Group actions of prime order on local normal rings

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Let $B$ be a Noetherian normal local ring and $G \subset \text{Aut}(B)$ be a cyclic group of local automorphisms of prime order. Let $A$ be the subring of $G$-invariants of $B$ and assume that $A$ is Noetherian. We prove that $B$ is a monogenous $A$-algebra if and only if the augmentation ideal of $B$ is principal. If in particular $B$ is regular, we prove that $A$ is regular if the augmentation ideal of $B$ is principal.

An important class of singularities is built by the famous Hirzebruch–Jung singularities. They arise by dividing out a finite cyclic group action on a smooth surface. Their resolution is well understood and has nice arithmetic properties related to continued fractions; see [Hirzebruch 1953; Jung 1908].

One can also look at such group actions from a purely algebraic point of view. So let $B$ be a regular local ring and $G$ a finite cyclic group of order $n$ acting faithfully on $B$ by local automorphisms. In the tame case, that is, the order of $G$ is prime to the characteristic of the residue field $k$ of $B$, there is a central result of J. P. Serre [1968] saying that the action is given by multiplying a suitable system of parameters $(y_1, \ldots, y_d)$ by roots of unity $y_i \mapsto \zeta^{n_i} \cdot y_i$ for $i = 1, \ldots, d$, where $\zeta$ is a primitive $n$-th root of unity. Moreover, the ring of invariants $A := B^G$ is regular if and only if $n_i \equiv 0 \mod n$ for $d - 1$ of the parameters. The latter is equivalent to the fact that $\text{rk}((\sigma - \text{id})|T) \leq 1$ for the action of $\sigma \in G$ on the tangent space $T := m_B/m_B^2$. For more details see [Bourbaki 1981, Chapter 5, ex. 7].

Only very little is known in the case of a wild group action, that is, when $\gcd(n, \text{char } k) > 1$. In this paper we will restrict ourselves to the case of $p$-cyclic group actions, that is, where $n = p$ is a prime number. We will present a sufficient condition for the ring of invariants $A$ to be regular. Our result is also valid in the tame case, that is, where $n$ is a prime different from char $k$. As the method of Serre depends on an intrinsic formula for writing down the action explicitly, we provide also an explicit formula for presenting $B$ as a free $A$-module if our condition is fulfilled.

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The interest in our problem arises from investigating the relationship between
the regular and the stable $R$-model of a smooth projective curve $X_K$ over the field
of fractions $K$ of a discrete valuation ring $R$. In general, the curve $X_K$ admits a
stable model $X'$ over a finite Galois extension $R \hookrightarrow R'$. Then the Galois group
$G = G(R'/R)$ acts on $X'$. Our result provides a means to construct a regular model
over $R$ by starting from the stable model $X'$. As a special case, we discuss in
Section 4 the situation where $X_K$ has good reduction after a Galois $p$-extension
$R \hookrightarrow R'$. In this case there is a criterion for when the quotient of the smooth model
is regular. We intend to work out more general situations in a further article.

1. The main result

In this paper we will study only local actions of a cyclic group $G$ of prime order $p$
on a normal local ring $B$. We fix a generator $\sigma$ of $G$ and obtain the augmentation map
$I := I_\sigma := \sigma - \text{id} : B \to B, \quad b \mapsto \sigma(b) - b.$
We introduce the $B$-ideal
$I_G := (I(b); \ b \in B) \subset B$
which is generated by the image $I(B)$. This ideal is called augmentation ideal. If
this ideal is generated by an element $I(y)$, we call $y$ an augmentation generator.
Note that this ideal does not depend on the chosen generator $\sigma$ of $G$. Moreover, if
$y$ is an augmentation generator with respect to a generator $\sigma$ of $G$, then $y$ is also
an augmentation generator for any other generator of $G$. Since $B$ is local, the ideal
$I_G$ is generated by an augmentation generator if $I_G$ is principal. Namely, $I_G/m_B I_G$
is a vector space over the residue field $k_B = B/m_B$ of $B$ of dimension 1. So it is
generated by the residue class of $I(y)$ for some $y \in B$, and hence, by Nakayama’s
lemma, $I_G$ is generated by $I(y)$.

Definition 1. An action of a group $G$ on a regular local ring $B$ by local automor-
phisms is called a pseudoreflection if there exists a system of parameters $(y_1, \ldots, y_d)$
of $B$ such that $y_2, \ldots, y_d$ are invariant under $G$.

Theorem 2. Let $B$ be a normal local ring with residue field $k_B := B/m_B$. Let $p$ be
a prime number and $G$ a $p$-cyclic group of local automorphisms of $B$. Let $I_G$ be the
augmentation ideal. Let $A$ be the ring of $G$-invariants of $B$. Consider the following
conditions:
(a) $I_G := B \cdot I(B)$ is principal.
(b) $B$ is a monogenous $A$-algebra.
(c) $B$ is a free $A$-module.
Then the following implications are true:

\[(a) \iff (b) \implies (c).\]

Assume, in addition, that \(B\) is regular. Consider the following conditions:

(d) \(A\) is regular.

(e) \(G\) acts as a pseudoreflection.

Then the condition (c) is equivalent to (d). Moreover if, in addition, the canonical map \(k_A \to k_B\) is an isomorphism, then condition (a) is equivalent to condition (e).

We start the proof of the theorem with several preparations.

**Remark 3.** For \(b_1, b_2, b \in B\), the following relations are true:

(i) \(I(b_1 \cdot b_2) = I(b_1) \cdot \sigma(b_2) + b_1 \cdot I(b_2)\).

(ii) \(I(b^n) = \left(\sum_{i=1}^{n} \sigma(b)^{i-1}b^{n-i}\right) \cdot I(b)\).

(iii) \(I\left(\frac{b_1}{b_2}\right) = I(b_1)b_2 - b_1I(b_2)\sigma(b_2)\) if \(b_2 \neq 0\).

**Proof.** (i) follows by a direct calculation and (ii) by induction from (i).

As for (iii), the formula (i) holds for elements in the field of fractions as well. Therefore,

\[I(b_1) = I\left(\frac{b_1}{b_2}\right) = I\left(\frac{b_1}{b_2}\right)\sigma(b_2) + \frac{b_1}{b_2}I(b_2),\]

and the formula follows. \(\square\)

To prove that (a) implies (b) we need a technical lemma.

**Lemma 4.** Let \(y \in B\) be an augmentation generator. Then set, inductively,

\[y_i^{(0)} := y^i \quad \text{for } i = 0, \ldots, p - 1,\]

\[y_i^{(1)} := I(y_i^{(0)})/I(y_1^{(0)}) \quad \text{for } i = 1, \ldots, p - 1,\]

\[y_i^{(n+1)} := I(y_i^{(n)})/I(y_{n+1}^{(n)}) \quad \text{for } i = n + 1, \ldots, p - 1.\]

Then

\[y_i^{(n)} = \sum_{0 \leq k_1 \leq \cdots \leq k_{i-n} \leq n} \prod_{j=1}^{i-n} \sigma^{k_j}(y) \quad \text{for } i = n, \ldots, p - 1,\]

and in particular,

\[y_n^{(n)} = 1, \quad y_{n+1}^{(n)} = \sum_{j=1}^{n+1} \sigma^{j-1}(y), \quad I(y_{n+1}^{(n)}) = \sigma^{n+1}(y) - y.\]

Furthermore, \(y_{n+1}^{(n)}\) is again an augmentation generator for \(n = 0, \ldots, p - 2\).
For the left hand side one computes

\[ y_1^{(1)} = \frac{I(y_1^{(0)})}{I(y_1^{(0)})} = \frac{I(y^i)}{I(y)} = \sum_{j=1}^{i} \sigma(y)^{j-1}y^{i-j} = \sum_{0 \leq k_1 \leq \cdots \leq k_i \leq n} \prod_{v=1}^{j-1} \sigma^{k_i}(y), \]

since the last sum can be viewed as a sum over an index \( j \) where \( i - j \) is the number of \( k_v \) equal to 0. In particular, the formulas are correct for \( y_1^{(1)} \) and \( y_2^{(1)} \). Moreover

\[ I(y_2^{(1)}) = I(\sigma(y) + y) = \sigma^2(y) - y. \]

Since \( \sigma^2 \) is generator of \( G \) for \( 2 < p \), the element \( y_2^{(1)} \) is an augmentation generator as well.

Now assume that the formulas are correct for \( n \). Since \( y_{n+1}^{(1)} \) is an augmentation generator, \( I(y_{n+1}^{(1)}) \) divides \( I(y_i^{(n)}) \) for \( i = n + 1, \ldots, p - 1 \). Then it remains to show, upon substituting the expressions from the lemma for \( y_i^{(n)} \) and \( y_i^{(n+1)} \), that

\[ I(y_i^{(n)}) = (\sigma^{n+1}(y) - y) \cdot y_i^{(n+1)} \quad \text{for} \quad i = n + 1, \ldots, p - 1. \]

For the left hand side one computes

\[
\text{LHS} = I \left( \sum_{0 \leq k_1 \leq \cdots \leq k_{i-n} \leq n} \prod_{j=1}^{i-n} \sigma^{k_j}(y) \right) = \sum_{0 \leq k_1 \leq \cdots \leq k_{i-n} \leq n} I \left( \prod_{j=1}^{i-n} \sigma^{k_j}(y) \right)
= \sum_{0 \leq k_1 \leq \cdots \leq k_{i-n} \leq n} \left( \prod_{j=1}^{i-n} \sigma^{k_j+1}(y) - \prod_{j=1}^{i-n} \sigma^{k_j}(y) \right)
= \sum_{1 \leq k_1 \leq \cdots \leq k_{i-n} \leq n+1} \prod_{j=1}^{i-n} \sigma^{k_j}(y) - \sum_{0 \leq k_1 \leq \cdots \leq k_{i-n} \leq n} \prod_{j=1}^{i-n} \sigma^{k_j}(y).
\]

Now all terms occurring in both sums cancel. These are the terms with \( k_{i-n} \leq n \) in the first sum and \( 1 \leq k_1 \) in the second sum.

For the right hand side one computes

\[
\text{RHS} = (\sigma^{n+1}(y) - y) \cdot \sum_{0 \leq k_1 \leq \cdots \leq k_{i-n+1} \leq n+1} \prod_{j=1}^{i-n-1} \sigma^{k_j}(y)
= \sum_{0 \leq k_1 \leq \cdots \leq k_{i-n} \leq n+1} \prod_{j=1}^{i-n} \sigma^{k_j}(y) - \sum_{0 \leq k_1 \leq \cdots \leq k_{i-n} \leq n+1} \prod_{j=1}^{i-n} \sigma^{k_j}(y).
\]
Both sides are seen to be equal. In particular we have
\[
y_{n+1}^{(n+1)} = 1,
\]
\[
y_{n+2}^{(n+1)} = \sum_{0 \leq k_1 \leq n+1} \prod_{j=1}^{n+1} \sigma^k(y) = \sum_{j=1}^{n+2} \sigma^j(y),
\]
\[
I(y_{n+2}^{(n+1)}) = \sigma^{n+2}(y) - y.
\]

So \(y_{n+2}^{(n+1)}\) is an augmentation generator for \(n + 2 < p\), since \(\sigma^{n+2}\) generates \(G\). This concludes the technical part.

\[\square\]

**Proposition 5.** Assume that the augmentation ideal \(I_G\) is principal and let \(y \in B\) be an augmentation generator. Then \(B\) decomposes into the direct sum

\[
B = A \cdot y^0 \oplus A \cdot y^1 \oplus \cdots \oplus A \cdot y^{p-1}.
\]

**Proof.** Since \(I(y) \neq 0\), the element \(y\) generates the field of fractions \(Q(B)\) over \(Q(A)\). Therefore

\[
Q(B) = Q(A) \cdot y^0 \oplus Q(A) \cdot y^1 \oplus \cdots \oplus Q(A) \cdot y^{p-1}.
\]

Then it suffices to show the following claim:

Let \(a, a_0, \ldots, a_{p-1} \in A\). Assume that \(a\) divides

\[
b = a_0 \cdot y^0 + a_1 \cdot y^1 + \cdots + a_{p-1} \cdot y^{p-1}.
\]

Then \(a\) divides \(a_0, a_1, \ldots, a_{p-1}\).

If \(b = a \cdot \beta\), then \(I(b) = a \cdot I(\beta)\). Since \(I(\beta) = \beta_1 \cdot I(y)\), we get \(I(b) = a \beta_1 \cdot I(y)\).

So we see that \(a\) divides \(I(b)/I(y) \in B\). Using the notation of Lemma 4, set

\[
b^{(0)} := b = a_0 \cdot y^0 + a_1 \cdot y^1 + \cdots + a_{p-1} \cdot y^{p-1}
\]

\[
b^{(1)} := \frac{I(b^{(0)})}{I(y)} = a_1 + a_2 \frac{I(y^2)}{I(y)} + \cdots + a_{p-1} \frac{I(y^{p-1})}{I(y)} = a_1 \cdot y^{(1)} + a_2 \cdot y^{(2)} + \cdots + a_{p-1} \cdot y^{(p-1)}
\]

\[
b^{(n)} := \frac{I(b^{(n-1)})}{I(y^{(n-1)})} = a_n \cdot y^{(n)} + a_{n+1} \cdot y^{(n+1)} + \cdots + a_{p-1} \cdot y^{(p-1)}.
\]

Due to the observation above, by induction \(a\) divides \(b^{(0)}, b^{(1)}, \ldots, b^{(p-1)}\), since \(y_{n+1}^{(n)}\) is an augmentation generator for \(n = 1, \ldots, p - 2\). So we obtain

\[
a \mid b^{(p-1)} = a_{p-1} \cdot y^{(p-1)}_{p-1} = a_{p-1}.
\]
Now proceeding downwards, one obtains
\[ a \mid b^{(p-2)} = a_{p-2} + a_{p-1} \cdot y^{(p-2)}_{p-1}, \quad \text{hence } a \mid a_{p-2}, \]
\[ a \mid b^{(n)} = a_n + a_{n+1} \cdot y^{(n)}_{n+1} + \cdots + a_{p-1} \cdot y^{(n)}_{p-1}, \quad \text{hence } a \mid a_n \]
for \( n = p - 1, p - 2, \ldots, 0. \) \( \square \)

**Proof of the first part of Theorem 2.** (a) \( \implies \) (b): This follows from Proposition 5.

(b) \( \implies \) (a): If \( B = A[y] \) is monogenous, then \( I_G = B \cdot I(y) \) is principal.

(b) \( \implies \) (c) is clear. Namely, if \( B = A[y] \), the minimal polynomial of \( y \) over the field of fraction is of degree \( p \) and the coefficients of this polynomial belong to \( A \). Then \( B \) has \( y^0, y^1, \ldots, y^{p-1} \) as an \( A \)-basis. \( \square \)

Next we do some preparations for proving the second part of the theorem where \( B \) is assumed to be regular.

**Proposition 6.** Keep the assumption of the second part of Theorem 2, namely that \( B \) is regular and that the canonical morphism \( k_A \xrightarrow{\sim} k_B \) is an isomorphism. Let \( (y_1, \ldots, y_d) \) be a generating system of the maximal ideal \( m_B \). Then the following assertions are true:

(i) \( I_G = B \cdot I(y_1) + \cdots + B \cdot I(y_d) \).

(ii) If the ideal \( I_G = B \cdot I(B) \) is principal, then there exists an index \( i \in \{1, \ldots, d\} \) with \( I_G = B \cdot I(y_i) \).

**Proof.** (i) Recall that \( A = B^G \) denotes the ring of invariants. Due to the assumption, we have \( B = A + m_B \), and hence, \( I(B) = I(m_B) \). Furthermore, we have
\[ m_B = m_B^2 + \sum_{i=1}^d A \cdot y_i. \]
Since \( I \) is \( A \)-linear, we get
\[ I(m_B) = I(m_B^2) + \sum_{i=1}^d A \cdot I(y_i). \]
Due to Remark 3, one knows \( I(m_B^2) \subseteq m_B \cdot I(m_B) \). So, one obtains
\[ I(m_B) \subseteq m_B \cdot I(m_B) + \sum_{i=1}^d B \cdot I(y_i). \]
Since \( B \) is local, Nakayama’s lemma yields
\[ I_G = B \cdot I(B) = B \cdot I(m_B) = \sum_{i=1}^d B \cdot I(y_i). \]
(ii) Since $I_G$ is principal, $I_G/m_B I_G$ is generated by one of the $I(y_i)$, and hence, again by Nakayama’s lemma, $I_G = B \cdot I(y_i)$ for a suitable $i \in \{1, \ldots, d\}$. □

Proof of the second part of Theorem 2. (c) $\implies$ (d) follows from [Matsumura 1980, Theorem 51]. Namely, $B$ is noetherian due to the definition of a regular ring. Since $A \to B$ is faithfully flat, $A$ is noetherian. Then one can apply [loc. cit.].

(d) $\implies$ (c) follows from [Serre 1965, IV, Prop. 22].

(a) $\implies$ (e): We assume that the canonical map $k_A \to k_B$ of the residue fields is an isomorphism. If $I_G$ is principal, one can choose an augmentation generator $y \in m_B$ that is part of a system of parameters $(y, y_2, \ldots, y_d)$ due to Proposition 6. Due to Proposition 5, we know that $B$ decomposes into the direct sum

$$B = A \cdot y^0 \oplus A \cdot y^1 \oplus \cdots \oplus A \cdot y^{p-1}.$$ 

Now we can represent

$$y_j = \sum_{i=0}^{p-1} a_{i,j} \cdot y^i \quad \text{for} \quad j = 2, \ldots, d.$$ 

Then, set

$$\tilde{y}_j := y_j - \sum_{i=1}^{p-1} a_{i,j} y^i = a_{0,j} \in A \cap m_B = m_A \quad \text{for} \quad j = 2, \ldots, d.$$ 

So $(y, \tilde{y}_2, \ldots, \tilde{y}_d)$ is a system of parameters of $B$ as well. Thus $G$ acts by a pseudoreflection.

(e) $\implies$ (a): If $G$ is a pseudoreflection, $I_G$ is generated by $I(y)$ due to Proposition 6, where $y, x_2, \ldots, x_p$ is a system of parameters with $x_i \in m_A$ for $i = 2, \ldots, p$ if $k_A = k_B$. □

2. An example

If $k_A \to k_B$ is not an isomorphism, the implication (e) $\implies$ (a) is false:

Example 7. Let $k$ be a field of positive characteristic $p$ and look at the polynomial ring $R := k[Z, Y, X_1, X_2]$ over $k$. We define a $p$-cyclic action of $G = \langle \sigma \rangle$ on $R$ by

$$\sigma|k := \text{id}_k, \quad \sigma(Z) = Z + X_1, \quad \sigma(Y) = Y + X_2, \quad \sigma(X_i) = X_i \quad \text{for} \quad i = 1, 2.$$ 

This is a well-defined action of order $p$, since $p \cdot X_i = 0$ for $i = 1, 2$, and it leaves the ideal $I := (Y, X_1, X_2)$ invariant. Furthermore, for any $g \in k[Z] - \{0\}$ the image is given by $\sigma(g) = g + I(g)$ with $I(g) \in X_1 \cdot k[Z, X_1]$.

Then consider the polynomial ring $S := k(Z)[Y, X_1, X_2]$ over the field of fractions $k(Z)$ of the polynomial ring $k[Z]$. Then $S$ has the maximal ideal $m = (Y, X_1, X_2)$.
Then set \( B := S_m = k(Z[Y, X_1, X_2])_{(Y, X_1, X_2)}. \) We can regard all these rings as subrings of the field of fractions of \( R: \)

\[ R \subset S \subset B \subset k(Z, Y, X_1, X_2). \]

Clearly, \( \sigma \) acts on \( R \), and hence it induces an action on its field of fractions; denote this action by \( \sigma \) as well. Then we claim that the restriction of \( \sigma \) to \( B \) induces an action on \( B \) by local automorphisms. For this, it suffices to show that for any \( g \in R - \mathcal{I} \) the image \( \sigma(g) \) does not belong to \( \mathcal{I} \). The latter is true, since \( \sigma(g) = g + I(g) \) with \( I(g) \in \mathcal{I} \). The augmentation ideal \( I_G = B \cdot X_1 + B \cdot X_2 \) is not principal although \( G \) acts through a pseudoreflection.

### 3. A conjecture

**Remark 8.** In the tame case \( p \neq \text{char}(k_B) \), the converse \((d) \Rightarrow (a)\) is also true due to the theorem of Serre, as explained in the introduction.

In the case of a wild group action, that is, \( p = \text{char}(k_B) \), it is not known whether the converse is true, but we conjecture it.

**Conjecture 9.** Let \( B \) be a regular local ring and let \( G \) be a \( p \)-cyclic group acting on \( B \) by local automorphisms. Then the following conditions are *conjectured* to be equivalent:

1. \( I_G \) is principal.
2. \( A := B^G \) is regular.

The implication \((1) \Rightarrow (2)\) was shown in Theorem 2. Of course the converse is true if \( \text{dim } A \leq 1 \). In higher dimension, the converse \((2) \Rightarrow (1)\) is uncertain, but it holds for small primes \( p \leq 3 \) as we explain now. Since \( A \) is regular, the ring \( B \) is a free \( A \)-module of rank \( p \); see [Serre 1965, IV, Proposition 22]. So,

\[ B/Bm_A^n \text{ is a free } A/m_A^n \text{-module of rank } p \text{ for any } n \in \mathbb{N}. \]  

\[(*)\]

In the case \( p = 2 \), the rank of \( m_B/Bm_A \) is 0 or 1. In the first case, \( k_B \) is an extension of degree \([k_B : k_A] = 2\) over \( k_A \) and \( m_B = Bm_A \). So there exists an element \( \beta \in B \) such that \( B/Bm_A \) is generated by the residue classes of 1 and \( \beta \). Due to Nakayama’s lemma, \( B = A[\beta] \) is monogenous, and hence, \( I_G \) is principal.

In the second case, where \( k_A \rightarrow k_B \) is an isomorphism, there exists an element \( \beta \in m_B \) such that \( m_B = B\beta + Bm_A \). Then \( G \) acts as a pseudoreflection, and hence, \( I_G \) is principal.

In the case \( p = 3 \) we claim that \( Bm_A \not\subset m_B^2 \).

If we assume the contrary \( Bm_A \subset m_B^2 \), then these ideals coincide; \( Bm_A = m_B^2 \). Namely, the rank of \( B/Bm_A \) as \( A/m_A \)-module is 3 and the rank of \( B/m_B^2 \) is at least 3 due to \( d := \text{dim } B \geq 2 \), so \( Bm_A = m_B^3 \). Therefore the length of \( B/Bm_A^n = B/m_B^n \). 

\[ (*) \]
is 3 times the length of $A/m_A^2$, which is $3 \cdot (\dim A + 1)$. On the other hand the rank of $B/m_B^4$ is equal to

$$(1 + \dim m_B/m_B^2) + \dim m_B^2/m_B^3 + \dim m_B^3/m_B^4 = \sum_{n=0}^{3} \binom{d+n-1}{d-1},$$

which is larger than $(1 + \dim m_A/m_A^2) + (1 + \dim m_A/m_A^2) + (1 + \dim m_A/m_A^2)$, since for $d \geq 2$ both

$$\binom{d+1}{d-1} = \frac{(d+1)d}{2} \geq 1 + d = 1 + \dim m_A/m_A^2$$

and

$$\binom{d+3-1}{d-1} = \frac{(d+2)(d+1)d}{2 \cdot 3} > 1 + d$$

hold. Here we used the formula for the number $\lambda_{n,d}$ of monomials $T_1^{m_1} \cdots T_d^{m_d}$ in $d$ variables of degree $n = m_1 + \cdots + m_d$:

$$\lambda_{n,d} = \binom{d+n-1}{d-1}.$$

So, using only the condition $(\ast)$ and proceeding by induction on $\dim(A)$, we see that there exists a system of parameters $\alpha_1, \ldots, \alpha_d$ of $A$ such that $\alpha_2, \ldots, \alpha_d$ is part of a system of parameters of $B$. In the case where $k_A \to k_B$ is an isomorphism, $G$ acts as a pseudoreflection, and hence $I_G$ is principal. If $k_A \to k_B$ is not an isomorphism, then we must have $m_B = Bm_A$; otherwise the rank of $B/m_B$ is at least 4. Since $[k_B : k_A] \leq 3$, the field extension $k_A \to k_B$ is monogenous, and hence $A \to B$ is monogenous due to the lemma of Nakayama.

### 4. Relationship between the regular and the stable model of a smooth curve

As explained in the introduction, our incentive to study the invariant rings under a $p$-cyclic group action stems from the study of the relationship between the regular and the stable model of a smooth projective curve over the field of fractions $K$ of a discrete valuation ring $R$. So let $R \leftrightarrow R'$ be a Galois extension of discrete valuation rings of prime order $p$ and let $\pi$ and $\pi'$ be uniformizers of $R$ and of $R'$, respectively. Denote by $K'$ the field of fractions of $R'$ and let $k$ and $k'$ be the residue fields of $R$ and $R'$, respectively. Assume that $k = k'$ is algebraically closed and that $\text{char}(k) = p$. Let $G$ be the Galois group of $R'$ over $R$.

In the tame case, the action can always be diagonalized and the invariant rings have the well-known Hirzebruch–Jung singularities. The tame case of higher dimension is also settled in [Edixhoven 1992, Proposition 3.5]. If the action of $G$ is wild, this is in general not the case and the situation becomes quite capricious.
For example, consider an elliptic curve $E$ over $K$ having good reduction over $K'$, and let $X'$ be the corresponding proper smooth $R'$-model of $E \otimes_K K'$. Then $G$ acts naturally on $X'$, and hence one can consider the quotient $Y = X'/G$, which is a normal proper flat $R$-model of $E$. Assume that $E$ has reduction of Kodaira type $I^*_0$ over $K$; see [Silverman 1986, Theorem 15.2]. Curves of this type exist, since elliptic curves with Kodaira type $I^*_0$ have integer $j$-invariant and thus potentially good reduction. Moreover, that a wild extension might be needed can be checked via Tate’s algorithm [1975]. Let $X$ be the minimal regular $R$-model of $E$. Then $X$ happens to be a minimal blowing-up of $Y$ and, in general, $Y$ has singularities that are not of Hirzebruch–Jung type, since the special fiber of $X$ contains components having three neighbors.

Our result now provides a tool to study the correspondence between $X$ and the singularities of $Y$ by looking at the group action $G$ on $X'$ and on $R'$-models $Z'$, which are obtained by blowing-up $G$-invariant centers of $X'$. On these models, one can study the augmentation ideal and thereby obtain statements about which components have to occur in a desingularization of $Y$ and in the regular model $X$, respectively. Since this analysis is beyond the scope of this article, we intend to explain this in greater detail in a further paper.

In the following we will look at Conjecture 9 in the case of relative curves.

**Proposition 10.** Keep the situation of above. Let $Y$ be an affine smooth relative curve over $R'$ such that its closed fiber $Y \otimes_{R'} k'$ is irreducible. Assume that $G$ acts on $Y \to \text{Spec}(R')$ equivariantly. Let $B := \mathcal{O}_Y(Y)$ be the coordinate ring of $Y$. Then the following assertions are equivalent:

1. The augmentation ideal $I_G$ is locally principal.
2. The ring $A := B^G$ of invariants is regular and $A/p$ is regular where $p = A \cap B \pi'$.

**Proof.** $(1) \Rightarrow (2)$. It follows from Theorem 2 that $A$ is regular. It remains to show that the special fiber is regular. For showing this, it is enough to prove it after the $\pi$-adic completion, since the group action extends to the completion, taking invariants commutes with completion, and regularity of $A/p$ can be checked after $\pi$-adic completion. So we may assume that $B$ is the coordinate ring of the associated formal completion of $Y$ with respect to its special fiber. So set

$$\mathfrak{P} := B\pi' \quad \text{and} \quad p := A \cap \mathfrak{P}.$$

Then we obtain a finite extension of discrete valuation rings $A_p \hookrightarrow B_{\mathfrak{P}}$. Namely, the localization with respect to $A - p$ yields a finite flat extension $A_p \hookrightarrow B_p$. Since $\mathfrak{P}$ is the unique prime ideal of $B$ lying above $p$, so $B_p$ is a local Dedekind ring, and hence we get $B_p = B_{\mathfrak{P}}$. Since $A$ is regular, and hence locally factorial, the ideal $p$ is locally principal. The extended ideal $B\mathfrak{P}$ is locally principal and a power of $\mathfrak{P}$ and, hence, globally a power of $\mathfrak{P}$, that is, $\mathfrak{P}^e = Bp$. The degree of the residue
extension is denoted by \( f := [Q(B/\mathfrak{P}) : Q(A/p)] \). Moreover we have \( p = e \cdot f \).

In the case \( f = p \) and \( e = 1 \) we have \( \mathfrak{P} = Bp \). Since \( A \hookrightarrow B \) is faithfully flat, so \( A/p \to B/\mathfrak{P} \) is faithfully flat as well. Then, due to [Matsumura 1980, Theorem 51], the ring \( A/p \) is regular.

In the case \( f = 1, e = p \), the ideal \( p \) contains the uniformizer \( \pi \) of \( R \). Since \( pB = \mathfrak{P}^p \) due to \( e = p \) and \( \mathfrak{P} = B\pi^\prime \) as \( Y \) is smooth over \( S \), we obtain by faithfully flat descent \( p = A\pi \). Therefore \( A \otimes_R k \) is reduced and hence geometrically reduced. Then \( A \) is the set of all \( G \)-invariant functions \( f \) on \( Y \) that are bounded by 1 and also \( B \) consists of all functions on \( Y \) that are bounded by 1; see [Bosch et al. 1984, 6.4.3/4]. Moreover, it follows from [loc. cit.] that \( A \otimes_R R' \) coincides with \( B \). Thus we see that \( A \otimes_R k = A \otimes_R R' \otimes_R k' = B \otimes_R k' \) is regular.

(2) \( \implies \) (1). For the converse implication, \( A \) is regular. Since \( B \) is regular as well, the extension \( A \to B \) is faithfully flat; see [Serre 1965, IV, Proposition 22].

As above, we have the finite extension of discrete valuation rings \( A_p \hookrightarrow B_\mathfrak{P} \) and its associated numbers \( e \) and \( f \). In the case, \( f = 1 \) and \( e = p \) the finite ring extension \( A/p \to B/\mathfrak{P} \) is birational, and hence an isomorphism as \( A/p \) is regular. So any local parameter of \( A/p \) gives rise to a local parameter of \( B/\mathfrak{P} \). Therefore, any maximal ideal of \( B \) is generated by a \( G \)-invariant element and \( \pi^\prime \). Therefore, \( I_G = B \cdot I(\pi^\prime) \) is principal.

Now consider the case \( f = p \) and \( e = 1 \). Since \( A \) is regular, the ideal \( p \) is locally principal. So we may assume that \( p = A\alpha \) is principal. Due to \( e = 1 \), we obtain \( \mathfrak{P} = B\alpha \). Since \( B/\mathfrak{P} \) is regular, any maximal ideal of \( B \) is generated by \( \alpha \) and a lifting of a local parameter of \( B/\mathfrak{P} \). Therefore, \( I_G \) is locally principal as it is generated by the \( I(\beta) \), where \( \beta \) is a lifting of the local parameter \( \beta \) of \( B/\mathfrak{P} \).  

\textbf{Conjecture 11.} In the case of an affine arithmetic surface, that is, \( Y \) is regular with irreducible special fiber, one conjectures that the following conditions are equivalent, where \( \mathfrak{P} \subset B \) is the prime ideal whose locus is the special fiber and \( p := A \cap \mathfrak{P} \):

1. \( I_G \) is locally principal and \( B/\mathfrak{P} \) is regular.
2. \( A \) is regular and \( A/p \) is regular.

The proof of the last proposition tells us that the implication (1) \( \implies \) (2) is true in the case \( f = p \) and \( e = 1 \). In the case \( f = 1 \) and \( e = p \), we used the fact that the formation of the ring of \( 1 \)-bounded functions is compatible with base change; this is true when the multiplicity is 1. But it is not clear if one only knows that both models \( A \) and \( B \) have the same multiplicity in the special fiber over their base rings.

The implication (2) \( \implies \) (1) is true in the case \( f = 1 \) and \( e = p \), as seen by the same arguments as given in Proposition 10. But the case \( f = p \) and \( e = 1 \), is uncertain, although in this case the multiplicity behaves well.
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