decomposition of F in (2-1), the preceding expression equals

$$\sum_{(C,D) \in F} \chi((C + D\delta)\mathfrak{b}) \sum_{i=1}^{m} B_1(x^i_{(C+D\delta)}) B_1(y^i_{(C+D\delta)}) + \frac{b_i}{2} B_2(x^i_{(C+D\delta)}).$$

If $((C + D\delta)\mathfrak{b}, q) \neq 1$ then $\chi((C + D\delta)\mathfrak{b}) = 0$. Thus we complete the proof. □

Remark 2.10. The summation running over $C, D \in [1, q]$ is actually supported on F. This is justified by the twist of the mod q Dirichlet character. Obviously, F depends on δ in K, but the twisted sum has an invariant form of δ and K. This is a subtle point in the proof of the main theorem where we deal with values of the Hecke L-function with respect to a family $(K_n, \chi_n, \mathfrak{b})$.

3. Proof of the main theorem

In this section, we compute special values of the Hecke L-function for a family of real quadratic fields. The computation is made using the expression for the L-value from the previous section. After the computation, it will be apparent that the linearity property comes from the shape of the continued fractions in the family. This will complete the proof of Theorem 1.3.

This gives a criterion that will recover several approaches of class number problems for some families of real quadratic fields.

Consider a family of real quadratic fields $K_n = \mathbb{Q}(\sqrt{d_n})$, where d_n is a positive square free integer. For a fixed Dirichlet character χ of modulus q, we associate a ray class character $\chi_n := \chi \circ N_{K_n/\mathbb{Q}}$ for each n. Let us fix an ideal $\mathfrak{b}_n$ of K_n for each n. Then we have a family of Hecke L-functions associated to $(K_n, \chi_n, \mathfrak{b}_n)$:

$$L_{K_n}(s, \chi_n, \mathfrak{b}_n) = \sum_{\mathfrak{a}} \frac{\chi_n(\mathfrak{a})}{N(\mathfrak{a})^s}$$

where $\mathfrak{a}$ ranges over integral ideals in the ray class represented by $\mathfrak{b}_n$.

Plan of the proof. Assume that

$$\mathfrak{b}_n^{-1} = [1, \delta(n)]$$

with $\delta(n) > 2$, $0 < \delta(n)' < 1$. As discussed in Proposition 2.4, $\delta(n)$ has a purely periodic minus continued fraction expansion:

$$\begin{aligned}\delta(n) &= ((b_0(n), b_1(n), \ldots, b_{m(n)-1}(n))) \\ &= b_0(n) - \cfrac{1}{b_1(n) - \cdots \cfrac{1}{b_{m(n)-1}(n) - \cfrac{1}{b_0(n) - \cdots}}},\end{aligned} \tag{3-1}$$

with $b_k(n) \geq 2$.

We extend the definition of $b_i(n)$ to all $i \in \mathbb{Z}$ by requiring that $b_{i+m(n)}(n) = b_i(n)$ for $i \in \mathbb{Z}$, and take $\delta_k(n) = ((b_k(n), b_{k+1}(n), \ldots, b_{k+m(n)-1}(n)))$. We define $\{A_k(n)\}_{k\in\mathbb{Z}}$ by

$$A_{-1}(n) = \delta(n), \quad A_0(n) = 1, \quad \ldots, \quad A_{k+1}(n) = A_k(n)/\delta_{k+1}(n).$$

Then for fixed C, D and n, there is a unique $(x^i_{C+D\delta(n)}, y^i_{C+D\delta(n)})$ such that

$$0 < x^i_{C+D\delta(n)} \leq 1, \quad 0 \leq y^i_{C+D\delta(n)} < 1, \tag{3-2}$$

and

$$x^i_{C+D\delta(n)} A_{i-1}(n) + y^i_{C+D\delta(n)} A_i(n) \in \frac{C + D\delta(n)}{q} + \mathfrak{b}_n^{-1}, \tag{3-3}$$

for each $i \in \mathbb{Z}$, as described in the previous section. This $(x^i_{C+D\delta(n)}, y^i_{C+D\delta(n)})$ satisfies Yamamoto's recursive relation (2-6) as follows:

$$x^{i+1}_{C+D\delta(n)} = \langle b_i(n) x^i_{C+D\delta(n)} + y^i_{C+D\delta(n)} \rangle, \quad y^{i+1}_{C+D\delta(n)} = 1 - x^i_{C+D\delta(n)}. \tag{3-4}$$

Now recall a standard conversion formula from a continued fraction expansion to a minus continued fraction expansion:

Lemma 3.1. *Let $\delta - 1$ be a purely periodic continued fraction*:

$$[\![a_0, a_1, \dots, a_{s-1}]\!].$$

Then the minus continued fraction expansion of δ is

$$((b_0, b_1, \dots, b_{m-1})),$$

where

$$b_i := \begin{cases} a_{2j} + 2 & \text{for } i = S_j, \\ 2 & \text{otherwise}, \end{cases}$$

where

$$S_j = \begin{cases} 0 & \text{for } j = 0, \\ S_{j-1} + a_{2j-1} & \text{for } j \geq 1, \end{cases}$$

and the period m is given by

$$m = \begin{cases} a_1 + a_3 + a_5 \cdots + a_{s-1} = S_{s/2} & \text{for even } s, \\ a_0 + a_1 + a_2 \cdots + a_{s-1} = S_s & \text{for odd } s. \end{cases}$$

Proof. (See [Zagier 1975, pp. 177, 178].) If s is an odd integer, the period m is

$$\sum_{i=1}^{s} a_{2i-1} = a_1 + a_3 + \cdots + a_{2s-1} = S_s.$$

Since a_i has period s, we find that

$$a_1 + a_3 + \cdots + a_{2s-1} = a_0 + a_1 + a_2 \cdots + a_{s-1} = \sum_{i=0}^{s-1} a_i. \qquad \square$$

For the family of $\delta(n) \in K$, we assumed that

$$\delta(n) - 1 = [\![a_0(n), a_1(n), a_2(n), \dots, a_{s-1}(n)]\!]$$

has the same period for every n.

Then $\delta(n)$ has a purely periodic minus continued fraction expansion

$$\delta(n) = ((b_0(n), b_1(n), \dots, b_{m(n)-1}(n))),$$

with $b_i(n)$, $S_j(n)$ and $m(n)$ defined in the same manner as in the previous lemma.

One should be aware that $m(n)$ varies with n, while the period s of the positive continued fraction is fixed.

From Proposition 2.9 and the recursion (3-4) for $(x^i_{C+D\delta(n)}, y^i_{C+D\delta(n)})$, we have

$$L_{K_n}(0, \chi_n, \mathfrak{b}_n) = \sum_{1\le C,D\le q} \Bigl(\chi_n((C + D\delta(n))\mathfrak{b}_n) \times \sum_{i=1}^{m(n)} \bigl(B_1(x^i_{C+D\delta(n)})B_1(y^i_{C+D\delta(n)}) + \frac{b_i(n)}{2} B_2(x^i_{C+D\delta(n)})\bigr)\Bigr). \tag{3-5}$$

To check the linear behavior, it suffices to show that

$$\sum_{i=1}^{m(n)} \bigl(B_1(x^i_{C+D\delta(n)})B_1(y^i_{C+D\delta(n)}) + \frac{b_i(n)}{2} B_2(x^i_{C+D\delta(n)})\bigr) \tag{3-6}$$

is linear in k with the coefficients depending only on r.

Because $b_i(n) = 2$ if $i \ne S_j(n)$ for every j, we can divide the sum above into two parts:

$$\sum_{l=1}^{s\mu(s)} \bigl(-B_1(x^{S_l(n)}_{C+D\delta(n)})B_1(x^{S_l(n)-1}_{C+D\delta(n)}) + \frac{a_{2l}(n)+2}{2} B_2(x^{S_l(n)}_{C+D\delta(n)})\bigr) + \sum_{l=0}^{s\mu(s)-1} \sum_{i=S_l(n)+1}^{S_{l+1}(n)-1} F(x^i_{C+D\delta(n)}, x^{i-1}_{C+D\delta(n)}), \tag{3-7}$$

where $\mu(s) = \frac{1}{2}$ or 1 for s even or odd, respectively, and

$$F(x, y) := -B_1(x)B_1(y) + B_2(x).$$

We will use the following fact to be proved later. Here and wherever there is no danger of misunderstanding, $x_i(n)$ means $x^i_{C+D\delta(n)}$ for fixed C, D.

Claim. *The sequence $\{x_i(n)\}$ is a piecewise arithmetic progression, in the sense that it satisfies these properties*:

1. $\{x_i(n)\}_{S_j(n)\le i\le S_{j+1}(n)}$ *is an arithmetic progression mod $\mathbb{Z}$ with common difference $\langle x_{S_j(n)+1}(n) - x_{S_j(n)}(n)\rangle$.*
2. $\{x_i(n)\}_{S_j(n)\le i\le S_{j+1}(n)}$ *has period q.*
3. $x_{S_j(n)}(n)$, $x_{S_j(n)-1}(n)$ *and* $x_{S_j(n)+1}(n)$ *are independent of k, where $n = qk + r$.*

Because of the constraint $a_i(n) = \alpha_i n + \beta_i$, the value of $\langle a_i(n)\rangle_q$ is independent of k for $n = qk + r$ and depends only on i and r. We can thus set

$$\gamma_i(r) := \langle a_i(n)\rangle_q, \tag{3-8}$$

where $n = qk + r$. In particular, $\gamma_i(r) = \langle a_i(r)\rangle_q$.

Since $\{F(x_i(n), x_{i-1}(n))\}_{S_j(n)+1\le i\le S_{j+1}(n)-1}$ has period q (item 2 of the claim), we obtain

$$\sum_{i=S_l(n)+1}^{S_{l+1}(n)-1} F(x_i(n), x_{i-1}(n))$$
$$= \sum_{i=S_l(n)+1}^{S_l(n)+\gamma_{2l+1}(r)-1} F(x_i(n), x_{i-1}(n)) + \kappa_{2l+1}(n) \sum_{i=S_l(n)+1}^{S_l(n)+q} F(x_i(n), x_{i-1}(n)),$$

where $a_i(n) = \kappa_i(n)q + \gamma_i(r)$ for an integer $\kappa_i(n)$. Written precisely,

$$\kappa_i(n) = \frac{a_i(n) - \gamma_i(r)}{q}. \tag{3-9}$$

Since

$$\alpha_i r + \beta_i = q\tau_i(r) + \gamma_i(r)$$

for some integer $\tau_i(r)$, we can write for $n = qk + r$

$$\kappa_i(n) = k\alpha_i + \tau_i(r) \tag{3-10}$$

Using 3, we see that $x_{S_l(n)}(n)$ and $x_{S_l(n)+1}(n)$ are determined by the residue r of n by q. A priori the sums

$$\sum_{i=S_l(n)+1}^{S_l(n)+\gamma_{2l+1}(r)-1} F(x_i(n), x_{i-1}(n)) \quad \text{and} \quad \sum_{i=S_l(n)+1}^{S_l(n)+q} F(x_i(n), x_{i-1}(n))$$

are completely determined by $x_{S_l(n)}(n)$ and $x_{S_l(n)+1}(n)$ and remain unchanged while k varies.

Thus we conclude:

Fact I. *For $n = qk + r$,*

$$\sum_{i=S_l(n)+1}^{S_{l+1}(n)-1} F(x_i(n), x_{i-1}(n))$$

is a linear function of k.

Using (3-9) and (3-10), we have

$$-B_1(x_{S_l(n)}(n))B_1(x_{S_l(n)-1}(n)) + \frac{a_{2l}(n)+2}{2} B_2(x_{S_l(n)}(n))$$
$$= -B_1(x_{S_l(n)}(n))B_1(x_{S_l(n)-1}(n)) + \frac{\alpha_{2l}qk + \tau_{2l}(r)q + \gamma_{2l}(r) + 2}{2} B_2(x_{S_l(n)}(n)).$$

Again using item 3 of the Claim we conclude:

Fact II. *For $n = qk + r$,*

$$-B_1(x_{S_l(n)}(n))B_1(x_{S_l(n)-1}(n)) + \frac{a_{2l}(n)+2}{2}B_2(x_{S_l(n)}(n))$$

is a linear function of k.

Additionally, we have:

Fact III. *s and $\mu(s)$ are independent of n.*

Together, Facts I, II and III imply that

$$\sum_{i=1}^{m(n)} -B_1(x_i(n))B_1(x_{i-1}(n)) + \frac{b_i(n)}{2}B_2(x_i(n)) \tag{3-11}$$

is linear in k and the coefficients are functions of r for fixed C, D.

There remains to prove properties 1, 2, and 3 of $\{x_i(n)\}$. We will also give a precise description of the expression (3-11) to finish the proof of Theorem 1.3.

Periodicity and invariance. We now prove the Claim above about the sequence $\{x_i(n)\}$.

Proposition 3.2. *For $j \ge 0$, $\{x_i(n)\}_{S_j(n)\le i\le S_{j+1}(n)}$ is an arithmetic progression mod $\mathbb{Z}$ with common difference $\langle x_{S_j(n)+1}(n) - x_{S_j(n)}(n)\rangle$.*

Proof. Since $b_i(n) = 2$ for $S_j(n)+1 \le i \le S_{j+1}(n)-1$, we have that

$$x_{i+1}(n) = \langle 2x_i(n) - x_{i-1}(n)\rangle.$$

This implies that for $S_j(n)+1 \le i \le S_{j+1}(n)-1$,

$$\langle x_{i+1}(n) - x_i(n)\rangle = \langle\langle 2x_i(n) - x_{i-1}(n)\rangle - x_i(n)\rangle = \langle x_i(n) - x_{i-1}(n)\rangle. \qquad \square$$

Lemma 3.3. *For $i \ge -1$, we have $qx_i(n) \in \mathbb{Z}$ and $0 < x_i(n) \le 1$.*

Proof. Since $A_0(n) = 1$ and $A_{-1}(n) = \delta(n)$, we find from (3-2), (3-3), and (3-4) that

$$x_0(n) = \left\langle \frac{D}{q} \right\rangle, \quad x_{-1}(n) = 1 - \frac{C}{q}.$$

We also note that $b_i(n) \in \mathbb{Z}$ for any $i \ge 0$. Thus (3-4) implies this lemma. $\square$

Proposition 3.4. *For $j \ge 0$ and $a_{2j+1}(n) \ge q$, $\{x_i(n)\}_{S_j(n)\le i\le S_{j+1}(n)}$ has period q. Explicitly, we have*

$$x_{S_j(n)+q+i}(n) = x_{S_j(n)+i}(n) \quad \textit{for } 0 \le i \le a_{2j+1}(n) - q.$$

Proof. Note that $\{x_i(n) \mod 1\}_{S_j(n)\le i\le S_{j+1}(n)}$ is an arithmetic progression. Thus

$$x_{S_j(n)+q+i}(n) = \big\langle x_{S_j(n)+i}(n) + q\,\langle x_{S_j(n)+i}(n) - x_{S_j(n)+i-1}(n)\rangle\big\rangle,$$

for $0 \le i \le a_{2j+1}(n) - q$. From Lemma 3.3, we find that

$$q\,\langle x_{S_j(n)+i}(n) - x_{S_j(n)+i-1}(n)\rangle \in \mathbb{Z}.$$

Thus

$$\langle x_{S_j(n)+i}(n) + q\,\langle x_{S_j(n)+i}(n) - x_{S_j(n)+i-1}(n)\rangle\rangle = \langle x_{S_j(n)+i}(n)\rangle.$$

Since $0 < x_{S_j(n)+i}(n) \le 1$, we finally have

$$\langle x_{S_j(n)+i}(n)\rangle = x_{S_j(n)+i}(n).$$

□

For $0 \le r \le q-1$, we define

$$\Gamma_j(r) := \begin{cases} 0 & \text{for } j = 0,\\ \Gamma_j(r) + \gamma_{2j-1}(r) & \text{for } j \ge 1 \end{cases},$$

where $\gamma_i(r)$ is defined as in (3-8). For $i \ge 0$, we put

$$c_i(r) = \begin{cases} \gamma_{2j}(r) + 2 & \text{for } i = \Gamma_j(r),\\ 2 & \text{otherwise.} \end{cases}$$

Consider a sequence $\{\nu^i_{CD}(r)\}_{i\ge -1}$ with the initial value and the recursion relation as follows:

$$\nu^{-1}_{CD}(r) = \frac{q-C}{q}, \quad \nu^0_{CD}(r) = \left\langle \frac{D}{q}\right\rangle$$

and

$$\nu^{i+1}_{CD}(r) = \big\langle c_i(r)\nu^i_{CD}(r) - \nu^{i-1}_{CD}(r)\big\rangle.$$

If C, D are fixed and clear from the context, we omit the subscript and abbreviate $\nu^i_{CD}(r)$ to $\nu_i(r)$.

Proposition 3.5. *Using the above notation, we have, for $j \ge 0$ and $n = qk + r$*

$$x_{S_j(n)+i}(n) = \nu_{\Gamma_j(r)+i}(r) \quad \textit{for } 0 \le i \le \gamma_{2j+1}(r)$$

Proof. We use induction on j.

When $j = 0$, $S_0(n) = \Gamma_0(r) = 0$. We need to show $x_i(n) = \nu_i(r)$ for $i \in [0, \gamma_1(r)]$. As we saw in the proof of Lemma 3.3,

$$x_0(n) = \left\langle \frac{D}{q}\right\rangle = \nu_0(r), \quad x_{-1}(n) = 1 - \frac{C}{q} = \nu_{-1}(r).$$

Since $a_0(n) - \gamma_0(r) \in q\mathbb{Z}$, using (3-4) and the recursive relation of $\nu_i(r)$, one can easily check that

$$x_1(n) = \Big\langle (a_0(n)+2)\Big\langle \frac{D}{q}\Big\rangle + \frac{C}{q}\Big\rangle = \big\langle (\gamma_0(r)+2)\nu_0(r) - \nu_{-1}(r)\big\rangle = \nu_1(r).$$

For $1 \le i \le \gamma_1(r) - 1$, $x_i(n)$ and $\nu_i(r)$ satisfy the same recursion relation

$$x_{i+1}(n) = \langle 2x_i(n) - x_{i-1}(n)\rangle, \quad \nu_{i+1}(r) = \langle 2\nu_i(r) - \nu_{i-1}(r)\rangle.$$

Thus we have $x_i(n) = \nu_i(r)$ for $0 \le i \le \gamma_1(r)$.

Now assume that the proposition holds true for $j < j_0$. From Proposition 3.4, we find that if $a_{2j_0-1}(n) \ge q$ then

$$x_{S_{j_0-1}(n)+q+i}(n) = x_{S_{j_0-1}(n)+i}(n) \text{ for } 0 \le i \le a_{2j_0-1}(n) - q. \tag{3-12}$$

Since $a_{2j_0-1}(n) - \gamma_{2j_0-1}(r) \in q\mathbb{Z}$, we obtain

$$\begin{aligned} x_{S_{j_0}(n)-1}(n) &= x_{S_{j_0-1}(n)+a_{2j_0-1}(n)-1}(n) = x_{S_{j_0-1}(n)+\gamma_{2j_0-1}(r)-1}(n) \\ &= \nu_{\Gamma j_0-1(r)+\gamma_{2j_0-1}(r)-1}(r) = \nu_{\Gamma_{j_0}(r)-1}(r) \end{aligned}$$

and

$$\begin{aligned} x_{S_{j_0}(n)}(n) &= x_{S_{j_0-1}(n)+a_{2j_0-1}(n)}(n) \\ &= x_{S_{j_0-1}(n)+\gamma_{2j_0-1}(r)}(n) = \nu_{\Gamma j_0-1(r)+\gamma_{2j_0-1}(r)}(r) = \nu_{\Gamma_{j_0}(r)}(r). \end{aligned}$$

Moreover from (3-4), we find that

$$\begin{aligned} x_{S_{j_0}(n)+1}(n) &= \langle (a_{2j_0}(n)+2)x_{S_{j_0}(n)}(n) - x_{S_{j_0}(n)-1}(n)\rangle \\ &= \langle (\gamma_{2j_0}(r)+2)\nu_{\Gamma j_0(r)}(r) - \nu_{\Gamma_{j_0}(r)-1}(r)\rangle = \nu_{\Gamma_{j_0}(r)+1}(r). \end{aligned}$$

Since

$$x_{i+1}(n) = \langle 2x_i(n) - x_{i-1}(n)\rangle \quad \text{for } S_{j_0}(n)+1 \le i \le S_{j_0+1}(n) - 1$$

and

$$\nu_{i+1}(r) = \langle 2\nu_i(r) - \nu_{i-1}(r)\rangle$$

for $\Gamma_{j_0}(r) + 1 \le i \le \Gamma_{j_0}(r) + \gamma_{2j_0+1}(r) - 1 = \Gamma_{j_0+1}(r) - 1$, we have

$$x_{S_{j_0}(n)+i}(n) = \nu_{\Gamma_{j_0}(r)+i}(r) \quad \text{for } 0 \le i \le \gamma_{2j_0+1}(r). \qquad \square$$

Summations. Next we express (3-11), that is,

$$\sum_{i=1}^{m(n)} -B_1(x^i_{C+D\delta(n)})B_1(x^{i-1}_{C+D\delta(n)}) + \frac{b_i(n)}{2}B_2(x^i_{C+D\delta(n)})$$

in terms of $\{\nu^i_{CD}(r)\}$.

Lemma 3.6. *Let $d_l(r) := \langle \nu_{\Gamma_l(r)+1}(r) - \nu_{\Gamma_l(r)}(r)\rangle$ and $[x]_1 := x - \langle x\rangle$. Then for $1 \le \gamma \le q$ and n such that $\gamma \le a_{2l+1}(n)$ and $n = qk + r$, we have*

$$\sum_{i=S_l(n)+1}^{S_l(n)+\gamma} (x_i(n) - x_{i-1}(n))^2 = \gamma d_l(r)^2 + (1 - 2d_l(r))[\nu_{\Gamma_l(r)}(r) + d_l(r)\gamma]_1$$

Proof. Since $0 < x_i(n) \le 1$, we have

$$-1 < x_i(n) - x_{i-1}(n) < 1.$$

Thus

$$x_i(n) - x_{i-1}(n) = \langle x_i(n) - x_{i-1}(n)\rangle + \psi_i(n),$$

where

$$\psi_i(n) = \begin{cases} -1 & \text{if } x_i(n) \le x_{i-1}(n), \\ 0 & \text{if } x_i(n) > x_{i-1}(n). \end{cases}$$

Since

$$\langle x_{i+1}(n) - x_i(n)\rangle = \big\langle \langle 2x_i(n) - x_{i-1}(n)\rangle - x_i(n)\big\rangle = \langle x_i(n) - x_{i-1}(n)\rangle$$

for $S_l(n) + 1 \le i \le S_{l+1}(n) - 1$, we have

$$\langle x_i(n) - x_{i-1}(n)\rangle = \langle x_{S_l(n)+1}(n) - x_{S_l(n)}(n)\rangle = \langle \nu_{\Gamma_l(r)+1}(r) - \nu_{\Gamma_l(r)}(r)\rangle = d_l(r).$$

Hence we have

$$x_i(n) - x_{i-1}(n) = d_l(r) + \psi_i(n).$$

Thus we obtain

$$\sum_{i=S_l(n)+1}^{S_l(n)+\gamma} (x_i(n) - x_{i-1}(n))^2 = \gamma d_l(r)^2 + (1 - 2d_l(r)) \sum_{i=S_l(n)+1}^{S_l(n)+\gamma} \psi_i(n)^2.$$

Note that the sum on the right equals the number of i's satisfying $x_i(n) \le x_{i-1}(n)$ for $S_l(n) + 1 \le i \le S_l(n) + \gamma$.

Therefore

$$\sum_{i=S_l(n)+1}^{S_l(n)+\gamma} \psi_i(n)^2 = [x_{S_l(n)}(n) + d_l(r)\gamma]_1 = [\nu_{\Gamma_l(r)}(r) + d_l(r)\gamma]_1. \qquad \square$$

For simplicity, we let

$$F(x, y) := -B_1(x)B_1(y) + B_2(x) = \left(x - \tfrac{1}{2}\right)\left(\tfrac{1}{2} - y\right) + x^2 - x + \tfrac{1}{6}.$$

Lemma 3.7. *If $l \geq 0$ and $a_{2l+1}(n) \geq q$, then*

$$\sum_{i=S_l(n)+1}^{S_l(n)+q} F(x_i(n), x_{i-1}(n)) = \tfrac{1}{12}\big[6\big(qd_l(r)^2 + (1-2d_l(r))[\nu_{\Gamma_l(r)}(r) + d_l(r)q]_1\big) - q\big].$$

And if $1 \leq \gamma \leq q-1$ and $a_{2l+1}(n) \geq \gamma$,

$$\sum_{i=S_l(n)+1}^{S_l(n)+\gamma} F(x_i(n), x_{i-1}(n)) = \tfrac{1}{12}\big[6\big(\gamma d_l(r)^2 + (1-2d_l(r))[\nu_{\Gamma_l(r)}(r) + d_l(r)\gamma]_1 + B_2(x_{S_l(n)+\gamma}(n)) - B_2(x_{S_l(n)}(n))\big) - \gamma\big],$$

where $B_2(x)$ is the second Bernoulli polynomial.

Proof. We note that

$$F(x,y) = \tfrac{1}{2}(x-y)^2 - \tfrac{1}{12} + \tfrac{1}{2}(B_2(x) - B_2(y)).$$

Thus

$$\sum_{i=S_l(n)+1}^{S_l(n)+\gamma} F(x_i(n), x_{i-1}(n)) = \sum_{i=S_l(n)+1}^{S_l(n)+\gamma} \big[\tfrac{1}{2}(x_i(n) - x_{i-1}(n))^2 - \tfrac{1}{12} + \tfrac{1}{2}(B_2(x_i(n)) - B_2(x_{i-1}(n)))\big].$$

We note that for $1 \leq \gamma \leq q-1$,

$$\sum_{i=S_l(n)+1}^{S_l(n)+\gamma} B_2(x_i(n)) - B_2(x_{i-1}(n)) = B_2(x_{S_l(n)+\gamma}(n)) - B_2(x_{S_l(n)}(n)).$$

and, from the periodicity of $x_i(n)$, we have that for $\gamma = q$

$$\sum_{i=S_l(n)+1}^{S_l(n)+q} B_2(x_i(n)) - B_2(x_{i-1}(n)) = 0. \qquad \square$$

Proposition 3.8. *Suppose $\delta(n) - 1 = [\![a_0(n), a_2(n), \ldots, a_{s-1}(n)]\!]$, $a_i(n) = \alpha_i n + \beta_i$ for $\alpha_i, \beta_i \in \mathbb{Z}$ and $a_i(r) = q\tau_i(r) + \gamma_i(r)$ for $\gamma_i(r) = \langle a_i(r)\rangle_q$. Let $d_{CD}^l(r) := \langle \nu_{CD}^{\Gamma_l(r)+1}(r) - \nu_{CD}^{\Gamma_l(r)}(r)\rangle$. Then, for $n = qk + r$, we have*

$$\sum_{i=1}^{m(n)} -B_1(x^i_{C+D\delta(n)})B_1(y^i_{C+D\delta(n)}) + \frac{b_i(n)}{2} B_2(x^i_{C+D\delta(n)}) = \tfrac{1}{12}(A_{CD}(r) + kB_{CD}(r)),$$

where

$$
\begin{aligned}
&A_{CD}(r)\\
&:= \sum_{l=1}^{s\mu(s)} -12B_1(\nu_{CD}^{\Gamma_l(r)}(r))B_1(\nu_{CD}^{\Gamma_l(r)-1}(r)) + 6(a_{2l}(r)+2)B_2(\nu_{CD}^{\Gamma_l(r)}(r))\\
&\quad + \sum_{l=0}^{s\mu(s)-1} \Big[6\Big((\gamma_{2l+1}(r)-1)d_{CD}^{l}(r)^2\\
&\qquad\qquad + (1-2d_{CD}^{l}(r))[\nu_{CD}^{\Gamma_l(r)}(r) + d_{CD}^{l}(r)(\gamma_{2l+1}(r)-1)]_1\\
&\qquad\qquad\qquad + B_2(\nu_{CD}^{\Gamma_{l+1}(r)-1}(r)) - B_2(\nu_{CD}^{\Gamma_l(r)}(r))\Big) - \gamma_{2l+1}(r)+1\\
&\qquad + \tau_{2l+1}(r)\big(6(qd_{CD}^{l}(r)^2 + (1-2d_{CD}^{l}(r))[\nu_{CD}^{\Gamma_l(r)}(r) + d_{CD}^{l}(r)q]_1) - q\big)\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
B_{CD}(r) := &\sum_{l=1}^{s\mu(s)} 6q\alpha_{2l}B_2(\nu_{CD}^{\Gamma_l(r)}(r))\\
&+ \sum_{l=0}^{s\mu(s)-1} \alpha_{2l+1}\big(6(qd_{CD}^{l}(r)^2 + (1-2d_{CD}^{l}(r))[\nu_{CD}^{\Gamma_l(r)} + d_{CD}^{l}(r)q]_1) - q\big).
\end{aligned}
$$

Proof. From (3-7), we have

$$
\begin{aligned}
&\sum_{i=1}^{m(n)} B_1(x_{C+D\delta(n)}^{i})B_1(y_{C+D\delta(n)}^{i}) + \frac{b_i(n)}{2}B_2(x_{C+D\delta(n)}^{i})\\
&= \sum_{l=1}^{s\mu(s)} \Big(-B_1(x_{C+D\delta(n)}^{S_l(n)})B_1(x_{C+D\delta(n)}^{S_l(n)-1}) + \frac{\alpha_{2l}qk+\tau_{2l}(r)q+\gamma_{2l}(r)+2}{2}B_2(x_{C+D\delta(n)}^{S_l(n)})\Big)\\
&\qquad + \sum_{l=0}^{s\mu(s)-1} \sum_{i=S_l(n)+1}^{\substack{S_l(n)+q\alpha_{2l+1}k\\ +q\tau_{2l+1}(r)+\gamma_{2l+1}(r)-1}} F(x_{C+D\delta(n)}^{i}, x_{C+D\delta(n)}^{i-1}).
\end{aligned}
$$

From Lemma 3.7, we have

$$
\begin{aligned}
&12 \sum_{i=S_l(n)+1}^{\substack{S_l(n)+q\alpha_{2l+1}k\\ +q\tau_{2l+1}(r)\gamma_{2l+1}(r)-1}} F(x_{C+D\delta(n)}^{i}, x_{C+D\delta(n)}^{i-1})\\
&= 12 \sum_{i=S_l(n)+1}^{S_l(n)+\gamma_{2l+1}(r)-1} F(x_{C+D\delta(n)}^{i}, x_{C+D\delta(n)}^{i-1})\\
&\qquad + 12(\alpha_{2l+1}k+\tau_{2l+1}(r)) \sum_{i=S_l(n)+1}^{S_l(n)+q} F(x_{C+D\delta(n)}^{i}, x_{C+D\delta(n)}^{i-1})
\end{aligned}
$$

$$= 6\Big((\gamma_{2l+1}(r)-1) d^l_{CD}(r)^2 + (1-2d^l_{CD}(r)) \big[\nu^{\Gamma_l(r)}_{CD}(r) + d^l_{CD}(r)(\gamma_{2l+1}(r)-1) \big]_1 \\ + B_2(x^{S_l(n)+\gamma_{2l+1}-1}_{C+D\delta(n)}) - B_2(x^{S_l(n)}_{C+D\delta(n)}) \Big) - (\gamma_{2l+1}(r)-1) \\ + (\alpha_{2l+1}k + \tau_{2l+1}(r)) \big(6(q d^l_{CD}(r)^2 + (1-2d^l_{CD}(r))[\nu^{\Gamma_l(r)}_{CD}(r) + d^l_{CD}(r) q]_1) - q \big).$$

Since

$$x^{S_l(n)}_{C+D\delta(n)} = \nu^{\Gamma_l(r)}_{CD}(r), \quad x^{S_l(n)-1}_{C+D\delta(n)} = \nu^{\Gamma_l(r)-1}_{CD}(r), \quad \text{and} \quad x^{S_l(n)+\gamma_{2l+1}(r)-1}_{C+D\delta(n)} = \nu^{\Gamma_{l+1}(r)-1}_{CD},$$

we complete the proof. □

End of the proof. Since $\nu^{\Gamma_l(r)}_{CD}(r)$, $\nu^{\Gamma_l(r)-1}_{CD}(r)$ and $d^l_{CD}(r)$ are in $\frac{1}{q}\mathbb{Z}$, we find that

$$q^2 A_{CD}(r), q^2 B_{CD}(r) \in \mathbb{Z}.$$

Moreover, we have

$$L_{K_n}(0, \chi_n, \mathfrak{b}_n) = \frac{1}{12q^2} \sum_{C,D} \chi_n(C + D\delta(n))(q^2 A_{CD}(r) + kq^2 B_{CD}(r)).$$

Since χ is a Dirichlet character of modulus q, if $n = qk + r$, we can write

$$\chi_n(\mathfrak{b}_n(C + D\delta(n))) = F_{CD}(r)$$

for a function F_{CD}. Note that, if K_r is defined,

$$\chi_n(\mathfrak{b}_n(C + D\delta(n))) = \chi_r(\mathfrak{b}_r(C + D\delta(r))) = F_{CD}(r).$$

(This expression does not make sense if K_r and $\delta(r)$ are undefined.)

If we set

$$A_\chi(r) := \sum_{C,D} F_{CD}(r) q^2 A_{CD}(r)$$

and

$$B_\chi(r) := \sum_{C,D} F_{CD}(r) q^2 B_{CD}(r),$$

we obtain the proof. □

4. Biró's method

Let K_n be a family of real quadratic fields such that the special value of the Hecke L-function at $s = 0$ has linearity. Biró [Biró 2003a; 2003b] developed a method using linearity to find the residue of n such that $h(K_n) = 1$ by certain primes. In this section, we sketch Biró's method.

Let $K_n = \mathbb{Q}(\sqrt{d}\,)$ for a square free integer $d = f(n)$ and D_n be the discriminant K_n. For an odd Dirichlet character $\chi : \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}^*$, let χ_n denote the ray class character defined as $\chi_n = \chi \circ N_{K_n} : I_n(q)/P_n(q)^+ \to \mathbb{C}^*$, and let $\chi_D = (\frac{D}{\cdot})$ denote the Kronecker character. Then the special value of the Hecke L-function at $s = 0$ has a factorization

$$L_{K_n}(0, \chi_n) = L(0, \chi)L(0, \chi\chi_{D_n}) = \left(\frac{1}{q}\sum_{a=1}^{q} a\chi(a)\right)\left(\frac{1}{qD_n}\sum_{b=1}^{qD_n} b\chi(b)\chi_{D_n}(b)\right).$$

Let $\mathfrak{b}_n = O_{K_n}$. Suppose that $L_{K_n}(0, \chi_n, \mathfrak{b}_n)$ is linear in the form

$$L_{K_n}(0, \chi_n, \mathfrak{b}_n) = \frac{1}{12q^2}(A_\chi(r) + kB_\chi(r))$$

for $A_\chi(r), B_\chi(r) \in \mathbb{Z}[\chi(1), \chi(2) \cdots \chi(q)]$. Let ϵ_n be the fundamental unit of K_n. From Proposition 2.2 in [Byeon and Lee 2011], we find that $L_{K_n}(0, \chi_n, \mathfrak{b}_n) = L_{K_n}(0, \chi_n, (\epsilon_n)\mathfrak{b}_n)$. Thus if the class number of K_n is one, then we have for $n = qk + r$

$$L_{K_n}(0, \chi_n) = \frac{c}{12q^2}(A_\chi(r) + kB_\chi(r))$$

where c is the number of narrow ideal classes.

Then we have

$$B_\chi(r)k + A_\chi(r) = \frac{12q}{c} \cdot \left(\sum_{a=1}^{q} a\chi(a)\right) \cdot \left(\frac{1}{qD_n}\sum_{b=1}^{qD_n} b\chi(b)\chi_{D_n}(b)\right).$$

Let L_χ be the cyclotomic field generated by the values of χ. Since

$$\frac{1}{qD_n}\sum_{b=1}^{qD_n} b\chi(b)\chi_{D_n}(b)$$

is integral in L_χ, for a prime ideal I of L_χ dividing $\sum_{a=1}^{q} a\chi(a)$, we have

$$B_\chi(r)k + A_\chi(r) \equiv 0 \pmod{I}.$$

And if I does not divide $B_\chi(r)$, then

$$k \equiv -\frac{A_\chi(r)}{B_\chi(r)} \pmod{I}.$$

Since $n = qk + r$, we have

$$n \equiv -q\frac{A_\chi(r)}{B_\chi(r)} + r \pmod{I}.$$

Moreover, if $O_{L_\chi}/I = \mathbb{Z}/p\mathbb{Z}$, the residue of n modulo p is expressed only in terms of $A_\chi(r)$, $B_\chi(r)$, and r as above.

We now list the necessary conditions on q and p:

Condition $(*)$. *q is an odd integer*; *p is an odd prime*; *χ is a character with conductor q*; *I is prime ideal in L_χ lying over p, with $I \mid \bigl(\sum_{a=1}^{q} a\chi(a)\bigr)$ and $O_{L_\chi}/I = \mathbb{Z}/p\mathbb{Z}$.*

When linearity holds, these conditions are independent of the family $\{K_n\}$.

Let S be the set of (q, p) satisfying Condition $(*)$. We partition S as follows:

$$S = \bigcup_{q \text{ odd integer}} S_q, \quad \text{where } S_q := \{(q, p) \in S\}.$$

Finally, for $(q, p) \in S$, we obtain the residue of $n = qk + r$ modulo p for which the class number of K_n is 1.

The above method has been used to find an upper bound on the discriminant of real quadratic fields with class number 1 in some families of Richaud–Degert type where the linearity criterion is satisfied [Biró 2003a; 2003b; Byeon et al. 2007; Lee 2009a]. This information, together with a properly developed class number one criteria for each case, could be used to solve the class number problems.

It is easily checked that the criterion is fulfilled by general families of Richaud–Degert type. Furthermore, there are abundant examples of families of real quadratic fields satisfying the linearity criterion [McLaughlin 2003]. For these, we have controlled behavior of the special values of the Hecke L-function at $s = 0$, and Biro's method is directly applicable in each case. We expect this method can be used to study many meaningful arithmetic problems for families of real quadratic fields, in addition to the class number problem.

5. A generalization

We conclude with a possible generalization of the linearity of the special value of the Hecke L-function. This generalization will be dealt in [Jun and Lee 2012].

As in the criterion for linearity, we set $K_n = \mathbb{Q}(\sqrt{f(n)}\,)$ and let $\mathfrak{b}_n$ an integral ideal of K_n. We assume $\mathfrak{b}_n^{-1} = [1, \delta(n)]$ for $\delta(n) - 1 = [a_1(n), a_2(n), \ldots, a_s(n)]$, with $a_i(x) \in \mathbb{Z}[x]$.

For a given conductor q, write $n = qk + r$ for $r = 0, 1, 2, \ldots, q-1$. Suppose $N = \max_i\{\deg(a_i(x))\}$. Then the special value of the partial ζ-function of the ray class of $\mathfrak{b}_n$ mod q at $s = 0$ can be written as

$$\zeta_{K_n,q}(0, (C + D\delta(n))\mathfrak{b}_n) = \frac{1}{12q^2}\bigl(A_0(r) + A_1(r)k + \cdots + A_N(r)k^N\bigr)$$

for some rational integers A_i depending only on r.

We have no application of this property in arithmetic, but it will be very interesting if one applies it in a similar fashion as Biró's method.

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