On fusion categories with few irreducible degrees

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We prove some results on the structure of certain classes of integral fusion categories and semisimple Hopf algebras under restrictions on the set of their irreducible degrees.

1. Introduction

Let $k$ be an algebraically closed field of characteristic zero. Let $\mathcal{C}$ be a fusion category over $k$. That is, $\mathcal{C}$ is a $k$-linear semisimple rigid tensor category with finitely many isomorphism classes of simple objects, finite-dimensional spaces of morphisms, and such that the unit object $1$ of $\mathcal{C}$ is simple.

For example, if $G$ is a finite group, then the categories $\text{Rep } G$ of its finite-dimensional representations and the category $\mathcal{C}(G, \omega)$ of $G$-graded vector spaces with associativity determined by the 3-cocycle $\omega$ are fusion categories over $k$. More generally, if $H$ is a finite-dimensional semisimple quasi-Hopf algebra over $k$, then the category $\text{Rep } H$ of its finite-dimensional representations is a fusion category.

Let $\text{Irr}(\mathcal{C})$ denote the set of isomorphism classes of simple objects in the fusion category $\mathcal{C}$. In analogy with the case of finite groups [Isaacs 1976], we shall use the notation $c.d.(\mathcal{C})$ to indicate the set

$$\text{c.d.}(\mathcal{C}) = \{\text{FPdim } x \mid x \in \text{Irr}(\mathcal{C})\}.$$

Here, $\text{FPdim } x$ denotes the Frobenius–Perron dimension of $x \in \text{Irr}(\mathcal{C})$. Notice that, when $\mathcal{C}$ is the representation category of a quasi-Hopf algebra, Frobenius–Perron dimensions coincide with the dimensions of the underlying vector spaces. In this case, we shall use the notation $\text{c.d.}(\mathcal{C}) = \text{c.d.}(H)$.

The positive real numbers $\text{FPdim } x, x \in \text{Irr}(\mathcal{C})$, will be called the irreducible degrees of $\mathcal{C}$.

The fusion categories that we shall consider in this paper are all integral, that is, the Frobenius–Perron dimensions of objects of $\mathcal{C}$ are (natural) integers. By [Etingof

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For a finite group $G$, the knowledge of the set $c.d.(G) = c.d.(kG)$ gives in some cases substantial information about the structure of $G$. It is known, for instance, that if $|c.d.(G)| \leq 3$, then $G$ is solvable.

On the other hand, if $|c.d.(G)| = 2$, say $c.d.(G) = \{1, m\}$, $m \geq 1$, then either $G$ has an abelian normal subgroup of index $m$ or else $G$ is nilpotent of class $\leq 3$. Furthermore, if $G$ is nonabelian, then $c.d.(G) = \{1, p\}$ for some prime number $p$, and if and only if $G$ contains an abelian normal subgroup of index $p$ or the center $Z(G)$ has index $p^3$; see [Isaacs 1976, Theorems 12.11, 12.14, and 12.15].

In the context of semisimple Hopf algebras, some results in the same spirit are known. A basic one is that of [Zhu 1993], which asserts that if $|c.d.(H)| \leq 3$, then $G(H^*)$ is not trivial; in other words, $H$ has nontrivial characters of degree 1. A similar result appears in [Natale 1999, Theorem 2.2.3].

Further results, leading to classification theorems in some specific cases, appear in [Izumi and Kosaki 2002] for Kac algebras, that is, Hopf $C^*$-algebras.

In this paper we consider the general problem of understanding the structure of a fusion category $\mathcal{C}$ from a knowledge of $c.d.(\mathcal{C})$. For instance, it is well known that $c.d.(\mathcal{C}) = \{1\}$ if and only if $\mathcal{C}$ is pointed, if and only if $\mathcal{C} \simeq \mathcal{C}(G, \omega)$, for some 3-cocycle $\omega$ on the group $G = G(\mathcal{C})$ of isomorphism classes of invertible objects of $\mathcal{C}$.

More specifically, we address the following question:

**Question 1.1.** Suppose $c.d.(\mathcal{C}) = \{1, p\}$, with $p$ a prime number. What can be said about the structure of $\mathcal{C}$?

We treat mostly structural questions regarding nilpotency and solvability, in the sense introduced in [Gelaki and Nikshych 2008] and [Etingof et al. 2011]. (A related question for semisimple Hopf algebras, that we shall not discuss in the present paper, was posed in [Natale 2011, Question 7.2].)

The notions of nilpotency and solvability of a fusion category are related to the corresponding notions for finite groups as follows: if $G$ is a finite group, then the category $\text{Rep} \ G$ is nilpotent or solvable if and only if $G$ is nilpotent or solvable, respectively. On the dual side, a pointed fusion category $\mathcal{C}(G, \omega)$ is always nilpotent, while it is solvable if and only if the group $G$ is solvable.

An important class of fusion categories, called weakly group-theoretical fusion categories, was introduced and studied in [Etingof et al. 2011]. This generalized in turn the notion of a group-theoretical fusion category of [Etingof et al. 2005]. By definition, $\mathcal{C}$ is group-theoretical if it is Morita equivalent to a pointed fusion category, and it is weakly group-theoretical if it is Morita equivalent to a nilpotent fusion category. Every nilpotent or solvable fusion category is weakly group-theoretical.
With regard to Question 1.1, consider, for instance, the case where \( \mathcal{C} = \text{Rep } H \), for a semisimple Hopf algebra \( H \). A result in this direction is known in the case \( p = 2 \). It is shown in [Bichon and Natale 2011, Corollary 6.6] that if \( H \) is a semisimple Hopf algebra such that \( \text{c.d.}(H) \subseteq \{1, 2\} \), then \( H \) is upper semisolvable. Moreover, \( H \) is necessarily cocommutative if \( G(H^*) \) is of order 2. The proof of these results relies on a refinement of [Nichols and Richmond 1996, Theorem 11] given in [Bichon and Natale 2011, Theorem 1.1].

In the context of Kac algebras, it is shown in [Izumi and Kosaki 2002, Theorem IX.8(iii)] that if \( \text{c.d.}(H^*) = \{1, p\} \) and, in addition, \( |G(H)| = p \), then \( H \) is a central abelian extension associated to an action of the cyclic group of order \( p \) on a nilpotent group. In the recent terminology introduced in [Gelaki and Nikshych 2008], this result implies that such a Kac algebra is nilpotent. See Remark 4.5.

The main results of this paper are summarized in the following theorem.

**Theorem 1.2.** Let \( \mathcal{C} \) be a fusion category over \( k \).

(i) (Proposition 7.1) Suppose \( \mathcal{C} \) is weakly group-theoretical and has odd dimension. Then \( \mathcal{C} \) is solvable.

Let \( p \) be a prime number.

(ii) (Theorem 7.3) Suppose that \( \mathcal{C} \) is braided odd-dimensional and that \( \text{c.d.}(\mathcal{C}) \subseteq \{p^m : m \geq 0\} \). Then \( \mathcal{C} \) is solvable.

(iii) Suppose \( \text{c.d.}(\mathcal{C}) \subseteq \{1, p\} \). Then \( \mathcal{C} \) is solvable in any of the following cases:

- (Corollary 5.4) \( \mathcal{C} \) is of the form \( \mathcal{C}(G, \omega, \mathbb{Z}_p, \alpha) \), that is, a group-theoretical fusion category [Etingof et al. 2005], and \( G(\mathcal{C}) \) is of order \( p \).
- (Theorem 6.2) \( \mathcal{C} \) is a near-group category [Siehler 2003].
- (Theorem 6.12) \( \mathcal{C} = \text{Rep } H \), where \( H \) is a semisimple quasitriangular Hopf algebra and \( p = 2 \).

(iv) Let \( H \) be a semisimple Hopf algebra such that \( \text{c.d.}(H) \subseteq \{1, p\} \). Then \( H^* \) is nilpotent in any of the following cases:

- (Proposition 4.8) \( |G(H^*)| = p \) and \( p \) divides \( |G(H)| \).
- (Proposition 4.9) \( |G(H^*)| = p \) and \( H \) is quasitriangular.
- (Proposition 4.12) \( H \) is of type \( (1, p; p, 1) \) as an algebra.

(v) Let \( H \) be a semisimple Hopf algebra such that \( \text{c.d.}(H) \subseteq \{1, 2\} \). Then:

- (Theorem 6.4) \( H \) is weakly group-theoretical, and, furthermore, it is group-theoretical if \( H = H_{\text{ad}} \).
- (Corollary 6.9) The group \( G(H) \) is solvable.
(vi) (Theorem 4.13) Let $H$ be a semisimple Hopf algebra of type $(1, p; p, 1)$ as an algebra. Then $H$ is isomorphic to a twisting of the group algebra $kN$, where either $p = 2$ and $N = S_3$ or $p = 2^{\alpha-1}$, $\alpha > 1$, and $N$ is the affine group of the field $\mathbb{F}_{2^\alpha}$.

The proof of part (i) is a consequence of the Feit–Thompson theorem [1963], which asserts that every finite group of odd order is solvable.

By [Natale 2011, Corollary 4.5], the semisimple Hopf algebras $H$ in part (iv) are lower semisolvable in the sense of [Montgomery and Witherspoon 1998].

The results on semisimple Hopf algebras $H$ with $c.d.(H) \subseteq \{1, 2\}$ rely on the results of [Bichon and Natale 2011]. We also make strong use of several results of [Gelaki and Nikshych 2008; Gelaki and Naidu 2009; Etingof et al. 2011] on weakly group-theoretical, solvable, and nilpotent fusion categories.

**Organization of the paper.** In Section 2 we recall the main notions and results relevant to the problem we consider. In particular, several properties of group-theoretical fusion categories and Hopf algebra extensions are discussed here. The results on nilpotency are contained in Sections 3 and 4. The strategy in these sections consists in reducing the problem to considering Hopf algebra extensions. Sections 5, 6, and 7 are devoted to the solvability question in different situations.

# 2. Preliminaries

2A. **Fusion categories.** A fusion category over $k$ is a $k$-linear semisimple rigid tensor category $\mathcal{C}$ with finitely many isomorphism classes of simple objects, finite-dimensional spaces of morphisms, and such that the unit object $1$ of $\mathcal{C}$ is simple. We refer the reader to [Bakalov and Kirillov 2001; Etingof et al. 2005] for basic definitions and facts concerning fusion categories. In particular, if $H$ is a semisimple (quasi-)Hopf algebra over $k$, then $\text{Rep} \ H$ is a fusion category.

A fusion subcategory of a fusion category $\mathcal{C}$ is a full tensor subcategory $\mathcal{C}' \subseteq \mathcal{C}$ such that if $X \in \mathcal{C}$ is isomorphic to a direct summand of an object of $\mathcal{C}'$, then $X \in \mathcal{C}'$. A fusion subcategory is necessarily rigid, so it is indeed a fusion category [Drinfeld et al. 2010, Corollary F.7(i)].

A pointed fusion category is a fusion category where all simple objects are invertible. A pointed fusion category is equivalent to the category $\mathcal{C}(G, \omega)$, of finite-dimensional $G$-graded vector spaces with associativity constraint determined by a cohomology class $\omega \in H^3(G, k^\times)$, for some finite group $G$. In other words, $\mathcal{C}(G, \omega)$ is the category of representations of the quasi-Hopf algebra $k^G$, with associator $\omega \in (k^G)^{\otimes 3}$.

The fusion subcategory generated by a collection $\mathcal{K}$ of objects of $\mathcal{C}$ is the smallest fusion subcategory containing $\mathcal{K}$.

If $\mathcal{C}$ is a fusion category, then the set of isomorphism classes of invertible objects of $\mathcal{C}$ forms a group, denoted $G(\mathcal{C})$. The fusion subcategory generated by the
invertible objects of \( \mathcal{C} \) is a fusion subcategory, denoted \( \mathcal{C}_{\mathrm{pt}} \); it is the maximal pointed subcategory of \( \mathcal{C} \).

Let \( \text{Irr}(\mathcal{C}) \) denote the set of isomorphism classes of simple objects in the fusion category \( \mathcal{C} \). The set \( \text{Irr}(\mathcal{C}) \) is a basis over \( \mathbb{Z} \) of the Grothendieck ring \( \mathcal{G}(\mathcal{C}) \).

**2B. Irreducible degrees.** For \( x \in \text{Irr}(\mathcal{C}) \), let \( \text{FPdim} \ x \) be its Frobenius–Perron dimension. The positive real numbers \( \text{FPdim} \ x, x \in \text{Irr}(\mathcal{C}) \), will be called the irreducible degrees of \( \mathcal{C} \). These extend to a ring homomorphism \( \text{FPdim} : \mathcal{G}(\mathcal{C}) \to \mathbb{R} \). When \( \mathcal{C} \) is the representation category of a quasi-Hopf algebra, Frobenius–Perron dimensions coincide with the dimensions of the underlying vector spaces.

The set of irreducible degrees of \( \mathcal{C} \) is defined as

\[
\text{c.d.}(\mathcal{C}) = \{ \text{FPdim} \ x \mid x \in \text{Irr}(\mathcal{C}) \}.
\]

The category \( \mathcal{C} \) is called integral if \( \text{c.d.}(\mathcal{C}) \subseteq \mathbb{N} \).

If \( X \) is any object of \( \mathcal{C} \), then its class \( x \) in \( \mathcal{G}(\mathcal{C}) \) decomposes as

\[
x = \sum_{y \in \text{Irr}(\mathcal{C})} m(y, x) y,
\]

where \( m(y, x) = \dim \text{Hom}(Y, X) \) is the multiplicity of \( Y \) in \( X \), if \( Y \) is an object representing the class \( y \in \text{Irr}(\mathcal{C}) \).

For all \( x, y, z \in \mathcal{G}(\mathcal{C}) \), we have:

\[
m(x, yz) = m(y^*, z x^*) = m(y, x z^*).
\]  

(2-1)

Let \( x \in \text{Irr}(\mathcal{C}) \). The stabilizer of \( x \) under left multiplication by elements of \( G(\mathcal{C}) \) in the Grothendieck ring will be denoted by \( G[x] \). So, an invertible element \( g \in G(\mathcal{C}) \) belongs to \( G[x] \) if and only if \( g x = x \).

In view of (2-1), for all \( x \in \text{Irr}(\mathcal{C}) \), we have

\[
G[x] = \{ g \in G(\mathcal{C}) : m(g, xx^*) > 0 \} = \{ g \in G(\mathcal{C}) : m(g, xx^*) = 1 \}.
\]

In particular, we have the following relation in \( \mathcal{G}(\mathcal{C}) \):

\[
xx^* = \sum_{g \in G[x]} g + \sum_{y \in \text{Irr}(\mathcal{C})} m(y, xx^*) y.
\]

**Remark 2.1.** An object \( g \in \mathcal{C} \) is invertible if and only if \( \text{FPdim} \ g = 1 \).

Suppose that \( \mathcal{C} \) is an integral fusion category with \( |\text{c.d.}(\mathcal{C})| = 2 \). That is, \( \text{c.d.}(\mathcal{C}) = \{1, d\} \) for some integer \( d > 1 \). We claim that \( d \) divides the order of \( G[x] \) for all \( x \in \text{Irr}(\mathcal{C}) \) with \( \text{FPdim} \ x > 1 \); in particular, \( d \) divides the order of \( G(\mathcal{C}) \), and thus \( G(\mathcal{C}) \neq 1 \).
Indeed, if \( x \in \text{Irr}(\mathcal{C}) \) with \( \text{FPdim} \, x = d \), we have the relation

\[
xx^* = \sum_{g \in G[x]} g + \sum_{y \in \text{Irr}(\mathcal{C})} \text{FPdim} \, y = d \, m(y, xx^*) y.
\]

The claim follows by taking Frobenius–Perron dimensions.

**2C. Semisimple Hopf algebras.** Let \( H \) be a semisimple Hopf algebra over \( k \). We next recall some of the terminology and conventions from [Natale 2007b] that will be used throughout this paper.

As an algebra, \( H \) is isomorphic to a direct sum of full matrix algebras

\[
H \simeq k^{(n)} \oplus \bigoplus_{i=1}^{r} M_{d_i}(k)^{(n_i)}, \quad (2-2)
\]

where \( n = |G(H^*)| \). The Nichols–Zoeller theorem [Nichols and Zoeller 1989] implies that \( n \) divides both \( \dim H \) and \( n_i d_i^2 \), for all \( i = 1, \ldots, r \).

If we have an isomorphism as in (2-2), we shall say that \( H \) is of type \( (1, n; d_1, n_1; \ldots; d_r, n_r) \) as an algebra. If \( H^* \) is of type \( (1, n; d_1, n_1; \ldots; d_r, n_r) \) as an algebra, we shall say that \( H \) is of type \( (1, n; d_1, n_1; \ldots; d_r, n_r) \) as a coalgebra.

Let \( V \) be an \( H \)-module. The character of \( V \) is the element \( \chi = \chi_V \in H^* \) defined by \( \chi(h) = \text{Tr}_V(h) \), for all \( h \in H \). For a character \( \chi \), its degree is the integer \( \deg \chi = \chi(1) = \dim V \). The character \( \chi_V \) is called irreducible if \( V \) is irreducible.

The set \( \text{Irr}(H) \) of irreducible characters of \( H \) spans a semisimple subalgebra \( R(H) \) of \( H^* \), called the character algebra of \( H \). It is isomorphic, under the map \( V \to \chi_V \), to the extension of scalars \( k \otimes_{\mathbb{Z}} \text{End}(\text{Rep} \, H) \) of the Grothendieck ring of the category \( \text{Rep} \, H \). In particular, there is an identification \( \text{Irr}(H) \simeq \text{Irr}(\text{Rep} \, H) \).

Under this identification, all properties listed in Section 2B hold true for characters.

In this context, we have \( G(\text{Rep} \, H) = G(H^*) \). The stabilizer of \( \chi \) under left multiplication by elements in \( G(H^*) \) will be denoted by \( G[\chi] \). By the Nichols–Zoeller theorem [Nichols and Zoeller 1989], we have that \( |G[\chi]| \) divides \( (\deg \chi)^2 \).

Following [Isaacs 1976, Chapter 12], we use the notation \( \text{c.d.}(H) = \text{c.d.}(\text{Rep} \, H) \). Hence,

\[
\text{c.d.}(H) = \{ \deg \chi \mid \chi \in \text{Irr}(H) \}.
\]

In particular, if \( H \) is of type \( (1, n; d_1, n_1; \ldots; d_r, n_r) \) as an algebra, then \( \text{c.d.}(H) = \{1, d_1, \ldots, d_r\} \).

There is a bijective correspondence between Hopf algebra quotients of \( H \) and standard subalgebras of \( R(H) \), that is, subalgebras spanned by irreducible characters of \( H \). This correspondence assigns to the Hopf algebra quotient \( H \to \widetilde{H} \) its character algebra \( R(\widetilde{H}) \subseteq R(H) \). See [Nichols and Richmond 1996].
2D. **Group-theoretical categories.** A group-theoretical fusion category is a fusion category Morita equivalent to a pointed fusion category $\mathcal{C}(G, \omega)$. Such a fusion category is equivalent to the category $\mathcal{C}(G, \omega, F, \alpha)$ of $k_\alpha F$-bimodules in $\mathcal{C}(G, \omega)$, where $G$ is a finite group, $\omega$ is a 3-cocycle on $G$, $F \subseteq G$ is a subgroup, and $\alpha \in C^2(F, k^\times)$ is a 2-cochain on $F$ such that $\omega|_F = d\alpha$. A semisimple Hopf algebra $H$ is called group-theoretical if the category $\text{Rep} H$ is group-theoretical.

Let $\mathcal{C} = \mathcal{C}(G, \omega, F, \alpha)$ be a group-theoretical fusion category. Let also $\Gamma$ be a subgroup of $G$, endowed with a 2-cocycle $\beta \in Z^2(\Gamma, k^\times)$, such that:

- The class $\omega|_\Gamma$ is trivial.
- $G = F\Gamma$.
- The class $\alpha|_{F\cap\Gamma}\beta^{-1}|_{F\cap\Gamma}$ is nondegenerate.

Then there is an associated semisimple Hopf algebra $H$, such that the category $\text{Rep} H$ is equivalent to $\mathcal{C}$. By [Ostrik 2003], equivalence classes of subgroups $\Gamma$ of $G$ satisfying the conditions above classify fiber functors $\mathcal{C} \rightarrow \text{Vec}$; these correspond to the distinct Hopf algebras $H$.

Let $\mathcal{C} = \mathcal{C}(G, \omega, F, \alpha)$ be a group-theoretical fusion category. The simple objects of $\mathcal{C}$ are classified by pairs $(s, U_s)$, where $s$ runs over a set of representatives of the double cosets of $F$ in $G$, that is, orbits of the action of $F$ in the space $F \setminus G$ of left cosets of $F$ in $G$, $F_s = F \cap sFs^{-1}$ is the stabilizer of $s \in F \setminus G$, and $U_s$ is an irreducible representation of the twisted group algebra $k_{\sigma_s}F_s$, that is, an irreducible projective representation of $F_s$ with respect to a certain 2-cocycle $\sigma_s$ determined by $\omega$; see [Gelaki and Naidu 2009, Theorem 5.1].

The irreducible representation $W(s, U_s)$ corresponding to such a pair $(s, U_s)$ has dimension

$$\dim W(s, U_s) = [F : F_s] \dim U_s. \quad (2-3)$$

**Corollary 2.2.** The irreducible degrees of $\mathcal{C}(G, \omega, F, \alpha)$ divide the order of $F$.

**Remark 2.3.** A group-theoretical category $\mathcal{C} = \mathcal{C}(G, \omega, F, \alpha)$ is an integral fusion category. An explicit construction of a quasi-Hopf algebra $H$ such that $\text{Rep} H \simeq \mathcal{C}$ was given in [Natale 2005].

As an algebra, $H$ is a crossed product $k^{F \setminus G}\#_\sigma k F$, where $F \setminus G$ is the space of left cosets of $F$ in $G$ with the natural action of $F$, and $\sigma$ is a certain 2-cocycle determined by $\omega$.

Irreducible representations of $H$, that is, simple objects of $\mathcal{C}$, can therefore be described using the results for group crossed products in [Montgomery and Witherspoon 1998]: this is done in [Natale 2005, Proposition 5.5].

By [Gelaki and Naidu 2009, Theorem 5.2], the group $G(\mathcal{C})$ of invertible objects of $\mathcal{C}$ fits into an exact sequence

$$1 \rightarrow \hat{F} \rightarrow G(\mathcal{C}) \rightarrow K \rightarrow 1, \quad (2-4)$$
where $K = \{ x \in N_G(F) : [\sigma_x] = 1 \}$.

**2E. Abelian extensions.** Suppose that $G = F \Gamma$ is an exact factorization of the finite group $G$, where $\Gamma$ and $F$ are subgroups of $G$. Equivalently, $F$ and $\Gamma$ form a matched pair of groups with the actions $\triangleleft \Gamma \times F \to \Gamma$ and $\triangleright: \Gamma \times F \to F$, defined by $sx = (x \triangleleft s)(x \triangleright s)$, $x \in F$, $s \in \Gamma$. In this case, $G$ is isomorphic to the group $F \rtimes \Gamma$ defined as follows: $F \rtimes \Gamma = F \times \Gamma$, with multiplication $(x, s)(t, y) = (x(s \triangleright y), (s \triangleleft y)t)$, for all $x, y \in F$, $s, t \in \Gamma$.

Let $\sigma \in Z^2(F, (k^F)^\times)$ and $\tau \in Z^2(\Gamma, (k^F)^\times)$ be normalized 2-cocycles with respect to the actions afforded, respectively, by $\triangleleft$ and $\triangleright$, subject to appropriate compatibility conditions [Masuoka 1999].

The bicrossed product $H = k^F \rtimes_{\sigma, \tau} kF$ associated to this data is a semisimple Hopf algebra. There is an abelian exact sequence

$$k \to k^F \to H \to kF \to k.$$  \hspace{1cm} \text{(2-5)}

Moreover, every Hopf algebra $H$ fitting into such an exact sequence can be described in this way. This gives a bijective correspondence between the equivalence classes of Hopf algebra extensions (2-5) associated to the matched pair $(F, \Gamma)$ and a certain abelian group $\text{Opext}(k^F, kF)$.

**Remark 2.4.** The Hopf algebra $H$ is group theoretical. In fact, by [Natale 2003, Section 4.2], we have an equivalence of fusion categories $\text{Rep} H \simeq \mathcal{C}(G, \omega, F, 1)$, where $\omega$ is the 3-cocycle on $G$ coming from the so-called Kac exact sequence.

Irreducible representations of $H$ are classified by pairs $(s, U_s)$, where $s$ runs over a set of representatives of the orbits of the action of $F$ in $\Gamma$, $F_s = F \cap sFs^{-1}$ is the stabilizer of $s \in \Gamma$, and $U_s$ is an irreducible representation of the twisted group algebra $k_{\sigma_s}F_s$, that is, an irreducible projective representation of $F_s$ with cocycle $\sigma_s$, where $\sigma_s(x, y) = \sigma(x, y)(s)$, $x, y \in F$, $s \in \Gamma$; see [Kashina et al. 2002].

Note that, for all $s \in \Gamma$, the restriction of $\sigma_s : F \times F \to k^\times$ to the stabilizer $F_s$ indeed defines a 2-cocycle on $F_s$.

The irreducible representation corresponding to such a pair $(s, U_s)$ is in this case of the form

$$W(s, U_s) := \text{Ind}^H_{k^F \otimes kF_s} s \otimes U_s.$$  \hspace{1cm} \text{(2-6)}

**2F. Quasitriangular Hopf algebras.** Let $H$ be a finite-dimensional Hopf algebra. Recall that $H$ is called quasitriangular if there exists an invertible element $R \in H \otimes H$, called an $R$-matrix, such that

$$(\Delta \otimes \text{id})(R) = R_{13}R_{23}, \quad (\epsilon \otimes \text{id})(R) = 1,$$

$$(\text{id} \otimes \Delta)(R) = R_{13}R_{12}, \quad (\text{id} \otimes \epsilon)(R) = 1,$$

$$\Delta^\text{cop}(h) = R\Delta(h)R^{-1} \quad \text{for all } h \in H.$$
The existence of an $R$-matrix (also called a *quasitriangular structure* in what follows) amounts to the category $\text{Rep} \, H$ being a braided tensor category; see [Bakalov and Kirillov 2001].

For instance, the group algebra $kG$ of a finite group $G$ is a quasitriangular Hopf algebra with $R = 1 \otimes 1$. On the other hand, the dual Hopf algebra $k^G$ admits a quasitriangular structure if and only if $G$ is abelian.

If it exists, a quasitriangular structure in a Hopf algebra $H$ need not be unique. Another example of a quasitriangular Hopf algebra is the Drinfeld double $D(H)$ of $H$, where $H$ is any finite-dimensional Hopf algebra. We have $D(H) = H^* \text{cop} \otimes H$ as coalgebras, with a canonical $R$-matrix $R = \sum_i h_i \otimes h_i$, where $(h_i)_i$ is a basis of $H$ and $(h^i)_i$ is the dual basis.

As braided tensor categories, $\text{Rep} \, D(H)_{\text{cop}} \otimes H = \text{Rep} \, H$ is equivalent to the center of the tensor category $\text{Rep} \, H$. Suppose $(H, R)$ is a quasitriangular Hopf algebra. There are Hopf algebra maps $f_R : H^* \text{cop} \to H$ and $f_{R_{21}} : H^* \to H^\text{cop}$ defined by

$$f_R(p) = p(R(1))R(2), \quad f_{R_{21}}(p) = p(R(2))R(1),$$

for all $p \in H^*$, where $R = R(1) \otimes R(2) \in H \otimes H$.

We shall denote $f_R(H^*) = H_+$ and $f_{R_{21}}(H^*) = H_-$, respectively. Hence $H_+$ and $H_-$ are Hopf subalgebras of $H$ and we have $H_+ \simeq (H^*)^\text{cop}$.

We shall also denote by $H_R = H_- H_+ = H_+ H_-$ the minimal quasitriangular Hopf subalgebra of $H$; see [Radford 1993].

By [Radford 1993, Theorem 2], the multiplication of $H$ determines a surjective Hopf algebra map $D(H_-) \to H_R$.

A quasitriangular Hopf algebra $(H, R)$ is called *factorizable* if the map $\Phi_R : H^* \to H$ is an isomorphism, where

$$\Phi_R(p) = p(Q(1))Q(2), \quad p \in H^*; \quad (2-7)$$

here, $Q = Q(1) \otimes Q(2) = R_{21} R \in H \otimes H$ [Reshetikhin and Semenov-Tian-Shansky 1988].

If on the other hand $\Phi_R = \epsilon 1$ (or equivalently, $R_{21} R = 1 \otimes 1$), then $(H, R)$ is called *triangular*. Finite-dimensional triangular Hopf algebras were completely classified in [Etingof and Gelaki 2003]. In particular, if $(H, R)$ is a semisimple quasitriangular Hopf algebra, then $H$ is isomorphic, as a Hopf algebra, to a twisting $(kG)^J$ of some finite group $G$.

It is well known that the Drinfeld double $(D(H), \mathcal{R})$ is indeed a *factorizable* quasitriangular Hopf algebra. We have $D(H)_+ = H$ and $D(H)_- = H^* \text{cop}$.

We shall use later on in this paper the following result about factorizable Hopf algebras. A categorical version is established in [Gelaki and Nikshych 2008].
**Theorem 2.5** [Schneider 2001, Theorem 2.3]. Let \((H, R)\) be a factorizable Hopf algebra. Then the map \(\Phi_R\) induces an isomorphism of groups \(G(H^*) \to G(H) \cap Z(H)\).

Note that we may identify \(G(D(H)) = G(H^*) \times G(H)\). Under this identification, **Theorem 2.5** gives us a group isomorphism
\[
G(D(H^*)) \to G(D(H)) \cap Z(D(H)),
\]
such that \(g \# f \mapsto f \# g\). See also [Radford 1993].

In particular, if \(f = \epsilon\), then \(g \in G(H) \cap Z(H)\), and also if \(g = 1\), then \(f \in G(H^*) \cap Z(H^*)\).

Suppose \((H, R)\) is a finite-dimensional quasitriangular Hopf algebra, and let \(D(H)\) be the Drinfeld double of \(H\). Then there is a surjective Hopf algebra map \(f : D(H) \to H\), such that \((f \otimes f) \mathcal{R} = R\). The map \(f\) is determined by \(f(p \otimes h) = f_R(p)h\), for all \(p \in H^*, h \in H\).

This corresponds to the canonical inclusion of the braided tensor category \(\text{Rep} H\) (with braiding determined by the action of the \(R\)-matrix) into its center.

In particular, in the case where \(H\) is quasitriangular, the group \(G(H^*)\) can be identified with a subgroup of \(G(D(H^*))\).

### 3. Nilpotency

Let \(G\) be a finite group. A **\(G\)-grading** of a fusion category \(\mathcal{C}\) is a decomposition of \(\mathcal{C}\) as a direct sum of full abelian subcategories \(\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g\), such that \(\mathcal{C}_g^* = \mathcal{C}_{g^{-1}}\) and the tensor product \(\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}\) maps \(\mathcal{C}_g \times \mathcal{C}_h\) to \(\mathcal{C}_{gh}\). The neutral component \(\mathcal{C}_e\) is thus a fusion subcategory of \(\mathcal{C}\).

The grading is called **faithful** if \(\mathcal{C}_g \neq 0\), for all \(g \in G\). In this case, \(\mathcal{C}\) is called a **\(G\)-extension** of \(\mathcal{C}_e\) [Etingof et al. 2011].

The following proposition is a consequence of [Gelaki and Nikshych 2008, Theorem 3.8].

**Proposition 3.1.** Let \(\mathcal{C} = \text{Rep} H\), where \(H\) is a semisimple Hopf algebra. Then a faithful \(G\)-grading on \(\mathcal{C}\) corresponds to a central exact sequence of Hopf algebras \(k \to k^G \to H \to \overline{H} \to k\), such that \(\text{Rep} \overline{H} = \mathcal{C}_e\).

Let \(\mathcal{C}\) be a fusion category and let \(\mathcal{C}_{\text{ad}}\) be the adjoint subcategory of \(\mathcal{C}\). That is, \(\mathcal{C}_{\text{ad}}\) is the fusion subcategory of \(\mathcal{C}\) generated by \(X \otimes X^*\), where \(X\) runs through the simple objects of \(\mathcal{C}\).

It is shown in [Gelaki and Nikshych 2008] that there is a canonical faithful grading on \(\mathcal{C}\): \(\mathcal{C} = \bigoplus_{g \in U(\mathcal{C})} \mathcal{C}_g\), called the **universal grading**, such that \(\mathcal{C}_e = \mathcal{C}_{\text{ad}}\). The group \(U(\mathcal{C})\) is called the **universal grading group** of \(\mathcal{C}\).
In the case where $\mathcal{C} = \text{Rep} \; H$, for a semisimple Hopf algebra $H$, $K = k^U(\mathcal{C})$ is the maximal central Hopf subalgebra of $H$ and $\mathcal{C}_{ad} = \text{Rep} \; H/\text{KK}^+$ [Gelaki and Nikshych 2008, Theorem 3.8].

Recall from [Gelaki and Nikshych 2008; Etingof et al. 2011] that a fusion category $\mathcal{C}$ is called (cyclically) nilpotent if there is a sequence of fusion categories

$$\mathcal{C}_0 = \text{Vec}, \; \mathcal{C}_1, \ldots, \mathcal{C}_n = \mathcal{C}$$

and a sequence $G_1, \ldots, G_n$ of finite (cyclic) groups such that $\mathcal{C}_i$ is faithfully graded by $G_i$ with trivial component $\mathcal{C}_{i-1}$.

The semisimple Hopf algebra $H$ is called nilpotent if the fusion category $\text{Rep} \; H$ is nilpotent [Gelaki and Nikshych 2008, Definition 4.4].

For instance, if $G$ is a finite group, then the dual group algebra $kG$ is always nilpotent. However, the group algebra $kG$ is nilpotent if and only if the group $G$ is nilpotent [Gelaki and Nikshych 2008, Remark 4.7(1)].

3A. Nilpotency of an abelian extension. It is shown in [Gelaki and Naidu 2009, Corollary 4.3] that a group-theoretical fusion category $\mathcal{C}(G, \omega, F, \alpha)$ is nilpotent if and only if the normal closure of $F$ in $G$ is nilpotent. On the other hand, this happens if and only if $F$ is nilpotent and subnormal in $G$, if and only if $F \subseteq \text{Fit}(G)$, where $\text{Fit}(G)$ is the Fitting subgroup of $G$, that is, the unique largest normal nilpotent subgroup of $G$ [Gelaki and Naidu 2009, §2.3].

Combined with Remark 2.4, this implies:

**Proposition 3.2.** Let $k \to k^\Gamma \to H \to kF \to k$ be an abelian exact sequence and let $G = F \vartriangleleft \Gamma$ be the associated factorizable group. Then $H$ is nilpotent if and only if $F \subseteq \text{Fit}(G)$.

An abelian exact sequence (2-5) is called central if the image of $k^\Gamma$ is a central Hopf subalgebra of $H$. It is called cocentral if the dual exact sequence is central.

The following facts are well known:

**Lemma 3.3.** Consider an abelian exact sequence (2-5).

(i) The sequence is central if and only if the action $\lhd: \Gamma \times F \to \Gamma$ is trivial. In this case, the group $G = F \bowtie \Gamma$ is a semidirect product $G \simeq F \rtimes \Gamma$ with respect to the action $\triangleright: \Gamma \times F \to F$.

(ii) The sequence is cocentral if and only if the action $\triangleright: \Gamma \times F \to F$ is trivial. In this case, the group $G = F \bowtie \Gamma$ is a semidirect product $G \simeq F \lefttharpoonup \Gamma$ with respect to the action $\lhd: \Gamma \times F \to \Gamma$.

**Remark 3.4.** Assume the exact sequence (2-5) is central. Then $F$ is a normal subgroup of $G$. It follows from Proposition 3.2 that $H$ is nilpotent if and only if $F$ is nilpotent.
4. On the nilpotency of a class of semisimple Hopf algebras

Let \( p \) be a prime number. We shall consider in this subsection a nontrivial semisimple Hopf algebra \( H \) fitting into an abelian exact sequence

\[
k \to k \mathbb{Z}_p \to H \to kF \to k.
\]  (4-1)

The main result of this subsection is Proposition 4.3 below.

Suppose that \( \mathcal{C} \) is any group-theoretical fusion category of the form \( \mathcal{C} = \mathcal{C}(G, \omega, \mathbb{Z}_p, \alpha) \) (note that we may assume that \( \alpha = 1 \)). In particular, \( p \) divides the order of \( G(\mathcal{C}) \). We also have \( \text{c.d.}(\mathcal{C}) \subseteq \{1, p\} \), by Corollary 2.2.

**Lemma 4.1.** Let \( \mathcal{C} = \mathcal{C}(G, \omega, \mathbb{Z}_p, \alpha) \). Assume that \( |G(\mathcal{C})| = p \). Then \( G \) is a Frobenius group with Frobenius complement \( \mathbb{Z}_p \).

**Proof.** The description of the irreducible representations of \( \mathcal{C} \) in Section 2D, combined with the assumption that \( |G(\mathcal{C})| = p \), implies that \( g\mathbb{Z}_pg^{-1} \cap \mathbb{Z}_p = \{e\} \), for all \( g \in G \setminus \mathbb{Z}_p \). (In particular, the action of \( \mathbb{Z}_p \) on \( \mathbb{Z}_p \setminus G \) has no fixed points \( s \neq e \).)

This condition says that \( G \) is a Frobenius group with Frobenius complement \( \mathbb{Z}_p \), as claimed. \( \square \)

**Remark 4.2.** Let \( G \) be a Frobenius group with Frobenius complement \( \mathbb{Z}_p \), as in Lemma 4.1. By the Frobenius theorem we have that the Frobenius kernel \( N \) is a normal subgroup of \( G \), such that \( G \) is a semidirect product \( G = N \rtimes \mathbb{Z}_p \). Moreover, \( N \) is a nilpotent group, by a theorem of Thompson. See [Isaacs 1976, Theorem 7.2; Robinson 1982, Theorem 10.5.6]. In fact, the Frobenius kernel \( N \) is equal to \( \text{Fit}(G) \), the Fitting subgroup of \( G \) [Robinson 1982, Exercise 10.5.8].

As a consequence we get the following:

**Proposition 4.3.** Consider the abelian exact sequence (4-1) and assume that \( |G(H)| = p \).

(i) The sequence is central, that is, \( G(H) \subseteq Z(H) \).

(ii) \( G = F \rtimes Z_p \) is a Frobenius group with kernel \( F \). In particular, \( F \) is nilpotent.

**Proof.** We follow the lines of the proof of [Izumi and Kosaki 2002, Proposition X.7(i)]. Consider the matched pair \( (F, \mathbb{Z}_p) \) associated to (4-1), as in Section 2E. Let \( G = F \rtimes \mathbb{Z}_p \) be the corresponding factorizable group.

We have an equivalence of fusion categories \( \text{Rep} H^* \simeq \mathcal{C}(G, \omega, \mathbb{Z}_p, 1) \); see Remark 2.4. Then \( \text{Rep} H^* \) is group-theoretical and, by assumption, \( G(\text{Rep} H^*) \) is of order \( p \). By Lemma 4.1, \( G \) is a Frobenius group with Frobenius complement \( \mathbb{Z}_p \). Therefore \( G \) is a semidirect product \( G = N \rtimes \mathbb{Z}_p \), where \( N = \text{Fit}(G) \) is a nilpotent subgroup (see Remark 4.2).
Since $|G(H)| = p$, then the action of $\mathbb{Z}_p$ on $F$ has no fixed points. It follows, after decomposing $F$ as a disjoint union of $\mathbb{Z}_p$-orbits, that $|F| = 1 \pmod{p}$. In particular, $|F|$ is not divisible by $p$. Then $F$ must map trivially under the canonical projection $G \to G/N$, that is, $F \subseteq N$. Hence $F = N$, because they have the same order. This shows (ii). Since $F$ is normal in $G$, we get (i) in view of Lemma 3.3. $\square$

**Corollary 4.4.** Let $k \to k^\mathbb{Z}_p \to H \to kF \to k$ be an abelian exact sequence such that $|G(H)| = p$. Then $H$ is nilpotent.

**Proof.** It follows from Proposition 4.3, in view of Remark 3.4. $\square$

**Remark 4.5.** In view of [Izumi and Kosaki 2002, Theorem IX.8(iii)], if $H$ is a Kac algebra with $\text{c.d.}(H^*) = \{1, p\}$ and $|G(H)| = p$, then $H$ is a central abelian extension associated to an action of the cyclic group of order $p$ on a nilpotent group. It follows from Corollary 4.4 that $H$ is a nilpotent Hopf algebra.

**Remark 4.6.** Note that the (dual) assumption that $\text{c.d.}(H) = \{1, p\}$ does not imply that $H$ is nilpotent in general. For example, take $H$ to be the group algebra of a nonabelian semidirect product $F \rtimes \mathbb{Z}_p$, where $F$ is an abelian group such that $(|F|, p) = 1$.

On the other hand, the assumption on $|G(H)|$ in Corollary 4.4 and Proposition 4.3 is essential. Namely, for all prime number $p$, there exist semisimple Hopf algebras $H$ with $\text{c.d.}(H^*) = \{1, p\}$ and such that $H$ is not nilpotent.

To see an example, consider a group $F$ with an automorphism of order $p$ and suppose $F$ is not nilpotent (take, for instance, $F = S_n$, a symmetric group, such that $n > 6$ is sufficiently large). Consider the corresponding action of $\mathbb{Z}_p$ on $F$ by group automorphisms and let $G = F \rtimes \mathbb{Z}_p$ be the semidirect product.

Then there is an associated (split) abelian exact sequence $k \to k\mathbb{Z}_p \to H \to kF \to k$, such that $H$ is not commutative and not cocommutative. Moreover, in view of Corollary 2.2, $\text{c.d.}(H^*) = \{1, p\}$. But, by Remark 3.4, $H$ is not nilpotent, because $F$ is not nilpotent by assumption.

**4A. Reduction to abelian extensions from character degrees.** In this subsection we consider the case where $\text{c.d.}(H) = \{1, p\}$ for some prime $p$ and $|G(H^*)| = p$. We treat the problem of deducing an abelian extension like (4-1) from this assumption.

It is known, for instance, that if $p = 2$, then the assumption implies that $H$ is cocommutative [Izumi and Kosaki 2002, Corollary IX.9; Bichon and Natale 2011, Proposition 6.8].

**Lemma 4.7.** If $\text{c.d.}(H^*) = \{1, p\}$ for some prime $p$, then $H/(kG(H))^+ H$ is a cocommutative coalgebra.

**Proof.** Let $\chi$ be an irreducible character of degree $p$. We have that

$$\chi \chi^* = \sum_{g \in G[\chi]} g + \sum_{\deg \lambda = p} \lambda.$$
So \( p \mid |G[\chi]| \). Therefore \(|G[\chi]|\) is either \( p = \deg \chi \) or \( p^2 \), because it divides \((\deg \chi)^2\).

Moreover, since \( \chi = g \chi \) for all \( g \in G[\chi] \), we have \( G[\chi]C = C \), where \( C \) is the simple subcoalgebra of \( H \) containing \( \chi \). Then it follows from [Natale 2007b, Remark 3.2.7] that \( C/(kG[\chi])^+C \) is a cocommutative coalgebra (indeed, \(|G[\chi]|\) is either \( p = \deg \chi \) or \( p^2 \), but in the last case, \( C/(kG[\chi])^+C \) is one-dimensional, hence also cocommutative). Then \( H/(kG(H))^+H \) is a cocommutative coalgebra, by [Natale 2007b, Corollary 3.3.2]. □

4B. Results for the type \((1, p; p, n)\). Let \( p \) be a prime number. In this subsection \( H \) will be a semisimple Hopf algebra such that \( c, d. (H) = \{1, p\} \) and \( |G(H^*)| = p \). Hence \( H \) is of type \((1, p; p, n)\) as an algebra.

**Proposition 4.8.** Suppose that \( p \) divides \(|G(H)|\). Then \( G(H^*) \subseteq Z(H^*) \) and \( H^* \) is nilpotent.

**Proof.** By assumption, there is a subgroup \( G \) of \( G(H) \) with \( |G| = p \) (that is, \( G \simeq \mathbb{Z}_p \)) and the Hopf algebra inclusion \( kG \to H \) induces the following sequence:

\[
\begin{align*}
\kG(H^*) & \xrightarrow{i} H^* \xrightarrow{\pi} kG, \\
\end{align*}
\]

with \( \pi \) surjective. Set \( A = kG(H^*) \) and \( B = kG \). By [Natale 2007b, Lemma 4.1.9], \( \pi \circ i : kG(H^*) \to kG \) is an isomorphism and \( H^* \simeq R#kG(H^*) \simeq R\#\mathbb{Z}_p \) is a biproduct, where \( R \simeq (H^*)^{co \pi} \) is a semisimple braided Hopf algebra over \( \mathbb{Z}_p \). The coalgebra \( R \) is cocommutative, by Lemma 4.7, because \( R \simeq H^*/H^*kG(H^*)^+ \) as coalgebras. Since \( p \nmid 1 + np = \dim R \) then by [Sommerhäuser 2002, Proposition 7.2], \( R \) is trivial. Therefore, by [Natale 2007b, Proposition 4.6.1], \( H^* \) fits into an abelian central exact sequence

\[
\begin{align*}
k & \to k\mathbb{Z}_p \to H^* \to R \to k.
\end{align*}
\]

Now, since the extension is abelian, there is a group \( F \) such that \( R \simeq kF \). It follows from Corollary 4.4 that \( H^* \) is nilpotent. □

**Proposition 4.9.** Suppose \( H \) is quasitriangular. Then \( G(H^*) \subseteq Z(H^*) \) and \( H^* \) is nilpotent.

**Proof.** Consider the Drinfeld double \( D(H) \). Since \( H \) is quasitriangular, \( G(H^*) \simeq \mathbb{Z}_p \) is isomorphic to a subgroup of \( G(D(H^*)) \). Then \( G(D(H^*)) \) has an element \( g \# f \) of order \( p \). We have

\[
G(D(H^*)) \simeq G(D(H)) \cap Z(D(H)) \subseteq G(D(H)) = G(H^*) \times G(H);
\]

see Section 2F.
In particular, the element $f \# g \in G(D(H)) \cap Z(D(H))$ is of order $p$. If $g$ is of order $p$, then the proposition follows from Proposition 4.8. Thus we may assume that $g = 1$. Then $f \in G(H^*) \cap Z(H^*)$ is of order $p$, implying that $G(H^*) \subseteq Z(H^*)$.

Therefore $H^*$ fits into an abelian central exact sequence

$$k \rightarrow k^\mathbb{Z}_p \rightarrow H^* \rightarrow kF \rightarrow k,$$

where $F$ is a finite group such that $kF \cong H^*/H^*(k^\mathbb{Z}_p)^+$, by Lemma 4.7. In view of the assumption on the algebra structure of $H$, Corollary 4.4 implies that $H^*$ is nilpotent, as claimed. □

4C. Results for the type $(1, p; p, 1)$. We next discuss the case where $H$ is of type $(1, p; p, 1)$ as an algebra (not necessarily quasitriangular). In particular, $\dim H = p(p + 1)$ is even.

Notice that under this assumption, the category $\text{Rep} H$ is a near-group category with fusion rule given by the group $G = G(H^*) \cong \mathbb{Z}_p$ and the integer $\kappa$ [Siehler 2003].

Let $\chi$ be the irreducible character of degree $p$. It follows that $\chi = \chi^*$ and $\chi g = \chi = g \chi$. Then

$$\chi^2 = \sum_{g \in G(H^*)} g + \kappa \chi.$$

Taking degrees in the equation above we obtain $p^2 = p + \kappa p$, which means that $\kappa = p - 1$.

We shall use the following proposition. A more general statement will be proved in Theorem 6.2.

**Proposition 4.10.** Suppose $H$ is of type $(1, p; p, 1)$ as an algebra. Then either

(i) $p = 2$ and $H \cong kS_3$, or

(ii) $p = 2^{\alpha - 1}^1$ and $\dim H = 2^{\alpha} p$.

In particular, $H$ is solvable.

**Proof.** By [Siehler 2003, Theorem 1.2], it follows that $G(H^*) \cong \mathbb{Z}_{q^{\alpha - 1}}$, for some prime $q$ and $\alpha \geq 1$. Therefore $p = q^{\alpha - 1}$. If $q > 2$, then $p = 2$, which implies $H \cong kS_3$ is cocommutative. If $q = 2$, then $p$ has the particular expression $p = 2^{\alpha - 1}$. Hence $\dim H$ equals 6 or $p(p + 1) = 2^{\alpha} p$. By Burnside’s theorem for fusion categories [Etingof et al. 2011, Theorem 1.6], $H$ is solvable. □

**Remark 4.11.** Let $p$ be a prime number such that $p = 2^{\alpha - 1}^1$, as in Proposition 4.10. Consider the affine group $N$ of the field $\mathbb{F}_{2^\alpha}$, that is, $N$ is the semidirect product $\mathbb{F}_{2^\alpha} \rtimes \mathbb{F}_{2^\alpha}^\times$ with respect to the natural action of $\mathbb{F}_{2^\alpha}^\times$ on $\mathbb{F}_{2^\alpha}$. Then the group $N$ has the prescribed algebra type (see [Siehler 2003, §4.1]).

\[^1\text{Such a prime number is called a Mersenne prime; in particular } \alpha \text{ must be prime.} \]
Furthermore, suppose $p$ is (any) prime number, and $N$ is a group whose group algebra has algebra type $(1, p; p, 1)$. Then $N$ has order $p(p + 1)$ and it follows from the main result of [Seitz 1968] that either $p = 2$ and $N \cong S_3$ or $p = 2^\alpha - 1$, $\alpha > 1$, and $N \cong \mathbb{F}_{2^\alpha} \rtimes \mathbb{F}_{2^\alpha}^\times$.

**Proposition 4.12.** Let $H$ be a semisimple Hopf algebra of type $(1, p; p, 1)$ as an algebra. Then $G(H^*) \subseteq Z(H^*)$ and $H^*$ is nilpotent.

**Proof.** We have just proved in Proposition 4.10 that under this hypothesis $H$ is solvable. Since $\text{Rep} \ D(H) \cong Z(\text{Rep} \ H)$, then $D(H)$ is also solvable [Etingof et al. 2011, Proposition 4.5(i)].

By [Etingof et al. 2011, Proposition 4.5(iv)], $D(H)$ has nontrivial representations of dimension 1, that is, $|\text{Rep} \ D(H^*)| \neq 1$. We have

$$G(D(H^*)) \cong G(D(H)) \cap Z(D(H)) \subseteq G(D(H)) = G(H^*) \times G(H);$$

see Section 2F.

We next argue as in the proof of Proposition 4.9. Consider an element $1 \neq f \# g \in G(D(H)) \cap Z(D(H))$. If $f = 1$, then $1 \neq g \in Z(H) \cap G(H)$. Therefore, $H^*$ fits into a cocentral extension $k \to K \to H^* \to k^{(g)} \to k$, where $K$ is a proper normal Hopf subalgebra. The assumption on the algebra structure of $H$ implies $K = kG(H^*)$. Thus $kG(H^*)$ is normal in $H^*$, and the extension is abelian, by Lemma 4.7. The proposition follows in this case from Proposition 4.3(i) and Corollary 4.4.

Thus we may assume that $f \neq 1$. In particular, $f$ has order $p$.

If $|f| = |g| = p = |G(H^*)|$, we have that $p | |G(H)|$. Then $G(H^*) \subseteq Z(H^*)$ and $H^*$ is nilpotent, by Proposition 4.8.

Otherwise, take $|g| = n$, with $p \neq n$. If $f^n = 1$, then $p$ divides $n$ and thus $p$ divides $|G(H)|$. As before, we are done by Proposition 4.8.

If $f^n \neq 1$, then $f^n \# 1 = (f^n \# g^n) = (f \# g)^n \in Z(D(H))$, which implies that $f^n \neq 1$ is central in $H^*$ and thus $G(H^*) \subseteq Z(H^*)$.

Therefore $H^*$ fits into an abelian central exact sequence

$$k \to k^{Z_p} \to H^* \to kF \to k,$$

where $F$ is a finite group such that $kF \cong H^*/H^*(k^{Z_p})^+$, by Lemma 4.7. In view of the assumption on the algebra structure of $H$, Corollary 4.4 implies that $H^*$ is nilpotent, as claimed. \qed

**Theorem 4.13.** Let $H$ be a semisimple Hopf algebra of type $(1, p, p, 1)$ as an algebra. Then either $p = 2$ and $H \cong kS_3$, or $H$ is isomorphic to a twisting of the group algebra $kN$, where $p = 2^\alpha - 1$, $\alpha > 1$, and $N$ is the affine group of the field $\mathbb{F}_{2^\alpha}$. 
Proof. If $p = 2$, then $\dim H = 6$ and the result follows from [Masuoka 1995]. So suppose that $p$ is odd. By Propositions 4.12 and 4.10, $H^*$ fits into an abelian central exact sequence $k \to k^Z_p \to H^* \to kF \to k$, where $F$ is a finite group of order $p + 1 = 2^\alpha$. Then the action $\cdot: \mathbb{Z}_p \times F \to \mathbb{Z}_p$ is trivial, while the action $\triangleright: \mathbb{Z}_p \times F \to F$ is determined by an automorphism $\varphi \in \text{Aut } F$ of order $p = 2^\alpha - 1$.

We first claim that the group $F$ must be abelian. By a result of P. Hall [Robinson 1982, (5.3.3)], since $F$ is a 2-group, the order of $\text{Aut } F$ divides the number $n2^{(\alpha - r)r}$, where $n = |\text{GL}(r, 2)|$ and $2^r$ equals the index in $F$ of the Frattini subgroup $\text{Frat}(F)$ (which is defined as the intersection of all the maximal subgroups of $F$ [Robinson 1982, p. 135]). In particular, we have $r \leq \alpha$.

Since the order of $\varphi$ divides the order of $\text{Aut } F$ and $|\text{GL}(r, 2)| = (2^r - 1)(2^r - 2) \ldots (2^r - 2^{r-1})$, it follows that the prime $p = 2^\alpha - 1$ divides $2^r - 1$, which means that $r = \alpha$ and, therefore, $\text{Frat}(F) = 1$.

Since $F$ is nilpotent (because it is a 2-group), a result of Wielandt [Robinson 1982, (5.2.16)] implies that $[F, F]$, the commutator subgroup of $F$, is a subgroup of the Frattini subgroup $\text{Frat}(F)$. As we have just shown, we have $\text{Frat}(F) = 1$ in this case. Thus $[F, F] = 1$ and therefore $F$ is abelian, as claimed.

Consider the split extension $B_0 = k^Z_p \# kF$ associated to the matched pair $(\mathbb{Z}_p, F)$. Since $F$ is abelian, $B_0$ (being a central extension) is commutative. This means that $B_0$ is isomorphic to $k^N$, where $N = F \rtimes \mathbb{Z}_p$.

Notice that $|F| = 2^\alpha$ is relatively prime to $p$. It follows from [Natale 2007a, Proposition 5.22] and [Masuoka 2002, Proposition 3.1] that $H^*$ is obtained from the split extension $B_0 = k^Z_p \# kF \simeq k^N$ by twisting the multiplication. Indeed, the element representing the class of $H^*$ in the group $\text{Opext}(kF, k^Z_p)$ is the image of an element of $H^2(F, k^\times)$ under the map $H^2(F, k^\times) \oplus H^2(\mathbb{Z}_p, k^\times) \simeq H^2(F, k^\times) \to \text{Opext}(kF, k^Z_p)$ in the Kac exact sequence [Masuoka 2002, Theorem 1.10]. Then the claim follows from [Masuoka 2002, Proposition 3.1]. Dualizing, we get that $H$ is a twisting of the group algebra of the group $N$.

Finally, the assumption on the algebra structure of $H$ implies that $N$ is one of the claimed groups. See Remark 4.11. □

Corollary 4.14. Let $H$ be a semisimple Hopf algebra of type $(1, p, p, 1)$ as an algebra. Then $\text{Rep } H \simeq \text{Rep } N$, where $N = \mathbb{S}_3$ or $N$ is the affine group of the field $\mathbb{F}_{2^\alpha}$, for some $\alpha > 1$.

5. Solvability

Recall from [Etingof et al. 2011] that a fusion category $\mathcal{C}$ is called weakly group-theoretical if it is Morita equivalent to a nilpotent fusion category. If, furthermore, $\mathcal{C}$ is Morita equivalent to a cyclically nilpotent fusion category, then $\mathcal{C}$ is called solvable.
In other words, \( \mathcal{C} \) is weakly group-theoretical (solvable) if there exists an indecomposable algebra \( A \) in \( \mathcal{C} \) such that the category \( {}^A\mathcal{C}_A \) of \( A \)-bimodules in \( \mathcal{C} \) is a (cyclically) nilpotent fusion category.

Note that a group-theoretical fusion category is weakly group-theoretical.

On the other hand, the condition on \( \mathcal{C} \) being solvable is equivalent to the existence of a sequence of fusion categories

\[
\mathcal{C}_0 = \text{Vec}_k, \; \mathcal{C}_1, \ldots, \mathcal{C}_n = \mathcal{C},
\]

such that \( \mathcal{C}_i \) is obtained from \( \mathcal{C}_{i-1} \) either by a \( G_i \)-equivariantization or as a \( G_i \)-extension, where \( G_1, \ldots, G_n \) are cyclic groups of prime order. See [Etingof et al. 2011, Proposition 4.4].

If \( G \) is a finite group and \( \omega \in \text{H}^3(G, k^\times) \), we have that the categories \( \mathcal{C}(G, \omega) \) and \( \text{Rep } G \) are solvable if and only if \( G \) is solvable.

Let us call a semisimple Hopf algebra \( H \) weakly group-theoretical or solvable if the category \( \text{Rep } H \) is weakly group-theoretical or solvable, respectively.

### 5A. Solvability of an abelian extension.

By [Etingof et al. 2011, Proposition 4.5(i)], solvability of a fusion category is preserved under Morita equivalence. Therefore, a group-theoretical fusion category \( \mathcal{C}(G, \omega, F, \alpha) \) is solvable if and only if the group \( G \) is solvable.

**Remark 5.1.** As a consequence of the Feit–Thompson theorem [1963], we get that if the order of \( G \) is odd, then \( \mathcal{C}(G, \omega, F, \alpha) \) is solvable. This fact generalizes to weakly group-theoretical fusion categories; see Proposition 7.1 below.

This implies the following characterization of the solvability of an abelian extension:

**Corollary 5.2.** Let \( H \) be a semisimple Hopf algebra fitting into an abelian exact sequence (2-5); then \( H \) is solvable if and only if \( G = F \bowtie \Gamma \) is solvable.

In particular, if \( H \) is solvable, then \( F \) and \( \Gamma \) are solvable.

A result of Wielandt [1958] implies that if the groups \( \Gamma \) and \( F \) are nilpotent, then \( G \) is solvable. As a consequence, we get the following:

**Corollary 5.3.** Suppose \( \Gamma \) and \( F \) are nilpotent. Then \( H \) is solvable.

Then, for instance, the abelian extensions in Proposition 4.3 are solvable.

Combining Corollary 5.3 with Lemma 4.1 and Remark 4.2, we get:

**Corollary 5.4.** Let

\[
\mathcal{C} = \mathcal{C}(G, \omega, \mathbb{Z}_p, \alpha).
\]

Assume that \( |G(\mathcal{C})| = p \). Then \( \mathcal{C} \) is solvable.
6. Solvability from character degrees

Let \( p \) be a prime number. We study in this section fusion categories \( \mathcal{C} \) such that \( \text{c.d.}(\mathcal{C}) = \{1, p\} \).

It is known that if \( G \) is a finite group, then this assumption implies that the group \( G \), and thus the category \( \text{Rep} \, G \), are solvable [Isaacs 1976].

**Remark 6.1.** If \( H \) is any semisimple Hopf algebra such that \( \text{c.d.}(H) = \{1, p\} \) and \( G \) is any finite group, then the tensor product Hopf algebra \( A = H \otimes k^G \) also satisfies that \( \text{c.d.}(A) = \{1, p\} \) (since the irreducible modules of \( A \) are tensor products of irreducible modules of \( H \) and \( k^G \)).

But \( A \) is not solvable unless \( G \) is solvable; indeed, \( k^G \) is a Hopf subalgebra as well as a quotient Hopf algebra of \( A \).

Our aim in this section is to prove some structural results on \( \mathcal{C} \), regarding solvability, under additional restrictions.

The following theorem generalizes Proposition 4.10.

**Theorem 6.2.** Let \( \mathcal{C} \) be a near-group fusion category such that \( \text{c.d.}(\mathcal{C}) = \{1, p\} \). Then \( \mathcal{C} \) is solvable.

**Proof.** In the notation of [Siehler 2003], let the fusion rules of \( \mathcal{C} \) be given by the pair \((G, \kappa)\), where \( G \) is the group of invertible objects of \( \mathcal{C} \) and \( \kappa \) is a nonnegative integer. Then \( \text{Irr}(\mathcal{C}) = G \cup \{m\} \), with the relation

\[
m^2 = \sum_{g \in G} g + \kappa m. \tag{6-1}
\]

The assumption on \( \text{c.d.}(\mathcal{C}) \) implies that \( \text{FPdim} \, m = p \). Hence \( \text{FPdim} \, \mathcal{C} = |G| + p^2 \), and since \( |G| = |G(\mathcal{C})| \) divides \( \text{FPdim} \, \mathcal{C} \), we get that \( |G| = p \) or \( p^2 \).

(Note that, taking Frobenius–Perron dimensions in (6-1), we get that \( G \neq 1 \).)

If \( |G| = p^2 \), then \( \kappa = 0 \) and \( \mathcal{C} \) is a Tambara–Yamagami category [Tambara and Yamagami 1998]. Furthermore, \( \mathcal{C} \) is a \( \mathbb{Z}_2 \)-extension of a pointed category \( \mathcal{C}(G, \omega) \). Then \( \mathcal{C} \) is solvable in this case, by [Etingof et al. 2011, Proposition 4.5(i)].

Suppose that \( |G| = p \). Then \( \kappa = p - 1 \). As in the proof of Proposition 4.10, using [Siehler 2003, Theorem 1.2], we get that \( \text{FPdim} \, \mathcal{C} = p(p + 1) \) equals 6 or \( p2^\alpha \). Then \( \mathcal{C} \) is solvable, by [Etingof et al. 2011, Theorem 1.6]. \( \square \)

Our next result is the following theorem, for \( \mathcal{C} = \text{Rep} \, H \), which is a consequence of Proposition 4.9. A stronger version of this result will be given in Section 7B, under additional dimension restrictions.

**Theorem 6.3.** Suppose \( H \) is of type \((1, p; p, n)\) as an algebra. Assume in addition that \( H \) is quasitriangular. Then \( H \) is solvable.
Proof. We have shown in Proposition 4.9 that $H^*$ is nilpotent. Moreover, by Lemma 4.7, $H$ fits into an abelian cocentral exact sequence

$$k \to k^F \to H \to k\mathbb{Z}_p \to k,$$

where $F$ is a nilpotent group. Therefore, $H$ is solvable, by Corollary 5.3. □

In the remainder of this section, we restrict ourselves to the case where $\mathcal{C} = \text{Rep} H$ for a semisimple Hopf algebra $H$.

6A. The case $p = 2$. Let $H$ be a semisimple Hopf algebra such that $\text{c.d.}(H) \subseteq \{1, 2\}$. By [Bichon and Natale 2011, Theorem 6.4], one of the following possibilities holds:

(i) there is a cocentral abelian exact sequence $k \to k^F \to H \to k\Gamma \to k$, where $F$ is a finite group and $\Gamma \simeq \mathbb{Z}_2^n$, $n \geq 1$, or

(ii) there is a central exact sequence $k \to k^U \to H \to B \to k$, where $B = H_{\text{ad}}$ is a proper Hopf algebra quotient, and $U = U(\text{Rep} H)$ is the universal grading group of the category of finite-dimensional $H$-modules.

In particular, if $H = H_{\text{ad}}$, then $H$ satisfies (i).

As a consequence of this result we have:

Theorem 6.4. Let $H$ be a semisimple Hopf algebra such that $\text{c.d.}(H) \subseteq \{1, 2\}$. Then $H$ is weakly group-theoretical.

Moreover, if $H = H_{\text{ad}}$, then $H$ is group-theoretical.

Proof. The assumption implies that $H$ satisfies (i) or (ii) above. If $H$ satisfies (i), then $H$ is group-theoretical, by Remark 2.4.

Otherwise, $H$ satisfies (ii), and then the category $\text{Rep} H$ is a $U$-extension of $\text{Rep} B$, in view of Proposition 3.1. By an inductive argument, we may assume that $B$ is weakly group-theoretical (note that $\text{c.d.}(B) \subseteq \{1, 2\}$). Therefore so is $H$, by [Etingof et al. 2011, Proposition 4.1]. □

We next discuss conditions that guarantee the solvability of $H$. The following result is proved in [Bichon and Natale 2011].

Proposition 6.5 [Bichon and Natale 2011, Proposition 6.8]. Suppose $H$ is of type $(1, 2; 2, n)$ as an algebra. Then $H$ is cocommutative.

The proposition implies that such a Hopf algebra $H$ is isomorphic to a group algebra $kG$ for some finite group $G$. By the assumption on the algebra structure of $H$, the group $G$, and then also $H$, are solvable.

The next lemma gives a sufficient condition for $H$ to be solvable.

Lemma 6.6. Suppose $\text{c.d.}(H) \subseteq \{1, 2\}$ and $H = H_{\text{ad}}$. Then $H$ is solvable if and only if the group $F$ in (i) is solvable.
Proof. Since \( H = H_{\text{ad}} \), then \( H \) satisfies (i). Therefore \( H \) is solvable if and only if the relevant factorizable group \( G = F \rtimes \Gamma \) is solvable, by Corollary 5.2. Also, since the sequence (i) is cocentral, then \( G \) is a semidirect product: \( G = F \rtimes \Gamma \). This proves the lemma.

Remark 6.7. Suppose that \( H \) has a faithful irreducible character \( \chi \) of degree 2, such that \( \chi^* = \chi^* \chi \). Then it follows from [Bichon and Natale 2011, Theorem 3.5] that \( H \) fits into a central abelian exact sequence \( k \to k^m \to H \to kT \to k \), for some polyhedral group \( T \) of even order and some \( m \geq 1 \). In particular, since \( c.d.(H) = \{1, 2\} \), then \( T \) is necessarily cyclic or dihedral (see, for instance, [Bichon and Natale 2011, p. 10] for a description of the polyhedral groups and their character degrees). Therefore \( H \) is solvable in this case.

The assumption on \( \chi \) is satisfied in the case where \( H \) is quasitriangular; hence the conclusion holds in this case. We shall show in the next subsection that every quasitriangular semisimple Hopf algebra with \( c.d.(H) \subseteq \{1, 2\} \) is also solvable.

We next prove some lemmas that will be useful in the next subsection.

Lemma 6.8. Suppose \( c.d.(H) \subseteq \{1, 2\} \) and let \( K \) be a Hopf subalgebra or quotient Hopf algebra of \( H \). Then \( c.d.(K) \subseteq \{1, 2\} \).

Proof. We only need to show the claim when \( K \subseteq H \) is a Hopf subalgebra. In this case, the statement follows from surjectivity of the restriction functor \( \text{Rep} H \to \text{Rep} K \).

The lemma has the following immediate consequence:

Corollary 6.9. If \( c.d.(H) \subseteq \{1, 2\} \), then the group \( G(H) \) is solvable.

Lemma 6.10. Suppose \( c.d.(H), c.d.(H^*) \subseteq \{1, 2\} \). Then \( H \) is solvable.

Proof. By induction on the dimension of \( H \).

Consider the universal grading group \( U \) of the category \( \text{Rep} H \). Then \( H^* \to kU \) is a quotient Hopf algebra and therefore \( c.d.(U) \subseteq \{1, 2\} \), by Lemma 6.8. This implies that the group \( U \) is solvable.

Suppose first \( H_{\text{ad}} \neq H \). In view of Lemma 6.8, we also have \( c.d.(H_{\text{ad}}), c.d.(H^*_{\text{ad}}) \subseteq \{1, 2\} \). By the inductive assumption \( H_{\text{ad}} \) is solvable. By [Etingof et al. 2011, Proposition 4.5(i)], \( H \) is solvable, since \( \text{Rep} H \) is a \( U \)-extension of \( \text{Rep} H_{\text{ad}} \).

It remains to consider the case where \( H_{\text{ad}} = H \). As pointed out at the beginning of this subsection, it follows from [Bichon and Natale 2011, Theorem 6.4] that in this case \( H \) satisfies condition (i), that is, \( H \) fits into a cocentral abelian exact sequence \( k \to k^\Gamma \to H \to k\Gamma \to k \), with \( |\Gamma| > 1 \) and \( \Gamma \) abelian.

In particular, \( k^\Gamma \subseteq H^* \) is a nontrivial central Hopf subalgebra, implying that \( H^* \neq H^*_\text{ad} \). The inductive assumption implies, as before, that \( H^*_\text{ad} \) and thus also \( H^* \) is solvable. Then \( H \) is too. \( \square \)
6B. The quasitriangular case. We shall assume in this subsection that $H$ is quasitriangular. Let $R \in H \otimes H$ be an $R$-matrix. We keep the notation of Section 2F.

Remark 6.11. Since the category $\text{Rep} H$ is braided, then the universal grading group $U = U(\text{Rep} H)$ is abelian (and, in particular, solvable).

The following is the main result of this subsection.

Theorem 6.12. Let $H$ be a quasitriangular semisimple Hopf algebra such that $\text{c.d.}(H) \subseteq \{1, 2\}$. Then $H$ is solvable.

Proof. If $\text{c.d.}(H) = \{1\}$, then $H$ is commutative and, because it is quasitriangular, isomorphic to the group algebra of an abelian group. Hence we may assume that $\text{c.d.}(H) = \{1, 2\}$.

Consider the Hopf subalgebras $H_+, H_- \subseteq H$. By Lemma 6.8, we have $\text{c.d.}(H_+), \text{c.d.}(H_-) \subseteq \{1, 2\}$. Then $\text{c.d.}(H_-), \text{c.d.}(H_+) \subseteq \{1, 2\}$, since $(H^*)^{\text{cop}} \simeq H_+$.

By Lemma 6.10, $H_-$ is solvable. Therefore the Drinfeld double $D(H_-)$ and its homomorphic image $H_R$ are also solvable.

We may thus assume that $H_R \subsetneq H$.

Observe that, being a quotient of $H$, $H_{\text{ad}}$ is also quasitriangular and satisfies $\text{c.d.}(H_{\text{ad}}) \subseteq \{1, 2\}$. Hence, by induction, we may also assume that $H = H_{\text{ad}}$, and, in particular, $G(H) \cap Z(H) = 1$. Indeed, $\text{Rep} H$ is a $U$-extension of $\text{Rep} H_{\text{ad}}$ and the group $U$ is abelian, as pointed out before.

Therefore $H$ fits into a cocentral abelian exact sequence $k \to k^F \to H \to k\Gamma \to k$, where $1 \neq \Gamma$ is elementary abelian of exponent 2.

In view of Lemma 6.6, it will be enough to show that the group $F$ is solvable. We have $\hat{\Gamma} \subseteq G(H^*) \cap Z(H^*)$. By [Radford 1992, Proposition 3],

$$f_{R_{21}}(G(H^*) \cap Z(H^*)) \subseteq G(H) \cap Z(H).$$

Hence we may assume that $f_{R_{21}}|\hat{\Gamma} = 1$ and similarly $f_R|\hat{\Gamma} = 1$. Thus $f_R$ and $f_{R_{21}}$ factorize through the quotient $H^*/H^*(k\hat{\Gamma})^+ \simeq kF$.

Therefore $H_+ = f_R(H^*)$ and $H_- = f_{R_{21}}(H^*)$ are cocommutative. (Then they are also commutative, since $H_+ \simeq H^*_{\text{cop}}$.) In particular, $H_R = H_+H_-$ is cocommutative. Hence $\Phi_R(H^*) \subseteq H_R \subseteq kG(H)$.

By [Natale 2006, Theorem 4.11], $K = \Phi_R(H^*)$ is a commutative (and cocommutative) normal Hopf subalgebra, which is necessarily solvable, since $H_R$ is. In addition, $\Phi_R(H^*) \simeq kT$, where $T \subseteq G(H)$ is an abelian subgroup [Natale 2006, Example 2.1], and there is an exact sequence of Hopf algebras

$$k \to kT \to H \xrightarrow{\pi} \overline{H} \to k,$$

where $\overline{H}$ is a certain (canonical) triangular Hopf algebra.

Since $\overline{H}$ is triangular, $\overline{H} \simeq (kL)^J$ is a twisting of the group algebra of some
finite group \( L \). Because \( \text{c.d.}(L) = \text{c.d.}(\overline{H}) \leq \{1, 2\} \), \( L \) must be solvable. Hence \( \overline{H} \) is solvable, since \( \text{Rep} \overline{H} \simeq \text{Rep} L \).

The map \( \pi : H \to \overline{H} \) induces, by restriction to the Hopf subalgebra \( k^F \subseteq H \), an exact sequence

\[
\begin{align*}
k &\to kT \cap k^F \to k^F \xrightarrow{\pi|_{k^F}} \pi(k^F) \to k.
\end{align*}
\]

We have \( kT \cap k^F = k^F \) and \( \pi(k^F) = k^S \), where \( F \) and \( S \) are a quotient and a subgroup of \( F \), respectively, in such a way that the exact sequence above corresponds to an exact sequence of groups

\[
1 \to S \to F \to F/\overline{F} \to 1.
\]

Now, \( F \) is abelian, because \( k^F = kT \cap k^F \) is cocommutative, and \( S \) is solvable, because \( k^S \) is a Hopf subalgebra of \( \overline{H} \). Therefore \( F \) is solvable. This implies that \( H \) is solvable and finishes the proof of the theorem.

\[\square\]

7. Odd-dimensional fusion categories

In this section, \( p \) will be a prime number. Let \( \mathcal{C} \) be a fusion category over \( k \). Recall that the set of irreducible degrees of \( \mathcal{C} \) was defined as

\[
\text{c.d.}(\mathcal{C}) = \{\text{FPdim } x \mid x \in \text{Irr } \mathcal{C}\}.
\]

The fusion categories that we shall consider in this section are all integral, that is, the Frobenius–Perron dimensions of objects of \( \mathcal{C} \) are (natural) integers. By [Etingof et al. 2005, Theorem 8.33], \( \mathcal{C} \) is isomorphic to the category of representations of some finite-dimensional semisimple quasi-Hopf algebra.

7A. Odd-dimensional weakly group-theoretical fusion categories. The following result is a consequence of the Feit–Thompson theorem [1963].

**Proposition 7.1.** Let \( \mathcal{C} \) be a weakly group-theoretical fusion category and assume that \( \text{FPdim } \mathcal{C} \) is an odd integer. Then \( \mathcal{C} \) is solvable.

Note that since \( \text{FPdim } \mathcal{C} \) is an odd integer, the fusion category \( \mathcal{C} \) is integral. See [Drinfeld et al. 2010, Corollary 2.22].

**Proof.** By definition, \( \mathcal{C} \) is Morita equivalent to a nilpotent fusion category. Then, by [Etingof et al. 2011, Proposition 4.5(i)], it will be enough to show that a nilpotent fusion category of odd Frobenius–Perron dimension is solvable. So, assume that \( \mathcal{C} \) is nilpotent, so that \( \mathcal{C} \) is a \( G \)-extension of a fusion subcategory \( \mathcal{C} \), with \( |G| > 1 \). In particular, \( \text{FPdim } \mathcal{C} = |G| \text{FPdim } \mathcal{C} \). Hence \( \text{FPdim } \mathcal{C} \) and the order of \( G \) are both odd, and \( \text{FPdim } \mathcal{C} < \text{FPdim } \mathcal{C} \). The proposition follows by induction, since \( G \) is solvable by the Feit–Thompson theorem; see [Etingof et al. 2011, Proposition 4.5(i)].

\[\square\]
7B. Braided fusion categories. We shall need the following lemma whose proof is contained in the proof of [Etingof et al. 2011, Proposition 6.2(i)]. We include a sketch of the argument for the sake of completeness.

**Lemma 7.2.** Let \( \mathcal{C} \) be a fusion category and let \( G \) be a finite group acting on \( \mathcal{C} \) by tensor autoequivalences. Assume \( c.d.(\mathcal{C}^G) \subseteq \{ p^m : m \geq 0 \}, \) where \( p \) is a prime number. Then \( c.d.(\mathcal{C}) \subseteq \{ p^m : m \geq 0 \}. \)

**Proof.** Regard \( \mathcal{C} \) as an indecomposable module category over itself via tensor product, and similarly for \( \mathcal{C}^G \). Let \( Y \) be a simple object of \( \mathcal{C} \). Since the forgetful functor \( \mathcal{C}^G \rightarrow \mathcal{C} \) is surjective, \( Y \) is a simple constituent of \( \mathcal{F}(X) \), for some simple object \( X \) of \( \mathcal{C}^G \).

Since \( F \) is a tensor functor, we have \( \text{FPdim } X = \text{FPdim } \mathcal{F}(X) \). By formula (7) in [Etingof et al. 2011, Proof of Proposition 6.2],

\[
\text{FPdim}(X) = \deg(\pi)[G : G_Y]\text{FPdim } Y,
\]

where \( G_Y \subseteq G \) is the stabilizer of \( Y \) and \( \pi \) is an irreducible representation of \( G_Y \) associated to \( X \). Therefore \( \text{FPdim } Y \) divides \( \text{FPdim } X \).

The assumption on \( \mathcal{C}^G \) implies that \( \text{FPdim } X \) is a power of \( p \). Then so is \( \text{FPdim } Y \). This proves the lemma. \( \square \)

**Theorem 7.3.** Let \( \mathcal{C} \) be a braided fusion category such that \( c.d.(\mathcal{C}) \subseteq \{ p^m : m \geq 0 \}, \) where \( p \) is a prime number. Assume that \( \text{FPdim } \mathcal{C} \) is odd. Then \( \mathcal{C} \) is solvable.

**Proof.** By induction on \( \text{FPdim } \mathcal{C} \). (The Frobenius–Perron dimension of a fusion subcategory of \( \mathcal{C} \) divides the dimension of \( \mathcal{C} \) [Etingof et al. 2005, Proposition 8.15], and the same is true for the Frobenius–Perron dimension of a fusion category \( \mathcal{D} \) such that there exists a surjective tensor functor \( \mathcal{C} \rightarrow \mathcal{D} \) [Etingof et al. 2005, Corollary 8.11]. Thus these fusion categories are odd-dimensional as well.) If \( c.d.(\mathcal{C}) = \{ 1 \} \), then \( \mathcal{C} \) is pointed. Then \( \mathcal{C} \simeq \mathcal{C}(G, \omega) \) for some abelian group \( G \) and some 3-cocycle \( \omega \) on \( G \). Then \( \mathcal{C} \) is solvable, by [Etingof et al. 2011, Proposition 4.5(ii)].

Suppose next that \( \mathcal{C} \) is not pointed. Then all noninvertible objects in \( \mathcal{C} \) have Frobenius–Perron dimension \( p^m \), for some \( m \geq 1 \). Consider the group \( G(\mathcal{C}) \) of invertible objects of \( \mathcal{C} \). Then \( G(\mathcal{C}) \) is abelian and \( G(\mathcal{C}) \neq 1 \), as follows by taking Frobenius–Perron dimensions in a decomposition of the tensor product \( X \otimes X^* \), for some simple noninvertible object \( X \).

Let us regard \( \mathcal{C} \) as a premodular fusion category with respect to its canonical spherical structure (as \( \text{FPdim } \mathcal{C} \) is an integer). Then \( \mathcal{C} \) is modularizable, in view of [Bruguières and Natale 2011, Lemma 7.2].

Let \( \widetilde{\mathcal{C}} \) be its modularization, which is a modular category over \( k \). Then \( \mathcal{C} \) is an equivariantization \( \mathcal{C} \simeq \mathcal{C}^G \) with respect to the action of a certain group \( G \) on \( \mathcal{C} \) [Bruguières 2000]. (Indeed, the modularization functor \( \mathcal{C} \rightarrow \widetilde{\mathcal{C}} \) gives rise to
an exact sequence of fusion categories \( \text{Rep} \ G \to \mathcal{C} \to \tilde{\mathcal{C}} \), which comes from an equivariantization; see [Bruguières and Natale 2011, Example 5.33].

By construction of \( G \), the category \( \text{Rep} \ G \) is the (tannakian) fusion subcategory of transparent objects in \( \mathcal{C} \). Therefore there is an embedding of braided fusion categories \( \text{Rep} \ G \subseteq \mathcal{C} \). In particular, the order of \( G \) is odd, implying that \( G \) is solvable.

By Lemma 7.2, \( c.d.(\tilde{\mathcal{C}}) \subseteq \{ p^m : m \geq 0 \} \). Then, by induction, and since an equivariantization of a solvable fusion category under the action of a solvable group is again solvable, we may and shall assume in what follows that \( \mathcal{C} = \tilde{\mathcal{C}} \) is modular.

It is shown in [Gelaki and Nikshych 2008, Theorem 6.2] that the universal grading group \( U(\mathcal{C}) \) is (abelian and) isomorphic to the group \( \hat{G}(\mathcal{C}) \) of characters of \( G(\mathcal{C}) \). In particular, \( U(\mathcal{C}) \neq 1 \). On the other hand, \( \mathcal{C} \) is a \( U(\mathcal{C}) \)-extension of its fusion subcategory \( \mathcal{C}_{\text{ad}} \). Since also \( c.d.(\mathcal{C}_{\text{ad}}) \subseteq \{ p^m : m \geq 0 \} \), then \( \mathcal{C}_{\text{ad}} \) is solvable, by induction. Therefore \( \mathcal{C} \) is solvable, as claimed. □

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