Cohomological invariants of algebraic tori

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Let $G$ be an algebraic group over a field $F$. As defined by Serre, a cohomological invariant of $G$ of degree $n$ with values in $\mathbb{Q}/\mathbb{Z}(j)$ is a functorial-in-$K$ collection of maps of sets $\text{Tors}_G(K) \to H^n(K, \mathbb{Q}/\mathbb{Z}(j))$ for all field extensions $K/F$, where $\text{Tors}_G(K)$ is the set of isomorphism classes of $G$-torsors over $\text{Spec} \ K$. We study the group of degree 3 invariants of an algebraic torus with values in $\mathbb{Q}/\mathbb{Z}(2)$. In particular, we compute the group $H^3_{nr}(F(S), \mathbb{Q}/\mathbb{Z}(2))$ of unramified cohomology of an algebraic torus $S$.

1. Introduction

Let $G$ be a linear algebraic group over a field $F$ (of arbitrary characteristic). The notion of an invariant of $G$ was defined in [Garibaldi et al. 2003] as follows. Consider the category $\text{Fields}_F$ of field extensions of $F$ and the functor

$$\text{Tors}_G : \text{Fields}_F \to \text{Sets}$$

taking a field $K$ to the set $\text{Tors}_G(K)$ of isomorphism classes of (right) $G$-torsors over $\text{Spec} \ K$. Let

$$H : \text{Fields}_F \to \text{Abelian Groups}$$

be another functor. An $H$-invariant of $G$ is then a morphism of functors

$$i : \text{Tors}_G \to H,$$

viewing $H$ with values in $\text{Sets}$, that is, a functorial in $K$ collection of maps of sets $\text{Tors}_G(K) \to H(K)$ for all field extensions $K/F$. We denote the group of $H$-invariants of $G$ by $\text{Inv}(G, H)$.

An invariant $i \in \text{Inv}(G, H)$ is called normalized if $i(I) = 0$ for the trivial $G$-torsor $I$. The normalized invariants form a subgroup $\text{Inv}(G, H)_{\text{norm}}$ of $\text{Inv}(G, H)$ and there is a natural isomorphism

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\[ \text{Inv}(G, H) \simeq H(F) \oplus \text{Inv}(G, H)_{\text{norm}}, \]

so it is sufficient to study normalized invariants.

Typically, \( H \) is a cohomological functor given by Galois cohomology groups with values in a fixed Galois module. Of particular interest to us is the functor \( H \) which takes a field \( K/F \) to the Galois cohomology group \( H^n(K, \mathbb{Q}/\mathbb{Z}(j)) \), where the coefficients \( \mathbb{Q}/\mathbb{Z}(j) \) are defined as follows. For a prime integer \( p \) different from the characteristic of \( F \), the \( p \)-component \( \mathbb{Q}_p/\mathbb{Z}_p(j) \) is the colimit over \( n \) of the étale sheaves \( \mu_{mp^n} \), where \( \mu_m \) is the sheaf of \( m \)-th roots of unity. In the case \( p = \text{char}(F) > 0 \), \( \mathbb{Q}_p/\mathbb{Z}_p(j) \) is defined via logarithmic de Rham–Witt differentials; see Section 3b.

We write \( \text{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j)) \) for the group of cohomological invariants of \( G \) of degree \( n \) with values in \( \mathbb{Q}/\mathbb{Z}(j) \).

The second cohomology group \( H^2(K, \mathbb{Q}/\mathbb{Z}(1)) \) is canonically isomorphic to the Brauer group \( \text{Br}(K) \) of the field \( K \). In Section 2c we prove (Theorem 2.4) that if \( G \) is a connected group (reductive if \( F \) is not perfect), then

\[ \text{Inv}(G, \text{Br})_{\text{norm}} \simeq \text{Pic}(G). \]

The group \( \text{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\text{norm}} \) for a semisimple simply connected group \( G \) has been studied by Rost; see [Garibaldi et al. 2003].

An essential object in the study of cohomological invariants is the notion of a classifying torsor: a \( G \)-torsor \( E \to X \) for a smooth variety \( X \) over \( F \) such that every \( G \)-torsor over an infinite field \( K/F \) is isomorphic to the pull-back of \( E \to X \) along a \( K \)-point of \( X \). If \( V \) is a generically free linear representation of \( G \) with a nonempty open subset \( U \subset V \) such that there is a \( G \)-torsor \( \pi : U \to X \), then \( \pi \) is classifying. Such representations exist (see Section 2b).

The generic fiber of \( \pi \) is the generic torsor over \( \text{Spec} F(X) \) attached to \( \pi \). Evaluation at the generic torsor yields a homomorphism

\[ \text{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j)) \to H^n(F(X), \mathbb{Q}/\mathbb{Z}(j)), \]

and in Section 3 we show that the image of this map is contained in the subgroup \( H^n_{\text{Zar}}(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))) \) of \( H^n(F(X), \mathbb{Q}/\mathbb{Z}(j)) \), where \( \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)) \) is the Zariski sheaf associated to the presheaf \( W \mapsto H^n(W, \mathbb{Q}/\mathbb{Z}(j)) \) of the étale cohomology groups. In fact, the image is contained in the subgroup \( H^n_{\text{Zar}}(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))_{\text{bal}} \) of balanced elements, that is, elements that have the same images under the pull-back homomorphisms with respect to the two projections \((U \times U)/G \to X\). Moreover, the balanced elements precisely describe the image and we prove (Theorem 3.4):

**Theorem A.** Let \( G \) be a smooth linear algebraic group over a field \( F \). We assume that \( G \) is connected if \( F \) is a finite field. Let \( E \to X \) be a classifying \( G \)-torsor with
E a $G$-rational variety such that $E(F) \neq \emptyset$. Then (1-1) yields an isomorphism
\[ \text{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j)) \simeq H^n_{\text{Zar}}(X, \mathbb{H}^n(\mathbb{Q}/\mathbb{Z}(j)))_{\text{bal}}. \]

At this point it is convenient to make use of a construction due to Totaro [1999]: because the Chow groups are homotopy invariant, the groups $\text{CH}^n(X)$ do not depend on the choice of the representation $V$ and the open set $U \subset V$ provided the codimension of $V \setminus U$ in $V$ is large enough. This leads to the notation $\text{CH}^n(BG)$, the Chow groups of the so-called classifying space $BG$, although $BG$ itself is not defined in this paper.

Unfortunately, the étale cohomology groups with values in $\mathbb{Q}_p/\mathbb{Z}_p(j)$, where $p = \text{char}(F) > 0$, are not homotopy invariant. In particular, we cannot use the theory of cycle modules of Rost [1996].

The main result of this paper is the exact sequence in Theorem 4.3 describing degree 3 cohomological invariants of an algebraic torus $T$. Writing $\widehat{T}_{\text{sep}}$ for the character lattice of $T$ over a separable closure of $F$ and $T^\circ$ for the dual torus, we prove our main result:

**Theorem B.** Let $T$ be an algebraic torus over a field $F$. Then there is an exact sequence
\[ 0 \to \text{CH}^2(BT)_{\text{tors}} \to H^1(F, T^0) \xrightarrow{\alpha} \text{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{\text{norm}} \to H^0(F, S^2(\widehat{T}_{\text{sep}}))/\text{Dec} \to H^2(F, T^0). \]

The homomorphism $\alpha$ is given by $\alpha(a)(b) = a_K \cup b$ for every $a \in H^1(F, T^0)$ and $b \in H^1(K, T)$ and every field extension $K/F$, where the cup-product is defined in (4-5), and $\text{Dec}$ is the subgroup of decomposable elements in the symmetric square $S^2(\widehat{T}_{\text{sep}})$ defined in Section A-II.

In the proof of the theorem we compute the group of balanced elements in the motivic cohomology group $H^4(BT, \mathbb{Z}(2))$ and relate it, using an exact sequence of B. Kahn and Theorem A, with the group of invariants $\text{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{\text{norm}}$.

We also prove that the torsion group $\text{CH}^2(BT)_{\text{tors}}$ is finite of exponent 2 (Theorem 4.7) and the last homomorphism in the sequence is also of exponent 2 (see the discussion before Theorem 4.13).

Moreover, if $p$ is an odd prime, the group $\text{Inv}^3(T, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\text{norm}}$, which is the $p$-primary component of $\text{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{\text{norm}}$, splits canonically into the direct sum of linear invariants (those that induce group homomorphisms from $\text{Tors}_T$ to $H^3$) and quadratic invariants, that is, the invariants $i$ such that the function $h(a, b) := i(a + b) - i(a) - i(b)$ is bilinear and $h(a, a) = 2i(a)$ for all $a$ and $b$. Furthermore, the groups of linear and quadratic invariants with values in $\mathbb{Q}_p/\mathbb{Z}_p(2)$ are canonically isomorphic to $H^1(F, T^\circ)[p]$ and $(H^0(F, S^2(\widehat{T}_{\text{sep}}))/\text{Dec})[p]$, respectively.
We also prove (Theorem 4.10) that the degree 3 invariants have control over the structure of all invariants. Precisely, the group \( \text{Inv}^3(T_K, \mathbb{Q}/\mathbb{Z}(2))_{\text{norm}} \) is trivial for all \( K/F \) if and only if \( T \) is special, that is, \( T \) has no nontrivial torsors over any field \( K/F \), which in particular means \( T \) has no nonconstant \( H \)-invariants for every functor \( H \).

Our motivation for considering invariants of tori comes from their connection with unramified cohomology (defined in Section 5). Specifically, this work began as an investigation of a problem posed by Colliot-Thélène [1995, p. 39]: for \( n \) prime to \( \text{char}(F) \) and \( i \geq 0 \), determine the unramified cohomology group \( H^i_{\text{nr}}(F(S), \mu_{2^n}) \), where \( F(S) \) is the function field of a torus \( S \) over \( F \). The connection is provided by Theorem 5.7 where we show that the unramified cohomology of a torus \( S \) is calculated by the invariants of an auxiliary torus:

**Theorem C.** Let \( S \) be a torus over \( F \) and let \( 1 \to T \to P \to S \to 1 \) be a flasque resolution of \( S \), that is, \( T \) is flasque and \( P \) is quasisplit. Then there is a natural isomorphism

\[
H^n_{\text{nr}}(F(S), \mathbb{Q}/\mathbb{Z}(j)) \simeq \text{Inv}^n(T, \mathbb{Q}/\mathbb{Z}(j)).
\]

By Theorem B and Theorem C, we have an exact sequence

\[
0 \to \text{CH}^2(BT)_{\text{tors}} \to H^1(F, T^0) \overset{\alpha}{\to} \overline{H}^3_{\text{nr}}(F(S), \mathbb{Q}/\mathbb{Z}(2)) \to H^0(F, S^2(\hat{T}_{\text{sep}}))/\text{Dec} \to H^2(F, T^0)
\]

describing the reduced third cohomology group

\[
\overline{H}^3_{\text{nr}}(F(S), \mathbb{Q}/\mathbb{Z}(2)) := H^3_{\text{nr}}(F(S), \mathbb{Q}/\mathbb{Z}(2))/H^3(F, \mathbb{Q}/\mathbb{Z}(2)).
\]

Moreover, for an odd prime \( p \), we have a canonical direct sum decomposition of the \( p \)-primary components:

\[
\overline{H}^3_{\text{nr}}(F(S), \mathbb{Q}_p/\mathbb{Z}_p(2)) = H^1(F, T^0)[p] \oplus (H^0(F, S^2(\hat{T}_{\text{sep}}))/\text{Dec})[p].
\]

Note that the torus \( S \) determines \( T \) up to multiplication by a quasisplit torus. If \( X \) is a smooth compactification of \( S \), one can take the torus \( T \) with \( \hat{T}_{\text{sep}} = \text{Pic}(X_{\text{sep}}) \); see [Colliot-Thélène and Sansuc 1977, §2].

In the present paper, \( F \) denotes a field of arbitrary characteristic, \( F_{\text{sep}} \) a separable closure of \( F \), and \( \Gamma \) the absolute Galois group \( \text{Gal}(F_{\text{sep}}/F) \) of \( F \).

The word “scheme” over a field \( F \) means a separated scheme over \( F \) and, following [Fulton 1984], a “variety” over \( F \) is an integral scheme of finite type over \( F \). If \( X \) is a scheme over \( F \) and \( L/F \) is a field extension then we write \( X_L \) for \( X \times_F \text{Spec} \ L \). When \( L = F_{\text{sep}} \) we write simply \( X_{\text{sep}} \).

A “linear algebraic group over \( F \)” is an affine group scheme of finite type over \( F \), not necessarily smooth.
2. Invariants of algebraic groups

2a. Definitions and basic properties. Let $G$ be a linear algebraic group over a field $F$. Consider the functor

$$\text{Tors}_G : \text{Fields}_F \rightarrow \text{Sets}$$

from the category of field extensions of $F$ to the category of sets taking a field $K$ to the set $\text{Tors}_G(K)$ of isomorphism classes of (right) $G$-torsors over $\text{Spec} K$. Note that if $G$ is a smooth group, then there is a natural bijection

$$\text{Tors}_G(K) \simeq H^1(K, G) := H^1(\text{Gal}(K_{\text{sep}}/K), G(K_{\text{sep}})).$$

Let $H : \text{Fields}_F \rightarrow \text{Abelian Groups}$ be a functor. We also view $H$ as a functor with values in $\text{Sets}$. Following [Garibaldi et al. 2003], we define an $H$-invariant of $G$ as a morphism of functors $\text{Tors}_G \rightarrow H$ from the category $\text{Fields}_F$ to $\text{Sets}$. All the $H$-invariants of $G$ form the abelian group of invariants $\text{Inv}(G, H)$. An invariant $i \in \text{Inv}(G, H)$ is called constant if there is an element $h \in H(F)$ such that $i(I) = h_K$ for every $G$-torsor $I \rightarrow \text{Spec} K$, where $h_K$ is the image of $h$ under natural map $H(F) \rightarrow H(K)$. The constant invariants form a subgroup $\text{Inv}(G, H)_{\text{const}}$ of $\text{Inv}(G, H)$ isomorphic to $H(F)$. An invariant $i \in \text{Inv}(G, H)$ is called normalized if $i(I) = 0$ for the trivial $G$-torsor $I$. The normalized invariants form a subgroup $\text{Inv}(G, H)_{\text{norm}}$ of $\text{Inv}(G, H)$ and we have the decomposition

$$\text{Inv}(G, H) = \text{Inv}(G, H)_{\text{const}} \oplus \text{Inv}(G, H)_{\text{norm}} \simeq H(F) \oplus \text{Inv}(G, H)_{\text{norm}},$$

so it suffices to determine the normalized invariants.

2b. Classifying torsors. Let $G$ be a linear algebraic group over a field $F$. A $G$-torsor $E \rightarrow X$ over a smooth variety $X$ over $F$ is called classifying if for every field extension $K/F$, with $K$ infinite, and for every $G$-torsor $I \rightarrow \text{Spec} K$, there is a point $x : \text{Spec} K \rightarrow X$ such that the torsor $I$ is isomorphic to the fiber $E(x)$ of $E \rightarrow X$ over $x$, that is, $I \simeq E(x) := x^*(E) = \text{Spec}(K) \times_X E$. The generic fiber $E_{\text{gen}} \rightarrow \text{Spec} F(X)$ of a classifying torsor is called a generic $G$-torsor; see [ibid., Part 1, §5.3].

If $V$ is a generically free linear representation of $G$ with a nonempty open subset $U \subset V$ such that there is a $G$-torsor $\pi : U \rightarrow X$, then $\pi$ is classifying; see [ibid., Part 1, §5.4]. We will write $U/G$ for $X$ and call $\pi$ a standard classifying $G$-torsor. Standard classifying $G$-torsors exist: we can embed $G$ into $U := \text{GL}_n, F$ for some $n$ as a closed subgroup. Then $U$ is an open subset in the affine space $M_n(F)$ on which $G$ acts linearly and the canonical morphism $U \rightarrow X := U/G$ is a $G$-torsor. Note that $U(F) \neq \varnothing$.

We say that a $G$-variety $Y$ is $G$-rational if there is an affine space $V$ with a linear $G$-action such that $Y$ and $V$ have $G$-isomorphic nonempty open $G$-invariant
subvarieties. Note that if \( U \to U/G \) is a standard classifying \( G \)-torsor, then \( U \) is a \( G \)-rational variety.

Let \( E \to X \) be a classifying \( G \)-torsor and let \( H : \text{Fields}_F \to \text{Abelian Groups} \) be a functor. Define the map

\[
\theta_G : \text{Inv}(G, H) \to H(F(X)), \quad i \mapsto i(E_{\text{gen}}),
\]

by sending an invariant to its value at the generic torsor \( E_{\text{gen}} \).

Consider the following property of the functor \( H \):

**Property 2.1.** The map \( H(K) \to H(K((t))) \) is injective for any field extension \( K / F \).

The following theorem, due to M. Rost, was proved in [Garibaldi et al. 2003, Part II, Theorem 3.3]. For completeness, we give a slightly modified proof in Section A-I.

**Theorem 2.2.** Let \( G \) be a smooth linear algebraic group over \( F \). If a functor \( H : \text{Fields}_F \to \text{Abelian Groups} \) has Property 2.1, then the map \( \theta_G \) is injective, that is, every \( H \)-invariant of \( G \) is determined by its value at the generic \( G \)-torsor.

Let \( G' \) be a (closed) subgroup of \( G \) over \( F \). The map of sets

\[
H^1(K, G') \to H^1(K, G)
\]

for every field extension \( K / F \) yields the restriction map

\[
\text{res} : \text{Inv}(G, H) \to \text{Inv}(G', H).
\]

Choose standard torsors \( \pi : U \to U/G \) and \( \pi' : U \to U/G' \) (for example, with \( U = \text{GL}_{n,F} \) as above). The pull-back of \( \pi \) with respect to the natural morphism \( \alpha : U/G' \to U/G \) is the push-forward of \( \pi' \) via the inclusion \( G' \hookrightarrow G \). It follows that the diagram

\[
\begin{array}{ccc}
\text{Inv}(G, H) & \xrightarrow{\text{res}} & \text{Inv}(G', H) \\
\theta_G \downarrow & & \theta_{G'} \downarrow \\
H(F(U/G)) & \xrightarrow{\alpha^*} & H(F(U/G'))
\end{array}
\]

is commutative.

**2c. The Brauer group invariants.** Let \( G \) be a smooth connected linear algebraic group over \( F \). Every cohomological invariant of \( G \) of degree 1 is constant by [Knus et al. 1998, Proposition 31.15]. In this section we study (degree 2) \( \text{Br} \)-invariants for the Brauer group functor \( K \mapsto \text{Br}(K) \). We assume that \( G \) is reductive if \( \text{char}(F) > 0 \).

**Lemma 2.3.** For any field extension \( K / F \) such that \( F \) is algebraically closed in \( K \), the natural map \( \text{Pic}(G) \to \text{Pic}(G_K) \) is an isomorphism.
Proof. We may assume that $G$ is reductive by factoring out the unipotent radical in the case that $F$ is perfect. There is an exact sequence (see [Colliot-Thélène 2004, Theorem 1.2])

$$1 \rightarrow C \rightarrow G' \rightarrow G \rightarrow 1$$

with $C$ a torus and $G'$ a reductive group with $\text{Pic}(G_L') = 0$ for any field extension $L/F$. Let $T$ be the factor group of $G'$ by the semisimple part. The result follows from the exact sequence [Sansuc 1981, Proposition 6.10] (note that $G$ is reductive if $L$ is not perfect)

$$\hat{T}(L) \rightarrow \hat{C}(L) \rightarrow \text{Pic}(G_L) \rightarrow \text{Pic}(G_L') = 0$$

with $L = F$ and $K$ since the groups $\hat{T}(F)$ and $\hat{C}(F)$ don’t change when $F$ is replaced by $K$. □

Since for any $G_K$-torsor $E \rightarrow \text{Spec}(K)$ over a field extension $K/F$ one has [Sansuc 1981, Proposition 6.10] the exact sequence

$$\text{Pic}(E) \rightarrow \text{Pic}(G_K) \xrightarrow{\delta} \text{Br}(K) \xrightarrow{\varepsilon} \text{Br}(E),$$

we obtain the homomorphism

$$\nu : \text{Pic}(G) \rightarrow \text{Inv}(G, \text{Br}),$$

which takes an element $\alpha \in \text{Pic}(G)$ to the invariant that sends a $G$-torsor $E$ over a field extension $K/F$ to $\delta(\alpha_K)$. If $E$ is a trivial torsor, that is, $E(K) \neq \emptyset$, then $\varepsilon$ is injective and hence $\delta = 0$. It follows that the invariant $\nu(\alpha)$ is normalized.

**Theorem 2.4.** Let $G$ be a smooth connected linear algebraic group over $F$. Assume that $G$ is reductive if $\text{char}(F) > 0$. Then the map $\nu : \text{Pic}(G) \rightarrow \text{Inv}(G, \text{Br})_{\text{norm}}$ is an isomorphism.

Proof. Choose a standard classifying $G$-torsor $U \rightarrow U/G$. Write $K$ for the function field $F(U/G)$ and let $U_{\text{gen}}$ be the generic $G$-torsor over $K$. Consider the commutative diagram

$$\text{Pic}(G) \xrightarrow{\nu} \text{Inv}(G, \text{Br})_{\text{norm}}$$

$$\downarrow j \quad \downarrow \theta_G$$

$$\text{Pic}(U_{\text{gen}}) \xrightarrow{\delta} \text{Pic}(G_K) \xrightarrow{\delta} \text{Br}(K) \xrightarrow{i} \text{Br}(K(U_{\text{gen}})),$$

where the bottom sequence is (2-2) for the $G$-torsor $U_{\text{gen}} \rightarrow \text{Spec}(K)$ followed by the injection $\text{Br}(U_{\text{gen}}) \rightarrow \text{Br}(K(U_{\text{gen}}))$ (see [Milne 1980, Chapter IV, Corollary 2.6]), and the map $\theta_G$ is evaluation at the generic torsor $U_{\text{gen}}$ given in (2-1) and is injective by Theorem 2.2. Since the generic torsor is split over $K(U_{\text{gen}})$, $\text{Im}(\theta_G) \subset \text{Ker}(i) = \text{Im}(\delta)$. By Lemma 2.3, $j$ is an isomorphism, hence $\nu$ is surjective.
Note that $U_{\text{gen}}$ is a localization of $U$, hence $\text{Pic}(U_{\text{gen}}) = 0$ as $\text{Pic}(U) = 0$. It follows that $\nu$ is injective.

An algebraic group $G$ over a field $F$ is called special if $H^1(K, G) = \{1\}$ for every field extension $K/F$, that is, all $G$-torsors over any field extension of $F$ are trivial.

**Corollary 2.5.** If the group $G$ is special, then $\text{Pic}(G) = 0$.

3. **Invariants with values in $\mathbb{Q}/\mathbb{Z}(j)$**

In this section we find a description for the group of cohomological invariants with values in $\mathbb{Q}/\mathbb{Z}(j)$ by identifying the image of the embedding $\theta_G$ in (2-1).

Let $G$ be a linear algebraic group over a field $F$, let $H \subset G$ be a subgroup and let $E \to X$ be a $G$-torsor. Suppose that $G/H$ is affine. Consider a $G$-action on $E \times (G/H)$ by $(e, gH)g = (eg, g^{-1}gH)$. By [Milne 1980, Theorem I.2.23], the affine $G$-equivariant projection $E \times (G/H) \to E$ descends to an affine morphism $Y \to X$. The (trivial right) $H$-torsor $E \times G \to E \times (G/H)$ descends to an $H$-torsor $E \to Y$. We will write $E/H$ for $Y$.

**Example 3.1.** Let $G$ be a linear algebraic group over a field $F$ and let $E \to X$ be a $G$-torsor. Then for every $n > 0$, $E^n := E \times_F \cdots \times_F E$ ($n$ times) is a $G^n$-torsor over $X^n$. Viewing $G$ as the diagonal subgroup of $G^n$, we have the $G$-torsor $E^n \to E^n/G$.

3a. **Balanced elements.** Let $G$ be a linear algebraic group over a field $F$. We assume that $G$ is connected if $F$ is finite. Let $E \to X$ be a $G$-torsor such that $E(F) \neq \emptyset$. We write $p_1$ and $p_2$ for the two projections $E^2/G = (E \times_F E)/G \to X$ (see Example 3.1).

**Lemma 3.2.** Let $K/F$ be a field extension and $x_1, x_2 \in X(K)$. Then the $G$-torsors $E(x_1)$ and $E(x_2)$ over $K$ are isomorphic if and only if there is a point $y \in (E^2/G)(K)$ such that $p_1(y) = x_1$ and $p_2(y) = x_2$.

**Proof.** “$\Rightarrow$”: By construction, we have $G$-equivariant morphisms $f_i : E(x_i) \to E$ for $i = 1, 2$. Choose an isomorphism $h : E(x_1) \xrightarrow{\sim} E(x_2)$ of $G$-torsors over $K$. The morphism $(f_1, f_2h) : E(x_1) \to E^2$ yields the required point $\text{Spec} K = E(x_1)/G \to E^2/G$.

“$\Leftarrow$”: The pull-back of $E \to X$ with respect to any projection $E^2/G \to X$ coincides with the $G$-torsor $E^2 \to E^2/G$, hence

$$E(x_1) = x_1^*(E) = y^*p_1^*(E) \simeq y^*(E^2) \simeq y^*p_2^*(E) = x_2^*(E) = E(x_2).$$

Let $H$ be a (contravariant) functor from the category of schemes over $F$ to the category of abelian groups. We have the two maps $p_i^* : H(X) \to H(E^2/G), i = 1, 2$. An element $h \in H(X)$ is called balanced if $p_1^*(h) = p_2^*(h)$. We write $H(X)_{\text{bal}}$ for the subgroup of balanced elements in $H(X)$. In other words, $H(X)_{\text{bal}} = h_0(H(E^*/G))$ in the notation of Section A-IV.
We can view $H$ as a (covariant) functor $\text{Fields}_F \to \text{Sets}$ taking a field $K$ to $H(K) := H(\text{Spec } K)$.

**Lemma 3.3.** Let $h \in H(X)_{\text{bal}}$ be a balanced element, $K/F$ a field extension and $I$ a $G$-torsor over $\text{Spec}(K)$. Let $x \in X(K)$ be a point such that $E(x) \simeq I$. Then the element $x^*(h)$ in $H(K)$ does not depend on the choice of $x$.

**Proof.** Let $x_1, x_2 \in X(K)$ be two points such that $E(x_1) \simeq E(x_2)$. By Lemma 3.2, there is a point $y \in (E^2/G)(K)$ such that $p_1(y) = x_1$ and $p_2(y) = x_2$. Therefore

$$x_1^*(h) = y^*(p_1^*(h)) = y^*(p_2^*(h)) = x_2^*(h).$$

It follows from Lemma 3.3 that if the torsor $E \to X$ is classifying with $E(F) \neq \emptyset$, then every element $h \in H(X)_{\text{bal}}$ determines an $H$-invariant $i_h$ of $G$ as follows. Let $I$ be a $G$-torsor over a field extension $K/F$. We claim that there is a point $x \in X(K)$ such that $E(x) \simeq I$. If $K$ is infinite, this follows from the definition of the classifying $G$-torsor. If $K$ is finite then all $G$-torsors over $K$ are trivial by [Lang 1956], as $G$ is connected. Since $E(K) \neq \emptyset$, we can take for $x$ the image in $X(K)$ of any point in $E(K)$. Defining $i_h(E) = x^*(h) \in H(K)$, we have a group homomorphism

$$H(X)_{\text{bal}} \to \text{Inv}(G, H), \quad h \mapsto i_h.$$

**3b. Cohomology with values in** $\mathbb{Q}/\mathbb{Z}(j)$. For every integer $j \geq 0$, the coefficients $\mathbb{Q}/\mathbb{Z}(j)$ are defined as the direct sum over all prime integers $p$ of the objects $\mathbb{Q}_p/\mathbb{Z}_p(j)$ in the derived category of sheaves of abelian groups on the big étale site of $\text{Spec } F$, where

$$\mathbb{Q}_p/\mathbb{Z}_p(j) = \colim_n \mu_{p^n}^j$$

if $p \neq \text{char } F$, with $\mu_{p^n}$ the sheaf of $(p^n)$-th roots of unity, and

$$\mathbb{Q}_p/\mathbb{Z}_p(j) = \colim_n W_n \Omega_{\log}^j[-j]$$

if $p = \text{char } F > 0$, with $W_n \Omega_{\log}^j$ the sheaf of logarithmic de Rham–Witt differentials; see [Illusie 1979, I.5.7; Kahn 1996].

We write $H^m(X, \mathbb{Q}/\mathbb{Z}(j))$ for the étale cohomology of a scheme $X$ with values in $\mathbb{Q}/\mathbb{Z}(j)$. Then

$$H^m(X, \mathbb{Q}/\mathbb{Z}(j)) \{p\} = \colim_n H^m(X, \mu_{p^n}^j)$$

if $p \neq \text{char } F$ and

$$H^m(X, \mathbb{Q}/\mathbb{Z}(j)) \{p\} = \colim_n H^{m-j}(X, W_n \Omega_{\log}^j)$$

if $p = \text{char } F > 0$. In the latter case, the group $W_n \Omega_{\log}^j(F)$ is canonically isomorphic to $K_j^M(F)/p^n K_j^M(F)$, where $K_j^M(F)$ is Milnor’s $K$-group of $F$ (see
[Bloch and Kato 1986, Corollary 2.8], hence by [Izquierdo 1991; Garibaldi et al. 2003, Part II, Appendix A], $H^s(F, W^i_n \Omega^j_{\log})$ is isomorphic to

$$H^s(F, K^M_j (F_{\text{sep}})/p^n K^M_j (F_{\text{sep}})) = \begin{cases} K^M_j (F)/p^n K^M_j (F) & \text{if } s = 0, \\ H^2(F, K^M_j (F_{\text{sep}}))p^n & \text{if } s = 1, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that in the case $p = \text{char } F > 0$, we have

$$H^m(F, \mathbb{Q}/\mathbb{Z}(j))\{p\} = \begin{cases} K^M_j (F) \otimes (\mathbb{Q}_p/\mathbb{Z}_p) & \text{if } m = j, \\ H^2(F, K^M_j (F_{\text{sep}}))\{p\} & \text{if } m = j + 1, \\ 0 & \text{otherwise.} \end{cases}$$

The motivic complexes $\mathbb{Z}(j)$, for $j = 0, 1, 2$, of étale sheaves on a smooth scheme $X$ were defined by S. Lichtenbaum [1987; 1990]. We write $H^*(X, \mathbb{Z}(j))$ for the étale (hyper)cohomology groups of $X$ with values in $\mathbb{Z}(j)$.

The complex $\mathbb{Z}(0)$ is equal to the constant sheaf $\mathbb{Z}$ and $\mathbb{Z}(1) = \mathbb{G}_{m, X}[-1]$, thus $H^n(X, \mathbb{Z}(1)) = H^{n-1}(X, \mathbb{G}_{m, X})$. In particular, $H^3(X, \mathbb{Z}(1)) = \text{Br}(X)$, the cohomological Brauer group of $X$. The complex $\mathbb{Z}(2)$ is concentrated in degrees 1 and 2 and there is a product map $\mathbb{Z}(1) \otimes^L \mathbb{Z}(1) \to \mathbb{Z}(2)$; see [Lichtenbaum 1987, Proposition 2.5].

The exact triangle in the derived category of étale sheaves

$$\mathbb{Z}(j) \to \mathbb{Q} \otimes \mathbb{Z}(j) \to \mathbb{Q}/\mathbb{Z}(j) \to \mathbb{Z}(j)[1]$$

yields the connecting homomorphism

$$H^i(X, \mathbb{Q}/\mathbb{Z}(j)) \to H^{i+1}(X, \mathbb{Z}(j)),$$

which is an isomorphism if $X = \text{Spec}(F)$ for a field $F$ and $i > j$ [Kahn 1993, Lemme 1.1].

Write $\mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))$ for the Zariski sheaf on a smooth scheme $X$ associated to the presheaf $U \mapsto H^n(U, \mathbb{Q}/\mathbb{Z}(j))$ of étale cohomology groups.

Let $G$ be a linear algebraic group over $F$. We assume that $G$ is connected if $F$ is a finite field and write $\text{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j))$ for the group of degree $n$ invariants of $G$ for the functor $K \mapsto H^n(K, \mathbb{Q}/\mathbb{Z}(j))$. Note that Property 2.1 holds for this functor by [Garibaldi et al. 2003, Part 2, Proposition A.9].

Choose a classifying $G$-torsor $E \to X$ with $E$ a $G$-rational variety such that $E(F) \neq \emptyset$. Applying the construction given in Section 3a to the functor $U \mapsto H^0_{\text{Zar}}(U, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))$, we get a homomorphism

$$\varphi : H^0_{\text{Zar}}(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))_{\text{bal}} \to \text{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j)).$$

**Theorem 3.4.** Let $G$ be a smooth linear algebraic group over a field $F$. We assume that $G$ is connected if $F$ is a finite field. Let $E \to X$ be a classifying $G$-torsor
with $E$ a $G$-rational variety such that $E(F) \neq \emptyset$. Then the homomorphism $\varphi$ is an isomorphism.

**Proof.** Let $E_{\text{gen}} \to F(X)$ be the generic fiber of the classifying $G$-torsor $E \to X$. Note that since the pull-back of $E \to X$ with respect to any of the two projections $E^2/G \to X$ coincides with the $G$-torsor $E^2 \to E^2/G$, the pull-backs of the generic $G$-torsor $E_{\text{gen}} \to \text{Spec } F(X)$ with respect to the two morphisms $\text{Spec } F(E^2/G) \to \text{Spec } F(X)$ induced by the projections are isomorphic. It follows that for every invariant $i \in \text{Inv}(G, H^*(\mathbb{Q}/\mathbb{Z}(j)))$ we have

$$p_1(i(E_{\text{gen}})) = i(p_1^*(E_{\text{gen}})) = i(p_2^*(E_{\text{gen}})) = p_2^*(i(E_{\text{gen}}))$$

in $H^*(F(E^2/G), \mathbb{Q}/\mathbb{Z}(j))$, that is, $i(E_{\text{gen}}) \in H^*(F(X), \mathbb{Q}/\mathbb{Z}(j))_{\text{bal}}$. By Proposition A.9, $\partial_x(h) = 0$ for every point $x \in X$ of codimension 1, hence

$$\theta_G(i) = i(E_{\text{gen}}) \in H^0_{\text{Zar}}(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))_{\text{bal}}$$

by Proposition A.10. By Theorem 2.2, $\theta_G$ is injective and by construction, the composition $\theta_G \circ \varphi$ is the identity. It follows that $\varphi$ is an isomorphism. \hfill \square

Write $H^0_{\text{Zar}}(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))$ for the factor group of $H^0_{\text{Zar}}(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))$ by the natural image of $H^n(F, \mathbb{Q}/\mathbb{Z}(j))$.

**Corollary 3.5.** The isomorphism $\varphi$ yields an isomorphism

$$H^0_{\text{Zar}}(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))_{\text{bal}} \xrightarrow{\sim} \text{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j))_{\text{norm}}.$$

### 4. Degree 3 invariants of algebraic tori

In this section we prove the main theorem that describes degree 3 invariants of an algebraic torus with values in $\mathbb{Q}/\mathbb{Z}(2)$.

**4a. Algebraic tori.** Let $F$ be a field and $\Gamma = \text{Gal}(F_{\text{sep}}/F)$ the absolute Galois group of $F$. An algebraic torus of dimension $n$ over $F$ is an algebraic group $T$ such that $T_{\text{sep}}$ is isomorphic to the product of $n$ copies of the multiplicative group $\mathbb{G}_m$; see [Colliot-Thélène and Sansuc 1977; Voskresenskiĭ 1998]. For an algebraic torus $T$ over a field $F$, we write $\widehat{T}_{\text{sep}}$ for the $\Gamma$-module of characters $\text{Hom}(T_{\text{sep}}, \mathbb{G}_m)$. The group $\widehat{T}_{\text{sep}}$ is a $\Gamma$-lattice, that is, a free abelian group of finite rank with a continuous $\Gamma$-action. The contravariant functor $T \mapsto \widehat{T}_{\text{sep}}$ is an antiequivalence between the category of algebraic tori and the category of $\Gamma$-lattices: the torus $T$ and the group $T(F)$ can be reconstructed from the lattice $\widehat{T}_{\text{sep}}$ by the formulas

$$T = \text{Spec}(F_{\text{sep}}[\widehat{T}_{\text{sep}}]^\Gamma),$$

$$T(F) = \text{Hom}_\Gamma(\widehat{T}_{\text{sep}}, F_{\text{sep}}^\times) = (\widehat{T}_{\text{sep}} \otimes F_{\text{sep}}^\times)^\Gamma,$$

where $\widehat{T}_{\text{sep}} = \text{Hom}(\widehat{T}_{\text{sep}}, \mathbb{Z})$. 
We write \( \hat{T} \) for the character group \( \text{Hom}_F(T, \mathbb{G}_m) = (\hat{T}_{\text{sep}})^\Gamma \) and \( T^\circ \) for the dual torus having character lattice \( \hat{T}_{\text{sep}}^\circ \).

A torus \( T \) is called quasisplit if \( T \) is isomorphic to the group of invertible elements of an étale \( F \)-algebra, or equivalently, the \( \Gamma \)-lattice \( \hat{T}_{\text{sep}} \) is permutation, that is, \( \hat{T}_{\text{sep}} \) has a \( \Gamma \)-invariant \( \mathbb{Z} \)-basis. An invertible torus is a direct factor of a quasisplit torus.

A torus \( T \) is called flasque or coflasque if \( H^1(L, \hat{T}_{\text{sep}}^\circ) = 0 \) or \( H^1(L, \hat{T}_{\text{sep}}) = 0 \), respectively, for every finite field extension \( L/F \). A flasque resolution of a torus \( S \) is an exact sequence of tori \( 1 \to T \to P \to S \to 1 \) with \( T \) flasque and \( P \) quasisplit. By [Colliot-Thélène and Sansuc 1977, §4; Voskresenskiĭ 1998, §4.7], the torus \( T \) in the flasque resolution is invertible if and only if \( S \) is a direct factor of a rational torus.

4b. Products. Let \( T \) be a torus over \( F \) and let \( \hat{T}(i) \) denote the complex \( \hat{T}_{\text{sep}} \otimes \mathbb{Z}(i) \) of étale sheaves over \( F \) for \( i = 0, 1, 2 \). Thus, \( \hat{T}(0) = \hat{T}_{\text{sep}} \) and

\[
\hat{T}(1) = (\hat{T}_{\text{sep}} \otimes F_{\text{sep}}^\times)[−1] = T^\circ(F_{\text{sep}})[−1].
\]

Let \( S \) and \( T \) be algebraic tori over \( F \) and let \( i \) and \( j \) be nonnegative integers with \( i + j \leq 2 \). For any smooth variety \( X \) over \( F \), we have the product map

\[
(\hat{S}_{\text{sep}} \otimes \hat{T}_{\text{sep}})^\Gamma \otimes H^p(X, \hat{S}^\circ(i)) \otimes H^q(X, \hat{T}^\circ(j)) \to H^{p+q}(X, \mathbb{Z}(i+j))
\]  

(4-1)

taking \( a \otimes b \otimes c \) to \( a \cup b \cup c \), via the canonical pairings between \( \hat{S}_{\text{sep}} \) and \( \hat{S}^\circ_{\text{sep}}, \hat{T}_{\text{sep}} \) and \( \hat{T}^\circ_{\text{sep}} \), and the product map \( \mathbb{Z}(i) \otimes \mathbb{Z}(j) \to \mathbb{Z}(i+j) \).

Recall that there is an isomorphism \( H^n(F, \mathbb{Q}(k)) \simeq H^{n-1}(F, \mathbb{Q}/\mathbb{Z}(k)) \) for \( n > k \).

In particular, we have the cup-product map

\[
(\hat{S}_{\text{sep}} \otimes \hat{T}_{\text{sep}})^\Gamma \otimes H^p(F, S) \otimes H^q(F, T) \to H^3(F, \mathbb{Q}/\mathbb{Z}(2))
\]  

(4-2)

if \( p + q = 2 \).

If \( S = T^\circ \) is the dual torus, then \( (\hat{S}_{\text{sep}} \otimes \hat{T}_{\text{sep}})^\Gamma = \text{End}_T(\hat{T}_{\text{sep}}) \) contains the canonical element \( 1_T \). We then have the product map

\[
H^p(X, \hat{T}(i)) \otimes H^q(X, \hat{T}^\circ(j)) \to H^{p+q}(X, \mathbb{Z}(i+j))
\]  

(4-3)

and in particular, the product maps

\[
H^1(F, \hat{T}_{\text{sep}}) \otimes H^1(F, T) \to H^2(F, \mathbb{Q}/\mathbb{Z}(1)) = \text{Br}(F),
\]  

(4-4)

\[
H^1(F, T^\circ) \otimes H^1(F, T) \to H^3(F, \mathbb{Q}/\mathbb{Z}(2)),
\]  

(4-5)

\[
H^2(F, T^\circ) \otimes H^0(F, T) \to H^3(F, \mathbb{Q}/\mathbb{Z}(2)),
\]  

(4-6)

taking \( a \otimes b \) to \( 1_T = a \cup b \) and applying (4-2).

As \( T \) is a commutative group, the set \( H^1(K, T) \) is an abelian group. An invariant \( i \in \text{Inv}(T, H) \) for a functor \( H \) is called linear if \( i_K : H^1(K, T) \to H(K) \) is a group
homomorphism for every $K/F$. In the next section we will see that a normalized degree 3 invariant of a torus need not be linear.

4c. Main theorem. Let $T$ be a torus over $F$ and choose a standard classifying $T$-torsor $U \to U/T$ such that the codimension of $V \setminus U$ in $V$ is at least 3. Such a torsor exists by [Edidin and Graham 1998, Lemma 9].

By [Sansuc 1981, Proposition 6.10], there is an exact sequence

$$F_{\text{sep}}[U]^\times / F_{\text{sep}}^\times \to \hat{T}_{\text{sep}} \to \text{Pic}((U/T)_{\text{sep}}) \to \text{Pic}(U_{\text{sep}}).$$

The codimension assumption implies that the side terms are trivial, hence the map $\hat{T}_{\text{sep}} \to \text{Pic}((U/T)_{\text{sep}})$ is an isomorphism. It follows that the classifying $T$-torsor $U \to U/T$ is universal in the sense of [Colliot-Thélène and Sansuc 1987a].

Write $K_*(F)$ for the (Quillen) $K$-groups of $F$ and $\mathcal{K}_*$ for the Zariski sheaf associated to the presheaf $U \mapsto K_*(U)$. Then the groups $H^2_{\text{zar}}(U/T, \mathcal{K}_2)$ are independent of the choice of the classifying torsor; see [Edidin and Graham 1998]. So we write $H^2_{\text{zar}}(BT, \mathcal{K}_2)$ for this group (see Section A-IV). As $T_{\text{sep}}$ is a split torus, by the Künneth formula (see Example A.5),

$$H^2_{\text{zar}}(BT_{\text{sep}}, \mathcal{K}_2) = \begin{cases} 
K_2(F_{\text{sep}}) & \text{if } n = 0, \\
\text{Pic}((U/T)_{\text{sep}}) \otimes F_{\text{sep}}^\times \otimes \hat{T}_{\text{sep}} \otimes F_{\text{sep}}^\times = T^\circ(F_{\text{sep}}) & \text{if } n = 1, \\
\text{CH}^2((U/T)_{\text{sep}}) = S^2(\hat{T}_{\text{sep}}) & \text{if } n = 2. 
\end{cases}$$

Applying the calculation of the $\mathcal{K}$-cohomology groups to the standard classifying $T$-torsor $U^i \to U^i/T$ for every $i > 0$ instead of $U \to U/T$, by Proposition B.3, we have the exact sequence

$$0 \to H^1(F, T^\circ) \xrightarrow{\alpha} \overline{H}^4(U^i/T, \mathbb{Z}(2)) \rightarrow \overline{H}^4((U^i/T)_{\text{sep}}, \mathbb{Z}(2))^\Gamma \to H^2(F, T^\circ), \quad (4-7)$$

where $\overline{H}^4(U^i/T, \mathbb{Z}(2))$ is the factor group of $H^4(U^i/T, \mathbb{Z}(2))$ by $H^4(F, \mathbb{Z}(2))$, the map $\alpha$ is given by $\alpha(a) = q^*(a) \cup [U^i]$ with $q : U^i/T \to \text{Spec } F$ the structure morphism, $[U^i]$ the class of the $T$-torsor $U^i \to U^i/T$ in $H^1(U^i/T, T)$, and the cup-product is taken for the pairing (B-6).

Taking the sequences (4-7) for all $i$ (see Section A-IV), we get the exact sequence of cosimplicial groups

$$0 \to H^1(F, T^\circ) \xrightarrow{\alpha} \overline{H}^4(U^*/T, \mathbb{Z}(2)) \to \overline{H}^4((U^*/T)_{\text{sep}}, \mathbb{Z}(2))^\Gamma \to H^2(F, T^\circ).$$

The first and the last cosimplicial groups in the sequence are constant, hence by Lemma A.2, the sequence

$$0 \to H^1(F, T^\circ) \xrightarrow{\alpha} \overline{H}^4(U/T, \mathbb{Z}(2))_{\text{bal}} \rightarrow \overline{H}^4((U/T)_{\text{sep}}, \mathbb{Z}(2))^\Gamma_{\text{bal}} \to H^2(F, T^\circ) \quad (4-8)$$
is exact as \( h_0(\overline{H}^4(U^*/T, \mathbb{Z}(2))) = H^4(U/T, \mathbb{Z}(2))_{\text{bal}} \).

The following theorem was proved by B. Kahn [1996, Theorem 1.1]:

**Theorem 4.1.** Let \( X \) be a smooth variety over \( F \). Then there is an exact sequence

\[
0 \to \text{CH}^2(X) \to H^4(X, \mathbb{Z}(2)) \to H^0_{\text{Zar}}(X, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2))) \to 0.
\]

By Theorem 4.1, there is an exact sequence of cosimplicial groups

\[
0 \to \text{CH}^2(U^*/T) \to H^4(U^*/T, \mathbb{Z}(2)) \to H^0_{\text{Zar}}(U^*/T, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2)))_{\text{bal}} \to 0.
\]

As the functor \( \text{CH}^2 \) is homotopy invariant, by Lemma A.4, the first group in the sequence is constant. In view of Lemma A.2, and following the notation for the \( \mathcal{H} \)-cohomology, the sequence

\[
0 \to \text{CH}^2(BT) \to \overline{H}^4(U/T, \mathbb{Z}(2))_{\text{bal}} \to \overline{H}^0_{\text{Zar}}(U^*/T, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2)))_{\text{bal}} \to 0 \quad (4\,-\,9)
\]

is exact. By Corollary 3.5, the last group in the sequence is canonically isomorphic to \( \text{Inv}(T, H^3(\mathbb{Q}/\mathbb{Z}(2)))_{\text{norm}} \).

As the torus \( T_{\text{sep}} \) is split, all the invariants of \( T_{\text{sep}} \) are trivial hence the sequence (4-9) over \( F_{\text{sep}} \) yields an isomorphism

\[
\overline{H}^4((U/T)_{\text{sep}}, \mathbb{Z}(2))_{\text{bal}} \cong \text{CH}^2(BT_{\text{sep}}) \cong S^2(\widehat{T}_{\text{sep}}). \quad (4\,-\,10)
\]

Combining (4-8), (4-9) and (4-10), we get the following diagram with an exact row and column:

\[
\begin{array}{ccccc}
0 & \to & \text{CH}^2(BT) & \to & \overline{H}^4(U/T, \mathbb{Z}(2))_{\text{bal}} & \to & \text{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{\text{norm}} & \to & 0.
\end{array}
\]

Write \( \text{Dec} = \text{Dec}(\widehat{T}_{\text{sep}}) \) for the subgroup of decomposable elements in \( S^2(\widehat{T}_{\text{sep}})^\Gamma \) (see Section A-II).

**Lemma 4.2.** The image of the homomorphism

\[
\text{CH}^2(BT) \to \text{CH}^2(BT_{\text{sep}})^\Gamma = S^2(\widehat{T}_{\text{sep}})^\Gamma
\]

in the diagram coincides with \( \text{Dec} \).
The homomorphism $\alpha$ is given by $\alpha(a)(b) = a \cup b$ for every $a \in H^1(F, T^0)$ and $b \in H^1(K, T)$ and every field extension $K/F$, where the cup-product is defined in (4-5).
Proof. The exactness of the sequence follows from the diagram before Lemma 4.2. It remains to describe the map $\alpha$. Take an $a \in H^1(F, T^0)$ and consider the invariant $i$ defined by $i(b) = a_K \cup b$, where the cup-product is given by (4-5). We need to prove that $i = \alpha(a)$. Choose a standard classifying $T$-torsor $U \to U/T$ and set $K = F(U/T)$. Let $U_{\text{gen}}$ be the generic fiber of the classifying torsor. By Theorem 2.2, it suffices to show that $i(U_{\text{gen}}) = \alpha(a)(U_{\text{gen}})$ over $K$. This follows from the description of the map $\alpha$ in the exact sequence (4-7). \hfill \square

Remark 4.4. In a similar (and much simpler) fashion one can describe degree 2 invariants of an algebraic torus $T$ with values in $\mathbb{Q}/\mathbb{Z}(1)$, that is, invariants with values in the Brauer group by computing the étale motivic cohomology group $H^3(U/T, \mathbb{Z}(1)) = H^2(U/T, \mathbb{G}_m) = \text{Br}(U/T)$ for a standard classifying $T$-torsor $U \to U/T$. One establishes canonical isomorphisms

$$H^1(F, \widehat{T}_{\text{sep}}) \simeq H^3(U/T, \mathbb{Z}(1))_{\text{bal}} \simeq \text{Inv}^2(T, \mathbb{Q}/\mathbb{Z}(1))_{\text{norm}} = \text{Inv}(T, \text{Br})_{\text{norm}}.$$ 

The composition takes an element $a \in H^1(F, \widehat{T}_{\text{sep}})$ to the invariant $b \mapsto a_K \cup b$ for $b \in H^1(K, T)$ and a field extension $K/F$. This description shows that every normalized Br-invariant of $T$ is linear.

4d. Torsion in $\text{CH}^2(BT)$. We investigate the group $\text{CH}^2(BT)_{\text{tors}}$, the first term of the exact sequence in Theorem 4.3.

Let $S$ be an algebraic torus over $F$. Using the Gersten resolution, [Quillen 1973, Proposition 5.8] we identify the group $H^0(S_{\text{sep}}, \mathcal{H}_2)$ with a subgroup in $K_2(F_{\text{sep}}(S))$. Set $\overline{H}^0(S_{\text{sep}}, \mathcal{H}_2) := H^0(S_{\text{sep}}, \mathcal{H}_2)/K_2(F_{\text{sep}})$. By [Garibaldi et al. 2003, Part 2, §5.7], we have an exact sequence

$$0 \to \widehat{S}_{\text{sep}} \otimes F_{\text{sep}}^\times \to \overline{H}^0(S_{\text{sep}}, \mathcal{H}_2) \xrightarrow{\lambda} \wedge^2 \widehat{S}_{\text{sep}} \to 0$$ (4-11)

of $\Gamma$-modules, where $\lambda((e^x, e^y)) = x \wedge y$ for $x, y \in \widehat{S}_{\text{sep}}$.

Consider the $\Gamma$-homomorphism

$$\gamma : \wedge^2 \widehat{S}_{\text{sep}} \to \overline{H}^0(S_{\text{sep}}, \mathcal{H}_2)$$

$$x \wedge y \mapsto \{e^x, e^y\} - \{e^y, e^x\}.$$ We have $\lambda \circ \gamma = 2 \cdot \text{Id}$, hence the connecting homomorphism

$$\vartheta : H^i(F, \wedge^2 \widehat{S}_{\text{sep}}) \to H^{i+1}(F, \widehat{S}_{\text{sep}} \otimes F_{\text{sep}}^\times)$$ (4-12)

does not satisfy $2 \vartheta = 0$.

Lemma 4.5. If $S$ is an invertible torus, then the sequence of $\Gamma$-modules (4-11) is split.
Proof. Suppose first that $S$ is quasisplit. Let \( \{x_1, x_2, \ldots, x_m\} \) be a permutation basis for $\hat{S}_{\text{sep}}$. Then the elements $x_i \wedge x_j$ for $i < j$ form a $\mathbb{Z}$-basis for $\wedge^2 \hat{S}_{\text{sep}}$. The map $\wedge^2 \hat{S}_{\text{sep}} \to \overline{H}^0(S_{\text{sep}}, \mathcal{H}_2)$, taking $x_i \wedge x_j$ to $(e^{x_i}, e^{x_j})$ is a splitting for $\gamma$.

In general, find a torus $S'$ such that $S \times S'$ is quasisplit. The desired splitting is the composition

$$
\wedge^2 \hat{S}_{\text{sep}} \to \wedge^2 \hat{S}_{\text{sep}} \times S'_{\text{sep}} \xrightarrow{\alpha} \overline{H}^0(S_{\text{sep}} \times S'_{\text{sep}}, \mathcal{H}_2) \xrightarrow{\beta} \overline{H}^0(S_{\text{sep}}, \mathcal{H}_2),
$$

where $\alpha$ is a splitting for the torus $S \times S'$ and $\beta$ is the pull-back map for the canonical inclusion $S \hookrightarrow S \times S'$.

Let

$$1 \to T \to P \to Q \to 1$$

be a coflasque resolution of $T$, that is, $P$ is a quasisplit torus and $Q$ is a coflasque torus; see [Colliot-Thélène and Sansuc 1977]. The torus $P$ is an open set in the affine space of a separable $F$-algebra on which $T$ acts linearly. Hence $P \to Q$ is a standard classifying $T$-torsor. By Theorem 2.2, the natural map

$$\theta_T : \text{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(F(Q), \mathbb{Q}/\mathbb{Z}(2))$$

is injective.

Consider the spectral sequence (B-10) for the variety $X = Q$. By [Garibaldi et al. 2003, Part 2, Corollary 5.6], we have $H^1(Q_{\text{sep}}, \mathcal{H}_2) = 0$. In view of Proposition B.4, we have an injective homomorphism

$$\beta : H^2(F, \overline{H}^0(Q_{\text{sep}}, \mathcal{H}_2)) \to \overline{H}^4(Q, \mathbb{Z}(2))$$

such that the composition of $\beta$ with the homomorphism

$$H^2(F, Q^\circ) \to H^2(F, \overline{H}^0(Q_{\text{sep}}, \mathcal{H}_2))$$

is given by the cup-product with the class of the identity in $H^0(Q, Q)$.

Lemma 4.6. For a coflasque torus $Q$, the group $\text{CH}^2(Q)$ is trivial.

Proof. By [Merkurjev and Panin 1997, Theorem 9.1], for every torus $Q$, the Grothendieck group $K_0(Q)$ is generated by the classes of the sheaves $i_*(P)$, where $P$ is an invertible sheaf on $Q_L, L/F$ a finite separable field extension and $i : Q_L \to Q$ is the natural morphism. By definition of a coflasque torus,

$$\text{Pic}(Q_L) = H^1(L, \hat{Q}_{\text{sep}}) = 0.$$ 

It follows that every invertible sheaf on $Q_L$ is trivial, hence $K_0(Q) = \mathbb{Z} \cdot 1$. Since the group $\text{CH}^2(Q)$ is generated by the second Chern classes of vector bundles on $Q$ [Esnault et al. 1998, Lemma C.3], we have $\text{CH}^2(Q) = 0$. 

\hfill \Box
It follows from Proposition A.10, Theorem 4.1, and Lemma 4.6 that the homomorphism
\[ \kappa : \bar{H}^4(Q, \mathbb{Z}(2)) \to \bar{H}^4(F(Q), \mathbb{Z}(2)) = \bar{H}^3(F(Q), \mathbb{Q}/\mathbb{Z}(2)) \] (4-14)
is injective.
Consider the diagram
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & Q^\circ(F_{\text{sep}}) & \longrightarrow & H^0(Q_{\text{sep}}, \mathcal{H}_2) & \longrightarrow & \wedge^2 \hat{Q}_{\text{sep}} & \longrightarrow & 0 \\
\downarrow s & & \downarrow t & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q^\circ(F_{\text{sep}}) & \longrightarrow & P^\circ(F_{\text{sep}}) & \longrightarrow & T^\circ(F_{\text{sep}}) & \longrightarrow & 0
\end{array}
\]
where \( s \) is the composition of the natural map \( \bar{H}^0(Q_{\text{sep}}, \mathcal{H}_2) \to \bar{H}^0(P_{\text{sep}}, \mathcal{H}_2) \) and a splitting of \( P^\circ(F_{\text{sep}}) \to \bar{H}^0(P_{\text{sep}}, \mathcal{H}_2) \) (see Lemma 4.5).
We have the following diagram
\[
\begin{array}{ccc}
H^1(F, T^\circ) & \xrightarrow{\alpha} & \text{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{\text{norm}} \\
\downarrow & \theta_T & \downarrow \theta_T \\
H^1(F, \bar{H}^0(Q_{\text{sep}}, \mathcal{H}_2)) & \xrightarrow{\varphi} & H^1(F, \wedge^2 \hat{Q}_{\text{sep}}) & \xrightarrow{\beta} & H^2(F, Q^\circ) & \xrightarrow{\sigma} & \bar{H}^3(F(Q), \mathbb{Q}/\mathbb{Z}(2))
\end{array}
\]
with the bottom sequence a complex, where \( \sigma \) is the composition of the maps in (4-13) and (4-14):
\[
H^2(F, Q^\circ) \xrightarrow{\psi} H^2(F, \bar{H}^0(Q_{\text{sep}}, \mathcal{H}_2)) \xrightarrow{\beta} \bar{H}^4(Q, \mathbb{Z}(2)) \xrightarrow{\kappa} \bar{H}^4(F(Q), \mathbb{Z}(2)) = \bar{H}^3(F(Q), \mathbb{Q}/\mathbb{Z}(2)),
\]
with \( \varphi \) and \( \psi \) given by Galois cohomology applied to the exact sequence (4-11) for the torus \( Q \). Note that the connecting map \( \partial_1 \) is injective as \( H^1(F, P^\circ) = 0 \) since \( P^\circ \) is a quasisplit torus. As \( 2\partial = 0 \) in (4-12), we have \( 2t^* = 0 \).

The commutativity of the triangle follows from the definition of \( t^* \). We claim that the square in the diagram is anticommutative. Note that \( \partial_2(\xi) = [P_{\text{gen}}] \), where \( \partial_2 : H^0(F, Q) \to H^1(F, T) \) is the connecting homomorphism, \( P_{\text{gen}} \) is the generic fiber of the morphism \( P \to Q \), and \( \xi \in H^0(K, Q) \) is the generic point of \( Q \) with \( K = F(Q) \). It follows from the description of the maps \( \alpha \) and \( \beta \) in (4-7) and (4-13), respectively, and Lemma A.1 that
\[
\sigma(\partial_1(a)) = \partial_1(a)\kappa \cup \xi = (-a_K) \cup \partial_2(\xi) = (-a_K) \cup [P_{\text{gen}}] = -\theta_T(\alpha(a))
\]
for every \( a \in H^1(F, T^\circ) \).

The maps \( \beta \) and \( \kappa \) are injective, hence the bottom sequence in the diagram is
exact. Thus, we have an exact sequence
\[ H^1(F, \overline{H}^0(Q_{sep}, \hat{\mathcal{M}}_2)) \to H^1(F, \wedge^2 \hat{Q}_{sep}) \to \ker(\alpha) \to 0 \]
and \(2 \cdot \ker(\alpha) = 2 \cdot \text{Im}(t^*) = 0\). Furthermore, \(\ker(\alpha) \cong \text{CH}^2(BT)_{\text{tors}}\) by Theorem 4.3 and the group \(H^1(F, \wedge^2 \hat{Q}_{sep})\) is finite as \(\wedge^2 \hat{Q}_{sep}\) is a lattice.

We have proved:

**Theorem 4.7.** Let \(1 \to T \to P \to Q \to 1\) be a coflasque resolution of a torus \(T\). Then there is an exact sequence
\[ H^1(F, \overline{H}^0(Q_{sep}, \mathcal{M}_2)) \to H^1(F, \wedge^2 \hat{Q}_{sep}) \to \text{CH}^2(BT)_{\text{tors}} \to 0. \]
Moreover, \(\text{CH}^2(BT)_{\text{tors}}\) is a finite group satisfying \(2 \cdot \text{CH}^2(BT)_{\text{tors}} = 0\).

**Corollary 4.8.** If \(T^\circ\) is a birational direct factor of a rational torus, or if \(T\) is split over a cyclic field extension, then \(\text{CH}^2(BT)_{\text{tors}} = 0\), that is, the map \(\alpha\) in Theorem 4.3 is injective.

**Proof.** The exact sequence \(1 \to Q^\circ \to P^\circ \to T^\circ \to 1\) is a flasque resolution of \(T^\circ\). If \(T^\circ\) is a birational direct factor of a rational torus, or if \(T\) is split over a cyclic field extension, the torus \(Q^\circ\), and hence \(Q\), is invertible; see Section 4a and [Voskresenskiĭ 1998, §4, Theorem 3]. By Lemma 4.5, the sequence (4-11) for the torus \(Q\) is split, hence the first map in Theorem 4.7 is surjective. \(\square\)

**Question 4.9.** Is \(\text{CH}^2(BT)_{\text{tors}}\) trivial for every torus \(T\)?

**4e. Special tori.** Let \(T\) be an algebraic torus over a field \(F\). The *tautological invariant* of the torus \(T^\circ \times T\) is the normalized invariant taking a pair
\[(a, b) \in H^1(K, T^\circ) \times H^1(K, T)\]
to the cup-product \(a \cup b \in H^3(K, Q/\mathbb{Z}(2))\) defined in (4-5).

The following theorem shows that if a torus \(T\) has only trivial degree 3 normalized invariants with values in \(Q/\mathbb{Z}(2)\) universally, that is, over all field extensions of \(F\), then \(T\) has no nonconstant invariants at all by the simple reason: every \(T\)-torsor over a field is trivial. Note that it follows from Theorem 2.4 that \(T\) has no degree 2 normalized invariants with values in \(Q/\mathbb{Z}(1)\) universally if and only if \(T\) is coflasque.

**Theorem 4.10.** Let \(T\) be an algebraic torus over a field \(F\). Then the following are equivalent:

1. \(\text{Inv}^3(T_K, Q/\mathbb{Z}(2))_{\text{norm}} = 0\) for every field extension \(K\) of \(F\).
2. The tautological invariant of the torus \(T^\circ \times T\) is trivial.
3. The torus \(T\) is invertible.
4. The torus \(T\) is special.
Proof. (1) ⇒ (2): Let $K/F$ be a field extension and $a \in H^1(K, T^\circ)$. By assumption, the degree 3 normalized invariant $i = \alpha(a)$ with values in $\mathbb{Q}/\mathbb{Z}(2)$, defined by $i(b) = a \cup b$ for every $b \in H^1(K, T)$, is trivial. In other words, the tautological invariant of the torus $T^\circ \times T$ is trivial.

(2) ⇒ (3): The image of the tautological invariant in the group

$$S^2(\hat{T}_{\text{sep}} \oplus \hat{T}_{\text{sep}})^\Gamma / \text{Dec}$$

is represented by the identity $1_\hat{T}$ in the direct factor $(\hat{T}_{\text{sep}} \otimes \hat{T}_{\text{sep}})^\Gamma = \text{End}_\Gamma(\hat{T}_{\text{sep}})$ of $S^2(\hat{T}_{\text{sep}} \oplus \hat{T}_{\text{sep}})^\Gamma$ (see Section A-II). The projection of Dec on the direct summand $(\hat{T}_{\text{sep}} \otimes \hat{T}_{\text{sep}})^\Gamma$ is generated by the traces $\text{Tr}(a \otimes b)$ for all open subgroups $\Gamma' \subset \Gamma$ and all $a \in (\hat{T}_{\text{sep}})^{\Gamma'}$ and $b \in (\hat{T}_{\text{sep}})^{\Gamma'}$. Hence $1_\hat{T} = \sum_i \text{Tr}(a_i \otimes b_i)$ for some open subgroups $\Gamma_i \subset \Gamma$, $a_i \in (\hat{T}_{\text{sep}})^{\Gamma_i}$ and $b_i \in (\hat{T}_{\text{sep}})^{\Gamma_i}$. If $P_i = \mathbb{Z}[\Gamma/\Gamma_i]$, then $a_i$ can be viewed as a $\Gamma'$-homomorphism $\hat{T} \to P_i$ and $b_i$ can be viewed as a $\Gamma'$-homomorphism $P_i \to \hat{T}$ such that the composition

$$\hat{T} \xrightarrow{(b_i)} P \xrightarrow{(a_i)} \hat{T},$$

where $P = \bigsqcup P_i$, is the identity. It follows that $\hat{T}$ is a direct summand of a permutation $\Gamma'$-module $P$ and hence $T$ is invertible.

(3) ⇒ (4): Obvious as every invertible torus is special.

(4) ⇒ (1): Obvious. \hfill \Box

Remark 4.11. The equivalence (3) ⇔ (4) was essentially proved in [Colliot-Thélène and Sansuc 1987b, Proposition 7.4].

4f. Linear and quadratic invariants. Let $T$ be a torus over $F$. By Theorem 4.3, we have a natural homomorphism to the group of linear invariants:

$$\alpha : H^1(F, T^\circ) \to \text{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{\text{lin}}.$$

Let $S$ and $T$ be algebraic tori over $F$. For every field extension $K/F$, the cup-product (4-2) yields a homomorphism

$$\varepsilon : (\hat{T}_{\text{sep}}^{\otimes 2})^\Gamma \to \text{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))$$

defined by $\varepsilon(a)(b) = a_K \cup b \cup b$ for $a \in (\hat{T}_{\text{sep}}^{\otimes 2})^\Gamma$ and $b \in H^1(K, T)$.

We say that an invariant $i \in \text{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))$ is quadratic if the function

$$h(a, b) := i(a + b) - i(a) - i(b)$$

is bilinear and $h(a, a) = 2i(a)$ for all $a$ and $b$. For example, the tautological invariant of the torus $T^\circ \times T$ in Section 4e is quadratic. We write $\text{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{\text{quad}}$ for the subgroup of all quadratic invariants in $\text{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))$. The image of $\varepsilon$ is contained in $\text{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{\text{quad}}$. 
Lemma 4.12. The composition of $\varepsilon$ with the map

$$\text{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow S^2(\hat{T}_{\text{sep}})^\Gamma / \text{Dec}$$

in Theorem 4.3 is induced by the natural homomorphism $\hat{T}_{\text{sep}}^\otimes S \rightarrow S^2(\hat{T}_{\text{sep}})$.

Proof. Let $U \rightarrow U/T =: X$ be a standard classifying $T$-torsor as in Section 4c. Consider the commutative diagram

$$
\begin{array}{ccc}
(\hat{T}_{\text{sep}}^\otimes)^2 \otimes H^1(X, T)^\otimes & \xrightarrow{\text{prod}} & \overline{H}^4(X, \mathbb{Z}(2))_{\text{bal}} \\
\downarrow & & \downarrow \\
\hat{T}_{\text{sep}}^\otimes \otimes H^1(X_{\text{sep}}, T)^\otimes & \xrightarrow{\text{prod}} & \overline{H}^4(X_{\text{sep}}, \mathbb{Z}(2))_{\text{bal}} \\
\eta \downarrow & & \downarrow \\
\hat{T}_{\text{sep}}^\otimes \otimes (\hat{T}_{\text{sep}}^\circ)^\otimes \otimes \hat{T}_{\text{sep}}^\otimes & \xrightarrow{\kappa} & S^2(\hat{T}_{\text{sep}})^\Gamma / \text{Dec}
\end{array}
$$

where the product maps are given by (4-1), $\eta$ identifies $H^1(X_{\text{sep}}, T) = \hat{T}_{\text{sep}}^\circ \otimes \text{Pic}(X_{\text{sep}})$ with $\hat{T}_{\text{sep}}^\circ \otimes \hat{T}_{\text{sep}}$ and $\kappa$ is given by the pairing between the first and second factors. Write $[U]$ for the class of the classifying torsor in $H^1(X, T)$. The image of $[U]$ in $H^1(X_{\text{sep}}, T_{\text{sep}}) = \hat{T}_{\text{sep}}^\circ \otimes \hat{T}_{\text{sep}} = \text{End}(\hat{T}_{\text{sep}})$ is the identity $1_{\hat{T}_{\text{sep}}}$. Hence for every $a \in (\hat{T}_{\text{sep}}^\otimes)^2$, the image of $a \otimes [U] \otimes [U]$ under the diagonal map in the diagram coincides with the canonical image of $a$ in $S^2(\hat{T}_{\text{sep}})^\Gamma / \text{Dec}$. $\square$

The composition of the map $S^2(\hat{T}_{\text{sep}})^\Gamma \rightarrow (\hat{T}_{\text{sep}}^\otimes)^2$ given by $a \cdot b \mapsto a \otimes b + b \otimes a$ with the natural map $(\hat{T}_{\text{sep}}^\otimes)^2 \rightarrow S^2(\hat{T}_{\text{sep}})^\Gamma$ is multiplication by 2. Then by Lemma 4.12, $2 \cdot S^2(\hat{T}_{\text{sep}})^\Gamma / \text{Dec}$ is contained in the image of the map

$$\text{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow S^2(\hat{T}_{\text{sep}})^\Gamma / \text{Dec}.$$

Theorem 4.3 then yields:

Theorem 4.13. Let $T$ be an algebraic torus over $F$. Then 2 times the homomorphism $S^2(\hat{T}_{\text{sep}})^\Gamma / \text{Dec} \rightarrow H^2(F, T^0)$ from Theorem 4.3 is trivial. If $p$ is an odd prime,

$$\text{Inv}^3(T, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\text{norm}} = \text{Inv}^3(T, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\text{lin}} \oplus \text{Inv}^3(T, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\text{quad}}$$

and there are natural isomorphisms $\text{Inv}^3(T, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\text{lin}} \simeq H^1(F, T^0)(p)$ and

$$\text{Inv}^3(T, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\text{quad}} \simeq (S^2(\hat{T}_{\text{sep}})^\Gamma / \text{Dec})(p).$$
Example 4.14. Let \( X = \{x_1, x_2, \ldots, x_n\} \) be a set of \( n \) elements with the natural action of the symmetric group \( S_n \). A continuous surjective group homomorphism \( \Gamma \to S_n \) yields a separable field extension \( L/F \) of degree \( n \). Consider the torus \( T = R_{L/F}(\mathbb{G}_{m, L})/\mathbb{G}_m \), where \( R_{L/F} \) is the Weil restriction; see [Voskresenskiĭ 1998, Chapter 1, §3.12]. Note that the generic maximal torus of the group \( \text{PGL}_n \) is of this form (see Section 5b). The character lattice \( \hat{T}_{\text{sep}} \) is the kernel of the augmentation homomorphism \( \mathbb{Z}[X] \to \mathbb{Z} \).

The dual torus \( T^\circ \) is the norm one torus \( R_{L/F}^{(1)}(\mathbb{G}_{m, L}) \). For every field extension \( K/F \), we have:

\[
H^1(K, T) = \text{Br}(KL/K), \quad H^1(K, T^\circ) = K^\times/N(KL)^\times,
\]

where \( KL := K \otimes L \), \( N \) is the norm map for the extension \( KL/K \) and

\[
\text{Br}(KL/K) = \text{Ker}(\text{Br}(K) \to \text{Br}(KL)).
\]

The pairing

\[
K^\times/N(KL)^\times \otimes \text{Br}(KL/K) \to H^3(F, \mathbb{Q}/\mathbb{Z}(2))
\]

defines linear degree 3 invariants of both \( T \) and \( T^\circ \).

We claim that \( S^2(\hat{T}_{\text{sep}})^\Gamma/\text{Dec} = 0 \) and \( S^2(\hat{T}_{\text{sep}}^\circ)^\Gamma/\text{Dec} = 0 \), that is, \( T \) and \( T^\circ \) have no nontrivial quadratic degree 3 invariants. We have \( \hat{T}_{\text{sep}} = \mathbb{Z}[X]/\mathbb{Z}N_X \), where \( N_X = \sum x_i \). The group \( S^2(\hat{T}_{\text{sep}}^\circ)^\Gamma \) is generated by \( S := \sum i<j x_i x_j \). As \( S \in \text{Dec} \), we have \( S^2(\hat{T}_{\text{sep}}^\circ)^\Gamma/\text{Dec} = 0 \).

Let \( D = \sum x_i^2 \) and \( E := \text{Qtr}(x_1 - x_2) = 2S - (n - 1)D \), where the quadratic map \( \text{Qtr} \) is defined in Section A-II. The group \( S^2(\hat{T}_{\text{sep}})^\Gamma \) is generated by \( E \) if \( n \) is even and by \( E/2 \) if \( n \) is odd. A computation shows that \( nE/2 = \text{Qtr}(nx_1 - N_X) \). It follows that the generator of \( S^2(\hat{T}_{\text{sep}})^\Gamma \) belongs to \( \text{Dec} \), hence \( S^2(\hat{T}_{\text{sep}}^\circ)^\Gamma/\text{Dec} \) is trivial.

Note that as the torus \( T \) is rational, it follows from Theorem 4.3 and Corollary 4.8 that \( \text{Inv}^3(T^\circ, \mathbb{Q}/\mathbb{Z}(2))_{\text{norm}} \simeq \text{Br}(L/F) \).

5. Unramified invariants

Let \( K/F \) be a field extension and \( v \) a discrete valuation of \( K \) over \( F \) with valuation ring \( O_v \). We say that an element \( a \in H^n(K, \mathbb{Q}/\mathbb{Z}(j)) \) is unramified with respect to \( v \) if \( a \) belongs to the image of the map \( H^n(O_v, \mathbb{Q}/\mathbb{Z}(j)) \to H^n(K, \mathbb{Q}/\mathbb{Z}(j)) \); see [Colliot-Thélène and Ojanguren 1989]. We write \( H^n_{\text{nr}}(K, \mathbb{Q}/\mathbb{Z}(j)) \) for the subgroup of the elements in \( H^n(K, \mathbb{Q}/\mathbb{Z}(j)) \) that are unramified with respect to all discrete valuations of \( K \) over \( F \). We have a natural homomorphism

\[
H^n(F, \mathbb{Q}/\mathbb{Z}(j)) \to H^n_{\text{nr}}(K, \mathbb{Q}/\mathbb{Z}(j)).
\]
A dominant morphism of varieties $Y \to X$ yields a homomorphism

$$H^p_{\text{nr}}(F(X), \mathbb{Q}/\mathbb{Z}(j)) \to H^p_{\text{nr}}(F(Y), \mathbb{Q}/\mathbb{Z}(j)).$$  \hspace{1cm} (5-2)

**Proposition 5.1.** Let $K/F$ be a purely transcendental field extension. Then the homomorphism (5-1) is an isomorphism.

*Proof.* The statement is well known for the $p$-components if $p \neq \text{char } F$; see, for example, [Colliot-Thélène and Ojanguren 1989, Proposition 1.2]. It suffices to consider the case $K = F(t)$ and prove the surjectivity of (5-1). The coniveau spectral sequence for the projective line $\mathbb{P}^1$ (see (A-1) in the Appendix) yields an exact sequence

$$H^n(\mathbb{P}^1, \mathbb{Q}/\mathbb{Z}(j)) \to H^n(K, \mathbb{Q}/\mathbb{Z}(j)) \to \bigoplus_{x \in \mathbb{P}^1} H^{n+1}_x(\mathbb{P}^1, \mathbb{Q}/\mathbb{Z}(j))$$

and, therefore, a surjective homomorphism $H^n(\mathbb{P}^1, \mathbb{Q}/\mathbb{Z}(j)) \to H^n_{\text{nr}}(K, \mathbb{Q}/\mathbb{Z}(j))$. By the projective bundle theorem (classical if $p \neq \text{char}(F)$ and [Gros 1985, Theorem 2.1.11] if $p = \text{char}(F) > 0$), we have

$$H^n(\mathbb{P}^1, \mathbb{Q}/\mathbb{Z}(j)) = H^n(F, \mathbb{Q}/\mathbb{Z}(j)) \oplus H^{n-2}(F, \mathbb{Q}/\mathbb{Z}(j - 1))t,$$

where $t$ is a generator of $H^2(\mathbb{P}^1, \mathbb{Z}(1)) = \text{Pic}(\mathbb{P}^1) = \mathbb{Z}$. As $t$ vanishes over the generic point of $\mathbb{P}^1$, the result follows. \hfill $\square$

Let $G$ be a linear algebraic group over $F$. Choose a standard classifying $G$-torsor $U \to U/G$. An invariant $i \in \text{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j))$ is called *unramified* if the image of $i$ under $\theta_G : \text{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j)) \to H^n(F(U/G), \mathbb{Q}/\mathbb{Z}(j))$ is unramified. This is independent of the choice of standard classifying torsor. Indeed, if $U' \to U'/G$ is another standard classifying $G$-torsor, then $(U \times V')/G \to U/G$ and $(V \times U')/G \to U'/G$ are vector bundles. Hence the field $F((U \times U')/G)$ is a purely transcendental extension of $F(U/G)$ and $F(U'/G)$ and by Proposition 5.1,

$$H^n_{\text{nr}}(F(U/G), \mathbb{Q}/\mathbb{Z}(j)) \simeq H^n_{\text{nr}}(F((U \times U')/G), \mathbb{Q}/\mathbb{Z}(j))$$

$$\simeq H^n_{\text{nr}}(F(U'/G), \mathbb{Q}/\mathbb{Z}(j)).$$

We write $H^n_{\text{nr}}(F(BG), \mathbb{Q}/\mathbb{Z}(j))$ for this common value and $\text{Inv}^n_{\text{nr}}(G, \mathbb{Q}/\mathbb{Z}(j))$ for the subgroup of unramified invariants. Similarly, we write $\text{Br}_{\text{nr}}(F(BG))$ for the *unramified Brauer group* $H^2_{\text{nr}}(F(BG), \mathbb{Q}/\mathbb{Z}(1))$.

**Proposition 5.2.** If $G'$ be a subgroup of $G$ and $i \in \text{Inv}^n_{\text{nr}}(G, \mathbb{Q}/\mathbb{Z}(j))$, then

$$\text{res}(i) \in \text{Inv}^n_{\text{nr}}(G', \mathbb{Q}/\mathbb{Z}(j)).$$
We have the projection $q$. We follow Totaro’s approach; see [Garibaldi et al. 2003, p. 99]. Consider the open \( (\cdot)^{\cdot} \). Applying the homomorphism (5-2) we see that \( \text{res} 
abla \) is unramified.

**Proposition 5.3.** Let \( G \) be a smooth linear algebraic group over a field \( F \). The map \( \text{Inv}^n_{\text{nr}}(G, \mathbb{Q}/\mathbb{Z}(j)) \rightarrow \text{H}^n_{\text{nr}}(F(BG), \mathbb{Q}/\mathbb{Z}(j)) \) induced by \( \theta_G \) is an isomorphism.

**Proof.** By Theorem 3.4, it suffices to show that

\[
\text{H}^n_{\text{nr}}(F(U/G), \mathbb{Q}/\mathbb{Z}(j)) \subset \text{H}^n(F(U/G), \mathbb{Q}/\mathbb{Z}(j))_{\text{bal}}.
\]

We follow Totaro’s approach; see [Garibaldi et al. 2003, p. 99]. Consider the open subscheme \( W \) of \( (U^2/G) \times \mathbb{A}^1 \) of all triples \((u, u', t)\) such that \((2-t)u+(t-1)u'\in U\).

We have the projection \( q : W \rightarrow U^2/G \), the morphisms \( f : W \rightarrow U/G \) defined by \( f(u, u', t) = (2-t)u+(t-1)u' \), and \( h_i : U^2/G \rightarrow W \) defined by \( h_i(u, u') = (u, u', i) \) for \( i = 1 \) and \( 2 \). The composition \( f \circ h_i \) is the projection \( p_i : U^2/G \rightarrow U/G \) and \( q \circ h_i \) is the identity of \( U^2/G \).

Let \( w_i \) be the generic point of the preimage of \( i \) with respect to the projection \( W \rightarrow \mathbb{A}^1 \) and write \( O_i \) for the local ring of \( W \) at \( w_i \). The morphisms \( q, f, \) and \( h_i \) yield \( F \)-algebra homomorphisms \( F(U^2/G) \rightarrow O_i, F(U/G) \rightarrow O_i \) and \( O_i \rightarrow F(U/G) \). Note that by Proposition A.11, we have

\[
\text{H}^n_{\text{nr}}(F(W), \mathbb{Q}/\mathbb{Z}(j)) \subset \text{H}^n(O_i, \mathbb{Q}/\mathbb{Z}(j)).
\]

In the commutative diagram

\[
\begin{array}{ccc}
\text{H}^n_{\text{nr}}(F(U/G), \mathbb{Q}/\mathbb{Z}(j)) & \xrightarrow{f^*} & \text{H}^n_{\text{nr}}(F(W), \mathbb{Q}/\mathbb{Z}(j)) \\
\downarrow & & \downarrow \\
\text{H}^n(F(U/G), \mathbb{Q}/\mathbb{Z}(j)) & \xrightarrow{q^*} & \text{H}^n(F(U^2/G), \mathbb{Q}/\mathbb{Z}(j)) \\
\downarrow & & \downarrow \\
\text{H}^n(F(U/G), \mathbb{Q}/\mathbb{Z}(j)) & \xrightarrow{q^*} & \text{H}^n(F(U^2/G), \mathbb{Q}/\mathbb{Z}(j)) \\
\downarrow & & \downarrow \\
\text{H}^n(F(U^2/G), \mathbb{Q}/\mathbb{Z}(j)) & \xrightarrow{p_i^*} & \text{H}^n(F(U^2/G), \mathbb{Q}/\mathbb{Z}(j)) \\
\end{array}
\]

the top right map \( q^* \) is an isomorphism by Proposition 5.1 since the field extension \( F(W)/F(U^2/G) \) is purely transcendental. It follows that the restriction of \( p_i^* \) on \( \text{H}^n_{\text{nr}}(F(U/G), \mathbb{Q}/\mathbb{Z}(j)) \) coincides with \( (q^*)^{-1} \circ f^* \) and hence is independent of \( i \).

\[\square\]

**5a. Unramified invariants of tori.**

**Proposition 5.4.** If \( T \) is a flasque torus, then every invariant in \( \text{Inv}^n(T, \mathbb{Q}/\mathbb{Z}(j)) \) is unramified.
Proof. Consider an exact sequence of tori \(1 \to T \to P \to Q \to 1\) with \(P\) quasisplit. Choose a smooth projective compactification \(X\) of \(Q\); see [Colliot-Thélène et al. 2005]. As \(T\) is flasque, by [Colliot-Thélène and Sansuc 1977, Proposition 9], there is a \(T\)-torsor \(E \to X\) extending the \(T\)-torsor \(P \to Q\). The torsor \(E\) is classifying and \(T\)-rational. Choose an invariant in \(\text{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j))\) and consider its image \(a\) in \(H^n(F(X), \mathbb{Q}/\mathbb{Z}(j))\) (see Theorem 3.4). We show that \(a\) is unramified with respect to every discrete valuation \(v\) on \(F(X)\) over \(F\); see [Colliot-Thélène 1995, Proposition 2.1.8]. By Proposition A.9, \(a\) is unramified with respect to the discrete valuation associated to every point \(x \in X\) of codimension 1, that is, \(\partial_x(a) = 0\).

As \(X\) is projective, the valuation ring \(O_v\) of the valuation \(v\) dominates a point \(x \in X\). It follows from Proposition A.11 that \(a\) belongs to the image of \(H^n(O_X, x, \mathbb{Q}/\mathbb{Z}(j)) \to H^n(F(X), \mathbb{Q}/\mathbb{Z}(j))\) and hence \(a\) is unramified with respect to \(v\). □

Let \(T\) be a torus over \(F\). By [Colliot-Thélène and Sansuc 1987b, Lemma 0.6], there is an exact sequence of tori \(1 \to T \to T' \to P \to 1\) with \(T'\) flasque and \(P\) quasisplit. The following theorem computes the unramified invariants of \(T\) in terms of the invariants of \(T'\).

**Theorem 5.5.** There is a natural isomorphism
\[
\text{Inv}^n_{nr}(T, \mathbb{Q}/\mathbb{Z}(j)) \cong \text{Inv}^n(T', \mathbb{Q}/\mathbb{Z}(j)).
\]

**Proof.** Choose an exact sequence \(1 \to T' \to P' \to S \to 1\) with \(P'\) a quasisplit torus. Let \(S'\) be the cokernel of the composition \(T \to T' \to P'\). We have an exact sequence \(1 \to P \to S' \to S \to 1\). As \(P\) is quasisplit, the latter exact sequence splits at the generic point of \(S\) and therefore, \(F(S')\) is a purely transcendental field extension of \(F(S)\). It follows from Propositions 5.1, 5.3, and 5.4 that
\[
\text{Inv}^n_{nr}(T, \mathbb{Q}/\mathbb{Z}(j)) \cong H^n_{nr}(F(S'), \mathbb{Q}/\mathbb{Z}(j)) \cong H^n_{nr}(F(S), \mathbb{Q}/\mathbb{Z}(j))
\]
\[
\cong \text{Inv}^n_{nr}(T', \mathbb{Q}/\mathbb{Z}(j)) = \text{Inv}^n(T', \mathbb{Q}/\mathbb{Z}(j)).
\] □

The following corollary is essentially equivalent to [Colliot-Thélène and Sansuc 1987b, Proposition 9.5] in the case when \(F\) is of zero characteristic.

**Corollary 5.6.** With notation as above, the isomorphism
\[
\text{Inv}(T, \text{Br}) \cong \text{Pic}(T) = H^1(F, \hat{T})
\]
identifies $\text{Inv}_{nr}(T, \text{Br})$ with the subgroup $H^1(F, \hat{T}')$ of $H^1(F, \hat{T})$ of all elements that are trivial when restricted to all cyclic subgroups of the decomposition group of $T$.

**Proof.** The description of $H^1(F, \hat{T}')$ as a subgroup of $H^1(F, \hat{T})$ is given in [Colliot-Thélène and Sansuc 1987b, Proposition 9.5], and this part of the proof is characteristic free. □

In view of Propositions 5.1 and 5.3 we can calculate the group of unramified cohomology for the function field of an arbitrary torus in terms of the invariants of a flasque torus:

**Theorem 5.7.** Let $S$ be a torus over $F$ and let $1 \to T \to P \to S \to 1$ be a flasque resolution of $S$, that is, $T$ is flasque and $P$ is quasisplit. Then there is a natural isomorphism

$$H^n_{nr}(F(S), \mathbb{Q}/\mathbb{Z}(j)) \simeq \text{Inv}^n(T, \mathbb{Q}/\mathbb{Z}(j)).$$

Note that the torus $S$ determines $T$ up to multiplication by a quasisplit torus. If $X$ is a smooth compactification of $S$, then one can take a torus $T$ with $\hat{T}_{\text{sep}} \simeq \text{Pic}(X_{\text{sep}})$; see [Colliot-Thélène and Sansuc 1977, Proposition 6; Voskresenskiǐ 1998, §4.6].

**Corollary 5.8.** A torus $S$ has no nonconstant unramified degree 3 cohomology with values in $\mathbb{Q}/\mathbb{Z}(2)$ universally, that is, $H^3_{nr}(K(S), \mathbb{Q}/\mathbb{Z}(2)) = H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ for any field extension $K/F$, if and only if $S$ is a direct factor of a rational torus.

**Proof.** If $S$ is a direct factor of a rational torus, then $S$ has no nonconstant unramified cohomology by Proposition 5.1.

Conversely, let $1 \to T \to P \to S \to 1$ be a flasque resolution of $S$. By Theorem 5.7, $H^3_{nr}(T_K, \mathbb{Q}/\mathbb{Z}(2))_{\text{norm}} = 0$ for every $K/F$. It follows from Theorem 4.10 that $T$ is invertible and hence $S$ is a factor of a rational torus (see Section 4a). □

Theorems 4.3, 5.7 and [Colliot-Thélène and Sansuc 1977, §2] yield the following proposition.

**Proposition 5.9.** Let $S$ be a torus over $F$ and let $1 \to T \to P \to S \to 1$ be a flasque resolution of $S$. Then we have an exact sequence

$$0 \to \text{CH}^2(BT)_{\text{tors}} \to H^1(F, T^0) \to \overline{H}^3_{nr}(F(S), \mathbb{Q}/\mathbb{Z}(2)) \to H^0(F, S^2(\hat{T}_{\text{sep}}))/\text{Dec} \to H^2(F, T^0).$$

For an odd prime $p$, there is a canonical direct sum decomposition

$$\overline{H}^3_{nr}(F(S), \mathbb{Q}_p/\mathbb{Z}_p(2)) = H^1(F, T^0)[p] \oplus (H^0(F, S^2(\hat{T}_{\text{sep}}))/\text{Dec})[p].$$

If $X$ is a smooth compactification of $S$, one can take the torus $T$ with $\hat{T}_{\text{sep}} = \text{Pic}(X_{\text{sep}})$. 
5b. The Brauer invariant for semisimple groups. The following theorem was proved by Bogomolov [1987, Lemma 5.7] in characteristic zero:

**Theorem 5.10.** Let $G$ be a (connected) semisimple group over a field $F$. Then $\text{Inv}_{\text{nr}}(G, \text{Br}) = \text{Inv}(G, \text{Br})_{\text{const}} = \text{Br}(F)$ and $\text{Br}_{\text{nr}}(F(BG)) = \text{Br}(F)$.

**Proof.** Let $G' \to G$ be a simply connected cover of $G$ and $C$ the kernel of $G' \to G$. By Theorem 2.4 and [Sansuc 1981, Lemme 6.9(iii)], we have

$$\text{Inv}(G, \text{Br})_{\text{norm}} = \text{Pic}(G) = \widehat{C}(F).$$

As the map $\widehat{C}(F) \to \widehat{C}(F_{\text{sep}})$ is injective, we can replace $F$ by $F_{\text{sep}}$ and assume that the group $G$ is split.

Consider the variety $\mathcal{T}$ of maximal tori in $G$ and the closed subscheme $\mathcal{X} \subset G \times \mathcal{T}$ of all pairs $(g, T)$ with $g \in T$. The generic fiber of the projection $\mathcal{X} \to \mathcal{T}$ is the generic torus $T_{\text{gen}}$ of $G$. Then $T_{\text{gen}}$ is a maximal torus of $G_K$, where $K := F(\mathcal{T})$.

Every maximal torus in $G$ is the factor torus of a maximal torus in $G'$ by $C$. It follows that the variety $\mathcal{T}'$ of maximal tori in $G'$ is naturally isomorphic to $\mathcal{T}$. Moreover, as the generic torus $T_{\text{gen}}'$ of $G'$ is a maximal torus of $G_K'$, we have $T_{\text{gen}} \simeq T_{\text{gen}}'/C_K$ and, therefore, an exact sequence of character groups

$$0 \to \widehat{T}_{\text{gen}} \to \widehat{T}_{\text{gen}}' \to \widehat{C}_K \to 0.$$

By Theorem 2.4, the composition of the natural homomorphism

$$\text{Inv}(G, \text{Br})_{\text{norm}} \to \text{Inv}(G_K, \text{Br})_{\text{norm}}$$

with the restriction $\text{Inv}(G_K, \text{Br})_{\text{norm}} \to \text{Inv}(T_{\text{gen}}, \text{Br})_{\text{norm}}$ can be identified with the natural composition $\text{Pic}(G) \to \text{Pic}(G_K) \to \text{Pic}(T_{\text{gen}})$ and hence with the connecting homomorphism $\widehat{C}(F) = \widehat{C}(K) \to H^1(K, \widehat{T}_{\text{gen}})$. Note that as $F = F_{\text{sep}}$, the decomposition group of $T_{\text{gen}}$ coincides with the Weyl group $W$ of $G$ by [Voskresenskiï 1988, Theorem 1], hence $H^1(K, \widehat{T}_{\text{gen}}) \simeq H^1(W, \widehat{T}_{\text{gen}})$.

Let $w$ be a Coxeter element in $W$.\(^1\) It is the product of reflections with respect to all simple roots (in some order). By [Humphreys 1990, Lemma, p. 76], 1 is not an eigenvalue of $w$ on the space of weights $\widehat{T}_{\text{gen}}' \otimes \mathbb{R}$. Let $W_0$ be the cyclic subgroup in $W$ generated by $w$. It follows that the first term in the exact sequence

$$(\widehat{T}_{\text{gen}}')^{W_0} \to \widehat{C}(K) \to H^1(W_0, \widehat{T}_{\text{gen}})$$

is trivial, that is, the second map is injective. Hence every nonzero character $\chi$ in $\widehat{C}(K)$ restricts to a nonzero element in $H^1(W_0, \widehat{T}_{\text{gen}})$. It follows that the image of $\chi$ in $H^1(W, \widehat{T}_{\text{gen}})$ is ramified by Corollary 5.6, hence so is $\chi$ by Proposition 5.2. □

\(^1\)We owe the idea to use the Coxeter element and the reference below to S. Garibaldi.
Appendix A: Generalities

A-I: Proof of Theorem 2.2. Suppose that \( i(E_{\text{gen}}) = 0 \) for an \( H \)-invariant \( i \) of \( G \). Let \( K/F \) be a field extension and \( I \to \text{Spec} \ K \) a \( G \)-torsor. We need to show that \( i(I) = 0 \) in \( H(K) \).

Suppose first that \( K \) is infinite. Find a point \( x \in X(K) \) such that \( I \) is isomorphic to the pull-back of the classifying torsor with respect to \( x \). Let \( x' \) be a rational point of \( X_K \) above \( x \) and write \( O \) for the local ring \( O_{X_K,x'} \). The \( K \)-algebra \( O \) is a regular local ring with residue field \( K \). Therefore, the completion \( \hat{O} \) is isomorphic to \( K[[t_1, t_2, \ldots, t_n]] \) over \( K \). Let \( L \) be the quotient field of \( \hat{O} \), a field extension of \( K(X) \). We have the following diagram of morphisms with a commutative square and three triangles:

\[
\begin{array}{ccc}
\text{Spec} \ K & \xrightarrow{x} & \text{Spec} \hat{O} \\
\downarrow & & \downarrow \\
\text{Spec} \ L & \to & \text{Spec} \hat{O} \\
\downarrow & & \downarrow \\
\text{Spec} K(X) & \to & \text{Spec} O
\end{array}
\]

The pull-back of the classifying torsor \( E \to X \) via \( \text{Spec} \ K(X) \to X \) is \( (E_{\text{gen}})_{K(X)} \). The \( G \)-torsor \( I \) is the pull-back of \( E \to X \) with respect to \( x \). Let \( \hat{E} \) be the pull-back of \( E \to X \) via \( \text{Spec} \hat{O} \to X \). Therefore, \( I \) is the pull-back of \( \hat{E} \). Since \( G \) is smooth, by a theorem of Grothendieck [Demazure and Grothendieck 1970, XXIV, Proposition 8.1], \( \hat{E} \) is the pull-back of \( I \) with respect to \( \text{Spec} \hat{O} \to \text{Spec}(K) \). It follows that \( I_L \cong (E_{\text{gen}})_{L} \) as torsors over \( L \). Hence the images of \( i(I) \) and \( i(E_{\text{gen}}) \) in \( H(L) \) are equal and therefore, \( i(I)_L = 0 \). By Property 2.1, we have \( i(I) = 0 \).

If \( K \) is finite, we replace \( F \) by \( F((t)) \) and \( K \) by \( K((t)) \). By the first part of the proof, \( i(I) \) belongs to the kernel of \( H(K) \to H(K((t))) \) and hence is trivial by Property 2.1 again.

A-II: Decomposable elements. Let \( \Gamma \) be a profinite group and \( A \) a \( \Gamma \)-lattice. Write \( A^\Gamma \) for the subgroup of \( \Gamma \)-invariant elements in \( A \). Let \( \Gamma' \subset \Gamma \) be an open subgroup and choose representatives \( \gamma_1, \gamma_2, \ldots, \gamma_n \) for the left cosets of \( \Gamma' \) in \( \Gamma \). We have the trace map \( \text{Tr} : A^\Gamma \to A^\Gamma \) defined by \( \text{Tr}(a) = \sum_{i=1}^n \gamma_i a \).

Let \( S^2(A) \) be the symmetric square of \( A \). Consider the quadratic trace map \( \text{Qtr} : A^{\Gamma'} \to S^2(A)^\Gamma \) defined by \( \text{Qtr}(a) = \sum_{i<j}(\gamma_i a)(\gamma_j a) \). Write Dec\( (A) \) for the subgroup of decomposable elements in \( S^2(A)^\Gamma \) generated by the square \( (A^\Gamma)^2 \) of \( A^\Gamma \) and the elements \( \text{Qtr}(a) \) for all open subgroups \( \Gamma' \subset \Gamma \) and all \( a \in A^{\Gamma'} \).
Let $B$ be another $\Gamma$-lattice. We write $\text{Dec}(A, B)$ for the subgroup of $(A \otimes B)^\Gamma$ generated by elements of the form $\text{Tr}(a \otimes b)$ for all open subgroups $\Gamma' \subset \Gamma$ and all $a \in A^{\Gamma'}$, $b \in B^{\Gamma'}$.

There is a natural isomorphism $S^2(A \oplus B) \simeq S^2(A) \oplus (A \otimes B) \oplus S^2(B)$. Moreover, the equality $\text{Qtr}(a + b) = \text{Qtr}(a) + (\text{Tr}(a) \otimes \text{Tr}(b) - \text{Tr}(a \otimes b)) + \text{Qtr}(b)$ yields the decomposition

$$\text{Dec}(A \oplus B) \simeq \text{Dec}(A) \oplus \text{Dec}(A, B) \oplus \text{Dec}(B).$$

**A-III: Cup-products.** Let $1 \to T \to P \to Q \to 1$ be an exact sequence of tori. We consider the connecting maps

$$\partial_1 : H^1(F, \hat{T}(i)) \to H^{p+1}(F, \hat{Q}(i))$$

for the exact sequence $0 \to \hat{Q}_{\text{sep}} \to \hat{P}_{\text{sep}} \to \hat{T}_{\text{sep}} \to 0$ of character $\Gamma$-lattices and

$$\partial_2 : H^q(F, \hat{Q}^o(j)) \to H^{q+1}(F, \hat{Q}^o(j))$$

for the dual sequence of lattices (see notation in Section 4b).

**Lemma A.1.** Let $a \in H^p(F, \hat{T}(i))$ and $b \in H^q(F, \hat{Q}^o(j))$ with $i + j \leq 2$. Then $\partial_1(a) \cup b = (-1)^{p+1}a \cup \partial_2(b)$ in $H^{p+q+1}(F, \mathbb{Z}(i + j))$, where the cup-product is defined in (4-3).

**Proof.** By [Cartan and Eilenberg 1999, Chapter V, Proposition 4.1], the elements $\partial_1(1_T)$ and $\partial_2(1_Q)$ in

$$H^1(F, \hat{T}_{\text{sep}} \otimes \hat{Q}_{\text{sep}}) = \text{Ext}_1^1(\hat{T}_{\text{sep}}, \hat{Q}_{\text{sep}})$$

differ by a sign. Write $\tau$ for the isomorphism induced by permutation of the factors. By the standard properties of the cup-product, we have

$$\partial_1(a) \cup b = 1_T \cup \partial_1(a) \cup b = \partial_1(1_T) \cup a \cup b = (-1)^{pq} \tau(\partial_1(1_T) \cup b \cup a)$$

$$= (-1)^{pq+1} \tau(\partial_2(1_Q) \cup b \cup a) = (-1)^{pq+1} \tau(1_Q \cup \partial_2(b) \cup a)$$

$$= (-1)^{p+1} \tau(1_Q \cup a \cup \partial_2(b) = (-1)^{p+1}a \cup \partial_2(b). \quad \square$$

**A-IV: Cosimplicial abelian groups.** Let $A^\bullet$ be a cosimplicial abelian group

$$A^0 \underbrace{\xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \xrightarrow{d^2} \cdots}_j$$

and write $h_*(A^\bullet)$ for the homology groups of the associated complex of abelian groups. In particular,

$$h_0(A^\bullet) = \text{Ker}[(d^0 - d^1) : A^0 \to A^1].$$
We say that the cosimplicial abelian group $A^\bullet$ is constant if for every $i$, all the coface maps $d_j : A^{i-1} \to A^i$, $j = 0, 1, \ldots, i$, are isomorphisms. In this case all the $d_j$ are equal as $d_j = s_j^{-1} = d_{j+1}$, where the $s_j$ are the codegeneracy maps. For a constant cosimplicial abelian group $A^\bullet$, we have $h_0(A^\bullet) = A^0$ and $h_i(A^\bullet) = 0$ for all $i > 0$. We will need the following straightforward statement.

**Lemma A.2.** Let $0 \to A^\bullet \to B^\bullet \to C^\bullet \to D^\bullet$ be an exact sequence of cosimplicial abelian groups with $A^\bullet$ a constant cosimplicial group. Then the sequence of groups $0 \to A^0 \to h_0(B^\bullet) \to h_0(C^\bullet) \to h_0(D^\bullet)$ is exact.

Let $H$ be a contravariant functor from the category of schemes over $F$ to the category of abelian groups. We say that $H$ is homotopy invariant if for every vector bundle $E \to X$ over $F$, the induced map $H(X) \to H(E)$ is an isomorphism.

For an integer $d > 0$ consider the following property of the functor $H$:

**Property A.3.** For every closed subscheme $Z$ of a scheme $X$ with $\text{codim}_X(Z) \geq d$, the natural homomorphism $H(X) \to H(X \setminus Z)$ is an isomorphism.

Let $G$ be a linear algebraic group over a field $F$ and choose a standard classifying $G$-torsor $U \to U/G$. Let $U^i$ denote the product of $i$ copies of $U$. We have the $G$-torsors $U^i \to U^i/G$.

Consider the cosimplicial abelian group $A^\bullet = H(U^\bullet/G)$ with $A^i = H(U^{i+1}/G)$ and coface maps $A^{i-1} \to A^i$ induced by the projections $U^{i+1}/G \to U^i/G$.

**Lemma A.4.** Let $H$ be a homotopy invariant functor satisfying Property A.3 for some $d$. Let $U \to U/G$ be a standard classifying $G$-torsor and $U'$ an open subset of a $G$-representation $V'$.

1. If $\text{codim}_{V'}(V' \setminus U') \geq d$, then the natural homomorphism

$$H(U/G) \to H((U \times U')/G)$$

is an isomorphism.

2. If $\text{codim}_{V}(V \setminus U) \geq d$, then the cosimplicial group $H(U^\bullet/G)$ is constant.

**Proof.** 1. The scheme $(U \times U')/G$ is an open subset of the vector bundle $(U \times V')/G$ over $U/G$ with complement of codimension at least $d$. The map in question is the composition $H(U/G) \to H((U \times V')/G) \to H((U \times U')/G)$ and both maps in the composition are isomorphisms since $H$ is homotopy invariant and satisfies Property A.3.

2. By the first part of the lemma applied to the $G$-torsor $U^i \to U^i/G$ and $U' = U$, the map $H(U^i/G) \to H(U^{i+1}/G)$ induced by a projection $U^{i+1}/G \to U^i/G$ is an isomorphism. □

By Lemma A.4, if $H$ is a homotopy invariant functor satisfying Property A.3 for some $d$, then the group $H(U/G)$ does not depend on the choice of the representation.
V and the open set U ⊂ V provided codim_V(V \ U) ≥ d. Following [Totaro 1999], we denote this group by H(BG).

Example A.5. The split torus T = (G_m)^n over F acts freely on the product U of n copies of A^{r+1} \ {0} with U/T ≃ (P^r)^n, that is, BT is “approximated” by the varieties (P^r)^n if “r ≫ 0.” We then have CH^*(BT) = S^*(\hat{T}), where S^* represents the symmetric algebra and \hat{T} is the character group of T; see [Edidin and Graham 1998, p. 607]. In particular, Pic(BT) = CH^1(BT) = \hat{T}. More generally, by the Künneth formula [Esnault et al. 1998, Proposition 3.7],

$$H^*_\text{Zar}(BT, \mathcal{H}_n) \simeq \text{CH}^*(BT) \otimes K_*(F) \simeq S^*(\hat{T}) \otimes K_*(F),$$

where K_n(F) is the Quillen K-group of F and \mathcal{H}_n is the Zariski sheaf associated to the presheaf U ↦ K_n(U).

A-V: Étale cohomology. For a scheme X and a closed subscheme Z ⊂ X we write H^*_Z(X, \mathbb{Q}/\mathbb{Z}(j)) for the étale cohomology group of X with support in Z and values in \mathbb{Q}/\mathbb{Z}(j) [Milne 1980, Chapter III, §1]. Write X^{(i)} for the set of points in X of codimension i. For a point x ∈ X^{(1)} set

$$H^*_x(X, \mathbb{Q}/\mathbb{Z}(j)) = \colim_{x ∈ U} H^*_x(U, \mathbb{Q}/\mathbb{Z}(j)),$$

where the colimit is taken over all open subsets U ⊂ X containing x. If X is a variety, write

$$\partial_x : H^*(F(X), \mathbb{Q}/\mathbb{Z}(j)) → H_x^{*+1}(X, \mathbb{Q}/\mathbb{Z}(j))$$

for the residue homomorphisms arising from the coniveau spectral sequence [Colliot-Thélène et al. 1997, 1.2]

$$E_1^{p,q} = \bigsqcup_{x ∈ X^{(p)}} H_x^{p+q}(X, \mathbb{Q}/\mathbb{Z}(j)) \Rightarrow H^{p+q}(X, \mathbb{Q}/\mathbb{Z}(j)). \quad \text{(A-1)}$$

Let f : Y → X be a dominant morphism of varieties over F, y ∈ Y^{(1)}, and x = f(y). If x ∈ X^{(1)}, there is a natural homomorphism

$$f_Y^* : H^*_y(X, \mathbb{Q}/\mathbb{Z}(j)) → H^*_x(Y, \mathbb{Q}/\mathbb{Z}(j)).$$

The following lemma is straightforward.

Lemma A.6. Let f : Y → X be a dominant morphism of varieties over F, y ∈ Y^{(1)} and x = f(y).

1. If x is the generic point of X, then the composition

$$H^*(F(X), \mathbb{Q}/\mathbb{Z}(j)) \xrightarrow{f^*} H^*(F(Y), \mathbb{Q}/\mathbb{Z}(j)) \xrightarrow{\partial_y} H^{*+1}_y(Y, \mathbb{Q}/\mathbb{Z}(j))$$

is trivial.
(2) If \( x \in X^{(1)} \), the diagram

\[
\begin{array}{ccc}
H^*(F(X), \mathbb{Q}/\mathbb{Z}(j)) & \xrightarrow{\partial_x} & H^{*+1}_x(X, \mathbb{Q}/\mathbb{Z}(j)) \\
\downarrow f^* & & \downarrow f^*_y \\
H^*(F(Y), \mathbb{Q}/\mathbb{Z}(j)) & \xrightarrow{\partial_y} & H^{*+1}_y(Y, \mathbb{Q}/\mathbb{Z}(j)).
\end{array}
\]

is commutative.

**Lemma A.7.** Let \( X \) be a geometrically irreducible variety, \( Z \subset X \) a closed subvariety of codimension 1, and \( x \) the generic point of \( Z \). Let \( P \) be a variety over \( F \) such that \( P(K) \) is dense in \( P \) for every field extension \( K/F \) with \( K \) infinite, and let \( y \) be the generic point of \( Z \times P \in Y := X \times P \). Then the homomorphism \( f^*_y : H^*_x(X, \mathbb{Q}/\mathbb{Z}(j)) \to H^*_y(Y, \mathbb{Q}/\mathbb{Z}(j)) \) induced by the projection \( f : Y \to X \) is injective.

**Proof.** Assume first that the field \( F \) is infinite. An element \( \alpha \in H^*_x(X, \mathbb{Q}/\mathbb{Z}(j)) \) is represented by an element \( h \in H^*_x(Z \cap U(U, \mathbb{Q}/\mathbb{Z}(j)) \) for a nonempty open set \( U \subset X \) containing \( x \). If \( \alpha \) belongs to the kernel of

\[
f^*_y : H^*_x(X, \mathbb{Q}/\mathbb{Z}(j)) \to H^*_y(Y, \mathbb{Q}/\mathbb{Z}(j)),
\]

then there is an open subset \( W \subset U \times P \) containing \( y \) such that \( h \) belongs to the kernel of the composition

\[
g : H^*_x(Z \cap U(U, \mathbb{Q}/\mathbb{Z}(j)) \to H^*_x(Z \cap U \times P(U \times P, \mathbb{Q}/\mathbb{Z}(j)) \to H^*_x(Z \cap U \times P(U \times P, \mathbb{Q}/\mathbb{Z}(j))).
\]

As \( F \) is infinite, by the assumption on \( P \), there is a rational point \( t \in P \) in the image of the dominant composition \( (Z \times P) \cap W \hookrightarrow Z \times P \to P \). We have \( (U \times t) \cap W = U' \times t \) for an open subset \( U' \subset U \) such that \( x \in U' \). Composing \( g \) with the homomorphism \( H^*_x(Z \times P \cap W(W, \mathbb{Q}/\mathbb{Z}(j)) \to H^*_x(U', \mathbb{Q}/\mathbb{Z}(j))) \) induced by the composition \( (U', Z \cap U') \to (W, (Z \times P) \cap W), u \mapsto (u, t) \), we see that \( h \) belongs to the kernel of the restriction homomorphism \( H^*_x(Z \cap U(U, \mathbb{Q}/\mathbb{Z}(j)) \to H^*_x(Z \cap U(U', \mathbb{Q}/\mathbb{Z}(j))) \), hence the image of \( \alpha \) in \( H^*_x(X, \mathbb{Q}/\mathbb{Z}(j)) \) is trivial.

Suppose now that \( F \) is a finite field. Choose a prime integer \( p \) and an infinite algebraic pro-\( p \)-extension \( L/F \). By the first part of the proof, the statement holds for the variety \( X_L \) over \( L \). By the restriction-corestriction argument, \( \text{Ker}(f^*_y) \) is a \( p \)-primary torsion group. Since this holds for every prime \( p \), we have \( \text{Ker}(f^*_y) = 0 \).

**Corollary A.8.** Let \( G \) be a linear algebraic group over \( F \), let \( E \to X \) be a \( G \)-torsor over a geometrically irreducible variety \( X \) with \( E \) a \( G \)-rational variety and consider the first projection \( p : E^2/G \to X \). Let \( x \in X \) and \( y \in E^2/G \) be points of
codimension 1 such that \( p(y) = x \). Then the homomorphism

\[
p_{\gamma}^*: H^*_x(X, \mathbb{Q}/\mathbb{Z}(j)) \to H^*_y(E^2/G, \mathbb{Q}/\mathbb{Z}(j))
\]

is injective.

**Proof.** Choose a linear \( G \)-space \( V \) and a nonempty \( G \)-variety \( U \) that is \( G \)-isomorphic to open subschemes of \( E \) and \( V \). We can replace the variety \( E^2/G \) by \((E \times U)/G\), an open subscheme in the vector bundle \((E \times V)/G\) over \( X \). Shrinking \( X \) around \( x \), we may assume that the vector bundle is trivial, that is, \((E \times U)/G\) is isomorphic to an open subscheme in \( X \times V \). The statement then follows from Lemma A.7. \( \square \)

**Proposition A.9.** In the conditions of Corollary A.8, let \( h \in H^*(F(X), \mathbb{Q}/\mathbb{Z}(j))_{\text{bal}} \). Then \( \partial_x(h) = 0 \) for every point \( x \in X \) of codimension 1.

**Proof.** Let \( y \in E^2/G \) be the point of codimension 1 such that \( p_1(y) = x \). As \( p_2(y) \) is the generic point of \( X \), by Lemma A.6(1), \( \partial_x(h') = 0 \), where \( h' = p_1^*(h) = p_2^*(h) \) in \( H^*(F(E^2/G), \mathbb{Q}/\mathbb{Z}(j)) \). It follows from Lemma A.6(2) that \( \partial_x(h) \) is in the kernel of \( (p_1)_x^*: H^*_x(X, \mathbb{Q}/\mathbb{Z}(j)) \to H^*_y(E^2/G, \mathbb{Q}/\mathbb{Z}(j)) \) and hence is trivial by Corollary A.8. \( \square \)

The sheaf \( \mathcal{H}^*(\mathbb{Q}/\mathbb{Z}(j)) \) defined in Section 3 has a flasque resolution related to the Cousin complex by [Colliot-Thélène et al. 1997, §2] (for the \( p \)-components with \( p \neq \text{char } F \)) and [Gros and Suwa 1988, Theorem 1.4] (for the \( p \)-component with \( p = \text{char } F > 0 \)):

\[
0 \to \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)) \to \bigsqcup_{x \in X^{(0)}} i_x^*H^*_x(X, \mathbb{Q}/\mathbb{Z}(j)) \to \bigsqcup_{x \in X^{(1)}} i_x^*H^*_{x+1}(X, \mathbb{Q}/\mathbb{Z}(j)) \to \cdots ,
\]

where \( i_x : \text{Spec } F(x) \to X \) are the canonical morphisms. In particular, we have:

**Proposition A.10.** Let \( X \) be a smooth variety over \( F \). The sequence

\[
0 \to H^0_{\text{Zar}}(X, \mathcal{H}^*(\mathbb{Q}/\mathbb{Z}(j))) \to H^*(F(X), \mathbb{Q}/\mathbb{Z}(j)) \xrightarrow{\partial} \bigsqcup_{x \in X^{(1)}} H^*_{x+1}(X, \mathbb{Q}/\mathbb{Z}(j)),
\]

where \( \partial = \bigsqcup \partial_x \), is exact.

**Proposition A.11.** Let \( X \) be a smooth variety over \( F \) and \( x \in X \). The sequence

\[
0 \to H^*(O_{X,x}, \mathbb{Q}/\mathbb{Z}(j))) \to H^*(F(X), \mathbb{Q}/\mathbb{Z}(j)) \xrightarrow{\partial} \bigsqcup_{x' \in X^{(1)}} H^*_{x'+1}(X, \mathbb{Q}/\mathbb{Z}(j))
\]

is exact.
Appendix B: Spectral sequences

B-I: Hochschild–Serre spectral sequence. Let
\[ \mathcal{A} \xrightarrow{W} \mathcal{B} \xrightarrow{V} \mathcal{C} \]
be additive left exact functors between abelian categories with enough injective objects. If \( W \) takes injective objects to \( V \)-acyclic ones, there is a spectral sequence
\[ E_2^{p,q} = R^p V(R^q W(A)) \Rightarrow R^{p+q}(VW)(A) \]
for every complex \( A \) in \( \mathcal{A} \) bounded from below.

We have exact triangles in the derived category of \( \mathcal{B} \):
\[ \tau_{\leq n}RW(A) \to RW(A) \to \tau_{\geq n+1}RW(A) \to \tau_{\leq n}RW(A)[1], \quad (B-1) \]
\[ \tau_{\leq n-1}RW(A) \to \tau_{\leq n}RW(A) \to R^nW(A)[-n] \to \tau_{\leq n-1}RW(A)[1]. \quad (B-2) \]

The filtration on \( R^n(VW)(A) \) is defined by
\[ F^jR^n(VW)(A) = \text{Im}(R^nV(\tau_{\leq n-j}RW(A)) \to R^nV(RW(A)) = R^n(VW)(A)). \]

As \( \tau_{\geq n+1}RW(A) \) is acyclic in degrees less than or equal to \( n \), the morphism
\[ R^nV(\tau_{\leq n}RW(A)) \to R^nV(RW(A)) = R^n(VW)(A) \]
is an isomorphism, in particular, \( F^0R^n(VW)(A) = R^n(VW)(A) \).

The edge homomorphism is defined as the composition
\[ R^n(VW)(A) \xrightarrow{\sim} R^nV(\tau_{\leq n}RW(A)) \to R^nV(R^nW(A)[-n]) = V(R^nW(A)). \]

Moreover, the kernel \( F^1R^n(VW)(A) \) of the edge homomorphism is isomorphic to \( R^nV(\tau_{\leq n-1}RW(A)) \). We define the morphism \( d_n \) as the composition
\[ d_n : F^1R^n(VW)(A) \to R^nV(R^{n-1}W(A)[-n+1]) = R^1V(R^{n-1}W(A)) = E_2^{1,n-1}. \]

B-II: First spectral sequence. Let \( X \) be a smooth variety over a field \( F \). We have the functors
\[ \text{Sheaves}_{\acute{e}t}(X) \xrightarrow{q_*} \text{Sheaves}_{\acute{e}t}(F) \xrightarrow{V} Ab, \]
where \( q_* \) is the push-forward map for the structure morphism \( q : X \to \text{Spec}(F) \) and \( V(M) = H^0(F, M) \).

Consider the Hochschild–Serre spectral sequence
\[ E_2^{p,q} = H^p(F, H^q(X_{\text{sep}}, \mathbb{Z}(2))) \Rightarrow H^{p+q}(X, \mathbb{Z}(2)). \quad (B-3) \]

Set \( \Delta(i) := Rq_*(\mathbb{Z}(i)) \) for \( i = 1 \) or 2. Then \( \Delta(i) \) is the complex of étale sheaves on \( F \) concentrated in degrees \( \geq 1 \). The \( j \)-th term \( F^jH^n(X, \mathbb{Z}(i)) \) of the filtration
on $H^n(X, \mathbb{Z}(i))$ coincides with the image of the canonical homomorphism

$$H^n(F, \tau_{\leq (n-j)} \Delta(i)) \to H^n(F, \Delta(i)) = H^n(X, \mathbb{Z}(i)).$$

Let $M$ be a $\Gamma$-lattice viewed as an étale sheaf over $F$. Note that there are canonical isomorphisms

$$H^*(F, M^\circ \otimes \Delta(i)) = \Ext^*_F(M, \Delta(i)) = \Ext^*_X(q^*M, \mathbb{Z}(i)), \quad (B-4)$$

where $M^\circ := \Hom(M, \mathbb{Z})$ is the dual lattice.

Consider also the following product map:

$$\mathbb{Z}(1) \otimes^L \Delta(1) \to Rq_*(q^* \mathbb{Z}(1) \otimes^L \mathbb{Z}(1)) \to Rq_*(\mathbb{Z}(1) \otimes^L \mathbb{Z}(1)) \to Rq_*(\mathbb{Z}(2)).$$

The complex $\mathbb{Z}(1) \otimes^L \tau_{\leq 2} \Delta(1)$ is trivial in degrees greater than 3, hence we have a commutative diagram

$$\begin{array}{ccc}
\mathbb{Z}(1) \otimes^L \tau_{\leq 2} \Delta(1) & \overset{\text{prod}}{\longrightarrow} & \tau_{\leq 3} Rq_*(\mathbb{Z}(2)) = \Delta(2) \\
\downarrow & & \downarrow \\
\mathbb{Z}(1) \otimes^L \Delta(1) & \overset{\text{prod}}{\longrightarrow} & Rq_*(\mathbb{Z}(2)) = \Delta(2).
\end{array}$$

There are canonical morphisms from (B-2):

$$h_2 : \tau_{\leq 2} \Delta(1)[2] \to H^2(X_{\text{sep}}, \mathbb{Z}(1)) \quad \text{and} \quad h_3 : \tau_{\leq 3} \Delta(2)[3] \to H^3(X_{\text{sep}}, \mathbb{Z}(2)).$$

Consider an element

$$\delta \in H^1(F, M \otimes F_{\text{sep}}^\times) = \Ext^1_F(M^\circ, \mathbb{G}_m, F) = \Ext^2_F(M^\circ, \mathbb{Z}(1)), $$

and view $\delta$ as a morphism $\delta : M^\circ \to \mathbb{Z}(1)[2]$ in $D^+(\text{Sheaves}_{\text{ét}}(F))$.

The following diagram

$$\begin{array}{ccc}
M^\circ \otimes \Delta(1)[2] & \overset{\delta \otimes 1}{\longrightarrow} & \mathbb{Z}(1) \otimes^L \Delta(1)[4] & \overset{\text{prod}}{\longrightarrow} & \Delta(2)[4] \\
(1 \otimes i_2)[2] & & (1 \otimes i_2)[4] & & (i_3)[4] \\
M^\circ \otimes \tau_{\leq 2} \Delta(1)[2] & \overset{\delta \otimes 1}{\longrightarrow} & \mathbb{Z}(1) \otimes^L \tau_{\leq 2} \Delta(1)[4] & \overset{\text{prod}}{\longrightarrow} & \tau_{\leq 3} \Delta(2)[4] \\
1 \otimes h_2 & & 1 \otimes h_2 & & h_3 \\
M^\circ \otimes H^2(X_{\text{sep}}, \mathbb{Z}(1)) & \overset{\delta \otimes 1}{\longrightarrow} & \mathbb{Z}(1) \otimes^L H^2(X_{\text{sep}}, \mathbb{Z}(1))[2] & \overset{\text{prod}}{\longrightarrow} & H^3(X_{\text{sep}}, \mathbb{Z}(2))[1]
\end{array}$$

where $i_2 : \tau_{\leq 2} \Delta(1) \to \Delta(1)$ and $i_3 : \tau_{\leq 3} \Delta(2) \to \Delta(2)$ are natural morphisms, is commutative.

By (B-4), we have

$$H^0(F, M^\circ \otimes \Delta(1)[2]) = \Ext^2_F(M, \Delta(1)) = \Ext^2_X(q^*M, \mathbb{Z}(1)).$$
Furthermore, the diagram above yields a commutative square

\[
\begin{array}{c}
\text{Ext}^2_X(q^*M, \mathbb{Z}(1)) \\
\downarrow d_2 \\
\text{Hom}_\Gamma(M, H^2(X_{\text{sep}}, \mathbb{Z}(1))) \\
\downarrow j \\
\text{Hom}_\Gamma(M, H^1(F, H^3(X_{\text{sep}}, \mathbb{Z}(2))))
\end{array}
\]

where \(d_2\) is the edge map coming from the spectral sequence

\[
\text{Ext}^p_F(M, H^q(X_{\text{sep}}, \mathbb{Z}(1))) \Rightarrow \text{Ext}^{p+q}_X(q^*M, \mathbb{Z}(1))
\]

and \(j\) coincides with the composition

\[
\text{Hom}_\Gamma(M, H^2(X_{\text{sep}}, \mathbb{Z}(1))) = H^0(F, M^\circ \otimes H^2(X_{\text{sep}}, \mathbb{Z}(1)))
\]

with \(\rho\) given by the product map.

Now suppose the group \(H^2(X_{\text{sep}}, \mathbb{Z}(1))\), which is canonically isomorphic to \(\text{Pic}(X_{\text{sep}})\), is a lattice. Let \(M = \text{Pic}(X_{\text{sep}})\) and consider the torus \(T\) over \(F\) with \(\hat{T}_{\text{sep}} = M\). It follows that

\[
\delta \in H^1(F, T^\circ) = H^1(F, \hat{T}_{\text{sep}} \otimes F_{\text{sep}}^\times) = H^2(F, \hat{T}_{\text{sep}} \otimes \mathbb{Z}(1)),
\]

where \(T^\circ\) is the dual torus. Note that \(\delta \cup 1_M = \delta\), where

\[
1_M \in H^0(F, M^\circ \otimes H^2(X_{\text{sep}}, \mathbb{Z}(1))) = \text{End}_\Gamma(M)
\]

is the identity.

The top map in the last diagram is given by the pairing

\[
H^1(X, T^0) \otimes H^1(X, T) \rightarrow F^1 H^4(X, \mathbb{Z}(2)),
\]

\[
a \otimes b \mapsto a \cup b,
\]

defined as the cup-product in (4-3),

\[
H^2(X, \hat{T}(1)) \otimes H^2(X, \hat{T}^\circ(1)) \rightarrow F^1 H^4(X, \mathbb{Z}(2)),
\]

if we identify \(\text{Ext}^1_X(q^*M, \mathbb{Z}(1))\) with \(H^2(X, \hat{T}^\circ(1)) = H^1(X, T)\).

In this case, the homomorphism

\[
\rho : H^1(F, T^\circ) \rightarrow H^1(F, H^3(X_{\text{sep}}, \mathbb{Z}(2)))
\]

is given by the product homomorphism

\[
T^\circ(F_{\text{sep}}) = F_{\text{sep}}^\times \otimes \hat{T}_{\text{sep}} = F_{\text{sep}}^\times \otimes \text{Pic}(X_{\text{sep}}) \rightarrow H^3(X_{\text{sep}}, \mathbb{Z}(2)).
\]
A $T$-torsor $E \to X$ is called universal if the class of $E$ in

$$H^1(X, T) = \text{Ext}_X^2(q^*M, \mathbb{Z}(1))$$

satisfies $d_2([E]) = 1_M$; see [Colliot-Thélène and Sansuc 1987a].

Commutativity of the previous diagram gives:

**Proposition B.1.** Let $X$ be a smooth variety over $F$ such that $\text{Pic}(X_{\text{sep}})$ is a lattice. Let $T$ be the torus over $F$ satisfying $T_{\text{sep}} = \text{Pic}(X_{\text{sep}})$ and let $E$ be a universal $T$-torsor over $X$ with the class $[E] \in H^1(X, T)$. Then for every $\delta \in H^1(F, T^\circ)$, we have

$$d_4(q^*(\delta) \cup [E]) = \rho(\delta),$$

where $d_4 : F^1H^4(X, \mathbb{Z}(2)) \to H^1(F, H^3(X_{\text{sep}}, \mathbb{Z}(2)))$ is the map induced by the Hochschild–Serre spectral sequence (B-3) and the cup-product is taken for the pairing (B-6).

**B-III: Second spectral sequence.** We assume that $H^3(X_{\text{sep}}, \mathbb{Z}(2)) = 0$, hence in particular $E^{0,3}_2 = 0$ in the spectral sequence (B-3) and so $E^{2,2}_\infty \subset E^{2,2}_2$. Therefore, we have a canonical map

$$e_4 : F^2H^4(X, \mathbb{Z}(2)) \to E^{2,2}_\infty \hookrightarrow E^{2,2}_2 = H^2(F, H^2(X_{\text{sep}}, \mathbb{Z}(2))).$$

Let $N$ be a $\Gamma$-lattice. Consider an element

$$\gamma \in H^2(F, N \otimes F_{\text{sep}}^\times) = \text{Ext}^2_F(N^\circ, \mathbb{G}_m, F) = \text{Ext}^3_F(N^\circ, \mathbb{Z}(1)),$$

and view $\gamma$ as a morphism $\gamma : N^\circ \to \mathbb{Z}(1)[3]$ in $D^+(\text{Sheaves}_{\text{ét}}(F))$.

As above, the commutative diagram

$$
\begin{array}{cccc}
N^\circ \otimes \Delta(1)[1] & \gamma \otimes 1 & \mathbb{Z}(1) \otimes L \Delta(1)[4] & \text{prod} \\
\downarrow (1 \otimes \iota_1)[1] & & \downarrow (1 \otimes \iota_1)[4] & \downarrow (i_2)[4] \\
N^\circ \otimes \tau_{\leq 1} \Delta(1)[1] & \gamma \otimes 1 & \mathbb{Z}(1) \otimes L \tau_{\leq 1} \Delta(1)[4] & \text{prod} \\
\downarrow 1 \otimes h_1 & & \downarrow 1 \otimes h_1 & \downarrow h_2 \\
N^\circ \otimes H^1(X_{\text{sep}}, \mathbb{Z}(1)) & \gamma \otimes 1 & \mathbb{Z}(1) \otimes L H^1(X_{\text{sep}}, \mathbb{Z}(1))[3] & \text{prod} \\
\downarrow i_1 & & \downarrow i_2 & \\
N^\circ \otimes H^1(X_{\text{sep}}, \mathbb{Z}(1)) & \gamma \otimes 1 & \mathbb{Z}(1) \otimes L H^1(X_{\text{sep}}, \mathbb{Z}(1))[3] & \text{prod} \\
\downarrow k & & \downarrow k & \\
\text{Ext}^1_X(q^*N, \mathbb{Z}(1)) & q^*(\gamma) \cup - & F^2H^4(X, \mathbb{Z}(2)) & \text{Hom}_{\Gamma}(N, H^1(X_{\text{sep}}, \mathbb{Z}(1))) \\
\downarrow d_1 & & \downarrow e_4 & \\
\text{Hom}_{\Gamma}(N, H^1(X_{\text{sep}}, \mathbb{Z}(1))) & k & H^2(F, H^2(X_{\text{sep}}, \mathbb{Z}(2)))
\end{array}
$$

where $i_1, i_2, h_1$ and $h_2$ are defined in a similar fashion as in Section B-II, yields a commutative square

$$
\begin{array}{ccc}
\text{Ext}^1_X(q^*N, \mathbb{Z}(1)) & q^*(\gamma) \cup - & F^2H^4(X, \mathbb{Z}(2)) \\
\downarrow d_1 & & \downarrow e_4 \\
\text{Hom}_{\Gamma}(N, H^1(X_{\text{sep}}, \mathbb{Z}(1))) & k & H^2(F, H^2(X_{\text{sep}}, \mathbb{Z}(2)))
\end{array}
$$
where \( d_1 \) is the edge map coming from the spectral sequence

\[
\text{Ext}^0_F(N, H^q(X_{\text{sep}}, \mathbb{Z}(1))) \Rightarrow \text{Ext}^{q+q}_X(q^*N, \mathbb{Z}(1))
\]

and \( k \) coincides with the composition

\[
\text{Hom}_0(N, H^1(X_{\text{sep}}, \mathbb{Z}(1))) = \text{H}^0(F, N^0 \otimes H^1(X_{\text{sep}}, \mathbb{Z}(1)))
\]

\[
\mu(x) = \gamma \cup x,
\]

\[
\text{with the last homomorphism given by the product map.}
\]

Suppose \( N \) is a \( \Gamma \)-lattice in \( F_{\text{sep}}[X]^\times \) such that the composition

\[
N \hookrightarrow F_{\text{sep}}[X]^\times \to F_{\text{sep}}[X]^\times / F_{\text{sep}}^\times
\]

is an isomorphism. Consider the torus \( Q \) with \( \hat{Q}_{\text{sep}} = N \), so that \( \gamma \in H^2(F, Q^\circ) \).

Note that \( \gamma \cup i_N = \gamma \), where

\[
i_N \in H^0(F, N^0 \otimes H^1(X_{\text{sep}}, \mathbb{Z}(1))) = \text{Hom}_\Gamma(N, F_{\text{sep}}[X]^\times)
\]

is the embedding.

The top map in the previous diagram is given by the pairing

\[
H^2(X, Q^0) \otimes H^0(X, Q) \to F^2 H^4(X, \mathbb{Z}(2)),
\]

\[
a \otimes b \mapsto a \cup b,
\]

defined as the cup-product in (4-3),

\[
H^3(X, \hat{Q}^\circ(1)) \otimes H^1(X, \hat{Q}^\circ(1)) \to H^4(X, \mathbb{Z}(2)),
\]

if we identify \( \text{Ext}^1_X(q^*N, \mathbb{Z}(1)) \) with \( H^1(X, \hat{Q}^\circ(1)) = H^0(X, Q) \).

The inclusion of \( \hat{Q}_{\text{sep}} \) into \( F_{\text{sep}}[X]^\times \) yields a morphism \( \varepsilon : X \to Q \) that can be viewed as an element of \( H^0(X, Q) \). Consider the map

\[
\mu : H^2(F, Q^\circ) \to H^2(F, H^2(X_{\text{sep}}, \mathbb{Z}(2)))
\]

given by composition with the product homomorphism

\[
Q^\circ(F_{\text{sep}}) = F_{\text{sep}}^\times \otimes \hat{Q}_{\text{sep}} \to F_{\text{sep}}^\times \otimes H^1(X_{\text{sep}}, \mathbb{Z}(1)) \to H^2(X_{\text{sep}}, \mathbb{Z}(2)).
\]

We have proved:

**Proposition B.2.** Let \( X \) be a smooth variety over \( F \) such that \( H^3(X_{\text{sep}}, \mathbb{Z}(2)) = 0 \).

Let \( N \) be a \( \Gamma \)-lattice in \( F_{\text{sep}}[X]^\times \) such that the composition

\[
N \hookrightarrow F_{\text{sep}}[X]^\times \to F_{\text{sep}}[X]^\times / F_{\text{sep}}^\times
\]
is an isomorphism. Let $Q$ be the torus over $F$ satisfying $\hat{Q}_{\text{sep}} = N$. Then for every $\gamma \in H^2(F, Q^o)$, we have

$$e_4(q^*(\gamma) \cup \varepsilon) = \mu(\gamma),$$

where $e_4 : F^2H^4(X, \mathbb{Z}(2)) \to H^2(F, H^2(X_{\text{sep}}, \mathbb{Z}(2)))$ is the map induced by the Hochschild–Serre spectral sequence (B-3) and the cup-product is taken for the pairing (B-8).

**B-IV: Relative étale cohomology.** Let $X$ be a smooth variety over $F$. Following B. Kahn [1996, §3], we define the relative étale cohomology groups as follows. Recall that $\Delta(i) = Rq_*(\mathcal{Z}(i))$ for $i = 1$ and 2, where $q : X \to \text{Spec}(F)$ is the structure morphism, and let $\Delta'(i)$ be the cone of the natural morphism $\mathcal{Z}(i) \to \Delta(i)$ in $D_+(\text{Sheaves}_{\text{ét}}(F))$. Define

$$H^*(X/F, \mathbb{Z}(2)) := H^*(F, \Delta'(2)).$$

(Note that our indexing is different from that in [Kahn 1996, §3].)

There is an infinite exact sequence

$$\cdots \to H^i(F, \mathbb{Z}(2)) \to H^i(X, \mathbb{Z}(2)) \to H^i(X/F, \mathbb{Z}(2)) \to H^{i+1}(F, \mathbb{Z}(2)) \to \cdots$$

If $X$ has a rational point, we have

$$H^i(X/F, \mathbb{Z}(2)) = H^i(X, \mathbb{Z}(2))/H^i(F, \mathbb{Z}(2)).$$

There is a Hochschild–Serre type spectral sequence [Kahn 1996, §3]

$$(B-10) \quad E_2^{p,q} = H^p(F, H^q(X_{\text{sep}}/F_{\text{sep}}, \mathbb{Z}(2))) \Rightarrow H^{p+q}(X/F, \mathbb{Z}(2)),$$

and we have by [Kahn 1996, Lemma 3.1] that

$$H^q(X_{\text{sep}}/F_{\text{sep}}, \mathbb{Z}(2)) = \begin{cases} 0 & \text{if } q \leq 0, \\ \text{uniquely divisible group} & \text{if } q = 1, \\ H^0_{\text{Zar}}(X_{\text{sep}}, \mathcal{O}_2) & \text{if } q = 2, \\ H^1_{\text{Zar}}(X_{\text{sep}}, \mathcal{O}_2) & \text{if } q = 3. \end{cases}$$

It follows that $E_2^{p,q} = 0$ if $q \leq 1$ and $p > 0$. Comparing the spectral sequences (B-3) and (B-10), by Proposition B.1 we have:

**Proposition B.3.** Let $X$ be a smooth variety over $F$ such that $X(F) \neq \emptyset$. If $H^0_{\text{Zar}}(X_{\text{sep}}, \mathcal{O}_2) = K_2(F_{\text{sep}})$, then the spectral sequence (B-10) yields an exact sequence

$$0 \to H^1(F, H^1_{\text{Zar}}(X_{\text{sep}}, \mathcal{O}_2)) \xrightarrow{\alpha} H^4(X, \mathbb{Z}(2)) \to H^4(X_{\text{sep}}, \mathbb{Z}(2))^\Gamma \to H^2(F, H^1_{\text{Zar}}(X_{\text{sep}}, \mathcal{O}_2)).$$
If, moreover, the group $\text{Pic}(X_{\text{sep}})$ is a lattice and $T$ is the torus over $F$ such that $\hat{T}_{\text{sep}} = \text{Pic}(X_{\text{sep}})$, then $\alpha(\rho(\delta)) = q^*(\delta) \cup [E]$ for every $\delta \in H^1(F, T^\circ)$, where $\rho$ is defined in (B-7) and $E$ is a universal $T$-torsor over $X$.

Comparing the spectral sequences (B-3) and (B-10), by Proposition B.2 we have:

**Proposition B.4.** Let $X$ be a smooth variety over $F$ such that $X(F) \neq \emptyset$. If $H^1_{\text{Zar}}(X_{\text{sep}}, \mathcal{H}_2) = 0$, then the spectral sequence (B-10) yields an exact sequence

$$0 \to H^2(F, \overline{H}^0_{\text{Zar}}(X_{\text{sep}}, \mathcal{H}_2)) \xrightarrow{\beta} \overline{H}^4(X, \mathbb{Z}(2)) \to \overline{H}^4(X_{\text{sep}}, \mathbb{Z}(2))^\Gamma \to H^3(F, \overline{H}^0_{\text{Zar}}(X_{\text{sep}}, \mathcal{H}_2)).$$

If $N$ is a $\Gamma$-lattice in $F_{\text{sep}}[X]^\times$ such that the composition

$$N \hookrightarrow F_{\text{sep}}[X]^\times \to F_{\text{sep}}[X]^\times / F_{\text{sep}}^\times$$

is an isomorphism and $Q$ is the torus over $F$ satisfying $\hat{Q}_{\text{sep}} = N$, then $\beta(\mu(\gamma)) = q^*(\gamma) \cup \epsilon$ for every $\gamma \in H^2(F, Q^\circ)$, where $\mu$ is defined in (B-9) and $\epsilon \in H^0(X, Q)$ is given by the inclusion of $\hat{Q}_{\text{sep}}$ into $F_{\text{sep}}[X]^\times$.

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Cohomological invariants of algebraic tori


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