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Cohomological invariants of algebraic tori

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Let G be an algebraic group over a field F. As defined by Serre, a cohomological invariant of G of degree n with values in $\mathbb{Q}/\mathbb{Z}(j)$ is a functorial-in-K collection of maps of sets $\mathrm{Tors}_G(K) \to H^n(K, \mathbb{Q}/\mathbb{Z}(j))$ for all field extensions K/F, where $\mathrm{Tors}_G(K)$ is the set of isomorphism classes of G-torsors over Spec K. We study the group of degree 3 invariants of an algebraic torus with values in $\mathbb{Q}/\mathbb{Z}(2)$. In particular, we compute the group $H^3_{\mathrm{nr}}(F(S), \mathbb{Q}/\mathbb{Z}(2))$ of unramified cohomology of an algebraic torus S.

1. Introduction

Let G be a linear algebraic group over a field F (of arbitrary characteristic). The notion of an *invariant* of G was defined in [Garibaldi et al. 2003] as follows. Consider the category Fields_F of field extensions of F and the functor

$$\mathrm{Tors}_G : \mathit{Fields}_F \to \mathit{Sets}$$

taking a field K to the set $\mathrm{Tors}_G(K)$ of isomorphism classes of (right) G-torsors over Spec K. Let

$$H : \mathit{Fields}_F \to \mathit{Abelian\ Groups}$$

be another functor. An H-*invariant* of G is then a morphism of functors

$$i : \mathrm{Tors}_G \to H,$$

viewing H with values in *Sets*, that is, a functorial in K collection of maps of sets $\mathrm{Tors}_G(K) \to H(K)$ for all field extensions K/F. We denote the group of H-invariants of G by $\mathrm{Inv}(G, H)$.

An invariant $i \in \mathrm{Inv}(G, H)$ is called *normalized* if $i(I) = 0$ for the trivial G-torsor I. The normalized invariants form a subgroup $\mathrm{Inv}(G, H)_{\mathrm{norm}}$ of $\mathrm{Inv}(G, H)$ and there is a natural isomorphism

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$$\mathrm{Inv}(G, H) \simeq H(F) \oplus \mathrm{Inv}(G, H)_{\mathrm{norm}},$$

so it is sufficient to study normalized invariants.

Typically, H is a cohomological functor given by Galois cohomology groups with values in a fixed Galois module. Of particular interest to us is the functor H which takes a field K/F to the Galois cohomology group $H^n(K, \mathbb{Q}/\mathbb{Z}(j))$, where the coefficients $\mathbb{Q}/\mathbb{Z}(j)$ are defined as follows. For a prime integer p different from the characteristic of F, the p-component $\mathbb{Q}_p/\mathbb{Z}_p(j)$ of $\mathbb{Q}/\mathbb{Z}(j)$ is the colimit over n of the étale sheaves $\mu_{p^n}^{\otimes j}$, where μ_m is the sheaf of m-th roots of unity. In the case $p = \mathrm{char}(F) > 0$, $\mathbb{Q}_p/\mathbb{Z}_p(j)$ is defined via logarithmic de Rham–Witt differentials; see Section 3b.

We write $\mathrm{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j))$ for the group of *cohomological invariants of G of degree n with values in* $\mathbb{Q}/\mathbb{Z}(j)$.

The second cohomology group $H^2(K, \mathbb{Q}/\mathbb{Z}(1))$ is canonically isomorphic to the Brauer group $\mathrm{Br}(K)$ of the field K. In Section 2c we prove (Theorem 2.4) that if G is a connected group (reductive if F is not perfect), then

$$\mathrm{Inv}(G, \mathrm{Br})_{\mathrm{norm}} \simeq \mathrm{Pic}(G).$$

The group $\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ for a semisimple simply connected group G has been studied by Rost; see [Garibaldi et al. 2003].

An essential object in the study of cohomological invariants is the notion of a *classifying torsor*: a G-torsor $E \to X$ for a smooth variety X over F such that every G-torsor over an infinite field K/F is isomorphic to the pull-back of $E \to X$ along a K-point of X. If V is a generically free linear representation of G with a nonempty open subset $U \subset V$ such that there is a G-torsor $\pi : U \to X$, then π is classifying. Such representations exist (see Section 2b).

The generic fiber of π is the *generic torsor* over $\mathrm{Spec}\, F(X)$ attached to π. Evaluation at the generic torsor yields a homomorphism

$$\mathrm{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j)) \to H^n(F(X), \mathbb{Q}/\mathbb{Z}(j)), \tag{1-1}$$

and in Section 3 we show that the image of this map is contained in the subgroup $H^0_{\mathrm{Zar}}(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))$ of $H^n(F(X), \mathbb{Q}/\mathbb{Z}(j))$, where $\mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))$ is the Zariski sheaf associated to the presheaf $W \mapsto H^n(W, \mathbb{Q}/\mathbb{Z}(j))$ of the étale cohomology groups. In fact, the image is contained in the subgroup $H^0_{\mathrm{Zar}}(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))_{\mathrm{bal}}$ of *balanced* elements, that is, elements that have the same images under the pull-back homomorphisms with respect to the two projections $(U \times U)/G \to X$. Moreover, the balanced elements precisely describe the image and we prove (Theorem 3.4):

Theorem A. *Let G be a smooth linear algebraic group over a field F. We assume that G is connected if F is a finite field. Let $E \to X$ be a classifying G-torsor with*

E a G-rational variety such that $E(F) \neq \varnothing$. Then (1-1) *yields an isomorphism* $\mathrm{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j)) \simeq H^0_{\mathrm{Zar}}(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))_{\mathrm{bal}}$.

At this point it is convenient to make use of a construction due to Totaro [1999]: because the Chow groups are homotopy invariant, the groups $\mathrm{CH}^n(X)$ do not depend on the choice of the representation V and the open set $U \subset V$ provided the codimension of $V \setminus U$ in V is large enough. This leads to the notation $\mathrm{CH}^n(BG)$, the Chow groups of the so-called *classifying space* BG, although BG itself is not defined in this paper.

Unfortunately, the étale cohomology groups with values in $\mathbb{Q}_p/\mathbb{Z}_p(j)$, where $p = \mathrm{char}(F) > 0$, are not homotopy invariant. In particular, we cannot use the theory of cycle modules of Rost [1996].

The main result of this paper is the exact sequence in Theorem 4.3 describing degree 3 cohomological invariants of an algebraic torus T. Writing $\widehat{T}_{\mathrm{sep}}$ for the character lattice of T over a separable closure of F and T° for the dual torus, we prove our main result:

Theorem B. *Let T be an algebraic torus over a field F. Then there is an exact sequence*

$$0 \to \mathrm{CH}^2(BT)_{\mathrm{tors}} \to H^1(F, T^0) \xrightarrow{\alpha} \mathrm{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \\ \to H^0(F, S^2(\widehat{T}_{\mathrm{sep}}))/\mathrm{Dec} \to H^2(F, T^0).$$

The homomorphism α is given by $\alpha(a)(b) = a_K \cup b$ for every $a \in H^1(F, T^0)$ and $b \in H^1(K, T)$ and every field extension K/F, where the cup-product is defined in (4-5), *and* Dec *is the subgroup of decomposable elements in the symmetric square $S^2(\widehat{T}_{\mathrm{sep}})$ defined in Section A-II.*

In the proof of the theorem we compute the group of balanced elements in the motivic cohomology group $H^4(BT, \mathbb{Z}(2))$ and relate it, using an exact sequence of B. Kahn and Theorem A, with the group of invariants $\mathrm{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$.

We also prove that the torsion group $\mathrm{CH}^2(BT)_{\mathrm{tors}}$ is finite of exponent 2 (Theorem 4.7) and the last homomorphism in the sequence is also of exponent 2 (see the discussion before Theorem 4.13).

Moreover, if p is an odd prime, the group $\mathrm{Inv}^3(T, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{norm}}$, which is the p-primary component of $\mathrm{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$, splits canonically into the direct sum of *linear* invariants (those that induce group homomorphisms from Tors_T to H^3) and *quadratic* invariants, that is, the invariants i such that the function $h(a, b) := i(a+b) - i(a) - i(b)$ is bilinear and $h(a, a) = 2i(a)$ for all a and b. Furthermore, the groups of linear and quadratic invariants with values in $\mathbb{Q}_p/\mathbb{Z}_p(2)$ are canonically isomorphic to $H^1(F, T^\circ)\{p\}$ and $(H^0(F, S^2(\widehat{T}_{\mathrm{sep}}))/\mathrm{Dec})\{p\}$, respectively.

We also prove (Theorem 4.10) that the degree 3 invariants have control over the structure of all invariants. Precisely, the group $\mathrm{Inv}^3(T_K, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ is trivial for all K/F if and only if T is *special*, that is, T has no nontrivial torsors over any field K/F, which in particular means T has no nonconstant H-invariants for every functor H.

Our motivation for considering invariants of tori comes from their connection with unramified cohomology (defined in Section 5). Specifically, this work began as an investigation of a problem posed by Colliot-Thélène [1995, p. 39]: for n prime to $\mathrm{char}(F)$ and $i \geq 0$, determine the unramified cohomology group $H^i_{\mathrm{nr}}(F(S), \mu_n^{\otimes(i-1)})$, where $F(S)$ is the function field of a torus S over F. The connection is provided by Theorem 5.7 where we show that the unramified cohomology of a torus S is calculated by the invariants of an auxiliary torus:

Theorem C. *Let S be a torus over F and let $1 \to T \to P \to S \to 1$ be a flasque resolution of S, that is, T is flasque and P is quasisplit. Then there is a natural isomorphism*

$$H^n_{\mathrm{nr}}(F(S), \mathbb{Q}/\mathbb{Z}(j)) \simeq \mathrm{Inv}^n(T, \mathbb{Q}/\mathbb{Z}(j)).$$

By Theorem B and Theorem C, we have an exact sequence

$$0 \to \mathrm{CH}^2(BT)_{\mathrm{tors}} \to H^1(F, T^0) \xrightarrow{\alpha} \overline{H}^3_{\mathrm{nr}}(F(S), \mathbb{Q}/\mathbb{Z}(2)) \\ \to H^0(F, S^2(\widehat{T}_{\mathrm{sep}}))/\,\mathrm{Dec} \to H^2(F, T^0)$$

describing the reduced third cohomology group

$$\overline{H}^3_{\mathrm{nr}}(F(S), \mathbb{Q}/\mathbb{Z}(2))) := H^3_{\mathrm{nr}}(F(S), \mathbb{Q}/\mathbb{Z}(2))/H^3(F, \mathbb{Q}/\mathbb{Z}(2)).$$

Moreover, for an odd prime p, we have a canonical direct sum decomposition of the p-primary components:

$$\overline{H}^3_{\mathrm{nr}}(F(S), \mathbb{Q}_p/\mathbb{Z}_p(2)) = H^1(F, T^0)\{p\} \oplus (H^0(F, S^2(\widehat{T}_{\mathrm{sep}}))/\,\mathrm{Dec})\{p\}.$$

Note that the torus S determines T up to multiplication by a quasisplit torus. If X is a smooth compactification of S, one can take the torus T with $\widehat{T}_{\mathrm{sep}} = \mathrm{Pic}(X_{\mathrm{sep}})$; see [Colliot-Thélène and Sansuc 1977, §2].

In the present paper, F denotes a field of arbitrary characteristic, F_{sep} a separable closure of F, and Γ the absolute Galois group $\mathrm{Gal}(F_{\mathrm{sep}}/F)$ of F.

The word "scheme" over a field F means a separated scheme over F and, following [Fulton 1984], a "variety" over F is an integral scheme of finite type over F. If X is a scheme over F and L/F is a field extension then we write X_L for $X \times_F \mathrm{Spec}\, L$. When $L = F_{\mathrm{sep}}$ we write simply X_{sep}.

A "linear algebraic group over F" is an affine group scheme of finite type over F, not necessarily smooth.

2. Invariants of algebraic groups

2a. ***Definitions and basic properties.*** Let G be a linear algebraic group over a field F. Consider the functor

$$\mathrm{Tors}_G : \mathit{Fields}_F \to \mathit{Sets}$$

from the category of field extensions of F to the category of sets taking a field K to the set $\mathrm{Tors}_G(K)$ of isomorphism classes of (right) G-torsors over $\operatorname{Spec} K$. Note that if G is a smooth group, then there is a natural bijection

$$\mathrm{Tors}_G(K) \simeq H^1(K, G) := H^1(\mathrm{Gal}(K_{\mathrm{sep}}/K), G(K_{\mathrm{sep}})).$$

Let $H : \mathit{Fields}_F \to \mathit{Abelian\ Groups}$ be a functor. We also view H as a functor with values in Sets. Following [Garibaldi et al. 2003], we define an *H-invariant of G* as a morphism of functors $\mathrm{Tors}_G \to H$ from the category Fields_F to Sets. All the H-invariants of G form the abelian group of invariants $\mathrm{Inv}(G, H)$.

An invariant $i \in \mathrm{Inv}(G, H)$ is called *constant* if there is an element $h \in H(F)$ such that $i(I) = h_K$ for every G-torsor $I \to \operatorname{Spec} K$, where h_K is the image of h under natural map $H(F) \to H(K)$. The constant invariants form a subgroup $\mathrm{Inv}(G, H)_{\mathrm{const}}$ of $\mathrm{Inv}(G, H)$ isomorphic to $H(F)$. An invariant $i \in \mathrm{Inv}(G, H)$ is called *normalized* if $i(I) = 0$ for the trivial G-torsor I. The normalized invariants form a subgroup $\mathrm{Inv}(G, H)_{\mathrm{norm}}$ of $\mathrm{Inv}(G, H)$ and we have the decomposition

$$\mathrm{Inv}(G, H) = \mathrm{Inv}(G, H)_{\mathrm{const}} \oplus \mathrm{Inv}(G, H)_{\mathrm{norm}} \simeq H(F) \oplus \mathrm{Inv}(G, H)_{\mathrm{norm}},$$

so it suffices to determine the normalized invariants.

2b. ***Classifying torsors.*** Let G be a linear algebraic group over a field F. A G-torsor $E \to X$ over a smooth variety X over F is called *classifying* if for every field extension K/F, with K infinite, and for every G-torsor $I \to \operatorname{Spec} K$, there is a point $x : \operatorname{Spec} K \to X$ such that the torsor I is isomorphic to the fiber $E(x)$ of $E \to X$ over x, that is, $I \simeq E(x) := x^*(E) = \operatorname{Spec}(K) \times_X E$. The generic fiber $E_{\mathrm{gen}} \to \operatorname{Spec} F(X)$ of a classifying torsor is called a *generic G-torsor*; see [ibid., Part 1, §5.3].

If V is a generically free linear representation of G with a nonempty open subset $U \subset V$ such that there is a G-torsor $\pi : U \to X$, then π is classifying; see [ibid., Part 1, §5.4]. We will write U/G for X and call π a *standard classifying G-torsor*. Standard classifying G-torsors exist: we can embed G into $U := \mathbf{GL}_{n,F}$ for some n as a closed subgroup. Then U is an open subset in the affine space $M_n(F)$ on which G acts linearly and the canonical morphism $U \to X := U/G$ is a G-torsor. Note that $U(F) \neq \varnothing$.

We say that a G-variety Y is *G-rational* if there is an affine space V with a linear G-action such that Y and V have G-isomorphic nonempty open G-invariant subvarieties. Note that if $U \to U/G$ is a standard classifying G-torsor, then U is a G-rational variety.

Let $E \to X$ be a classifying G-torsor and let $H : \mathit{Fields}_F \to \mathit{Abelian\ Groups}$ be a functor. Define the map

$$\theta_G : \mathrm{Inv}(G, H) \to H(F(X)), \quad i \mapsto i(E_{\mathrm{gen}}), \tag{2-1}$$

by sending an invariant to its value at the generic torsor E_{gen}.

Consider the following property of the functor H:

Property 2.1. The map $H(K) \to H(K((t)))$ is injective for any field extension K/F.

The following theorem, due to M. Rost, was proved in [Garibaldi et al. 2003, Part II, Theorem 3.3]. For completeness, we give a slightly modified proof in Section A-I.

Theorem 2.2. *Let G be a smooth linear algebraic group over F. If a functor $H : \mathit{Fields}_F \to \mathit{Abelian\ Groups}$ has Property 2.1, then the map θ_G is injective, that is, every H-invariant of G is determined by its value at the generic G-torsor.*

Let G' be a (closed) subgroup of G over F. The map of sets

$$H^1(K, G') \to H^1(K, G)$$

for every field extension K/F yields the *restriction map*

$$\mathrm{res} : \mathrm{Inv}(G, H) \to \mathrm{Inv}(G', H).$$

Choose standard torsors $\pi : U \to U/G$ and $\pi' : U \to U/G'$ (for example, with $U = \mathbf{GL}_{n,F}$ as above). The pull-back of π with respect to the natural morphism $\alpha : U/G' \to U/G$ is the push-forward of π' via the inclusion $G' \hookrightarrow G$. It follows that the diagram

$$\begin{tikzcd}
\mathrm{Inv}(G, H) \arrow[r, "\mathrm{res}"] \arrow[d, "\theta_G"'] & \mathrm{Inv}(G', H) \arrow[d, "\theta_{G'}"'] \\
H(F(U/G)) \arrow[r, "\alpha^*"] & H(F(U/G'))
\end{tikzcd}$$

is commutative.

2c. ***The Brauer group invariants.*** Let G be a smooth connected linear algebraic group over F. Every cohomological invariant of G of degree 1 is constant by [Knus et al. 1998, Proposition 31.15]. In this section we study (degree 2) Br-invariants for the Brauer group functor $K \mapsto \mathrm{Br}(K)$. We assume that G is reductive if $\mathrm{char}(F) > 0$.

Lemma 2.3. *For any field extension K/F such that F is algebraically closed in K, the natural map $\mathrm{Pic}(G) \to \mathrm{Pic}(G_K)$ is an isomorphism.*

Proof. We may assume that G is reductive by factoring out the unipotent radical in the case that F is perfect. There is an exact sequence (see [Colliot-Thélène 2004, Theorem 1.2])

$$1 \to C \to G' \to G \to 1$$

with C a torus and G' a reductive group with $\mathrm{Pic}(G'_L) = 0$ for any field extension L/F. Let T be the factor group of G' by the semisimple part. The result follows from the exact sequence [Sansuc 1981, Proposition 6.10] (note that G is reductive if L is not perfect)

$$\widehat{T}(L) \to \widehat{C}(L) \to \mathrm{Pic}(G_L) \to \mathrm{Pic}(G'_L) = 0$$

with $L = F$ and K since the groups $\widehat{T}(F)$ and $\widehat{C}(F)$ don't change when F is replaced by K. □

Since for any G_K-torsor $E \to \mathrm{Spec}(K)$ over a field extension K/F one has [Sansuc 1981, Proposition 6.10] the exact sequence

$$\mathrm{Pic}(E) \to \mathrm{Pic}(G_K) \xrightarrow{\delta} \mathrm{Br}(K) \xrightarrow{\varepsilon} \mathrm{Br}(E), \tag{2-2}$$

we obtain the homomorphism

$$\nu : \mathrm{Pic}(G) \to \mathrm{Inv}(G, \mathrm{Br}),$$

which takes an element $\alpha \in \mathrm{Pic}(G)$ to the invariant that sends a G-torsor E over a field extension K/F to $\delta(\alpha_K)$. If E is a trivial torsor, that is, $E(K) \neq \varnothing$, then ε is injective and hence $\delta = 0$. It follows that the invariant $\nu(\alpha)$ is normalized.

Theorem 2.4. *Let G be a smooth connected linear algebraic group over F. Assume that G is reductive if $\mathrm{char}(F) > 0$. Then the map $\nu : \mathrm{Pic}(G) \to \mathrm{Inv}(G, \mathrm{Br})_{\mathrm{norm}}$ is an isomorphism.*

Proof. Choose a standard classifying G-torsor $U \to U/G$. Write K for the function field $F(U/G)$ and let U_{gen} be the generic G-torsor over K. Consider the commutative diagram

$$\begin{tikzcd}
 & \mathrm{Pic}(G) \arrow[r, "\nu"] \arrow[d, "j"'] & \mathrm{Inv}(G, \mathrm{Br})_{\mathrm{norm}} \arrow[d, "\theta_G"'] & \\
\mathrm{Pic}(U_{\mathrm{gen}}) \arrow[r] & \mathrm{Pic}(G_K) \arrow[r, "\delta"] & \mathrm{Br}(K) \arrow[r, "i"] & \mathrm{Br}(K(U_{\mathrm{gen}})),
\end{tikzcd}$$

where the bottom sequence is (2-2) for the G-torsor $U_{\mathrm{gen}} \to \mathrm{Spec}(K)$ followed by the injection $\mathrm{Br}(U_{\mathrm{gen}}) \to \mathrm{Br}(K(U_{\mathrm{gen}}))$ (see [Milne 1980, Chapter IV, Corollary 2.6]), and the map θ_G is evaluation at the generic torsor U_{gen} given in (2-1) and is injective by Theorem 2.2. Since the generic torsor is split over $K(U_{\mathrm{gen}})$, $\mathrm{Im}(\theta_G) \subset \mathrm{Ker}(i) = \mathrm{Im}(\delta)$. By Lemma 2.3, j is an isomorphism, hence ν is surjective.

Note that U_{gen} is a localization of U, hence $\operatorname{Pic}(U_{\mathrm{gen}}) = 0$ as $\operatorname{Pic}(U) = 0$. It follows that ν is injective. □

An algebraic group G over a field F is called *special* if $H^1(K, G) = \{1\}$ for every field extension K/F, that is, all G-torsors over any field extension of F are trivial.

Corollary 2.5. *If the group G is special, then* $\operatorname{Pic}(G) = 0$.

3. Invariants with values in $\mathbb{Q}/\mathbb{Z}(j)$

In this section we find a description for the group of cohomological invariants with values in $\mathbb{Q}/\mathbb{Z}(j)$ by identifying the image of the embedding θ_G in (2-1).

Let G be a linear algebraic group over a field F, let $H \subset G$ be a subgroup and let $E \to X$ be a G-torsor. Suppose that G/H is affine. Consider a G-action on $E \times (G/H)$ by $(e, g'H)g = (eg, g^{-1}g'H)$. By [Milne 1980, Theorem I.2.23], the affine G-equivariant projection $E \times (G/H) \to E$ descends to an affine morphism $Y \to X$. The (trivial right) H-torsor $E \times G \to E \times (G/H)$ descends to an H-torsor $E \to Y$. We will write E/H for Y.

Example 3.1. Let G be a linear algebraic group over a field F and let $E \to X$ be a G-torsor. Then for every $n > 0$, $E^n := E \times_F \cdots \times_F E$ (n times) is a G^n-torsor over X^n. Viewing G as the diagonal subgroup of G^n, we have the G-torsor $E^n \to E^n/G$.

3a. ***Balanced elements.*** Let G be a linear algebraic group over a field F. We assume that G is connected if F is finite. Let $E \to X$ be a G-torsor such that $E(F) \neq \varnothing$. We write p_1 and p_2 for the two projections $E^2/G = (E \times_F E)/G \to X$ (see Example 3.1).

Lemma 3.2. *Let K/F be a field extension and $x_1, x_2 \in X(K)$. Then the G-torsors $E(x_1)$ and $E(x_2)$ over K are isomorphic if and only if there is a point $y \in (E^2/G)(K)$ such that $p_1(y) = x_1$ and $p_2(y) = x_2$.*

Proof. "⇒": By construction, we have G-equivariant morphisms $f_i : E(x_i) \to E$ for $i = 1, 2$. Choose an isomorphism $h : E(x_1) \xrightarrow{\sim} E(x_2)$ of G-torsors over K. The morphism $(f_1, f_2h) : E(x_1) \to E^2$ yields the required point $\operatorname{Spec} K = E(x_1)/G \to E^2/G$.

"⇐": The pull-back of $E \to X$ with respect to any projection $E^2/G \to X$ coincides with the G-torsor $E^2 \to E^2/G$, hence

$$E(x_1) = x_1^*(E) = y^* p_1^*(E) \simeq y^*(E^2) \simeq y^* p_2^*(E) = x_2^*(E) = E(x_2). \qquad \square$$

Let H be a (contravariant) functor from the category of schemes over F to the category of abelian groups. We have the two maps $p_i^* : H(X) \to H(E^2/G)$, $i = 1, 2$. An element $h \in H(X)$ is called *balanced* if $p_1^*(h) = p_2^*(h)$. We write $H(X)_{\mathrm{bal}}$ for the subgroup of balanced elements in $H(X)$. In other words, $H(X)_{\mathrm{bal}} = h_0(H(E^\bullet/G))$ in the notation of Section A-IV.

We can view H as a (covariant) functor $\mathit{Fields}_F \to \mathit{Sets}$ taking a field K to $H(K) := H(\operatorname{Spec} K)$.

Lemma 3.3. *Let $h \in H(X)_{\mathrm{bal}}$ be a balanced element, K/F a field extension and I a G-torsor over* $\operatorname{Spec}(K)$. *Let $x \in X(K)$ be a point such that $E(x) \simeq I$. Then the element $x^*(h)$ in $H(K)$ does not depend on the choice of x.*

Proof. Let $x_1, x_2 \in X(K)$ be two points such that $E(x_1) \simeq E(x_2)$. By Lemma 3.2, there is a point $y \in (E^2/G)(K)$ such that $p_1(y) = x_1$ and $p_2(y) = x_2$. Therefore

$$x_1^*(h) = y^*(p_1^*(h)) = y^*(p_2^*(h)) = x_2^*(h). \qquad \square$$

It follows from Lemma 3.3 that if the torsor $E \to X$ is classifying with $E(F) \neq \varnothing$, then every element $h \in H(X)_{\mathrm{bal}}$ determines an H-invariant i_h of G as follows. Let I be a G-torsor over a field extension K/F. We claim that there is a point $x \in X(K)$ such that $E(x) \simeq I$. If K is infinite, this follows from the definition of the classifying G-torsor. If K is finite then all G-torsors over K are trivial by [Lang 1956], as G is connected. Since $E(K) \neq \varnothing$, we can take for x the image in $X(K)$ of any point in $E(K)$. Defining $i_h(E) = x^*(h) \in H(K)$, we have a group homomorphism

$$H(X)_{\mathrm{bal}} \to \operatorname{Inv}(G, H), \quad h \mapsto i_h.$$

3b. *Cohomology with values in $\mathbb{Q}/\mathbb{Z}(j)$.* For every integer $j \geq 0$, the coefficients $\mathbb{Q}/\mathbb{Z}(j)$ are defined as the direct sum over all prime integers p of the objects $\mathbb{Q}_p/\mathbb{Z}_p(j)$ in the derived category of sheaves of abelian groups on the big étale site of $\operatorname{Spec} F$, where

$$\mathbb{Q}_p/\mathbb{Z}_p(j) = \operatorname*{colim}_n \mu_{p^n}^{\otimes j}$$

if $p \neq \operatorname{char} F$, with μ_{p^n} the sheaf of (p^n)-th roots of unity, and

$$\mathbb{Q}_p/\mathbb{Z}_p(j) = \operatorname*{colim}_n W_n\Omega^j_{\log}[-j]$$

if $p = \operatorname{char} F > 0$, with $W_n\Omega^j_{\log}$ the sheaf of *logarithmic de Rham–Witt differentials*; see [Illusie 1979, I.5.7; Kahn 1996].

We write $H^m(X, \mathbb{Q}/\mathbb{Z}(j))$ for the étale cohomology of a scheme X with values in $\mathbb{Q}/\mathbb{Z}(j)$. Then

$$H^m(X, \mathbb{Q}/\mathbb{Z}(j))\{p\} = \operatorname*{colim}_n H^m(X, \mu_{p^n}^{\otimes j})$$

if $p \neq \operatorname{char} F$ and

$$H^m(X, \mathbb{Q}/\mathbb{Z}(j))\{p\} = \operatorname*{colim}_n H^{m-j}(X, W_n\Omega^j_{\log})$$

if $p = \operatorname{char} F > 0$. In the latter case, the group $W_n\Omega^j_{\log}(F)$ is canonically isomorphic to $K_j^M(F)/p^n K_j^M(F)$, where $K_j^M(F)$ is Milnor's K-group of F (see

[Bloch and Kato 1986, Corollary 2.8]), hence by [Izhboldin 1991; Garibaldi et al. 2003, Part II, Appendix A], $H^s(F, W_n\Omega^j_{\log})$ is isomorphic to

$$H^s(F, K_j^M(F_{\mathrm{sep}})/p^n K_j^M(F_{\mathrm{sep}})) = \begin{cases} K_j^M(F)/p^n K_j^M(F) & \text{if } s = 0, \\ H^2(F, K_j^M(F_{\mathrm{sep}}))_{p^n} & \text{if } s = 1, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that in the case $p = \operatorname{char} F > 0$, we have

$$H^m(F, \mathbb{Q}/\mathbb{Z}(j))\{p\} = \begin{cases} K_j^M(F) \otimes (\mathbb{Q}_p/\mathbb{Z}_p) & \text{if } m = j, \\ H^2(F, K_j^M(F_{\mathrm{sep}}))\{p\} & \text{if } m = j+1, \\ 0 & \text{otherwise.} \end{cases}$$

The *motivic complexes* $\mathbb{Z}(j)$, for $j = 0, 1, 2$, of étale sheaves on a smooth scheme X were defined by S. Lichtenbaum [1987; 1990]. We write $H^*(X, \mathbb{Z}(j))$ for the étale (hyper)cohomology groups of X with values in $\mathbb{Z}(j)$.

The complex $\mathbb{Z}(0)$ is equal to the constant sheaf $\mathbb{Z}$ and $\mathbb{Z}(1) = \mathbb{G}_{m,X}[-1]$, thus $H^n(X, \mathbb{Z}(1)) = H^{n-1}(X, \mathbb{G}_{m,X})$. In particular, $H^3(X, \mathbb{Z}(1)) = \mathrm{Br}(X)$, the cohomological Brauer group of X. The complex $\mathbb{Z}(2)$ is concentrated in degrees 1 and 2 and there is a product map $\mathbb{Z}(1) \otimes^L \mathbb{Z}(1) \to \mathbb{Z}(2)$; see [Lichtenbaum 1987, Proposition 2.5].

The exact triangle in the derived category of étale sheaves

$$\mathbb{Z}(j) \to \mathbb{Q} \otimes \mathbb{Z}(j) \to \mathbb{Q}/\mathbb{Z}(j) \to \mathbb{Z}(j)[1]$$

yields the connecting homomorphism

$$H^i(X, \mathbb{Q}/\mathbb{Z}(j)) \to H^{i+1}(X, \mathbb{Z}(j)),$$

which is an isomorphism if $X = \operatorname{Spec}(F)$ for a field F and $i > j$ [Kahn 1993, Lemme 1.1].

Write $\mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))$ for the Zariski sheaf on a smooth scheme X associated to the presheaf $U \mapsto H^n(U, \mathbb{Q}/\mathbb{Z}(j))$ of étale cohomology groups.

Let G be a linear algebraic group over F. We assume that G is connected if F is a finite field and write $\mathrm{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j))$ for the group of *degree n invariants* of G for the functor $K \mapsto H^n(K, \mathbb{Q}/\mathbb{Z}(j))$. Note that Property 2.1 holds for this functor by [Garibaldi et al. 2003, Part 2, Proposition A.9].

Choose a classifying G-torsor $E \to X$ with E a G-rational variety such that $E(F) \neq \varnothing$. Applying the construction given in Section 3a to the functor $U \mapsto H^0_{\mathrm{Zar}}(U, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))$, we get a homomorphism

$$\varphi : H^0_{\mathrm{Zar}}(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))_{\mathrm{bal}} \to \mathrm{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j)).$$

Theorem 3.4. *Let G be a smooth linear algebraic group over a field F. We assume that G is connected if F is a finite field. Let $E \to X$ be a classifying G-torsor* *with E a G-rational variety such that $E(F) \neq \varnothing$. Then the homomorphism φ is an isomorphism.*

Proof. Let $E_{\mathrm{gen}} \to F(X)$ be the generic fiber of the classifying G-torsor $E \to X$. Note that since the pull-back of $E \to X$ with respect to any of the two projections $E^2/G \to X$ coincides with the G-torsor $E^2 \to E^2/G$, the pull-backs of the generic G-torsor $E_{\mathrm{gen}} \to \operatorname{Spec} F(X)$ with respect to the two morphisms $\operatorname{Spec} F(E^2/G) \to \operatorname{Spec} F(X)$ induced by the projections are isomorphic. It follows that for every invariant $i \in \operatorname{Inv}(G, H^*(\mathbb{Q}/\mathbb{Z}(j)))$ we have

$$p_1^*(i(E_{\mathrm{gen}})) = i(p_1^*(E_{\mathrm{gen}})) = i(p_2^*(E_{\mathrm{gen}})) = p_2^*(i(E_{\mathrm{gen}}))$$

in $H^*(F(E^2/G), \mathbb{Q}/\mathbb{Z}(j))$, that is, $i(E_{\mathrm{gen}}) \in H^*(F(X), \mathbb{Q}/\mathbb{Z}(j))_{\mathrm{bal}}$. By Proposition A.9, $\partial_x(h) = 0$ for every point $x \in X$ of codimension 1, hence

$$\theta_G(i) = i(E_{\mathrm{gen}}) \in H^0_{\mathrm{Zar}}(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))_{\mathrm{bal}}$$

by Proposition A.10. By Theorem 2.2, θ_G is injective and by construction, the composition $\theta_G \circ \varphi$ is the identity. It follows that φ is an isomorphism. □

Write $\overline{H}^0_{\mathrm{Zar}}(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))$ for the factor group of $H^0_{\mathrm{Zar}}(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))$ by the natural image of $H^n(F, \mathbb{Q}/\mathbb{Z}(j))$.

Corollary 3.5. *The isomorphism φ yields an isomorphism*

$$\overline{H}^0_{\mathrm{Zar}}(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))_{\mathrm{bal}} \xrightarrow{\sim} \operatorname{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j))_{\mathrm{norm}}.$$

4. Degree 3 invariants of algebraic tori

In this section we prove the main theorem that describes degree 3 invariants of an algebraic torus with values in $\mathbb{Q}/\mathbb{Z}(2)$.

4a. *Algebraic tori.* Let F be a field and $\Gamma = \operatorname{Gal}(F_{\mathrm{sep}}/F)$ the absolute Galois group of F. An *algebraic torus* of dimension n over F is an algebraic group T such that T_{sep} is isomorphic to the product of n copies of the multiplicative group $\mathbb{G}_m$; see [Colliot-Thélène and Sansuc 1977; Voskresenskiĭ 1998]. For an algebraic torus T over a field F, we write $\widehat{T}_{\mathrm{sep}}$ for the Γ-module of characters $\operatorname{Hom}(T_{\mathrm{sep}}, \mathbb{G}_m)$. The group $\widehat{T}_{\mathrm{sep}}$ is a Γ*-lattice*, that is, a free abelian group of finite rank with a continuous Γ-action. The contravariant functor $T \mapsto \widehat{T}_{\mathrm{sep}}$ is an antiequivalence between the category of algebraic tori and the category of Γ-lattices: the torus T and the group $T(F)$ can be reconstructed from the lattice $\widehat{T}_{\mathrm{sep}}$ by the formulas

$$T = \operatorname{Spec}(F_{\mathrm{sep}}[\widehat{T}_{\mathrm{sep}}]^{\Gamma}),$$
$$T(F) = \operatorname{Hom}_{\Gamma}(\widehat{T}_{\mathrm{sep}}, F_{\mathrm{sep}}^{\times}) = (\widehat{T}^{\circ}_{\mathrm{sep}} \otimes F_{\mathrm{sep}}^{\times})^{\Gamma},$$

where $\widehat{T}^{\circ}_{\mathrm{sep}} = \operatorname{Hom}(\widehat{T}_{\mathrm{sep}}, \mathbb{Z})$.

We write $\widehat{T}$ for the character group $\mathrm{Hom}_F(T, \mathbb{G}_m) = (\widehat{T}_{\mathrm{sep}})^\Gamma$ and T° for the *dual torus* having character lattice $\widehat{T}^\circ_{\mathrm{sep}}$.

A torus T is called *quasisplit* if T is isomorphic to the group of invertible elements of an étale F-algebra, or equivalently, the Γ-lattice $\widehat{T}_{\mathrm{sep}}$ is permutation, that is, $\widehat{T}_{\mathrm{sep}}$ has a Γ-invariant $\mathbb{Z}$-basis. An *invertible* torus is a direct factor of a quasisplit torus.

A torus T is called *flasque* or *coflasque* if $H^1(L, \widehat{T}^\circ_{\mathrm{sep}}) = 0$ or $H^1(L, \widehat{T}_{\mathrm{sep}}) = 0$, respectively, for every finite field extension L/F. A *flasque resolution* of a torus S is an exact sequence of tori $1 \to T \to P \to S \to 1$ with T flasque and P quasisplit. By [Colliot-Thélène and Sansuc 1977, §4; Voskresenskiĭ 1998, §4.7], the torus T in the flasque resolution is invertible if and only if S is a direct factor of a rational torus.

4b. ***Products.*** Let T be a torus over F and let $\widehat{T}(i)$ denote the complex $\widehat{T}_{\mathrm{sep}} \otimes \mathbb{Z}(i)$ of étale sheaves over F for $i = 0, 1, 2$. Thus, $\widehat{T}(0) = \widehat{T}_{\mathrm{sep}}$ and

$$\widehat{T}(1) = (\widehat{T}_{\mathrm{sep}} \otimes F^\times_{\mathrm{sep}})[-1] = T^\circ(F_{\mathrm{sep}})[-1].$$

Let S and T be algebraic tori over F and let i and j be nonnegative integers with $i + j \le 2$. For any smooth variety X over F, we have the product map

$$(\widehat{S}_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}})^\Gamma \otimes H^p(X, \widehat{S}^\circ(i)) \otimes H^q(X, \widehat{T}^\circ(j)) \to H^{p+q}(X, \mathbb{Z}(i+j)) \tag{4-1}$$

taking $a \otimes b \otimes c$ to $a \cup b \cup c$, via the canonical pairings between $\widehat{S}_{\mathrm{sep}}$ and $\widehat{S}^\circ_{\mathrm{sep}}$, $\widehat{T}_{\mathrm{sep}}$ and $\widehat{T}^\circ_{\mathrm{sep}}$, and the product map $\mathbb{Z}(i) \otimes^L \mathbb{Z}(j) \to \mathbb{Z}(i+j)$.

Recall that there is an isomorphism $H^n(F, \mathbb{Z}(k)) \simeq H^{n-1}(F, \mathbb{Q}/\mathbb{Z}(k))$ for $n > k$. In particular, we have the cup-product map

$$(\widehat{S}_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}})^\Gamma \otimes H^p(F, S) \otimes H^q(F, T) \to H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \tag{4-2}$$

if $p + q = 2$.

If $S = T^\circ$ is the dual torus, then $(\widehat{S}_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}})^\Gamma = \mathrm{End}_\Gamma(\widehat{T}_{\mathrm{sep}})$ contains the canonical element 1_T. We then have the product map

$$H^p(X, \widehat{T}(i)) \otimes H^q(X, \widehat{T}^\circ(j)) \to H^{p+q}(X, \mathbb{Z}(i+j)) \tag{4-3}$$

and in particular, the product maps

$$H^1(F, \widehat{T}_{\mathrm{sep}}) \otimes H^1(F, T) \to H^2(F, \mathbb{Q}/\mathbb{Z}(1)) = \mathrm{Br}(F), \tag{4-4}$$

$$H^1(F, T^\circ) \otimes H^1(F, T) \to H^3(F, \mathbb{Q}/\mathbb{Z}(2)), \tag{4-5}$$

$$H^2(F, T^\circ) \otimes H^0(F, T) \to H^3(F, \mathbb{Q}/\mathbb{Z}(2)), \tag{4-6}$$

taking $a \otimes b$ to $1_T \cup a \cup b$ and applying (4-2).

As T is a commutative group, the set $H^1(K, T)$ is an abelian group. An invariant $i \in \mathrm{Inv}(T, H)$ for a functor H is called *linear* if $i_K : H^1(K, T) \to H(K)$ is a group homomorphism for every K/F. In the next section we will see that a normalized degree 3 invariant of a torus need not be linear.

4c. ***Main theorem.*** Let T be a torus over F and choose a standard classifying T-torsor $U \to U/T$ such that the codimension of $V \setminus U$ in V is at least 3. Such a torsor exists by [Edidin and Graham 1998, Lemma 9].

By [Sansuc 1981, Proposition 6.10], there is an exact sequence

$$F_{\mathrm{sep}}[U]^\times / F_{\mathrm{sep}}^\times \to \widehat{T}_{\mathrm{sep}} \to \mathrm{Pic}((U/T)_{\mathrm{sep}}) \to \mathrm{Pic}(U_{\mathrm{sep}}).$$

The codimension assumption implies that the side terms are trivial, hence the map $\widehat{T}_{\mathrm{sep}} \to \mathrm{Pic}((U/T)_{\mathrm{sep}})$ is an isomorphism. It follows that the classifying T-torsor $U \to U/T$ is universal in the sense of [Colliot-Thélène and Sansuc 1987a].

Write $K_*(F)$ for the (Quillen) K-groups of F and $\mathcal{K}_*$ for the Zariski sheaf associated to the presheaf $U \mapsto K_*(U)$. Then the groups $H^n_{\mathrm{Zar}}(U/T, \mathcal{K}_2)$ are independent of the choice of the classifying torsor; see [Edidin and Graham 1998]. So we write $H^n_{\mathrm{Zar}}(BT, \mathcal{K}_2)$ for this group (see Section A-IV). As T_{sep} is a split torus, by the Künneth formula (see Example A.5),

$$H^n_{\mathrm{Zar}}(BT_{\mathrm{sep}}, \mathcal{K}_2) = \begin{cases} K_2(F_{\mathrm{sep}}) & \text{if } n = 0, \\ \mathrm{Pic}((U/T)_{\mathrm{sep}}) \otimes F_{\mathrm{sep}}^\times = \widehat{T}_{\mathrm{sep}} \otimes F_{\mathrm{sep}}^\times = T^\circ(F_{\mathrm{sep}}) & \text{if } n = 1, \\ \mathrm{CH}^2((U/T)_{\mathrm{sep}}) = S^2(\widehat{T}_{\mathrm{sep}}) & \text{if } n = 2. \end{cases}$$

Applying the calculation of the $\mathcal{K}$-cohomology groups to the standard classifying T-torsor $U^i \to U^i/T$ for every $i > 0$ instead of $U \to U/T$, by Proposition B.3, we have the exact sequence

$$0 \to H^1(F, T^\circ) \xrightarrow{\alpha} \overline{H}^4(U^i/T, \mathbb{Z}(2)) \to \overline{H}^4((U^i/T)_{\mathrm{sep}}, \mathbb{Z}(2))^\Gamma \to H^2(F, T^\circ), \tag{4-7}$$

where $\overline{H}^4(U^i/T, \mathbb{Z}(2))$ is the factor group of $H^4(U^i/T, \mathbb{Z}(2))$ by $H^4(F, \mathbb{Z}(2))$, the map α is given by $\alpha(a) = q^*(a) \cup [U^i]$ with $q : U^i/T \to \operatorname{Spec} F$ the structure morphism, $[U^i]$ the class of the T-torsor $U^i \to U^i/T$ in $H^1(U^i/T, T)$, and the cup-product is taken for the pairing (B-6).

Taking the sequences (4-7) for all i (see Section A-IV), we get the exact sequence of cosimplicial groups

$$0 \to H^1(F, T^\circ) \xrightarrow{\alpha} \overline{H}^4(U^\bullet/T, \mathbb{Z}(2)) \to \overline{H}^4((U^\bullet/T)_{\mathrm{sep}}, \mathbb{Z}(2))^\Gamma \to H^2(F, T^\circ).$$

The first and the last cosimplicial groups in the sequence are constant, hence by Lemma A.2, the sequence

$$0 \to H^1(F, T^\circ) \xrightarrow{\alpha} \overline{H}^4(U/T, \mathbb{Z}(2))_{\mathrm{bal}} \to \overline{H}^4((U/T)_{\mathrm{sep}}, \mathbb{Z}(2))^\Gamma_{\mathrm{bal}} \to H^2(F, T^\circ) \tag{4-8}$$

is exact as $h_0(\overline{H}^4(U^\bullet/T, \mathbb{Z}(2))) = H^4(U/T, \mathbb{Z}(2))_{\mathrm{bal}}$.

The following theorem was proved by B. Kahn [1996, Theorem 1.1]:

Theorem 4.1. *Let X be a smooth variety over F. Then there is an exact sequence*

$$0 \to \mathrm{CH}^2(X) \to H^4(X, \mathbb{Z}(2)) \to H^0_{\mathrm{Zar}}(X, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2))) \to 0.$$

By Theorem 4.1, there is an exact sequence of cosimplicial groups

$$0 \to \mathrm{CH}^2(U^\bullet/T) \to \overline{H}^4(U^\bullet/T, \mathbb{Z}(2)) \to \overline{H}^0_{\mathrm{Zar}}(U^\bullet/T, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2))) \to 0.$$

As the functor CH^2 is homotopy invariant, by Lemma A.4, the first group in the sequence is constant. In view of Lemma A.2, and following the notation for the $\mathcal{K}$-cohomology, the sequence

$$0 \to \mathrm{CH}^2(BT) \to \overline{H}^4(U/T, \mathbb{Z}(2))_{\mathrm{bal}} \to \overline{H}^0_{\mathrm{Zar}}(U/T, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2)))_{\mathrm{bal}} \to 0 \tag{4-9}$$

is exact. By Corollary 3.5, the last group in the sequence is canonically isomorphic to $\mathrm{Inv}(T, H^3(\mathbb{Q}/\mathbb{Z}(2)))_{\mathrm{norm}}$.

As the torus T_{sep} is split, all the invariants of T_{sep} are trivial hence the sequence (4-9) over F_{sep} yields an isomorphism

$$\overline{H}^4((U/T)_{\mathrm{sep}}, \mathbb{Z}(2))_{\mathrm{bal}} \simeq \mathrm{CH}^2(BT_{\mathrm{sep}}) \simeq S^2(\widehat{T}_{\mathrm{sep}}). \tag{4-10}$$

Combining (4-8), (4-9) and (4-10), we get the following diagram with an exact row and column:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & H^1(F, T^0) \arrow[d, "\alpha"'] \arrow[dr] & & \\
0 \arrow[r] & \mathrm{CH}^2(BT) \arrow[r] \arrow[dr] & \overline{H}^4(U/T, \mathbb{Z}(2))_{\mathrm{bal}} \arrow[r] \arrow[d] & \mathrm{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \arrow[r] & 0. \\
 & & S^2(\widehat{T}_{\mathrm{sep}})^\Gamma \arrow[d] & & \\
 & & H^2(F, T^0) & &
\end{tikzcd}$$

Write $\mathrm{Dec} = \mathrm{Dec}(\widehat{T}_{\mathrm{sep}})$ for the subgroup of *decomposable elements* in $S^2(\widehat{T}_{\mathrm{sep}})^\Gamma$ (see Section A-II).

Lemma 4.2. *The image of the homomorphism*

$$\mathrm{CH}^2(BT) \to \mathrm{CH}^2(BT_{\mathrm{sep}})^\Gamma = S^2(\widehat{T}_{\mathrm{sep}})^\Gamma$$

in the diagram coincides with Dec.

Proof. Consider the Grothendieck ring $K_0(BT)$ of the category of T-equivariant vector bundles over $\mathrm{Spec}(F)$, or equivalently, of the category of finite dimensional linear representations of T. If T is split, every linear representation of T is a direct sum of one-dimensional representations. Therefore, there is an isomorphism between the group ring $\mathbb{Z}[\widehat{T}]$ of all formal finite sums $\sum_{x\in\widehat{T}} a_x e^x$ and $K_0(BT)$, taking e^x with $x\in\widehat{T}$ to the class of the 1-dimensional representation given by x. In general, for every torus T, we have $K_0(BT_{\mathrm{sep}})=\mathbb{Z}[\widehat{T}_{\mathrm{sep}}]$ and $K_0(BT)=\mathbb{Z}[\widehat{T}_{\mathrm{sep}}]^\Gamma=K_0(BT_{\mathrm{sep}})^\Gamma$; see [Merkurjev and Panin 1997, page 136]. The group $\mathbb{Z}[\widehat{T}_{\mathrm{sep}}]^\Gamma$ is generated by the sums $\sum_{i=1}^n e^{\gamma_i x}$, where $\gamma_1,\gamma_2,\dots,\gamma_n$ are representatives of the left cosets of an arbitrary open subgroup Γ' in Γ and $x\in(\widehat{T}_{\mathrm{sep}})^{\Gamma'}$.

The equivariant Chern classes were defined in [Edidin and Graham 1998, §2.4]. The first Chern class $c_1 : K_0(BT_{\mathrm{sep}})\to \mathrm{CH}^1(BT_{\mathrm{sep}})=\widehat{T}_{\mathrm{sep}}$ takes e^x to x. In the diagram

$$\begin{tikzcd}
\mathbb{Z}[\widehat{T}_{\mathrm{sep}}]^\Gamma \arrow[r, equal] \arrow[d] & K_0(BT) \arrow[r, "c_2"] \arrow[d] & \mathrm{CH}^2(BT) \arrow[d] & \\
\mathbb{Z}[\widehat{T}_{\mathrm{sep}}] \arrow[r, equal] & K_0(BT_{\mathrm{sep}}) \arrow[r, "c_2"] & \mathrm{CH}^2(BT_{\mathrm{sep}}) \arrow[r, equal] & S^2(\widehat{T}_{\mathrm{sep}})
\end{tikzcd}$$

the second Chern class maps c_2 are surjective by [Esnault et al. 1998, Lemma C.3]. It follows from the formula $c_2(a+b)=c_2(a)+c_1(a)c_1(b)+c_2(b)$ that the composition

$$\mathbb{Z}[\widehat{T}_{\mathrm{sep}}]^\Gamma=K_0(BT)\to K_0(BT_{\mathrm{sep}})\xrightarrow{c_2}\mathrm{CH}^2(BT_{\mathrm{sep}})=S^2(\widehat{T}_{\mathrm{sep}})\to S^2(\widehat{T}_{\mathrm{sep}})/(\widehat{T})^2$$

is a homomorphism and its image is generated by the elements (see Section A-II)

$$c_2\Big(\sum_{i=1}^n e^{\gamma_i x}\Big)=\sum_{i<j}(\gamma_i x)(\gamma_j x)=\mathrm{Qtr}(x). \qquad \square$$

By the restriction-corestriction argument, the kernel of the homomorphism

$$\mathrm{CH}^2(BT)\to\mathrm{CH}^2(BT_{\mathrm{sep}})^\Gamma=S^2(\widehat{T}_{\mathrm{sep}})^\Gamma$$

coincides with the torsion subgroup $\mathrm{CH}^2(BT)_{\mathrm{tors}}$ in $\mathrm{CH}^2(BT)$.

The following theorem describes degree 3 invariants of an algebraic torus with values in $\mathbb{Q}/\mathbb{Z}(2)$:

Theorem 4.3. *Let T be an algebraic torus a field F. Then there is an exact sequence*

$$0\to\mathrm{CH}^2(BT)_{\mathrm{tors}}\to H^1(F,T^0)\xrightarrow{\alpha}\mathrm{Inv}^3(T,\mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}\\ \to S^2(\widehat{T}_{\mathrm{sep}})^\Gamma/\,\mathrm{Dec}\to H^2(F,T^0).$$

The homomorphism α is given by $\alpha(a)(b)=a_K\cup b$ for every $a\in H^1(F,T^0)$ and $b\in H^1(K,T)$ and every field extension K/F, where the cup-product is defined in (4-5).

Proof. The exactness of the sequence follows from the diagram before Lemma 4.2. It remains to describe the map α. Take an $a \in H^1(F, T^0)$ and consider the invariant i defined by $i(b) = a_K \cup b$, where the cup-product is given by (4-5). We need to prove that $i = \alpha(a)$. Choose a standard classifying T-torsor $U \to U/T$ and set $K = F(U/T)$. Let U_{gen} be the generic fiber of the classifying torsor. By Theorem 2.2, it suffices to show that $i(U_{\mathrm{gen}}) = \alpha(a)(U_{\mathrm{gen}})$ over K. This follows from the description of the map α in the exact sequence (4-7). □

Remark 4.4. In a similar (and much simpler) fashion one can describe degree 2 invariants of an algebraic torus T with values in $\mathbb{Q}/\mathbb{Z}(1)$, that is, invariants with values in the Brauer group by computing the étale motivic cohomology group $H^3(U/T, \mathbb{Z}(1)) = H^2(U/T, \mathbb{G}_m) = \mathrm{Br}(U/T)$ for a standard classifying T-torsor $U \to U/T$. One establishes canonical isomorphisms

$$H^1(F, \widehat{T}_{\mathrm{sep}}) \simeq \overline{H}^3(U/T, \mathbb{Z}(1))_{\mathrm{bal}} \simeq \mathrm{Inv}^2(T, \mathbb{Q}/\mathbb{Z}(1))_{\mathrm{norm}} = \mathrm{Inv}(T, \mathrm{Br})_{\mathrm{norm}}.$$

The composition takes an element $a \in H^1(F, \widehat{T}_{\mathrm{sep}})$ to the invariant $b \mapsto a_K \cup b$ for $b \in H^1(K, T)$ and a field extension K/F. This description shows that every normalized Br-invariant of T is linear.

4d. *Torsion in* $\mathbf{CH^2}(\boldsymbol{BT})$. We investigate the group $\mathrm{CH}^2(BT)_{\mathrm{tors}}$, the first term of the exact sequence in Theorem 4.3.

Let S be an algebraic torus over F. Using the Gersten resolution, [Quillen 1973, Proposition 5.8] we identify the group $H^0(S_{\mathrm{sep}}, \mathcal{K}_2)$ with a subgroup in $K_2(F_{\mathrm{sep}}(S))$. Set $\overline{H}^0(S_{\mathrm{sep}}, \mathcal{K}_2) := H^0(S_{\mathrm{sep}}, \mathcal{K}_2)/K_2(F_{\mathrm{sep}})$. By [Garibaldi et al. 2003, Part 2, §5.7], we have an exact sequence

$$0 \to \widehat{S}_{\mathrm{sep}} \otimes F_{\mathrm{sep}}^{\times} \to \overline{H}^0(S_{\mathrm{sep}}, \mathcal{K}_2) \xrightarrow{\lambda} \textstyle\bigwedge^2 \widehat{S}_{\mathrm{sep}} \to 0 \tag{4-11}$$

of Γ-modules, where $\lambda(\{e^x, e^y\}) = x \wedge y$ for $x, y \in \widehat{S}_{\mathrm{sep}}$.

Consider the Γ-homomorphism

$$\begin{aligned} \gamma : \textstyle\bigwedge^2 \widehat{S}_{\mathrm{sep}} &\to \overline{H}^0(S_{\mathrm{sep}}, \mathcal{K}_2) \\ x \wedge y &\mapsto \{e^x, e^y\} - \{e^y, e^x\}. \end{aligned}$$

We have $\lambda \circ \gamma = 2 \cdot \mathrm{Id}$, hence the connecting homomorphism

$$\partial : H^i(F, \textstyle\bigwedge^2 \widehat{S}_{\mathrm{sep}}) \to H^{i+1}(F, \widehat{S}_{\mathrm{sep}} \otimes F_{\mathrm{sep}}^{\times}) \tag{4-12}$$

satisfies $2\partial = 0$.

Lemma 4.5. *If S is an invertible torus, then the sequence of Γ-modules* (4-11) *is split.*

Proof. Suppose first that S is quasisplit. Let $\{x_1, x_2, \dots, x_m\}$ be a permutation basis for $\widehat{S}_{\mathrm{sep}}$. Then the elements $x_i \wedge x_j$ for $i < j$ form a $\mathbb{Z}$-basis for $\bigwedge^2 \widehat{S}_{\mathrm{sep}}$. The map $\bigwedge^2 \widehat{S}_{\mathrm{sep}} \to \overline{H}^0(S_{\mathrm{sep}}, \mathcal{K}_2)$, taking $x_i \wedge x_j$ to $\{e^{x_i}, e^{x_j}\}$ is a splitting for γ.

In general, find a torus S' such that $S \times S'$ is quasisplit. The desired splitting is the composition

$$\bigwedge^2 \widehat{S}_{\mathrm{sep}} \to \bigwedge^2 \widehat{S_{\mathrm{sep}} \times S'_{\mathrm{sep}}} \xrightarrow{\alpha} \overline{H}^0(S_{\mathrm{sep}} \times S'_{\mathrm{sep}}, \mathcal{K}_2) \xrightarrow{\beta} \overline{H}^0(S_{\mathrm{sep}}, \mathcal{K}_2),$$

where α is a splitting for the torus $S \times S'$ and β is the pull-back map for the canonical inclusion $S \hookrightarrow S \times S'$. $\square$

Let

$$1 \to T \to P \to Q \to 1$$

be a coflasque resolution of T, that is, P is a quasisplit torus and Q is a coflasque torus; see [Colliot-Thélène and Sansuc 1977]. The torus P is an open set in the affine space of a separable F-algebra on which T acts linearly. Hence $P \to Q$ is a standard classifying T-torsor. By Theorem 2.2, the natural map

$$\theta_T : \mathrm{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(F(Q), \mathbb{Q}/\mathbb{Z}(2))$$

is injective.

Consider the spectral sequence (B-10) for the variety $X = Q$. By [Garibaldi et al. 2003, Part 2, Corollary 5.6], we have $H^1(Q_{\mathrm{sep}}, \mathcal{K}_2) = 0$. In view of Proposition B.4, we have an injective homomorphism

$$\beta : H^2(F, \overline{H}^0(Q_{\mathrm{sep}}, \mathcal{K}_2)) \to \overline{H}^4(Q, \mathbb{Z}(2)) \tag{4-13}$$

such that the composition of β with the homomorphism

$$H^2(F, Q^\circ) \to H^2(F, \overline{H}^0(Q_{\mathrm{sep}}, \mathcal{K}_2))$$

is given by the cup-product with the class of the identity in $H^0(Q, Q)$.

Lemma 4.6. *For a coflasque torus Q, the group $\mathrm{CH}^2(Q)$ is trivial.*

Proof. By [Merkurjev and Panin 1997, Theorem 9.1], for every torus Q, the Grothendieck group $K_0(Q)$ is generated by the classes of the sheaves $i_*(P)$, where P is an invertible sheaf on Q_L, L/F a finite separable field extension and $i : Q_L \to Q$ is the natural morphism. By definition of a coflasque torus,

$$\mathrm{Pic}(Q_L) = H^1(L, \widehat{Q}_{\mathrm{sep}}) = 0.$$

It follows that every invertible sheaf on Q_L is trivial, hence $K_0(Q) = \mathbb{Z} \cdot 1$. Since the group $\mathrm{CH}^2(Q)$ is generated by the second Chern classes of vector bundles on Q [Esnault et al. 1998, Lemma C.3], we have $\mathrm{CH}^2(Q) = 0$. $\square$

It follows from Proposition A.10, Theorem 4.1, and Lemma 4.6 that the homomorphism

$$\kappa : \overline{H}^4(Q, \mathbb{Z}(2)) \to \overline{H}^4(F(Q), \mathbb{Z}(2)) = \overline{H}^3(F(Q), \mathbb{Q}/\mathbb{Z}(2)) \tag{4-14}$$

is injective.

Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & Q^\circ(F_{\mathrm{sep}}) \arrow[r] \arrow[d, equal] & \overline{H}^0(Q_{\mathrm{sep}}, \mathcal{K}_2) \arrow[r] \arrow[d, "s"'] & \bigwedge^2 \widehat{Q}_{\mathrm{sep}} \arrow[r] \arrow[d, "t"'] & 0 \\
0 \arrow[r] & Q^\circ(F_{\mathrm{sep}}) \arrow[r] & P^\circ(F_{\mathrm{sep}}) \arrow[r] & T^\circ(F_{\mathrm{sep}}) \arrow[r] & 0
\end{tikzcd}$$

where s is the composition of the natural map $\overline{H}^0(Q_{\mathrm{sep}}, \mathcal{K}_2) \to \overline{H}^0(P_{\mathrm{sep}}, \mathcal{K}_2)$ and a splitting of $P^\circ(F_{\mathrm{sep}}) \to \overline{H}^0(P_{\mathrm{sep}}, \mathcal{K}_2)$ (see Lemma 4.5).

We have the following diagram

$$\begin{tikzcd}
& & H^1(F, T^\circ) \arrow[r, "\alpha"] \arrow[d, hook, "\partial_1"] & \mathrm{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \arrow[d, hook, "\theta_T"] \\
H^1(F, \overline{H}^0(Q_{\mathrm{sep}}, \mathcal{K}_2)) \arrow[r, "\varphi"] & H^1(F, \bigwedge^2 \widehat{Q}_{\mathrm{sep}}) \arrow[ur, "t^*"] \arrow[r, "\partial"] & H^2(F, Q^\circ) \arrow[r, "\sigma"] & \overline{H}^3(F(Q), \mathbb{Q}/\mathbb{Z}(2))
\end{tikzcd}$$

with the bottom sequence a complex, where σ is the composition of the maps in (4-13) and (4-14):

$$H^2(F, Q^0) \xrightarrow{\psi} H^2(F, \overline{H}^0(Q_{\mathrm{sep}}, \mathcal{K}_2)) \xrightarrow{\beta} \overline{H}^4(Q, \mathbb{Z}(2))$$
$$\xrightarrow{\kappa} \overline{H}^4(F(Q), \mathbb{Z}(2)) = \overline{H}^3(F(Q), \mathbb{Q}/\mathbb{Z}(2)),$$

with φ and ψ given by Galois cohomology applied to the exact sequence (4-11) for the torus Q. Note that the connecting map ∂_1 is injective as $H^1(F, P^\circ) = 0$ since P° is a quasisplit torus. As $2\partial = 0$ in (4-12), we have $2t^* = 0$.

The commutativity of the triangle follows from the definition of t^*. We claim that the square in the diagram is anticommutative. Note that $\partial_2(\xi) = [P_{\mathrm{gen}}]$, where $\partial_2 : H^0(F, Q) \to H^1(F, T)$ is the connecting homomorphism, P_{gen} is the generic fiber of the morphism $P \to Q$, and $\xi \in H^0(K, Q)$ is the generic point of Q with $K = F(Q)$. It follows from the description of the maps α and β in (4-7) and (4-13), respectively, and Lemma A.1 that

$$\sigma(\partial_1(a)) = \partial_1(a)_K \cup \xi = (-a_K) \cup \partial_2(\xi) = (-a_K) \cup [P_{\mathrm{gen}}] = -\theta_T(\alpha(a))$$

for every $a \in H^1(F, T^\circ)$.

The maps β and κ are injective, hence the bottom sequence in the diagram is exact. Thus, we have an exact sequence

$$H^1(F, \overline{H}^0(Q_{\mathrm{sep}}, \mathcal{K}_2)) \to H^1(F, \textstyle\bigwedge^2 \widehat{Q}_{\mathrm{sep}}) \to \mathrm{Ker}(\alpha) \to 0$$

and $2\cdot\mathrm{Ker}(\alpha) = 2\cdot\mathrm{Im}(t^*) = 0$. Furthermore, $\mathrm{Ker}(\alpha) \simeq \mathrm{CH}^2(BT)_{\mathrm{tors}}$ by Theorem 4.3 and the group $H^1(F, \bigwedge^2 \widehat{Q}_{\mathrm{sep}})$ is finite as $\bigwedge^2 \widehat{Q}_{\mathrm{sep}}$ is a lattice.

We have proved:

Theorem 4.7. *Let $1 \to T \to P \to Q \to 1$ be a coflasque resolution of a torus T. Then there is an exact sequence*

$$H^1(F, \overline{H}^0(Q_{\mathrm{sep}}, \mathcal{K}_2)) \to H^1(F, \textstyle\bigwedge^2 \widehat{Q}_{\mathrm{sep}}) \to \mathrm{CH}^2(BT)_{\mathrm{tors}} \to 0.$$

Moreover, $\mathrm{CH}^2(BT)_{\mathrm{tors}}$ *is a finite group satisfying* $2 \cdot \mathrm{CH}^2(BT)_{\mathrm{tors}} = 0$.

Corollary 4.8. *If T° is a birational direct factor of a rational torus, or if T is split over a cyclic field extension, then* $\mathrm{CH}^2(BT)_{\mathrm{tors}} = 0$, *that is, the map α in Theorem 4.3 is injective.*

Proof. The exact sequence $1 \to Q^\circ \to P^\circ \to T^\circ \to 1$ is a flasque resolution of T°. If T° is a birational direct factor of a rational torus, or if T is split over a cyclic field extension, the torus Q°, and hence Q, is invertible; see Section 4a and [Voskresenskiĭ 1998, §4, Theorem 3]. By Lemma 4.5, the sequence (4-11) for the torus Q is split, hence the first map in Theorem 4.7 is surjective. □

Question 4.9. Is $\mathrm{CH}^2(BT)_{\mathrm{tors}}$ trivial for every torus T?

4e. ***Special tori.*** Let T be an algebraic torus over a field F. The *tautological invariant* of the torus $T^\circ \times T$ is the normalized invariant taking a pair

$$(a, b) \in H^1(K, T^\circ) \times H^1(K, T)$$

to the cup-product $a \cup b \in H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ defined in (4-5).

The following theorem shows that if a torus T has only trivial degree 3 normalized invariants with values in $\mathbb{Q}/\mathbb{Z}(2)$ universally, that is, over all field extensions of F, then T has no nonconstant invariants at all by the simple reason: every T-torsor over a field is trivial. Note that it follows from Theorem 2.4 that T has no degree 2 normalized invariants with values in $\mathbb{Q}/\mathbb{Z}(1)$ universally if and only if T is coflasque.

Theorem 4.10. *Let T be an algebraic torus over a field F. Then the following are equivalent*:

(1) $\mathrm{Inv}^3(T_K, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} = 0$ *for every field extension K of F.*

(2) *The tautological invariant of the torus $T^\circ \times T$ is trivial.*

(3) *The torus T is invertible.*

(4) *The torus T is special.*

Proof. (1) $\Rightarrow$ (2): Let K/F be a field extension and $a \in H^1(K, T^\circ)$. By assumption, the degree 3 normalized invariant $i = \alpha(a)$ with values in $\mathbb{Q}/\mathbb{Z}(2)$, defined by $i(b) = a \cup b$ for every $b \in H^1(K, T)$, is trivial. In other words, the tautological invariant of the torus $T^\circ \times T$ is trivial.

(2) $\Rightarrow$ (3): The image of the tautological invariant in the group

$$S^2(\widehat{T}^\circ_{\mathrm{sep}} \oplus \widehat{T}_{\mathrm{sep}})^\Gamma / \operatorname{Dec}$$

is represented by the identity $1_{\widehat{T}}$ in the direct factor $(\widehat{T}^\circ_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}})^\Gamma = \operatorname{End}_\Gamma(\widehat{T}_{\mathrm{sep}})$ of $S^2(\widehat{T}^\circ_{\mathrm{sep}} \oplus \widehat{T}_{\mathrm{sep}})^\Gamma$ (see Section A-II). The projection of Dec on the direct summand $(\widehat{T}^\circ_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}})^\Gamma$ is generated by the traces $\operatorname{Tr}(a \otimes b)$ for all open subgroups $\Gamma' \subset \Gamma$ and all $a \in (\widehat{T}^\circ_{\mathrm{sep}})^{\Gamma'}$ and $b \in (\widehat{T}_{\mathrm{sep}})^{\Gamma'}$. Hence $1_{\widehat{T}} = \sum_i \operatorname{Tr}(a_i \otimes b_i)$ for some open subgroups $\Gamma_i \subset \Gamma$, $a_i \in (\widehat{T}^\circ)^{\Gamma_i}$ and $b_i \in (\widehat{T})^{\Gamma_i}$. If $P_i = \mathbb{Z}[\Gamma/\Gamma_i]$, then a_i can be viewed as a Γ-homomorphism $\widehat{T} \to P_i$ and b_i can be viewed as a Γ-homomorphism $P_i \to \widehat{T}$ such that the composition

$$\widehat{T} \xrightarrow{(b_i)} P \xrightarrow{(a_i)} \widehat{T},$$

where $P = \coprod P_i$, is the identity. It follows that $\widehat{T}$ is a direct summand of a permutation Γ-module P and hence T is invertible.

(3) $\Rightarrow$ (4): Obvious as every invertible torus is special.

(4) $\Rightarrow$ (1): Obvious. $\square$

Remark 4.11. The equivalence (3) $\Leftrightarrow$ (4) was essentially proved in [Colliot-Thélène and Sansuc 1987b, Proposition 7.4].

4f. ***Linear and quadratic invariants.*** Let T be a torus over F. By Theorem 4.3, we have a natural homomorphism to the group of linear invariants:

$$\alpha : H^1(F, T^\circ) \to \operatorname{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{lin}}.$$

Let S and T be algebraic tori over F. For every field extension K/F, the cup-product (4-2) yields a homomorphism

$$\varepsilon : (\widehat{T}^{\otimes 2}_{\mathrm{sep}})^\Gamma \to \operatorname{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))$$

defined by $\varepsilon(a)(b) = a_K \cup b \cup b$ for $a \in (\widehat{T}^{\otimes 2}_{\mathrm{sep}})^\Gamma$ and $b \in H^1(K, T)$.

We say that an invariant $i \in \operatorname{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))$ is *quadratic* if the function

$$h(a, b) := i(a + b) - i(a) - i(b)$$

is bilinear and $h(a, a) = 2i(a)$ for all a and b. For example, the tautological invariant of the torus $T^\circ \times T$ in Section 4e is quadratic. We write $\operatorname{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{quad}}$ for the subgroup of all quadratic invariants in $\operatorname{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))$. The image of ε is contained in $\operatorname{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{quad}}$.

Lemma 4.12. *The composition of ε with the map*

$$\mathrm{Inv}^3(T,\mathbb{Q}/\mathbb{Z}(2)) \to S^2(\widehat{T}_{\mathrm{sep}})^{\Gamma}/\,\mathrm{Dec}$$

in Theorem 4.3 is induced by the natural homomorphism $\widehat{T}_{\mathrm{sep}}^{\otimes 2} \to S^2(\widehat{T}_{\mathrm{sep}})$.

Proof. Let $U \to U/T =: X$ be a standard classifying T-torsor as in Section 4c. Consider the commutative diagram

$$\begin{tikzcd}
(\widehat{T}_{\mathrm{sep}}^{\otimes 2})^{\Gamma} \otimes H^1(X,T)^{\otimes 2} \arrow[r, "\mathrm{prod}"] \arrow[d] & \overline{H}^4(X,\mathbb{Z}(2))_{\mathrm{bal}} \arrow[r] \arrow[d] & \mathrm{Inv}^3(T,\mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \\
\widehat{T}_{\mathrm{sep}}^{\otimes 2} \otimes H^1(X_{\mathrm{sep}},T)^{\otimes 2} \arrow[r, "\mathrm{prod}"] \arrow[d, "\eta"', "\wr"] & \overline{H}^4(X_{\mathrm{sep}},\mathbb{Z}(2))_{\mathrm{bal}} \arrow[d, "\wr"] & \\
\widehat{T}_{\mathrm{sep}}^{\otimes 2} \otimes (\widehat{T}_{\mathrm{sep}}^{\circ})^{\otimes 2} \otimes \widehat{T}_{\mathrm{sep}}^{\otimes 2} \arrow[r, "\kappa"] & S^2(\widehat{T}_{\mathrm{sep}})^{\Gamma}/\,\mathrm{Dec} &
\end{tikzcd}$$

where the product maps are given by (4-1), η identifies

$$H^1(X_{\mathrm{sep}},T) = \widehat{T}_{\mathrm{sep}}^{\circ} \otimes \mathrm{Pic}(X_{\mathrm{sep}})$$

with $\widehat{T}_{\mathrm{sep}}^{\circ} \otimes \widehat{T}_{\mathrm{sep}}$ and κ is given by the pairing between the first and second factors. Write $[U]$ for the class of the classifying torsor in $H^1(X,T)$. The image of $[U]$ in

$$H^1(X_{\mathrm{sep}},T_{\mathrm{sep}}) = \widehat{T}_{\mathrm{sep}}^{\circ} \otimes \widehat{T}_{\mathrm{sep}} = \mathrm{End}(\widehat{T}_{\mathrm{sep}})$$

is the identity $1_{\widehat{T}_{\mathrm{sep}}}$. Hence for every $a \in (\widehat{T}_{\mathrm{sep}}^{\otimes 2})^{\Gamma}$, the image of $a \otimes [U] \otimes [U]$ under the diagonal map in the diagram coincides with the canonical image of a in $S^2(\widehat{T}_{\mathrm{sep}})^{\Gamma}/\,\mathrm{Dec}$. □

The composition of the map $S^2(\widehat{T}_{\mathrm{sep}})^{\Gamma} \to (\widehat{T}_{\mathrm{sep}}^{\otimes 2})^{\Gamma}$ given by $a \cdot b \mapsto a \otimes b + b \otimes a$ with the natural map $(\widehat{T}_{\mathrm{sep}}^{\otimes 2})^{\Gamma} \to S^2(\widehat{T}_{\mathrm{sep}})^{\Gamma}$ is multiplication by 2. Then by Lemma 4.12, $2 \cdot S^2(\widehat{T}_{\mathrm{sep}})^{\Gamma}/\,\mathrm{Dec}$ is contained in the image of the map

$$\mathrm{Inv}^3(T,\mathbb{Q}/\mathbb{Z}(2)) \to S^2(\widehat{T}_{\mathrm{sep}})^{\Gamma}/\,\mathrm{Dec}\,.$$

Theorem 4.3 then yields:

Theorem 4.13. *Let T be an algebraic torus over F. Then 2 times the homomorphism $S^2(\widehat{T}_{\mathrm{sep}})^{\Gamma}/\,\mathrm{Dec} \to H^2(F,T^0)$ from Theorem 4.3 is trivial. If p is an odd prime,*

$$\mathrm{Inv}^3(T,\mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{norm}} = \mathrm{Inv}^3(T,\mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{lin}} \oplus \mathrm{Inv}^3(T,\mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{quad}}$$

and there are natural isomorphisms $\mathrm{Inv}^3(T,\mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{lin}} \simeq H^1(F,T^{\circ})\{p\}$ and

$$\mathrm{Inv}^3(T,\mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{quad}} \simeq (S^2(\widehat{T}_{\mathrm{sep}})^{\Gamma}/\,\mathrm{Dec})\{p\}.$$

Example 4.14. Let $X = \{x_1, x_2, \dots, x_n\}$ be a set of n elements with the natural action of the symmetric group S_n. A continuous surjective group homomorphism $\Gamma \to S_n$ yields a separable field extension L/F of degree n. Consider the torus $T = R_{L/F}(\mathbb{G}_{m,L})/\,\mathbb{G}_m$, where $R_{L/F}$ is the Weil restriction; see [Voskresenskiĭ 1998, Chapter 1, §3.12]. Note that the generic maximal torus of the group $\mathbf{PGL}_n$ is of this form (see Section 5b). The character lattice $\widehat{T}_{\mathrm{sep}}$ is the kernel of the augmentation homomorphism $\mathbb{Z}[X] \to \mathbb{Z}$.

The dual torus T° is the norm one torus $R^{(1)}_{L/F}(\mathbb{G}_{m,L})$. For every field extension K/F, we have:

$$H^1(K, T) = \mathrm{Br}(KL/K), \qquad H^1(K, T^\circ) = K^\times/N(KL)^\times,$$

where $KL := K \otimes L$, N is the norm map for the extension KL/K and

$$\mathrm{Br}(KL/K) = \mathrm{Ker}(\mathrm{Br}(K) \to \mathrm{Br}(KL)).$$

The pairing

$$K^\times/N(KL)^\times \otimes \mathrm{Br}(KL/K) \to H^3(F, \mathbb{Q}/\mathbb{Z}(2))$$

defines linear degree 3 invariants of both T and T°.

We claim that $S^2(\widehat{T}_{\mathrm{sep}})^\Gamma/\,\mathrm{Dec} = 0$ and $S^2(\widehat{T}^\circ_{\mathrm{sep}})^\Gamma/\,\mathrm{Dec} = 0$, that is, T and T° have no nontrivial quadratic degree 3 invariants. We have $\widehat{T}^\circ_{\mathrm{sep}} = \mathbb{Z}[X]/\mathbb{Z}N_X$, where $N_X = \sum x_i$. The group $S^2(\widehat{T}^\circ_{\mathrm{sep}})^\Gamma$ is generated by $S := \sum_{i<j} x_i \cdot x_j$. As $S \in \mathrm{Dec}$, we have $S^2(\widehat{T}^\circ_{\mathrm{sep}})^\Gamma/\,\mathrm{Dec} = 0$.

Let $D = \sum x_i^2$ and $E := \mathrm{Qtr}(x_1 - x_2) = 2S - (n-1)D$, where the quadratic map Qtr is defined in Section A-II. The group $S^2(\widehat{T}_{\mathrm{sep}})^\Gamma$ is generated by E if n is even and by $E/2$ if n is odd. A computation shows that $nE/2 = \mathrm{Qtr}(nx_1 - N_X)$. It follows that the generator of $S^2(\widehat{T}_{\mathrm{sep}})^\Gamma$ belongs to Dec, hence $S^2(\widehat{T}_{\mathrm{sep}})^\Gamma/\,\mathrm{Dec}$ is trivial.

Note that as the torus T is rational, it follows from Theorem 4.3 and Corollary 4.8 that $\mathrm{Inv}^3(T^\circ, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \simeq \mathrm{Br}(L/F)$.

5. Unramified invariants

Let K/F be a field extension and v a discrete valuation of K over F with valuation ring O_v. We say that an element $a \in H^n(K, \mathbb{Q}/\mathbb{Z}(j))$ is *unramified with respect to* v if a belongs to the image of the map $H^n(O_v, \mathbb{Q}/\mathbb{Z}(j)) \to H^n(K, \mathbb{Q}/\mathbb{Z}(j))$; see [Colliot-Thélène and Ojanguren 1989]. We write $H^n_{\mathrm{nr}}(K, \mathbb{Q}/\mathbb{Z}(j))$ for the subgroup of the elements in $H^n(K, \mathbb{Q}/\mathbb{Z}(j))$ that are unramified with respect to all discrete valuations of K over F. We have a natural homomorphism

$$H^n(F, \mathbb{Q}/\mathbb{Z}(j)) \to H^n_{\mathrm{nr}}(K, \mathbb{Q}/\mathbb{Z}(j)). \tag{5-1}$$

A dominant morphism of varieties $Y \to X$ yields a homomorphism

$$H^n_{\mathrm{nr}}(F(X), \mathbb{Q}/\mathbb{Z}(j)) \to H^n_{\mathrm{nr}}(F(Y), \mathbb{Q}/\mathbb{Z}(j)). \tag{5-2}$$

Proposition 5.1. *Let K/F be a purely transcendental field extension. Then the homomorphism* (5-1) *is an isomorphism.*

Proof. The statement is well known for the p-components if $p \neq \operatorname{char} F$; see, for example, [Colliot-Thélène and Ojanguren 1989, Proposition 1.2]. It suffices to consider the case $K = F(t)$ and prove the surjectivity of (5-1). The coniveau spectral sequence for the projective line $\mathbb{P}^1$ (see (A-1) in the Appendix) yields an exact sequence

$$H^n(\mathbb{P}^1, \mathbb{Q}/\mathbb{Z}(j)) \to H^n(K, \mathbb{Q}/\mathbb{Z}(j)) \to \coprod_{x \in \mathbb{P}^1} H^{n+1}_x(\mathbb{P}^1, \mathbb{Q}/\mathbb{Z}(j))$$

and, therefore, a surjective homomorphism $H^n(\mathbb{P}^1, \mathbb{Q}/\mathbb{Z}(j)) \to H^n_{\mathrm{nr}}(K, \mathbb{Q}/\mathbb{Z}(j))$. By the projective bundle theorem (classical if $p \neq \operatorname{char}(F)$ and [Gros 1985, Theorem 2.1.11] if $p = \operatorname{char}(F) > 0$), we have

$$H^n(\mathbb{P}^1, \mathbb{Q}/\mathbb{Z}(j)) = H^n(F, \mathbb{Q}/\mathbb{Z}(j)) \oplus H^{n-2}(F, \mathbb{Q}/\mathbb{Z}(j-1))t,$$

where t is a generator of $H^2(\mathbb{P}^1, \mathbb{Z}(1)) = \operatorname{Pic}(\mathbb{P}^1) = \mathbb{Z}$. As t vanishes over the generic point of $\mathbb{P}^1$, the result follows. □

Let G be a linear algebraic group over F. Choose a standard classifying G-torsor $U \to U/G$. An invariant $i \in \operatorname{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j))$ is called *unramified* if the image of i under $\theta_G : \operatorname{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j)) \to H^n(F(U/G), \mathbb{Q}/\mathbb{Z}(j))$ is unramified. This is independent of the choice of standard classifying torsor. Indeed, if $U' \to U'/G$ is another standard classifying G-torsor, then $(U \times V')/G \to U/G$ and $(V \times U')/G \to U'/G$ are vector bundles. Hence the field $F((U \times U')/G)$ is a purely transcendental extension of $F(U/G)$ and $F(U'/G)$ and by Proposition 5.1,

$$\begin{aligned} H^n_{\mathrm{nr}}(F(U/G), \mathbb{Q}/\mathbb{Z}(j)) &\simeq H^n_{\mathrm{nr}}(F((U \times U')/G), \mathbb{Q}/\mathbb{Z}(j)) \\ &\simeq H^n_{\mathrm{nr}}(F(U'/G), \mathbb{Q}/\mathbb{Z}(j)). \end{aligned}$$

We write $H^n_{\mathrm{nr}}(F(BG), \mathbb{Q}/\mathbb{Z}(j))$ for this common value and $\operatorname{Inv}^n_{\mathrm{nr}}(G, \mathbb{Q}/\mathbb{Z}(j))$ for the subgroup of unramified invariants. Similarly, we write $\operatorname{Br}_{\mathrm{nr}}(F(BG))$ for the *unramified Brauer group* $H^2_{\mathrm{nr}}(F(BG), \mathbb{Q}/\mathbb{Z}(1))$.

Proposition 5.2. *If G' be a subgroup of G and $i \in \operatorname{Inv}^n_{\mathrm{nr}}(G, \mathbb{Q}/\mathbb{Z}(j))$, then*

$$\operatorname{res}(i) \in \operatorname{Inv}^n_{\mathrm{nr}}(G', \mathbb{Q}/\mathbb{Z}(j)).$$

Proof. It is shown in Section 2b that there is a surjective morphism $X' \to X$ of the respective classifying varieties of G' and G, such that $\theta_G(i)_{F(X')} = \theta_{G'}(\mathrm{res}(i))$. Applying the homomorphism (5-2) we see that $\mathrm{res}(i)$ is unramified. □

Proposition 5.3. *Let G be a smooth linear algebraic group over a field F. The map* $\mathrm{Inv}^n_{\mathrm{nr}}(G, \mathbb{Q}/\mathbb{Z}(j)) \to H^n_{\mathrm{nr}}(F(BG), \mathbb{Q}/\mathbb{Z}(j))$ *induced by θ_G is an isomorphism.*

Proof. By Theorem 3.4, it suffices to show that

$$H^n_{\mathrm{nr}}(F(U/G), \mathbb{Q}/\mathbb{Z}(j)) \subset H^n(F(U/G), \mathbb{Q}/\mathbb{Z}(j))_{\mathrm{bal}}.$$

We follow Totaro's approach; see [Garibaldi et al. 2003, p. 99]. Consider the open subscheme W of $(U^2/G)\times\mathbb{A}^1$ of all triples (u, u', t) such that $(2-t)u+(t-1)u' \in U$. We have the projection $q : W \to U^2/G$, the morphisms $f : W \to U/G$ defined by $f(u, u', t) = (2-t)u+(t-1)u'$, and $h_i : U^2/G \to W$ defined by $h_i(u, u') = (u, u', i)$ for $i = 1$ and 2. The composition $f \circ h_i$ is the projection $p_i : U^2/G \to U/G$ and $q \circ h_i$ is the identity of U^2/G.

Let w_i be the generic point of the preimage of i with respect to the projection $W \to \mathbb{A}^1$ and write O_i for the local ring of W at w_i. The morphisms q, f, and h_i yield F-algebra homomorphisms $F(U^2/G) \to O_i$, $F(U/G) \to O_i$ and $O_i \to F(U/G)$. Note that by Proposition A.11, we have

$$H^n_{\mathrm{nr}}(F(W), \mathbb{Q}/\mathbb{Z}(j)) \subset H^n(O_i, \mathbb{Q}/\mathbb{Z}(j)).$$

In the commutative diagram

$$\begin{tikzcd}
H^n_{\mathrm{nr}}(F(U/G), \mathbb{Q}/\mathbb{Z}(j)) \arrow[r, "f^*"] \arrow[d, hook] & H^n_{\mathrm{nr}}(F(W), \mathbb{Q}/\mathbb{Z}(j)) \arrow[d, hook] & H^n_{\mathrm{nr}}(F(U^2/G), \mathbb{Q}/\mathbb{Z}(j)) \arrow[l, "q^*"', "\sim"] \arrow[d, hook] \\
H^n(F(U/G), \mathbb{Q}/\mathbb{Z}(j)) \arrow[r, "f^*"] \arrow[dr, "p_i^*"'] & H^n(O_i, \mathbb{Q}/\mathbb{Z}(j)) \arrow[d, "h_i^*"] & H^n(F(U^2/G), \mathbb{Q}/\mathbb{Z}(j)) \arrow[l, "q^*"'] \arrow[dl, equal] \\
 & H^n(F(U^2/G), \mathbb{Q}/\mathbb{Z}(j)) &
\end{tikzcd}$$

the top right map q^* is an isomorphism by Proposition 5.1 since the field extension $F(W)/F(U^2/G)$ is purely transcendental. It follows that the restriction of p_i^* on $H^n_{\mathrm{nr}}(F(U/G), \mathbb{Q}/\mathbb{Z}(j))$ coincides with $(q^*)^{-1} \circ f^*$ and hence is independent of i. □

5a. *Unramified invariants of tori.*

Proposition 5.4. *If T is a flasque torus, then every invariant in* $\mathrm{Inv}^n(T, \mathbb{Q}/\mathbb{Z}(j))$ *is unramified.*

Proof. Consider an exact sequence of tori $1 \to T \to P \to Q \to 1$ with P quasisplit. Choose a smooth projective compactification X of Q; see [Colliot-Thélène et al. 2005]. As T is flasque, by [Colliot-Thélène and Sansuc 1977, Proposition 9], there is a T-torsor $E \to X$ extending the T-torsor $P \to Q$. The torsor E is classifying and T-rational. Choose an invariant in $\mathrm{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j))$ and consider its image a in $H^n(F(X), \mathbb{Q}/\mathbb{Z}(j))_{\mathrm{bal}}$ (see Theorem 3.4). We show that a is unramified with respect to every discrete valuation v on $F(X)$ over F; see [Colliot-Thélène 1995, Proposition 2.1.8]. By Proposition A.9, a is unramified with respect to the discrete valuation associated to every point $x \in X$ of codimension 1, that is, $\partial_x(a) = 0$.

As X is projective, the valuation ring O_v of the valuation v dominates a point $x \in X$. It follows from Proposition A.11 that a belongs to the image of

$$H^n(O_{X,x}, \mathbb{Q}/\mathbb{Z}(j)) \to H^n(F(X), \mathbb{Q}/\mathbb{Z}(j)).$$

As the local ring $O_{X,x}$ is a subring of O_v, a belongs to the image of

$$H^n(O_v, \mathbb{Q}/\mathbb{Z}(j)) \to H^n(F(X), \mathbb{Q}/\mathbb{Z}(j))$$

and hence a is unramified with respect to v. □

Let T be a torus over F. By [Colliot-Thélène and Sansuc 1987b, Lemma 0.6], there is an exact sequence of tori $1 \to T \to T' \to P \to 1$ with T' flasque and P quasisplit. The following theorem computes the unramified invariants of T in terms of the invariants of T'.

Theorem 5.5. *There is a natural isomorphism*

$$\mathrm{Inv}^n_{\mathrm{nr}}(T, \mathbb{Q}/\mathbb{Z}(j)) \simeq \mathrm{Inv}^n(T', \mathbb{Q}/\mathbb{Z}(j)).$$

Proof. Choose an exact sequence $1 \to T' \to P' \to S \to 1$ with P' a quasisplit torus. Let S' be the cokernel of the composition $T \to T' \to P'$. We have an exact sequence $1 \to P \to S' \to S \to 1$. As P is quasisplit, the latter exact sequence splits at the generic point of S and therefore, $F(S')$ is a purely transcendental field extension of $F(S)$. It follows from Propositions 5.1, 5.3, and 5.4 that

$$\begin{aligned}\mathrm{Inv}^n_{\mathrm{nr}}(T, \mathbb{Q}/\mathbb{Z}(j)) &\simeq H^n_{\mathrm{nr}}(F(S'), \mathbb{Q}/\mathbb{Z}(j)) \simeq H^n_{\mathrm{nr}}(F(S), \mathbb{Q}/\mathbb{Z}(j)) \\ &\simeq \mathrm{Inv}^n_{\mathrm{nr}}(T', \mathbb{Q}/\mathbb{Z}(j)) = \mathrm{Inv}^n(T', \mathbb{Q}/\mathbb{Z}(j)).\end{aligned}$$

□

The following corollary is essentially equivalent to [Colliot-Thélène and Sansuc 1987b, Proposition 9.5] in the case when F is of zero characteristic.

Corollary 5.6. *With notation as above, the isomorphism*

$$\mathrm{Inv}(T, \mathrm{Br}) \xrightarrow{\sim} \mathrm{Pic}(T) = H^1(F, \widehat{T})$$

identifies $\mathrm{Inv}_{\mathrm{nr}}(T, \mathrm{Br})$ *with the subgroup* $H^1(F, \widehat{T}')$ *of* $H^1(F, \widehat{T})$ *of all elements that are trivial when restricted to all cyclic subgroups of the decomposition group of* T.

Proof. The description of $H^1(F, \widehat{T}')$ as a subgroup of $H^1(F, \widehat{T})$ is given in [Colliot-Thélène and Sansuc 1987b, Proposition 9.5], and this part of the proof is characteristic free. □

In view of Propositions 5.1 and 5.3 we can calculate the group of unramified cohomology for the function field of an arbitrary torus in terms of the invariants of a flasque torus:

Theorem 5.7. *Let* S *be a torus over* F *and let* $1 \to T \to P \to S \to 1$ *be a flasque resolution of* S, *that is,* T *is flasque and* P *is quasisplit. Then there is a natural isomorphism*

$$H^n_{\mathrm{nr}}(F(S), \mathbb{Q}/\mathbb{Z}(j)) \simeq \mathrm{Inv}^n(T, \mathbb{Q}/\mathbb{Z}(j)).$$

Note that the torus S determines T up to multiplication by a quasisplit torus. If X is a smooth compactification of S, then one can take a torus T with $\widehat{T}_{\mathrm{sep}} \simeq \mathrm{Pic}(X_{\mathrm{sep}})$; see [Colliot-Thélène and Sansuc 1977, Proposition 6; Voskresenskiĭ 1998, §4.6].

Corollary 5.8. *A torus* S *has no nonconstant unramified degree* 3 *cohomology with values in* $\mathbb{Q}/\mathbb{Z}(2)$ *universally, that is,* $H^3_{\mathrm{nr}}(K(S), \mathbb{Q}/\mathbb{Z}(2)) = H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ *for any field extension* K/F, *if and only if* S *is a direct factor of a rational torus.*

Proof. If S is a direct factor of a rational torus, then S has no nonconstant unramified cohomology by Proposition 5.1.

Conversely, let $1 \to T \to P \to S \to 1$ be a flasque resolution of S. By Theorem 5.7, $\mathrm{Inv}^3(T_K, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} = 0$ for every K/F. It follows from Theorem 4.10 that T is invertible and hence S is a factor of a rational torus (see Section 4a). □

Theorems 4.3, 5.7 and [Colliot-Thélène and Sansuc 1977, §2] yield the following proposition.

Proposition 5.9. *Let* S *be a torus over* F *and let* $1 \to T \to P \to S \to 1$ *be a flasque resolution of* S. *Then we have an exact sequence*

$$\begin{aligned} 0 \to \mathrm{CH}^2(BT)_{\mathrm{tors}} \to H^1(F, T^0) \to \overline{H}^3_{\mathrm{nr}}(F(S), \mathbb{Q}/\mathbb{Z}(2)) \\ \to H^0(F, S^2(\widehat{T}_{\mathrm{sep}}))/\operatorname{Dec} \to H^2(F, T^0). \end{aligned}$$

For an odd prime p, *there is a canonical direct sum decomposition*

$$\overline{H}^3_{\mathrm{nr}}(F(S), \mathbb{Q}_p/\mathbb{Z}_p(2)) = H^1(F, T^0)\{p\} \oplus (H^0(F, S^2(\widehat{T}_{\mathrm{sep}}))/\operatorname{Dec})\{p\}.$$

If X *is a smooth compactification of* S, *one can take the torus* T *with* $\widehat{T}_{\mathrm{sep}} = \mathrm{Pic}(X_{\mathrm{sep}})$.

5b. ***The Brauer invariant for semisimple groups.*** The following theorem was proved by Bogomolov [1987, Lemma 5.7] in characteristic zero:

Theorem 5.10. *Let G be a (connected) semisimple group over a field F. Then* $\mathrm{Inv}_{\mathrm{nr}}(G,\mathrm{Br})=\mathrm{Inv}(G,\mathrm{Br})_{\mathrm{const}}=\mathrm{Br}(F)$ *and* $\mathrm{Br}_{\mathrm{nr}}(F(BG))=\mathrm{Br}(F)$.

Proof. Let $G'\to G$ be a simply connected cover of G and C the kernel of $G'\to G$. By Theorem 2.4 and [Sansuc 1981, Lemme 6.9(iii)], we have

$$\mathrm{Inv}(G,\mathrm{Br})_{\mathrm{norm}}=\mathrm{Pic}(G)=\widehat{C}(F).$$

As the map $\widehat{C}(F)\to\widehat{C}(F_{\mathrm{sep}})$ is injective, we can replace F by F_{sep} and assume that the group G is split.

Consider the variety $\mathcal{T}$ of maximal tori in G and the closed subscheme $\mathcal{X}\subset G\times\mathcal{T}$ of all pairs (g,T) with $g\in T$. The generic fiber of the projection $\mathcal{X}\to\mathcal{T}$ is the *generic torus* T_{gen} of G. Then T_{gen} is a maximal torus of G_K, where $K:=F(\mathcal{T})$.

Every maximal torus in G is the factor torus of a maximal torus in G' by C. It follows that the variety $\mathcal{T}'$ of maximal tori in G' is naturally isomorphic to $\mathcal{T}$. Moreover, as the generic torus T'_{gen} of G' is a maximal torus of G'_K, we have $T_{\mathrm{gen}}\simeq T'_{\mathrm{gen}}/C_K$ and, therefore, an exact sequence of character groups

$$0\to\widehat{T}_{\mathrm{gen}}\to\widehat{T}'_{\mathrm{gen}}\to\widehat{C}_K\to 0.$$

By Theorem 2.4, the composition of the natural homomorphism

$$\mathrm{Inv}(G,\mathrm{Br})_{\mathrm{norm}}\to\mathrm{Inv}(G_K,\mathrm{Br})_{\mathrm{norm}}$$

with the restriction $\mathrm{Inv}(G_K,\mathrm{Br})_{\mathrm{norm}}\to\mathrm{Inv}(T_{\mathrm{gen}},\mathrm{Br})_{\mathrm{norm}}$ can be identified with the natural composition $\mathrm{Pic}(G)\to\mathrm{Pic}(G_K)\to\mathrm{Pic}(T_{\mathrm{gen}})$ and hence with the connecting homomorphism $\widehat{C}(F)=\widehat{C}(K)\to H^1(K,\widehat{T}_{\mathrm{gen}})$. Note that as $F=F_{\mathrm{sep}}$, the decomposition group of T_{gen} coincides with the Weyl group W of G by [Voskresenskiĭ 1988, Theorem 1], hence $H^1(K,\widehat{T}_{\mathrm{gen}})\simeq H^1(W,\widehat{T}_{\mathrm{gen}})$.

Let w be a Coxeter element in W.[1] It is the product of reflections with respect to all simple roots (in some order). By [Humphreys 1990, Lemma, p. 76], 1 is not an eigenvalue of w on the space of weights $\widehat{T'}_{\mathrm{gen}}\otimes\mathbb{R}$. Let W_0 be the cyclic subgroup in W generated by w. It follows that the first term in the exact sequence

$$(\widehat{T'}_{\mathrm{gen}})^{W_0}\to\widehat{C}(K)\to H^1(W_0,\widehat{T}_{\mathrm{gen}})$$

is trivial, that is, the second map is injective. Hence every nonzero character χ in $\widehat{C}(K)$ restricts to a nonzero element in $H^1(W_0,\widehat{T}_{\mathrm{gen}})$. It follows that the image of χ in $H^1(W,\widehat{T}_{\mathrm{gen}})$ is ramified by Corollary 5.6, hence so is χ by Proposition 5.2. □

[1] We owe the idea to use the Coxeter element and the reference below to S. Garibaldi.

Appendix A: Generalities

A-I: ***Proof of Theorem 2.2.*** Suppose that $i(E_{\mathrm{gen}}) = 0$ for an H-invariant i of G. Let K/F be a field extension and $I \to \operatorname{Spec} K$ a G-torsor. We need to show that $i(I) = 0$ in $H(K)$.

Suppose first that K is infinite. Find a point $x \in X(K)$ such that I is isomorphic to the pull-back of the classifying torsor with respect to x. Let x' be a rational point of X_K above x and write O for the local ring $O_{X_K,x'}$. The K-algebra O is a regular local ring with residue field K. Therefore, the completion $\widehat{O}$ is isomorphic to $K[[t_1, t_2, \dots, t_n]]$ over K. Let L be the quotient field of $\widehat{O}$, a field extension of $K(X)$. We have the following diagram of morphisms with a commutative square and three triangles:

$$\begin{tikzcd}
 & \operatorname{Spec} K \arrow[d, shift left] \arrow[dr, "x"] & \\
\operatorname{Spec} L \arrow[ur] \arrow[r] \arrow[d] & \operatorname{Spec} \widehat{O} \arrow[u, shift left] \arrow[r] \arrow[d] & X \\
\operatorname{Spec} K(X) \arrow[r] & \operatorname{Spec} O \arrow[ur] &
\end{tikzcd}$$

The pull-back of the classifying torsor $E \to X$ via $\operatorname{Spec} K(X) \to X$ is $(E_{\mathrm{gen}})_{K(X)}$. The G-torsor I is the pull-back of $E \to X$ with respect to x. Let $\widehat{E}$ be the pull-back of $E \to X$ via $\operatorname{Spec} \widehat{O} \to X$. Therefore, I is the pull-back of $\widehat{E}$. Since G is smooth, by a theorem of Grothendieck [Demazure and Grothendieck 1970, XXIV, Proposition 8.1], $\widehat{E}$ is the pull-back of I with respect to $\operatorname{Spec} \widehat{O} \to \operatorname{Spec}(K)$. It follows that $I_L \simeq (E_{\mathrm{gen}})_L$ as torsors over L. Hence the images of $i(I)$ and $i(E_{\mathrm{gen}})$ in $H(L)$ are equal and therefore, $i(I)_L = 0$. By Property 2.1, we have $i(I) = 0$.

If K is finite, we replace F by $F((t))$ and K by $K((t))$. By the first part of the proof, $i(I)$ belongs to the kernel of $H(K) \to H(K((t)))$ and hence is trivial by Property 2.1 again.

A-II: ***Decomposable elements.*** Let Γ be a profinite group and A a Γ-lattice. Write A^Γ for the subgroup of Γ-invariant elements in A. Let $\Gamma' \subset \Gamma$ be an open subgroup and choose representatives $\gamma_1, \gamma_2, \dots, \gamma_n$ for the left cosets of Γ' in Γ. We have the *trace map* $\mathrm{Tr} : A^{\Gamma'} \to A^\Gamma$ defined by $\mathrm{Tr}(a) = \sum_{i=1}^n \gamma_i a$.

Let $S^2(A)$ be the symmetric square of A. Consider the *quadratic trace map* $\mathrm{Qtr} : A^{\Gamma'} \to S^2(A)^\Gamma$ defined by $\mathrm{Qtr}(a) = \sum_{i<j} (\gamma_i a)(\gamma_j a)$. Write $\mathrm{Dec}(A)$ for the subgroup of *decomposable elements* in $S^2(A)^\Gamma$ generated by the square $(A^\Gamma)^2$ of A^Γ and the elements $\mathrm{Qtr}(a)$ for all open subgroups $\Gamma' \subset \Gamma$ and all $a \in A^{\Gamma'}$.

Let B be another Γ-lattice. We write $\mathrm{Dec}(A, B)$ for the subgroup of $(A \otimes B)^{\Gamma}$ generated by elements of the form $\mathrm{Tr}(a \otimes b)$ for all open subgroups $\Gamma' \subset \Gamma$ and all $a \in A^{\Gamma'}$, $b \in B^{\Gamma'}$.

There is a natural isomorphism $S^2(A \oplus B) \simeq S^2(A) \oplus (A \otimes B) \oplus S^2(B)$. Moreover, the equality $\mathrm{Qtr}(a+b) = \mathrm{Qtr}(a) + (\mathrm{Tr}(a) \otimes \mathrm{Tr}(b) - \mathrm{Tr}(a \otimes b)) + \mathrm{Qtr}(b)$ yields the decomposition

$$\mathrm{Dec}(A \oplus B) \simeq \mathrm{Dec}(A) \oplus \mathrm{Dec}(A, B) \oplus \mathrm{Dec}(B).$$

A-III: *Cup-products.* Let $1 \to T \to P \to Q \to 1$ be an exact sequence of tori. We consider the connecting maps

$$\partial_1 : H^p(F, \widehat{T}(i)) \to H^{p+1}(F, \widehat{Q}(i))$$

for the exact sequence $0 \to \widehat{Q}_{\mathrm{sep}} \to \widehat{P}_{\mathrm{sep}} \to \widehat{T}_{\mathrm{sep}} \to 0$ of character Γ-lattices and

$$\partial_2 : H^q(F, \widehat{Q}^{\circ}(j)) \to H^{q+1}(F, \widehat{T}^{\circ}(j))$$

for the dual sequence of lattices (see notation in Section 4b).

Lemma A.1. *Let $a \in H^p(F, \widehat{T}(i))$ and $b \in H^q(F, \widehat{Q}^{\circ}(j))$ with $i + j \leq 2$. Then $\partial_1(a) \cup b = (-1)^{p+1} a \cup \partial_2(b)$ in $H^{p+q+1}(F, \mathbb{Z}(i+j))$, where the cup-product is defined in* (4-3).

Proof. By [Cartan and Eilenberg 1999, Chapter V, Proposition 4.1], the elements $\partial_1(1_T)$ and $\partial_2(1_Q)$ in

$$H^1(F, \widehat{T}^{\circ}_{\mathrm{sep}} \otimes \widehat{Q}_{\mathrm{sep}}) = \mathrm{Ext}^1_{\Gamma}(\widehat{T}_{\mathrm{sep}}, \widehat{Q}_{\mathrm{sep}})$$

differ by a sign. Write τ for the isomorphism induced by permutation of the factors. By the standard properties of the cup-product, we have

$$\begin{aligned}\partial_1(a) \cup b = 1_T \cup \partial_1(a) \cup b &= \partial_1(1_T) \cup a \cup b = (-1)^{pq} \tau(\partial_1(1_T) \cup b \cup a) \\ &= (-1)^{pq+1} \tau(\partial_2(1_Q) \cup b \cup a) = (-1)^{pq+1} \tau(1_Q \cup \partial_2(b) \cup a) \\ &= (-1)^{p+1} 1_Q \cup a \cup \partial_2(b) = (-1)^{p+1} a \cup \partial_2(b).\end{aligned}$$

□

A-IV: *Cosimplicial abelian groups.* Let $A^{\bullet}$ be a cosimplicial abelian group

$$A^0 \underset{d^1}{\overset{d^0}{\rightrightarrows}} A^1 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; A^2 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; \cdots$$

and write $h_*(A^{\bullet})$ for the homology groups of the associated complex of abelian groups. In particular,

$$h_0(A^{\bullet}) = \mathrm{Ker}\left[(d^0 - d^1) : A^0 \to A^1\right].$$

We say that the cosimplicial abelian group $A^\bullet$ is *constant* if for every i, all the coface maps $d_j : A^{i-1} \to A^i$, $j = 0, 1, \dots, i$, are isomorphisms. In this case all the d_j are equal as $d_j = s_j^{-1} = d_{j+1}$, where the s_j are the codegeneracy maps. For a constant cosimplicial abelian group $A^\bullet$, we have $h_0(A^\bullet) = A^0$ and $h_i(A^\bullet) = 0$ for all $i > 0$. We will need the following straightforward statement.

Lemma A.2. *Let* $0 \to A^\bullet \to B^\bullet \to C^\bullet \to D^\bullet$ *be an exact sequence of cosimplicial abelian groups with* $A^\bullet$ *a constant cosimplicial group. Then the sequence of groups* $0 \to A^0 \to h_0(B^\bullet) \to h_0(C^\bullet) \to h_0(D^\bullet)$ *is exact.*

Let H be a contravariant functor from the category of schemes over F to the category of abelian groups. We say that H is *homotopy invariant* if for every vector bundle $E \to X$ over F, the induced map $H(X) \to H(E)$ is an isomorphism.

For an integer $d > 0$ consider the following property of the functor H:

Property A.3. For every closed subscheme Z of a scheme X with $\operatorname{codim}_X(Z) \geq d$, the natural homomorphism $H(X) \to H(X \setminus Z)$ is an isomorphism.

Let G be a linear algebraic group over a field F and choose a standard classifying G-torsor $U \to U/G$. Let U^i denote the product of i copies of U. We have the G-torsors $U^i \to U^i/G$.

Consider the cosimplicial abelian group $A^\bullet = H(U^\bullet/G)$ with $A^i = H(U^{i+1}/G)$ and coface maps $A^{i-1} \to A^i$ induced by the projections $U^{i+1}/G \to U^i/G$.

Lemma A.4. *Let H be a homotopy invariant functor satisfying Property A.3 for some d. Let $U \to U/G$ be a standard classifying G-torsor and U' an open subset of a G-representation V'.*

1. *If* $\operatorname{codim}_{V'}(V' \setminus U') \geq d$, *then the natural homomorphism*
$$H(U/G) \to H((U \times U')/G)$$
is an isomorphism.
2. *If* $\operatorname{codim}_V(V \setminus U) \geq d$, *then the cosimplicial group* $H(U^\bullet/G)$ *is constant.*

Proof. 1. The scheme $(U \times U')/G$ is an open subset of the vector bundle $(U \times V')/G$ over U/G with complement of codimension at least d. The map in question is the composition $H(U/G) \to H((U \times V')/G) \to H((U \times U')/G)$ and both maps in the composition are isomorphisms since H is homotopy invariant and satisfies Property A.3.

2. By the first part of the lemma applied to the G-torsor $U^i \to U^i/G$ and $U' = U$, the map $H(U^i/G) \to H(U^{i+1}/G)$ induced by a projection $U^{i+1}/G \to U^i/G$ is an isomorphism. □

By Lemma A.4, if H is a homotopy invariant functor satisfying Property A.3 for some d, then the group $H(U/G)$ does not depend on the choice of the representation

V and the open set $U \subset V$ provided $\operatorname{codim}_V(V \setminus U) \geq d$. Following [Totaro 1999], we denote this group by $H(BG)$.

Example A.5. The split torus $T = (\mathbb{G}_m)^n$ over F acts freely on the product U of n copies of $\mathbb{A}^{r+1} \setminus \{0\}$ with $U/T \simeq (\mathbb{P}^r)^n$, that is, BT is "approximated" by the varieties $(\mathbb{P}^r)^n$ if "$r \gg 0$." We then have $\mathrm{CH}^*(BT) = S^*(\widehat{T})$, where S^* represents the symmetric algebra and $\widehat{T}$ is the character group of T; see [Edidin and Graham 1998, p. 607]. In particular, $\mathrm{Pic}(BT) = \mathrm{CH}^1(BT) = \widehat{T}$. More generally, by the Künneth formula [Esnault et al. 1998, Proposition 3.7],

$$H^*_{\mathrm{Zar}}(BT, \mathcal{K}_*) \simeq \mathrm{CH}^*(BT) \otimes K_*(F) \simeq S^*(\widehat{T}) \otimes K_*(F),$$

where $K_n(F)$ is the Quillen K-group of F and $\mathcal{K}_n$ is the Zariski sheaf associated to the presheaf $U \mapsto K_n(U)$.

A-V: *Étale cohomology.* For a scheme X and a closed subscheme $Z \subset X$ we write $H^*_Z(X, \mathbb{Q}/\mathbb{Z}(j))$ for the étale cohomology group of X with support in Z and values in $\mathbb{Q}/\mathbb{Z}(j)$ [Milne 1980, Chapter III, §1]. Write $X^{(i)}$ for the set of points in X of codimension i. For a point $x \in X^{(1)}$ set

$$H^*_x(X, \mathbb{Q}/\mathbb{Z}(j)) = \operatorname*{colim}_{x \in U} H^*_{\overline{\{x\}} \cap U}(U, \mathbb{Q}/\mathbb{Z}(j)),$$

where the colimit is taken over all open subsets $U \subset X$ containing x. If X is a variety, write

$$\partial_x : H^*(F(X), \mathbb{Q}/\mathbb{Z}(j)) \to H^{*+1}_x(X, \mathbb{Q}/\mathbb{Z}(j))$$

for the residue homomorphisms arising from the *coniveau spectral sequence* [Colliot-Thélène et al. 1997, 1.2]

$$E_1^{p,q} = \coprod_{x \in X^{(p)}} H_x^{p+q}(X, \mathbb{Q}/\mathbb{Z}(j)) \Rightarrow H^{p+q}(X, \mathbb{Q}/\mathbb{Z}(j)). \qquad \text{(A-1)}$$

Let $f : Y \to X$ be a dominant morphism of varieties over F, $y \in Y^{(1)}$, and $x = f(y)$. If $x \in X^{(1)}$, there is a natural homomorphism

$$f^*_y : H^*_x(X, \mathbb{Q}/\mathbb{Z}(j)) \to H^*_y(Y, \mathbb{Q}/\mathbb{Z}(j)).$$

The following lemma is straightforward.

Lemma A.6. *Let $f : Y \to X$ be a dominant morphism of varieties over F, $y \in Y^{(1)}$ and $x = f(y)$.*

(1) *If x is the generic point of X, then the composition*

$$H^*(F(X), \mathbb{Q}/\mathbb{Z}(j)) \xrightarrow{f^*} H^*(F(Y), \mathbb{Q}/\mathbb{Z}(j)) \xrightarrow{\partial_y} H^{*+1}_y(Y, \mathbb{Q}/\mathbb{Z}(j))$$

is trivial.

(2) *If $x \in X^{(1)}$, the diagram*

$$\begin{tikzcd} H^*(F(X), \mathbb{Q}/\mathbb{Z}(j)) \arrow[r, "\partial_x"] \arrow[d, "f^*"'] & H^{*+1}_x(X, \mathbb{Q}/\mathbb{Z}(j)) \arrow[d, "f^*_y"'] \\ H^*(F(Y), \mathbb{Q}/\mathbb{Z}(j)) \arrow[r, "\partial_y"] & H^{*+1}_y(Y, \mathbb{Q}/\mathbb{Z}(j)). \end{tikzcd}$$

is commutative.

Lemma A.7. *Let X be a geometrically irreducible variety, $Z \subset X$ a closed subvariety of codimension 1, and x the generic point of Z. Let P be a variety over F such that $P(K)$ is dense in P for every field extension K/F with K infinite, and let y be the generic point of $Z \times P$ in $Y := X \times P$. Then the homomorphism $f^*_y : H^*_x(X, \mathbb{Q}/\mathbb{Z}(j)) \to H^*_y(Y, \mathbb{Q}/\mathbb{Z}(j))$ induced by the projection $f : Y \to X$ is injective.*

Proof. Assume first that the field F is infinite. An element $\alpha \in H^*_x(X, \mathbb{Q}/\mathbb{Z}(j))$ is represented by an element $h \in H^*_{Z\cap U}(U, \mathbb{Q}/\mathbb{Z}(j))$ for a nonempty open set $U \subset X$ containing x. If α belongs to the kernel of

$$f^*_y : H^*_x(X, \mathbb{Q}/\mathbb{Z}(j)) \to H^*_y(Y, \mathbb{Q}/\mathbb{Z}(j)),$$

then there is an open subset $W \subset U \times P$ containing y such that h belongs to the kernel of the composition

$$g : H^*_{Z\cap U}(U, \mathbb{Q}/\mathbb{Z}(j)) \to H^*_{(Z\cap U)\times P}(U \times P, \mathbb{Q}/\mathbb{Z}(j)) \to H^*_{(Z\times P)\cap W}(W, \mathbb{Q}/\mathbb{Z}(j)).$$

As F is infinite, by the assumption on P, there is a rational point $t \in P$ in the image of the dominant composition $(Z \times P) \cap W \hookrightarrow Z \times P \to P$. We have $(U \times t) \cap W = U' \times t$ for an open subset $U' \subset U$ such that $x \in U'$. Composing g with the homomorphism $H^*_{(Z\times P)\cap W}(W, \mathbb{Q}/\mathbb{Z}(j)) \to H^*_{Z\cap U'}(U', \mathbb{Q}/\mathbb{Z}(j))$ induced by the morphism $(U', Z\cap U') \to (W, (Z\times P)\cap W)$, $u \mapsto (u, t)$, we see that h belongs to the kernel of the restriction homomorphism $H^*_{Z\cap U}(U, \mathbb{Q}/\mathbb{Z}(j)) \to H^*_{Z\cap U'}(U', \mathbb{Q}/\mathbb{Z}(j))$, hence the image of α in $H^*_x(X, \mathbb{Q}/\mathbb{Z}(j))$ is trivial.

Suppose now that F is a finite field. Choose a prime integer p and an infinite algebraic pro-p-extension L/F. By the first part of the proof, the statement holds for the variety X_L over L. By the restriction-corestriction argument, $\mathrm{Ker}(f^*_y)$ is a p-primary torsion group. Since this holds for every prime p, we have $\mathrm{Ker}(f^*_y) = 0$. □

Corollary A.8. *Let G be a linear algebraic group over F, let $E \to X$ be a G-torsor over a geometrically irreducible variety X with E a G-rational variety and consider the first projection $p : E^2/G \to X$. Let $x \in X$ and $y \in E^2/G$ be points of* *codimension* 1 *such that* $p(y) = x$. *Then the homomorphism*

$$p_y^* : H_x^*(X, \mathbb{Q}/\mathbb{Z}(j)) \to H_y^*(E^2/G, \mathbb{Q}/\mathbb{Z}(j))$$

is injective.

Proof. Choose a linear G-space V and a nonempty G-variety U that is G-isomorphic to open subschemes of E and V. We can replace the variety E^2/G by $(E \times U)/G$, an open subscheme in the vector bundle $(E \times V)/G$ over X. Shrinking X around x, we may assume that the vector bundle is trivial, that is, $(E \times U)/G$ is isomorphic to an open subscheme in $X \times V$. The statement then follows from Lemma A.7. □

Proposition A.9. *In the conditions of Corollary A.8, let* $h \in H^*(F(X), \mathbb{Q}/\mathbb{Z}(j))_{\mathrm{bal}}$. *Then* $\partial_x(h) = 0$ *for every point* $x \in X$ *of codimension* 1.

Proof. Let $y \in E^2/G$ be the point of codimension 1 such that $p_1(y) = x$. As $p_2(y)$ is the generic point of X, by Lemma A.6(1), $\partial_y(h') = 0$, where $h' = p_1^*(h) = p_2^*(h)$ in $H^*(F(E^2/G), \mathbb{Q}/\mathbb{Z}(j))$. It follows from Lemma A.6(2) that $\partial_x(h)$ is in the kernel of $(p_1)_y^* : H_x^*(X, \mathbb{Q}/\mathbb{Z}(j)) \to H_y^*(E^2/G, \mathbb{Q}/\mathbb{Z}(j))$ and hence is trivial by Corollary A.8. □

The sheaf $\mathcal{H}^*(\mathbb{Q}/\mathbb{Z}(j))$ defined in Section 3 has a flasque resolution related to the Cousin complex by [Colliot-Thélène et al. 1997, §2] (for the p-components with $p \neq \operatorname{char} F$) and [Gros and Suwa 1988, Theorem 1.4] (for the p-component with $p = \operatorname{char} F > 0$):

$$0 \to \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)) \to \coprod_{x \in X^{(0)}} i_{x*} H_x^n(X, \mathbb{Q}/\mathbb{Z}(j)) \to \coprod_{x \in X^{(1)}} i_{x*} H_x^{n+1}(X, \mathbb{Q}/\mathbb{Z}(j)) \to \cdots,$$

where $i_x : \operatorname{Spec} F(x) \to X$ are the canonical morphisms. In particular, we have:

Proposition A.10. *Let X be a smooth variety over F. The sequence*

$$0 \to H^0_{\mathrm{Zar}}(X, \mathcal{H}^*(\mathbb{Q}/\mathbb{Z}(j))) \to H^*(F(X), \mathbb{Q}/\mathbb{Z}(j)) \xrightarrow{\partial} \coprod_{x \in X^{(1)}} H_x^{*+1}(X, \mathbb{Q}/\mathbb{Z}(j)),$$

where $\partial = \coprod \partial_x$, *is exact.*

Proposition A.11. *Let X be a smooth variety over F and $x \in X$. The sequence*

$$0 \to H^*(O_{X,x}, \mathbb{Q}/\mathbb{Z}(j))) \to H^*(F(X), \mathbb{Q}/\mathbb{Z}(j)) \xrightarrow{\partial} \coprod_{\substack{x' \in X^{(1)} \\ x' \in \overline{\{x\}}}} H_{x'}^{*+1}(X, \mathbb{Q}/\mathbb{Z}(j))$$

is exact.

Appendix B: Spectral sequences

B-I: ***Hochschild–Serre spectral sequence.*** Let

$$\mathscr{A} \xrightarrow{W} \mathscr{B} \xrightarrow{V} \mathscr{C}$$

be additive left exact functors between abelian categories with enough injective objects. If W takes injective objects to V-acyclic ones, there is a spectral sequence

$$E_2^{p,q} = R^p V(R^q W(A)) \Rightarrow R^{p+q}(VW)(A)$$

for every complex A in $\mathscr{A}$ bounded from below.

We have exact triangles in the derived category of $\mathscr{B}$:

$$\tau_{\le n} RW(A) \to RW(A) \to \tau_{\ge n+1} RW(A) \to \tau_{\le n} RW(A)[1], \tag{B-1}$$

$$\tau_{\le n-1} RW(A) \to \tau_{\le n} RW(A) \to R^n W(A)[-n] \to \tau_{\le n-1} RW(A)[1]. \tag{B-2}$$

The filtration on $R^n(VW)(A)$ is defined by

$$F^j R^n(VW)(A) = \mathrm{Im}(R^n V(\tau_{\le(n-j)} RW(A)) \to R^n V(RW(A)) = R^n(VW)(A)).$$

As $\tau_{\ge n+1} RW(A)$ is acyclic in degrees less than or equal to n, the morphism

$$R^n V(\tau_{\le n} RW(A)) \to R^n V(RW(A)) = R^n(VW)(A)$$

is an isomorphism, in particular, $F^0 R^n(VW)(A) = R^n(VW)(A)$.

The *edge* homomorphism is defined as the composition

$$R^n(VW)(A) \xrightarrow{\sim} R^n V(\tau_{\le n} RW(A)) \to R^n V(R^n W(A)[-n]) = V(R^n W(A)).$$

Moreover, the kernel $F^1 R^n(VW)(A)$ of the edge homomorphism is isomorphic to $R^n V(\tau_{\le n-1} RW(A))$. We define the morphism d_n as the composition

$$d_n : F^1 R^n(VW)(A) \to R^n V(R^{n-1} W(A)[-n+1]) = R^1 V(R^{n-1} W(A)) = E_2^{1,n-1}.$$

B-II: ***First spectral sequence.*** Let X be a smooth variety over a field F. We have the functors

$$\mathrm{Sheaves}_{\text{ét}}(X) \xrightarrow{q_*} \mathrm{Sheaves}_{\text{ét}}(F) \xrightarrow{V} Ab,$$

where q_* is the push-forward map for the structure morphism $q : X \to \mathrm{Spec}(F)$ and $V(M) = H^0(F, M)$.

Consider the Hochschild–Serre spectral sequence

$$E_2^{p,q} = H^p(F, H^q(X_{\mathrm{sep}}, \mathbb{Z}(2)) \Rightarrow H^{p+q}(X, \mathbb{Z}(2)). \tag{B-3}$$

Set $\Delta(i) := Rq_*(\mathbb{Z}(i))$ for $i = 1$ or 2. Then $\Delta(i)$ is the complex of étale sheaves on F concentrated in degrees ≥ 1. The j-th term $F^j H^n(X, \mathbb{Z}(i))$ of the filtration on $H^n(X, \mathbb{Z}(i))$ coincides with the image of the canonical homomorphism

$$H^n(F, \tau_{\le(n-j)}\Delta(i)) \to H^n(F, \Delta(i)) = H^n(X, \mathbb{Z}(i)).$$

Let M be a Γ-lattice viewed as an étale sheaf over F. Note that there are canonical isomorphisms

$$H^*(F, M^\circ \otimes \Delta(i)) = \mathrm{Ext}^*_F(M, \Delta(i)) = \mathrm{Ext}^*_X(q^*M, \mathbb{Z}(i)), \tag{B-4}$$

where $M^\circ := \mathrm{Hom}(M, \mathbb{Z})$ is the dual lattice.

Consider also the following product map:

$$\mathbb{Z}(1) \otimes^L \Delta(1) \to Rq_*(q^*\mathbb{Z}(1) \otimes^L \mathbb{Z}(1)) \to Rq_*(\mathbb{Z}(1) \otimes^L \mathbb{Z}(1)) \to Rq_*(\mathbb{Z}(2)).$$

The complex $\mathbb{Z}(1) \otimes^L \tau_{\le 2}\Delta(1)$ is trivial in degrees greater than 3, hence we have a commutative diagram

$$\begin{tikzcd}
\mathbb{Z}(1) \otimes^L \tau_{\le 2}\Delta(1) \arrow[r, "\mathrm{prod}"] \arrow[d] & \tau_{\le 3}Rq_*(\mathbb{Z}(2)) \arrow[r, phantom, "="] \arrow[d] & \tau_{\le 3}\Delta(2) \\
\mathbb{Z}(1) \otimes^L \Delta(1) \arrow[r, "\mathrm{prod}"] & Rq_*(\mathbb{Z}(2)) \arrow[r, phantom, "="] & \Delta(2).
\end{tikzcd}$$

There are canonical morphisms from (B-2):

$$h_2 : \tau_{\le 2}\Delta(1)[2] \to H^2(X_{\mathrm{sep}}, \mathbb{Z}(1)) \quad \text{and} \quad h_3 : \tau_{\le 3}\Delta(2)[3] \to H^3(X_{\mathrm{sep}}, \mathbb{Z}(2)).$$

Consider an element

$$\delta \in H^1(F, M \otimes F^\times_{\mathrm{sep}}) = \mathrm{Ext}^1_F(M^\circ, \mathbb{G}_{m,F}) = \mathrm{Ext}^2_F(M^\circ, \mathbb{Z}(1)),$$

and view δ as a morphism $\delta : M^\circ \to \mathbb{Z}(1)[2]$ in $D^+(\mathrm{Sheaves}_{\text{ét}}(F))$.

The following diagram

$$\begin{tikzcd}
M^\circ \otimes \Delta(1)[2] \arrow[r, "\delta\otimes 1"] & \mathbb{Z}(1) \otimes^L \Delta(1)[4] \arrow[r, "\mathrm{prod}"] & \Delta(2)[4] \\
M^\circ \otimes \tau_{\le 2}\Delta(1)[2] \arrow[r, "\delta\otimes 1"] \arrow[u, "(1\otimes i_2)[2]"] \arrow[d, "1\otimes h_2"'] & \mathbb{Z}(1) \otimes^L \tau_{\le 2}\Delta(1)[4] \arrow[r, "\mathrm{prod}"] \arrow[u, "(1\otimes i_2)[4]"] \arrow[d, "1\otimes h_2"'] & \tau_{\le 3}\Delta(2)[4] \arrow[u, "(i_3)[4]"] \arrow[d, "h_3"'] \\
M^\circ \otimes H^2(X_{\mathrm{sep}}, \mathbb{Z}(1)) \arrow[r, "\delta\otimes 1"] & \mathbb{Z}(1) \otimes^L H^2(X_{\mathrm{sep}}, \mathbb{Z}(1))[2] \arrow[r, "\mathrm{prod}"] & H^3(X_{\mathrm{sep}}, \mathbb{Z}(2))[1]
\end{tikzcd}$$

where $i_2 : \tau_{\le 2}\Delta(1) \to \Delta(1)$ and $i_3 : \tau_{\le 3}\Delta(2) \to \Delta(2)$ are natural morphisms, is commutative.

By (B-4), we have

$$H^0(F, M^\circ \otimes \Delta(1)[2]) = \mathrm{Ext}^2_F(M, \Delta(1)) = \mathrm{Ext}^2_X(q^*M, \mathbb{Z}(1)).$$

Furthermore, the diagram above yields a commutative square

$$\begin{tikzcd}
\operatorname{Ext}^2_X(q^*M, \mathbb{Z}(1)) \arrow[r, "q^*(\delta)\cup -"] \arrow[d, "d_2"'] & F^1H^4(X, \mathbb{Z}(2)) \arrow[d, "d_4"'] \\
\operatorname{Hom}_\Gamma(M, H^2(X_{\mathrm{sep}}, \mathbb{Z}(1)) \arrow[r, "j"] & H^1(F, H^3(X_{\mathrm{sep}}, \mathbb{Z}(2))),
\end{tikzcd}$$

where d_2 is the edge map coming from the spectral sequence

$$\operatorname{Ext}^p_F(M, H^q(X_{\mathrm{sep}}, \mathbb{Z}(1))) \Rightarrow \operatorname{Ext}^{p+q}_X(q^*M, \mathbb{Z}(1)) \tag{B-5}$$

and j coincides with the composition

$$\begin{aligned}\operatorname{Hom}_\Gamma(M, H^2(X_{\mathrm{sep}}, \mathbb{Z}(1)) &= H^0(F, M^\circ \otimes H^2(X_{\mathrm{sep}}, \mathbb{Z}(1)))\\ &\xrightarrow{\delta\cup -} H^1(F, F^\times_{\mathrm{sep}} \otimes H^2(X_{\mathrm{sep}}, \mathbb{Z}(1))) \xrightarrow{\rho} H^1(F, H^3(X_{\mathrm{sep}}, \mathbb{Z}(2))),\end{aligned}$$

with ρ given by the product map.

Now suppose the group $H^2(X_{\mathrm{sep}}, \mathbb{Z}(1))$, which is canonically isomorphic to $\operatorname{Pic}(X_{\mathrm{sep}})$, is a lattice. Let $M = \operatorname{Pic}(X_{\mathrm{sep}})$ and consider the torus T over F with $\widehat{T}_{\mathrm{sep}} = M$. It follows that

$$\delta \in H^1(F, T^\circ) = H^1(F, \widehat{T}_{\mathrm{sep}} \otimes F^\times_{\mathrm{sep}}) = H^2(F, \widehat{T}_{\mathrm{sep}} \otimes \mathbb{Z}(1)),$$

where T° is the dual torus. Note that $\delta \cup 1_M = \delta$, where

$$1_M \in H^0(F, M^\circ \otimes H^2(X_{\mathrm{sep}}, \mathbb{Z}(1))) = \operatorname{End}_\Gamma(M)$$

is the identity.

The top map in the last diagram is given by the pairing

$$\begin{gathered}H^1(X, T^0) \otimes H^1(X, T) \to F^1H^4(X, \mathbb{Z}(2)),\\ a \otimes b \mapsto a \cup b,\end{gathered} \tag{B-6}$$

defined as the cup-product in (4-3),

$$H^2(X, \widehat{T}(1)) \otimes H^2(X, \widehat{T}^\circ(1)) \to F^1H^4(X, \mathbb{Z}(2)),$$

if we identify $\operatorname{Ext}^2_X(q^*M, \mathbb{Z}(1))$ with $H^2(X, \widehat{T}^\circ(1)) = H^1(X, T)$.

In this case, the homomorphism

$$\rho : H^1(F, T^\circ) \to H^1(F, H^3(X_{\mathrm{sep}}, \mathbb{Z}(2))) \tag{B-7}$$

is given by the product homomorphism

$$T^\circ(F_{\mathrm{sep}}) = F^\times_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}} = F^\times_{\mathrm{sep}} \otimes \operatorname{Pic}(X_{\mathrm{sep}}) \to H^3(X_{\mathrm{sep}}, \mathbb{Z}(2)).$$

A T-torsor $E \to X$ is called *universal* if the class of E in

$$H^1(X,T) = \mathrm{Ext}^2_X(q^*M, \mathbb{Z}(1))$$

satisfies $d_2([E]) = 1_M$; see [Colliot-Thélène and Sansuc 1987a].

Commutativity of the previous diagram gives:

Proposition B.1. *Let X be a smooth variety over F such that* $\mathrm{Pic}(X_{\mathrm{sep}})$ *is a lattice. Let T be the torus over F satisfying $\widehat{T}_{\mathrm{sep}} = \mathrm{Pic}(X_{\mathrm{sep}})$ and let E be a universal T-torsor over X with the class $[E] \in H^1(X,T)$. Then for every $\delta \in H^1(F, T^\circ)$, we have*

$$d_4(q^*(\delta) \cup [E]) = \rho(\delta),$$

where $d_4 : F^1H^4(X, \mathbb{Z}(2)) \to H^1(F, H^3(X_{\mathrm{sep}}, \mathbb{Z}(2)))$ is the map induced by the Hochschild–Serre spectral sequence (B-3) *and the cup-product is taken for the pairing* (B-6).

B-III: ***Second spectral sequence.*** We assume that $H^3(X_{\mathrm{sep}}, \mathbb{Z}(2)) = 0$, hence in particular $E_2^{0,3} = 0$ in the spectral sequence (B-3) and so $E_\infty^{2,2} \subset E_2^{2,2}$. Therefore, we have a canonical map

$$e_4 : F^2H^4(X, \mathbb{Z}(2)) \to E_\infty^{2,2} \hookrightarrow E_2^{2,2} = H^2(F, H^2(X_{\mathrm{sep}}, \mathbb{Z}(2))).$$

Let N be a Γ-lattice. Consider an element

$$\gamma \in H^2(F, N \otimes F_{\mathrm{sep}}^\times) = \mathrm{Ext}^2_F(N^\circ, \mathbb{G}_{m,F}) = \mathrm{Ext}^3_F(N^\circ, \mathbb{Z}(1)),$$

and view γ as a morphism $\gamma : N^\circ \to \mathbb{Z}(1)[3]$ in $D^+(\mathrm{Sheaves}_{\text{ét}}(F))$.

As above, the commutative diagram

$$\begin{tikzcd}
N^\circ \otimes \Delta(1)[1] \arrow[r, "\gamma\otimes 1"] & \mathbb{Z}(1) \otimes^L \Delta(1)[4] \arrow[r, "\mathrm{prod}"] & \Delta(2)[4] \\
N^\circ \otimes \tau_{\le 1}\Delta(1)[1] \arrow[r, "\gamma\otimes 1"] \arrow[u, "(1\otimes i_1)[1]"] \arrow[d, "1\otimes h_1"'] & \mathbb{Z}(1) \otimes^L \tau_{\le 1}\Delta(1)[4] \arrow[r, "\mathrm{prod}"] \arrow[u, "(1\otimes i_1)[4]"] \arrow[d, "1\otimes h_1"'] & \tau_{\le 2}\Delta(2)[4] \arrow[u, "(i_2)[4]"] \arrow[d, "h_2"'] \\
N^\circ \otimes H^1(X_{\mathrm{sep}}, \mathbb{Z}(1)) \arrow[r, "\gamma\otimes 1"] & \mathbb{Z}(1) \otimes^L H^1(X_{\mathrm{sep}}, \mathbb{Z}(1))[3] \arrow[r, "\mathrm{prod}"] & H^2(X_{\mathrm{sep}}, \mathbb{Z}(2))[2]
\end{tikzcd}$$

where i_1, i_2, h_1 and h_2 are defined in a similar fashion as in Section B-II, yields a commutative square

$$\begin{tikzcd}
\mathrm{Ext}^1_X(q^*N, \mathbb{Z}(1)) \arrow[r, "q^*(\gamma) \cup -"] \arrow[d, "d_1"'] & F^2H^4(X, \mathbb{Z}(2)) \arrow[d, "e_4"] \\
\mathrm{Hom}_\Gamma(N, H^1(X_{\mathrm{sep}}, \mathbb{Z}(1)) \arrow[r, "k"] & H^2(F, H^2(X_{\mathrm{sep}}, \mathbb{Z}(2)))
\end{tikzcd}$$

where d_1 is the edge map coming from the spectral sequence

$$\operatorname{Ext}^p_F(N, H^q(X_{\mathrm{sep}}, \mathbb{Z}(1))) \Rightarrow \operatorname{Ext}^{p+q}_X(q^*N, \mathbb{Z}(1))$$

and k coincides with the composition

$$\begin{aligned}\operatorname{Hom}_\Gamma(N, H^1(X_{\mathrm{sep}}, \mathbb{Z}(1)) &= H^0(F, N^\circ \otimes H^1(X_{\mathrm{sep}}, \mathbb{Z}(1)))\\ &\xrightarrow{\gamma\,\cup\,-} H^2(F, F^\times_{\mathrm{sep}} \otimes H^1(X_{\mathrm{sep}}, \mathbb{Z}(1))) \to H^2(F, H^2(X_{\mathrm{sep}}, \mathbb{Z}(2)))\end{aligned}$$

with the last homomorphism given by the product map.

Suppose N is a Γ-lattice in $F_{\mathrm{sep}}[X]^\times$ such that the composition

$$N \hookrightarrow F_{\mathrm{sep}}[X]^\times \to F_{\mathrm{sep}}[X]^\times / F^\times_{\mathrm{sep}}$$

is an isomorphism. Consider the torus Q with $\widehat{Q}_{\mathrm{sep}} = N$, so that $\gamma \in H^2(F, Q^\circ)$.

Note that $\gamma \cup i_N = \gamma$, where

$$i_N \in H^0(F, N^\circ \otimes H^1(X_{\mathrm{sep}}, \mathbb{Z}(1))) = \operatorname{Hom}_\Gamma(N, F_{\mathrm{sep}}[X]^\times)$$

is the embedding.

The top map in the previous diagram is given by the pairing

$$\begin{gathered} H^2(X, Q^0) \otimes H^0(X, Q) \to F^2H^4(X, \mathbb{Z}(2)),\\ a \otimes b \mapsto a \cup b, \end{gathered} \tag{B-8}$$

defined as the cup-product in (4-3),

$$H^3(X, \widehat{Q}(1)) \otimes H^1(X, \widehat{Q}^\circ(1))) \to H^4(X, \mathbb{Z}(2)),$$

if we identify $\operatorname{Ext}^1_X(q^*N, \mathbb{Z}(1))$ with $H^1(X, \widehat{Q}^\circ(1)) = H^0(X, Q)$.

The inclusion of $\widehat{Q}_{\mathrm{sep}}$ into $F_{\mathrm{sep}}[X]^\times$ yields a morphism $\varepsilon : X \to Q$ that can be viewed as an element of $H^0(X, Q)$. Consider the map

$$\mu : H^2(F, Q^\circ) \to H^2(F, H^2(X_{\mathrm{sep}}, \mathbb{Z}(2))) \tag{B-9}$$

given by composition with the product homomorphism

$$Q^\circ(F_{\mathrm{sep}}) = F^\times_{\mathrm{sep}} \otimes \widehat{Q}_{\mathrm{sep}} \to F^\times_{\mathrm{sep}} \otimes H^1(X_{\mathrm{sep}}, \mathbb{Z}(1)) \to H^2(X_{\mathrm{sep}}, \mathbb{Z}(2)).$$

We have proved:

Proposition B.2. *Let X be a smooth variety over F such that $H^3(X_{\mathrm{sep}}, \mathbb{Z}(2)) = 0$. Let N be a Γ-lattice in $F_{\mathrm{sep}}[X]^\times$ such that the composition*

$$N \hookrightarrow F_{\mathrm{sep}}[X]^\times \to F_{\mathrm{sep}}[X]^\times / F^\times_{\mathrm{sep}}$$

is an isomorphism. Let Q be the torus over F satisfying $\widehat{Q}_{\mathrm{sep}} = N$. Then for every $\gamma \in H^2(F, Q^\circ)$, we have

$$e_4(q^*(\gamma) \cup \varepsilon) = \mu(\gamma),$$

where $e_4 : F^2H^4(X, \mathbb{Z}(2)) \to H^2(F, H^2(X_{\mathrm{sep}}, \mathbb{Z}(2)))$ is the map induced by the Hochschild–Serre spectral sequence (B-3) *and the cup-product is taken for the pairing* (B-8).

B-IV: ***Relative étale cohomology.*** Let X be a smooth variety over F. Following B. Kahn [1996, §3], we define the relative étale cohomology groups as follows. Recall that $\Delta(i) = Rq_*(\mathbb{Z}(i))$ for $i = 1$ and 2, where $q : X \to \mathrm{Spec}(F)$ is the structure morphism, and let $\Delta'(i)$ be the cone of the natural morphism $\mathbb{Z}(i) \to \Delta(i)$ in $D_+(\mathrm{Sheaves}_{\text{ét}}(F))$. Define

$$H^*(X/F, \mathbb{Z}(2)) := H^*(F, \Delta'(2)).$$

(Note that our indexing is different from that in [Kahn 1996, §3].)

There is an infinite exact sequence

$$\cdots \to H^i(F, \mathbb{Z}(2)) \to H^i(X, \mathbb{Z}(2)) \to H^i(X/F, \mathbb{Z}(2)) \to H^{i+1}(F, \mathbb{Z}(2)) \to \cdots$$

If X has a rational point, we have

$$H^i(X/F, \mathbb{Z}(2)) = \overline{H}^i(X, \mathbb{Z}(2)) := H^i(X, \mathbb{Z}(2))/H^i(F, \mathbb{Z}(2)).$$

There is a Hochschild–Serre type spectral sequence [Kahn 1996, §3]

$$E_2^{p,q} = H^p(F, H^q(X_{\mathrm{sep}}/F_{\mathrm{sep}}, \mathbb{Z}(2))) \Rightarrow H^{p+q}(X/F, \mathbb{Z}(2)), \tag{B-10}$$

and we have by [Kahn 1996, Lemma 3.1] that

$$H^q(X_{\mathrm{sep}}/F_{\mathrm{sep}}, \mathbb{Z}(2)) = \begin{cases} 0 & \text{if } q \le 0, \\ \text{uniquely divisible group} & \text{if } q = 1, \\ \overline{H}^0_{\mathrm{Zar}}(X_{\mathrm{sep}}, \mathcal{K}_2) & \text{if } q = 2, \\ H^1_{\mathrm{Zar}}(X_{\mathrm{sep}}, \mathcal{K}_2) & \text{if } q = 3. \end{cases}$$

It follows that $E_2^{p,q} = 0$ if $q \le 1$ and $p > 0$. Comparing the spectral sequences (B-3) and (B-10), by Proposition B.1 we have:

Proposition B.3. *Let X be a smooth variety over F such that $X(F) \neq \varnothing$. If $H^0_{\mathrm{Zar}}(X_{\mathrm{sep}}, \mathcal{K}_2) = K_2(F_{\mathrm{sep}})$, then the spectral sequence* (B-10) *yields an exact sequence*

$$0 \to H^1(F, H^1_{\mathrm{Zar}}(X_{\mathrm{sep}}, \mathcal{K}_2)) \xrightarrow{\alpha} \overline{H}^4(X, \mathbb{Z}(2)) \\ \to \overline{H}^4(X_{\mathrm{sep}}, \mathbb{Z}(2))^\Gamma \to H^2(F, H^1_{\mathrm{Zar}}(X_{\mathrm{sep}}, \mathcal{K}_2)).$$

If, moreover, the group $\mathrm{Pic}(X_{\mathrm{sep}})$ *is a lattice and* T *is the torus over* F *such that* $\widehat{T}_{\mathrm{sep}} = \mathrm{Pic}(X_{\mathrm{sep}})$, *then* $\alpha(\rho(\delta)) = q^*(\delta) \cup [E]$ *for every* $\delta \in H^1(F, T^\circ)$, *where* ρ *is defined in* (B-7) *and* E *is a universal* T*-torsor over* X.

Comparing the spectral sequences (B-3) and (B-10), by Proposition B.2 we have:

Proposition B.4. *Let* X *be a smooth variety over* F *such that* $X(F) \neq \varnothing$. *If* $H^1_{\mathrm{Zar}}(X_{\mathrm{sep}}, \mathcal{K}_2) = 0$, *then the spectral sequence* (B-10) *yields an exact sequence*

$$0 \to H^2(F, \overline{H}^0_{\mathrm{Zar}}(X_{\mathrm{sep}}, \mathcal{K}_2)) \xrightarrow{\beta} \overline{H}^4(X, \mathbb{Z}(2)) \\ \to \overline{H}^4(X_{\mathrm{sep}}, \mathbb{Z}(2))^\Gamma \to H^3(F, \overline{H}^0_{\mathrm{Zar}}(X_{\mathrm{sep}}, \mathcal{K}_2)).$$

If N *is a* Γ*-lattice in* $F_{\mathrm{sep}}[X]^\times$ *such that the composition*

$$N \hookrightarrow F_{\mathrm{sep}}[X]^\times \to F_{\mathrm{sep}}[X]^\times / F_{\mathrm{sep}}^\times$$

is an isomorphism and Q *is the torus over* F *satisfying* $\widehat{Q}_{\mathrm{sep}} = N$, *then* $\beta(\mu(\gamma)) = q^*(\gamma) \cup \varepsilon$ *for every* $\gamma \in H^2(F, Q^\circ)$, *where* μ *is defined in* (B-9) *and* $\varepsilon \in H^0(X, Q)$ *is given by the inclusion of* $\widehat{Q}_{\mathrm{sep}}$ *into* $F_{\mathrm{sep}}[X]^\times$.

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