Triangulable $\mathcal{O}_E$-analytic $(\varphi_q,\Gamma)$-modules of rank 2

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The theory of $(\varphi_q, 0)$-modules is a generalization of Fontaine’s theory of $(\varphi, \Gamma)$-modules, which classifies $G_F$-representations on $\mathcal{O}_F$-modules and $F$-vector spaces for any finite extension $F$ of $\mathbb{Q}_p$. In this paper following Colmez’s method we classify triangulable $\mathcal{O}_F$-analytic $(\varphi_q, 0)$-modules of rank 2. In the process we establish two kinds of cohomology theories for $\mathcal{O}_F$-analytic $(\varphi_q, 0)$-modules. Using them, we show that if $D$ is an étale $\mathcal{O}_F$-analytic $(\varphi_q, 0)$-module such that $D^\varphi_q = 1, q = 0$ (i.e., $V^G_F = 0$, where $V$ is the Galois representation attached to $D$), then any overconvergent extension of the trivial representation of $G_F$ by $V$ is $\mathcal{O}_F$-analytic. In particular, contrary to the case of $F = \mathbb{Q}_p$, there are representations of $G_F$ that are not overconvergent.

Introduction

This paper depends heavily on the theory of $(\varphi, \Gamma)$-modules for Lubin–Tate extensions, a generalization of Fontaine’s theory of $(\varphi, \Gamma)$-modules. The existence of this generalization was more or less implicit in [Fontaine 1990; Colmez 2002]. See also [Fourquaux 2005; Scholl 2006, Remark 2.3.1]. Kisin and Ren [2009] provided details, where $(\varphi, \Gamma)$-modules for Lubin–Tate extensions are called $(\varphi_q, \Gamma)$-modules.

To recall this theory, let $F$ be a finite extension of $\mathbb{Q}_p$, $\mathcal{O}_F$ the ring of integers in $F$, and $\pi$ a uniformizer of $\mathcal{O}_F$. Fix an algebraic closure of $F$ denoted by $\overline{F}$, and put $G_F = \text{Gal}(\overline{F}/F)$. Let $k_F$ be the residue field of $F$ and set $q = \#k_F$. Let $W = W(k_F)$ be the ring of Witt vectors over $k_F$. Then $F_0 := W[1/p]$ is the maximal absolutely unramified subfield of $F$. Let $\mathcal{F}$ be a Lubin–Tate group over $F$ corresponding to the uniformizer $\pi$. Then $\mathcal{F}$ is a formal $\mathcal{O}_F$-module. Let $X$ be a local coordinate on $\mathcal{F}$. Then the formal Hopf algebra $\mathcal{O}_\mathcal{F}$ may be identified with $\mathcal{O}_F[[X]]$. For any $a \in \mathcal{O}_F$, let $[a]_\mathcal{F} \in \mathcal{O}_F[[X]]$ be the power series giving the endomorphism $a$ of $\mathcal{F}$. If $n \geq 1$, let $F_n \subset \overline{F}$ be the subfield generated by the $\pi^n$-torsion points of $\mathcal{F}$. Write $F_\infty = \bigcup_n F_n$, $\Gamma = \text{Gal}(\overline{F}/F_\infty)$ and $G_{F_\infty} = \text{Gal}(\overline{F}/F_{\infty})$. For any integer $n \geq 0$, let $\Gamma_n \subset \Gamma$ be the subgroup $\text{Gal}(F_{\infty}/F_n)$. Let $T\mathcal{F}$ be the Tate module of $\mathcal{F}$. It

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is a free $\mathcal{O}_F$-module of rank 1. The action of $G_F$ on $T \mathcal{F}$ factors through $\Gamma$ and induces an isomorphism $\chi_f : \Gamma \to \mathcal{O}_F^\times$. For any $a \in \mathcal{O}_F^\times$ we write $\sigma_a := \chi_f^{-1}(a)$. Using the periods of $T \mathcal{F}$, one can construct a ring $\mathcal{O}_\mathcal{E}$ with actions of $\varphi_q = \varphi_{\log p, q}$ and $\Gamma$. We will recall the construction in Section 1. Kisin and Ren [2009] defined étale $(\varphi_q, \Gamma)$-modules over $\mathcal{O}_\mathcal{E}$ and classified $G_F$-representations on $\mathcal{O}_F$-modules in terms of these modules.

Here we are interested in triangulable $\mathcal{O}_F$-analytic $(\varphi_q, \Gamma)$-modules over a Robba ring $\mathcal{R}_L$, where $L$ is a finite extension of $F$. A triangulable $(\varphi_q, \Gamma)$-module over $\mathcal{R}_L$ means a $(\varphi_q, \Gamma)$-module $D$ that has a filtration consisting of $(\varphi_q, \Gamma)$-submodules $0 = D_0 \subset D_1 \subset \cdots \subset D_d = D$ such that $D_i/D_{i-1}$ is free of rank 1 over $\mathcal{R}_L$.

In the spirit of [Colmez 2008] on the classification of triangulable $(\varphi, \Gamma)$-modules of rank 2, in the present paper we will classify triangulable $\mathcal{O}_F$-analytic $(\varphi_q, \Gamma)$-modules over $\mathcal{R}_L$ of rank 2. One motivation for doing this is our belief that under the hypothetical $p$-adic local Langlands correspondence these $(\varphi_q, \Gamma)$-modules should correspond to certain unitary principal series of $GL_2(F)$. Colmez [2010a] and Liu, Xie, and Zhang [Liu et al. 2012] determined the spaces of locally analytic representations of $GL_2(\mathbb{Q}_p)$, based on this kind of $(\varphi, \Gamma)$-module. Our computations of dimensions of $\text{Ext}_\mathcal{R}_L^1$ match those of [Kohlhaase 2011] on extensions of locally analytic representations. Nakamura [2009] gave a generalization of Colmez’s work in another direction. But we think that Nakamura’s point of view is probably not the best one for applications to the $p$-adic local Langlands correspondence.

For our purpose we consider two kinds of cohomology theories for $\mathcal{O}_F$-analytic $(\varphi_q, \Gamma)$-modules.

For a $(\varphi_q, \Gamma)$-module $D$ over $\mathcal{R}_L$, we define $H^\bullet(D)$ by the cohomology of the semigroup $\varphi_q^n \times \Gamma$ as in [Colmez 2010a]. Then the first cohomology group $H^1(D)$ is isomorphic to $\text{Ext}(\mathcal{R}_L, D)$, the $L$-vector space of extensions of $\mathcal{R}_L$ by $D$ in the category of $(\varphi_q, \Gamma)$-modules.

If $D$ is $\mathcal{O}_F$-analytic, we consider the complex

$$C^\bullet_{\varphi_q, \nabla}(D) : \quad 0 \to D \xrightarrow{f_1} D \oplus D \xrightarrow{f_2} D \to 0,$$

where $f_1 : D \to D \oplus D$ is the map defined as $m \mapsto ((\varphi_q - 1)m, \nabla m)$, and $f_2 : D \oplus D \to D$ is $(m, n) \mapsto \nabla m - (\varphi_q - 1)n$. The operator $\nabla$ is defined in Section 1C. Put $H^i_{\varphi_q, \nabla}(D) := H^i(C^\bullet_{\varphi_q, \nabla}(D))$, for $i = 0, 1, 2$. Each of these modules admits a $\Gamma$-action. We set $H^i_{\text{an}}(D) = H^i_{\varphi_q, \nabla}(D)^\Gamma$.

**Theorem 0.1.** Let $D$ be an $\mathcal{O}_F$-analytic $(\varphi_q, \Gamma)$-module over $\mathcal{R}_L$. Then there is a natural isomorphism $\text{Ext}_\mathcal{R}_L^1(\mathcal{R}_L, D) \to H^1_\text{an}(D)$, where $\text{Ext}_\mathcal{R}_L^1(\mathcal{R}_L, D)$ is the $L$-vector space that consists of extensions of $\mathcal{R}_L$ by $D$ in the category of $\mathcal{O}_F$-analytic $(\varphi_q, \Gamma)$-modules.
The proof of Theorem 0.1 is given in Section 4; it is due to the referee, and is much simpler than that in our original version.

**Theorem 0.2.** Let $D$ be an $\mathcal{O}_F$-analytic $(\varphi_q, \Gamma)$-module over $\mathcal{R}_L$. The codimension of $\text{Ext}_{\text{an}}(\mathcal{R}_L, D)$ in $\text{Ext}(\mathcal{R}_L, D)$ is $\lfloor [F : \mathbb{Q}_p] - 1 \rfloor \dim_L D\varphi_q = 1, \Gamma = 1$. In particular, if $D\varphi_q = 1, \Gamma = 1 = 0$, then $\text{Ext}_{\text{an}}(\mathcal{R}_L, D) = \text{Ext}(\mathcal{R}_L, D)$.

To prove this, we will construct a (noncanonical) projection from $\text{Ext}(\mathcal{R}_L, D)$ onto $\text{Ext}_{\text{an}}(\mathcal{R}_L, D)$ whose kernel is of dimension $\lfloor [F : \mathbb{Q}_p] - 1 \rfloor \dim_L D\varphi_q = 1, \Gamma = 1$.

If $V$ is an overconvergent $L$-representation of $G_F$ (in the sense of Definition 1.4), $\Delta$ is the $(\varphi_q, \Gamma)$-module over $\mathcal{E}_L$ attached to $V$, and $D = \mathcal{R}_L \otimes_{\mathcal{E}_L} \Delta$, then $\text{Ext}(\mathcal{R}_L, D)$ measures the set of extensions of the trivial representation by $V$ that are overconvergent (see Proposition 1.5 and Proposition 1.6). Theorem 0.2 tells us that if $\mathcal{V}^G = D\varphi_q = 1, \Gamma = 1 = 0$, then any such extension is $\mathcal{O}_F$-analytic.

Let $\mathcal{F}(L)$ (resp. $\mathcal{F}_{\text{an}}(L)$) be the set of continuous (resp. locally $F$-analytic) characters $\delta : F^\times \to \mathbb{L}^\times$. Let $\delta_{\text{unr}}$ denote the character of $F^\times$ such that $\delta_{\text{unr}}(\pi) = q^{-1}$ and $\delta_{\text{unr}}|_{\mathcal{O}_F^\times} = 1$. Then $\delta_{\text{unr}}$ is a locally $F$-analytic character. If $\delta \in \mathcal{F}(L)$, let $\mathcal{R}_L(\delta)$ be the $(\varphi_q, \Gamma)$-module over $\mathcal{R}_L$ of rank 1 that has a basis $e_{\delta}$ such that $\varphi_q(e_{\delta}) = \delta(\pi)e_{\delta}$ and $\sigma_{\delta}(e_{\delta}) = \delta(a)e_{\delta}$. If $\delta \in \mathcal{F}_{\text{an}}(L)$, then $\mathcal{R}_L(\delta)$ is $\mathcal{O}_F$-analytic.

For locally $F$-analytic characters we have the following:

**Theorem 0.3.** For any $\delta \in \mathcal{F}_{\text{an}}(L)$, we have

$$
\dim_L H^1_{\text{an}}(\mathcal{R}_L(\delta)) = \begin{cases}
2 & \text{if } \delta = x^{-i}, i \in \mathbb{N} \text{ or } x^i \delta_{\text{unr}}, i \in \mathbb{Z}_+,
1 & \text{otherwise},
\end{cases}
$$

$$
\dim_L H^1(\mathcal{R}_L(\delta)) = \begin{cases}
[F : \mathbb{Q}_p] + 1 & \text{if } \delta = x^{-i}, i \in \mathbb{N},
2 & \text{if } \delta = x^i \delta_{\text{unr}}, i \in \mathbb{Z}_+,
1 & \text{otherwise}.
\end{cases}
$$

For the proof of Theorem 0.3 we follow Colmez’s method. Colmez [2008] used the theory of $p$-adic Fourier transform for $\mathbb{Z}_p$. For our case we use the $p$-adic Fourier transform for $\mathcal{O}_F$ developed by Schneider and Teitelbaum [2001] instead. But this transform can not be applied to our situation directly because, except for the case of $F = \mathbb{Q}_p$, it is defined over $\mathbb{C}_p$ and can not be defined over any finite extension $L$ of $F$. We overcome this difficulty by applying it to $\mathcal{R}_{C_p}$ and then descending certain results to $\mathcal{R}_L$. As a result, we obtain that if $\delta_1$ and $\delta_2$ are in $\mathcal{F}_{\text{an}}(L)$, then $\mathcal{R}_L(\delta_1)^\psi = 0$ and $\mathcal{R}_L(\delta_2)^\psi = 0$ are isomorphic to each other as $L[\Gamma]$-modules. This is exactly what we need. In fact, we will show that $S_{\delta} := (\mathcal{R}_L e_{\delta}/\mathcal{R}_L e_{\delta})^\psi = 0, \Gamma = 1$ is 1-dimensional over $L$ for any $\delta \in \mathcal{F}_{\text{an}}(L)$, and that $H^1_{\text{an}}(\mathcal{R}_L(\delta))$ is isomorphic to $S_{\delta}$ when $v_{\pi}(\delta(\pi)) < 1 - v_{\pi}(q)$ and $\delta$ is not of the form $x^i$.

For characters that are not locally $F$-analytic we have the following:

**Theorem 0.4.** For any $\delta \in \mathcal{F}(L) \setminus \mathcal{F}_{\text{an}}(I)$ we have $H^1(\mathcal{R}_L(\delta)) = 0$. Consequently, every extension of $\mathcal{R}_L$ by $\mathcal{R}_L(\delta)$ splits.
To state our result on the classification, we need some parameter spaces. These
parameter spaces are analogues of Colmez’s parameter spaces [Colmez 2008]. Let \( \mathcal{S} \) be the analytic variety over \( \mathcal{F}_{\text{an}}(L) \times \mathcal{F}_{\text{an}}(L) \) whose fiber over \((\delta_1, \delta_2)\) is isomorphic to \( \text{Proj}(H^1(\delta_1 \delta_2^{-1}), \mathcal{S}_{\text{an}} \) the analytic variety over \( \mathcal{F}_{\text{an}}(L) \times \mathcal{F}_{\text{an}}(L) \) whose fiber over \((\delta_1, \delta_2)\) is isomorphic to \( \text{Proj}(H^1(\delta_1 \delta_2^{-1})) \). There is a natural inclusion \( \mathcal{F}_{\text{an}} \hookrightarrow \mathcal{S} \). Let \( \mathcal{S}^+, \mathcal{F}^+, \mathcal{F}^{\text{ord}}, \mathcal{F}^{\text{cris}}, \mathcal{F}^{\text{st}}, \mathcal{F}^{\text{ncl}} \) and \( \mathcal{S}^{\text{ncl}} \) be the subsets of \( \mathcal{S} \) defined in Section 6. We can assign to any \( s \in \mathcal{S} \) (resp. \( s \in \mathcal{F}_{\text{an}} \)) a triangulable (resp. triangulable and \( \mathcal{O}_F \)-analytic) \((\varphi_q, \Gamma)\)-module \( D(s) \).

**Theorem 0.5.**

(a) For \( s \in \mathcal{S} \), \( D(s) \) is of slope zero if and only if \( s \) is in \( \mathcal{S}^+ - \mathcal{S}^{\text{ncl}} \);
\( D(s) \) is of slope zero and the Galois representation attached to \( D(s) \) is irreducible if and only if \( s \) is in \( \mathcal{S}^+ - (\mathcal{S}^{\text{ord}} \cup \mathcal{S}^{\text{ncl}}) \); \( D(s) \) is of slope zero and \( \mathcal{O}_F \)-analytic if and only if \( s \) is in \( \mathcal{S}^+ - \mathcal{S}^{\text{ncl}} \).

(b) Let \( s = (\delta_1, \delta_2, \mathcal{L}) \) and \( s' = (\delta_1', \delta_2', \mathcal{L}') \) be in \( \mathcal{S}^+ - \mathcal{S}^{\text{ncl}} \). If \( \delta_1 = \delta_1' \), then \( D(s) \cong D(s') \) if and only if \( s = s' \). If \( \delta_1 \neq \delta_1' \), then \( D(s) \cong D(s') \) if and only if \( s, s' \in \mathcal{S}^{\text{cris}} \cup \mathcal{S}^{\text{ord}} \), with \( \delta_1' = x^{\mathcal{L}} \delta_2, \delta_2' = x^{-\mathcal{L}} \delta_2 \).

In the case when \( F = \mathbb{Q}_p \), this becomes Colmez’s result [Colmez 2008]. The proof of Theorem 0.5 will be given at the end of Section 6.

We give another application of Theorem 0.3. In the case of \( F = \mathbb{Q}_p \) — the cyclotomic extension case — Cherbonnier and Colmez [1998] showed that all representations of \( G_{\mathbb{Q}_p} \) are overconvergent. But our following result shows that this is not the case when \([F : \mathbb{Q}_p] \geq 2\).

**Theorem 0.6.** Suppose that \([F : \mathbb{Q}_p] \geq 2\). Then there exist 2-dimensional \( \mathcal{L} \)-

representations of \( G_F \) that are not overconvergent (in the sense of Definition 1.4).

By Kedlaya’s theorem [2004], any \((\varphi_q, \Gamma)\)-module of slope zero \( D(s) \) in Theorem

0.5(a) comes from a 2-dimensional \( \mathcal{L} \)-representation of \( G_F \) that is overconvergent.

We outline the structure of this paper. We recall Fontaine’s rings, the theory of

\((\varphi_q, \Gamma)\)-modules and the relation between \((\varphi_q, \Gamma)\)-modules and Galois representations in Section 1A and Section 1B, and then define \( \mathcal{O}_F \)-analytic \((\varphi_q, \Gamma)\)-modules over the Robba ring \( \mathcal{R}_L \) in Section 1C. We define \( \psi \) in Section 2A, and study the properties of \( \partial \) and Res in Section 2B. In Section 3A we extend \( \psi \) to \( \mathcal{R}_{C_p} \), in Section 3B we define operators \( m_\alpha \) on \( \mathcal{R}_{C_p} \), and then in Section 3C we study the \( \Gamma \)-action on \( \mathcal{R}_L(\delta) \psi = 0 \) for all \( \delta \in \mathcal{F}_{\text{an}}(L) \). The cohomology theories for \( \mathcal{O}_F \)-analytic

\((\varphi_q, \Gamma)\)-modules are given in Section 4. In Section 5 we compute \( H^1(\mathcal{R}_L(\delta)) \) and \( H^1(\mathcal{R}_L(\delta)) \) for all \( \delta \in \mathcal{F}_{\text{an}}(L) \). After providing preliminary lemmas in Section 5A, we compute \( H^0(\partial \delta) \) for all \( \delta \in \mathcal{F}(L) \) in Section 5B and \( H^1(\partial \delta) \) for all \( \delta \in \mathcal{F}_{\text{an}}(L) \) satisfying \( \nu_\gamma(\partial(\delta(\pi))) < 1 - \nu_\gamma(q) \) in Section 5C. For the purpose of computing \( H^1(\partial \delta) \) for all \( \delta \in \mathcal{F}_{\text{an}}(L) \), we construct a transition map \( \partial : H^1(\chi^{-1} \delta) \to H^1(\delta) \), which is done in Section 5D. The computation of \( H^1(\delta) \) is given in Section 5E.
In Section 5F we define two maps \( \iota_k \) and \( \iota_{k,an} \). Applying results in Section 5, we classify triangulable \( \mathcal{O}_F \)-analytic \((\varphi_q, \Gamma)\)-modules in Section 6.

1. \((\varphi_q, \Gamma)\)-modules and \( \mathcal{O}_F \)-analytic \((\varphi_q, \Gamma)\)-modules

In this section we recall the theory of \((\varphi_q, \Gamma)\)-modules built in [Colmez 2002; Fourquaux 2005; Kisin and Ren 2009]. We keep using notation from the introduction.

1A. The rings of formal series. Put \( \tilde{E}^+ = \varprojlim \mathcal{O}_F/p \) with the transition maps given by Frobenius, and let \( \tilde{E} \) be the fractional field of \( \tilde{E}^+ \). We may also identify \( \tilde{E}^+ \) with \( \varprojlim \mathcal{O}_F/\pi \) with the transition maps given by the \( q \)-Frobenius \( \varphi_q = \varphi_{\log, q} \). Evaluation of \( X \) at \( \pi^{-\infty} \)-torsion points induces a map \( \iota : TF \to \tilde{E}^+ \). Precisely, if \( (v_n)_{n \geq 0} \in TF \), then \( \iota(v) = (v_n(X) + \pi^m \mathcal{O}_F)_{n \geq 0} \).

Let \( \{ \cdot \} \) be the unique lifting map \( \tilde{E}^+ \to W(\tilde{E})_F := W(\tilde{E}) \otimes_{\mathcal{O}_F} \mathcal{O}_F \) such that \( \varphi_q(x) = [\pi]_F((x)) \) (see [Colmez 2002, Lemma 9.3]). When \( F \) is the cyclotomic Lubin–Tate group \( \mathbb{G}_m \), we have \( \{ x \} = [1 + x] - 1 \), where \( [1 + x] \) is the Teichmüller lifting of \( 1 + x \). This map respects the action of \( G_F \). If \( (v_n)_{n \geq 0} \in TF \), then there is an embedding \( \mathcal{O}_F[[u_F]] \hookrightarrow W(\tilde{E})_F \) sending \( u_F \) to \( \{ \iota(v) \} \) which identifies \( \mathcal{O}_F[[u_F]] \) with a \( G_F \)-stable and \( \varphi_q \)-stable subring of \( W(\tilde{E})_F \). The \( G_F \)-action on \( \mathcal{O}_F[[u_F]] \) factors through \( \Gamma \). By [Colmez 2002, Lemma 9.3] we have

\[
\varphi_q(u_F) = [\pi]_F(u_F), \quad \sigma_a(u_F) = [a]_F(u_F).
\]

In the case of \( F = \mathbb{G}_m \), \( u_F \) is denoted by \( T \) in [Colmez 2008]. Here \( T \) is used to denote the Tate module of a Lubin–Tate group.

Let \( \mathcal{O}_E \) be the \( \pi \)-adic completion of \( \mathcal{O}_F[[u_F]][1/u_F] \). Then \( \mathcal{O}_E \) is a complete discrete valuation ring with uniformizer \( \pi \) and residue field \( k_F((u_F)) \). The topology induced by this valuation is called the strong topology. Usually we consider the weak topology on \( \mathcal{O}_E \), i.e., the topology with \( \{ \pi^i \mathcal{O}_E + u_F^j \mathcal{O}_F[[u_F]] \} : i, j \in \mathbb{N} \), as a fundamental system of open neighborhoods of \( 0 \). Let \( E \) be the field of fractions of \( \mathcal{O}_E \). Let \( E^+ \) be the subring \( \mathcal{O}_E \otimes_{\mathcal{O}_F} \mathcal{O}_F[[u_F]] \) of \( E \).

For any \( r \in \mathbb{R}_+ \cup \{ +\infty \} \), let \( E^{[0,r]} \) be the ring of Laurent series \( f = \sum_{i \in \mathbb{Z}} a_i u_F^i \) with coefficients in \( F \) that are convergent on the annulus \( 0 < v_F(a_i) \leq r \). For any \( 0 < s \leq r \) we define the valuation \( v^{[s]} \) on \( E^{[0,r]} \) by

\[
v^{[s]}(f) = \inf_{i \in \mathbb{Z}} (v_F(a_i) + is) \in \mathbb{R} \cup \{ \pm \infty \}.
\]

We equip \( E^{[0,r]} \) with the Fréchet topology defined by the family of valuations \( \{ v^{[s]} : 0 < s \leq r \} \). Then \( E^{[0,r]} \) is complete. We equip the Robba ring \( \mathcal{R} := \bigcup_{r > 0} E^{[0,r]} \) with the inductive limit topology. The subring of \( \mathcal{R} \) consisting of Laurent series of the form \( \sum_{i \geq 0} a_i u_F^i \) is denoted by \( \mathcal{R}^+ \).
Furthermore, with the inductive limit topology. The resulting topology on \( \mathcal{E} \) and in \( \mathcal{E}^+ \), with the inductive limit topology. The resulting topology on \( \mathcal{E}^{(0,r)} \) is called the weak topology [Colmez 2010b]. Note that the restriction of the weak topology to the subset

\[
\left\{ f \in \mathcal{E}^{(0,r)} : a_i = 0 \text{ if } i \geq 0 \right\}
\]

coincides with the topology defined by the valuation \( v^{[r]} \), and its restriction to \( \mathcal{E}^+ \) coincides with the weak topology on \( \mathcal{E}^+ \). Then we equip \( \mathcal{E}^+ = \bigcup_{r > 0} \mathcal{E}^{(0,r)}[1/u_{\mathcal{F}}] \) with the inductive limit topology.

We extend the actions of \( \varphi_q \) and \( \Gamma \) on \( \mathcal{O}_\mathcal{F}[u_{\mathcal{F}}] \) to \( \mathcal{E}^+ \), \( \mathcal{O}_\mathcal{E} \), \( \mathcal{E} \), \( \mathcal{E}^+ \) and \( \mathcal{R} \) continuously.

Put \( t_{\mathcal{F}} = \log_{\mathcal{F}}(u_{\mathcal{F}}) \), where \( \log_{\mathcal{F}} \) is the logarithmic of \( \mathcal{F} \). Then \( t_{\mathcal{F}} \) is in \( \mathcal{R} \) but not in \( \mathcal{E}^+ \). When \( \mathcal{F} = \mathbb{G}_m \), \( t_{\mathcal{F}} \) coincides with the usual \( t \) in [Colmez 2008]. Note that \( \varphi_q(t_{\mathcal{F}}) = \pi t_{\mathcal{F}} \) and \( \sigma_a(t_{\mathcal{F}}) = at_{\mathcal{F}} \) for any \( a \in \mathcal{O}_\mathcal{F}^\times \). Put \( Q = Q(u_{\mathcal{F}}) = [\pi]_{\mathcal{F}}(u_{\mathcal{F}})/u_{\mathcal{F}} \).

We have the following analogue of [Berger 2004, Lemma I.3.2].

**Lemma 1.1.** If \( I \) is a \( \Gamma \)-stable principal ideal of \( \mathcal{R}^+ \), then \( I \) is generated by an element of the form

\[
u_{\mathcal{F}} \prod_{n=0}^{+\infty} \left( \varphi_q^n(Q(u_{\mathcal{F}})/Q(0)) \right)_{j_{n+1}}.
\]

(1-1)

Furthermore, if \( \mathcal{R}^+ \cdot \varphi_q(I) \subseteq I \), then the sequence \( \{j_n\}_{n \geq 0} \) is decreasing, and if \( \mathcal{R}^+ \cdot \varphi_q(I) \supseteq I \), then the sequence \( \{j_n\}_{n \geq 0} \) is increasing.

**Proof.** The argument is similar to the proof of [Berger 2004, Lemma I.3.2]. Let \( f(u_{\mathcal{F}}) \) be a generator of \( I \). Put \( V_{\rho}(I) = \{ z \in \mathbb{C}_{\rho} : f(z) = 0, 0 \leq |z| \leq \rho \} \) for any \( \rho \in (0, 1) \). If \( I \) is stable by \( \Gamma \), then \( V_{\rho}(I) \) is stable by \([a]_{\mathcal{F}} \) for any \( a \in \mathcal{O}_\mathcal{F}^\times \). As \( V_{\rho}(I) \) is finite, for any \( z \in V_{\rho}(I) \) there must be some element \( a \in \mathcal{O}_\mathcal{F}^\times \), \( a \neq 1 \) such that \([a]_{\mathcal{F}}(z) = z \). Note that \([\pi]_{\mathcal{F}}(z) \) satisfies \([a]_{\mathcal{F}}([\pi]_{\mathcal{F}}(z)) = [\pi]_{\mathcal{F}}(z)\) if \([a]_{\mathcal{F}}(z) = z \). But the cardinal number of the set \( \{ z \in \mathbb{C}_{\rho} : [a]_{\mathcal{F}}(z) = z, \rho \leq |z| \} \) is finite. Thus for any \( z \in V_{\rho}(I) \) there exists a positive integer \( m = m(\rho) \) such that \([\pi]^m]_{\mathcal{F}}(z) = 0 \). Therefore \( I \) is generated by an element of the form (1-1).

The last assertion is easy to prove.

**Corollary 1.2.** We have \( t_{\mathcal{F}} = \left( u_{\mathcal{F}} \prod_{n \geq 0} \varphi_q^n(Q(u_{\mathcal{F}})/Q(0)) \right) \) in the ring \( \mathcal{R}^+ \).
Proof. Because the ideal \((t_F)\) is \(\Gamma\)-invariant and \(\mathcal{R}^+ \cdot \varphi_q(t_F) = (t_F)\), by Lemma 1.1 there exists \(j \in \mathbb{N}\) such that \((t_F) = (u_F^j \prod_{n \geq 0} \varphi_q^n(Q(u_F)/Q(0)))\). Since \((t_F/u_F) \equiv 1 \mod u_F\mathcal{R}^+\) we obtain \(j = 1\). \(\Box\)

If \(\mathcal{F}'\) is another Lubin–Tate group over \(F\) corresponding to \(\pi\), by the theory of Lubin–Tate groups there exists a unique continuous ring isomorphism \(\eta_{\mathcal{F},\mathcal{F}'}: O_{\mathcal{F}'} \to O_{\mathcal{F},\mathcal{F}'}\) such that \(\eta_{\mathcal{F},\mathcal{F}'}(u_{\mathcal{F}}) = u_{\mathcal{F}'} + \text{higher degree terms in } O_F[u_{\mathcal{F}}]\)

such that \(\eta_{\mathcal{F},\mathcal{F}'} \circ [a]_{\mathcal{F}} = [a]_{\mathcal{F}'} \circ \eta_{\mathcal{F},\mathcal{F}'}\) for all \(a \in O_F\). We extend \(\eta_{\mathcal{F},\mathcal{F}'}\) to isomorphisms

\[
O_{\mathcal{F}'} \sim \to O_{\mathcal{F},\mathcal{F}'}, \quad \mathcal{C}_{\mathcal{F}'} \sim \to \mathcal{C}_{\mathcal{F},\mathcal{F}'}, \quad \mathcal{C}_{\mathcal{F}} \sim \to \mathcal{C}_{\mathcal{F}'}, \quad \mathcal{C}_{\mathcal{F}} \sim \to \mathcal{C}_{\mathcal{F}'}^{\dagger}, \quad \mathcal{R}_{\mathcal{F}} \sim \to \mathcal{R}_{\mathcal{F}'},
\]

By abuse of notation these isomorphisms are again denoted by \(\eta_{\mathcal{F},\mathcal{F}'}\).

Let \(\ell_u = \log u_F\) be a variable over \(\mathcal{R}[1/t_F]\). We extend the \(\varphi_q, \Gamma\)-actions to \(\mathcal{R}[1/t_F, \ell_u]\) by

\[
\varphi_q(\ell_u) = q\ell_u + \log \frac{\pi_F(u_F)}{u_F^{q F}}, \quad \sigma_a(\ell_u) = \ell_u + \log \frac{[a]_{\mathcal{F}}(u_F)}{u_F}.
\]

1B. Galois representations and \((\varphi_q, \Gamma)\)-modules. Let \(L\) be a finite extension of \(F\). Let \(\text{Rep}_L G_F\) be the category of finite-dimensional \(L\)-vector spaces \(V\) equipped with a linear action of \(G_F\).

If \(A\) is any of \(\mathcal{C}^+, \mathcal{C}, \mathcal{C}^{\dagger}, \mathcal{R}\), we put \(A_L = A \otimes_F L\). Then we extend the \(\varphi_q, \Gamma\)-actions on \(A\) to \(A_L\) by \(L\)-linearity. Let \(R\) denote any of \(\mathcal{C}_L, \mathcal{C}_L'^\dagger\) and \(\mathcal{R}_L\). For a \((\varphi_q, \Gamma)\)-module over \(R\), we mean a free \(R\)-module \(D\) of finite rank together with continuous semilinear actions of \(\varphi_q\) and \(\Gamma\) commuting with each other such that \(\varphi_q\) sends a basis of \(D\) to a basis of \(D\). When \(R = \mathcal{C}_L\), we say that \(D\) is \(\text{étale}\) if \(D\) has a \(\varphi_q\)-stable \(\mathcal{O}_{q,L}\)-lattice \(M\) such that the linear map \(\varphi_q^* M \to M\) is an isomorphism. When \(R = \mathcal{C}_L^{\dagger}\), we say that \(D\) is \(\text{étale}\) if \(\mathcal{C}_L \otimes \mathcal{C}_L'^\dagger D\) is \(\text{étale}\). When \(R = \mathcal{R}_L\), we say that \(D\) is \(\text{étale}\) or \(\text{of slope } 0\) if there exists an \(\text{étale} (\varphi_q, \Gamma)\)-module \(\Delta\) over \(\mathcal{C}_L\) such that \(D = \mathcal{R}_L \otimes \mathcal{C}_L'^\dagger \Delta\). Let \(\text{Mod}_L^{\varphi_q, \Gamma, \text{ét}}\) be the category of \(\text{étale} (\varphi_q, \Gamma)\)-modules over \(R\).

Put \(\widetilde{B} = W(\mathcal{E})_F[1/\pi]\). Let \(B\) be the completion of the maximal unramified extension of \(\mathcal{E}\) in \(\widetilde{B}\) for the \(\pi\)-adic topology. Both \(\widetilde{B}\) and \(B\) admit actions of \(\varphi_q\) and \(G_F\). We have \(B^{G_{F_{\infty}}} = \mathcal{E}\).

For any \(V \in \text{Rep}_L G_F\), put \(D_{\mathcal{E}}(V) = (B \otimes_F V)^{G_{F_{\infty}}}\). For any \(D \in \text{Mod}_L^{\varphi_q, \Gamma, \text{ét}}\), put \(V(D) = (B \otimes_E D)^{\varphi_q=1}\).

Theorem 1.3 [Kisin and Ren 2009, Theorem 1.6]. The functors \(V\) and \(D_{\mathcal{E}}\) are quasi-inverse equivalences of categories between \(\text{Mod}_L^{\varphi_q, \Gamma, \text{ét}}\) and \(\text{Rep}_L G_F\).
As usual, let $\tilde{B}^+$ be the subring of $\tilde{B}$ consisting of overconvergent elements, and put $B^+ = B \cap \tilde{B}^+$. Then $(B^+)^{G_F} = \varnothing^+$.

**Definition 1.4.** If $V$ is an $L$-representation of $G_F$, we say that $V$ is overconvergent if $D_{\varnothing^+}(V) := (B^+ \otimes_F V)^{G_F}$ contains a basis of $D_{\varnothing}(V)$.

When $F = \mathbb{Q}_p$, according to the Cherbonnier–Colmez theorem [1998], all $L$-representations are overconvergent. But in general this is not true. For details, see Remark 5.21.

**Proposition 1.5.** (a) If $\Delta$ is an étale $(\varphi_q, \Gamma)$-module over $\varnothing_1^+$, then

$$V(\varnothing_1 \otimes \varnothing_1^+ \Delta) = (B^+ \otimes \varnothing_1^+ \Delta)^{\varphi_q=1}.$$

(b) The functor $\Delta \mapsto \varnothing_1 \otimes \varnothing_1^+ \Delta$ is a fully faithful functor from the category $\text{Mod}_{\varnothing_1 \otimes \varnothing_1^+}^{\varphi_q, \Gamma, \text{et}}$ to the category $\text{Mod}_{\varnothing_1}^{\varphi_q, \Gamma, \text{et}}$.

(c) The functor $D_{\varnothing^+}$ is an equivalence of categories between the category of overconvergent $L$-representations of $G_F$ and $\text{Mod}_{\varnothing_1}^{\varphi_q, \Gamma, \text{et}}$.

**Proof.** Without loss of generality we may assume that $L = F$. Put $\tilde{B}_{Q_p} = W(\varnothing)[1/p]$ and $\tilde{B}^+_Q = \tilde{B}_{Q_p} \cap \tilde{B}^+$. The technique of almost étale descent as in [Berger and Colmez 2008] allows us to show that the functor $\Delta \mapsto \tilde{B}_{Q_p} \otimes_{\tilde{B}_{Q_p}} \Delta$ from the category of étale $(\varphi, G_F)$-modules over $\tilde{B}^+_Q$ to the category of étale $(\varphi, G_F)$-modules over $\tilde{B}_{Q_p}$ is an equivalence. For any $(\varphi_q, G_F)$-module $D$ over $\tilde{B}^+$ (resp. $\tilde{B}$), we can attach a $(\varphi, G_F)$-module $D$ over $\tilde{B}^+_Q$ (resp. $\tilde{B}_{Q_p}$) to $D$ by letting $D = \bigoplus_{i=0}^{f-1} \varphi^{i*}(D)$ with the map

$$\varphi^*(D) = \bigoplus_{i=1}^f \varphi^{i*}(D) \rightarrow \bigoplus_{i=0}^{f-1} \varphi^{i*}(D) = D$$

that sends $\varphi^{i*}(D)$ identically to $\varphi^{i*}(D)$ for $i = 1, \ldots, f-1$ and sends $\varphi^{f*}(D) = \varphi^*_q(D)$ to $D$ using $\varphi_q$. Here $f = \log_p q$. Thus the functor $\alpha : \Delta \mapsto \tilde{B} \otimes_{\tilde{B}^+} \Delta$ from the category of étale $(\varphi_q, G_F)$-modules over $\tilde{B}^+$ to the category of étale $(\varphi_q, G_F)$-modules over $\tilde{B}$ is an equivalence. Now let $\Delta$ be an étale $(\varphi_q, \Gamma)$-module over $\varnothing^+$, and put $V = V(\varnothing \otimes \varnothing^+ \Delta)$. As $\alpha(\tilde{B}^+ \otimes_F V) = \tilde{B} \otimes_{\tilde{B}^+} V = \tilde{B} \otimes_{\tilde{B}^+} \Delta = \alpha(\tilde{B}^+ \otimes_{\varnothing^+} \Delta)$, we have $\tilde{B}^+ \otimes_F V = \tilde{B}^+ \otimes_{\varnothing^+} \Delta$. Thus $V$ is contained in $\tilde{B}^+ \otimes_{\varnothing^+} \Delta \cap B \otimes_{\varnothing^+} \Delta = \tilde{B}^+ \otimes_{\varnothing^+} \Delta$, and $V = (B^+ \otimes_{\varnothing^+} \Delta)^{\varphi_q=1}$. This proves (a).

Next we prove (b). Let $\Delta_1$ and $\Delta_2$ be two objects in $\text{Mod}_{\varnothing_1}^{\varphi_q, \Gamma, \text{et}}$. What we have to show is that the natural map

$$\text{Hom}_{\text{Mod}_{\varnothing_1}^{\varphi_q, \Gamma, \text{et}}}(\Delta_1, \Delta_2) \rightarrow \text{Hom}_{\text{Mod}_{\varnothing_1}^{\varphi_q, \Gamma, \text{et}}}(\varnothing \otimes \varnothing_1^+ \Delta_1, \varnothing \otimes \varnothing_1^+ \Delta_2)$$

is an isomorphism. For this we reduce the problem to showing that

$$(\tilde{\Delta}_1 \otimes \varnothing_1^+ \Delta_2)^{\varphi_q=1, \Gamma=1} \rightarrow (\varnothing \otimes \varnothing_1^+ (\tilde{\Delta}_1 \otimes \varnothing_1^+ \Delta_2))^{\varphi_q=1, \Gamma=1}$$
is an isomorphism. Here $\tilde{\Delta}_1$ is the $\mathcal{E}^\dagger$-module of $\mathcal{E}^\dagger$-linear maps from $\Delta_1$ to $\mathcal{E}^\dagger$, which is equipped with a natural étale $(\varphi_q, \Gamma)$-module structure. We have

$$(\mathcal{E} \otimes \mathcal{E}^\dagger (\tilde{\Delta}_1 \otimes \mathcal{E}^\dagger \Delta_2)) \varphi_q=1, \Gamma=1 = (B \otimes \mathcal{E}^\dagger (\tilde{\Delta}_1 \otimes \mathcal{E}^\dagger \Delta_2)) \varphi_q=1, G_F=1$$

$$= V(\mathcal{E} \otimes \mathcal{E}^\dagger (\tilde{\Delta}_1 \otimes \mathcal{E}^\dagger \Delta_2)) \varphi_q=1, G_F=1$$

$$= (B^\dagger \otimes \mathcal{E}^\dagger (\tilde{\Delta}_1 \otimes \mathcal{E}^\dagger \Delta_2)) \varphi_q=1, G_F=1$$

$$= (\tilde{\Delta}_1 \otimes \mathcal{E}^\dagger \Delta_2) \varphi_q=1, \Gamma=1. \quad (1-2)$$

Finally, (c) follows from (a), (b) and Theorem 1.3. □

**Proposition 1.6.** The functor $\Delta \mapsto \mathcal{R}_L \otimes \mathcal{E}^\dagger L$ is an equivalence of categories between $\operatorname{Mod}^{\varphi_q, \Gamma, \tilde{\mathcal{E}}}$ and $\operatorname{Mod}^{\varphi_q, \Gamma, \mathcal{E}^\dagger}$. Proof. Let $D$ be an étale $(\varphi_q, \Gamma)$-module over $\mathcal{R}_L$. By Kedlaya’s slope filtration theorem [2004], there exists a unique $\varphi_q$-stable $\mathcal{E}^\dagger_L$-submodule $\Delta$ of $D$ that is étale as a $\varphi_q$-module such that $D = \mathcal{R}_L \otimes \mathcal{E}^\dagger L \Delta$. For any $\gamma \in \Gamma$, $\gamma(\Delta)$ also has this property. Thus, by uniqueness of $\Delta$, we have $\gamma(\Delta) = \Delta$. This means that $\Delta$ is $\Gamma$-invariant. □

**1C. $\mathcal{O}_F$-analytic $(\varphi_q, \Gamma)$-modules.** For any $r \geq s > 0$, let $v^{[s, r]}$ be the valuation defined by $v^{[s, r]}(f) = \inf_{r \leq z \leq s} v_p(f(z))$. Note that

$$v^{[s, r]}(f) = \inf_{z \in \mathbb{C}_p, s \leq v_p(z) \leq r} v_p(f(z)).$$

**Lemma 1.7.** For any $r > s > 0$, there exists a sufficiently large integer $n = n(s, r)$ such that, if $\gamma \in \Gamma_n$, then we have $v^{[s, r]}((1 - \gamma)z) \geq v^{[s, r]}(z) + 1$ for all $z \in \mathcal{E}^{[0, r]}_L$.

**Proof.** It suffices to consider $z = u^k_F$, $k \in \mathbb{Z}$. If $k \geq 0$, then

$$\gamma(u^k_F) - u^k_F = u^k_F \left( \frac{\gamma(u^k_F)}{u^k_F} - 1 \right) \left( \frac{\gamma(u^{k-1}_F)}{u^{k-1}_F} + \cdots + 1 \right)$$

and

$$\gamma(u^{-k}_F) - u^{-k}_F = u^{-k}_F \left( \frac{u^k_F}{\gamma(u^k_F)} - 1 \right) \left( \frac{u^{k-1}_F}{\gamma(u^{k-1}_F)} + \cdots + 1 \right).$$

As $v^{[s, r]}(yz) \geq v^{[s, r]}(y) + v^{[s, r]}(z)$, the lemma follows from the fact that $\gamma(u_F)/u_F$ approaches 1 as $\gamma \to 1$. □

Let $D$ be an object in $\operatorname{Mod}^{\varphi_q, \Gamma, \tilde{\mathcal{E}}}_{/\mathcal{R}_L}$. We choose a basis $\{e_1, \ldots, e_d\}$ of $D$ and write $D^{[0, r]} = \bigoplus_{i=1}^d \mathcal{E}^{[0, r]}_L \cdot e_i$. Our definition of $D^{[0, r]}$ depends on the choice of $\{e_1, \ldots, e_d\}$; however, if $\{e'_1, \ldots, e'_d\}$ is another basis, then

$$\bigoplus_{i=1}^d \mathcal{E}^{[0, r]}_L \cdot e_i = \bigoplus_{i=1}^d \mathcal{E}^{[0, r]}_L \cdot e'_i.$$
for sufficiently small $r > 0$. When $r > 0$ is sufficiently small, $D^{[0,r]}$ is stable under $\Gamma$. By Lemma 1.7 and the continuity of the $\Gamma$-action on $D^{[0,r]}$, the series
\[
\log \gamma = \sum_{i=1}^{\infty} (\gamma - 1)^i (-1)^{i-1} / i
\]
converges on $D^{[0,r]}$ when $\gamma \to 1$. It follows that the map
\[
d\Gamma : \text{Lie} \Gamma \to \text{End}_L D^{[0,r]}, \quad \beta \mapsto \log(\exp \beta)
\]
is well defined for sufficiently small $\beta$, and we extend it to all of $\text{Lie} \Gamma$ by $\mathbb{Z}_p$-linearity. As a result, we obtain a $\mathbb{Z}_p$-linear map $d\Gamma_D : \text{Lie} \Gamma \to \text{End}_L D$. For any $\beta \in \text{Lie} \Gamma$, $d\Gamma_{\mathbb{Z}_L}(\beta)$ is a derivation of $\mathcal{R}_L$ and $d\Gamma_D(\beta)$ is a differential operator over $d\Gamma_{\mathbb{Z}_L}(\beta)$, which means that for any $a \in \mathcal{R}_L$, $m \in D$ and $\beta \in \text{Lie} \Gamma$ we have
\[
d\Gamma_D(\beta)(am) = d\Gamma_{\mathbb{Z}_L}(\beta)(a)m + a \cdot d\Gamma_D(\beta)(m).
\]
(1.3)

The isomorphism $\chi_{F} : \Gamma \to \mathcal{O}_{F}^\times$ induces an $\mathcal{O}_{F}$-linear isomorphism $\text{Lie} \Gamma \to \mathcal{O}_{F}$. We will identify $\text{Lie} \Gamma$ with $\mathcal{O}_{F}$ via this isomorphism.

We say that $D$ is $\mathcal{O}_{F}$-analytic if the map $d\Gamma_D$ is not only $\mathbb{Z}_p$-linear, but also $\mathcal{O}_{F}$-linear. If $D$ is $\mathcal{O}_{F}$-analytic, the operator $d\Gamma_{D}(\beta)/\beta$, $\beta \in \mathcal{O}_{F}$, $\beta \neq 0$, does not depend on the choice of $\beta$. The resulting operator is denoted by $\nabla_D$ or just $\nabla$ if there is no confusion. Note that the $\Gamma$-action on $\mathcal{R}_L$ is $\mathcal{O}_{F}$-analytic and by [Kisin and Ren 2009, Lemma 2.1.4]
\[
\nabla = t_{F} \cdot \partial \frac{dF_{F}}{\partial Y}(u_{F}, 0) \cdot d/du_{F},
\]
(1.4)

where $F_{F}(X, Y)$ is the formal group law of $F$. Put $\partial = (\partial F_{F}/\partial Y)(u_{F}, 0) \cdot d/du_{F}$. From the relation $\sigma_{a}(t_{F}) = at_{F}$ we obtain $\nabla t_{F} = t_{F}$ and $\partial \nabla = 1$. When $F = \mathbb{G}_{m}$, $\nabla$ and $\partial$ are already defined in [Berger 2002]. In this case $F_{F}(X, Y) = X + Y + XY$ and so $\partial = (1 + u_{F}) \cdot d/du_{F}$.

We end this section by classification of $(\varphi_{q}, \Gamma)$-modules over $\mathcal{R}_{L}$ of rank 1.

Let $\mathcal{J}(L)$ be the set of continuous characters $\delta : F^\times \to L^\times$ and $\mathcal{J}_{\text{an}}(L)$ the subset of locally $F$-analytic characters. If $\delta$ is in $\mathcal{J}_{\text{an}}(L)$, the quotient $\log \delta(a)/\log a$, for $a \in \mathcal{O}_{F}^\times$ (which makes sense when $\log a \neq 0$) does not depend on $a$. This number, denoted by $w_{\delta}$, is called the weight of $\delta$. Clearly $w_{\delta} = 0$ if and only if $\delta$ is locally constant; $w_{\delta}$ is in $\mathbb{Z}$ if and only if $\delta$ is locally algebraic.

If $\delta \in \mathcal{J}(L)$, let $\mathcal{R}_{L}(\delta)$ be the $(\varphi_{q}, \Gamma)$-module over $\mathcal{R}_{L}$ (of rank 1) that has a basis $e_{\delta}$ such that $\varphi_{q}(e_{\delta}) = \delta(\pi)e_{\delta}$ and $\sigma_{a}(e_{\delta}) = \delta(a)e_{\delta}$. It is easy to check that, if $\delta \in \mathcal{J}_{\text{an}}(L)$, then $\mathcal{R}_{L}(\delta)$ is $\mathcal{O}_{F}$-analytic and $\nabla_{\delta} = \nabla_{\mathcal{R}_{L}(\delta)} = t_{F} \partial + w_{\delta}$ (more precisely $\nabla_{\delta}(ze_{\delta}) = (t_{F} \partial z + w_{\delta}z)e_{\delta}$). If $\mathcal{R}_{L}(\delta)$ is étale, that is, $v_{\rho}(\delta(\pi)) = 0$, we will use $L(\delta)$ to denote the Galois representation attached to $\mathcal{R}_{L}(\delta)$. 
Remark 1.8. All 1-dimensional $L$-representations of $G_F$ are overconvergent. In fact, such a representation comes from a character of $F^\times$ and thus is of the form $L(\delta)$.

Proposition 1.9. Let $D$ be a $(\varphi_q, \Gamma)$-module over $\mathcal{R}_L$ of rank 1. Then there exists a character $\delta \in \mathcal{H}(L)$ such that $D$ is isomorphic to $\mathcal{R}_L(\delta)$. Furthermore, $D$ is $\mathcal{O}_F$-analytic if and only if $\delta \in \mathcal{H}_{an}(L)$.

Proof. The argument is similar to the proof of [Colmez 2008, Proposition 3.1]. We first reduce to the case that $D$ is étale. Then by Proposition 1.6 there exists an étale $(\varphi_q, \Gamma)$-module $\Delta$ over $\mathcal{E}^\dagger_L$ such that $D = \mathcal{R}_L \otimes_{\mathcal{E}^\dagger_L} \Delta$. Now the first assertion follows from Proposition 1.5 and Remark 1.8. The second assertion is obvious. \qed

2. The operators $\psi$ and $\vartheta$

2A. The operator $\psi$. We define an operator $\psi$ and study its properties.

Note that $\{u_{x}^{i}\}_{0 \leq i \leq q-1}$ is a basis of $\mathcal{E}_L$ over $\varphi_q(\mathcal{E}_L)$. So $\mathcal{E}_L$ is a field extension of $\varphi_q(\mathcal{E}_L)$ of degree $q$. Put $\text{tr} = \text{tr}_{\mathcal{E}_L/\varphi_q(\mathcal{E}_L)}$.

Lemma 2.1. (a) There is a unique operator $\psi : \mathcal{E}_L \to \mathcal{E}_L$ such that $\varphi_q \circ \psi = q^{-1}\text{tr}$. (b) For any $a, b \in \mathcal{E}_L$ we have $\psi(\varphi_q(a)b) = a\psi(b)$. In particular, $\psi \circ \varphi_q = \text{id}$. (c) $\psi$ commutes with $\Gamma$.

Proof. Assertion (a) follows from the fact that $\varphi_q$ is injective. Assertion (b) follows from the relation

$$
\varphi_q(\psi(\varphi_q(a)b)) = \text{tr}(\varphi_q(a)b)/q = \varphi_q(a)\text{tr}(b)/q = \varphi_q(a)\varphi_q(\psi(b)) = \varphi_q(\psi(a)(b))
$$

and the injectivity of $\varphi_q$. As $\varphi_q$ commutes with $\Gamma$, $\varphi_q(\mathcal{E}_L)$ is stable under $\Gamma$. Thus $\gamma \circ \text{tr} \circ \gamma^{-1} = \text{tr}$ for all $\gamma \in \Gamma$. This ensures that $\psi$ commutes with $\Gamma$. Assertion (c) follows. \qed

We first compute $\psi$ in the case of the special Lubin–Tate group.

Proposition 2.2. Suppose that $\mathcal{F}$ is the special Lubin–Tate group.

(a) If $\ell \geq 0$, then $\psi(u_{x}^{i}) = \sum_{i=0}^{[\ell/q]} a_{\ell,i} u_{x}^{i}$ with $v_\pi(a_{\ell,i}) \geq [\ell/q] + 1 - i - v_\pi(q)$.

(b) If $\ell < 0$, then $\psi(u_{x}^{i}) = \sum_{i=\ell}^{[\ell/q]} b_{\ell,i} u_{x}^{i}$ with $v_\pi(b_{\ell,i}) \geq [\ell/q] + 1 - i - v_\pi(q)$.

Proof. First we prove (a) by induction on $\ell$. As the minimal polynomial of $u_{x}^{i}$ is $X^q + \pi X - (u_{x}^{i} + \pi u_{x})$, by Newton’s formula we have

$$
\text{tr}(u_{x}^{i}) = \begin{cases} 
0 & \text{if } 1 \leq i \leq q-2, \\
(1-q)\pi & \text{if } i = q-1.
\end{cases}
$$

It follows that

$$
\psi(u_{x}^{i}) = \begin{cases} 
0 & \text{if } 1 \leq i \leq q-2, \\
(1-q)\pi/q & \text{if } i = q-1.
\end{cases}
$$
Thus the assertion holds when $0 \leq \ell \leq q - 1$. Now we assume that $\ell = j \geq q$ and the assertion holds when $0 \leq \ell \leq j - 1$. We have
\[
\psi(u^\ell_{\cal F}) = \psi((u^{q-1}_{\cal F} + \pi u^q_{\cal F})u^{\ell-q}_{\cal F}) - \psi(\pi u^q_{\cal F}) = u^\ell_{\cal F}\psi(u^{\ell-q}_{\cal F}) - \pi \psi(u^{\ell-q+1}_{\cal F})
\]
\[
= \sum_{i=1}^{[\ell/q]} a_{\ell-q,i-1} u^\ell_{\cal F} - \sum_{i=0}^{[(\ell+1)/q]-1} \pi a_{\ell-q+1,i} u^\ell_{\cal F}.
\]
Thus $a_{\ell,i} = a_{\ell,q,i-1} - \pi a_{\ell-q+1,i}$. By the inductive assumption we have
\[
v_\pi(a_{\ell,q,i-1}) \geq [(\ell - q)/q] + 1 - (i - 1) - v_\pi(q) = [\ell/q] + 1 - i - v_\pi(q)
\]
and
\[
v_\pi(a_{\ell-q+1,i}) \geq [(\ell - q + 1)/q] + 1 - i - v_\pi(q) \geq [\ell/q] - i - v_\pi(q).
\]
It follows that $v_\pi(a_{\ell,i}) \geq [\ell/q] + 1 - i - v_\pi(q)$.

Next we prove (b). We have
\[
\psi(u^\ell_{\cal F}) = \psi\left(\frac{(u^{q-1}_{\cal F} + \pi)^{-\ell}}{\varphi_q(u^\ell_{\cal F})}\right) = \psi\left(\sum_{j=0}^{[\ell]} \left(\frac{-\ell}{j}\right)_{\ell} u^{(q-1)_\ell} \pi^{-\ell-j}\right)
\]
\[
= \sum_{j=0}^{[\ell]} \sum_{j=0}^{[\ell]} \left(\frac{-\ell}{j}\right)_{\ell} \pi^{-\ell-j} a_{j(q-1),i} u^{\ell+j}_{\cal F}
\]
\[
= \sum_{j=0}^{[\ell]} \sum_{j=0}^{[\ell]} \left(\frac{-\ell}{j}\right)_{\ell} \pi^{-\ell-j} a_{j(q-1),i-\ell} \cdot u^\ell_{\cal F}.
\]
Here, \(\left(\frac{-\ell}{j}\right)_{\ell} = \frac{(-\ell)!}{j!(\ell-j)!}\). Thus $b_{\ell,i} = \sum_{j=0}^{[\ell]} \left(\frac{-\ell}{j}\right)_{\ell} \pi^{-\ell-j} a_{j(q-1),i-\ell}$. Since
\[
v_\pi(\pi^{-\ell-j} a_{j(q-1),i-\ell}) \geq -\ell - j + \left(\left[j(q-1)/q\right] + 1 - (i-\ell) - v_\pi(q)\right)
\]
\[
= [-j/q] + 1 - i - v_\pi(q)
\]
\[
\geq [\ell/q] + 1 - i - v_\pi(q),
\]
we obtain $v_\pi(b_{\ell,i}) \geq [\ell/q] + 1 - i - v_\pi(q)$. 

Let $\mathbb{E}_L^-$ be the subset of $\mathbb{E}_L$ consisting of elements of the form $\sum_{i \leq -1} a_i u^i_{\cal F}$.

**Corollary 2.3.** Suppose that $\mathcal{F}$ is the special Lubin–Tate group. Then $\psi(\mathbb{E}_L^-) \subset \mathbb{E}_L^-$. 

**Proof.** This follows directly from Proposition 2.2. 

**Proposition 2.4.** (a) $\psi(\mathbb{E}_L^+) = \mathbb{E}_L^+$, $\psi(\mathcal{O}_{\mathbb{E}_L^+}) \subset \frac{\pi}{q} \mathcal{O}_{\mathbb{E}_L^+}$ and $\psi(\mathcal{O}_{\mathbb{E}_L}) \subset \frac{\pi}{q} \mathcal{O}_{\mathbb{E}_L}$.

(b) $\psi$ is continuous for the weak topology on $\mathbb{E}_L^+$. 

(c) $\mathbb{E}_L^+$ is stable under $\psi$, and the restriction of $\psi$ on $\mathbb{E}_L^+$ is continuous for the weak topology of $\mathbb{E}_L^+$. 


(d) If $f \in \mathcal{E}_L^{(0,r)}$, then the sequence $\left(\frac{2}{\pi} \psi\right)^n(f)$, $n \in \mathbb{N}$, is bounded in $\mathcal{E}_L^{(0,r)}$ for the weak topology.

**Proof.** Let $\mathcal{F}_0$ be the special Lubin–Tate group over $F$ corresponding to $\pi$. Observe that $\psi_\mathcal{F} = \eta_{\mathcal{F}_0,\mathcal{F}}^{-1} \psi_\mathcal{F}_0 \eta_{\mathcal{F}_0,\mathcal{F}}$. Since $\eta_{\mathcal{F}_0,\mathcal{F}}(u_{\mathcal{F}_0})$ equals $u_{\mathcal{F}}$ times a unit in $\mathcal{O}_F[u_{\mathcal{F}}]$, we have

$$
\eta_{\mathcal{F}_0,\mathcal{F}}(\mathcal{O}_{\mathcal{E}_{\mathcal{F}_0,L}}[1/u_{\mathcal{F}_0}]) = \mathcal{O}_{\mathcal{E}_{\mathcal{F},L}}[1/u_{\mathcal{F}}] \quad \text{for any } r > 0,
$$

and $\eta_{\mathcal{F}_0,\mathcal{F}}$ respects the valuation $v^{[0,r]}$. Thus $\eta_{\mathcal{F}_0,\mathcal{F}} : \mathcal{E}_{\mathcal{F}_0,L} \to \mathcal{E}_{\mathcal{F},L}$ is a topological isomorphism. It follows that $\mathcal{E}_{\mathcal{F},L} \subset \mathcal{E}_{\mathcal{F}_0,L}$ and its inverse are continuous for the weak topology. Similarly $\eta_{\mathcal{F}_0,\mathcal{F}} : \mathcal{E}_{\mathcal{F}_0,L} \to \mathcal{E}_{\mathcal{F},L}$ and its inverse are continuous for the weak topology. Hence we only need to consider the case of the special Lubin–Tate group. Assertions (a) and (b) follow from Proposition 2.2. For (c) we only need to show that, for any $r > 0$, we have $\psi(\mathcal{E}_L^{[0,r]}) \subset \mathcal{E}_L^{[0,r]}$ and the restriction $\psi : \mathcal{E}_L^{[0,r]} \to \mathcal{E}_L^{[0,r]}$ is continuous. By (b) the restriction of $\psi$ to $\mathcal{E}_L^+$ is continuous. By Proposition 2.2(b) and Corollary 2.3, if $f$ is in $\mathcal{E}_L^- \cap \mathcal{E}_L^{[0,r]}$, then $\psi(f)$ is in $\mathcal{E}_L^-$ and $v^{[r]}(\psi(f)) \geq v^{[r]}(f) + v_p(\pi/q)$. Thus $\psi : \mathcal{E}_L^- \cap \mathcal{E}_L^{[0,r]} \to \mathcal{E}_L^- \cap \mathcal{E}_L^{[0,r]}$ is continuous, which proves (c). As $(q/p)\psi(\mathcal{O}_{\mathcal{E}_L^+}) \subset \mathcal{O}_{\mathcal{E}_L^+}^+$ and $v^{[r]}((q/p)\psi(f)) \geq v^{[r]}(f)$ for any $f \in \mathcal{E}_L^- \cap \mathcal{E}_L^{[0,r]}$, (d) follows.

Next we extend $\psi$ to $\mathcal{R}_L$.

**Proposition 2.5.** We can extend $tr$ continuously to $\mathcal{R}_L$. The resulting operator $tr$ satisfies $tr|_{\varphi_q(\mathcal{R}_L)} = q \cdot \text{id}$ and $tr(\mathcal{R}_L) = \varphi_q(\mathcal{R}_L)$.

**Proof.** Let $\mathcal{E}_L^{[0,\infty)}$ denote the subset of $\mathcal{E}_L$ consisting of $f \in \mathcal{E}_L$ of the form

$$
\sum_{n > -\infty} a_n u_{\mathcal{F}}^n.
$$

If $f \in \mathcal{E}_L^{[0,\infty)}$, then $tr(f) = \sum_{n \in \ker[\pi]_{\mathcal{F}}} f(u_{\mathcal{F}} + \eta)$. If $f \in \mathcal{E}_L^{[0,\infty)}$, then $v_p(\eta) \geq \frac{1}{(q-1)e_F}$, where $e_F = [F : F_0]$. Thus, if $r$ and $s \in \mathbb{R}_+$ satisfy $1/(r(q-1)e_F) > r > s$, the morphisms $u_{\mathcal{F}} \mapsto u_{\mathcal{F}} + \eta$ (where $\eta \in \ker[\pi]_{\mathcal{F}}$) keep the annulus $\{z \in \mathbb{C}_p : p^{-r} \leq |z| \leq p^{-s}\}$ stable. So for any $f \in \mathcal{E}_L^{[0,\infty)}$ we have $v^{[s,r]}(f(u_{\mathcal{F}} + \eta)) = v^{[s,r]}(f)$ and $v^{[s,r]}(tr(f)) \geq v^{[s,r]}(f)$. Hence there exists a unique continuous operator $Tr : \mathcal{R}_L \to \mathcal{R}_L$ such that $Tr(f) = tr(f)$ for any $f \in \mathcal{E}_L^{[0,\infty)}$. (For any $f \in \mathcal{R}_L$, choosing a positive real number $r$ such that $f \in \mathcal{E}_L^{[0,r]}$, we can find a sequence $\{f_i\}_{i \geq 1}$ in $\mathcal{E}_L^{[0,\infty)}$ such that $f_i \to f$ in $\mathcal{E}_L^{[0,r]}$; then $\{tr(f_i)\}_{i \geq 1}$ is a Cauchy sequence in $\mathcal{E}_L^{[s,r]}$ for any $s$ satisfying $0 < s \leq r$, and we let $Tr(f)$ be their limit in $\mathcal{E}_L^{[0,r]}$; it is easy to show that $Tr(f)$ does not depend on any choice.) From the continuity of $Tr$ we obtain that $Tr|_{\mathcal{E}_L^+} = tr$ and $Tr|_{\varphi_q(\mathcal{R}_L)} = q \cdot \text{id}$. By Lemma 2.6 below, $\varphi_q : \mathcal{R}_L \to \mathcal{R}_L$ is strict and thus has a closed image. Since $\mathcal{E}_L^+$ is dense in $\mathcal{R}_L$ and $\text{Tr}(\mathcal{E}_L^+) = \varphi_q(\mathcal{E}_L^+) \subset \varphi_q(\mathcal{R}_L)$, we have $\text{Tr}(\mathcal{R}_L) \subseteq \varphi_q(\mathcal{R}_L)$. □
Lemma 2.6. If \( \frac{q}{(q-1)e_F} > r \geq s > 0 \) and \( f \in \mathcal{E}_L^{[0,r]} \), then we have

- \( v^{[s,r]}(\gamma(f)) = v^{[s,r]}(f) \) for all \( \gamma \in \Gamma \);
- \( v^{[s,r]}(\varphi_q(f)) = v^{[q,s,qr]}(f) \) if \( r < 1/(q-1)e_F \).

**Proof.** Since \( [\chi_F(\gamma)]_F(u_F) \in u_F\mathcal{O}_F[[u_F]] \), we have \( v_p([\chi_F(\gamma)]_F(z)) \geq v_p(z) \) for all \( z \in \mathbb{C}_p \) such that \( v_p(z) > 0 \). By the same reason we have \( v_p([\chi_F(\gamma^{-1})]_F(z)) \geq v_p(z) \) and thus \( v_p([\chi_F(\gamma)]_F(z)) \leq v_p(z) \). So \( v_p([\chi_F(\gamma)]_F(z)) = v_p(z) \).

If \( z \in \mathbb{C}_p \) satisfies

\[
p^{-\frac{1}{(q-1)e_F}} < p^{-r} \leq |z| \leq p^{-s} < 1,
\]

then \( v_p([\pi]_F(z)) = q v_p(z) \). Thus, the image by \( z \mapsto [\pi]_F(z) \) of the annulus \( \{ z \in \mathbb{C}_p : p^{-r} \leq |z| \leq p^{-s} \} \) is inside the annulus \( \{ z \in \mathbb{C}_p : p^{-qr} \leq |z| \leq p^{-qs} \} \). Conversely, if \( w \in \mathbb{C}_p \) is such that \( p^{-qr} \leq |w| \leq p^{-qs} \), then \( v_p(w) < q/(q-1)e_F \).

The Newton polygon of the polynomial \(-w + [\pi]_F(u_F)\) shows that this polynomial has \( q \) roots of valuation \( \frac{1}{q} v_p(w) \).

If \( z \in \mathbb{C}_p \) is such a root, we have \( p^{-r} \leq |z| \leq p^{-s} \).

Thus, the image of the annulus \( p^{-r} \leq |z| \leq p^{-s} \) is the annulus \( p^{-qr} \leq |z| \leq p^{-qs} \). \( \square \)

We define \( \psi : \mathbb{R}_L \to \mathbb{R}_L \) by \( \psi = \frac{1}{q} \varphi_q^{-1} \circ \text{tr} \).

**Lemma 2.7.** If \( q/(q-1)e_F > r \geq s > 0 \) and \( f \in \mathcal{E}_L^{[0,r]} \), then

\[ v^{[s,r]}(\psi(f)) \geq v^{[s/q,r/q]}(f) - v_p(q). \]

**Proof.** By Lemma 2.6 it suffices to show that

\[ v^{[s/q,r/q]}(\varphi_q(\psi(f))) = v^{[s/q,r/q]}(q^{-1}\text{tr}(f)) \geq v^{[s/q,r/q]}(f) - v_p(q). \]

But this follows from Proposition 2.5 and its proof. \( \square \)

As a consequence, \( \psi : \mathbb{R}_L \to \mathbb{R}_L \) is continuous.

**Corollary 2.8.** (a) \( \{u_{F_i}^j\}_{0 \leq i \leq q-1} \) is a basis of \( \mathcal{E}_L^\dagger \) over \( \varphi_q(\mathcal{E}_L^\dagger) \), and

\[ \text{tr}|_{\mathcal{E}_L^\dagger} = \text{tr}|_{\mathcal{E}_L^\dagger/\varphi_q(\mathcal{E}_L^\dagger)}. \]

(b) \( \{u_{F_i}^j\}_{0 \leq i \leq q-1} \) is a basis of \( \mathbb{R}_L \) over \( \varphi_q(\mathbb{R}_L) \).

**Proof.** Let \( \{b_i\}_{0 \leq i \leq q-1} \) be the dual basis of \( \{u_{F_i}^j\}_{0 \leq i \leq q-1} \) relative to \( \text{tr}|_{\mathcal{E}_L/\varphi_q(\mathcal{E}_L)} \). Let \( B \) be the inverse of the matrix \( (\text{tr}(u_{F_i}^j))_{i,j} \). Then \( B \in \text{GL}_q(\mathcal{E}_L^\dagger) \) and

\[ (b_0, b_1, \ldots, b_{q-1})^t = B(1, u_{F_i}^j, \ldots, u_{F_i}^{q-1})^t. \]

So \( b_0, b_1, \ldots, b_{q-1} \) are in \( \mathcal{E}_L^\dagger \). Then \( f = \sum_{i=0}^{q-1} u_{F_i}^j \psi(b_i f) \) for any \( f \in \mathcal{E}_L, \mathcal{E}_L^\dagger \) or \( \mathbb{R}_L \). (For the former two cases, this follows from the definition of \( \{b_i\}_{0 \leq i \leq q-1} \); for the last case, we apply the continuity of \( \psi \).) Thus \( \{u_{F_i}^j\}_{0 \leq i \leq q-1} \) generate \( \mathcal{E}_L^\dagger \) (resp. \( \mathbb{R}_L \)) over \( \varphi_q(\mathcal{E}_L^\dagger) \) (resp. \( \varphi_q(\mathbb{R}_L) \)). In either case, to prove the independence of
We have Lemma 2.11. □ as desired.

We apply the above to \((\varphi_q, \Gamma)\)-modules.

**Proposition 2.9.** If \(D\) is a \((\varphi_q, \Gamma)\)-module over \(R\) where \(R = \mathcal{E}_L, \mathcal{E}_L^\dagger\) or \(\mathcal{R}_L\), then there is a unique operator \(\psi : D \to D\) such that

\[
\psi(a\varphi_q(x)) = \psi(a)x \quad \text{and} \quad \psi(\varphi_q(a)x) = a\psi(x)
\]

for any \(a \in R\) and \(x \in D\). Moreover \(\psi\) commutes with \(\Gamma\).

**Proof.** Let \(\{e_1, e_2, \ldots, e_d\}\) be a basis of \(D\) over \(R\). By the definition of \((\varphi_q, \Gamma)\)-modules, \(\{\varphi_q(e_1), \varphi_q(e_2), \ldots, \varphi_q(e_d)\}\) is also a basis of \(D\). For any \(m \in D\), writing \(m = a_1\varphi_q(e_1) + a_2\varphi_q(e_2) + \cdots + a_d\varphi_q(e_d)\), we put

\[
\psi(m) = \psi(a_1)e_1 + \psi(a_2)e_2 + \cdots + \psi(a_d)e_d.
\]

Then \(\psi\) satisfies (2.1). It is easy to prove the uniqueness of \(\psi\). Observe that for any \(\gamma \in \Gamma\), \(\gamma \psi \gamma^{-1}\) also satisfies (2.1). Thus \(\gamma \psi \gamma^{-1} = \psi\) by uniqueness of \(\psi\). This means that \(\psi\) commutes with \(\Gamma\). □

**2B. The operator \(\partial\) and the map Res.** Recall that \(\partial = (\partial F_\mathcal{F}/\partial Y)(u_\mathcal{F}, 0) \cdot d/du_\mathcal{F}\). So \(d \mathcal{F}_\mathcal{F} = (\partial F_\mathcal{F}/\partial Y)(u_\mathcal{F}, 0) du_\mathcal{F}\) and \((d \mathcal{F}_\mathcal{F}/du_\mathcal{F}) = (\partial F_\mathcal{F}/\partial Y)(u_\mathcal{F}, 0)^{-1}\).

**Lemma 2.10.** If \(r \geq s > 0\) and \(f \in \mathcal{R}_L^{[0, r]}\), then \(v^{[s, r]}(\partial f) \geq v^{[s, r]}(f) - r\).

**Proof.** Observe that \(v^p((\partial F_\mathcal{F}/\partial Y)(z, 0)) = 0\) for all \(z\) in the disk \(|z| < 1\). Thus \(v^{[s, r]}(\partial f) = v^{[s, r]}(df/du_\mathcal{F})\). Write \(f = \sum_{n \in \mathbb{Z}} a_n u^{n}_\mathcal{F}\). Then we have

\[
v^{[s, r]}\left(\frac{df}{du_\mathcal{F}}\right) = \inf_{r > v^p(z) \geq s} v_p(na_n z^{n-1})
\]

\[
\geq \inf_{r > v_p(z) \geq s, n \in \mathbb{Z}} \left( v_p(a_n) + n v_p(z) - v_p(z) \right)
\]

\[
\geq \inf_{r > v_p(z) \geq s, n \in \mathbb{Z}} \left( v_p(a_n) + n v_p(z) \right) - r \geq v^{[s, r]}(f) - r,
\]

as desired. □

**Lemma 2.11.** We have

\[
\partial \circ \sigma_a = a \sigma_a \circ \partial, \quad \partial \circ \varphi_q = \pi \varphi_q \circ \partial, \quad \partial \circ \psi = \pi^{-1} \psi \circ \partial.
\]

**Proof.** From the definition of \(\nabla\) we see that \(\nabla = t_\mathcal{F}_{\Gamma}\partial\) commutes with \(\Gamma, \varphi_q\) and \(\psi\). Hence the lemma follows from the equalities

\[
\sigma_a(t_\mathcal{F}) = at_\mathcal{F}, \quad \varphi_q(t_\mathcal{F}) = \pi t_\mathcal{F}, \quad \psi(t_\mathcal{F}) = \psi(\pi^{-1} \varphi_q(t_\mathcal{F})) = \pi^{-1} t_\mathcal{F}.
\]

□
Let \( \text{res} : \mathcal{R}_L \to L \) be the residue map \( \text{res}(\sum_{i \in \mathbb{Z}} a_i u_i \partial u_i) = a_{-1} \), and let \( \text{Res} : \mathcal{R}_L \to L \) be the map defined by \( \text{Res}(f) = \text{res}(f \, \partial u) \).

**Proposition 2.12.** We have the exact sequence

\[
0 \to L \to \mathcal{R}_L \to \mathcal{R}_L^\text{Res} \to L \to 0,
\]

where \( L \to \mathcal{R}_L \) is the inclusion map.

**Proof.** The kernel of \( \partial \) is just the kernel of \( \text{d}/\text{d} u \) and thus is \( L \). For any \( a \in L \) we have \( \text{Res}((a/u) \cdot (\text{d} u/\text{d} u)^{-1}) = a \), which implies that \( \text{Res} \) is surjective. If \( f = \partial g \), then \( f \, \partial u = dg \) and so \( \text{Res}(f) = \text{res}(dg) = 0 \). It follows that \( \text{Res} \circ \partial = 0 \). Conversely, if \( f \in \mathcal{R}_L \) satisfies \( \text{Res}(f) = 0 \), then \( f \) can be written as

\[
f = \left( \frac{\text{d} u}{\partial u} \right)^{-1} \cdot \sum_{i \neq -1} a_i u_i^{i+1}.
\]

Put \( g = \sum_{i \neq 0} a_i u_i^{i+1} \). Then \( f = \partial g \). \( \square \)

**Proposition 2.13.** (a) \( \text{Res} \circ \sigma_a = a^{-1} \text{Res} \).

(b) \( \text{Res} \circ \varphi_q = (q/\pi) \text{Res} \) and \( \text{Res} \circ \psi = (\pi/q) \text{Res} \).

**Proof.** First we prove (a). Let \( g \) be in \( \mathcal{R}_L \) and put \( f = \partial g \). By Lemma 2.11 we have

\[
\sigma_a(f) = \sigma_a \circ \partial(g) = a^{-1} \partial(\sigma_a(g)), \quad \psi(f) = \psi \circ \partial(g) = \pi \partial(\psi(g)).
\]

Thus by Proposition 2.12 we have \( \text{Res} \circ \sigma_a = a^{-1} \text{Res} = 0 \) and \( \text{Res} \circ \psi = \frac{\pi}{q} \text{Res} = 0 \) on \( \partial \mathcal{R}_L \). From

\[
\sigma_a(1/u) = \frac{1}{\overline{a} u} \equiv \frac{1}{au} \mod \mathcal{R}_L^+, \n\]

we see that \( \text{Res} \circ \sigma_a(1/u) = a^{-1} \text{Res}(1/u) \). Assertion (a) follows.

To prove \( \text{Res} \circ \psi = (\pi/q) \text{Res} \), without loss of generality we suppose that \( \mathcal{F} \) is the special Lubin–Tate group. In this case \( \psi(1/u) = \pi/(qu) \), and so \( \text{Res}(\psi(1/u)) = (\pi/q) \text{Res}(1/u) \). It follows that \( \text{Res} \circ \psi = (\pi/q) \text{Res} \). Finally we have \( \text{Res}(\varphi_q(z)) = (q/\pi) \text{Res}(\psi(\varphi_q(z))) = (q/\pi) \text{Res}(z) \) for any \( z \in \mathcal{R}_L \). In other words, \( \text{Res} \circ \varphi_q = (q/\pi) \text{Res} \). \( \square \)

Using \( \text{Res} \) we can define a pairing \( \{ \cdot, \cdot \} : \mathcal{R}_L \times \mathcal{R}_L \to L \) by \( \{ f, g \} = \text{Res}(fg) \).

**Proposition 2.14.** The pairing \( \{ \cdot, \cdot \} \) is perfect and induces a continuous isomorphism from \( \mathcal{R}_L \) to its dual. Moreover we have

\[
\{ \sigma_a(f), \sigma_a(g) \} = a^{-1} \{ f, g \}, \quad \{ \varphi_q(f), \varphi_q(g) \} = \frac{q}{\pi} \{ f, g \}, \quad \{ f, \psi(g) \} = \frac{\pi}{q} \{ \varphi_q(f), g \}.
\]

**Proof.** The first assertion follows from [Colmez 2010d, Remark I.1.5]; the formulas from Proposition 2.13. \( \square \)
3. Operators on $\mathcal{R}_C$

3A. The operator $\psi$ on $\mathcal{R}_C$. First we define $\mathcal{R}_C$. For any $r \geq 0$, let

$$\mathcal{E}_{C}^{[0,r]} := \mathcal{E}_{C}^{[0,r]} \hat{\otimes}_F \mathbb{C}_p$$

be the topological tensor product, i.e., the Hausdorff completion of the projective tensor product $[\mathcal{E}_{C}^{[0,r]} \otimes_F \mathbb{C}_p]$ (see [Schneider 2002]). Then $\mathcal{E}_{C}^{[0,r]}$ is the ring of Laurent series $f = \sum_{i \in \mathbb{Z}} a_i u_F^i$ with coefficients in $\mathbb{C}_p$ that are convergent on the annulus $0 < v_p(u_F) \leq r$. We also write $\mathcal{R}_C^+$ for $\mathcal{E}_{C}^{[0,+,\infty]}$. Then we define $\mathcal{R}_C$ to be the inductive limit $\lim_{r \to 0} \mathcal{E}_{C}^{[0,r]}$.

We recall how the $p$-adic Fourier theory of [Schneider and Teitelbaum 2001] shows that $\mathcal{R}_C^+$ is isomorphic to the ring $\mathcal{D}(\mathcal{O}_F, \mathbb{C}_p)$ of $\mathbb{C}_p$-valued locally $F$-analytic distributions on $\mathcal{O}_F$. From that reference we know that there exists a rigid analytic group variety $\mathfrak{X}$ such that $\mathfrak{X}(L)$, for any extension $L \subseteq \mathbb{C}_p$ of $F$, is the set of locally $F$-analytic characters. For $\lambda \in \mathcal{D}(\mathcal{O}_F, L)$, put $F_{\lambda}(\chi) = \lambda(\chi)$, $\chi \in \mathfrak{X}(L)$. Then $F_{\lambda}$ is in $\mathcal{O}(\mathfrak{X}/L)$, and the map $\mathcal{D}(\mathcal{O}_F, L) \rightarrow \mathcal{O}(\mathfrak{X}/L)$, $\lambda \mapsto F_{\lambda}$, is an isomorphism of $L$-Fréchet algebras.

Let $\mathcal{F}'$ be the $p$-divisible group dual to $\mathcal{F}$ and $T \mathcal{F}'$ the Tate module of $\mathcal{F}'$. Then $T \mathcal{F}'$ is a free $\mathcal{O}_F$-module of rank 1; the Galois action on $T \mathcal{F}'$ is given by the continuous character $\tau := \chi_{\text{cyc}} \cdot \chi_f^{-1}$, where $\chi_{\text{cyc}}$ is the cyclotomic character. By Cartier duality, we obtain a Galois equivariant pairing $\langle \cdot , \cdot \rangle : \mathcal{F}(\mathbb{C}_p) \otimes_{\mathcal{O}_F} T \mathcal{F}' \rightarrow B_1(\mathbb{C}_p)$, where $B_1(\mathbb{C}_p)$ is the multiplicative group $\{ z \in \mathbb{C}_p : |z - 1| < 1 \}$. Fixing a generator $t'$ of $T \mathcal{F}'$, we obtain a map $\mathcal{F}(\mathbb{C}_p) \rightarrow B_1(\mathbb{C}_p)$. As a formal series, this morphism can be written as $\beta_{\mathcal{F}}(X) := \exp(\Omega \log_{\mathcal{F}}(X))$ for some $\Omega \in \mathbb{C}_p$, and it lies in $1 + X \mathcal{O}_{C_\mathcal{F}}[[X]]$. Moreover, we have

$$v_p(\Omega) = \frac{1}{p - 1} - \frac{1}{(q - 1)e_F}$$

(see the appendix of [Schneider and Teitelbaum 2001] or [Colmez 1993]) and $\sigma(\Omega) = \tau(\sigma)\Omega$ for all $\sigma \in G_F$. Using $\langle \cdot , \cdot \rangle$ we obtain an isomorphism of rigid analytic group varieties

$$\kappa : \mathcal{F}(\mathbb{C}_p) \tilde{ightarrow} \mathfrak{X}(\mathbb{C}_p), \quad z \mapsto \kappa_z(i) := \langle t', [i]_{\mathcal{F}}(z) \rangle = \beta_{\mathcal{F}}([i]_{\mathcal{F}}(z)).$$

Passing to global sections, we obtain the desired isomorphism

$$\mathcal{D}(\mathcal{O}_F, \mathbb{C}_p) \cong \mathcal{O}(\mathfrak{X}/\mathbb{C}_p) \cong \mathcal{R}_C^+.$$

We extend $\varphi_q$, $\psi$ and the $\Gamma$-action $\mathbb{C}_p$-linearly and continuously to $\mathcal{R}_C$. By continuity we have $\psi(\varphi_q(f)g) = f \psi(g)$ for any $f, g \in \mathcal{R}_C$. All these actions keep $\mathcal{R}_C^+$ invariant.
Lemma 3.1. We have
\[
\sigma_a(\beta_{\mathcal{F}}([i]_{\mathcal{F}})) = \beta_{\mathcal{F}}([ai]_{\mathcal{F}}),
\]
\[
\varphi_q(\beta_{\mathcal{F}}([i]_{\mathcal{F}})) = \beta_{\mathcal{F}}([\pi i]_{\mathcal{F}}),
\]
\[
\psi(\beta_{\mathcal{F}}([i]_{\mathcal{F}})) = \begin{cases} 
0 & \text{if } i \not\in \pi \mathcal{O}_F, \\
\beta_{\mathcal{F}}([i/\pi]_{\mathcal{F}}) & \text{if } i \in \pi \mathcal{O}_F,
\end{cases}
\]
\[
\partial(\beta_{\mathcal{F}}([i]_{\mathcal{F}})) = i\Omega\beta_{\mathcal{F}}([i]_{\mathcal{F}}).
\]

Proof. The formulae for \(\sigma_a\) and \(\varphi_q\) are obvious. The formula for \(\partial\) follows from
\[
\partial \exp(i\Omega \log_{\mathcal{F}}(u_{\mathcal{F}})) = \exp(i\Omega \log_{\mathcal{F}}(u_{\mathcal{F}})) \cdot \partial(i\Omega \tau_{\mathcal{F}}) = i\Omega \exp(i\Omega \log_{\mathcal{F}}(u_{\mathcal{F}})).
\]

If \(i \in \pi \mathcal{O}_F\), then \(\psi(\beta_{\mathcal{F}}([i]_{\mathcal{F}})) = \psi \circ \varphi_q(\beta_{\mathcal{F}}([i/\pi]_{\mathcal{F}})) = \beta_{\mathcal{F}}([i/\pi]_{\mathcal{F}})\). For any \(i \not\in \pi \mathcal{O}_F\), we have
\[
\psi(\beta_{\mathcal{F}}([i]_{\mathcal{F}})) = \frac{1}{q} \varphi_q^{-1} \left( \sum_{\eta \in \ker[\pi]_{\mathcal{F}}} \beta_{\mathcal{F}}([i]_{\mathcal{F}})(u_{\mathcal{F}} + \tau_{\mathcal{F}} \eta) \right)
= \frac{1}{q} \varphi_q^{-1} \left( \beta_{\mathcal{F}}([i]_{\mathcal{F}}) \sum_{\eta \in \ker[\pi]_{\mathcal{F}}} \beta_{\mathcal{F}}([i]_{\mathcal{F}})(\eta) \right) = 0
\]
because \([\beta_{\mathcal{F}}([i]_{\mathcal{F}})(\eta) : \eta \in \ker[\pi]_{\mathcal{F}}) = [\beta_{\mathcal{F}}(\eta) : \eta \in \ker[\pi]_{\mathcal{F}}]\) take values in the set of \(p\)-th roots of unity and each of these \(p\)-th roots of unity appears \(q/p\) times. □

The isomorphism \(\mathcal{R}_{\mathbb{C}_p}^+ \cong \mathfrak{D}(\mathcal{O}_F, \mathbb{C}_p)\) transfers the actions of \(\varphi_q\), \(\psi\) and \(\Gamma\) to \(\mathfrak{D}(\mathcal{O}_F, \mathbb{C}_p)\).

Lemma 3.2. For any \(\mu \in \mathfrak{D}(\mathcal{O}_F, \mathbb{C}_p)\), we have
\[
\sigma_a(\mu)(f) = \mu(f(a \cdot)), \quad \varphi_q(\mu)(f) = \mu(f(\pi \cdot)).
\]

Proof. Note that the action of \(\varphi_q\) and \(\Gamma\) on \(\mathcal{R}_{\mathbb{C}_p}^+\) comes, by passing to global sections, from the \((\varphi_q, \Gamma)\)-action on \(\mathcal{F}(\mathbb{C}_p)\) with \(\varphi_q = [\pi]_{\mathcal{F}}\) and \(\sigma_a = [a]_{\mathcal{F}}\). The isomorphism \(\kappa\) transfers the action to \(\mathfrak{X}(\mathbb{C}_p): \varphi_q(\chi)(x) = \chi(\pi x)\) and \(\sigma_a(\chi)(x) = \chi(a x)\). Passing to global sections yields what we want. □

Lemma 3.3. The family \((\beta_{\mathcal{F}}([i]_{\mathcal{F}}))_{i \in \mathcal{O}_F/\pi}\) is a basis of \(\mathcal{R}_{\mathbb{C}_p}\) over \(\varphi_q(\mathcal{R}_{\mathbb{C}_p})\). Moreover, if
\[
f = \sum_{i \in \mathcal{O}_F/\pi} \beta_{\mathcal{F}}([i]_{\mathcal{F}}) \varphi_q(f_i),
\]
then the terms of the sum do not depend on the choice of the liftings \(i\), and
\[
f_i = \psi(\beta_{\mathcal{F}}([-i]_{\mathcal{F}}) f).
Proof. What we need to show is that

$$f = \sum_{i \in \mathcal{O}_F/\pi} \beta_F([i]_F) \cdot \varphi_q \circ \psi(\beta_F([-i]_F)f)$$

(3-2)

for all $f \in \mathcal{R}_{\mathbb{C}_p}$. Indeed, (3-2) implies that $\{\beta_F([i]_F)\}_{i \in \mathcal{O}_F/\pi}$ generate $\mathcal{R}_{\mathbb{C}_p}$ over $\varphi_q(\mathcal{R}_{\mathbb{C}_p})$. On the other hand, if

$$f = \sum_{i \in \mathcal{O}_F/\pi} \beta_F([i]_F) \varphi_q(f_i),$$

using (3-1) we obtain $f_i = \psi(\beta_F([-i]_F)f)$, which implies the linear independence of $\{\beta_F([i]_F)\}_{i \in \mathcal{O}_F/\pi}$ over $\varphi_q(\mathcal{R}_{\mathbb{C}_p})$. As the map

$$f \mapsto \sum_{i \in \mathcal{O}_F/\pi} \beta_F([i]_F) \cdot \varphi_q \circ \psi(\beta_F([-i]_F)f)$$

is $\varphi_q(\mathcal{R}_{\mathbb{C}_p})$-linear and continuous, we only need to prove (3-2) for a subset that topologically generates $\mathcal{R}_{\mathbb{C}_p}$ over $\varphi_q(\mathcal{R}_{\mathbb{C}_p})$. For example, $\{u^i_F\}_{0 \leq i \leq q-1}$ is such a subset. So it is sufficient to prove (3-2) for $f \in \mathcal{R}_{\mathbb{C}_p}^+$. For any $i \in \mathcal{O}_F$, let $\delta_i$ be the Dirac distribution such that $\delta_i(f) = f(i)$. Then $\kappa^*(\delta_i) = \beta_F([i]_F)$. Indeed, we have

$$\kappa^*(\delta_i)(z) = \delta_i(z) = \kappa(z) = \beta_F([i]_F(z)).$$

It is easy to see that $(\delta_i)_{i \in \mathcal{O}_F/\pi}$ is a basis of $\mathcal{D}(\mathcal{O}_F, \mathbb{C}_p)$ over $\varphi_q(\mathcal{D}(\mathcal{O}_F, \mathbb{C}_p))$. Thus every $f \in \mathcal{R}_{\mathbb{C}_p}^+$ can be written uniquely in the form $f = \sum_{i \in \mathcal{O}_F/\pi} \beta_F([i]_F) \varphi_q(f_i)$ with $f_i \in \mathcal{R}_{\mathbb{C}_p}^+$. As observed above, from (3-1) we deduce that $f_i = \psi(\beta_F([-i]_F)f)$.

Next we define operators $\text{Res}_U$, analogous to the operators defined in [Colmez 2010d].

For any $f \in \mathcal{R}_{\mathbb{C}_p}$, $i \in \mathcal{O}_F$ and integer $m \geq 0$, put

$$\text{Res}_{i+\pi^m \mathcal{O}_F}(f) = \beta_F([i]_F) \varphi_q^m \circ \psi^m(\beta_F([-i]_F)f).$$

Lemma 3.3 says that

$$f = \sum_{i \in \mathcal{O}_F/\pi} \text{Res}_{i+\pi^m \mathcal{O}_F}(f).$$

This implies that the operators $\text{Res}_{i+\pi^m \mathcal{O}_F}$ are well defined (i.e., independent of the choice of $i$ in the ball $i + \pi^m \mathcal{O}_F$). Applying Lemma 3.3 recursively we get

$$f = \sum_{i \in \mathcal{O}_F/\pi^m} \text{Res}_{i+\pi^m \mathcal{O}_F}(f).$$

Finally, if $U$ is a compact open subset of $\mathcal{O}_F$, it is a finite disjoint union of balls $i_k + \pi^{m_k} \mathcal{O}_F$. Define $\text{Res}_U = \sum_k \text{Res}_{i_k+\pi^{m_k} \mathcal{O}_F}$. The map $\text{Res}_U : \mathcal{R}_{\mathbb{C}_p} \to \mathcal{R}_{\mathbb{C}_p}$ does
not depend on the choice of these balls, and we have \( \text{Res}_{\mathcal{O}_F} = 1 \), \( \text{Res}_\emptyset = 0 \) and \( \text{Res}_{\mathcal{O}_F \cup U'} + \text{Res}_{\mathcal{O}_F \cap U'} = \text{Res}_U + \text{Res}_{U'} \).

**3B. The operator \( m_\alpha \).** Let \( \alpha : \mathcal{O}_F \to \mathbb{C}_p \) be a locally \((F\!)-\)analytic function. In this subsection, we define an operator \( m_\alpha : \mathcal{R}_{\mathbb{C}_p} \to \mathcal{R}_{\mathbb{C}_p} \) similar to the one defined in [Colmez 2010c, V.2].

Since \( \alpha \) is a locally analytic function on \( \mathcal{O}_F \), there is an integer \( m \geq 0 \) such that

\[
\alpha(x) = \sum_{n=0}^{+\infty} a_{i,n} (x - i)^n \quad \text{for all } x \in i + \pi^m \mathcal{O}_F,
\]

with \( a_{i,n} = (1/n!)(d^n/dx^n)\alpha(x)|_{x=i} \). Let \( \ell \geq m \) be an integer. Define

\[
m_\alpha(f) = \sum_{\tilde{i} \in \mathcal{O}_F/\pi^\ell} \beta_{\tilde{i}}([i]_{\mathcal{T}}) \left( \varphi_q^\ell \circ \left( \sum_{n=0}^{+\infty} a_{i,n} \pi^{\ell n} \Omega^{-n} \partial^n \right) \circ \psi^\ell \right) \left( \beta_{\tilde{i}}([-i]_{\mathcal{T}}) \cdot f \right).
\]

(Formally, this definition can be seen as saying that \( m_\alpha = \alpha(\Omega^{-1} \partial) \)). According to Lemmas 2.6, 2.7 and 2.10, if \( r < 1/(q^{\ell-1}(q - 1)e_F) \) then we have

\[
v^{[s,r]}((\varphi_q^\ell \circ \Omega^{-n} \partial^n \circ \psi^\ell)(g)) \geq -nq^\ell r - nv_p(\Omega) + v^{[s,r]}(g) - \ell v_p(q),
\]

and thus \( \sum_{n=0}^{+\infty} a_{n,i} \pi^{\ell n} (\varphi_q^\ell \circ \Omega^{-n} \partial^n \circ \psi^\ell)(g) \) converges when \( \ell \) and \( r \) satisfy

\[
\frac{\ell}{e_F} - q^{\ell}r - \frac{1}{p - 1} + \frac{1}{(q - 1)e_F} \geq \frac{m}{e_F}.
\]

If we choose \( \ell > m + e_F/(p - 1) - 1/(q - 1) \) and \( r \) close enough to 0, then this condition is satisfied. Hence, we have indeed defined a continuous operator \( m_\alpha : \mathcal{R}_{\mathbb{C}_p} \to \mathcal{R}_{\mathbb{C}_p} \).

Now, let us prove that \( m_\alpha(f) \) neither depend on the choice of \( \ell \), nor on that of the liftings \( i \) for \( \tilde{i} \in \mathcal{O}_F/\pi^\ell \). By linearity and continuity, we may assume that \( f = 1_{i + \pi^m \mathcal{O}_F}(x - i)^k \). Note that we have

\[
a_{i + \pi^m v, n} = \binom{k}{n} \pi^{(k-n)m} v^{k-n}.
\]

It suffices to show that

\[
\sum_{\tau \in \mathcal{O}_F/\pi^{\ell-m}} \beta_{\tau}([\pi^m v]_{\mathcal{T}}) \left( \varphi_q^\ell \circ \left( \sum_{n=0}^{k} a_{i + \pi^m v, n} \pi^{\ell n} \Omega^{-n} \partial^n \right) \circ \psi^\ell \right) \left( \beta_{\tau}([-\pi^m v]_{\mathcal{T}}) \cdot f \right)
\]

\[
= \left( \varphi_q^m \circ (\pi^{mk} \Omega^{-k} \partial^k) \circ \psi^m \right) f,
\]

and for it is sufficient to prove that
\[
\sum_{\tau \in \mathcal{O}_F / \pi^{\ell+m}} \beta_\tau([v]_F) \left( \varphi^{m} - m \circ \left( \sum_{n=0}^{k} a_i + \pi^{m} \Omega^{n} \partial^n \right) \circ \psi^{m} \right) (\beta_\tau([-v]_F) \cdot f)
= \pi^m \Omega^{-k} \partial^k f.
\]

As
\[
\sum_{n=0}^{k} a_i + \pi^{m} \Omega^{n} \partial^n = \sum_{n=0}^{k} \left[ \begin{array}{c} k \\ n \end{array} \right] \pi^{(k-n)m} v^{k-n} \cdot \pi^{\ell n} \Omega^{-n} \partial^n
= \pi^m \left( \pi^{\ell - m} \Omega^{-1} \partial + v \right)^k,
\]
it suffices to prove that
\[
\Omega^{-k} \partial^k f = \sum_{\tau \in \mathcal{O}_F / \pi^{\ell+m}} \beta_\tau([v]_F) \left( \varphi^{m} - m \circ \left( \pi^{\ell-m} \Omega^{-1} \partial + v \right)^k \circ \psi^{m} \right) (\beta_\tau([-v]_F) \cdot f).
\]

Since \((\pi^{\ell-m} \Omega^{-1} \partial + v)^k \circ \psi^{m} = \psi^{m} \circ (\Omega^{-1} \partial + v)^k\) and
\[
(\Omega^{-1} \partial + v)(\beta_\tau([-v]_F) \cdot f) = \beta_\tau([-v]_F) \Omega^{-1} \partial f
\]
(which follows from Lemma 3.1), the problem reduces to proving
\[
f = \sum_{\tau \in \mathcal{O}_F / \pi^{\ell-m}} \beta_\tau([v]_F) \left( \varphi^{m} - m \circ \psi^{m} \right) (\beta_\tau([-v]_F) \cdot f).
\]

But this can be deduced from Lemma 3.1 and Lemma 3.3.

**Lemma 3.4.** If \(a, \beta : \mathcal{O}_F \to \mathbb{C}_p\) are locally analytic functions, then \(m_a \circ m_\beta = m_{a\beta}\).

**Proof.** We can choose \(\ell\) sufficiently large, so that the same value can be used to define \(m_a(f)\) and \(m_\beta(f)\). Since \(\psi^{\ell} \circ \varphi^{\ell} = 1\), the equality in the lemma reduces to the expression of the product of two power series. \(\square\)

**Lemma 3.5.**

- \(m_1 = \text{id}\).
- If \(U\) is a compact open subset of \(\mathcal{O}_F\), then \(\text{Res}_U = m_{1_U}\).
- If \(\lambda \in \mathbb{C}_p\), then \(m_{\lambda a} = \lambda m_a\).
- \(\varphi_q \circ m_a = m_{x \mapsto 1_{\pi} \mathcal{O}_F (x) a (\pi^{-1} x)} \circ \varphi_q\).
- \(\psi \circ m_a = m_{x \mapsto a (\pi^{-1} x)} \circ \psi\).
- For any \(a \in \mathcal{O}_F^\times\), we have \(\sigma_a \circ m_a = m_{x \mapsto a (\pi^{-1} x)} \circ \sigma_a\).
- \(\mathcal{H}_c^+\) is stable under \(m_a\).

**Proof.** These are easy consequences of the definition of \(m_a\). \(\square\)
Remark 3.6. The notation \( m_\alpha \) stands for “multiply by \( \alpha \)”: for any \( \mu \in \mathcal{D}(\mathcal{O}_F, \mathbb{C}_p) \) we have \( m_\alpha \kappa^*(F_\mu) = \kappa^*(F_{\alpha \mu}) \), where \( \alpha \mu \) is the distribution such that \( (\alpha \mu)(f) = \mu(\alpha f) \) for all locally \( F \)-analytic function \( f \).

The operator \( m_\alpha \) has been defined over \( \mathcal{R}_{\mathbb{C}_p} \) using a period \( \epsilon \in \mathbb{C}_p \) that is transcendental over \( F \). However, in some cases, it is possible to construct related operators over \( \mathcal{R}_L \), for \( L \) smaller than \( \mathbb{C}_p \). This is done using the following lemma.

**Lemma 3.7.** Let \( \sigma \) be in \( G_L \). Consider the action of \( \sigma \) over \( \mathcal{R}_{\mathbb{C}_p} \) given by

\[
 f^\sigma(u_F) = \sum_{n \in \mathbb{Z}} \sigma(a_n) u_F^n \quad \text{if} \quad f(u_F) = \sum_{n \in \mathbb{Z}} a_n u_F^n \in \mathcal{R}_{\mathbb{C}_p}.
\]

Then \( m_\alpha(f)^\sigma = m_\beta(f^\sigma) \) for \( \beta(x) = \sigma\left( \frac{\chi_F(\sigma)}{\chi_{\mathbb{G}_m}(\sigma)} x \right) \).

**Proof.** This can be deduced easily from the definition of \( m_\alpha \) and the action of \( \sigma \) on \( \Omega \). \( \square \)

3C. The \( L[\Gamma] \)-module \( \mathcal{R}_L(\delta)^\psi = 0 \). Let \( \delta : F^\times \to L^\times \) be a locally \( F \)-analytic character. Then the map \( x \mapsto \mathbb{1}_{\mathcal{O}_F^\times}(x) \delta(x) \) is locally analytic on \( \mathcal{O}_F \). Thus, we have an operator \( m_{\mathbb{1}_{\mathcal{O}_F^\times}}(\delta) \) on \( \mathcal{R}_{\mathbb{C}_p} \).

**Lemma 3.8.** Let \( f \) be in \( \mathcal{R}_L \). If \( m_{\mathbb{1}_{\mathcal{O}_F^\times}}(\delta)(f) = \sum_{n \in \mathbb{Z}} a_n u_F^n \in \mathcal{R}_{\mathbb{C}_p} \), the coefficients \( a_n \) are all on the same line of the \( L \)-vector space \( \mathbb{C}_p \). Moreover, this line does not depend on \( f \).

**Proof.** Let \( \sigma \) be in \( G_L \). From Lemma 3.7 and Lemma 3.5 we see that

\[
 m_{\mathbb{1}_{\mathcal{O}_F^\times}}(\delta)(f)^\sigma = \delta\left( \frac{\chi_F(\sigma)}{\chi_{\mathbb{G}_m}(\sigma)} \right) m_{\mathbb{1}_{\mathcal{O}_F^\times}}(\delta)(f),
\]

and thus \( \sigma(a_n) = \delta\left( \frac{\chi_F(\sigma)}{\chi_{\mathbb{G}_m}(\sigma)} \right) a_n \) for all \( n \).

The Ax–Sen–Tate theorem (see [Ax 1970] or [Le Borgne 2010], for example) says that \( \mathbb{C}_p^{G_L} = L \). Hence,

\[
 \left\{ z \in \mathbb{C}_p : \sigma(z) = \delta\left( \frac{\chi_F(\sigma)}{\chi_{\mathbb{G}_m}(\sigma)} \right) z \text{ for all } \sigma \in G_L \right\}
\]

is an \( L \)-vector subspace of \( \mathbb{C}_p \) with dimension 0 or 1, which proves the lemma. \( \square \)

Since

\[
 m_{\mathbb{1}_{\mathcal{O}_F^\times}}(\delta) \circ m_{\mathbb{1}_{\mathcal{O}_F^\times}}^{-1} = \text{Res}_{\mathcal{O}_F^\times} = 1 - \varphi_q \circ \psi
\]

is not null, there is a unique \( L \)-line in \( \mathbb{C}_p \) (which depends only on \( \delta \)) in which all the coefficients of the series \( m_{\mathbb{1}_{\mathcal{O}_F^\times}}(\delta)(f) \), for \( f \in \mathcal{R}_L \), lie. Choose some nonzero \( a_\delta \) on this line.
Since \( \varphi_q \circ \psi \circ m_{1_{\mathcal{O}_F}} \delta = m_{1_{\mathcal{O}_F}} \delta \) and \( \varphi_q \) is injective, \( m_{1_{\mathcal{O}_F}} \delta(f) \) is in \( \mathcal{R}_{\mathcal{C}_p}^L \).

**Lemma 3.9.** Define

\[
M_\delta : \mathcal{R}_L^\psi = 0 \to \mathcal{R}_L^\psi = 0, \quad f \mapsto a_\delta^{-1} m_{1_{\mathcal{O}_F}} \delta(f).
\]

(These maps are defined up to homothety, with ratio in \( L \), because of the choice of constants \( a_\delta \)).

- \( M_1 \) is a homothety (with ratio in \( L^\times \)) of \( \mathcal{R}_L^\psi = 0 \).
- \( M_{\delta_1} \circ M_{\delta_2} = M_{\delta_1 \delta_2} \), up to homothety.
- \( M_\delta \) is a bijection, and its inverse is \( M_{\delta^{-1}} \) up to homothety.
- For all \( \gamma \in \Gamma \), we have \( \delta(\gamma) \gamma \circ M_\delta = M_\delta \circ \gamma \).
- \( (\mathcal{R}_L^+) \psi = 0 \) is stable under \( M_\delta \).

**Proof.** This follows from Lemma 3.5 and the equalities

\[
\text{Im Res}_{\mathcal{O}_F^\times} = \text{Ker Res}_{\mathcal{O}_F} = \mathcal{R}_{\mathcal{C}_p}^L.
\]

If \( \delta \) is in \( J_{\text{an}}(L) \), we put \( \mathcal{R}_L^+(\delta) = \mathcal{R}_L(\delta) / \mathcal{R}_L^+(\delta) \). Since \( \mathcal{R}_L^+(\delta) \) is \( \varphi_q, \psi, \Gamma \)-stable, \( \mathcal{R}_L^-(\delta) \) also has \( \varphi_q, \psi, \Gamma \)-actions.

**Lemma 3.10.** We have an exact sequence

\[
0 \to \mathcal{R}_L^+(\delta) \psi = 0 \to \mathcal{R}_L(\delta) \psi = 0 \to \mathcal{R}_L^-(\delta) \psi = 0 \to 0.
\]

**Proof.** This follows from the snake lemma and the surjectivity of the map

\[
\psi : \mathcal{R}_L^+(\delta) \to \mathcal{R}_L^+(\delta).
\]

Observe that \( \mathcal{R}_L(\delta) \psi = 0 = \mathcal{R}_L^+(\delta) \psi = 0 \cdot e_\delta \) and \( \mathcal{R}_L^+(\delta) \psi = 0 = (\mathcal{R}_L^+) \psi = 0 \cdot e_\delta \). As \( \psi \) commutes with \( \Gamma \), \( \mathcal{R}_L(\delta) \psi = 0 \), \( \mathcal{R}_L^+(\delta) \psi = 0 \) and \( \mathcal{R}_L^-(\delta) \psi = 0 \) are all \( \Gamma \)-invariant.

**Proposition 3.11.** Let \( \delta_1 \) and \( \delta_2 \) be two locally \( F \)-analytic characters \( F^\times \to L^\times \).
Then as \( L[\Gamma] \)-modules, \( \mathcal{R}_L(\delta_1) \psi = 0 \) is isomorphic to \( \mathcal{R}_L(\delta_2) \psi = 0 \), \( \mathcal{R}_L^+(\delta_1) \psi = 0 \) is isomorphic to \( \mathcal{R}_L^+(\delta_2) \psi = 0 \), and \( \mathcal{R}_L^-(\delta_1) \psi = 0 \) is isomorphic to \( \mathcal{R}_L^-(\delta_2) \psi = 0 \).

**Proof.** All of the isomorphisms in question are induced by \( M_{\delta_1^{-1} \delta_2} \).

**Proposition 3.12.** The map \( \partial \) induces \( \Gamma \)-equivariant isomorphisms

\[
(\mathcal{R}_L(\delta)) \psi = 0 \to (\mathcal{R}_L(x \delta)) \psi = 0,
(\mathcal{R}_L^+(\delta)) \psi = 0 \to (\mathcal{R}_L^+(x \delta)) \psi = 0,
(\mathcal{R}_L^-(\delta)) \psi = 0 \to (\mathcal{R}_L^-(x \delta)) \psi = 0.
\]

**Proof.** We first show that the maps in question are bijective. For this we only need to consider the case of \( \delta = 1 \). Since \( \text{Ker} \partial = L \), \( \partial \) is injective on \( \mathcal{R}_L^\psi = 0 \). For any
As before, let $z \in R_L^{\psi=0}$, $\text{Res}(z) = (q/\pi)\text{Res}(\psi(z)) = 0$. Thus by Proposition 2.12 there exists $z' \in R_L$ such that $\partial z' = z$. As $\partial(\psi(z')) = \frac{1}{2}\psi(\partial z') = 0$, we have $\psi(z) = c$ for some $c \in L$. Then $z - c \in R_L^{\psi=0}$ and $\partial(z' - c) = z$. This shows that the map $R_L^{\psi=0} \rightarrow R_L^{\psi=0}$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0$, $\psi(z)$ is bijective. It is clear that, for any $\psi(z) = 0
Proof. Note that $T_1 = (\mathcal{R}_L^+/\mathcal{R}_L^{+1} t_F)^\Gamma = 1, \psi = 0$. As $\mathcal{R}_L^+$ is a Fréchet–Stein algebra, from the decomposition of the ideal $(t_F)$ given by Corollary 1.2 we obtain an isomorphism

$$ j : \mathcal{R}_L^+/\mathcal{R}_L^{+1} t_F \to \mathcal{R}_L^+/((\pi)]_F(u_F)) \times \prod_{n \geq 1} \mathcal{R}_L^+/(\varphi^n_q(Q)). \tag{3-4} $$

The operator $\psi$ induces maps

$$ \psi_0 : \mathcal{R}_L^+/((\pi)]_F(u_F)) \to \mathcal{R}_L^+/\mathcal{R}_L^{+1} u_F \quad \text{and} \quad \psi_n : \mathcal{R}_L^+/(\varphi^n_q(Q)) \to \mathcal{R}_L^+/(\varphi^{n-1}_q(Q)), $$

for $n \geq 1$. Thus $j((\mathcal{R}_L^+/\mathcal{R}_L^{+1} t_F)^\Gamma = 1, \psi = 0)$ is exactly the subset of the codomain of (3-4) consisting of $(y_n)_{n \geq 0}$ such that $y_0 \in (\mathcal{R}_L^+/((\pi)]_F(u_F)))^\Gamma$, $\psi_0(y_0) = 0$, and

$$ y_n \in (\mathcal{R}_L^+/(\varphi^n_q(Q)))^\Gamma, \quad \psi_n(y_n) = 0, \quad \text{for all } n \geq 1. $$

If $n \geq 1$, then $\mathcal{R}_L^+/(\varphi^n_q(Q))$ is a finite extension of $F$ and the action of $\Gamma$ factors through the whole Galois group of this extension. Thus $(\mathcal{R}_L^+/(\varphi^n_q(Q)))^\Gamma = F$ and $(\mathcal{R}_L^+/(\varphi^n_q(Q)))^\Gamma = L$. Since $\psi_n(a) = a$ for any $a \in L$, $(\mathcal{R}_L^+/(\varphi^n_q(Q)))^\Gamma \cap \ker(\psi_n) = 0$ for any $n \geq 1$. Similarly, $((\pi)]_F(u_F)))^\Gamma = (\mathcal{R}_L^+/((\pi)]_F(u_F)))^\Gamma \times (\mathcal{R}_L^+/(Q))^\Gamma$ has dimension 2 over $L$. As $\psi_0(1) = 1$ and the image $\mathcal{R}_L^+/\mathcal{R}_L^{+1} u_F$ of $\psi_0$ has dimension 1 over $L$, the kernel of $\psi_0|_{(\mathcal{R}_L^+/(\pi]_F(u_F)))^\Gamma}$ is of dimension 1. It follows that $T_1 = (\mathcal{R}_L^+/\mathcal{R}_L^{+1} t_F)^\Gamma = 1, \psi = 0$ is of dimension 1.

\textbf{Corollary 3.18.} \quad \dim_L S_1 = 1.

\textbf{Proof.} The map $\nabla$ injects $S_1$ into $T_1$ with image of dimension 1. \hfill \square

\textbf{Remark 3.19.} If $z \in T_1$ is nonzero, then any lifting $\tilde{z} \in \mathcal{R}_L^+$ of $z$ is not in $u_F\mathcal{R}_L^+$, or equivalently $\tilde{z}|_{u_F} \neq 0$. We only need to verify this for the special Lubin–Tate group. In this case, $\mathcal{R}_L^+/((\pi)]_F(u_F)) = \bigoplus_{i=1}^{q-1} Lu_F^i$. We have

$$ (\mathcal{R}_L^+/(\pi)]_F(u_F)))^\Gamma = L \oplus Lu_F^{q-1}. $$

Indeed, an element of $\mathcal{R}_L^+/((\pi)]_F(u_F))$ is fixed by $\Gamma$ if and only if it is fixed by the operators $z \mapsto \sigma_\xi(z)$ with $\xi \in \mu_{q-1}$; but $\sigma_\xi(u_F) = [\xi]_F(u_F) = \xi u_F$, so $\sigma_\xi(u_F^i) = \xi^i u_F$ for any $i \in \mathbb{N}$. Then $(\mathcal{R}_L^+/((\pi)]_F(u_F)))^\Gamma = 1, \psi = 0 = L \cdot (u_F^{q-1} - (1 - q)\pi/q)$.

\textbf{Proposition 3.20.} For any $\delta \in \mathcal{J}_{an}(L)$, $\dim_L S_{\delta} = \dim_L T_{\delta} = 1$ and the map $\nabla_{\delta}$ is an isomorphism.

\textbf{Proof.} Use Corollary 3.14, Lemma 3.16, Lemma 3.17 and Corollary 3.18. \hfill \square

\section{4. Cohomology theories for $(\varphi_q, \Gamma)$-modules}

For a $(\varphi_q, \Gamma)$-module $D$ over $\mathcal{R}_L$, the $(\varphi_q, \Gamma)$-module structure induces an action of the semigroup $G^+ := \varphi^\mathbb{N} \times \Gamma$ on $D$. Following [Colmez 2010a] we define $H^\bullet(D)$
as the cohomology of the semigroup $G^+$. Let $C^\bullet(G^+, D)$ be the complex

$$0 \to C^0(G^+, D) \xrightarrow{d_1} C^1(G^+, D) \xrightarrow{d_2} \ldots,$$

where $C^0(G^+, D) = D$, $C^n(G^+, D)$ for $n \geq 1$ is the set of continuous functions from $(G^+)^n$ to $D$, and $d_{n+1}$ is the differential

$$d_{n+1}c(g_0, \ldots, g_n) =
$$

$$g_0 \cdot c(g_1, \ldots, g_n) + \sum_{i=1}^{n-1} (-1)^{i+1} c(g_0, \ldots, g_i g_{i+1}, \ldots, g_n) + (-1)^{n+1} c(g_0, \ldots, g_{n-1}).$$

Then $H^i(D) = H^i(C^\bullet(G^+, D))$.

If $D_1$ and $D_2$ are $(\varphi_q, \Gamma)$-modules over $\mathcal{R}_L$, we use $\text{Ext}(D_1, D_2)$ to denote the set, in fact an $L$-vector space, of extensions of $D_1$ by $D_2$ in the category of $(\varphi_q, \Gamma)$-modules over $\mathcal{R}_L$.

We construct a natural map $\Theta^D : \text{Ext}(\mathcal{R}_L, D) \to H^1(D)$ for any $(\varphi_q, \Gamma)$-module $D$. Let $\tilde{D}$ be an extension of $\mathcal{R}_L$ by $D$. Let $e \in \tilde{D}$ be a lifting of $1 \in \mathcal{R}_L$. Then $g \mapsto g(e) - e$, $g \in G^+$, is a 1-cocycle, and induces an element of $H^1(D)$ independent of the choice of $e$. Thus we obtain the desired map

$$\Theta^D : \text{Ext}(\mathcal{R}_L, D) \to H^1(D).$$

**Proposition 4.1.** For any $(\varphi_q, \Gamma)$-module $D$ over $\mathcal{R}_L$, $\Theta^D$ is an isomorphism.

**Proof.** Let $\tilde{D}$ be an extension of $\mathcal{R}_L$ by $D$ in the category of $(\varphi_q, \Gamma)$-modules whose image under $\Theta^D$ is zero. Let $e \in \tilde{D}$ be a lifting of $1 \in \mathcal{R}_L$. As the image of $g \mapsto g(e) - e$, $g \in G^+$, in $H^1(D)$ is zero, there exists some $d \in D$ such that $(g - 1)e = (g - 1)d$ for all $g \in G^+$. Then $g(e - d) = e - d$ for all $g \in G^+$. Thus $\tilde{D} = D \oplus \mathcal{R}_L(e - d)$ as a $(\varphi_q, \Gamma)$-module. This proves the injectivity of $\Theta^D$. Next we prove the surjectivity of $\Theta^D$. Given a 1-cocycle $g \mapsto c(g) \in D$, correspondingly we can extend the $(\varphi_q, \Gamma)$-module structure on $D$ to the $\mathcal{R}_L$-module $\tilde{D} = D \oplus \mathcal{R}_Le$ such that $\varphi_q(e) = e + c(\varphi_q)$ and $\gamma(e) = e + c(\gamma)$ for $\gamma \in \Gamma$. □

If $D_1$ and $D_2$ are $\mathcal{O}_F$-analytic $(\varphi_q, \Gamma)$-modules over $\mathcal{R}_L$, we use $\text{Ext}_\text{an}(D_1, D_2)$ to denote the $L$-vector space of extensions of $D_1$ by $D_2$ in the category of $\mathcal{O}_F$-analytic $(\varphi_q, \Gamma)$-modules over $\mathcal{R}_L$. We will introduce another cohomology theory $H^\ast\text{an}(-)$, wherein for any $\mathcal{O}_F$-analytic $(\varphi_q, \Gamma)$-module $D$ the first cohomology group $H^1\text{an}(D)$ coincides with $\text{Ext}_\text{an}(\mathcal{R}_L, D)$.

If $D$ is $\mathcal{O}_F$-analytic, we consider the complex

$$C^\bullet_{\varphi_q, \Gamma}(D) : \quad 0 \to D \xrightarrow{f_1} D \oplus D \xrightarrow{f_2} D \to 0,$$
where \( f_1 : D \to D \oplus D \) is the map \( m \mapsto ((q - 1)m, \nabla m) \) and \( f_2 : D \oplus D \to D \) is \((m, n) \mapsto \nabla m - (q - 1)n \). As \( f_1 \) and \( f_2 \) are \( \Gamma \)-equivariant, \( \Gamma \) acts on the cohomology groups \( H^i_{\psi \nabla}(D) := H^i(C^*_{\psi \nabla}(D)), i = 0, 1, 2 \). Put \( H^i_{\text{an}}(D) := H^i_{\psi \nabla}(D) \Gamma \).

By a simple calculation we obtain
\[
H^0(D) = H^0_{\text{an}}(D) = D^{\psi q = 1, \Gamma = 1}.
\]

Note that \( D^{\psi q = 1} \) is finite-dimensional over \( L \), and so is \( H^0(D) \). If \( D \) is étale and if \( V \) is the \( L \)-linear Galois representation of \( G_F \) attached to \( D \), then
\[
H^0(D) = H^0_{\text{an}}(D) = H^0(G_F, V) = V^{G_F}.
\]

We introduce some convenient notation. Put \( Z^1_{\psi \nabla}(D) := \ker(f_2) \) and \( B^1(D) := \text{im}(f_1) \). For any \((m_1, n_1) \) and \((m_2, n_2) \) in \( Z^1_{\psi \nabla}(D) \), we write \((m_1, n_1) \sim (m_2, n_2) \) if \((m_1 - m_2, n_1 - n_2) \in B^1(D) \). Put
\[
Z^1(D) := \{(m, n) \in Z^1_{\psi \nabla}(D) : (m, n) \sim \gamma(m, n) \text{ for any } \gamma \in \Gamma \}.
\]

Then \( H^1_{\text{an}}(D) = Z^1(D)/B^1(D) \).

Let \( \tilde{D} \) be an \( \mathcal{O}_F \)-analytic extension of \( \mathcal{R}_L \) by \( D \). Let \( e \in \tilde{D} \) be a lifting of \( 1 \in \mathcal{R}_L \). Then \((q - 1)e, \nabla \tilde{D}e \) belongs to \( Z^1(D) \) and induces an element of \( H^1_{\text{an}}(D) \) independent of the choice of \( e \). In this way we obtain a map
\[
\Theta^D_{\text{an}} : \text{Ext}_{\text{an}}(\mathcal{R}_L, D) \to H^1_{\text{an}}(D).
\]

**Theorem 4.2** (\( = \) Theorem 0.1). For any \( \mathcal{O}_F \)-analytic \((\varphi, \Gamma)\)-module \( D \) over \( \mathcal{R}_L \), \( \Theta^D_{\text{an}} \) is an isomorphism.

The proof below is due to the referee and is much simpler than that in our original version.

**Proof.** First we show that \( \Theta^D_{\text{an}} \) is injective. Let \( \tilde{D} \) be an \( \mathcal{O}_F \)-analytic extension of \( \mathcal{R}_L \) by \( D \) whose image under \( \Theta^D_{\text{an}} \) is zero. Let \( e \in \tilde{D} \) be a lifting of \( 1 \in \mathcal{R}_L \). As the image of \((q - 1)e, \nabla \tilde{D}e \) in \( H^1_{\psi \nabla, \nabla}(D) \) is zero, there exists some \( d \in D \) such that \((q - 1)e = (q - 1)d \) and \( \nabla \tilde{D}e = \nabla \tilde{D}d \). Then \( e - d \) is in \( \tilde{D}^{\psi = 1, \nabla = 0} \). The \( \Gamma \)-action on \( \tilde{D}^{\psi = 1, \nabla = 0} \) is locally constant and thus is semisimple. So \( 1 \in \mathcal{R}_L \) has a lifting \( e' \in \tilde{D}^{\psi = 1, \nabla = 0} \) fixed by \( \Gamma \). This proves the injectivity of \( \Theta^D_{\text{an}} \).

Next we prove the surjectivity of \( \Theta^D_{\text{an}} \).

Let \( z \) be in \( H^1_{\text{an}}(D) \) and let \((x, y) \) represent \( z \), so that \( \nabla x = (q - 1)y \). The invariance of \( z \) by \( \Gamma \) ensures the existence of \( y_\sigma \in D \) for each \( \sigma \in \Gamma \) such that \((\sigma - 1)(x, y) = ((q - 1)\sigma y, \nabla y_\sigma) \). As \( y_\sigma \) is unique up to an element of \( D^{\psi = 0, \nabla = 0} \), the 2-cocycle \( y_{\sigma, \tau} = y_{\sigma \tau} - \sigma y_\tau + y_\sigma \) takes values in \( D^{\psi = 0, \nabla = 0} \). If \( z = 0 \), then there exists \( a \in D \) such that \( x = (q - 1)a \) and \( y = \nabla a \). We have \( \nabla(y_\sigma - (\sigma - 1)a) = 0 \).

In other words, we can write \( y_\sigma = (\sigma - 1)a + a_\sigma \) with \( a_\sigma \in D^{\psi = 0, \nabla = 0} \). Then
\[ y_{\sigma, \tau} = a_{\sigma \tau} - \sigma a_{\tau} + a_{\sigma} \] and thus \( y_{*, *} \) is a coboundary. So we obtain a map \( H^1_{an}(D) \to H^2(\Gamma, D^{\phi_q=1, \nabla=0}) \).

We will show that the image of \( z \) by this map is zero. Fix a basis \( \{e_1, \ldots, e_d\} \) of \( D \) over \( \mathbb{R}_L \). Let \( r > 0 \) be sufficiently small such that the matrices of \( \phi_q \) and \( \sigma \in \Gamma \) relative to \( \{e_i\}_{i=1}^d \) are all in \( GL_d(\mathbb{C}^{[0,r]}_L) \). Put \( D^{[0,r]} = \bigoplus_{i=1}^d \mathbb{C}^{[0,r]}_L e_i \); if \( s \in (0, r] \) put \( D^{[s,r]} = \bigoplus_{i=1}^d \mathbb{C}^{[s,r]}_L e_i \). Then \( D^{[0,r]} \) and \( D^{[s,r]} \) are stable by \( \Gamma \). As the matrix of \( \phi_q \) is invertible in \( M_d(\mathbb{C}_L^{[0,r]}) \), \( \{\phi_q(e_i)\}_{i=1}^d \) is also a basis of \( D^{[0,r]} \). Shrinking \( r \) if necessary we may assume that \( \phi_q \) maps \( D^{[s,r]} \to D^{[s/q,r/q]} \); we may also suppose that \( x \) and \( y \) are in \( D^{[0,r]} \), and that \( t_{x} \in \mathbb{C}^{[0,r]}_L \). By the relation \( \nabla = t_{x} \partial \) on \( \mathbb{C}^{[0,r]}_L \), Lemma 2.10 and the fact that \( \nabla \) is a differential operator, that is, satisfies a relation similar to (1-3), we can show that the action of \( \Gamma \) induces a bounded infinitesimal action \( \nabla \) on the Banach space \( D^{[s,r]} \). We leave this to the reader. Let us denote \( \ell(\sigma) = \log(\chi_{x}(\sigma)) \). For \( \sigma \) close enough to 1 (depending on \( D \) and \( s, r \)) the series of operators

\[ E(\sigma) = \ell(\sigma) + \frac{\ell(\sigma)^2}{2} \nabla + \frac{\ell(\sigma)^3}{3!} \nabla^2 + \cdots \]

converges on \( D^{[s,r]} \) and also on \( D^{[s/q,r/q]} \). Note that for \( \sigma \) close enough to 1 we have \( \sigma = \exp(\ell(\sigma)\nabla) \) on \( D^{[s/q,r/q]} \). Let \( \Gamma' \) be an open subgroup of \( \Gamma \) such that for \( \sigma \in \Gamma' \) the above two facts hold. Then for \( \sigma \in \Gamma' \) we have

\[ (\phi_q - 1)(E(\sigma)y) = E(\sigma)(\phi_q - 1)y = E(\sigma)\nabla x = \nabla E(\sigma)x = (\sigma - 1)x. \quad (4-1) \]

Note that \( \phi_q(E(\sigma)y) \) is in \( D^{[s/q,r/q]} \). So by (4-1) we have

\[ E(\sigma)y \in D^{[s/q,r/q]} \cap D^{[s,r]} = D^{[s/q,r]} \]

if \( s \) is chosen such that \( s < r/q \). Doing this repeatedly we will obtain \( E(\sigma)y \in D^{[0,r]} \). Taking \( y_{\sigma, \tau} = E(\sigma)y \) for \( \sigma \in \Gamma' \) we will have \( y_{\sigma, \tau} = 0 \) for \( \sigma, \tau \in \Gamma' \). In other words, the restriction to \( \Gamma' \) of the image of \( z \) in \( H^2(\Gamma, D^{\phi_q=1, \nabla=0}) \) is 0. Since \( \Gamma \setminus \Gamma' \) is finite and \( D^{\phi_q=1, \nabla=0} \) is a \( \mathbb{Q} \)-vector space, the image of \( z \) is itself 0. So we can modify \( y_{\sigma} \) by an element of \( D^{\phi_q=1, \nabla=0} \) such that \( y_{\sigma, \tau} \) is identically 0. But this means that \( (\sigma - 1)y_{\tau} = (\tau - 1)y_{\sigma} \), so the 1-cocycle \( \phi_q \mapsto x, \sigma \mapsto y_{\sigma} \) defines an element of \( H^1(D) \), hence also an extension of \( \mathbb{R}_L \) by \( D \).

We will show that the resulting extension in fact belongs to \( \text{Ext}^1_{an}(\mathbb{R}_L, D) \). As \( \Gamma \) is locally constant on \( D^{\phi_q=1, \nabla=0} \), shrinking \( \Gamma' \) if necessary we may assume that \( \Gamma' \) acts trivially on \( D^{\phi_q=1, \nabla=0} \). Then \( \sigma \mapsto y_{\sigma} - E(\sigma)y \) is a continuous homomorphism from \( \Gamma' \) to \( D^{\phi_q=1, \nabla=0} \). Note that any continuous homomorphism from \( \Gamma' \) to \( D^{\phi_q=1, \nabla=0} \) can be extended to \( \Gamma \). Thus \( y_{\sigma} - E(\sigma)y = \lambda(\sigma) \) for some \( \lambda \in \text{Hom}(\Gamma, D^{\phi_q=1, \nabla=0}) \) and all \( \sigma \in \Gamma' \). If \( S \) is a set of representatives of \( \Gamma / \Gamma' \) in \( \Gamma \), the map

\[ T_S = \frac{1}{|\Gamma : \Gamma'|} \sum_{\sigma \in S} \sigma \]
is the identity on $H^1_{\mathrm{an}}(D)$ and a projection from $D^{\varphi_q=1, \nabla \neq 0}$ to $H^0(D)$; moreover it commutes with $\varphi_q$, $\nabla$ and $\Gamma$. This means that we can apply $T_S$ to $(x, y)$ and $y_\sigma$; then we have $y_\sigma - E(\sigma)y = \lambda(\sigma)$ for some $\lambda \in \mathrm{Hom}(\Gamma, H^0(D))$ and all $\sigma \in \Gamma'$. As $\sigma \mapsto E(\sigma)y$ is analytic, the extension in question is $\mathcal{O}_F$-analytic. 

As above, let $\mathrm{Hom}(\Gamma, H^0(D))$ be the set of continuous homomorphisms of groups from $\Gamma$ to $H^0(D)$. An element $h : \Gamma \rightarrow H^0(D)$ of this set is said to be \textit{locally analytic} if $h(\exp(a\beta)) = ah(\exp \beta)$ for all $a \in \mathcal{O}_F$ and $\beta \in \mathrm{Lie}\Gamma$. Let $\mathrm{Hom}_{\mathrm{an}}(\Gamma, H^0(D))$ be the subset of $\mathrm{Hom}(\Gamma, H^0(D))$ consisting of locally analytic homomorphisms. We have natural injections

$$\mathrm{Hom}_{\mathrm{an}}(\Gamma, H^0(D)) \rightarrow \mathrm{Ext}_{\mathrm{an}}^1(\mathcal{R}_L, D) \quad \text{and} \quad \mathrm{Hom}(\Gamma, H^0(D)) \rightarrow \mathrm{Ext}^1(\mathcal{R}_L, D).$$

\textbf{Theorem 4.3.} Assume that $D$ is an $\mathcal{O}_F$-analytic $(\varphi_q, \Gamma)$-module over $\mathcal{R}_L$. Then we have an exact sequence

$$0 \rightarrow \mathrm{Hom}_{\mathrm{an}}(\Gamma, H^0(D)) \rightarrow \mathrm{Hom}(\Gamma, H^0(D)) \oplus \mathrm{Ext}_{\mathrm{an}}^1(\mathcal{R}_L, D) \rightarrow \mathrm{Ext}^1(\mathcal{R}_L, D) \rightarrow 0.$$

For the proof we introduce an auxiliary cohomology theory. Let $\gamma$ be an element of $\Gamma$ of infinite order, i.e., $\log(\chi_{\mathcal{F}}(\gamma)) \neq 0$. We consider the complex

$$C^\bullet_{\varphi_q, \gamma}(D) : \quad 0 \rightarrow D \xrightarrow{g_1} D \oplus D \xrightarrow{g_2} D \rightarrow 0,$$

where $g_1 : D \rightarrow D \oplus D$ is the map $m \mapsto ((\varphi_q - 1)m, (\gamma - 1)m)$ and $g_2 : D \oplus D \rightarrow D$ is $(m, n) \mapsto (\gamma - 1)m - (\varphi_q - 1)n$. As $g_1$ and $g_2$ are $\Gamma$-equivariant, $\Gamma$ acts on $H^i_{\varphi_q, \gamma}(D) := H^i(C^\bullet_{\varphi_q, \gamma}(D))$, $i = 0, 1, 2$. Put $H^i_{\mathrm{an}, \gamma}(D) := H^i_{\varphi_q, \gamma}(D)^\Gamma$. A simple calculation shows that $H^0_{\mathrm{an}, \gamma}(D) = H^0_{\mathrm{an}}(D)$.

For any $\gamma \in \Gamma$ we use $\overline{\{\gamma\}}$ to denote the closed subgroup of $\Gamma$ topologically generated by $\gamma$. If $\gamma$ is of infinite order and if $D$ is an $\mathcal{R}_L$-module together with a semilinear $\overline{\{\gamma\}}$-action, let $\nabla_{\gamma}$ be the operator on $D$ that can be written as $\lim_{\gamma'}(\log(\gamma')/\log(\chi_{\mathcal{F}}(\gamma')))$ formally, where $\gamma'$ runs through all elements of $\overline{\{\gamma\}}$ with $\log(\chi_{\mathcal{F}}(\gamma')) \neq 0$. (For a precise definition we only need to imitate the definition of $\nabla$.)

Let $\widetilde{D}$ be an $\mathcal{O}_F$-analytic extension of $\mathcal{R}_L$ by $D$. Let $e \in \widetilde{D}$ be a lifting of $1 \in \mathcal{R}_L$. Then $((\varphi_q - 1)e, (\gamma - 1)e)$ induces an element of $H^1_{\mathrm{an}, \gamma}(D)$ independent of the choice of $e$. This yields a map $\Theta^D_{\mathrm{an}, \gamma} : \mathrm{Ext}_{\mathrm{an}}(\mathcal{R}_L, D) \rightarrow H^1_{\mathrm{an}, \gamma}(D)$. Given an element of $H^1_{\mathrm{an}, \gamma}(D)$, we can attach to it an extension $\widetilde{D}$ of $\mathcal{R}_L$ by $D$ in the category of free $\mathcal{R}_L$-modules of finite rank together with semilinear actions of $\varphi_q$ and $\overline{\{\gamma\}}$. Let $e \in \widetilde{D}$ be a lifting of $1 \in \mathcal{R}_L$. Then $((\varphi_q - 1)e, \nabla_{\gamma} e)$ belongs to $Z^1(D)$ and induces an element of $H^1_{\mathrm{an}}(D)$ independent of the choice of $e$. This gives a map $\gamma^D_{\mathrm{an}, \gamma} : H^1_{\mathrm{an}, \gamma}(D) \rightarrow H^1_{\mathrm{an}}(D)$. Observe that $\gamma^D_{\mathrm{an}, \gamma} \circ \Theta^D_{\mathrm{an}, \gamma} = \Theta^D_{\mathrm{an}}$. By an argument similar to the proof of the injectivity of $\Theta^D_{\mathrm{an}, \gamma}$ and
\( \gamma_{an,y}^D \) are injective. Hence it follows from Theorem 4.2 that \( \Theta_{an,y}^D \) and \( \gamma_{an,y}^D \) are isomorphisms.

If \( c \) is a 1-cocycle representing an element \( z \) of \( H^1(D) \), then \( (c(\varphi_q), c(\gamma)) \) induces an element in \( H^1_{an,y}(D) \) which only depends on \( z \). This yields a map \( \gamma^D_\gamma : H^1(D) \to H^1_{an,y}(D) \). Hence, \( \Theta_{an,y}^D : \text{Ext}_{an}(\mathcal{R}_L, D) \to H^1_{an,y}(D) \) extends to a map \( \text{Ext}(\mathcal{R}_L, D) \to H^1_{an,y}(D) \), which will also be denoted by \( \Theta_{an,y}^D \). We have the following commutative diagram:

\[
\begin{array}{ccc}
\text{Ext}(\mathcal{R}_L, D) & \xrightarrow{\Theta^D} & H^1(D) \\
\downarrow \sim \theta_{an,y}^D & & \downarrow \gamma^D_\gamma \\
\text{Ext}_{an}(\mathcal{R}_L, D) & \xrightarrow{\Theta_{an,y}^D} & H^1_{an,y}(D)
\end{array}
\tag{4-2}
\]

The composition \( (\Theta_{an,y}^{-1} \circ \gamma^D_\gamma \circ \Theta^D) \) is a projection from \( \text{Ext}(\mathcal{R}_L, D) \) to \( \text{Ext}_{an}(\mathcal{R}_L, D) \), which depends on \( \gamma \).

**Proof of Theorem 4.3.** We only need to prove the surjectivity of

\[ \text{Hom}(\Gamma, H^0(D)) \oplus \text{Ext}^1_{an}(\mathcal{R}_L, D) \to \text{Ext}^1(\mathcal{R}_L, D). \]

Let \( \tilde{D} \) be in \( \text{Ext}^1(\mathcal{R}_L, D) \). Without loss of generality we may assume that the image of \( \tilde{D} \) by the projection \( \Theta_{an,y}^{-1} \circ \gamma^D_\gamma \circ \Theta^D \) is zero. Let \( e \in \tilde{D} \) be a lifting of \( 1 \in \mathcal{R}_L \). Then let \( c \) be the 1-cocycle defined by \( \varphi_q \mapsto (\varphi_q - 1)e \), \( \sigma \mapsto (\sigma - 1)e \) for \( \sigma \in \Gamma \), so that \( \tilde{c} \), the class of \( c \) in \( H^1(D) \), corresponds to \( \tilde{D} \). So the image of \( \tilde{c} \) by the map \( \gamma^D_\gamma \) is zero. This means that there exists \( d \in D \) such that \( (\varphi_q - 1)d = c(\varphi_q) \) and \( (\gamma - 1)d = c(\gamma) \). Replacing \( e \) by \( e - d \), we may assume that \( c(\varphi_q) = c(\gamma) = 0 \). Then for any \( \sigma \in \Gamma \), we have \( (\varphi_q - 1)c(\sigma) = (\sigma - 1)c(\varphi_q) = 0 \) and \( (\gamma - 1)c(\sigma) = (\sigma - 1)c(\gamma) = 0 \). This means that \( c(\sigma) \in D(\varphi_q = 1, \gamma = 1) \). Note that \( M := D(\varphi_q = 1, \gamma = 1) \) is of finite rank over \( L \). We write \( M = H^0(\Gamma, D) \oplus \bigoplus_j M_j \) as a \( \Gamma \)-module, where each \( M_j \) is an irreducible \( \Gamma \)-module. Write \( c = c' + \sum_j c_j \) by this decomposition. Observe that \( c' \) and \( c_j \) are all 1-cocycles. As \( M_j \) is irreducible and the \( \Gamma \)-action on \( M_j \) is nontrivial, there exists some \( \gamma_j \in \Gamma \) such that \( \gamma_j - 1 \) is invertible on \( M_j \). Then there exists \( m_j \in M_j \) such that \( c_j(\gamma_j) = (\gamma_j - 1)m_j \). A simple calculation shows that \( c_j(\sigma) = (\sigma - 1)m_j \) for all \( \sigma \in \Gamma \). Replacing \( e \) by \( e - \sum_j m_j \), we may assume that \( c = c' \). Then \( c(\varphi_q) = 0 \) and \( c|_\Gamma \) is a homomorphism from \( \Gamma \) to \( H^0(D) \).

**Corollary 4.4** (= Theorem 0.2). \( \text{Ext}_{an}(\mathcal{R}_L, D) \) is of codimension

\[ ([F : \mathbb{Q}_p] - 1) \dim_L H^0(D) \]

in \( \text{Ext}(\mathcal{R}_L, D) \). In particular, if \( H^0(D) = 0 \), then \( \text{Ext}_{an}(\mathcal{R}_L, D) = \text{Ext}(\mathcal{R}_L, D) \); in other words, all extensions of \( \mathcal{R}_L \) by \( D \) are \( \mathcal{O}_F \)-analytic.
Proof. This follows from Theorem 4.3 and the equalities \( \dim_L \text{Hom}(\Gamma, H^0(D)) = [F : \mathbb{Q}_p] \dim_L H^0(D) \) and \( \dim_L \text{Hom}_{\mathbb{Q}_p}(\Gamma, H^0(D)) = \dim_L H^0(D) \).

5. Computation of \( H^1_{\text{an}}(\delta) \) and \( H^1(\delta) \)

In the case of \( F = \mathbb{Q}_p \), Colmez [2008] computed \( H^1 \) for not necessarily étale \((\varphi, \Gamma)\)-modules of rank 1 over the Robba ring. In this case, Liu [2008] computed \( H^2 \) for this kind of \((\varphi, \Gamma)\)-modules, and used it and Colmez’s result to build analogues, for not necessarily étale \((\varphi, \Gamma)\)-modules over the Robba ring, of the Euler–Poincaré characteristic formula and Tate local duality. Later, Chenevier [2013] obtained the Euler–Poincaré characteristic formula for families of triangulable \((\varphi, \Gamma)\)-modules and some related results.

In this section we compute \( H^1_{\text{an}}(\delta) = H^1_{\text{an}}(\mathcal{R}_L(\delta)) \) (for \( \delta \in \mathcal{I}_{\text{an}}(L) \)) and \( H^1(\delta) = H^1(\mathcal{R}_L(\delta)) \) (for \( \delta \in \mathcal{I}(L) \)) following Colmez’s approach. In Sections 5B and 5E we assume that \( \delta \) is in \( \mathcal{I}(L) \); in Sections 5C, 5D and 5F we assume that \( \delta \) is in \( \mathcal{I}_{\text{an}}(L) \).

5A. Preliminary lemmas.

Lemma 5.1. (a) If \( \alpha \in L^\times \) is not of the form \( \pi^{-i}, i \in \mathbb{N} \), then \( \alpha \varphi_q - 1 : \mathcal{R}_L^+ \to \mathcal{R}_L^+ \) is an isomorphism.

(b) If \( \alpha = \pi^{-i} \) with \( i \in \mathbb{N} \), then the kernel of \( \alpha \varphi_q - 1 : \mathcal{R}_L^+ \to \mathcal{R}_L^+ \) is \( L \cdot t_F^i \), and \( \alpha \in \mathcal{R}_L^+ \) is in the image of \( \alpha \varphi_q - 1 \) if and only if \( \partial^i a |_{u_F} = 0 \). Further, \( \alpha \varphi_q - 1 \) is bijective on the subset \( \{ a \in \mathcal{R}_L^+ : \partial^i a |_{u_F} = 0 \} \).

Proof. The argument is similar to the proof of [Colmez 2008, Lemma A.1]. If \( k > -v_\pi(\alpha) \), then \( -\sum_{n=0}^{+\infty} (\alpha \varphi_q)^n \) is the continuous inverse of \( \alpha \varphi_q - 1 \) on \( u_F^k \mathcal{R}_L^+ \).

The assertions follow from the fact that \( \mathcal{R}_L^+ = \bigoplus_{i=0}^{k-1} L \cdot t_F^i + u_F^k \mathcal{R}_L^+ \) and the formula \( \varphi_q(t_F^i) = \pi^i t_F^i \). We just need to remark that \( \partial^i a |_{u_F} = 0 \) if and only if \( a \) is in \( \bigoplus_{i=0}^{k-1} L t_F^i + u_F^{i+1} \mathcal{R}_L^+ \).

Lemma 5.2. If \( \alpha \in L \) satisfies \( v_\pi(\alpha) < 1 - v_\pi(q) \), then for any \( b \in \mathcal{E}_L^+ \) there exists \( c \in \mathcal{E}_L^+ \) such that \( b' = b - (\alpha \varphi_q - 1)c \) is in \( \mathcal{E}_L^+ \).

Proof. By Proposition 2.4(d), \( c = \sum_{k=1}^{+\infty} \alpha^{-k} \psi^k(b) \) is convergent in \( \mathcal{E}_L^+ \). It is easy to check that \( \alpha c - \psi(c) = \psi(b) \), which proves the lemma.

Corollary 5.3. If \( \alpha \in L \) satisfies \( v_\pi(\alpha) < 1 - v_\pi(q) \), then for any \( b \in \mathcal{R}_L \) there exists \( c \in \mathcal{R}_L \) such that \( b' = b - (\alpha \varphi_q - 1)c \) is in \( \mathcal{E}_L^+ \).

Proof. Let \( k \) be an integer \( > -v_\pi(\alpha) \). By Lemma 5.1, there exists \( c_1 \in \mathcal{R}_L \) such that \( b - (\alpha \varphi_q - 1)c_1 \) is of the form \( \sum_{i < k} a_i u_F^i \) and thus is in \( \mathcal{E}_L^+ \). Then we apply Lemma 5.2.

Lemma 5.4. If \( \alpha \in L \) satisfies \( v_\pi(\alpha) < 1 - v_\pi(q) \), and if \( z \in \mathcal{R}_L \) satisfies \( \psi(z) - \alpha z \in \mathcal{R}_L^+ \), then \( z \in \mathcal{R}_L^+ \).
Proof. Write \( z \) in the form \( \sum_{k \in \mathbb{Z}} a_k u_k^\pm \) and put \( y = \sum_{k \leq -1} a_k u_k^\pm \in \mathfrak{E}_L^\pm \). If \( y \neq 0 \), multiplying \( z \) by a scalar in \( L \) we may suppose that \( \inf_{k \leq -1} v_p(a_k) = 0 \). Then

\[
y - \alpha^{-1} \psi(y) = \alpha^{-1}(\alpha z - \psi(z)) + \sum_{k \geq 0} a_k (\alpha^{-1} \psi(u_k^\pm) - u_k^\pm)
\]

belongs to \( \mathcal{O}_{\mathcal{E}_L^\pm} \cap \mathfrak{R}_L^+ = \mathcal{O}_L[[u_L]] \). But this is a contradiction since \( y - \alpha^{-1} \psi(y) \equiv y \mod \pi \). Hence \( y = 0 \). \( \square \)

**Corollary 5.5.** If \( \alpha \in L \) satisfies \( v_\pi(\alpha) < 1 - v_\pi(q) \) and if \( z \in \mathfrak{R}_L \) is such that \( (\alpha \varphi_q - 1)z \in \mathfrak{R}_L^{\varphi_q=0} \), then \( z \) is in \( \mathfrak{R}_L^+ \).

**Proof.** We have \( \psi(z) - \alpha z = \psi(z - \alpha \varphi_q(z)) = 0 \). Then we apply Lemma 5.4. \( \square \)

**5B. Computation of \( H^0(\delta) \).** Recall that if \( \delta \in \mathcal{F}_{\text{an}}(L) \), then \( H^0_{\text{an}}(\delta) = H^0(\delta) \).

**Proposition 5.6.** Let \( \delta \) be in \( \mathcal{F}(L) \).

(a) If \( \delta \) is not of the form \( x^{-i} \) with \( i \in \mathbb{N} \), then \( H^0(\delta) = 0 \).

(b) If \( i \in \mathbb{N} \), then \( H^0(x^{-i}) = Lt^i_x \).

**Proof.** Observe that

\[
\mathfrak{R}_L^-(\delta)^{\varphi_q=1} = (\mathfrak{R}_L^-(\delta)^{\varphi_q=1} \cdot e_\delta = 0,
\]

where \( \mathfrak{R}_L^-(\delta) = \mathfrak{R}_L(\delta)/\mathfrak{R}_L^+(\delta) \). Thus \( \mathfrak{R}_L^-(\delta)^{\varphi_q=1, \Gamma=1} = \mathfrak{R}_L^+(\delta)^{\varphi_q=1, \Gamma=1} \). If \( \delta(\pi) \) is not of the form \( \pi^{-i} \), with \( i \in \mathbb{N} \), by Lemma 5.1(a) we have \( \mathfrak{R}_L^+(\delta)^{\varphi_q=1} = 0 \) and so \( \mathfrak{R}_L^+(\delta)^{\varphi_q=1} = 0 \). If \( \delta(\pi) = \pi^{-i} \), then

\[
\mathfrak{R}_L^+(\delta)^{\varphi_q=1, \Gamma=1} = \begin{cases} Lt^i_x \cdot e_\delta \quad & \text{if } \delta = x^{-i}, \\ 0 & \text{otherwise}, \end{cases}
\]

as desired. \( \square \)

**Corollary 5.7.** If \( \delta_1 \) and \( \delta_2 \) are two different characters in \( \mathcal{F}(L) \), then \( \mathfrak{R}_L(\delta_1) \) is not isomorphic to \( \mathfrak{R}_L(\delta_2) \).

**Proof.** We only need to show that \( \mathfrak{R}_L(\delta_1 \delta_2^{-1}) \) is not isomorphic to \( \mathfrak{R}_L \). By Proposition 5.6, \( \mathfrak{R}_L(\delta_1 \delta_2^{-1}) \) is not generated by \( H^0(\delta_1 \delta_2^{-1}) \), but \( \mathfrak{R}_L \) is generated by \( H^0(1) \). Thus \( \mathfrak{R}_L(\delta_1 \delta_2^{-1}) \) is not isomorphic to \( \mathfrak{R}_L \). \( \square \)

**5C. Computation of \( H^1_{\text{an}}(\delta) \) for \( \delta \in \mathcal{F}_{\text{an}}(L) \) with \( v_\pi(\delta(\pi)) < 1 - v_\pi(q) \).** Until the end of Section 5 we will write \( \mathfrak{R}_L(\delta) \) as \( \mathfrak{R}_L \) with the twisted \( (\varphi_q, \Gamma) \)-action given by

\[
\varphi_q;\delta(x) = \delta(\pi) \varphi_q(x), \quad \sigma_a;\delta(x) = \delta(a) \sigma_a(x).
\]

Recall that \( \nabla_\delta = t_x \partial + w_\delta \). Write \( \delta(\sigma_a) = \delta(a) \).
Lemma 5.8. Suppose that $\delta \in \mathcal{I}_{\text{an}}(L)$ satisfies $v_\pi(\delta(\pi)) < 1 - v_\pi(q)$. For any $(a, b) \in Z^1_{\varphi_q, \nabla}(\delta)$, there exists $(m, n) \in Z^1_{\varphi_q, \nabla}(\delta)$ with $m \in (\mathcal{E}^+_L)_{\psi=0}$ and $n \in \mathcal{R}^+_L$ such that $(a, b) \sim (m, n)$.

Proof. As $v_\pi(\delta(\pi)) < 1 - v_\pi(q)$, by Corollary 5.3 there exists $c \in \mathcal{R}_L$ such that

$$m = a - (\delta(\pi)\varphi_q - 1)c$$

is in $(\mathcal{E}^+_L)_{\psi=0}$. Put $n = b - \nabla_\delta c$. Then $(m, n)$ is in $Z^1_{\varphi_q, \nabla}(\delta)$ and $(m, n) \sim (a, b)$. As $(\delta(\pi)\varphi_q - 1)n = \nabla_\delta m = t_{\delta} \partial m + w_\delta m$ is in $\mathcal{R}^+_L$, by Corollary 5.5, $n$ in $\mathcal{R}^+_L$. □

Lemma 5.9. Suppose that $v_\pi(\delta(\pi)) < 1 - v_\pi(q)$ and $\delta$ is not of the form $x^{-i}$. Let $(m, n)$ be in $Z^1_{\varphi_q, \nabla}(\delta)$ with $m \in (\mathcal{E}^+_L)_{\psi=0}$ and $n \in \mathcal{R}^+_L$. Then $(m, n)$ is in $B^1(\delta)$ if and only if

- $m \in (\mathcal{E}^+_L)_{\psi=0}$ when $\delta(\pi)$ is not of the form $\pi^{-i}$, $i \in \mathbb{N}$;
- $m \in (\mathcal{E}^+_L)_{\psi=0}$ and $\partial^i m|_{u_{\delta}=0} = 0$ when $\delta(\pi) = \pi^{-i}$ and $w_\delta \neq -i$ for some $i \in \mathbb{N}$;
- $m \in (\mathcal{E}^+_L)_{\psi=0}$ and $\partial^i m|_{u_{\delta}=0} = \partial^i n|_{u_{\delta}=0} = 0$ when $\delta(\pi) = \pi^{-i}$ and $w_\delta = -i$ for some $i \in \mathbb{N}$.

Proof. We only prove the assertion for the case that $\delta(\pi) = \pi^{-i}$ and $w_\delta \neq -i$ for some $i \in \mathbb{N}$. The arguments for the other two cases are similar.

If $(m, n)$ is in $B^1(\delta)$, then there exists $z \in \mathcal{R}_L$ such that $(\delta(\pi)\varphi_q - 1)z = m$ and $\nabla_\delta z = n$. Since $m$ is in $\mathcal{R}^+_L$, by Corollary 5.5 we have $z \in \mathcal{R}^+_L$. It follows that $m$ is in $\mathcal{R}^+_L \cap \mathcal{E}^+_L = \mathcal{E}^+_L$. By Lemma 5.1(b), we have $\partial^i m|_{u_{\delta}=0} = 0$.

Now we assume that $m$ is in $\mathcal{E}^+_L$ and $\partial^i m|_{u_{\delta}=0} = 0$. By Lemma 5.1(b), there exists $z \in \mathcal{R}^+_L$ with $\partial^j z|_{u_{\delta}=0} = 0$ such that $(\delta(\pi)\varphi_q - 1)z = m$. Then

$$(\delta(\pi)\varphi_q - 1)(\nabla_\delta z - n) = \nabla_\delta(\delta(\pi)\varphi_q - 1)z - (\delta(\pi)\varphi_q - 1)n$$

$$= \nabla_\delta m - (\delta(\pi)\varphi_q - 1)n = 0.$$Again by Lemma 5.1(b), we have $\nabla_\delta z - n = c t_{\delta}^j$ for some $c \in L$. Put $z' = z - c t_{\delta}^j/(w_\delta + i)$. Then $(\delta(\pi)\varphi_q - 1)z' = m$ and $\nabla_\delta z' = n$. Hence $(m, n)$ is in $B^1(\delta)$. □

Recall that $S_{\delta} = \mathcal{R}^-_L(\delta)\Gamma=1,\psi=0$.

Proposition 5.10. Suppose that $v_\pi(\delta(\pi)) < 1 - v_\pi(q)$.

(a) If $\delta$ is not of the form $x^{-i}$, then $H^1_{\text{an}}(\delta)$ is isomorphic to the $L$-vector space $S_{\delta}$ and is 1-dimensional.

(b) If $\delta = x^{-i}$, then $H^1_{\text{an}}(\delta)$ is 2-dimensional over $L$ and is generated by the images of $(t_{\delta}^j, 0)$ and $(0, t_{\delta}^j)$. 
Proof. For (a) we only consider the case that $\delta(\pi) = \pi^{-i}$ and $w_\delta = -i$ for some $i \in \mathbb{N}$. The arguments for the other cases are similar. As $\delta \neq x^{-i}$, there exists an element $\gamma_1 \in \Gamma$ of infinite order such that $\delta(\gamma_1) \neq \chi_\pi(\gamma_1)^{-i}$.

We give two useful facts: for any $z \in \mathcal{H}_L^\dagger$, $\partial^i z|_{u_x=0} = 0$ if and only if $\partial^i(\delta(\gamma_1)\gamma_1 - 1)z|_{u_x=0} = 0$; if $\partial^i z|_{u_x=0} = 0$, then $\partial^i(\delta(\gamma)\gamma - 1)z|_{u_x=0} = 0$ for any $\gamma \in \Gamma$. Both of these two facts follow from Lemma 5.1(b). We will use them freely below.

Let $(m, n)$ be in $Z^1(\delta)$ with $m \in (\mathfrak{H}_L^\dagger)^{\psi=0}$ and $n \in \mathfrak{R}_L^+$. For any $\gamma \in \Gamma$, since $\gamma(m, n) - (m, n) \in B^1(\delta)$, by Lemma 5.9, $(\delta(\gamma)\gamma - 1)m$ is in $\mathfrak{R}_L^+$; i.e., the image of $m$ in $\mathfrak{R}_L^-(\delta)$ belongs to $\delta$.

We will show that, for any $m$, there exists a lifting $m \in (\mathfrak{H}_L^\dagger)^{\psi=0}$ such that $\partial^i(\delta(\gamma)\gamma - 1)m|_{u_x=0} = 0$ for all $\gamma \in \Gamma$. Let $m' \in (\mathfrak{H}_L^\dagger)^{\psi=0}$ be an arbitrary lifting of $m$. Assume that $\partial^i(\delta(\gamma_1)\gamma_1 - 1)m'|_{u_x=0} = c$. Put

$$m = m' - \frac{1}{i!} \delta(\gamma_1)\chi_\pi(\gamma_1)^{i-1}.$$

Then $\partial^i(\delta(\gamma_1)\gamma_1 - 1)m|_{u_x=0} = 0$ and thus $\partial^i \nabla_\delta m|_{u_x=0} = 0$. Hence, by Lemma 5.1(b) there exists $n \in \mathfrak{R}_L^+$ with $\partial^i n|_{u_x=0} = 0$ such that $(\delta(\pi)\varphi_q - 1)n = \nabla_\delta m$. This means that $(m, n) \in Z^1(\delta)$.

Now let $(m_1, n_1)$ and $(m_2, n_2)$ be two elements of $Z^1(\delta)$ with $m_1, m_2 \in (\mathfrak{H}_L^\dagger)^{\psi=0}$ and $n_1, n_2 \in \mathfrak{R}_L^+$. By Lemma 5.9,

$$\partial^i(\delta(\gamma_1)\gamma_1 - 1)m_1|_{u_x=0} = \partial^i(\delta(\gamma_1)\gamma_1 - 1)m_2|_{u_x=0} = \partial^i(\delta(\gamma_1)\gamma_1 - 1)n_1|_{u_x=0} = \partial^i(\delta(\gamma_1)\gamma_1 - 1)n_2|_{u_x=0} = 0.$$

Suppose that the image of $m_1$ in $\delta$ coincides with that of $m_2$, which implies that $m_1 - m_2 \in \mathfrak{H}_L^\dagger$. From

$$\partial^i(\delta(\gamma_1)\gamma_1 - 1)(m_1 - m_2)|_{u_x=0} = \partial^i(\delta(\gamma_1)\gamma_1 - 1)(n_1 - n_2)|_{u_x=0} = 0,$$

we obtain $\partial^i(m_1 - m_2)|_{u_x=0} = \partial^i(n_1 - n_2)|_{u_x=0} = 0$. Thus $(m_1, n_1) \sim (m_2, n_2)$.

Combining all of the above discussions, we obtain an isomorphism $S_\delta \sim H_{\text{an}}^1(\delta)$. Then by Proposition 3.20, $\dim_L H_{\text{an}}^1(\delta) = \dim_L S_\delta = 1$. 


Next we prove (b). Again let \((m, n)\) be in \(Z^1(\delta)\) with \(m \in (\mathcal{E}_L^+)_{\psi=0}\) and \(n \in \mathcal{R}_L^+\). Then the image of \(m\) in \(\mathcal{R}_L^-\), denoted by \(\tilde{m}\), is in \(S_\delta\). We show that \(m\) in fact belongs to \((\mathcal{R}_L^+)_{\psi=0}\), i.e., \(\tilde{m} = 0\). By Corollary 3.15, \(\partial_i : S_\delta \to S_1\) is an isomorphism. So we only need to prove that the image of \(\partial_i m\) in \(S_1\) is zero. By Remark 3.19, it suffices to show that \(\nabla \partial_i m|_{u_F}=0\). But \(\nabla \partial_i m = \partial^i \nabla \delta m\). Since \(\nabla \delta m = (\delta(\pi)\varphi_q - 1)n\), by Lemma 5.1(b) we have \(\partial^i \nabla \delta m|_{u_F}=0\).

Write \(m = at^i_F + m'\) with \(a \in L\) and \(m' \in \mathcal{R}_L^+\) satisfying \(\partial^i m'|_{u_F}=0\). By Lemma 5.1(b) there exists \(z \in \mathcal{R}_L^+\) such that \((\delta(\pi)\varphi_q - 1)z = m'\). Then \((m, n) \sim (aL_F, n - \nabla \delta z)\). Thus we may suppose that \(m = aL_F\). Then

\[
(\delta(\pi)\varphi_q - 1)n = \nabla \delta (aL_F) = 0.
\]

So, by Lemma 5.1(b), we have \(n = bt^i_F\) for some \(b \in L\). Suppose that \((at^i_F, bt^i_F)\) is in \(B^1(\delta)\). Then there exists \(z \in \mathcal{R}_L\) such that \((\delta(\pi)\varphi_q - 1)z = at^i_F\) and \(\nabla \delta z = bt^i_F\). So \(\psi(z) - \delta(\pi)z = \psi((1-\delta(\pi)\varphi_q)z) = \psi(-at^i_F) \in \mathcal{R}_L^+\). By Lemma 5.4 we get \(z \in \mathcal{R}_L^+\). By Lemma 5.1(b) again we have \(a = 0\) and \(z \in Lt^i_F\). Then \(bt^i_F = \nabla \delta z = 0\). 

5D. \(\partial : H^1_{\mathcal{R}_\delta,\nabla}(x^{-1}\delta) \to H^1_{\mathcal{R}_\delta,\nabla}(\delta)\) and \(\partial : H^1_{\mathcal{R}_\delta,\nabla}(x^{-1}\delta) \to H^1_{\mathcal{R}_\delta,\nabla}(\delta)\). Observe that, if \((m, n)\) is in \(Z^1_{\mathcal{R}_\delta,\nabla}(x^{-1}\delta)\) (resp. \(B^1(x^{-1}\delta)\)), then \((\partial m, \partial n)\) is in \(Z^1_{\mathcal{R}_\delta,\nabla}(\delta)\) (resp. \(B^1(\delta)\)). Thus we have a map \(\partial : H^1_{\mathcal{R}_\delta,\nabla}(x^{-1}\delta) \to H^1_{\mathcal{R}_\delta,\nabla}(\delta)\). Further, the map is \(\Gamma\)-equivariant and thus induces a map \(\partial : H^1_{\mathcal{R}_\delta,\nabla}(x^{-1}\delta) \to H^1_{\mathcal{R}_\delta,\nabla}(\delta)\).

Put

\[
\bar{Z}^1_{\mathcal{R}_\delta,\nabla}(\delta) := \{(m, n) \in Z^1_{\mathcal{R}_\delta,\nabla}(\delta) : \text{Res}(m) = \text{Res}(n) = 0\},
\]

\[
\bar{B}^1(\delta) := \{(m, n) \in B^1(\delta) : \text{Res}(m) = \text{Res}(n) = 0\}.
\]

Then \(\bar{H}^1_{\mathcal{R}_\delta,\nabla}(\delta) := Z^1_{\mathcal{R}_\delta,\nabla}(\delta)/\bar{B}^1_{\mathcal{R}_\delta,\nabla}(\delta)\) is a subspace of \(H^1_{\mathcal{R}_\delta,\nabla}(\delta)\).

**Lemma 5.11.** If \(\delta(\pi) \neq \pi/q\) or \(w_\delta \neq 1\), then for any \((m, n) \in Z^1_{\mathcal{R}_\delta,\nabla}(\delta)\), there exists \((m_1, n_1) \in \bar{Z}^1_{\mathcal{R}_\delta,\nabla}(\delta)\) such that \((m, n) \sim (m_1, n_1)\), and so \(H^1_{\mathcal{R}_\delta,\nabla}(\delta) = \bar{H}^1_{\mathcal{R}_\delta,\nabla}(\delta)\).

**Proof.** Let \((m, n)\) be in \(Z^1_{\mathcal{R}_\delta,\nabla}(\delta)\). Then \(\nabla \delta m = (\delta(\pi)\varphi_q - 1)n\). If \(\delta(\pi) \neq \pi/q\), by Proposition 2.13 and the definition of \(\text{Res}\) we have

\[
\text{Res}\left(m - (\delta(\pi)\varphi_q - 1)\left(\text{Res}(m)\left(\frac{df_F/du_F}{\delta(\pi)\frac{df}{\pi} - 1}u_F\right)^{-1}\right)\right) = 0.
\]

Replacing \((m, n)\) by

\[
\left(m - (\delta(\pi)\varphi_q - 1)\left(\text{Res}(m)\left(\frac{df_F/du_F}{\delta(\pi)\frac{df}{\pi} - 1}u_F\right)^{-1}\right), n - \nabla \delta \left(\text{Res}(m)\left(\frac{df_F/du_F}{\delta(\pi)\frac{df}{\pi} - 1}u_F\right)^{-1}\right)\right),
\]

we may assume that \(\text{Res}(m) = 0\). Then

\[
\left(\frac{q}{\pi} \delta(\pi) - 1\right)\text{Res}(n) = \text{Res}\left((\delta(\pi)\varphi_q - 1)n\right) = \text{Res}(\nabla \delta m)
\]

\[
= \text{Res}(\partial(t_F m) + (w_\delta - 1)m) = (w_\delta - 1)\text{Res}(m) = 0,
\]
and so Res$(n) = 0$.

The argument for the case of $w_δ \neq 1$ is similar. \hfill \Box

The map $δ : H^1_{φ,q}(x^{-1}δ) → H^1_{φ,q}(δ)$ factors through $δ : H^1_{φ,q}(x^{-1}δ) → \overline{H}^1_{φ,q}(δ)$, since $\text{Res} ∘ δ = 0$.

**Lemma 5.12.** (a) If $δ(π) \neq π$ or $w_δ = 1$, then $δ : H^1_{φ,q}(x^{-1}δ) → \overline{H}^1_{φ,q}(δ)$ is surjective.

(b) If $δ(π) = π$ and $w_δ = 1$, then we have an exact sequence of $Γ$-modules

$$H^1_{φ,q}(x^{-1}δ) \xrightarrow{δ} \overline{H}^1_{φ,q}(δ) \to L(x^{-1}δ) → 0.$$  

**Proof.** Let $(m, n)$ be in $\widetilde{Z}^1_{φ,q}(δ)$. Then there exist $m'$ and $n'$ such that $∂m' = m$ and $∂n' = n$. Then $∇_{x^{-1}δ}m' - (π^{-1}δ(π)φ_q - 1)n' = c$ is in $L$. If $δ(π) \neq π$, we replace $n'$ by $n' + c/(π^{-1}δ(π) - 1)$. If $w_δ \neq 1$, we replace $m'$ by $m' - c/(w_δ - 1)$. Then $(m', n')$ is in $Z^1_{φ,q}(x^{-1}δ)$. This proves (a). When $δ(π) = π$ and $w_δ = 1$, $∇m' - (φ_q - 1)n'$ does not depend on the choice of $m'$ and $n'$. This induces a map $\overline{H}^1_{φ,q}(δ) → L$ whose kernel is exactly $∂H^1_{φ,q}(x^{-1}δ)$. We show that $\overline{H}^1_{φ,q}(δ) → L$ is surjective. Put $m' = \log(φ_q(u_f)/u_f^q)$. A simple calculation shows that

$$∇m' = \left(\frac{t_f \cdot [π \frac{t_f}{f(u_f)}]}{π \frac{t_f}{u_f}} - q \frac{t_f}{u_f}\right) \partial u_f \equiv (1 - q) \mod u_f R_L^+.$$  

Thus by Lemma 5.1(b) there exists $n' ∈ u_f R_L^+$ such that $(φ_q - 1)n' = ∇m' - (1 - q)$. Put $m = ∂m'$ and $n = ∂n'$. Then $(m, n)$ is in $\widetilde{Z}^1_{φ,q}(δ)$, whose image in $L$ is nonzero. The $Γ$-action on $\overline{H}^1_{φ,q}(δ)$ induces an action on $L$. From

$$(δ(a)σ_a(m), δ(a)σ_a(n)) = (δ(a^{-1}δ(a)σ_a(m')), δ(a^{-1}δ(a)σ_a(n')))$$

and

$$∇(a^{-1}δ(a)σ_a(m')) - (φ_q - 1)(a^{-1}δ(a)σ_a(n')) = a^{-1}δ(a)σ_a(∇m' - (φ_q - 1)n') \equiv a^{-1}δ(a)(1 - q) \mod u_f R_L^+,$$

we see that the induced action comes from the character $x^{-1}δ$. \hfill \Box

**Sublemma 5.13.** Let $a, b$ be in $L$. If $(a, b)$ is in $Z^1_{φ,q}(x^{-1}δ)$ but not in $B^1(x^{-1}δ)$, then $δ(π) = π$ and $w_δ = 1$.

**Proof.** If $δ(π) \neq π$, then $(a, b) ~ \left(0, b - \frac{∇_{x^{-1}δ}}{π^{-1}δ(π) - 1}a\right)$. So

$$(π^{-1}δ(π) - 1)\left(b - \frac{∇_{x^{-1}δ}}{π^{-1}δ(π) - 1}a\right) = (π^{-1}δ(π)φ_q - 1)\left(b - \frac{∇_{x^{-1}δ}}{π^{-1}δ(π) - 1}a\right) = 0.$$  

As $δ(π) \neq π$, we have $b - \frac{∇_{x^{-1}δ}}{π^{-1}δ(π) - 1}a = 0$. Similarly, if $w_δ \neq 1$, then $(a, b)$ is
in $Z_{\psi_q, \nabla}^1(x^{-1}\delta)$ if and only if $(a, b) \sim (0, 0)$. \hfill \square

Recall that $\delta_{\text{unr}}$ is the character of $F^x$ such that $\delta_{\text{unr}}(\pi) = q^{-1}$ and $\delta_{\text{unr}}|_{\mathcal{O}_F} = 1$.

**Sublemma 5.14.** The pair 

$$(m, n) := \left( \frac{1}{q} \log \frac{\phi_q(u_F)}{u_F^q}, \frac{t_F \partial u_F}{u_F} \right)$$

induces a nonzero element of $H^1_{\text{un}}(\delta_{\text{m}})$.

**Proof.** Note that $m = (\delta_{\text{unr}}(\pi) \psi_q - 1) \log u_F$ and $n = \nabla \log u_F$. Thus $(m, n)$ is in $Z_{\psi_q, \nabla}^1(\delta_{\text{unr}})$. For any $\gamma \in \Gamma$ we have $\gamma(m, n) \sim (m, n)$. Indeed,

$$\gamma(m, n) - (m, n) = \left( (\delta_{\text{unr}}(\pi) \psi_q - 1) \log \frac{\gamma(u_F)}{u_F}, \nabla \log \frac{\gamma(u_F)}{u_F} \right).$$

So $(m, n)$ is in $Z_1^1(\delta_{\text{unr}})$. We show that $(m, n)$ is not in $B^1(\delta_{\text{unr}})$. Otherwise there exists $z \in \mathfrak{R}_L$ such that $m = (\delta_{\text{unr}}(\pi) \psi_q - 1)z$ and $n = \nabla z$. This implies that $\nabla(\log u_F - z) = 0$, or equivalently $\log u_F - z$ is in $L$, a contradiction. \hfill \square

**Corollary 5.15.** If $\delta(\pi) = \pi/q$ and $w_\delta = 1$, then $(\frac{1}{q} \log(\psi_q(u_F)/u_F^q), t_F \partial u_F/u_F)$ is in $Z_{\psi_q, \nabla}^1(x^{-1}\delta)$ but not in $B^1(x^{-1}\delta)$.

**Lemma 5.16.**

(a) If $\delta(\pi) \neq \pi$, $\pi/q$ or if $w_\delta \neq 1$, then we have an exact sequence of $\Gamma$-modules

$$0 \to L(x^{-1}\delta) \oplus L(x^{-1}\delta) \to H_{\psi_q, \nabla}^1(x^{-1}\delta) \overset{\delta}{\to} \overline{H}_{\psi_q, \nabla}^1(\delta).$$

(b) If $\delta(\pi) = \pi$ and $w_\delta = 1$, then we have an exact sequence of $\Gamma$-modules

$$0 \to L(x^{-1}\delta) \to H_{\psi_q, \nabla}^1(x^{-1}\delta) \overset{\delta}{\to} \overline{H}_{\psi_q, \nabla}^1(\delta).$$

(c) If $\delta(\pi) = \pi/q$ and $w_\delta = 1$, then we have an exact sequence of $\Gamma$-modules

$$0 \to L(x^{-1}\delta) \to H_{\psi_q, \nabla}^1(x^{-1}\delta) \overset{\delta}{\to} \overline{H}_{\psi_q, \nabla}^1(\delta).$$

**Proof.** Let $(m, n)$ be in $Z_{\psi_q, \nabla}^1(x^{-1}\delta)$, and suppose that $(\partial m, \partial n) \in \overline{B}_1^1(\delta)$. Let $z$ be an element of $\mathfrak{R}_L$ such that $(\delta(\pi) \psi_q - 1)z = \partial m$ and $\nabla z = \partial n$. If $\text{Res}(z) = 0$, then there exists $z' \in \mathfrak{R}_L$ such that $\partial z' = z$. Then $m - (\delta(\pi) \psi_q - 1)z'$ and $n - \nabla x^{-1}\delta z'$ are in \{(a, b) : a, b \in L\}, i.e., $(m, n)$ is in $B^1(x^{-1}\delta) \oplus L(0, 1) \oplus L(1, 0)$.

If either $\delta(\pi) \neq \frac{\pi}{q}$ or $w_\delta \neq 1$, we always have $\text{Res}(z) = 0$. Indeed, this follows from

$$(\delta(\pi) \frac{q}{\pi} - 1)\text{Res}(z) = \text{Res}(\delta(\pi) \psi_q - 1)z) = \text{Res}(\partial m) = 0$$

and

$$(w_\delta - 1)\text{Res}(z) = \text{Res}(\partial(t_F z) + (w_\delta - 1)z) = \text{Res}(\nabla z) = \text{Res}(\partial n) = 0.$$
we obtain that \( \dim L(0, 1) \oplus L(1, 0) \oplus L\left( \frac{1}{q} \log \frac{u_F}{u_F^q}, \frac{t_F}{u_F} \partial u_F \right) \).

Now our lemma follows from Sublemma 5.13 and Corollary 5.15. \( \square \)

**Proposition 5.17.** (a) If \( \delta(\pi) \neq \pi, \pi/q \) or if \( w_\delta \neq 1 \), then

\[ \partial : H^1_{\eta, \nu}(x^{-1} \delta) \rightarrow H^1_{\eta, \nu}(\delta) \]

is an isomorphism of \( \Gamma \)-modules.

(b) If \( \delta(\pi) = \pi \) and \( w_\delta = 1 \), then we have an exact sequence of \( \Gamma \)-modules

\[ 0 \rightarrow L(x^{-1} \delta) \oplus L(x^{-1} \delta) \rightarrow H^1_{\eta, \nu}(x^{-1} \delta) \rightarrow H^1_{\eta, \nu}(\delta) \rightarrow L(x^{-1} \delta) \rightarrow 0. \]

(c) If \( \delta(\pi) = \pi/q \) and \( w_\delta = 1 \), then we have an exact sequence of \( \Gamma \)-modules

\[ 0 \rightarrow L(x^{-1} \delta) \rightarrow H^1_{\eta, \nu}(x^{-1} \delta) \rightarrow H^1_{\eta, \nu}(\delta) \rightarrow L(x^{-1} \delta) \oplus L(x^{-1} \delta) \rightarrow 0. \]

**Proof.** Assertions (a) and (b) follow from Lemmas 5.11, 5.12 and 5.16. Based on these lemmas, for (c) we only need to show that we have an exact sequence of \( \Gamma \)-modules

\[ 0 \rightarrow H^1_{\eta, \nu}(\delta) \rightarrow H^1_{\eta, \nu}(\delta) \rightarrow L(x^{-1} \delta) \oplus L(x^{-1} \delta) \rightarrow 0, \]

where \( \text{Res} \) is induced by \((m, n) \mapsto (\text{Res}(m), \text{Res}(n))\), which is \( \Gamma \)-equivariant by Proposition 2.13. Here we prove this under the assumption that \( q \) is not a power of \( \pi \). We will see in Section 5F that it also holds without this assumption. Put \( m_1 = 1/u_F \). Then \( \nabla \psi m_1 = \frac{t_F}{u_F} \partial (1/u_F) + 1/u_F = \partial (t_F/u_F) \) is in \( \mathcal{R}_L^+ \). As \( q \) is not a power of \( \pi \), the map \( \frac{z}{q} \psi q - 1 : \mathcal{R}_L^+ \rightarrow \mathcal{R}_L^+ \) is an isomorphism. Let \( n_1 \) be the unique solution of \( (\frac{z}{q} \psi q - 1) n_1 = t_F \partial m_1 + m_1 \) in \( \mathcal{R}_L^+ \). Then \( c_1 = (m_1, n_1) \) is in \( Z^1_{\eta, \nu}(\delta) \) and \( \text{Res}(m_1, n_1) = (1, 0) \neq 0 \). For any \( \ell \in \mathbb{N} \) we choose a root \( \xi_\ell \) of \( Q_\ell = \varphi^{-1}_q(Q) \). For any \( f(u_F) \in \mathcal{R}_L^+ \), the value of \( f \) at \( \xi_\ell \) is an element \( f(\xi_\ell) \) in \( L \otimes F \ell \). By (3-4) there exists an element \( z \in \mathcal{R}_L^+ \) whose value at \( \xi_\ell \) is \( 1 \otimes \log \xi_\ell \). Put \( m_2 = t_F^{-1}(q^{-1} \psi q - 1)(\log u_F - z) \) and \( n_2 = \partial (\log u_F - z) \). Then \( (m_2, n_2) \) is in \( Z^1_{\eta, \nu}(\delta) \) and \( \text{Res}(n_2) = 1 \). \( \square \)

**Proposition 5.18.** (a) If \( \delta \neq x, x \delta\text{unr}, \) then \( \partial : H^1_{an}(x^{-1} \delta) \rightarrow H^1_{an}(\delta) \) is an isomorphism.

(b) If \( \delta = x, \) then \( \partial : H^1_{an}(x^{-1} \delta) \rightarrow H^1_{an}(\delta) \) is zero, and \( \dim_L H^1_{an}(\delta) = 1. \)

(c) If \( \delta = x \delta\text{unr}, \) then \( \partial : H^1_{an}(x^{-1} \delta) \rightarrow H^1_{an}(\delta) \) is zero, and \( \dim_L H^1_{an}(\delta) = 2. \)

**Proof.** We apply Proposition 5.17. There is nothing to prove for the case that \( \delta(\pi) \neq \pi, \pi/q \) or \( w_\delta \neq 1 \). Combining the assertions in this case and Proposition 5.10 we obtain that \( \dim_L H^1_{an}(\delta\text{unr}) = 1. \) This fact is useful below.
Next we consider the case of \( \delta(\pi) = \pi/q \) and \( w_\delta = 1 \). The argument for the case of \( \delta(\pi) = \pi \) and \( w_\delta = 1 \) is similar.

Let \( M \) be the image of \( \partial : H^1_{\varphi_q, \nabla}(x^{-1}\delta) \to H^1_{\varphi_q, \nabla}(x) \). Then we have two short exact sequences of \( \Gamma \)-modules

\[
0 \to L(x^{-1}\delta) \to H^1_{\varphi_q, \nabla}(x^{-1}\delta) \xrightarrow{\partial} M \to 0
\]

and

\[
0 \to M \to H^1_{\varphi_q, \nabla}(\delta) \to L(x^{-1}\delta) \oplus L(x^{-1}\delta) \to 0.
\]

We will show that taking \( \Gamma \)-invariants yields two exact sequences

\[
0 \to L(x^{-1}\delta)\Gamma \to H^1_{\text{an}}(x^{-1}\delta) \xrightarrow{\partial} M^\Gamma \to 0
\]

and

\[
0 \to M^\Gamma \to H^1_{\text{an}}(\delta) \to L(x^{-1}\delta)\Gamma \oplus L(x^{-1}\delta)\Gamma \to 0.
\]

If the \( \Gamma \)-actions on \( H^1_{\varphi_q, \nabla}(x^{-1}\delta) \) and \( H^1_{\varphi_q, \nabla}(\delta) \) are semisimple, then there is nothing to prove. However we will avoid this by an alternative argument. It suffices to prove the surjectivity of \( H^1_{\varphi_q, \nabla}(x^{-1}\delta)\Gamma \to M^\Gamma \) and \( H^1_{\varphi_q, \nabla}(\delta)\Gamma \to L(x^{-1}\delta)\Gamma \oplus L(x^{-1}\delta)\Gamma \).

The latter follows from the proof of Proposition 5.17. In fact, if \( \delta = x\delta_{\text{unr}} \), then \((m_1, n_1)\) and \((m_2, n_2)\) constructed there are in \( Z^1(\delta) \). Now let \( c \) be any element of \( M^\Gamma \). Then the preimage \( \partial^{-1}(Lc) \) is two-dimensional over \( L \) and \( \Gamma \)-invariant.

From the definition of \( H^1_{\varphi_q, \nabla} \), we obtain that the induced \( \nabla \)-action on \( \partial^{-1}(Lc) \) is zero and thus \( \partial^{-1}(Lc) \) is a semisimple \( \Gamma \)-module, as wanted.

If \( \delta = x\delta_{\text{unr}} \), then \( \dim_L L(x^{-1}\delta)\Gamma = \dim_L H^1_{\text{an}}(x^{-1}\delta) = 1 \), and so \( M^\Gamma = 0 \). Thus \( \partial : H^1_{\text{an}}(x^{-1}\delta) \to H^1_{\text{an}}(\delta) \) is zero and \( \dim_L H^1_{\text{an}}(\delta) = 2 \). If \( \delta \neq x\delta_{\text{unr}} \), then \( \partial : H^1_{\text{an}}(x^{-1}\delta) \to H^1_{\text{an}}(\delta) \) is an isomorphism since both \( H^1_{\text{an}}(x^{-1}\delta) \to M^\Gamma \) and \( M^\Gamma \to H^1_{\text{an}}(\delta) \) are isomorphisms. \( \square \)

5E. Dimension of \( H^1(\delta) \) for \( \delta \in \mathcal{H}(L) \).

**Theorem 5.19.** (= Theorem 0.3) Let \( \delta \) be in \( \mathcal{H}_{\text{an}}(L) \).

(a) If \( \delta \) is not of the form \( x^{-i} \) with \( i \in \mathbb{N} \), or the form \( x^i\delta_{\text{unr}} \) with \( i \in \mathbb{Z}_+ \), then \( H^1_{\text{an}}(\delta) \) and \( H^1(\delta) \) are 1-dimensional over \( L \).

(b) If \( \delta = x^i\delta_{\text{unr}} \) with \( i \in \mathbb{Z}_+ \), then \( H^1_{\text{an}}(\delta) \) and \( H^1(\delta) \) are 2-dimensional over \( L \).

(c) If \( \delta = x^{-i} \) with \( i \in \mathbb{N} \), then \( H^1_{\text{an}}(\delta) \) is 2-dimensional over \( L \) and \( H^1(\delta) \) is \((d+1)\)-dimensional over \( L \), where \( d = [F : \mathbb{Q}_p] \).

**Proof.** The assertions for \( H^1_{\text{an}}(\delta) \) follow from Propositions 5.10 and 5.18. By Proposition 5.6 we have

\[
\dim_L \mathcal{R}_L(\delta)_{\varphi_q=1, \Gamma=1} = \begin{cases} 1 & \text{if } \delta = x^{-i} \text{ with } i \in \mathbb{N}, \\ 0 & \text{otherwise}. \end{cases}
\]

So the assertions for \( H^1(\delta) \) come from those for \( H^1_{\text{an}}(\delta) \) and Corollary 4.4. \( \square \)
When \( \delta = x^{-i} \) with \( i \in \mathbb{N}, \) \( H^1_{\text{an}}(\delta) \) is generated by the classes of \((t_F^i, 0)\) and \((0, t_F^i)\). Let \( \rho_i \) \((i = 1, \ldots, d)\) be a basis of \( \text{Hom}(\Gamma, L t_F^i) \). Then the class of the 1-cocycle \( c_0 \) with \( c_0(\varphi_q) = t_F^i \) and \( c_0|\Gamma = 0 \), and the classes of 1-cocycles \( c_i \) with \( c_i(\varphi_q) = 0 \) and \( c_i|\Gamma = \rho_i \) \((i = 1, \ldots, d)\), form a basis of \( H^1(\delta) \).

**Theorem 5.20.** \((= \text{Theorem 0.4})\) If \( \delta \in \mathcal{J}(L) \) is not locally \( F \)-analytic, then \( H^1(\delta) = 0 \).

**Proof.** As the maps \( \gamma - 1, \gamma \in \Gamma \), are null on \( H^1(\delta) \), by the definition of \( H^1 \), so are the maps \( d\Gamma_{R_L(\delta)}(\beta), \beta \in \text{Lie} \Gamma \) and the differences \( \beta^{-1}d\Gamma_{R_L(\delta)}(\beta) - \beta'^{-1}d\Gamma_{R_L(\delta)}(\beta') \). Not that \( \beta^{-1}d\Gamma_{R_L(\delta)}(\beta) - \beta'^{-1}d\Gamma_{R_L(\delta)}(\beta') \) are \( R_L \)-linear on \( R_L(\delta) \). So

\[
\beta^{-1}d\Gamma_{R_L(\delta)}(\beta) - \beta'^{-1}d\Gamma_{R_L(\delta)}(\beta')
\]

are multiplications by scalars in \( L \), since \( \beta^{-1}d\Gamma_{R_L(\delta)}(\beta)e_\delta - \beta'^{-1}d\Gamma_{R_L(\delta)}(\beta')e_\delta \) is in \( L e_\delta \). If the intersection of their kernels is null, then the cohomology \( H^1(\delta) \) vanishes. Thus, either the intersection of their kernels is 0 and so the cohomology vanishes, or they are all null and \( \delta \) is of the form \( x \mapsto x^w \) for \( x \) close to 1 with \( w = \frac{\log \delta(\beta)}{\log \beta} \) for \( \beta \) close to 1 (i.e., \( \delta \) is locally \( F \)-analytic).

**Remark 5.21.** Suppose that \([F : \mathbb{Q}_p]\geq 2\). Let \( \delta \neq 1 \) be a character of \( F^\times \) with \( \delta(\pi) \in \mathcal{O}_L^\times \), and let \( L(\delta) \) be the \( L \)-representation of \( G_F \) induced by \( \delta \). Suppose that \( \delta \neq x^2 \delta_{\text{uni}} \) when \([F : \mathbb{Q}_p] = 2\). Combining Theorem 5.19 and the Euler–Poincaré characteristic formula [Tate 1963] we obtain that there exist Galois representations in \( \text{Ext}(L, L(\delta)) \) that are not overconvergent. Theorem 5.20 tells us that if further \( \delta \) is not locally analytic, then there is no nontrivial overconvergent extension of \( L \) by \( L(\delta) \).

**5F. The maps** \( t_k : H^1(\delta) \to H^1(x^{-k}\delta) \) **and** \( t_{k,\text{an}} : H^1_{\text{an}}(\delta) \to H^1_{\text{an}}(x^{-k}\delta) \). Let \( k \) be a positive integer.

**Proposition 5.22.** Let \( \delta \) be in \( \mathcal{J}_{\text{an}}(L) \).

(a) If \( w_\delta \notin \{1 - k, \ldots, 0\} \), then \( H^0_{\text{an}}(R_L(\delta)/t_F^kR_L(\delta)) = 0 \).

(b) If \( w_\delta \in \{1 - k, \ldots, 0\} \), then \( H^0_{\text{an}}(R_L(\delta)/t_F^kR_L(\delta)) \) is a 1-dimensional \( L \)-vector space.

**Proof.** We have \( R_L^+/t_F^kR_L^+ = R_L^+/u_F^k \times \prod_{n=1}^\infty R_L^+/(\varphi_q^{n-1}(Q))^k \). As \( \Gamma \)-modules, \( R_L^+/u_F^k = \bigoplus_{i=0}^{k-1} L t_F^i \) and \( R_L^+/(\varphi_q^k(Q))^k = \bigoplus_{i=0}^{k-1} (L \otimes_F F_n)t_F^i \). Thus as a \( \Gamma \)-module, \( R_L^+/t_F^kR_L^+ \) is isomorphic to \( \bigoplus_{i=0}^{k-1} (R_L^+/R_L^+t_F^i) \otimes_L L t_F^i \). Note that the natural map \( R_L^+/R_L^+t_F^i \to R_L/R_L t_F^i \) is surjective. Furthermore, two sequences \((y_n)_{n \geq 0}\) and \((z_n)_{n \geq 0}\) in \( R_L^+/R_L^+u_F^k \times \prod_{n=1}^\infty R_L^+/(\varphi_q^{n-1}(Q))^k \) have the same image in \( R_L/R_L t_F^i \), if and only if there exists \( N > 0 \) such that \( y_n = z_n \) when \( n \geq N \).

Since the action of \( \Gamma \) on \( (R_L^+/R_L^+t_F^i) t_F^i \) twisted by the character \( x^{-i} \) is smooth, (a) follows.
For (b) we only need to consider the case of \( w_\delta = 0 \) and \( k = 1 \). The operator \( \varphi_q \) induces injections \( \mathcal{R}_L^+/\varphi_q^n(Q) \rightarrow \mathcal{R}_L^+/\varphi_{q+1}(Q) \) denoted by \( \varphi_{q,n} \). The action of \( \varphi_q \) on \( \mathcal{R}_L/\mathcal{R}_Lt_\mathcal{F} \) is given by \( \varphi_q(y_n)n = (\varphi_q(y_n))_{n+1} \). For any \( n \geq 0 \), the \( \Gamma \)-action on \( L \otimes_F F_n \) factors through \( \Gamma/\Gamma_n \), and the resulting \( \Gamma/\Gamma_n \)-module \( L \otimes_F F_n \) is isomorphic to the regular one. Thus for any discrete character \( \delta \) of \( \Gamma \), \( \dim_L(L \otimes_F F_n)_{\Gamma=\delta^{-1}} = 1 \) when \( n \) is sufficiently large. Then from the fact that the \( \varphi_{q,n} \) \( (n \geq 1) \) are injective, we obtain \( \dim_L(\mathcal{R}_L/t_\mathcal{F}_\mathcal{R}_L)_{\Gamma=\delta^{-1},\varphi_q=\delta(\pi)^{-1}} = 1 \).

**Corollary 5.23.** Let \( \delta \) be in \( \mathcal{F}_{an}(L) \).

(a) If \( w_\delta \notin \{1, \ldots, k\} \), then \( H_{an}^0(t_\mathcal{F}^k\mathcal{R}_L(\delta)/\mathcal{R}_L(\delta)) = 0 \).

(b) If \( w_\delta \in \{1, \ldots, k\} \), then \( H_{an}^0(t_\mathcal{F}^k\mathcal{R}_L(\delta)/\mathcal{R}_L(\delta)) \) is a 1-dimensional \( L \)-vector space.

Note that \( \mathcal{R}_L(x^{-k}) \) is canonically isomorphic to \( t_\mathcal{F}^k\mathcal{R}_L(\delta) \). When \( k \geq 1 \), the inclusion \( \mathcal{R}_L(\delta) \hookrightarrow t_\mathcal{F}^k\mathcal{R}_L(\delta) \) induces maps \( t_{k, an} : H_{an}^1(\delta) \rightarrow H_{an}^1(x^{-k}\delta) \) and \( t_k : H^1(\delta) \rightarrow H^1(x^{-k}\delta) \). If \( \gamma \in \Gamma \) is of infinite order, then we have this commutative diagram:

\[
\begin{array}{ccc}
H^1(\delta) & \xrightarrow{t_k} & H^1(x^{-k}\delta) \\
\downarrow\gamma^k_{an,y} \circ \gamma^s_y & & \downarrow\gamma^k_{an,y} \circ \gamma^s_y \circ \gamma^{-k}_y \\
H_{an}^1(\delta) & \xrightarrow{t_{k, an}} & H_{an}^1(x^{-k}\delta). \\
\end{array}
\]  

(5-1)

**Lemma 5.24.** We have the exact sequence

\[
0 \rightarrow H_{an}^0(\delta) \rightarrow H_{an}^0(x^{-k}\delta) \rightarrow H_{an}^0(t_\mathcal{F}^k\mathcal{R}_L(\delta)/\mathcal{R}_L(\delta)) \rightarrow H_{an}^1(\delta) \xrightarrow{t_{k, an}} H_{an}^1(x^{-k}\delta).
\]

**Proof.** From the short exact sequence

\[
0 \rightarrow \mathcal{R}_L(\delta) \rightarrow \mathcal{R}_L(x^{-k}\delta) \rightarrow \mathcal{R}_L(x^{-k}\delta)/\mathcal{R}_L(\delta) \rightarrow 0,
\]  

(5-2)

we deduce an exact sequence

\[
0 \rightarrow H_{\varphi_q,\mathcal{V}}^0(\delta) \rightarrow H_{\varphi_q,\mathcal{V}}^0(x^{-k}\delta) \rightarrow H_{\varphi_q,\mathcal{V}}^0(t_\mathcal{F}^k\mathcal{R}_L(\delta)/\mathcal{R}_L(\delta)) \rightarrow H_{\varphi_q,\mathcal{V}}^1(\delta) \rightarrow H_{\varphi_q,\mathcal{V}}^1(x^{-k}\delta). 
\]  

(5-3)

Being finite-dimensional, \( H_{\varphi_q,\mathcal{V}}^0(\delta) \) and \( H_{\varphi_q,\mathcal{V}}^0(x^{-k}\delta) \) are semisimple \( \Gamma \)-modules; since \( t_\mathcal{F}^k\mathcal{R}_L(\delta)/\mathcal{R}_L(\delta) \) is a semisimple \( \Gamma \)-module, so is \( H_{\varphi_q,\mathcal{V}}^0(t_\mathcal{F}^k\mathcal{R}_L(\delta)/\mathcal{R}_L(\delta)) \). Hence, taking \( \Gamma \)-invariants of each term in (5-3), we obtain the desired exact sequence.

**Proposition 5.25.** Let \( \delta \) be in \( \mathcal{F}_{an}(L) \), \( k \in \mathbb{Z}_+ \). If \( w_\delta \notin \{1, \ldots, k\} \), then \( t_{k, an} \) and \( t_k \) are isomorphisms.
Proof. We only prove the assertion for $\iota_{k,\text{an}}$. The proof of the assertion for $\iota_k$ is similar. By Theorem 5.19, $\dim_L H^1_{\text{an}}(\delta) = \dim_L H^1_{\text{an}}(x^{-k}\delta)$ when $w_\delta \notin \{1, \ldots, k\}$. Combining Lemma 5.24 with the fact that $H^0_{\text{an}}(t_{F,k}^*\mathcal{R}_L(\delta)/\mathcal{R}_L(\delta)) = 0$ and that $\dim_L H^1_{\text{an}}(\delta) = \dim_L H^1_{\text{an}}(x^{-k}\delta)$, we obtain the assertion. \qed

We assign to any nonzero $c \in H^1_{\text{an}}(\delta)$ an $\mathcal{L}$-invariant in $P^1(L) = L \cup \{\infty\}$. In the case of $\delta = x^{-k}$ with $k \in \mathbb{N}$, put $\mathcal{L}((at^k, bt^k)) = a/b$. If $\delta = x\delta_{\text{unr}}$, then any $c \in H^1_{\text{an}}(\delta)$ can be written as

$$c = t_{F}^{-1}((q^{-1}\varphi_q - 1)(\lambda G(1, 1) + \mu(\log u_{\mathcal{L}} - z)), t_{F}^{-1}\partial(\lambda G(1, 1) + \mu(\log u_{\mathcal{L}} - z)))$$

with $\lambda, \mu \in L$. Here $G(1, 1)$ is an element of $\mathcal{R}_L$ which induces a basis of $(\mathcal{R}_L/t_{F,L}^\delta)^\Gamma$ and whose value at $\xi_n$ is $1 \otimes 1 \in L \otimes F_n$ when $n$ is large enough; $z$ is an element of $\mathcal{R}_L$ whose value at $\xi_n$ is $1 \otimes \log(\xi_n) \in L \otimes F_n$ for any $n$. We put $\mathcal{L}(c) = -(e_F(q^{-1} - 1)/q) \cdot (\lambda/\mu)$. In the case of $\delta = x^k\delta_{\text{unr}}$ with $k \geq 2$, for any $c \in H^1_{\text{an}}(x^k\delta_{\text{unr}})$, put $\mathcal{L}(c) = \mathcal{L}(\iota_{k-1,\text{an}}(c))$. In the case that $\delta$ is not of the form $x^{-k}$ with $k \in \mathbb{N}$ or the form $x^k\delta_{\text{unr}}$ with $k \in \mathbb{Z}_+$, we put $\mathcal{L}(c) = \infty$.

**Proposition 5.26.** Let $\delta$ be in $\mathcal{L}_{\text{an}}(L)$, $k \in \mathbb{Z}_+$.

(a) If $w_\delta \in \{1, \ldots, k\}$ and if $\delta \neq x^{w_\delta}, x^{w_\delta}\delta_{\text{unr}}$, then $\iota_{k,\text{an}}$ and $\iota_k$ are zero.

(b) If $\delta = x^{w_\delta}\delta_{\text{unr}}$ with $1 \leq w_\delta \leq k$, then $\iota_{k,\text{an}}$ and $\iota_k$ are surjective, and the kernel of $\iota_{k,\text{an}}$ is the 1-dimensional subspace $\{c \in H^1_{\text{an}}(\delta): c = 0\}$ or $\mathcal{L}(c) = \infty$.

(c) If $\delta = x^{w_\delta}$ with $1 \leq w_\delta \leq k$, then $\iota_{k,\text{an}}$ and $\iota_k$ are injective, and the image of $\iota_{k,\text{an}}$ is $\{c \in H^1_{\text{an}}(x^{-k}\delta): c = 0\}$ or $\mathcal{L}(c) = \infty$.

**Proof.** We will use Lemma 5.24 frequently without mentioning it.

First we prove (a). From $\dim_L H^0_{\text{an}}(t_{F,k}^*\mathcal{R}_L(\delta)/\mathcal{R}_L(\delta)) = \dim_L H^1_{\text{an}}(\delta) = 1$ and $H^0_{\text{an}}(x^{-k}\delta) = 0$ we obtain the assertion for $\iota_{k,\text{an}}$. The assertion for $\iota_k$ follows from this and the commutative diagram (5-1), where the two vertical maps are isomorphisms.

Next we prove (b). From $H^0_{\text{an}}(x^{-k}\delta) = 0$, $\dim_L H^0_{\text{an}}(t_{F,k}^*\mathcal{R}_L(\delta)/\mathcal{R}_L(\delta)) = 1$, $\dim_L H^1_{\text{an}}(\delta) = 2$, and $\dim_L H^1_{\text{an}}(x^{-k}\delta) = 1$, we obtain the surjectivity of $\iota_{k,\text{an}}$. The surjectivity of $\iota_k$ follows from this and the commutative diagram (5-1), where the two vertical maps are isomorphisms. We show that if $c \in H^1_{\text{an}}(\delta)$ satisfies $\mathcal{L}(c) = \infty$, then $\iota_{k,\text{an}}(c) = 0$. As $\mathcal{L}(\iota_{w_\delta-1,\text{an}}(c)) = \infty$ and $\iota_{k,\text{an}} = \iota_{k+1-w_\delta,\text{an}}\iota_{w_\delta-1,\text{an}}$, we reduce to the case of $\delta = x\delta_{\text{unr}}$. In this case, $c = t_{F}^{-1}\lambda((q^{-1}\varphi_q - 1)G(1, 1), \nabla G(1, 1))$ with $\lambda \in L$. Thus $\iota_{1,\text{an}}(c) = \lambda((q^{-1}\varphi_q - 1)G(1, 1), \nabla G(1, 1)) \sim (0, 0)$. Hence $\iota_{k,\text{an}}(c) = 0$ for any integer $k \geq 1$.

Finally we prove (c). From the equalities $H^0_{\text{an}}(\delta) = 0$ and $\dim_L H^0_{\text{an}}(x^{-k}\delta) = \dim_L H^0_{\text{an}}(t_{F,k}^*\mathcal{R}_L(\delta)/\mathcal{R}_L(\delta)) = 1$, we obtain the injectivity of $\iota_{k,\text{an}}$. The injectivity of $\iota_k$ follows from this and the commutative diagram (5-1), where the vertical map $\Upsilon^\delta_{\text{an}} \circ \Upsilon^\delta_f$ is an isomorphism. For the second assertion, let $(m, n)$ be in $\mathbb{Z}^1(x^{w_\delta})$. 

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Thus it is unique up to a nonzero multiple and induces an isomorphism from

\[ D \] such that

\[ \text{Proj} \] (\( \phi \)

analytic)

\[ \text{Proj} \] to \( \text{Proj} \). Colmez [2008] classified 2-dimensional trianguline representations of the Galois group \( G_{\mathbb{Q}_p} \). Generalizing his work, Nakamura [2009] classified 2-dimensional trianguline representations of the Galois group of a \( p \)-adic local field that is finite over \( \mathbb{Q}_p \).

In this section we classify triangulable \( \mathcal{O}_F \)-analytic \( (\varphi_q, \Gamma) \)-modules of rank 2 following Colmez’s method. First we recall the definition.

**Definition 6.1.** A \( (\varphi_q, \Gamma) \)-module over \( \mathcal{R}_L \) is called **triangulable** if there exists a filtration of \( D \) consisting of \( (\varphi_q, \Gamma) \)-submodules \( 0 = D_0 \subset D_1 \subset \cdots \subset D_d = D \) such that \( D_i / D_{i-1} \) is free of rank 1 over \( \mathcal{R}_L \).

Note that if \( D \) is \( \mathcal{O}_F \)-analytic, then so is \( D_i / D_{i-1} \) for any \( i \).

If \( \delta_1, \delta_2 \in \mathcal{I}_\text{an}(L) \), then \( \text{Ext}(\mathcal{R}_L(\delta_2), \mathcal{R}_L(\delta_1)) \) is isomorphic to \( \text{Ext}(\mathcal{R}_L, \mathcal{R}_L(\delta_1\delta_2^{-1})) \) or \( H^1(\delta_1\delta_2^{-1}) \). The isomorphism only depends on the choices of \( e_{\delta_1} \), \( e_{\delta_2} \) and \( e_{\delta_1\delta_2^{-1}} \). Thus it is unique up to a nonzero multiple and induces an isomorphism from \( \text{Proj}(\text{Ext}(\mathcal{R}_L(\delta_2), \mathcal{R}_L(\delta_1))) \) to \( \text{Proj}(H^1(\delta_1\delta_2^{-1})) \) independent of the choices of \( e_{\delta_1} \), \( e_{\delta_2} \) and \( e_{\delta_1\delta_2^{-1}} \). Similarly, there is a natural isomorphism from

\[ \text{Proj}(\text{Ext}_\text{an}(\mathcal{R}_L(\delta_2), \mathcal{R}_L(\delta_1))) \]

to \( \text{Proj}(H^1_\text{an}(\delta_1\delta_2^{-1})) \). Hence the set of triangulable (resp. triangulable and \( \mathcal{O}_F \)-analytic) \( (\varphi_q, \Gamma) \)-modules \( D \) of rank 2 satisfying the following two conditions is classified by \( \text{Proj}(H^1(\delta_1\delta_2^{-1})) \) (resp. \( \text{Proj}(H^1_\text{an}(\delta_1\delta_2^{-1})) \)):

- \( \mathcal{R}_L(\delta_1) \) is a saturated \( (\varphi_q, \Gamma) \)-submodule of \( D \) and \( \mathcal{R}_L(\delta_2) \) is the quotient module.

- \( D \) is not isomorphic to \( \mathcal{R}_L(\delta_1) \oplus \mathcal{R}_L(\delta_2) \).
Let $\mathcal{F}^{an} = \mathcal{F}^{an}(L)$ be the analytic variety obtained by blowing up $(\delta_1, \delta_2) \in \mathcal{F}_{an}(L) \times \mathcal{F}_{an}(L)$ along the subvarieties $\delta_1 \delta_2^{-1} = x^i \delta_{unr}$ for $i \in \mathbb{Z}_+$ and the subvarieties $\delta_1 \delta_2^{-1} = x^{-i}$ for $i \in \mathbb{N}$. The fiber over the point $(\delta_1, \delta_2)$ is isomorphic to $\text{Proj}(H^1_{an}(\delta_1 \delta_2^{-1}))$. Similarly, let $\mathcal{F} = \mathcal{F}(L)$ be the analytic variety over $\mathcal{F}_{an}(L) \times \mathcal{F}_{an}(L)$ whose fiber over $(\delta_1, \delta_2)$ is isomorphic to $\text{Proj}(H^1(\delta_1 \delta_2^{-1}))$. The inclusions $\text{Ext}_{an}(\mathcal{R}_L(\delta_1), \mathcal{R}_L(\delta_2)) \hookrightarrow \text{Ext}(\mathcal{R}_L(\delta_1), \mathcal{R}_L(\delta_2))$ for $\delta_1, \delta_2 \in \mathcal{F}_{an}(L)$ induce a natural injective map $\mathcal{F}^{an} \hookrightarrow \mathcal{F}$. We write points of $\mathcal{F}$ (resp. $\mathcal{F}^{an}$) in the form $(\delta_1, \delta_2, c)$ with $c \in \text{Proj}(H^1(\delta_1 \delta_2^{-1}))$ (resp. $c \in \text{Proj}(H^1_{an}(\delta_1 \delta_2^{-1}))$). If $(\delta_1, \delta_2, c) \in \mathcal{F}$ is in the image of $\mathcal{F}^{an}$, for our convenience we use $c_{an}$ to denote the element in $\text{Proj}(H^1_{an}(\delta_1 \delta_2^{-1}))$ corresponding to $c$. For $(\delta_1, \delta_2, c) \in \mathcal{F}^{an}$, since the $L$-invariant induces an inclusion $\text{Proj}(H^1_{an}(\delta_1 \delta_2^{-1})) \hookrightarrow \mathbb{P}^1(L)$, we also use $(\delta_1, \delta_2, L(c))$ to denote $(\delta_1, \delta_2, c)$.

If $s \in \mathcal{F}$, we assign to $s$ the invariant $w(s) \in \mathbb{L}$ by $w(s) = w_{\delta_1} - w_{\delta_2}$. Let $\mathcal{F}_+$ be the subset of $\mathcal{F}$ consisting of elements $s \in \mathcal{F}$ with

$$v_\pi(\delta_1(\pi)) + v_\pi(\delta_2(\pi)) = 0, \quad v_\pi(\delta_1(\pi)) \geq 0.$$ 

If $s \in \mathcal{F}_+$, we assign to $s$ the invariant $u(s) \in \mathbb{Q}_+$ by

$$u(s) = v_\pi(\delta_1(\pi)) = -v_\pi(\delta_2(\pi)).$$

Put $\mathcal{F}_0 = \{s \in \mathcal{F}_+ \mid u(s) = 0\}$ and $\mathcal{F}_* = \{s \in \mathcal{F}_+ \mid u(s) > 0\}$. Then $\mathcal{F}_+$ is the disjoint union of $\mathcal{F}_0$ and $\mathcal{F}_*$. For $? \in \{+, 0, *\}$ we put $\mathcal{F}^{an}_? = \mathcal{F}^{an} \cap \mathcal{F}_?$. We decompose the set $\mathcal{F}^{an}$ as $\mathcal{F}^{an}_? = \mathcal{F}^{an}_{?g} \cup \mathcal{F}^{an}_{?c} \cup \mathcal{F}^{an}_{?st} \cup \mathcal{F}^{an}_{?ord} \cup \mathcal{F}^{an}_{?ncl}$, where

$$\mathcal{F}^{an}_{?g} = \{s \in \mathcal{F}^? \mid w(s) \text{ is not an integer } \geq 1\},$$

$$\mathcal{F}^{an}_{?c} = \{s \in \mathcal{F}^? \mid w(s) \text{ is an integer } \geq 1, u(s) < w(s), L = \infty\},$$

$$\mathcal{F}^{an}_{?st} = \{s \in \mathcal{F}^? \mid w(s) \text{ is an integer } \geq 1, u(s) < w(s), L \neq \infty\},$$

$$\mathcal{F}^{an}_{?ord} = \{s \in \mathcal{F}^? \mid w(s) \text{ is an integer } \geq 1, u(s) = w(s)\},$$

$$\mathcal{F}^{an}_{?ncl} = \{s \in \mathcal{F}^? \mid w(s) \text{ is an integer } \geq 1, u(s) > w(s)\}.$$ 

Note that $\mathcal{F}^{an}_{?ord}$ and $\mathcal{F}^{an}_{?ncl}$ are empty.

Let $D$ be an extension of $\mathcal{R}_L(\delta_2)$ by $\mathcal{R}_L(\delta_1)$. For any $k \in \mathbb{N}$, the preimage of $t^k_{\delta_2}\mathcal{R}_L(\delta_2)$ is a $(\varphi_q, \Gamma)$-submodule of $D$, which is denoted by $D'$. Then $D'$ is an extension of $\mathcal{R}_L(x^k \delta_2)$ by $\mathcal{R}_L(\delta_1)$. If $D$ is $\mathcal{O}_F$-analytic, then so is $D'$.

Lemma 6.2. (a) The class of $D'$ in $H^1(\delta_1 \delta_2^{-1} x^{-k})$ coincides with $\varpi_k(c)$ up to a nonzero multiple, where $c$ is the class of $D$ in $H^1(\delta_1 \delta_2^{-1})$.

(b) If $D$ is $\mathcal{O}_F$-analytic, the class of $D'$ in $H^1_{an}(\delta_1 \delta_2^{-1} x^{-k})$ coincides with $\varpi_k, an(c)$ up to a nonzero multiple, where $c$ is the class of $D$ in $H^1_{an}(\delta_1 \delta_2^{-1})$.

Proof. We only prove (b). The proof of (a) is similar. Let $e$ be a basis of $\mathcal{R}_L(\delta_2)$ such that $\varphi_q(e) = \delta_2(\pi)e$ and $\sigma_a e = \delta_2(a)e$. Let $\tilde{e}$ be a lifting of $e$ in $D$. The class
of \( D \), or the same, \( c \), coincides with the class of \(((\delta_2(\pi)^{-1}\varphi_q - 1)\tilde{e}, (\nabla - w_{\delta_2})\tilde{e})\) up to a nonzero multiple. Similarly, up to a nonzero multiple, the class of \( D' \) coincides with the class of

\[
((\pi^{-k}\delta_2(\pi)^{-1}\varphi_q - 1)(t_{\pi}^k\tilde{e}), (\nabla - w_{\delta_2} - k)(t_{\pi}^k\tilde{e}))
\]

\[
= (t_{\pi}^k(\delta_2(\pi)^{-1}\varphi_q - 1)\tilde{e}, t_{\pi}^k(\nabla - w_{\delta_2})\tilde{e}),
\]

which is exactly \( t_{k,an}(c) \).

**Proposition 6.3.** Put \( D = D(s) \) with \( s = (\delta_1, \delta_2, c) \in \mathcal{I} \). The following two conditions are equivalent:

(a) \( D(s) \) has a \((\varphi_q, \Gamma)\)-submodule \( M \) of rank 1 such that \( M \cap R_L(\delta_1) = 0 \).

(b) \( s \) is in \( \mathcal{I}_{an} \) and satisfies \( w(s) \in \mathbb{Z}_+ \), \( \delta_1\delta_2^{-1} \neq x^{w(s)} \) and \( L(c_{an}) = \infty \).

Among all such \( M \) there exists a unique one, \( M_{sat} \), that is saturated; \( M_{sat} \) is isomorphic to \( R_L(x^{w(s)}\delta_2) \). For any \( M \) that satisfies condition (a), there exists some \( i \in \mathbb{N} \) such that \( M = t_{i}^{k}M_{sat} \).

**Proof.** Assume that \( D(s) \) satisfies (a). Since the intersection of \( M \) and \( R_L(\delta_1) \) is zero, the image of \( M \) in \( R_L(\delta_2) \) is a nonzero \((\varphi_q, \Gamma)\)-submodule of \( R_L(\delta_2) \), and so must be of the form \( t_{\pi}^kR_L(\delta_2) \) with \( k \in \mathbb{N} \). Since \( D(s) \) does not split, we have \( k \geq 1 \). The preimage of \( t_{\pi}^kR_L(\delta_2) \) in \( D \) is exactly \( M \oplus R_L(\delta_1) \). Since \( M \oplus R_L(\delta_1) \) splits, by Lemma 6.2 we have \( t_{k}(c) = 0 \). By Proposition 5.26 this happens only if \( w(s) \in \{1, \ldots, k\} \) and \( \delta_1\delta_2^{-1} \neq x^{w(s)} \). Note that, when \( w(s) \in \{1, \ldots, k\} \) and \( \delta_1\delta_2^{-1} \neq x^{w(s)} \), \( D(s) \) is automatically \( \mathcal{O}_F \)-analytic. Again by Proposition 5.26 we obtain \( L(c_{an}) = \infty \). This proves (a)\( \Leftrightarrow \) (b).

If (a) holds, then the preimage of \( t_{\pi}^{w(s)}R_L(\delta_2) \) splits as \( R_L(\delta_1) \oplus M_0 \), where \( M_0 \) is not included in \( t_{\pi}D(s) \). Otherwise, the preimage of \( t_{\pi}^{w(s)-1}R_L(\delta_2) \) will split, which contradicts Proposition 5.26. Let \( e_1 \) (resp. \( e_2, e \)) be a basis of \( R_L(\delta_1) \) (resp. \( R_L(\delta_2), M_0 \)) such that \( Le_1 \) (resp. \( Le_2, Le \)) is stable under \( \varphi_q \) and \( \Gamma \). Let \( \tilde{e}_2 \) be a lifting of \( e_2 \). Write \( e = ae_1 + b\tilde{e}_2 \). Then \( a \notin t_{\pi}R_L \) and \( b \in t_{\pi}^{w(s)}R_L \). Observe that the ideal \( I \) generated by \( a \) and \( t_{\pi}^{w(s)} \) satisfies \( \varphi_q(I) = I \) and \( \gamma(I) = I \) for all \( \gamma \in \Gamma \). Thus by Lemma 1.1, \( I = R_L \). It follows that \( M_0 \) is saturated. If \( M \) is another \((\varphi_q, \Gamma)\)-submodule of \( D(s) \) such that \( M \cap R_L(\delta_1) = 0 \), then the image of \( M \) in \( R_L(\delta_2) \) is \( t_{\pi}^{k}R_L(\delta_2) \) for some integer \( k \geq w(s) \). Then \( M \subset R_L(\delta_1) \oplus M_0 \). Since \( \delta_1 \neq \delta_2x^{w(s)} \), \( R_L(\delta_1) \) has no nonzero \((\varphi_q, \Gamma)\)-submodule isomorphic to \( R_L(x^{\delta_2}) \). It follows that \( M \subset M_0 \) and thus \( M = t_{i}^{k-w(s)}M_0 \).
Let $s = (\delta_1, \delta_2, c)$ and $s' = (\delta_1', \delta_2', c')$ be in $\mathcal{F}(L)$.

(a) If $\delta_1 = \delta_1'$, then $D(s) \cong D(s')$ if and only if $s = s'$.

(b) If $\delta_1 \neq \delta_1'$, then $D(s) \cong D(s')$ if and only if $s$ and $s'$ are in $\mathcal{F}^{\text{an}}$ and satisfy $w(s) \in \mathbb{Z}_+, \delta_1' = x^{w(s)}\delta_2', \delta_2' = x^{-w(s)}\delta_1$ and $\mathcal{L}(c_{\text{an}}) = \mathcal{L}(c'_{\text{an}}) = \infty$.

**Proof.** Assertion (a) is clear. We prove (b). Since $D(s) \cong D(s')$, there exists a $(\varphi_q, \Gamma)$-submodule $M$ of $D(s)$ such that $M \cong \mathcal{R}_L(\delta_1')$ and $D(s)/M \cong \mathcal{R}_L(\delta_2')$. Since both $\mathcal{R}_L(\delta_1)$ and $M$ are saturated $(\varphi_q, \Gamma)$-submodules of $D$, $\mathcal{R}_L(\delta_1) \cap M = 0$. By Proposition 6.3 we have $w(s) \in \mathbb{Z}_+, \delta_1\delta_2^{-1} \neq x^{w(s)}$, $\mathcal{L}(c_{\text{an}}) = \infty$ and $\delta_1' = x^{w(s)}\delta_2$. Similarly, $\delta_1 = x^{w(s')}\delta_2'$. As $\delta_1\delta_2 = \delta_1'\delta_2'$, we have $w(s) = w(s')$.

**Proposition 6.6.** Let $s = (\delta_1, \delta_2, c)$ be in $\mathcal{F}$. Then $D(s)$ is of slope zero if and only if $s \in \mathcal{F}^+ - \mathcal{F}^{\text{cnc}}$, $D(s)$ is of slope zero and the Galois representation attached to $D(s)$ is irreducible if and only if $s$ is in $\mathcal{F}^+ - (\mathcal{F}^{\text{ord}} \cup \mathcal{F}^{\text{cnc}})$; $D(s)$ is of slope zero and $\mathcal{O}_F$-analytic if and only if $s \in \mathcal{F}^{\text{an}} - \mathcal{F}^{\text{cnc}}$.

**Proof.** By Kedlaya’s slope filtration theorem, $D(s)$ is of slope zero if and only if $v_{\pi}(\delta_1(\pi)\delta_2(\pi)) = 0$ and $D(s)$ has no $(\varphi_q, \Gamma)$-submodule of rank 1 that is of slope $< 0$. In particular, if $D(s)$ is of slope zero, then $v_{\pi}(\delta_1(\pi)) \geq 0$ and thus $s \in \mathcal{F}^+$. Hence we only need to consider the case of $s \in \mathcal{F}^+$. Assume that $D(s)$ has a $(\varphi_q, \Gamma)$-submodule of rank 1, say $M$, that is of slope $< 0$. Then the intersection of $M$ and $\mathcal{R}_L(\delta_1)$ is zero. By Proposition 6.3, we may suppose that $M$ is saturated. By Corollary 6.4, this happens if and only if $s$ is in $\mathcal{F}^{\text{an}}$ and satisfies $w(s) \in \mathbb{Z}_+, \delta_1\delta_2^{-1} \neq x^{w(s)}$, $\mathcal{L}(c_{\text{an}}) = \infty$ and $w(s) < u(s)$. Note that $\delta_1\delta_2^{-1} \neq x^{w(s)}$ and $\mathcal{L}(c_{\text{an}}) = \infty$ automatically hold when $0 < w(s) < u(s)$. The first assertion follows. Similarly, $D(s)$ has a saturated $(\varphi_q, \Gamma)$-submodule of rank 1 that is of slope zero if and only if $u(s) = 0$ or $u(s) = w(s)$. By Proposition 1.5(c) and Remark 1.8, we know that the Galois representation attached to an étale $(\varphi_q, \Gamma)$-module $D$ over $\mathcal{R}_L$ of rank 2 is irreducible if and only if $D$ has no étale $(\varphi_q, \Gamma)$-submodule of rank 1. This shows the second assertion. The third assertion follows from the first one.

**Proof of Theorem 0.5.** Assertion (a) follows from Proposition 6.6, and (b) follows from Corollary 6.5.

**Remark 6.7.** Let $s \neq s'$ be as in Theorem 0.5(b). Then $s \in \mathcal{F}_{\ast}^{\text{cnc}}$ if and only if $s' \in \mathcal{F}_{\ast}^{\text{cnc}}$, $s \in \mathcal{F}_{\ast}^{\text{ord}}$ if and only if $s' \in \mathcal{F}_{\ast}^{\text{ord}}$, $s \in \mathcal{F}_{\ast}^{\text{st}}$ if and only if $s' \in \mathcal{F}_{\ast}^{\text{st}}$.

**Remark 6.8.** By an argument similar to that in [Colmez 2008] one can show that if $s$ is in $\mathcal{F}_{\ast}^{\text{cnc}}$ (resp. $\mathcal{F}_{\ast}^{\text{ord}}, \mathcal{F}_{\ast}^{\text{st}}$), then $D(s)$ comes from a crystalline (resp. ordinary, semistable but noncrystalline) $L$-representation twisted by a character.
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