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# The image of complex conjugation in $l$-adic representations associated to automorphic forms

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If $F^+$ is a totally real field, if $n$ is an odd integer and if $\Pi$ is a regular, algebraic, essentially self-dual, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{F^+})$, then we calculate the image of any complex conjugation under the $l$-adic representations $r_{l,\iota}(\Pi)$ associated to $\Pi$.

## Introduction

Let $F^+$ denote a totally real number field and fix an isomorphism $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. It is known that to a regular, algebraic, essentially self-dual, cuspidal automorphic representation $\Pi$ of $\mathrm{GL}_n(\mathbb{A}_{F^+})$ one can associate a continuous semisimple Galois representation

$$r_{l,\iota}(\Pi) : \mathrm{Gal}(\overline{F}^+/F^+) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l).$$

(For the definition of "regular, algebraic, essentially self-dual, cuspidal" see the start of Section 1.) This representation is known to be de Rham and its Hodge–Tate numbers are known. (They can be simply calculated from the infinitesimal character of $\pi_\infty$.) For all finite places $v$ of $F^+$ not dividing $l$ one can calculate the Frobenius semisimplification of the restriction of $r_{l,\iota}(\Pi)$ to a decomposition group above $v$ in terms of $\pi_v$ via the local Langlands correspondence. This uniquely (in fact, over) determines $r_{l,\iota}(\Pi)$. (See [Shin 2011; Clozel et al. 2011; Caraiani 2010; Chenevier and Harris 2011].) The representation $r_{l,\iota}(\Pi)$ is conjectured to be irreducible. This is known if $\Pi$ is discrete series at some finite place [Taylor and Yoshida 2007]. Moreover $r_{l,\iota}(\Pi)^\vee \cong r_{l,\iota}(\Pi) \otimes \mu$ for some character $\mu$ of $\mathrm{Gal}(\overline{F}^+/F^+)$ which is either totally odd (takes the value $-1$ on all complex conjugations) or totally even (takes the value $+1$ on all complex conjugations).

Frank Calegari raised the question as to whether, for an infinite place $v$ of $F^+$ one can calculate the conjugacy class of $r_{l,\iota}(\Pi)(c_v)$, where $c_v \in \mathrm{Gal}(\overline{F}^+/F^+)$ is a

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complex conjugation for $v$. This conjugacy class has order two, so it is semisimple with eigenvalues $\pm 1$. The problem is to determine how many $+1$'s and how many $-1$'s occur. Because $\Pi$ was assumed to be regular, we expect that the number of $+1$'s and $-1$'s differ by at most one:

$$|\operatorname{tr} r_{l,\iota}(\Pi)(c_v)| \leq 1.$$

As we know the determinant of $r_{l,\iota}(\Pi)$ this would completely determine the conjugacy class of $r_{l,\iota}(\Pi)(c_v)$.

If $\mu$ is totally odd then [Bellaïche and Chenevier 2011] shows that $n$ is even and that $r_{l,\iota}(\Pi)$ preserves an alternating pairing up to multiplier $\mu$. In this case, because $\mathrm{GSp}_n(\overline{\mathbb{Q}}_l)$ has a unique conjugacy class of elements of order two and multiplier $-1$, we see that $\operatorname{tr} r_{l,\iota}(\Pi)(c_v) = 0$ for all $v|\infty$. So the problem lies in the case that $\mu$ is totally even, i.e., that $r_{l,\iota}(\Pi)$ preserves an orthogonal pairing up to multiplier $\mu$.

In this paper we will prove this conjecture in the case $n$ is odd:

**Proposition 1.** *Suppose that $F^+$ is a totally real field, that $n$ is an odd positive integer and that $\Pi$ a regular, algebraic, essentially self-dual, cuspidal automorphic representation $\Pi$ of $\mathrm{GL}_n(\mathbb{A}_{F^+})$. Suppose also that $r_{l,\iota}(\Pi)$ is irreducible. If*

$$c \in \mathrm{Gal}(\overline{F}^+/F^+)$$

*is a complex conjugation (for some embedding $\overline{F}^+ \hookrightarrow \mathbb{C}$) then*

$$|\operatorname{tr} r_{l,\iota}(\Pi)(c)| \leq 1.$$

We believe that essentially the same method works if $n$ is even and $\Pi$ is discrete series at a finite place, though we haven't taken the trouble to write the argument down in this case. (One would work with the construction of $r_{l,\iota}(\Pi)$ given in [Harris and Taylor 2001] rather than that given in [Shin 2011].) However we do not see how to treat the general case when $n$ is even. When $r_{l,\iota}(\Pi)$ is reducible one can calculate the trace of $r(c)$ for some representation of $r$ of $\mathrm{Gal}(\overline{F}^+/F^+)$ with the same restriction to $\mathrm{Gal}(\overline{F}^+/F)$, but this does not seem to be very helpful.

The construction of $r_{l,\iota}(\Pi)$ is via piecing together twists of representations of $\mathrm{Gal}(\overline{F}^+/F)$ which arise in the cohomology of unitary group Shimura varieties, as $F$ runs over certain imaginary CM fields. For none of these twisted restrictions does complex conjugation make sense. For an infinite place of $F$ one can assign a natural sign to the representations of $\mathrm{Gal}(\overline{F}^+/F)$ that arise in the cohomology of these Shimura varieties, because they are essentially conjugate self-dual. (See [Clozel et al. 2008] or [Bellaïche and Chenevier 2011].) As Calegari has stressed this sign is not related to the image of complex conjugation in our representation of $\mathrm{Gal}(\overline{F}^+/F^+)$. This latter image only makes sense for the Galois representations coming from certain automorphic forms on the unitary groups, namely those that arise from an automorphic form on $\mathrm{GL}_n(\mathbb{A}_{F^+})$ by some functoriality.

In the case that $n$ is odd the unitary groups employed by Shin [2011] have rank $n$ and we are able to use the moduli theoretic interpretation of its Shimura variety to write descent data to the maximal totally real subfield of $F$. This descent data does not commute with the action of the finite adelic points of the unitary group. However in the special case of an automorphic representation $\pi$ which arises by functoriality from an automorphic form on $\mathrm{GL}_n$ over a totally real field we are able to show that, up to twist, this descent data preserves the $\pi^\infty$ isotypical component of the cohomology, and hence gives a geometric realization of $r_{l,\iota}(\Pi)(c_v)$. Because of its geometric construction, $r_{l,\iota}(\Pi)(c_v)$ also makes sense in the world of variations of Hodge structures. Finally we can appeal to the fact that the Hodge structure corresponding to $r_{l,\iota}(\Pi)$ is regular (i.e., each $h^{p,q} \leq 1$) to show that $|\mathrm{tr}\, r_{l,\iota}(\Pi)(c_v)| \leq 1$.

In the case that $n$ is even and $\Pi$ is not discrete series at any finite place, [Shin 2011] realizes twists of $r_{l,\iota}(\Pi)|_{\mathrm{Gal}(\bar{F}^+/F)}$ in the cohomology of the Shimura varieties for unitary groups of rank $n+1$. One takes the $\pi^\infty$ isotypic component of the cohomology for an unstable automorphic representation $\pi$ of the unitary group, which one constructs from $\Pi$ using the theory of endoscopy. In this case our descent data relates the $\pi^\infty$ isotypic component of the cohomology, not to itself, but to a twist of the $(\pi')^\infty$ isotypic component for a second unstable automorphic representation $\pi'$ of the unitary group also arising from $\Pi$. (This $\pi'$ is not even nearly equivalent to a twist of $\pi$.) This does not seem to be helpful.

***Notation.*** Let us establish some notation that we will use throughout the paper.

If $\rho$ is a representation $\kappa_\rho$ will denote its central character.

If $F$ is a $p$-adic field with valuation $v$ then $F^{\mathrm{nr}}$ will denote its maximal unramified extension and $\mathrm{Frob}_v \in \mathrm{Gal}(F^{\mathrm{nr}}/F)$ will denote geometric Frobenius. Moreover $\mathrm{Art}_F : F^\times \to \mathrm{Gal}(\bar{F}/F)^{\mathrm{ab}}$ will denote the Artin map (normalized to take uniformizers to geometric Frobenius elements). Suppose that $V/\overline{\mathbb{Q}}_l$ is a finite-dimensional vector space and that

$$r : \mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}(V)$$

is a continuous homomorphism. If either $l \neq p$ or $l = p$ and $V$ is de Rham (i.e., $\dim_{\overline{\mathbb{Q}}_l}(V \otimes_{\tau,F} B_{\mathrm{DR}})^{\mathrm{Gal}(\bar{F}/F)} = \dim_{\overline{\mathbb{Q}}_l} V$ for all continuous embeddings $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$) then we may associate to $r$ a Weil–Deligne representation $\mathrm{WD}(r)$ of the Weil group $W_K$ of $K$ over $\overline{\mathbb{Q}}_l$. In the case $l \neq p$ the Weil–Deligne representation $\mathrm{WD}(r)$ determines $r$ up to equivalence. (See for instance [Taylor and Yoshida 2007, Section 1] for details.) If $(r, N)$ is a Weil–Deligne representation of $W_K$ then we will let $(r, N)^{\text{F-ss}} = (r^{\mathrm{ss}}, N)$ denote the Frobenius semisimplification of $(r, N)$. We will write $\mathrm{rec}_F$ for the local Langlands correspondence — a bijection from irreducible smooth representations of $\mathrm{GL}_n(F)$ over $\mathbb{C}$ to $n$-dimensional Frobenius semisimple Weil–Deligne representations of the Weil group $W_F$ of $F$. (See the Introduction or

Section VII.2 of [Harris and Taylor 2001].) Recall that if $\chi$ is a character of $F^\times$ then $\mathrm{rec}(\chi) = \chi \circ \mathrm{Art}_F^{-1}$.)

If $F = \mathbb{R}$ or $\mathbb{C}$ we will write $\mathrm{Art}_F : F^\times \twoheadrightarrow \mathrm{Gal}(\bar{F}/F)$. If $F = \mathbb{R}$ then we will denote by $c$ the nontrivial element of $\mathrm{Gal}(\bar{F}/F)$ and denote by sgn the unique surjection $F^\times \twoheadrightarrow \{\pm 1\}$.

If $F$ is a number field then

$$\mathrm{Art}_F = \prod_v \mathrm{Art}_{F_v} : \mathbb{A}_F^\times / \overline{F^\times (F_\infty^\times)^0} \xrightarrow{\sim} \mathrm{Gal}(\bar{F}/F)^{\mathrm{ab}}$$

will denote the Artin map. If $v$ is a real place of $F$ then we will let $c_v$ denote the image of $c \in \mathrm{Gal}(\bar{F}_v/F_v)$ in $\mathrm{Gal}(\bar{F}/F)$. Thus $c_v$ is well defined up to conjugacy. Suppose that

$$\chi : \mathbb{A}_F^\times / F^\times \to \mathbb{C}^\times$$

is a continuous character for which there exists $a \in \mathbb{Z}^{\mathrm{Hom}(F,\mathbb{C})}$ such that

$$\chi|_{(F_\infty^\times)^0} : x \mapsto \prod_{\tau \in \mathrm{Hom}(F,\mathbb{C})} (\tau x)^{a_\tau}$$

(i.e., an algebraic grossencharacter). Suppose also that $\iota : \bar{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. Then we define

$$r_{l,\iota}(\chi) : \mathrm{Gal}(\bar{F}/F) \to \bar{\mathbb{Q}}_l^\times$$

to be the continuous character such that

$$\iota\Bigg( (r_{l,\iota}(\chi) \circ \mathrm{Art}_F)(x) \prod_{\tau \in \mathrm{Hom}(F,\mathbb{C})} (\iota^{-1}\tau)(x_l)^{-a_\tau} \Bigg) = \chi(x) \prod_{\tau \in \mathrm{Hom}(F,\mathbb{C})} (\tau x)^{-a_\tau}.$$

## 1. Statement of the main result

Now let $F^+$ be a totally real field. By a *RAESDC* (regular, algebraic, essentially self dual, cuspidal) automorphic representation $\pi$ of $\mathrm{GL}_n(\mathbb{A}_{F^+})$ we mean a cuspidal automorphic representation such that

- $\pi^\vee \cong \pi \otimes (\chi \circ \det)$ for some continuous character $\chi : \mathbb{A}_{F^+}^\times / (F^+)^\times \to \mathbb{C}^\times$ with $\chi_v(-1)$ independent of $v|\infty$, and
- $\pi_\infty$ has the same infinitesimal character as some irreducible algebraic representation of the restriction of scalars from $F^+$ to $\mathbb{Q}$ of $\mathrm{GL}_n$.

Note that $\chi$ is necessarily algebraic. Also, if $n$ is odd and $\pi^\vee \cong \pi \otimes (\chi \circ \det)$, then $\chi_v(-1)$ is necessarily independent of $v|\infty$, in fact it is necessarily 1 for all such $v$.

If $F^+$ is totally real we will write $(\mathbb{Z}^n)^{\mathrm{Hom}(F^+,\mathbb{C}),+}$ for the set of $a = (a_{\tau,i}) \in (\mathbb{Z}^n)^{\mathrm{Hom}(F^+,\mathbb{C})}$ satisfying

$$a_{\tau,1} \geq \cdots \geq a_{\tau,n}.$$

If $F^{+\prime}/F^+$ is a finite totally real extension we define $a_{F^{+\prime}} \in (\mathbb{Z}^n)^{\mathrm{Hom}(F^{+\prime},\mathbb{C}),+}$ by

$$(a_{F^{+\prime}})_{\tau,i} = a_{\tau|_{F^+},i}.$$

If $a \in (\mathbb{Z}^n)^{\mathrm{Hom}(F^+,\mathbb{C}),+}$, let $\Xi_a$ denote the irreducible algebraic representation of $\mathrm{GL}_n^{\mathrm{Hom}(F^+,\mathbb{C})}$ which is the tensor product over $\tau$ of the irreducible representations of $\mathrm{GL}_n$ with highest weights $a_\tau$. We will say that a RAESDC automorphic representation $\pi$ of $\mathrm{GL}_n(\mathbb{A}_{F^+})$ has *weight $a$* if $\pi_\infty$ has the same infinitesimal character as $\Xi_a^\vee$.

Fix once and for all an isomorphism $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. The following theorem is proved in [Shin 2011] (see also [Clozel et al. 2011]). (This is not explicitly stated in [Shin 2011], but see Remark 7.6 of that reference. For the last sentence see [Taylor and Yoshida 2007].)

**Theorem 1.1.** *Let $F_0^+$ be a totally real field and let $n$ be an odd positive integer. Let $a \in (\mathbb{Z}^n)^{\mathrm{Hom}(F_0^+,\mathbb{C}),+}$. Suppose further that $\Pi$ is a RAESDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{F_0^+})$ of weight $a$. Specifically suppose that $\Pi^\vee \cong \Pi\chi$ where $\chi : \mathbb{A}_{F_0^+}^\times/(F_0^+)^\times \to \mathbb{C}^\times$ and $\chi_v(-1)$ is independent of $v|\infty$. Then there is a continuous semisimple representation*

$$r_{l,\iota}(\Pi) : \mathrm{Gal}(\overline{F}_0^+/F_0^+) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$$

*with the following properties.*

(1) *For every prime $v \nmid l$ of $F_0^+$ we have*

$$\mathrm{WD}(r_{l,\iota}(\Pi)|_{\mathrm{Gal}(\overline{F}_{0,v}^+/F_{0,v}^+)})^{\text{F-ss}} = r_l(\iota^{-1}\,\mathrm{rec}(\Pi_v \otimes |\det|_v^{(1-n)/2}).$$

(2) $r_{l,\iota}(\Pi)^\vee = r_{l,\iota}(\Pi)\epsilon^{n-1} r_{l,\iota}(\chi)$.

(3) $\det r_{l,\iota}(\Pi) = r_{l,\iota}(\kappa_\Pi)\epsilon_l^{n(1-n)/2}$.

(4) *If $v|l$ is a prime of $F_0^+$ then the restriction $r_{l,\iota}(\Pi)|_{\mathrm{Gal}(\overline{F}_{0,v}^+/F_{0,v}^+)}$ is de Rham. Moreover, if $\Pi_v$ is unramified, if $(F_{0,v}^+)^0$ denotes the maximal unramified subextension of $F_{0,v}^+/\mathbb{Q}_l$ and if $\tau : (F_{0,v}^+)^0 \hookrightarrow \overline{\mathbb{Q}}_l$ then $r_{l,\iota}(\Pi)|_{\mathrm{Gal}(\overline{F}_{0,v}^+/F_{0,v}^+)}$ is crystalline and the characteristic polynomial of $\phi^{[(F_{0,v}^+)^0:\mathbb{Q}_l]}$ on*

$$(r_{l,\iota}(\Pi) \otimes_{\tau,(F_{0,v}^+)^0} B_{\mathrm{cris}})^{\mathrm{Gal}(\overline{F}_{0,v}^+/F_{0,v}^+)}$$

*equals the characteristic polynomial of*

$$\iota^{-1}\,\mathrm{rec}_{F_{0,v}^+}(\Pi_v \otimes |\det|_v^{(1-n)/2})(\mathrm{Frob}_v).$$

(5) *If $v|l$ is a prime of $F_0^+$ and if $\tau : F_0^+ \hookrightarrow \overline{\mathbb{Q}}_l$ lies above $v$ then*

$$\dim_{\overline{\mathbb{Q}}_l} \mathrm{gr}^i (r_{l,\iota}(\Pi) \otimes_{\tau,F_{0,v}} B_{\mathrm{DR}})^{\mathrm{Gal}(\overline{F}_{0,v}^+/F_{0,v}^+)} = 0$$

*unless* $i = a_{\iota\tau,j} + n - j$ *for some* $j = 1, \ldots, n$ *in which case*

$$\dim_{\overline{\mathbb{Q}}_l} \mathrm{gr}^i (r_{l,\iota}(\Pi) \otimes_{\tau, F_{0,v}^+} B_{\mathrm{DR}})^{\mathrm{Gal}(\overline{F}_{0,v}^+/F_{0,v}^+)} = 1.$$

(6) *If* $\Pi$ *is discrete series at some finite place then* $r_{l,\iota}(\Pi)$ *is irreducible.*

The purpose of this paper is to calculate $r_{l,\iota}(\Pi)(c_v)$ for any infinite place $v$ of $F_0^+$.

**Proposition 1.2.** *Keep the notation and assumptions of the above theorem and suppose that* $r_{l,\iota}(\Pi)$ *is irreducible.* (*In particular we are assuming that* $n$ *is odd.*) *Let* $v$ *denote an infinite place of* $F_0^+$. *Then*

$$r_{l,\iota}(\Pi)(c_v)$$

*is semisimple with eigenvalues* 1 *of multiplicity* $(n + \kappa_{\Pi,v}(-1))/2$ *and* $-1$ *with multiplicity* $(n - \kappa_{\Pi,v}(-1))/2$.

## 2. A geometric realization of complex conjugation

We must recall some of the construction of $r_{l,\iota}(\Pi)$ and explain how the action of complex conjugation can be constructed geometrically.

***The basic set-up.*** There is a constant $\alpha \in \mathbb{Z}$ such that $a_{\tau,j} + a_{\tau,n+1-j} = \alpha$ for all $j = 1, \ldots, n$ and all $\tau : F_0^+ \hookrightarrow \mathbb{C}$. Thus

$$\chi|_{((F_{0,\infty}^+)^\times)^0} = \mathbf{N}^{\alpha}_{F_0^+/\mathbb{Q}}.$$

Shin shows that one can choose

- a soluble Galois totally real extension $F^+/F_0^+$,
- an imaginary quadratic field $E$ in which $l$ splits,
- an embedding $\tau_0 : F = F^+E \hookrightarrow \mathbb{C}$,
- a continuous character

$$\phi : \mathbb{A}_F^\times / F^\times \to \mathbb{C}^\times,$$

- a continuous character

$$\psi : \mathbb{A}_E^\times / E^\times \to \mathbb{C}^\times,$$

with the following properties.

- $[F^+ : \mathbb{Q}]$ is even and $> 2$.
- If Ram denotes the set of (finite) rational primes above which any of $F$, $\Pi$, $\phi$, or $\psi$ ramifies, then every prime of $F^+$ above a prime of Ram splits in $F$.
- $r_{l,\iota}(\Pi)|_{\mathrm{Gal}(\overline{F}/F)}$ remains irreducible.

- $\phi\phi^c = \chi_F$ and $\phi|_{F_\infty^\times} = \prod_\tau \tau^{-\beta_\tau}$ where $\beta_\tau + \beta_{\tau c} = -\alpha$.
- $\psi^c/\psi = (\kappa_\Pi|_{\mathbb{A}^\times}^{[F^+:F_0^+]} \circ \mathbf{N}_{E/\mathbb{Q}})\phi|_{\mathbb{A}_E^\times}^n$.
- $\psi_\infty = \tau_0^{-\epsilon}(\tau_0 \circ c)^{-\epsilon'}$ with $\epsilon, \epsilon' \in \mathbb{Z}$.
- $\psi$ is unramified at the prime of $E$ above $l$ corresponding to $\iota^{-1} \circ \tau_0$.

Let $V = F^n$ and let

$$\langle\ ,\ \rangle : V \times V \to \mathbb{Q}$$

be a nondegenerate alternating bilinear form such that

$$\langle xv, w\rangle = \langle v, {}^c x w\rangle$$

for all $x \in F$ and $v, w \in V$. Let $G$ be the reductive subgroup of $\mathrm{GL}(V/F)$ consisting of elements which preserve $\langle\ ,\ \rangle$ up to a $\mathbb{G}_m$-multiple and let $\nu : G \to \mathbb{G}_m$ denote the multiplier character. We may, and do, suppose that $V$ is chosen so that

- $G$ is quasisplit at all finite places;
- if $\tau : F \hookrightarrow \mathbb{C}$ satisfies $\tau|_E = \tau_0|_E$ then the Hermitian form on $V \otimes_{F,\tau} \mathbb{C}$ defined by
  $$(v, w) \mapsto \langle v, iw\rangle$$
  has a maximal positive definite subspace of dimension 0 if $\tau \neq \tau_0$ and 1 if $\tau = \tau_0$.

(See [Shin 2011, Lemma 5.1].) There is an identification of $G \times_{\mathbb{Q}} E$ with the product of $\mathrm{GL}_1$ and the restriction of scalars from $F$ to $E$ of $\mathrm{GL}_n$. The map sends $g$ to the product of its multiplier and its action on the direct summand $V \otimes_{E,1} E$ of $V \otimes_{\mathbb{Q}} E = V \otimes_{E,1} E \oplus V \otimes_{E,c} E$.

***The group G.*** Letting $\ker^1(\mathbb{Q}, G)$ denote the kernel of

$$H^1(\mathbb{Q}, G) \to \prod_v H^1(\mathbb{Q}_v, G),$$

using the fact that $n$ is odd, we see from [Kottwitz 1992, Section 8] that there is an identification

$$\ker^1(\mathbb{Q}, G) \cong ((F^+)^\times \cap (\mathbb{A}^\times \mathbf{N}_{F/F^+}\mathbb{A}_F^\times))/\mathbb{Q}^\times(\mathbf{N}_{F/F^+}F^\times).$$

As $F/F^+$ is unramified at all finite primes we see that $\mathbf{N}_{F/F^+}\mathbb{A}_F^\times \supset \widehat{\mathbb{Z}}^\times \mathbb{R}_{>0}^\times$ so that $\mathbb{A}^\times \mathbf{N}_{F/F^+}\mathbb{A}_F^\times = \mathbb{Q}^\times \mathbf{N}_{F/F^+}\mathbb{A}_F^\times$. Because $(F^+)^\times \cap \mathbf{N}_{F/F^+}\mathbb{A}_F^\times = \mathbf{N}_{F/F^+}F^\times$ we conclude that

$$\ker^1(\mathbb{Q}, G) \cong \mathbb{Q}^\times((F^+)^\times \cap \mathbf{N}_{F/F^+}\mathbb{A}_F^\times)/\mathbb{Q}^\times(\mathbf{N}_{F/F^+}F^\times) = \{1\}.$$

It follows from the proof of Lemma 3.1 of [Shin 2011] that the Tamagawa number $\tau(G)=2$.

Let $T$ denote the quotient of $G$ by its derived subgroup. Then we may identify $T$ by

$$T(R)=\{(x,y)\in R^\times\times(R\otimes_{\mathbb{Q}}F)^\times:\ x^n={}^cy\,y\}$$

for any $\mathbb{Q}$-algebra $R$. The quotient map $d:G\to T$ sends $g$ to $(\nu(g),\det g)$. Also let $Z$ denote the centre of $G$ so that

$$Z(R)=\{(x,y)\in R^\times\times(R\otimes_{\mathbb{Q}}F)^\times:\ x={}^cy\,y\}$$

for any $\mathbb{Q}$-algebra $R$. The map $d|_Z$ sends $(x,y)$ to $(x,y^n)$ and the map $\nu|_Z$ sends $(x,y)$ to $x$. Note that $Z\times E$ can be identified with the product of $\mathbb{G}_m$ with the restriction of scalars from $F$ to $E$ of $\mathbb{G}_m$ and the norm map sends $(a,b)$ to $(a\,{}^ca,\ {}^cab/{}^cb)$. Then

$$\nu: Z(\mathbb{A})/Z(\mathbb{Q})(\mathbf{N}_{E/\mathbb{Q}}Z(\mathbb{A}_E))\xrightarrow{\sim}\mathbb{A}^\times/\mathbb{Q}^\times(\mathbf{N}_{E/\mathbb{Q}}\mathbb{A}_E^\times)\cong\mathrm{Gal}(E/\mathbb{Q}).$$

[To see this note that the left hand side is

$$\{y\in\mathbb{A}_F^\times:\ y\,{}^cy\in\mathbb{A}^\times\}/\mathbb{A}_E^\times\{y\in F^\times:\ y\,{}^cy\in\mathbb{Q}^\times\}\{y/{}^cy:\ y\in\mathbb{A}_F^\times\}.$$

As $\{y/{}^cy:\ y\in\mathbb{A}_F^\times\}=\mathbb{A}_F^{\mathbf{N}_{F/F^+}=1}$ we see that the group in the previous displayed equations maps isomorphically under $\nu=\mathbf{N}_{F/F^+}$ to

$$\begin{aligned}&(\mathbb{A}^\times\cap\mathbf{N}_{F/F^+}\mathbb{A}_F^\times)/(\mathbf{N}_{E/\mathbb{Q}}\mathbb{A}_E^\times)(\mathbb{Q}^\times\cap\mathbf{N}_{F/F^+}F^\times)\\ \cong\ &(\mathbb{A}^\times\cap\mathbf{N}_{F/F^+}\mathbb{A}_F^\times)/((\mathbf{N}_{E/\mathbb{Q}}\mathbb{A}_E^\times)\mathbb{Q}^\times\cap\mathbf{N}_{F/F^+}\mathbb{A}_F^\times).\end{aligned}$$

There is a natural injection from here to $\mathbb{A}^\times/(\mathbf{N}_{E/\mathbb{Q}}\mathbb{A}_E^\times)\mathbb{Q}^\times$. It only remains to see that this map is surjective, i.e., that

$$\mathbb{A}^\times/\mathbb{Q}^\times(\mathbf{N}_{E/\mathbb{Q}}\mathbb{A}_E^\times)(\mathbb{A}^\times\cap\mathbf{N}_{F/F^+}\mathbb{A}_F^\times)=\{1\}.$$

However as $F/F^+$ is everywhere unramified we have that

$$(\mathbb{A}^\times\cap\mathbf{N}_{F/F^+}\mathbb{A}_F^\times)\supset\widehat{\mathbb{Z}}^\times\times\mathbb{R}_{>0}^\times,$$

while $\mathbb{A}^\times=\mathbb{Q}^\times\widehat{\mathbb{Z}}^\times\mathbb{R}_{>0}^\times$. ]

***The involution $I$.*** We can choose a $\mathbb{Q}$-linear map $I:V\to V$ such that

- $I(xv)={}^cx\,I(v)$ for all $x\in F$ and $v\in V$;
- $\langle Iv,Iw\rangle=-\langle v,w\rangle$ for all $v,w\in V$;
- $I^2=1$.

[To see this note that with respect to a suitable basis we have

$$\langle v, w\rangle = \mathrm{tr}_{F/\mathbb{Q}}({}^t v D^c w)$$

for some diagonal matrix $D$ with ${}^c D = -D$. With respect to such a basis we can take $I$ to simply be complex conjugation on coordinates.] The choice of $I$ gives rise to an automorphism # of $G$ of order two:

$$g^\# = IgI.$$

Note that

$$\nu \circ \# = \nu$$

and that

$$\det g^\# = {}^c \det g.$$

If we identify $G \times E$ with the product of $\mathbb{G}_m$ and the restriction of scalars from $F$ to $E$ of $\mathrm{GL}_n$ then # differs by composition with an inner automorphism from the automorphism:

$$(x, g) \mapsto (x, x\,{}^t g^{-1}).$$

***Base change from $G(\mathbb{A}^\infty)$ to $(\mathbb{A}_E^\infty)^\times \times \mathrm{GL}_n(\mathbb{A}_F^\infty)$.*** As in [Harris and Taylor 2001, Section VI.2] we can define the base change $\mathrm{BC}(\tilde{\pi})$ of an irreducible admissible representation $\tilde{\pi}$ of $G(\mathbb{A}^\infty)$ which is unramified at a place $v$ of $\mathbb{Q}$, unless all primes of $F^+$ above $v$ split in $F$. The base change lift, $\mathrm{BC}(\tilde{\pi})$, is an irreducible admissible representation of $(\mathbb{A}_E^\infty)^\times \times \mathrm{GL}_n(\mathbb{A}_F^\infty)$. Note that if $\delta_{E/\mathbb{Q}}$ denotes the nontrivial character of $\mathbb{A}^\times/\mathbb{Q}^\times \mathbf{N}_{E/\mathbb{Q}}\mathbb{A}_E^\times$ then

$$\mathrm{BC}(\tilde{\pi}) = \mathrm{BC}(\tilde{\pi} \otimes (\delta_{E/\mathbb{Q}} \circ \nu)).$$

Also note that $\tilde{\pi}$ and $\tilde{\pi} \otimes (\delta_{E/\mathbb{Q}} \circ \nu)$ have different central characters and so can not be isomorphic. (Recall that

$$\nu : Z(\mathbb{A}^\infty) \twoheadrightarrow (\mathbb{A}^\infty)^\times \cap \mathbf{N}_{F/F^+}(\mathbb{A}_F^\infty)^\times \supset \widehat{\mathbb{Z}}^\times,$$

and that $\delta_{E/\mathbb{Q}}$ is ramified at some finite prime.) We have that

$$\kappa_{\mathrm{BC}(\tilde{\pi})} = \kappa_{\tilde{\pi}} \circ \mathbf{N},$$

where $\mathbf{N}$ denotes the norm map $Z(\mathbb{A}_E^\infty) \to Z(\mathbb{A}^\infty)$. If

$$\mathrm{BC}(\tilde{\pi}) = (\tilde{\phi}, \widetilde{\Pi})$$

then

$$\mathrm{BC}(\tilde{\pi}^\#) = (\tilde{\phi}\kappa_{\widetilde{\Pi}}|_{(\mathbb{A}_E^\infty)^\times}, \widetilde{\Pi}^\vee)$$

and

$$\kappa_{\tilde{\pi}^\#} = \kappa_{\tilde{\pi}} \kappa_{\widetilde{\Pi}}^c|_{Z(\mathbb{A}^\infty)},$$

where we think of $Z(\mathbb{A}^\infty) \subset (\mathbb{A}_F^\infty)^\times$.

Define

$$\begin{array}{rcl} \omega : T(\mathbb{A})/T(\mathbb{Q}) & \to & \mathbb{C}^\times \\ (x,y) & \mapsto & \phi^c(y)^{-1}\kappa_{\Pi,F^+}(x)^{-1}. \end{array}$$

Note that

$$\omega^{\#}\omega = 1.$$

With the functorialities of the previous paragraph the next lemma is easy to verify.

**Lemma 2.1.** *Suppose that $\tilde{\pi}$ is as in the previous paragraph and that*

$$\mathrm{BC}(\tilde{\pi}) = (\psi^\infty, \Pi_F\phi).$$

*Then*

(1) $\kappa_{\tilde{\pi}^{\#}\otimes(\omega^\infty\circ d)} = \kappa_{\tilde{\pi}}$;

(2) $\mathrm{BC}(\tilde{\pi}^{\#}\otimes(\omega^\infty\circ d)) = \mathrm{BC}(\tilde{\pi})$;

(3) *and there exists an automorphism $A_{\tilde{\pi}}$ of the underlying space of $\tilde{\pi}$ such that*

$$A_{\tilde{\pi}}\tilde{\pi}(g) = \tilde{\pi}(g^{\#})\omega(d(g))A_{\tilde{\pi}}$$

*for all $g \in G(\mathbb{A}^\infty)$ and $A_{\tilde{\pi}}^2 = 1$. Moreover $A_{\tilde{\pi}}$ is unique up to sign.*

***Weights.*** We identify $G \times_{\mathbb{Q}} \mathbb{C}$ with

$$\mathbb{G}_m \times \prod_{\tau\in \mathrm{Hom}_{E,\tau_0}(F,\mathbb{C})} \mathrm{GL}(V\otimes_{F,\tau}\mathbb{C}),$$

where $\mathrm{Hom}_{E,\tau_0}(F,\mathbb{C})$ denotes the set of embeddings $\tau : F \hookrightarrow \mathbb{C}$ with $\tau|_E = \tau_0|_E$. The identification sends $g$ to its multiplier and its push forward to each $\mathrm{GL}(V\otimes_{F,\tau}\mathbb{C})$. Let $\xi$ denote the irreducible representations of $G\times_{\mathbb{Q}}\mathbb{C}$ with highest weights $(b_0; b_{\tau,i})_{\tau|_E=\tau_0|_E}$, where

- $b_0 = \epsilon$;
- $b_{\tau,i} = a_{\tau|_{F_0^+},i} + \beta_\tau$.

Then $\xi^{\#}$ has highest weights

$$\big(b_0 + \sum_{\tau\in\mathrm{Hom}_{E,\tau_0}(F,\mathbb{C}),i} b_{\tau,i}; -b_{\tau,n+1-i}\big)_{\tau\in\mathrm{Hom}_{E,\tau_0}(F,\mathbb{C});\ i=1,\ldots,n}.$$

Also let $\zeta$ be the irreducible representation with highest weights

$$\big(-n\big([F^+:\mathbb{Q}]\alpha/2 + \sum_{\tau\in\mathrm{Hom}_{E,\tau_0}(F,\mathbb{C})}\beta_\tau\big); \alpha+2\beta_\tau\big)_{\tau\in\mathrm{Hom}_{E,\tau_0}(F,\mathbb{C});\ i=1,\ldots,n}.$$

Then

- $\zeta$ is one-dimensional;
- $\xi^{\#}\otimes\zeta \cong \xi$;
- $\zeta^{\#}\cong\zeta^\vee$;
- and $\omega|_{T(\mathbb{R})} = \zeta^{-1}$.

***Shimura varieties.*** Let $U$ denote an open compact subgroup of $G(\mathbb{A}^\infty)$. Consider the functor $\mathfrak{X}_U$ from connected, locally noetherian $F$-schemes with a specified geometric point to sets, which sends a pair $(S, \bar{s})$ to the set of equivalence classes of 4-tuples

$$(A, i, \lambda, \bar{\eta})$$

where

(1) $A/S$ is an abelian scheme of relative dimension $n$;

(2) $i : F \hookrightarrow \mathrm{End}^0(A/S)$ is such that for all $x \in F$ we have
$$\mathrm{tr}(x|_{\mathrm{Lie}\, A}) = x - {}^c x + n\, \mathrm{tr}_{F/E}\, {}^c x;$$

(3) $\lambda : A \to A^\vee$ is a polarization such that $i(x)^\vee \circ \lambda = \lambda \circ i({}^c x)$ for all $x \in F$;

(4) $\bar{\eta}$ is a $\pi_1(S, \bar{s})$-invariant $U$-orbit of $\mathbb{A}_F^\infty$-isomorphisms $\eta : V \otimes \mathbb{A}^\infty \xrightarrow{\sim} VA_{\bar{s}}$ such that for some isomorphism $\eta_0 : \mathbb{A}^\infty \xrightarrow{\sim} \mathbb{A}^\infty(1)$ and for all $v, w \in V \otimes \mathbb{A}^\infty$ we have
$$\langle \eta v, \eta w \rangle_\lambda = \eta_0 \langle v, w \rangle,$$
where $\langle\ ,\ \rangle_\lambda$ denotes the $\lambda$-Weil pairing.

Two 4-tuples $(A, i, \lambda, \bar{\eta})$ and $(A', i', \lambda', \bar{\eta}')$ are considered equivalent if there is an isogeny

$$\gamma : A \to A'$$

such that

(1) $\gamma i(x) = i'(x)\gamma$ for all $x \in F$,

(2) $\gamma^\vee \lambda' \gamma \in \mathbb{Q}^\times \lambda$,

(3) and $(V\gamma_{\bar{s}}) \circ \bar{\eta} = \bar{\eta}'$.

This functor is canonically independent of the choice of base point $\bar{s}$ and so can be considered as a functor from connected, locally noetherian $F$-schemes to sets. It can be extended to all locally noetherian $F$-schemes by setting

$$\mathfrak{X}_U(S_1 \amalg S_2) = \mathfrak{X}_U(S_1) \times \mathfrak{X}_U(S_2).$$

(See for instance [Harris and Taylor 2001, Section III.1] for more details. We are using $\mathrm{End}^0(A/S)$ to denote $\mathrm{End}(A/S) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $VA_{\bar{s}}$ for $(\lim_{\leftarrow N} A[N](k(\bar{s}))) \otimes_{\mathbb{Z}} \mathbb{Q}$, where $k(\bar{s})$ denotes the residue field of $\bar{s}$.)

If $U$ is sufficiently small then $\mathfrak{X}_U$ is represented by an abelian scheme

$$\mathcal{A}_U / X_U / \operatorname{Spec} F.$$

If $V \subset U$ is an open subgroup there is a natural map $X_V \to X_U$ such that $\mathcal{A}_U$ pulls back to $\mathcal{A}_V$. The inverse system of the $X_U$'s carries a natural action of $G(\mathbb{A}^\infty)$, as does the inverse system of the $\mathcal{A}_U$'s. If $V$ is a normal open subgroup of $U$

then $U$ acts on $X_V$ and induces an isomorphism between $U/V$ and $\mathrm{Gal}(X_V/X_U)$. Thus $\iota^{-1}\xi$ gives a representation of $U$ and hence a lisse $\overline{\mathbb{Q}}_l$-sheaf $\mathcal{L}_\xi$ on $X_U$. The $\overline{\mathbb{Q}}_l$-vector space

$$H^i(X, \mathcal{L}_\xi) = \lim_{\to U} H^i(X_U \times \bar{F}, \mathcal{L}_\xi)$$

has an action of $G(\mathbb{A}^\infty) \times \mathrm{Gal}(\bar{F}/F)$. It is admissible and semisimple as a $G(\mathbb{A}^\infty)$-module. If $U$ is an open, compact subgroup of $G(\mathbb{A}^\infty)$ then

$$H^i(X, \mathcal{L}_\xi)^U = H^i(X_U \times \bar{F}, \mathcal{L}_\xi)$$

is a continuous representation of $\mathrm{Gal}(\bar{F}/F)$ on a finite-dimensional $\overline{\mathbb{Q}}_l$-vector space.

The pull back $X_U \times_{F,c} F$ represents the functor $\mathfrak{X}'_U$ defined exactly as $\mathfrak{X}_U$ except that the condition

$$\mathrm{tr}(x|_{\mathrm{Lie}\, A}) = x - {}^c x + n\, \mathrm{tr}_{F/E}\, {}^c x$$

is replaced by the condition

$$\mathrm{tr}(x|_{\mathrm{Lie}\, A}) = {}^c x - x + n\, \mathrm{tr}_{F/E}\, x.$$

There is a map of functors $\mathfrak{X}_U \to \mathfrak{X}'_U$ which sends $(A, i, \lambda, \bar{\eta})$ to $(A, i \circ c, \lambda, \overline{\eta \circ I})$. This induces an $F$-linear map $X_U \to X_U \times_{F,c} F$ and hence a $c$-linear map, which we will also denote $I$,

$$\begin{array}{ccc} X_U & \xrightarrow{I} & X_U \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, F & \xrightarrow{c} & \mathrm{Spec}\, F. \end{array}$$

We have

- $I^2 = 1$;
- $IgI = g^\#$ for $g \in G(\mathbb{A}^\infty)$;
- and a natural isomorphism $I^*\mathcal{L}_\xi \otimes \mathcal{L}_\zeta \cong \mathcal{L}_\xi$, i.e.,

$$I^*\mathcal{L}_\xi \cong \mathcal{L}_{\xi^\#}. \tag{2-1}$$

Thus $I$ provides a way to descend the system of the $X_U$ to $F^+$; however this descended system of varieties no longer has an action of $G(\mathbb{A}^\infty)$ defined over $F^+$.

***Complex points and connected components.*** We will need to consider the complex uniformization of $X_U \times_{F,\tau} \mathbb{C}$ for every homomorphism $\tau : F \hookrightarrow \mathbb{C}$. So suppose $\tau : F \hookrightarrow \mathbb{C}$. There is a nondegenerate alternating form

$$\langle\ ,\ \rangle_\tau : V \times V \to \mathbb{Q}$$

such that

$$\langle xv, w\rangle_\tau = \langle v, {}^c x w\rangle_\tau$$

for all $x \in F$ and $v, w \in V$ and such that

- there is an isomorphism $j_\tau : (V \otimes_{\mathbb{Q}} \mathbb{A}^\infty, \langle\ ,\ \rangle) \xrightarrow{\sim} (V \otimes_{\mathbb{Q}} \mathbb{A}^\infty, \langle\ ,\ \rangle_\tau)$ as $\mathbb{A}_F^\infty$-modules with alternating $\mathbb{A}^\infty$-bilinear pairing;
- if $\tau' : F \hookrightarrow \mathbb{C}$ satisfies $\tau'|_E = \tau|_E$ then the Hermitian form on $V \otimes_{F,\tau'} \mathbb{C}$ defined by
$$(v, w) \mapsto \langle v, iw\rangle_\tau$$
has a maximal positive definite subspace of dimension 0 if $\tau' \neq \tau$ and 1 if $\tau' = \tau$.

Let $G_\tau$ denote the group of symplectic $F$-linear similitudes for $(V, \langle\ ,\ \rangle_\tau)$ and $G_{\tau,1}$ the kernel of the multiplier character $G_\tau \to \mathbb{G}_m$. Note that $G_\tau \times_{\mathbb{Q}} \mathbb{A}^\infty \cong G \times_{\mathbb{Q}} \mathbb{A}^\infty$ and that $G_\tau / G_{\tau,1} \xrightarrow{\sim} T$. Choose a $\mathbb{Q}$-linear map $I_\tau : V \to V$ such that

- $I_\tau(xv) = {}^c x I_\tau(v)$ for all $x \in F$ and $v \in V$;
- $\langle I_\tau v, I_\tau w\rangle = -\langle v, w\rangle$ for all $v, w \in V$;
- $I_\tau^2 = 1$.

We may, and shall, take $\langle\ ,\ \rangle_{\tau_0} = \langle\ ,\ \rangle$ and $I_{\tau_0} = I$.

Let $\Omega_\tau$ denote the set of homomorphisms
$$h : \mathbb{C} \to \mathrm{End}_{F \otimes_{\mathbb{Q}} \mathbb{R}}(V \otimes_{\mathbb{Q}} \mathbb{R})$$
such that

- $\langle h(z)v, w\rangle_\tau = \langle v, h({}^c z)w\rangle_\tau$ for all $z \in \mathbb{C}$ and $v, w \in V \otimes \mathbb{R}$,
- $\langle v, h(i)v\rangle_\tau \geq 0$ for all $v \in V$.

Then $\Omega_\tau$ forms a single conjugacy class for $G_{\tau,1}(\mathbb{R})$ [Kottwitz 1992, Lemma 4.3]. This gives $\Omega_\tau$ a topology (the quotient topology) and, as the group $G_{\tau,1}(\mathbb{R})$ is connected, we see that $\Omega_\tau$ is connected. There are $G(\mathbb{A}^\infty)$-equivariant homeomorphisms (see [Kottwitz 1992, Section 8], for example)
$$G_\tau(\mathbb{Q}) \backslash (G(\mathbb{A}^\infty)/U \times \Omega_\tau) \xrightarrow{\sim} (X_U \times_{F,\tau} \mathbb{C})(\mathbb{C}).$$

Let $\Lambda$ be a $\mathbb{Z}$-lattice in $V$. The map sends $(g, h)$ to a the equivalence class of a four-tuple $(A, i, \lambda, \bar{\eta})$, which is determined as follows. The abelian variety $A$ is characterized by the complex uniformization $A(\mathbb{C}) = (V \otimes_{\mathbb{Q}} \mathbb{R})/\Lambda$ with the complex structure coming from $h$. The map $i$ arises from the natural action of $F$ on $V \otimes_{\mathbb{Q}} \mathbb{R}$ and the (quasi)polarization $\lambda$ corresponds to the Riemann form $\langle\ ,\ \rangle_\tau$. Note that $VA$ is naturally identified with $V \otimes_{\mathbb{Q}} \mathbb{A}^\infty$. The level structure $\bar{\eta}$ is the class of $j_\tau \circ g$. Under $I \times c_\tau$ this is taken to $({}^c A, i \circ c, \lambda, \bar{\eta} \circ I)$, which has analytic uniformization as $(V \otimes_{\mathbb{Q}} \mathbb{R})/\Lambda$ but with the complex structure coming from $h \circ c$. The $F$ action is the complex conjugate of the usual one. The Riemann form is sent to its negative and the level structure is $j_\tau \circ g \circ I$. The map $I \otimes 1_{\mathbb{R}}$ shows that this is isomorphic to the abelian variety with additional structure corresponding to $((j_\tau^{-1} I_\tau j_\tau I) g^\#, I_\tau h I_\tau) \in G(\mathbb{A}^\infty) \times \Omega_\tau$. Set $s_\tau = j_\tau^{-1} I_\tau j_\tau I \in G(\mathbb{Q})$ and note that $s_\tau^\# s_\tau = 1$.

We conclude that there is a bijection $\varsigma_\tau$ :

$$\pi_0(X_U \times_F \overline{F}) \cong \pi_0(X_U \times_{F,\tau} \mathbb{C})(\mathbb{C}) \cong G_\tau(\mathbb{Q}) \backslash G_\tau(\mathbb{A}^\infty)/U \xrightarrow{\sim} T(\mathbb{Q}) \backslash T(\mathbb{A}^\infty)/d(U).$$

(For the bijectivity of the third map, which is given by $d$, see [Milne 2005, Theorem 5.17] and the discussion following it.) Write $\varsigma$ for $\varsigma_{\tau_0}$. The map $\varsigma_\tau$ is $G(\mathbb{A}^\infty)$-equivariant. It is also $I \times c_\tau$ equivariant if we let $I \times c_\tau$ act on $T(\mathbb{Q}) \backslash T(\mathbb{A}^\infty)/d(U)$ via $t \mapsto d(s_\tau) t^\#$. Note that because of the $G(\mathbb{A}^\infty)$ equivariance we must have $\varsigma_\tau = u_\tau \varsigma$ for some $u_\tau \in T(\mathbb{A})$. Thus we see that

- $\varsigma(Cg) = d(g)\varsigma(C)$ for all $C \in \pi_0(X_U \times_F \overline{F})$ and all $g \in G(\mathbb{A}^\infty)$,
- and for any infinite place $v$ of $\overline{F}$ there is an $s_v \in T(\mathbb{A})$ such that $\varsigma((I \times c_v)x) = s_v \varsigma(x)^\#$ and $s_v s_v^\# = 1$.

(If $v|_F$ arises from $\tau : F \hookrightarrow \mathbb{C}$ then $s_v = d(s_\tau) u_\tau^\# u_\tau^{-1}$.)

We wish to also know the $\mathrm{Gal}(\overline{F}/F)$-equivariance of $\varsigma$. Note that the $X_U$ are the canonical models for the Shimura varieties $\mathrm{Sh}_U(G, [h^{-1}])$. (See [Kottwitz 1992, Section 8] and note that $\ker^1(\mathbb{Q}, G) = (0)$.) Define a map

$$r : \mathbb{A}_F^\times \to T(\mathbb{A}_E) \xrightarrow{\mathbf{N}_{E/\mathbb{Q}}} T(\mathbb{A})$$

where the first map sends

$$x \mapsto (\mathbf{N}_{F/E} x, x)^{-1}.$$

Note that $r \circ \mathrm{Art}_F^{-1}$ is a well defined map

$$(r \circ \mathrm{Art}_F^{-1}) : \mathrm{Gal}(\overline{F}/F) \to T(\mathbb{A})/T(\mathbb{Q})T(\mathbb{R}).$$

Then according to [Milne 2005, Section 13] we have

$$\varsigma(\sigma x) = (r \circ \mathrm{Art}_F^{-1})(\sigma)\varsigma(x)$$

for all $x \in \pi_0(X_U \times_F \overline{F})$ and all $\sigma \in \mathrm{Gal}(\overline{F}/F)$.

***$H^0$ of sheaves on our Shimura varieties.*** Let $\tilde{\xi}$ be the irreducible representation of $G \times \mathbb{C}$ which has highest weight $(\tilde{b}_0, \tilde{b}_{\tau,i})_{\tau|_E = \tau_0|_E}$. The description of the previous section allows us to calculate $H^0(X_U \times \overline{F}, \mathcal{L}_{\tilde{\xi}})$. It will be (0) unless $\tilde{b}_{\tau,i} = \tilde{b}_\tau$ is independent of $i$. In this case $\tilde{\xi}$ factors through a map $T \times \mathbb{C} \to \mathbb{G}_m$ which we will also denote $\tilde{\xi}$. We can then identify $H^0(X_U \times \overline{F}, \mathcal{L}_{\tilde{\xi}})$ with the space of functions

$$f : T(\mathbb{A})/T(\mathbb{R})T(\mathbb{Q}) \to \overline{\mathbb{Q}}_l$$

such that

$$f(tu) = (\iota^{-1}\tilde{\xi})(u_l)^{-1} f(t)$$

for all $t \in T(\mathbb{A})$ and all $u \in d(U)$. The action of $G(\mathbb{A}^\infty)$ is via

$$(gf)(t) = (\iota^{-1}\tilde{\xi})(g_l) f(td(g))$$

and the action of $\mathrm{Gal}(\bar{F}/F)$ is via

$$(\sigma f)(t) = f((r \circ \mathrm{Art}_F^{-1})(\sigma)t).$$

The map that sends $f$ to $\tilde{f}$ defined by

$$\tilde{f}(t) = (\iota^{-1} \circ \tilde{\xi})(t_\infty)^{-1}(\iota^{-1}\tilde{\xi})(t_l) f(t),$$

establishes an isomorphism between $H^0(X_U \times \bar{F}, \mathcal{L}_{\tilde{\xi}})$ and the space of functions $\tilde{f} : T(\mathbb{A})/T(\mathbb{Q})d(U) \to \overline{\mathbb{Q}}_l$ such that

$$\tilde{f}(tu_\infty) = (\iota^{-1} \circ \tilde{\xi})(u_\infty)^{-1} \tilde{f}(t)$$

for all $t \in T(\mathbb{A})$ and $u_\infty \in T(\mathbb{R})$. Now the action of $G(\mathbb{A}^\infty)$ is via right translation $((g\tilde{f})(t) = \tilde{f}(td(g)))$ and the action of $\mathrm{Gal}(\bar{F}/F)$ is via

$$(\sigma \tilde{f})(t) = (\iota^{-1} \circ \tilde{\xi})(s_\infty)(\iota^{-1}\tilde{\xi})(s_l)^{-1} \tilde{f}(st)$$

where $s$ is a lift of $(r \circ \mathrm{Art}_F^{-1})(\sigma)$ to $T(\mathbb{A})$. From this it follows that we can write

$$H^0(X, \mathcal{L}_{\tilde{\xi}}) = \bigoplus_{\tilde{\omega}} \overline{\mathbb{Q}}_l v_{\tilde{\omega}}$$

where $\tilde{\omega}$ runs over continuous characters

$$T(\mathbb{A})/T(\mathbb{Q}) \to \mathbb{C}^\times$$

such that $\tilde{\omega}|_{T(\mathbb{R})} = \tilde{\xi}^{-1}$, and where:

- the action of $G(\mathbb{A}^\infty)$ on $v_{\tilde{\omega}}$ is via $\iota^{-1} \circ \tilde{\omega} \circ d$;
- the action of $\mathrm{Gal}(\bar{F}/F)$ on $v_{\tilde{\omega}}$ is via $r_{l,\iota}(\tilde{\omega} \circ r)$;
- and, if $v$ is an infinite place of $\bar{F}$, then $(I \times c_v)v_{\tilde{\omega}} \in \overline{\mathbb{Q}}_l v_{\tilde{\omega}^\#}$.

In particular cupping with $v_{\delta_{E/\mathbb{Q}} \circ \nu} \in H^0(X, \overline{\mathbb{Q}}_l)$ we see that

$$\mathrm{Hom}_{G(\mathbb{A}^\infty)}(\iota^{-1}\pi, H^i(X, \mathcal{L}_\xi)) \cong \mathrm{Hom}_{G(\mathbb{A}^\infty)}(\iota^{-1}(\pi \otimes (\delta_{E/\mathbb{Q}} \circ \nu)), H^i(X, \mathcal{L}_\xi)).$$

If $v$ is a place of $\bar{F}$ above infinity then $I \times c_v$ defines a map $X_U \times_F \bar{F} \to X_U \times_F \bar{F}$, which in turn induces a map

$$H^i(X, \mathcal{L}_\xi) \to H^i(X, \mathcal{L}_{\xi^\#}).$$

Composing this with the cup product with $\omega(s_v)^{-1/2}\upsilon_\omega \in H^0(X, \mathcal{L}_\zeta)$, we get a map

$$I_v : H^i(X, \mathcal{L}_\xi) \to H^i(X, \mathcal{L}_\xi),$$

such that

- $I_v g I_v = g^\#(\iota^{-1} \circ \omega \circ d)(g)$ for $g \in G(\mathbb{A}^\infty)$;
- and $I_v \sigma I_v = (c_v \sigma c_v) r_{l,\iota}((\psi_F \phi)^c/(\psi_F \phi))(\sigma)$ for $\sigma \in \mathrm{Gal}(\bar{F}/F)$.

***Galois representations.*** Shin shows that

- $\bigoplus_{\mathrm{BC}(\tilde{\pi})=(\psi^\infty, \Pi_F^\infty \otimes \phi^\infty)} \mathrm{Hom}_{G(\mathbb{A}^\infty)}(\iota^{-1}\tilde{\pi}, H^i(X, \mathcal{L}_\xi)) \neq (0)$ if and only if $i = n-1$;
- $$\bigoplus_{\mathrm{BC}(\tilde{\pi})=(\psi^\infty, \Pi_F^\infty \otimes \phi^\infty)} \mathrm{Hom}_{G(\mathbb{A}^\infty)}(\iota^{-1}\tilde{\pi}, H^{n-1}(X, \mathcal{L}_\xi))^{\mathrm{ss}} \cong r_{l,\iota}(\Pi)|^\vee_{\mathrm{Gal}(\bar{F}/F)} \otimes r_{l,\iota}((\psi_F^{-1}\phi^{-1})^2).$$

(See in particular Theorem 6.4, Corollary 6.5 and the proof of Lemma 3.1 of [Shin 2011]. The sums run over $\tilde{\pi}$ which only ramify above rational primes $v$, such that all places of $F^+$ above $v$ split in $F$.) From the irreducibility of $r_{l,\iota}(\Pi)|_{\mathrm{Gal}(\bar{F}/F)}$ we see that at most two $\tilde{\pi}$'s can contribute to the latter sum. On the other hand if $\tilde{\pi}$ contributes so does $\tilde{\pi} \otimes (\delta_{E/\mathbb{Q}} \circ \nu)$, because one can cup with $\upsilon_{\delta_{E/\mathbb{Q}} \circ \nu}$. Thus exactly two $\tilde{\pi}$'s contribute. Choose one of them and from now on reserve the notation $\pi$ for this one. Thus we have the following.

- Suppose that $\tilde{\pi}$ is an irreducible representation of $G(\mathbb{A}^\infty)$ and $j \in \mathbb{Z}_{\geq 0}$ such that
  - if $\tilde{\pi}$ is ramified above a rational prime $v$, then all places of $F^+$ above $v$ split in $F$;
  - $\mathrm{BC}(\tilde{\pi}) = (\psi^\infty, \Pi_F^\infty \otimes \phi^\infty)$;
  - and $\mathrm{Hom}_{G(\mathbb{A}^\infty)}(\iota^{-1}\tilde{\pi}, H^j(X, \mathcal{L}_\xi)) \neq (0)$.

  Then $j = n-1$ and $\tilde{\pi} \cong \pi$ or $\pi \otimes (\delta_{E/\mathbb{Q}} \circ \nu)$.
- $\mathrm{Hom}_{G(\mathbb{A}^\infty)}(\iota^{-1}\pi, H^{n-1}(X, \mathcal{L}_\xi)) \otimes r_{l,\iota}(\psi_F \phi) \cong r_{l,\iota}(\Pi)|^\vee_{\mathrm{Gal}(\bar{F}/F)}$.
- $\mathrm{Hom}_{G(\mathbb{A}^\infty)}(\iota^{-1}(\pi \otimes (\delta_{E/\mathbb{Q}} \circ \nu)), H^{n-1}(X, \mathcal{L}_\xi)) \otimes r_{l,\iota}(\psi_F \phi) \cong r_{l,\iota}(\Pi)|^\vee_{\mathrm{Gal}(\bar{F}/F)}$.

If $v$ is an infinite place of $\bar{F}$ then the map

$$f \mapsto I_v \circ f \circ A_\pi$$

induces a map $\tilde{c}_v$ on

$$\mathrm{Hom}_{G(\mathbb{A}^\infty)}(\iota^{-1}\pi, H^{n-1}(X, \mathcal{L}_\xi)) \otimes r_{l,\iota}(\psi_F \phi)$$

such that

$$\tilde{c}_v \circ \sigma \circ \tilde{c}_v = (c_v \sigma c_v)$$

for all $\sigma \in \mathrm{Gal}(\bar{F}/F)$. Because $r_{l,\iota}(\Pi)|^{\vee}_{\mathrm{Gal}(\bar{F}/F)}$ is irreducible, we conclude that $\tilde{c}_v$ corresponds to a scalar multiple of $r_{l,\iota}(\Pi)^{\vee}(c_v)$. We can, and shall, replace $\tilde{c}_v$ by a scalar multiple so that $\tilde{c}_v^2 = 1$, so that $\tilde{c}_v = \pm r_{l,\iota}(\Pi)^{\vee}(c_v)$. We finally have our geometric realization of $r_{l,\iota}(\Pi)(c_v)$. To prove our proposition it suffices to check that the trace of $\tilde{c}_v$ on

$$\mathrm{Hom}_{G(\mathbb{A}^{\infty})}(\iota^{-1}\pi, H^{n-1}(X, \mathcal{L}_{\xi}))$$

is $\pm 1$. This we will do in the next section by working with the variations of Hodge structure analogue of our $l$-adic sheaves.

## 3. Calculation of the trace of $\tilde{c}_v$

We must recall an alternative construction of the sheaves $\mathcal{L}_{\xi}$, $\mathcal{L}_{\xi^{\#}}$ and $\mathcal{L}_{\zeta}$, which will make sense also for variations of Hodge structures. First we recall the theory of Young symmetrizers.

***Young symmetrizers.*** Let $k$ denote a field of characteristic 0 and let $\mathcal{C}$ denote a Tannakian category over $k$ in the terminology of [Deligne 1990]. Suppose that $e = (e_1, \dots, e_n) \in \mathbb{Z}^n$ satisfies $e_1 \geq e_2 \geq \cdots \geq e_n \geq 0$. Let $S_e$ denote the symmetric group on the set $\mathcal{T}_e$ of pairs of integers $(i, j)$ with $1 \leq i \leq n$ and $1 \leq j \leq e_i$. Let $S_e^+$ denote the subgroup of $S_e$ consisting of elements $\sigma$ with $\sigma(i, j) = (i, j')$ some $j'$ and let $S_e^-$ denote the subgroup of $S_e$ consisting of elements $\sigma$ with $\sigma(i, j) = (i', j)$ for some $i'$. Further we set

$$A_e^{\pm} = \sum_{\sigma \in S_e^{\pm}} (\pm)^{\sigma} \sigma \in \mathbb{Q}[S_e],$$

where $(+)^{\sigma} = 1$ and $(-)^{\sigma}$ denotes the sign of $\sigma$. Note that $(A_e^{\pm})^2 = (\# S_e^{\pm}) A_e^{\pm}$ and $(A_e^+ A_e^-)^2 = m(e)(A_e^+ A_e^-)$ and $(A_e^- A_e^+)^2 = m(e)(A_e^- A_e^+)$ for some nonzero integer $m(e)$ [Fulton and Harris 1991, Theorem 4.3]. If $W$ is an object of $\mathcal{C}$ we define

$$\mathcal{S}_e(W) = W^{\otimes \mathcal{T}_e} A_e^+ A_e^-,$$

where $S_e$ acts on $W^{\otimes \mathcal{T}_e}$ from the right by

$$(\otimes_{t \in \mathcal{T}_e} w_t) h = \otimes_{t \in \mathcal{T}_e} w_{ht}.$$

Then $\mathcal{S}_e$ is a functor from $\mathcal{C}$ to itself. Note that $\mathcal{S}_{(1,\dots,1)}(W) = \bigwedge^n W$. Right multiplication by $A_e^+$ defines an isomorphism

$$\mathcal{S}_e(W) \xrightarrow{\sim} W^{\otimes \mathcal{T}_e} A_e^- A_e^+,$$

with inverse given by right multiplication by $m(e)^{-1}A_e^-$. Thus we get natural isomorphisms

$$\mathcal{S}_e(W)^\vee = (W^{\otimes \mathcal{T}_e} A_e^+ A_e^-)^\vee \xrightarrow{\sim} (W^\vee)^{\otimes \mathcal{T}_e} A_e^- A_e^+ \xrightarrow{\sim} \mathcal{S}_e(W^\vee).$$

Let $e' = (e_1+1, \dots, e_n+1)$. Let

$$\iota : \mathcal{T}_{e'} \xrightarrow{\sim} \mathcal{T}_{(1,\dots,1)} \amalg \mathcal{T}_e$$

be the bijection which sends $(i, 1)$ to $(i, 1)$ in the first part and, if $j > 1$, sends $(i, j)$ to $(i, j-1)$ in the second part. Then $\iota$ induces an isomorphism

$$\iota^* : W^{\otimes n} \otimes W^{\otimes \mathcal{T}_e} \to W^{\otimes \mathcal{T}_{e'}}.$$

Note that

$$A_{e'}^+ \circ \iota^* \circ (A_{(1,\dots,1)}^- \otimes A_e^- A_e^+) = (\# S_e^+)(A_{e'}^- A_{e'}^+) \circ \iota^*$$

so that we get a natural surjection

$$\left(\bigwedge^n W\right) \otimes \mathcal{S}_e(W) \xrightarrow{\sim} W^{\otimes n} A_{(1,\dots,1)}^- \otimes W^{\otimes \mathcal{T}_e} A_e^- A_e^+ \twoheadrightarrow W^{\otimes \mathcal{T}_{e'}} A_{e'}^- A_{e'}^+ \xrightarrow{\sim} \mathcal{S}'_{e'}(W),$$

where the middle map is $A_{e'}^+ \circ \iota^*$. If $W$ has rank $n$ then this map is an isomorphism. (This can be checked after applying a fibre functor where one can either count dimension, or use the fact that the map is $\mathrm{GL}(W)$ equivariant and $\left(\bigwedge^n W\right) \otimes \mathcal{S}_e(W)$ is an irreducible $\mathrm{GL}(W)$-module.) Thus for any $e = (e_1, \dots, e_n) \in (\mathbb{Z}^n)^+$ and any $W$ of rank $n$ we can define

$$\mathcal{S}_e(W) = \mathcal{S}_{e'}(W) \otimes \left(\bigwedge^n W\right)^{\otimes -f}$$

where $f \in \mathbb{Z}$ satisfies $f \geq -e_n$ and where $e' = (e_1+f, \dots, e_n+f)$. We see that up to natural isomorphism this does not depend on the choice of $f$.

**Lemma 3.1.** *If $e \in (\mathbb{Z}^n)^+$ equals $(e_1, \dots, e_n)$ set $e^* = (-e_n, \dots, -e_1) \in (\mathbb{Z}^n)^+$. If $W$ has rank $n$ then there are natural isomorphisms*

$$\mathcal{S}_{e+(f,f,\dots,f)}(W) \cong \mathcal{S}_e(W) \otimes \mathcal{S}_{(f,f,\dots,f)}(W)$$

*and*

$$\mathcal{S}_e(W) \cong \mathcal{S}_{e^*}(W^\vee).$$

*Proof.* The first assertion has already been proved so we turn to the second. We may reduce to the case $e_n \geq 0$ and we may choose $f \in \mathbb{Z}_{\geq e_1}$. Set $e' = (f-e_n, \dots, f-e_1)$. Then it will suffice to show that

$$\mathcal{S}_e(W) \cong \mathcal{S}_{e'}(W)^\vee \otimes \left(\bigwedge^n W\right)^{\otimes f}.$$

It even suffices to find a nontrivial natural map

$$\mathcal{S}_e(W) \otimes \mathcal{S}_{e'}(W) \to \left(\bigwedge^n W\right)^{\otimes f} = (W^{\otimes \mathcal{T}_{(f,\dots,f)}}) A_{(f,\dots,f)}^-.$$

(For this then gives a nontrivial natural map $\mathcal{S}_e(W) \to \mathcal{S}_{e'}(W)^\vee \otimes \left(\bigwedge^n W\right)^{\otimes f}$, which we can check is an isomorphism after applying a fibre functor, in which case the left and right hand sides become irreducible $\mathrm{GL}(W)$-modules.) To this end let $\iota$ denote the bijection

$$\iota : \mathcal{T}_{(f,\dots,f)} \xrightarrow{\sim} \mathcal{T}_e \amalg \mathcal{T}_{e'}$$

which sends $(i, j)$ to $(i, j)$ if $j \leq e_i$ and to $(n+1-i, f+1-i)$ if $j > e_i$, and let $\iota^*$ denote the induced map

$$W^{\otimes \mathcal{T}_e} \otimes W^{\bigotimes \mathcal{T}_{e'}} \xrightarrow{\sim} W^{\bigotimes \mathcal{T}_{(f,\dots,f)}}.$$

Then we consider the map

$$A^-_{(f,\dots,f)} \circ \iota^* : \mathcal{S}_e(W) \otimes \mathcal{S}_{e'}(W) \to \mathcal{S}_{(f,\dots,f)}(W).$$

We must show that if $W$ has rank $n$ then this map is nontrivial. We can reduce this to the case of $\overline{\mathbb{Q}}$-vector spaces by applying a fibre functor. In this case let $w_1, \dots, w_n$ be a basis of $W$. Consider the element

$$x = (\otimes_{\mathcal{T}_e} u_t) A^-_e \otimes (\otimes_{\mathcal{T}_{e'}} v_t) A^-_{e'} \in W^{\otimes \mathcal{T}_e} \otimes W^{\otimes \mathcal{T}_{e'}}$$

where $u_{(i,j)} = w_i$ and $v_{(i,j)} = w_{n+1-i}$. Then

$$\begin{aligned}(\iota^* x) A^-_{(f,\dots,f)} &= \left(\prod_{i=1}^f (\#\{j :\ e_j < i\})!(\#\{j :\ e_j \geq i\})!\right) (\otimes_{\mathcal{T}_{(f,\dots,f)}} x_t) A^-_{(f,\dots,f)} \\ &\neq 0,\end{aligned}$$

where $x_{(i,j)} = w_i$. The lemma follows. □

***The relative cohomology of $\mathcal{A}/X_U$.*** If $\varpi$ denotes the projection map from the universal abelian variety $\mathcal{A}$ to $X_U$ then we decompose

$$R^1 \varpi_* \overline{\mathbb{Q}}_l = \bigoplus_{\tau \in \mathrm{Hom}(F,\mathbb{C})} \mathcal{L}_\tau$$

where $\mathcal{L}_\tau$ is the subsheaf of $R^1 \varpi_* \overline{\mathbb{Q}}_l$ where the action of $F$ coming from the endomorphisms of the universal abelian variety is via $\iota^{-1}\tau$. The sheaves $\mathcal{L}_\tau$ on the inverse system of the $X_U$'s carry a natural action of $G(\mathbb{A}^\infty)$ (coming from the action of $G(\mathbb{A}^\infty)$ on the inverse system of the $\mathcal{A}/X_U$. Let $\mathrm{Std}_\tau$ denote the representation of $G \times_{\mathbb{Q}} \mathbb{C}$ on $V \otimes_{F,\tau} \mathbb{C}$, so that $\mathrm{Std}_{\tau c} \cong \nu\, \mathrm{Std}_\tau^\vee$. Then $\mathcal{L}_\tau \cong \mathcal{L}_{\mathrm{Std}_\tau^\vee}$ with the $G(\mathbb{A}^\infty)$-actions. We also define an action of $G(\mathbb{A}^\infty)$ on the sheaves $\overline{\mathbb{Q}}_l(1)$ by letting $g : g^* \overline{\mathbb{Q}}_l(1) \to \overline{\mathbb{Q}}_l(1)$ be $\nu(g_l)^{-1}$ times the canonical map. Then $\mathcal{L}_{\nu^m} \cong \overline{\mathbb{Q}}_l(m)$ with the $G(\mathbb{A}^\infty)$-actions. Moreover the Weil pairing gives $G(\mathbb{A}^\infty)$-equivariant isomorphisms

$$\mathcal{L}_\tau \cong \mathcal{L}^\vee_{\tau c} \otimes \overline{\mathbb{Q}}_l(-1)$$

corresponding to $\mathcal{L}_{\mathrm{Std}_\tau^\vee} \cong \mathcal{L}_{\mathrm{Std}_{\tau c}} \otimes \mathcal{L}_{\nu^{-1}}$.

Suppose that $\tilde{\xi}$ is an irreducible representation of $G \times_{\mathbb{Q}} \mathbb{C}$ with highest weight $(\tilde{b}_0, \tilde{b}_{\tau,i})_{\tau|_E=\tau_0|_E}$. Then we see that

$$\mathcal{L}_{\tilde{\xi}} \cong \Big( \bigotimes_{\tau|_E=\tau_0|_E} \mathcal{S}_{(\tilde{b}_{\tau,1},\ldots,\tilde{b}_{\tau,n})}(\mathcal{L}_\tau^\vee) \Big) \otimes \overline{\mathbb{Q}}_l(\tilde{b}_0),$$

with their $G(\mathbb{A}^\infty)$-actions.

Note that there are natural isomorphisms $I^*\mathcal{L}_\tau \cong \mathcal{L}_{\tau c}$ and hence, by Lemma 3.1, natural isomorphisms

$$\begin{aligned} I^*\Big( \bigotimes_{\tau|_E=\tau_0|_E} \mathcal{S}_{(\tilde{b}_{\tau,1},\ldots,\tilde{b}_{\tau,n})}(\mathcal{L}_\tau^\vee) \Big) &\otimes \overline{\mathbb{Q}}_l(\tilde{b}_0) \\ &\cong \Big( \bigotimes_{\tau|_E=\tau_0|_E} \mathcal{S}_{(\tilde{b}_{\tau,1},\ldots,\tilde{b}_{\tau,n})}(\mathcal{L}_{\tau c}^\vee) \Big) \otimes \overline{\mathbb{Q}}_l(\tilde{b}_0) \\ &\cong \Big( \bigotimes_{\tau|_E=\tau_0|_E} \mathcal{S}_{(\tilde{b}_{\tau,1},\ldots,\tilde{b}_{\tau,n})}(\mathcal{L}_\tau(1)) \Big) \otimes \overline{\mathbb{Q}}_l(\tilde{b}_0) \\ &\cong \Big( \bigotimes_{\tau|_E=\tau_0|_E} \mathcal{S}_{(-\tilde{b}_{\tau,n},\ldots,-\tilde{b}_{\tau,1})}(\mathcal{L}_\tau^\vee) \Big) \otimes \overline{\mathbb{Q}}_l\Big(\tilde{b}_0 + \sum_{\tau|_E=\tau_0|_E} \sum_i b_{\tau,i}\Big). \end{aligned}$$

This isomorphism coincides up to scalar multiples with our previous isomorphism $I^*\mathcal{L}_{\tilde{\xi}} \cong \mathcal{L}_{\tilde{\xi}^\#}$ of (2-1).

***Betti realizations.*** Fix $\sigma : \bar{F} \hookrightarrow \mathbb{C}$ which gives rise to our infinite place $v$ of $\bar{F}$ and suppose that $\sigma|_E = \tau_0|_E$. Set $X_{U,\sigma}(\mathbb{C})$ to be the complex manifold $(X_U \times_{F,\sigma} \mathbb{C})(\mathbb{C})$. If $\tau : F \hookrightarrow \mathbb{C}$ let $L_\tau$ denote the maximal subsheaf of $R^1\varpi_*\mathbb{C}$ on $X_{U,\sigma}(\mathbb{C})$ where the action of $F$ from endomorphisms of the universal abelian variety is via $\tau$. The system of locally constant sheaves $L_\tau$ have a natural action of $G(\mathbb{A}^\infty)$. Also let $\mathbb{C}(1)$ denote the constant sheaf and endow the system of sheaves $\mathbb{C}(1)/X_{U,\sigma}(\mathbb{C})$ with an action of $G(\mathbb{A}^\infty)$ by letting $g : g^*\mathbb{C}(1) \to \mathbb{C}(1)$ be $|\nu(g)|^{-1}$ times the natural map. Then

$$L_\tau \cong L_{\tau c}^\vee \otimes \mathbb{C}(-1).$$

If $\tilde{\xi}$ is the irreducible representation of $G \times_{\mathbb{Q}} \mathbb{C}$ with highest weight $(\tilde{b}_0, \tilde{b}_{\tau,i})_{\tau|_E=\tau_0|_E}$, then we define a locally constant sheaf of finite-dimensional $\mathbb{C}$-vector spaces $L_{\tilde{\xi}}$ on $X_{U,\sigma}(\mathbb{C})$ as

$$\Big( \bigotimes_{\tau|_E=\tau_0|_E} \mathcal{S}_{(\tilde{b}_{\tau,1},\ldots,\tilde{b}_{\tau,n})}(L_\tau^\vee) \Big) \otimes \mathbb{C}(\tilde{b}_0).$$

Then $L_{\tilde{\xi}}$ is the locally constant sheaf associated to the pull back of $\mathcal{L}_{\tilde{\xi}}$ to $X_U \times_{F,\sigma} \mathbb{C}$, thought of as a sheaf of $\mathbb{C}$-vector spaces via $\iota^{-1}$. This correspondence is $G(\mathbb{A}^\infty)$-equivariant. Note that by Lemma 3.1 if $\tilde{\xi}'$ is one-dimensional then

$$L_{\tilde{\xi}} \otimes L_{\tilde{\xi}'} \xrightarrow{\sim} L_{\tilde{\xi} \otimes \tilde{\xi}'}.$$

Let ${}^c X_{U,\sigma}(\mathbb{C})$ denote the complex conjugate complex manifold of $X_{U,\sigma}(\mathbb{C})$, that is, the same topological space but with complex conjugate charts. Then $I \times c$ induces an isomorphism

$$I \times c : X_{U,\sigma}(\mathbb{C}) \xrightarrow{\sim} {}^c X_{U,\sigma}(\mathbb{C}).$$

As we described above in the l-adic setting, Lemma 3.1 together with the isomorphisms $L_\tau \cong L_{\tau c}^\vee \otimes \mathbb{C}(-1)$ gives rise to an isomorphism

$$(I \times c)^* L_{\tilde{\xi}} \cong L_{\tilde{\xi}^\#}$$

compatible with the corresponding isomorphism in the $l$-adic setting ($I^* \mathcal{L}_{\tilde{\xi}} \cong \mathcal{L}_{\tilde{\xi}^\#}$).

We set

$$H^i(X_\sigma(\mathbb{C}), L_{\tilde{\xi}}) = \lim_{\rightarrow U} H^i(X_{U,\sigma}(\mathbb{C}), L_{\tilde{\xi}})$$

which is naturally a $G(\mathbb{A}^\infty)$-module and which satisfies

$$H^i(X_\sigma(\mathbb{C}), L_{\tilde{\xi}}) \cong H^i(X, \mathcal{L}_{\tilde{\xi}}) \otimes_{\overline{\mathbb{Q}}_l, \iota} \mathbb{C}$$

as $\mathbb{C}[G(\mathbb{A}^\infty)]$-modules. Again as in the l-adic setting we have a decomposition

$$H^0(X_\sigma(\mathbb{C}), L_\zeta) = \bigoplus_{\widetilde{\omega}} \mathbb{C} v_{\widetilde{\omega}, B},$$

where $\widetilde{\omega}$ runs over continuous characters

$$T(\mathbb{A})/T(\mathbb{Q}) \to \mathbb{C}^\times$$

with $\widetilde{\omega}|_{T(\mathbb{R})} = \zeta^{-1}$, and where $G(\mathbb{A}^\infty)$ acts on $v_{\widetilde{\omega}, B}$ via $\widetilde{\omega} \circ d$. If we define

$$I_{v,B} : H^i(X_\sigma(\mathbb{C}), L_\xi) \to H^i(X_\sigma(\mathbb{C}), L_\xi)$$

to be the composite

$$H^i(X_\sigma(\mathbb{C}), L_\xi) \xrightarrow{I \times c} H^i(X_\sigma(\mathbb{C}), L_{\xi^\#}) \xrightarrow{\cup v_{\omega,B}} H^i(X_\sigma(\mathbb{C}), L_\xi).$$

Then under the isomorphism $H^i(X_\sigma(\mathbb{C}), L_\xi) \cong H^i(X, \mathcal{L}_\xi) \otimes_{\overline{\mathbb{Q}}_l, \iota} \mathbb{C}$, this map $I_{v,B}$ corresponds to a scalar multiple of the previous map $I_v \otimes 1$.

Again we can define a map $\tilde{c}_{v,B}$ on

$$\mathrm{Hom}_{G(\mathbb{A}^\infty)}(\pi, H^{n-1}(X_\sigma(\mathbb{C}), L_\xi)) \cong \mathbb{C}^n$$

to be the map which sends

$$f \mapsto I_{v,B} \circ f \circ A_\pi.$$

Then $\tilde{c}_{v,B}$ corresponds to a scalar multiple of the map $\tilde{c}_v$ previously defined on $\mathrm{Hom}_{G(\mathbb{A}^\infty)}(\iota^{-1}\pi, H^{n-1}(X, \mathcal{L}_\xi))$. Rescaling $\tilde{c}_{v,B}$ we may, and shall, suppose that $\tilde{c}_{v,B}^2 = 1$, in which case it corresponds to $\pm\tilde{c}_v$. Then it suffices to show that the trace of $\tilde{c}_{v,B}$ is $\pm 1$.

***Variation of Hodge structures I: generalities.*** We begin with a rather lengthy reminder about variations of pure Hodge structures on complex manifolds. We do this because we have not found a single clear reference for all the material we need, although it is all standard.

Recall that a (pure) $\mathbb{R}$-Hodge structure of weight $w$ is a finite-dimensional $\mathbb{R}$-vector space $H$ together with a decreasing, exhaustive and separated filtration $\mathrm{Fil}^i$ on the $\mathbb{C}$-vector space $H \otimes_{\mathbb{R}} \mathbb{C}$ such that

$$H \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Fil}^i (H \otimes_{\mathbb{R}} \mathbb{C}) \oplus (1 \otimes c)\, \mathrm{Fil}^{w-1-i} (H \otimes_{\mathbb{R}} \mathbb{C})$$

for all $i$. In this case $H \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_i H^{i,w-i}$, where

$$H^{i,w-i} = (\mathrm{Fil}^i\, H \otimes_{\mathbb{R}} \mathbb{C}) \cap (1 \otimes c)(\mathrm{Fil}^{w-i}\, H \otimes_{\mathbb{R}} \mathbb{C}).$$

By a polarization on $(H, \{\mathrm{Fil}^i\})$ we mean a perfect bilinear pairing

$$\langle\ ,\ \rangle : H \times H \to \mathbb{R}$$

such that the $\langle\ ,\ \rangle$-orthogonal complement of $\mathrm{Fil}^i\, H \otimes_{\mathbb{R}} \mathbb{C}$ is $\mathrm{Fil}^{w-1-i}\, H \otimes_{\mathbb{R}} \mathbb{C}$ and such that the following property holds. Define a sesquilinear pairing $(\ ,\ )$ on $H \otimes_{\mathbb{R}} \mathbb{C}$ by extending $\langle\ ,\ \rangle$ to a $\mathbb{C}$-bilinear pairing on $H \otimes \mathbb{C}$ and defining

$$(x, y) = \sqrt{-1}^{\,-w} \langle x, (1 \otimes c) y \rangle.$$

Note that $(\ ,\ )$ restricts to a perfect sesquilinear pairing on each $H^{i,w-i}$. We require that $(\ ,\ )$ is Hermitian (i.e., $(y, x) = c(x, y)$) and that the restriction of $(-1)^i(\ ,\ )$ to $H^{i,w-i}$ is positive definite. If $\phi : (H_1, \{\mathrm{Fil}_1^i\}) \to (H_2, \{\mathrm{Fil}_2^i\})$ is a map of $\mathbb{R}$-Hodge structures (i.e., a linear map $\phi : H_1 \to H_2$ such that $\phi \otimes 1$ maps $\mathrm{Fil}^i\, H_1 \otimes_{\mathbb{R}} \mathbb{C}$ to $\mathrm{Fil}^i\, H_2 \otimes_{\mathbb{R}} \mathbb{C}$ for all $i$) then

$$(\phi \otimes 1)(\mathrm{Fil}^i\, H_1 \otimes_{\mathbb{R}} \mathbb{C}) = (\mathrm{Fil}^i\, H_2 \otimes_{\mathbb{R}} \mathbb{C}) \cap (\phi(H_1) \otimes_{\mathbb{R}} \mathbb{C})$$

for all $i$. It follows that the category of $\mathbb{R}$-Hodge structures of weight $w$ is an abelian category. The restriction of a polarization to a subobject is again a polarization and the orthogonal complement of the subobject is again a subobject. It follows that the full subcategory of polarizable pure Hodge structures is also (semisimple) abelian. The direct sums of over all integers $w$ of the abelian category of $\mathbb{R}$-Hodge structures of weight $w$ and of the abelian category of polarizable $\mathbb{R}$-Hodge structures of weight $w$ are Tannakian. We will refer to them as the categories of pure $\mathbb{R}$-Hodge structures and of pure polarizable $\mathbb{R}$-Hodge structures; although strictly speaking their objects are not pure, but direct sums of pure objects.

A (pure) $\mathbb{C}$-Hodge structure of weight $w$ is a $\mathbb{C}$-vector space $H$ together with two decreasing, exhaustive and separated filtrations $\mathrm{Fil}^i$ and $\overline{\mathrm{Fil}}^i$ on $H$ such that $H = \mathrm{Fil}^i\, H \oplus \overline{\mathrm{Fil}}^{w-1-i} H$ for all $i$. If $\mathbb{H} = (H, \{\mathrm{Fil}^i\}, \{\overline{\mathrm{Fil}}^i\})$ is a $\mathbb{C}$-Hodge structure of weight $w$ then we define the underlying $\mathbb{R}$-Hodge structure to be

$$(H, \{\mathrm{Fil}^i\, H \oplus \overline{\mathrm{Fil}}^i H\}),$$

where

$$H \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} H \oplus H \supset \mathrm{Fil}^i\, H \oplus \overline{\mathrm{Fil}}^i H$$

is given by $x \otimes a \mapsto (ax, ({}^c a)x)$. This establishes an equivalence of categories between $\mathbb{C}$-Hodge structures of weight $w$ and $\mathbb{R}$-Hodge structures of weight $w$ with an action of $\mathbb{C}$. If $\mathbb{H} = (H, \{\mathrm{Fil}^i\}, \{\overline{\mathrm{Fil}}^i\})$ is a $\mathbb{C}$-Hodge structure of weight $\mathbb{R}$ then $H = \bigoplus H^{i,w-i}$, where $H^{i,w-i} = \mathrm{Fil}^i\, H \cap \overline{\mathrm{Fil}}^{w-i} H$. By a polarization on $\mathbb{H}$ we mean a perfect Hermitian pairing

$$(\ ,\ ) : H \times H \to \mathbb{C},$$

such that for all $i$ the orthogonal complement of $\mathrm{Fil}^i\, H$ is $\overline{\mathrm{Fil}}^{w-1-i} H$ and the restriction of $(-1)^i(\ ,\ )$ to $H^{i,w-i}$ is positive definite. This is equivalent to a polarization $\langle\ ,\ \rangle$ of the underlying $\mathbb{R}$-Hodge structure such that

$$\langle ax, y\rangle = \langle x, ({}^c a)y\rangle$$

for all $a \in \mathbb{C}$ and $x, y \in H$. The equivalence is given by

$$\langle x, y\rangle = \mathrm{Re}\ \sqrt{-1}^{\,w}(x, y).$$

The category of polarizable $\mathbb{C}$-Hodge structures of weight $w$ is the full subcategory of the category of $\mathbb{C}$-Hodge structures of weight $w$ whose objects are those that admit a polarization. It is closed under taking subobjects and quotients. By the category of (polarizable) pure $\mathbb{C}$-Hodge structures we mean the direct sum over $w$ of the categories of (polarizable) $\mathbb{C}$-Hodge structures of weight $w$. They are Tannakian categories. (Again objects of these categories are not strictly speaking pure, but the direct sum of pure objects of different weights.)

If $(H, \{\mathrm{Fil}^i\})$ is an $\mathbb{R}$-Hodge structure of weight $w$ then we define

$$(H, \{\mathrm{Fil}^i\}) \otimes \mathbb{C} = (H \otimes_{\mathbb{R}} \mathbb{C}, \{\mathrm{Fil}^i\}, \{(1 \otimes c)\, \mathrm{Fil}^i\}),$$

a $\mathbb{C}$-Hodge structure of weight $w$. If $(H, \{\mathrm{Fil}^i\})$ is polarizable, so is $(H, \{\mathrm{Fil}^i\}) \otimes \mathbb{C}$. (Define $(x \otimes a, y \otimes b) = \sqrt{-1}^{\,-w} a({}^c b)\langle x, y\rangle$.)

If $\mathbb{H} = (H, \{\mathrm{Fil}^i\}, \{\overline{\mathrm{Fil}}^i\})$ is a $\mathbb{C}$-Hodge structure we define its complex conjugate ${}^c\mathbb{H} = (H, \{\overline{\mathrm{Fil}}^i\}, \{\mathrm{Fil}^i\})$.

Recall also that a variation of $\mathbb{R}$-Hodge structures $\mathbb{H}$ of weight $w$ on a complex manifold $Y$ is a pair $(H, \{\mathrm{Fil}^i\})$, where $H$ is a locally constant sheaf of finite-dimensional $\mathbb{R}$-vector spaces, where $\{\mathrm{Fil}^i\}$ is an exhaustive, separated, decreasing filtration of $H \otimes_{\mathbb{R}} \mathcal{O}_Y$ by local $\mathcal{O}_Y$-direct summands, such that

- the pull back of $\mathbb{H}$ to any point of $Y$ is a pure $\mathbb{C}$-Hodge structure of weight $w$,
- and $1 \otimes d : \mathrm{Fil}^i(H \otimes_{\mathbb{R}} \mathcal{O}_Y) \to (\mathrm{Fil}^{i-1}(H \otimes_{\mathbb{R}} \mathcal{O}_Y)) \otimes_{\mathcal{O}_Y} \Omega^1_Y$.

If $\phi : \mathbb{H}_1 \to \mathbb{H}_2$ is a morphism of variation of $\mathbb{R}$-Hodge structures of weight $w$ on $Y$ then $(\phi \otimes 1)\,\mathrm{Fil}^i(H_1 \otimes_{\mathbb{R}} \mathcal{O}_Y) = ((\phi H_1) \otimes_{\mathbb{R}} \mathcal{O}_Y) \cap \mathrm{Fil}^i(H_2 \otimes_{\mathbb{R}} \mathcal{O}_Y)$. It follows that the category of variations of $\mathbb{R}$-Hodge structures of weight $w$ on $Y$ is abelian. By a polarization on $\mathbb{H}$ we mean a perfect bilinear pairing

$$\langle\ ,\ \rangle : H \times H \to \mathbb{R}$$

whose pull-back to any point of $Y$ is a polarization. The full subcategory of the category of variations of $\mathbb{R}$-Hodge structures of weight $w$ on $Y$ consisting of polarizable objects is a semisimple abelian subcategory closed under taking subobjects and quotients. By the category of (polarizable) pure variations of $\mathbb{R}$-Hodge structures on $Y$ we mean the direct sum over $w$ of the categories of (polarizable) variations of $\mathbb{R}$-Hodge structures of weight $w$ on $Y$. They are Tannakian categories. (Again objects of these categories are not strictly speaking pure, but the direct sum of pure objects of different weights.)

The pull back of a (polarizable) variation of $\mathbb{R}$-Hodge structures of weight $w$ by any morphism is clearly again a (polarizable) variation of $\mathbb{R}$-Hodge structures of weight $w$. If $Y$ is a compact Kähler manifold and $\mathbb{H}$ is a polarizable variation of $\mathbb{R}$-Hodge structures of weight $w$ on $Y$ then $H^i(Y, H)$ has a natural structure of a polarizable $\mathbb{R}$-Hodge structure of weight $i + w$ [Zucker 1979, Theorem (2.9)]. More precisely, we define $\Omega^\bullet(\mathbb{H})$ to be the complex

$$H \otimes_{\mathbb{R}} \mathcal{O}_Y \to H \otimes_{\mathbb{R}} \Omega^1_Y \to H \otimes_{\mathbb{R}} \Omega^2_Y \to \cdots,$$

and filter it by setting $\mathrm{Fil}^i\,\Omega^\bullet(\mathbb{H})$ to be the subcomplex

$$\mathrm{Fil}^i(H \otimes_{\mathbb{R}} \mathcal{O}_Y) \to \mathrm{Fil}^{i-1}(H \otimes_{\mathbb{R}} \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} \Omega^1_Y \to \mathrm{Fil}^{i-2}(H \otimes_{\mathbb{R}} \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} \Omega^2_Y \to \cdots.$$

Then the spectral sequence

$$E_1^{i,j} = \mathbb{H}^{i+j}(Y, \mathrm{gr}^i\,\Omega^\bullet(\mathbb{H})) \Rightarrow \mathbb{H}^{i+j}(Y, \Omega^\bullet(\mathbb{H})) \cong H^{i+j}(Y, H) \otimes_{\mathbb{R}} \mathbb{C}$$

degenerates at $E_1$ and defines the (Hodge) filtration on $H^i(Y, H) \otimes_{\mathbb{R}} \mathbb{C}$.

If $f : X \to Y$ is a smooth family of compact Kähler manifolds over a complex manifold $Y$ then $R^i f_* \mathbb{R}$ is naturally a polarizable variation of $\mathbb{R}$-Hodge structures of weight $i$. (See the Introduction and first two sections of [Zucker 1979].) More precisely, let $\Omega^\bullet_{X/Y}$ denote the complex

$$\mathcal{O}_X \to \Omega^1_{X/Y} \to \Omega^2_{X/Y} \to \cdots$$

and let $\mathrm{Fil}^i\,\Omega^\bullet_{X/Y}$ denote the subcomplex

$$\Omega^i_{X/Y} \to \Omega^{i+1}_{X/Y} \to \cdots$$

Then the filtration on $(R^i f_*\mathbb{R})\otimes\mathcal{O}_Y \cong \mathbb{R}^i f_*\Omega^\bullet_{X/Y}$ is the one induced by the spectral sequence

$$E_1^{i,j} = R^j f_*\Omega^i_{X/Y} \Rightarrow \mathbb{R}^{i+j} f_*\Omega^\bullet_{X/Y} \cong R^{i+j} f_*\mathbb{R} \otimes_{\mathbb{R}} \mathcal{O}_Y.$$

If moreover $Y$ is a compact Kähler manifold then the Leray spectral sequence

$$E_2^{i,j} = H^i(Y, R^j f_*\mathbb{R}) \Rightarrow H^{i+j}(X, \mathbb{R})$$

degenerates at $E_2$ and the $\mathbb{R}$-Hodge structure on $H^i(Y, R^j f_*\mathbb{R})$ is compatible with the $\mathbb{R}$-Hodge structure on $H^{i+j}(X, \mathbb{R})$ [Zucker 1979, Proposition (2.16)].

By a variation of $\mathbb{C}$-Hodge structures $\mathbb{H}$ of weight $w$ on a complex manifold $Y$ we mean a triple $(H, \{\mathrm{Fil}^i\}, \{\overline{\mathrm{Fil}}^i\})$, where $H$ is a locally constant sheaf of finite-dimensional $\mathbb{C}$-vector spaces, $\{\mathrm{Fil}^i\}$ is an exhaustive, separated, decreasing filtration of $H \otimes_{\mathbb{C}} \mathcal{O}_Y$ by local $\mathcal{O}_Y$-direct summands, and $\{\overline{\mathrm{Fil}}^i\}$ is an exhaustive, separated, decreasing filtration of $H\otimes_{\mathbb{C}}\mathcal{O}_{^cY}$ by local $\mathcal{O}_{^cY}$-direct summands such that

- the pull back of $\mathbb{H}$ to any point of $Y$ is a pure $\mathbb{C}$-Hodge structure of weight $w$,
- $1\otimes d : \mathrm{Fil}^i(H \otimes_{\mathbb{C}} \mathcal{O}_Y) \to (\mathrm{Fil}^{i-1}(H \otimes_{\mathbb{C}} \mathcal{O}_Y)) \otimes_{\mathcal{O}_Y} \Omega^1_Y$,
- and $1\otimes d : \overline{\mathrm{Fil}}^i(H \otimes_{\mathbb{C}} \mathcal{O}_{^cY}) \to (\overline{\mathrm{Fil}}^{i-1}(H \otimes_{\mathbb{C}} \mathcal{O}_{^cY})) \otimes_{\mathcal{O}_{^cY}} \Omega^1_{^cY}$.

(Recall that $^cY$ denote the same underlying topological space as $Y$ but with complex conjugate charts.) If $\mathbb{H}$ is a variation of $\mathbb{C}$-Hodge structures of weight $w$ on $Y$ then $(H, \{\mathrm{Fil}^i \oplus(1 \otimes c)\overline{\mathrm{Fil}}^i\})$ is a variation of $\mathbb{R}$-Hodge structures of weight $w$ on $Y$, where we think of $\mathrm{Fil}^i \oplus(1 \otimes c)\overline{\mathrm{Fil}}^i$ contained in

$$(H \otimes_{\mathbb{C}} \mathcal{O}_Y) \oplus (1 \otimes c)(H \otimes_{\mathbb{C}} \mathcal{O}_{^cY}) = (H \otimes_{\mathbb{C}} \mathcal{O}_Y) \oplus (H \otimes_{\mathbb{C},c} \mathcal{O}_Y) = H \otimes_{\mathbb{R}} \mathcal{O}_Y.$$

This establishes an equivalence of categories between variations of $\mathbb{C}$-Hodge structures of weight $w$ on $Y$ and variations of $\mathbb{R}$-Hodge structures of weight $w$ on $Y$ together with an action of $\mathbb{C}$. Thus the category of variations of $\mathbb{C}$-Hodge structures of weight $w$ on $Y$ is abelian. By the category of pure variations of $\mathbb{C}$-Hodge structures of weight $w$ on $Y$ we mean the direct sum over $w$ of the categories of variations of $\mathbb{C}$-Hodge structures of weight $w$. It is a Tannakian category. (Again the objects are not strictly speaking pure, but the direct sum of pure objects of different weights.)

By a polarization of a variation of $\mathbb{C}$-Hodge structures of weight $w$ on $Y$ we mean a perfect Hermitian pairing

$$( \, , \, ) : H \times H \to \mathbb{C}$$

such that the pull back to any point of $Y$ is a polarization. The category of polarizable $\mathbb{C}$-Hodge structures of weight $w$ on $Y$ is equivalent to the category of $\mathbb{R}$-Hodge structures of weight $w$ on $Y$ together with an action of $\mathbb{C}$, which admit a polarization for which the adjoint of any $a \in \mathbb{C}$ is ${}^c a$. Thus the category of polarizable variations of $\mathbb{C}$-Hodge structures of weight $w$ on $Y$ is a full abelian subcategory of the category of variations of $\mathbb{C}$-Hodge structures of weight $w$ on $Y$ and is closed under subobjects and quotients. By the category of pure polarizable variations of $\mathbb{C}$-Hodge structures of weight $w$ on $Y$ we mean the direct sum over $w$ of the categories of variations of $\mathbb{C}$-Hodge structures of weight $w$. It is again a Tannakian category. (And again the objects are not strictly speaking pure, but the direct sum of pure objects of different weights.)

If $(H, \{\mathrm{Fil}^i\})$ is a variation $\mathbb{R}$-Hodge structures of weight $w$ on $Y$ then we define

$$(H, \{\mathrm{Fil}^i\}) \otimes \mathbb{C} = (H \otimes_{\mathbb{R}} \mathbb{C}, \{\mathrm{Fil}^i\}, \{(1 \otimes c)\, \mathrm{Fil}^i\}),$$

a variation of $\mathbb{C}$-Hodge structures of weight $w$ on $Y$. If $(H, \{\mathrm{Fil}^i\})$ is polarizable then so is $(H, \{\mathrm{Fil}^i\}) \otimes \mathbb{C}$. (Define $(x \otimes a, y \otimes b) = \sqrt{-1}^{\,-w} a({}^c b)\langle x, y\rangle$.)

If $\mathbb{H} = (H, \{\mathrm{Fil}^i\}, \{\overline{\mathrm{Fil}}^i\})$ is a variation of $\mathbb{C}$-Hodge structures of weight $w$ on $Y$ we define its complex conjugate ${}^c\mathbb{H} = (H, \{\overline{\mathrm{Fil}}^i\}, \{\mathrm{Fil}^i\})$.

The pull back of a (polarizable) variation of $\mathbb{C}$-Hodge structures of weight $w$ by any morphism is clearly again a (polarizable) variation of $\mathbb{C}$-Hodge structures of weight $w$. If $Y$ is a compact Kähler manifold and $\mathbb{H}$ is a polarizable variation of $\mathbb{C}$-Hodge structures of weight $w$ on $Y$ then $H^i(Y, H)$ has a natural structure of a polarizable $\mathbb{C}$-Hodge structure of weight $i + w$). More precisely, define $\Omega^\bullet_Y(\mathbb{H})$ to be the complex

$$H \otimes_{\mathbb{C}} \mathcal{O}_Y \to H \otimes_{\mathbb{C}} \Omega^1_Y \to H \otimes_{\mathbb{C}} \Omega^2_Y \to \cdots$$

filtered by setting $\mathrm{Fil}^i\, \Omega^\bullet_Y(\mathbb{H})$ to be the subcomplex

$$\mathrm{Fil}^i(H \otimes_{\mathbb{C}} \mathcal{O}_Y) \to \mathrm{Fil}^{i-1}(H \otimes_{\mathbb{C}} \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} \Omega^1_Y \to \mathrm{Fil}^{i-2}(H \otimes_{\mathbb{C}} \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} \Omega^2_Y \to \cdots;$$

similarly $\Omega^\bullet_{{}^cY}(\mathbb{H})$ is the complex

$$H \otimes_{\mathbb{C}} \mathcal{O}_{{}^cY} \to H \otimes_{\mathbb{C}} \Omega^1_{{}^cY} \to H \otimes_{\mathbb{C}} \Omega^2_{c}Y \to \cdots$$

with $\mathrm{Fil}^i\, \Omega^\bullet_{{}^cY}(\mathbb{H}))$ the subcomplex

$$\overline{\mathrm{Fil}}^i(H \otimes_{\mathbb{C}} \mathcal{O}_{{}^cY}) \to \overline{\mathrm{Fil}}^{i-1}(H \otimes_{\mathbb{C}} \mathcal{O}_{{}^cY}) \otimes_{\mathcal{O}_{{}^cY}} \Omega^1_{{}^cY} \to \overline{\mathrm{Fil}}^{i-2}(H \otimes_{\mathbb{C}} \mathcal{O}_{{}^cY}) \otimes_{\mathcal{O}_{{}^cY}} \Omega^2_{{}^cY} \ldots.$$

Then the spectral sequences

$$E_1^{i,j} = \mathbb{H}^{i+j}(Y, \mathrm{gr}^i\, \Omega^\bullet_Y(\mathbb{H})) \Rightarrow \mathbb{H}^{i+j}(Y, \Omega^\bullet_Y(\mathbb{H})) \cong H^{i+j}(Y, H)$$

and

$$\bar{E}_1^{i,j} = \mathbb{H}^{i+j}({}^cY, \mathrm{gr}^i\, \Omega^\bullet_{{}^cY}(\mathbb{H})) \Rightarrow \mathbb{H}^{i+j}(Y, \Omega^\bullet_{{}^cY}(\mathbb{H})) \cong H^{i+j}(Y, H)$$

degenerate at $E_1$ and define the (Hodge) filtrations on $H^i(Y, H)$. (This can be easily deduced from the corresponding facts for variations of $\mathbb{R}$-Hodge structures.)

If $f : X \to Y$ is a smooth family of compact Kähler manifolds over a complex manifold $Y$ then $R^i f_*\mathbb{C}$ is naturally a polarizable variation of $\mathbb{C}$-Hodge structures of weight $i$. More precisely, the filtrations on $(R^i f_*\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_Y \cong \mathbb{R}^i f_*\Omega^\bullet_{X/Y}$ and $(R^i f_*\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{{}^cY} \cong \mathbb{R}^i f_*\Omega^\bullet_{{}^cX/{}^cY}$ are the ones induced by the spectral sequences

$$E_1^{i,j} = R^j f_*\Omega^i_{X/Y} \Rightarrow \mathbb{R}^{i+j} f_*\Omega^\bullet_{X/Y} \cong R^{i+j} f_*\mathbb{C} \otimes_{\mathbb{C}} \mathcal{O}_Y$$

and

$$\bar{E}_1^{i,j} = R^j f_*\Omega^i_{{}^cX/{}^cY} \Rightarrow \mathbb{R}^{i+j} f_*\Omega^\bullet_{{}^cX/{}^cY} \cong R^{i+j} f_*\mathbb{C} \otimes_{\mathbb{C}} \mathcal{O}_{{}^cY}.$$

If moreover $Y$ is a compact Kähler manifold then the Leray spectral sequence

$$E_2^{i,j} = H^i(Y, R^j f_*\mathbb{C}) \Rightarrow H^{i+j}(X, \mathbb{C})$$

degenerates at $E_2$ and the $\mathbb{C}$-Hodge structure on $H^i(Y, R^j f_*\mathbb{C})$ is compatible with the $\mathbb{C}$-Hodge structure on $H^{i+j}(X, \mathbb{C})$. (Again this is all easily deduced from the case of $\mathbb{R}$-Hodge structures.)

For example $\mathbb{C}(m)$ is the variation of pure $\mathbb{C}$-Hodge structures of weight $-2m$ with underlying locally constant sheaf $\mathbb{C}$ and with $\mathrm{Fil}^i = (0)$ and $\overline{\mathrm{Fil}}^i = (0)$ for $i > -m$, but with $\mathrm{Fil}^i$ and $\overline{\mathrm{Fil}}^i$ everything for $i \le m$.

If $\mathbb{H} = (H, \{\mathrm{Fil}^i\}, \{\overline{\mathrm{Fil}}^i\})$ is a variation of pure $\mathbb{C}$-Hodge structures of weight $w$ on $Y$ we define a variation pure $\mathbb{C}$-Hodge structures $\mathbb{H}\{j_1, j_2\}$ of weight $w + j_1 + j_2$ on $Y$ by setting $H\{j_1, j_2\} = H$ and

$$\mathrm{Fil}^i\, H\{j_1, j_2\} \otimes_{\mathbb{C}} \mathcal{O}_Y = \mathrm{Fil}^{i-j_1}\, H \otimes_{\mathbb{C}} \mathcal{O}_Y,$$
$$\overline{\mathrm{Fil}}^i H\{j_1, j_2\} \otimes_{\mathbb{C}} \mathcal{O}_{{}^cY} = \overline{\mathrm{Fil}}^{i-j_2} H \otimes_{\mathbb{C}} \mathcal{O}_{{}^cY}.$$

Thus $\mathbb{C}(j) = \mathbb{C}(0)\{-j, -j\}$.

***Variation of Hodge structures II.*** We will give $\mathbb{C}(j)$ (the constant variation of pure $\mathbb{C}$-Hodge structures of weight $-2j$ on $X_{U,\sigma}(\mathbb{C})$) an action of $G(\mathbb{A}^\infty)$ by letting $g : g^*\mathbb{C}(j) \to \mathbb{C}(j)$ be $|\nu(g)^{-j}|$ times the natural map. If $\mathbb{H}/X_{U,\sigma}(\mathbb{C})$ is a collection of variations of pure $\mathbb{C}$-Hodge structures with an action of $G(\mathbb{A}^\infty)$ we will give $\mathbb{H}\{j_1, j_2\}$ the action induced from the one on $\mathbb{H}$. Thus the actions of $G(\mathbb{A}^\infty)$ on $\mathbb{C}(j)$ and $\mathbb{C}(0)\{-j, -j\}$ are different.

$R^1\varpi_*\mathbb{C}$ is a variation of pure $\mathbb{C}$-Hodge structures of weight 1 on $X_{U,\sigma}(\mathbb{C})$ and we can decompose

$$R^1\varpi_*\mathbb{C} = \bigoplus_{\tau\in \mathrm{Hom}(F,\mathbb{C})} \mathbb{L}_\tau$$

where $\mathbb{L}_\tau$ is a variation of pure $\mathbb{C}$-Hodge structures of weight 1 extending $L_\tau$. The projective system of variations of pure $\mathbb{C}$-Hodge structures $\mathbb{L}_\tau/X_{U,\sigma}(\mathbb{C})$ as $U$ varies has an action of $G(\mathbb{A}^\infty)$. We have $G(\mathbb{A}^\infty)$-equivariant isomorphisms

$$\mathbb{L}_\tau \cong \mathbb{L}_{\tau c}^\vee \otimes \mathbb{C}(-1).$$

Also, if $\sigma, \tau \in \mathrm{Hom}_{E,\tau_0}(F,\mathbb{C})$ then

$$\left(\bigwedge^n \mathbb{L}_\tau\right)/X_{U,\sigma}(\mathbb{C})$$

is noncanonically isomorphic to $\mathbb{C}\{0,n\}$ if $\sigma\neq\tau$ and to $\mathbb{C}\{1,n-1\}$ if $\sigma=\tau$. This identification is not $G(\mathbb{A}^\infty)$-equivariant.

For $\tilde{\xi}$ an irreducible representation of $G\times_{\mathbb{Q}}\mathbb{C}$ with highest weight $(\tilde{b}_0, \tilde{b}_{\tau,i})$, we can then define a variation of pure $\mathbb{C}$-Hodge structures $\mathbb{L}_{\tilde{\xi}}$ of weight

$$-2\tilde{b}_0 - \sum_{\tau|_E=\tau_0|_E}\sum_i \tilde{b}_{\tau,i}$$

extending $L_{\tilde{\xi}}$ by

$$\mathbb{L}_{\tilde{\xi}} = \left(\bigotimes_{\tau|_E=\tau_0|_E} \mathcal{S}_{(\tilde{b}_{\tau,1},\dots,\tilde{b}_{\tau,n})}(\mathbb{L}_\tau^\vee)\right)\otimes\mathbb{C}(\tilde{b}_0).$$

Again the system $\mathbb{L}_{\tilde{\xi}}/X_{U,\sigma}(\mathbb{C})$ has an action of $G(\mathbb{A}^\infty)$. Again by Lemma 3.1 we see that if $\tilde{\xi}'$ is one-dimensional then there is a natural isomorphism

$$\mathbb{L}_{\tilde{\xi}}\otimes\mathbb{L}_{\tilde{\xi}'} \xrightarrow{\sim} \mathbb{L}_{\tilde{\xi}\otimes\tilde{\xi}'}.$$

We set

$$H^i(X_\sigma(\mathbb{C}),\mathbb{L}_{\tilde{\xi}}) = \lim_{\rightarrow U} H^i(X_{U,\sigma}(\mathbb{C}),\mathbb{L}_{\tilde{\xi}}).$$

It is a direct limit of pure $\mathbb{C}$-Hodge structures with an action of $G(\mathbb{A}^\infty)$, such that the fixed subspace of any open subgroup of $G(\mathbb{A}^\infty)$ is a (finite-dimensional) pure $\mathbb{C}$-Hodge structure of weight $w = i - 2\tilde{b}_0 - \left(\sum_{\tau|_E=\tau_0|_E}\sum_j \tilde{b}_{\tau,j}\right)$.

If $\tilde{b}_{\tau,j}=\tilde{b}_\tau$ is independent of $j$ for all $\tau\in\mathrm{Hom}_{E,\tau_0}(F,\mathbb{C})$ and if $\sigma|_E=\tau_0|_E$ then

$$\mathbb{L}_{\tilde{\xi}} \cong \mathbb{C}(0)\{-\tilde{b}_\sigma-\tilde{b}_0, \tilde{b}_\sigma-\tilde{b}_0-n\sum_{\tau\in\mathrm{Hom}_{E,\tau_0}(E,\mathbb{C})}\tilde{b}_\tau\}$$

noncanonically on $X_{U,\sigma}(\mathbb{C})$. If

$$\widetilde{\omega}: T(\mathbb{A})/T(\mathbb{Q}) \longrightarrow \mathbb{C}^\times$$

is a continuous character with $\widetilde{\omega}|_{T(\mathbb{R})} = \tilde{\xi}^{-1}$ then $v_{\widetilde{\omega},B}$ spans a sub pure $\mathbb{C}$-Hodge structure of $H^0(X_\sigma(\mathbb{C}), \mathbb{L}_{\tilde{\xi}})$ isomorphic to

$$\mathbb{C}(0)\{-\tilde{b}_\sigma - \tilde{b}_0, \tilde{b}_\sigma - \tilde{b}_0 - n \sum_{\tau \in \mathrm{Hom}_{E,\tau_0}(E,\mathbb{C})} \tilde{b}_\tau\}.$$

The choice of $\widetilde{\omega}$ fixes an equivariant isomorphism

$$\mathbb{L}_{\tilde{\xi}} \cong \mathbb{C}(0)\{-\tilde{b}_\sigma - \tilde{b}_0, \tilde{b}_\sigma - \tilde{b}_0 - n \sum_{\tau \in \mathrm{Hom}_{E,\tau_0}(E,\mathbb{C})} \tilde{b}_\tau\}(\widetilde{\omega} \circ d).$$

The map $(I \times c) : X_{U,\sigma}(\mathbb{C}) \to {}^c X_{U,\sigma}(\mathbb{C})$ lifts to a map $\mathcal{A}_\sigma(\mathbb{C}) \to {}^c\mathcal{A}_\sigma(\mathbb{C})$. We deduce that there is a natural isomorphism

$$(I \times c)^* \mathbb{L}_\tau \cong {}^c\mathbb{L}_{\tau c},$$

and hence applying Lemma 3.1 and the isomorphism $\mathbb{L}_\tau \cong \mathbb{L}_{\tau c}^\vee \otimes \mathbb{C}(-1)$ we get natural isomorphisms

$$(I \times c)^* \mathbb{L}_{\tilde{\xi}} \cong {}^c\mathbb{L}_{\tilde{\xi}^\#}$$

extending our previous isomorphism $(I \times c)^* L_{\tilde{\xi}} \cong L_{\tilde{\xi}^\#}$. Thus we get maps

$$H^i(X_\sigma(\mathbb{C}), \mathbb{L}_{\tilde{\xi}}) \to H^i({}^c X_\sigma(\mathbb{C}), {}^c\mathbb{L}_{\tilde{\xi}^\#}) \cong {}^c H^i(X_\sigma(\mathbb{C}), \mathbb{L}_{\tilde{\xi}^\#}).$$

Now suppose that $\sigma|_E = \tau_0|_E$. The line $\mathbb{C} v_{\omega,B}$ is a subpure $\mathbb{C}$-Hodge structure of $H^0({}^c X_\sigma(\mathbb{C}), {}^c\mathbb{L}_\zeta)$ isomorphic to $\mathbb{C}\{\gamma, -\gamma\}$ with

$$\gamma = \alpha + 2\beta_\sigma - n \sum_{\tau \in \mathrm{Hom}_{E,\tau_0}(F,\mathbb{C})} (\beta_\tau + \alpha/2).$$

Thus the cup product map

$$\cup v_{\omega,B} : {}^c\mathbb{L}_{\xi^\#} \to ({}^c\mathbb{L}_\xi)\{-\gamma, \gamma\}$$

is a map of variations of pure $\mathbb{C}$-Hodge structures. Thus the map

$$I_{v,B} : H^i(X_\sigma(\mathbb{C}), L_\xi) \to H^i(X_\sigma(\mathbb{C}), L_\xi)$$

extends to a map of pure $\mathbb{C}$-Hodge structures

$$I_{v,B} : H^i(X_\sigma(\mathbb{C}), \mathbb{L}_\xi) \to ({}^c H^i(X_\sigma(\mathbb{C}), \mathbb{L}_\xi))\{-\gamma, \gamma\},$$

or to a map of pure $\mathbb{C}$-Hodge structures

$$I_{v,B} : H^i(X_\sigma(\mathbb{C}), \mathbb{L}_\xi)\{\epsilon + \beta_\sigma, \epsilon' - \alpha - \beta_\sigma\} \to {}^c(H^i(X_\sigma(\mathbb{C}), \mathbb{L}_\xi)\{\epsilon + \beta_\sigma, \epsilon' - \alpha - \beta_\sigma\}).$$

(Note that $\epsilon' - \alpha - \beta_\sigma - (\epsilon + \beta_\sigma) = -\alpha - 2\beta_\sigma + n \sum_{\tau \in \mathrm{Hom}_{E,\tau_0}(F,\mathbb{C})} (\beta_\tau + \alpha/2) = -\gamma$.)

If we set

$$\mathbb{H} = \mathrm{Hom}_{G(\mathbb{A}^\infty)}(\pi, H^{n-1}(X_\sigma(\mathbb{C}), \mathbb{L}_\xi))\{\epsilon + \beta_\sigma, \epsilon' - \alpha - \beta_\sigma\},$$

then $\mathbb{H}$ is a pure $\mathbb{C}$-Hodge structure of weight $w = n-1-\alpha \in 2\mathbb{Z}$. We see that $\tilde{c}_{v,B}$ extends to a map of pure $\mathbb{C}$-Hodge structures:

$$\tilde{c}_{v,B} : \mathbb{H} \to {}^c\mathbb{H}$$

with $\tilde{c}_{v,B}^2 = 1$. Moreover we see that $\tilde{c}_{v,B}$ interchanges $\mathrm{Fil}^{w/2-1}\,\mathbb{H}$ and $\overline{\mathrm{Fil}}^{\,w/2-1}\mathbb{H}$, and that these two spaces have trivial intersection. We deduce that

$$\begin{aligned} |\operatorname{tr} \tilde{c}_{v,B}| &\le n - 2\dim_{\mathbb{C}} \mathrm{Fil}^{w/2-1}\,\mathbb{H} \\ &= \dim_{\mathbb{C}} \overline{\mathrm{Fil}}^{\,w/2}\mathbb{H} - \dim_{\mathbb{C}} \mathrm{Fil}^{w/2-1}\,\mathbb{H} \\ &= \dim_{\mathbb{C}} \mathrm{Fil}^{w/2}\,\mathbb{H} - \dim_{\mathbb{C}} \mathrm{Fil}^{w/2-1}\,\mathbb{H} \\ &= \dim_{\mathbb{C}} \mathrm{gr}^{w/2}\,\mathbb{H} = \mathrm{gr}^{w/2-\epsilon-\beta_\sigma} \mathrm{Hom}_{G(\mathbb{A}^\infty)}(\pi, H^{n-1}(X_\sigma(\mathbb{C}), \mathbb{L}_\xi)). \end{aligned}$$

Cupping with $v_{\delta_{E/\mathbb{Q}}\circ\nu,B}$ shows that

$$\begin{aligned} &\dim_{\mathbb{C}} \mathrm{gr}^{w/2-\epsilon-\beta_\sigma} \mathrm{Hom}_{G(\mathbb{A}^\infty)}(\pi, H^{n-1}(X_\sigma(\mathbb{C}), \mathbb{L}_\xi)) \\ &\qquad = \dim_{\mathbb{C}} \mathrm{gr}^{w/2-\epsilon-\beta_\sigma} \mathrm{Hom}_{G(\mathbb{A}^\infty)}(\pi \otimes (\delta_{E/\mathbb{Q}} \circ \nu), H^{n-1}(X_\sigma(\mathbb{C}), \mathbb{L}_\xi)). \end{aligned}$$

Thus it suffices to show that

$$\dim_{\mathbb{C}} \bigoplus_{\mathrm{BC}(\tilde{\pi}) = (\psi^\infty, \Pi_F^\infty \otimes \phi^\infty)} \mathrm{gr}^{w/2-\epsilon-\beta_\sigma} \mathrm{Hom}_{G(\mathbb{A}^\infty)}(\pi, H^{n-1}(X_\sigma(\mathbb{C}), \mathbb{L}_\xi)) \le 2.$$

However the proof of Corollary 6.7 of [Shin 2011] shows this. (Note that the constant $C_G = \tau(G)\#\ker^1(\mathbb{Q}, G)$ of [Shin 2011] in our case equals 2.) So we have finally completed the proof of Proposition 1.2.

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