Algebra & Number Theory

Volume 7

2013

No. 8

Kernels for products of L-functions

Nikolaos Diamantis and Cormac O'Sullivan

msp

ALGEBRA AND NUMBER THEORY 7:8 (2013)
dx.doi.org/10.2140/ant.2013.7.1883

Kernels for products of L-functions

Nikolaos Diamantis and Cormac O'Sullivan

The Rankin–Cohen bracket of two Eisenstein series provides a kernel yielding products of the periods of Hecke eigenforms at critical values. Extending this idea leads to a new type of Eisenstein series built with a double sum. We develop the properties of these series and their nonholomorphic analogs and show their connection to values of L-functions outside the critical strip.

1. Introduction

Rankin [1952] introduced the fruitful idea of expressing the product of two critical values of the L-function of a weight-k Hecke eigenform f for $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ in terms of the Petersson scalar product of f and a product of Eisenstein series:

$$\langle E_{k_1}E_{k_2}, f\rangle = (-1)^{k_1/2}2^{3-k}\frac{k_1k_2}{B_{k_1}B_{k_2}}L^*(f,1)L^*(f,k_2) \tag{1-1}$$

for $k=k_1+k_2$, the Bernoulli numbers B_j and the completed, entire L-function of f,

$$L^*(f,s) := \frac{\Gamma(s)}{(2\pi)^s}\sum_{m=1}^{\infty}\frac{a_f(m)}{m^s} = \int_0^\infty f(iy)y^{s-1}\,dy.$$

Zagier [1977, p. 149] extended (1-1) to get

$$\langle [E_{k_1},E_{k_2}]_n, f\rangle = (-1)^{k_1/2}(2\pi i)^n 2^{3-k}\binom{k-2}{n}\frac{k_1k_2}{B_{k_1}B_{k_2}}L^*(f,n+1)L^*(f,n+k_2), \tag{1-2}$$

where $k=k_1+k_2+2n$ and $[g_1,g_2]_n$ stands for the *Rankin–Cohen bracket of index n* given by

$$[g_1,g_2]_n := \sum_{r=0}^{n}(-1)^r\binom{k_1+n-1}{n-r}\binom{k_2+n-1}{r}g_1^{(r)}g_2^{(n-r)}. \tag{1-3}$$

The *periods* of f in the critical strip are the numbers

$$L^*(f,1), L^*(f,2), \dots, L^*(f,k-1). \tag{1-4}$$

MSC2010: primary 11F67; secondary 11F03, 11F37.
Keywords: L-functions, noncritical values, Rankin–Cohen brackets, Eichler–Shimura–Manin theory.

Zagier [1977, §5] and Kohnen and Zagier [1984] proved important results of the Eichler–Shimura–Manin theory on the algebraicity of these critical values using (1-2). We describe this in more depth in Sections 2C and 8A.

On the face of it, the techniques of [Zagier 1977], employing (1-2), apply only to critical values; an extension to noncritical values, $L^*(f, j)$ for integers $j \leqslant 0$ or $j \geq k$, would seem to require Rankin–Cohen brackets of negative index n or holomorphic Eisenstein series of negative weight, neither of which are defined. Analyzing the structure of the Rankin–Cohen bracket of two Eisenstein series in Section 6 reveals a natural construction, which we call a *double Eisenstein series*:[1]

$$\sum_{\substack{\gamma,\delta\in\Gamma_\infty\backslash\Gamma \\ \gamma\delta^{-1}\not\in\Gamma_\infty}} (c_{\gamma\delta^{-1}})^l j(\gamma, z)^{-k_1} j(\delta, z)^{-k_2}, \tag{1-5}$$

where, for $\gamma \in \Gamma$, we write

$$\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \quad \text{and} \quad j(\gamma, z) := c_\gamma z + d_\gamma.$$

By comparison, the usual holomorphic Eisenstein series is

$$E_k(z) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j(\gamma, z)^{-k}. \tag{1-6}$$

The double Eisenstein series (1-5) converges to a weight-$(k_1 + k_2)$ cusp form when $l < k_1 - 2, k_2 - 2$. For negative integers l, it behaves as a Rankin–Cohen bracket of negative index; see Proposition 2.4. This allows us to further generalize (1-1) and (1-2), and in Section 8, we characterize the field containing an arbitrary value of an L-function in terms of double Eisenstein series and their Fourier coefficients. In the interesting paper [Cohen et al. 1997], Rankin–Cohen brackets are linked to operations on automorphic pseudodifferential operators and may also be reinterpreted in this framework allowing for more general indices.

An extension of Zagier's kernel formula (1-2) in the nonholomorphic direction is given in Section 9C. There we show that the holomorphic double Eisenstein series have nonholomorphic counterparts:

$$\sum_{\substack{\gamma,\delta\in\Gamma_\infty\backslash\Gamma \\ \gamma\delta^{-1}\not\in\Gamma_\infty}} |c_{\gamma\delta^{-1}}|^{-s-s'} \operatorname{Im}(\gamma z)^s \operatorname{Im}(\delta z)^{s'}. \tag{1-7}$$

These weight-0 functions possess analytic continuations and functional equations resembling those for the classical nonholomorphic Eisenstein series. As kernels, they produce products of L-functions for *Maass cusp forms*; see Theorem 2.9. The main motivation for this construction was its potential use in the rapidly developing

[1]In the context of multiple zeta functions, the authors in [Gangl et al. 2006] give a different definition of "double Eisenstein series". See also [Deninger 1995], for example, for distinct "double Eisenstein–Kronecker series".

study of periods of Maass forms [Bruggeman et al. 2013; Lewis and Zagier 2001; Manin 2010; Mühlenbruch 2006]. In developing the properties of (1-7), we require a certain kernel $\mathcal{K}(z; s, s')$ as defined in (9-1). It is interesting to note that Diaconu and Goldfeld [2007] needed exactly the same series for their results on second moments of $L^*(f, s)$; see Section 9A.

2. Statement of main results

2A. *Preliminaries.* Our notation is as in [Diamantis and O'Sullivan 2010]. In all sections but two, Γ is the modular group $\mathrm{SL}(2, \mathbb{Z})$ acting on the upper half-plane $\mathbb{H}$. The definitions we give for double Eisenstein series extend easily to more general groups, so in Section 4, we prove their basic properties for Γ an arbitrary Fuchsian group of the first kind, and in Section 10, we see how some of our main results are valid in this general context.

Let $S_k(\Gamma)$ be the $\mathbb{C}$-vector space of holomorphic, weight-k cusp forms for Γ and $M_k(\Gamma)$ the space of modular forms. These spaces are acted on by the Hecke operators T_m; see (3-6). Let $\mathcal{B}_k$ be the unique basis of S_k consisting of Hecke eigenforms normalized to have first Fourier coefficient 1. We assume throughout this paper that $f \in \mathcal{B}_k$. Since $\langle T_m f, f\rangle = \langle f, T_m f\rangle$, it follows that all the Fourier coefficients of f are real, and hence, $\overline{L^*(f, s)} = L^*(f, \bar{s})$. Also, recall the functional equation

$$L^*(f, k-s) = (-1)^{k/2} L^*(f, s). \tag{2-1}$$

We summarize some standard properties of the nonholomorphic Eisenstein series; see for example [Iwaniec 2002, Chapters 3 and 6]. Throughout this paper, we use the variables $z = x + iy \in \mathbb{H}$ and $s = \sigma + it \in \mathbb{C}$.

Definition 2.1. For $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the weight-0, *nonholomorphic Eisenstein series* is

$$E(z, s) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma z)^s = \frac{y^s}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} |cz+d|^{-2s}. \tag{2-2}$$

Let $\theta(s) := \pi^{-s}\Gamma(s)\zeta(2s)$. Then $E(z, s)$ has a Fourier expansion [Iwaniec 2002, Theorem 3.4], which we may write in the form

$$E(z, s) = y^s + \frac{\theta(1-s)}{\theta(s)} y^{1-s} + \sum_{m \neq 0} \phi(m, s)|m|^{-1/2} W_s(mz), \tag{2-3}$$

where $W_s(mz) = 2(|m|y)^{1/2} K_{s-1/2}(2\pi|m|y)e^{2\pi i m x}$ is the Whittaker function for $z \in \mathbb{H}$ and also $\theta(s)\phi(m, s) = \sigma_{2s-1}(|m|)|m|^{1/2-s}$. As usual, $\sigma_s(m) := \sum_{d|m} d^s$ is the divisor function.

For the weight-k ($k \in 2\mathbb{Z}$) nonholomorphic Eisenstein series, generalizing (2-2), set

$$E_k(z,s) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma z)^s \left(\frac{j(\gamma,z)}{|j(\gamma,z)|} \right)^{-k}. \tag{2-4}$$

Then (2-4) converges to an analytic function of $s \in \mathbb{C}$ and a smooth function of $z \in \mathbb{H}$ for $\mathrm{Re}(s) > 1$. Also $y^{-k/2} E_k(z,s)$ has weight k in z. Define the *completed nonholomorphic Eisenstein series* as

$$E^*_k(z,s) := \theta_k(s) E_k(z,s) \quad \text{for } \theta_k(s) := \pi^{-s}\Gamma(s+|k|/2)\zeta(2s). \tag{2-5}$$

With (2-3), we see that $E(z,s)$ has a meromorphic continuation to all $s \in \mathbb{C}$. The same is true of $E_k(z,s)$; see [Diamantis and O'Sullivan 2010, §2.1] for example. We have the functional equations

$$\theta(s/2) = \theta((1-s)/2), \tag{2-6}$$

$$E^*_k(z,s) = E^*_k(z,1-s). \tag{2-7}$$

2B. ***Holomorphic double Eisenstein series.*** Define the subgroup

$$B := \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \,\middle|\, n \in \mathbb{Z} \right\} \subset \mathrm{SL}(2,\mathbb{Z}). \tag{2-8}$$

Then Γ_∞, the subgroup of $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ fixing ∞, is $B \cup -B$. For $\gamma \in \Gamma_\infty \backslash \Gamma$, the quantities c_γ, d_γ and $j(\gamma,z)$ are only defined up to sign (though even powers are well-defined). For $\gamma \in B \backslash \Gamma$, there is no ambiguity in the signs of c_γ, d_γ and $j(\gamma,z)$.

Definition 2.2. Let $z \in \mathbb{H}$ and $w \in \mathbb{C}$. For integers $k_1, k_2 \geq 3$, we define the *double Eisenstein series*

$$\boldsymbol{E}_{k_1,k_2}(z,w) := \sum_{\substack{\gamma,\delta \in B\backslash\Gamma \\ c_{\gamma\delta^{-1}} > 0}} (c_{\gamma\delta^{-1}})^{w-1} j(\gamma,z)^{-k_1} j(\delta,z)^{-k_2}. \tag{2-9}$$

This series is well-defined and converges to a holomorphic function of z that is a weight-($k = k_1 + k_2$) cusp form for $\mathrm{Re}(w) < k_1 - 1, k_2 - 1$, as we see in Proposition 4.2. It vanishes identically when k_1 and k_2 have different parity.

Let k be even. To get the most general kernel, with $s \in \mathbb{C}$ set

$$\boldsymbol{E}_{s,k-s}(z,w) := \sum_{\substack{\gamma,\delta \in B\backslash\Gamma \\ c_{\gamma\delta^{-1}} > 0}} (c_{\gamma\delta^{-1}})^{w-1} \left(\frac{j(\gamma,z)}{j(\delta,z)} \right)^{-s} j(\delta,z)^{-k}. \tag{2-10}$$

In the usual convention, for $\rho \in \mathbb{C}$ with $\rho \neq 0$, write

$$\rho = |\rho| e^{i \arg(\rho)} \quad \text{for } -\pi < \arg(\rho) \leqslant \pi$$

and

$$\rho^s = |\rho|^s e^{i \arg(\rho) s} \quad \text{for } s \in \mathbb{C}. \tag{2-11}$$

Note that

$$c_{\gamma\delta^{-1}} = \begin{vmatrix} c_\gamma & d_\gamma \\ c_\delta & d_\delta \end{vmatrix} > 0 \implies \frac{j(\gamma, z)}{j(\delta, z)} \in \mathbb{H} \quad \text{for } z \in \mathbb{H},$$

and so $(j(\gamma, z)/j(\delta, z))^{-s}$ in (2-10) is well-defined and a holomorphic function of $s \in \mathbb{C}$ and $z \in \mathbb{H}$. Proposition 4.2 shows that $\boldsymbol{E}_{s,k-s}(z, w)$ converges absolutely and uniformly on compact sets for which $2 < \sigma < k-2$ and $\mathrm{Re}(w) < \sigma - 1, k-1-\sigma$.

Define the *completed double Eisenstein series* as

$$\boldsymbol{E}^*_{s,k-s}(z, w) \tag{2-12}$$
$$:= \left[\frac{e^{si\pi/2}\Gamma(s)\Gamma(k-s)\Gamma(k-w)\zeta(1-w+s)\zeta(1-w+k-s)}{2^{3-w}\pi^{k+1-w}\Gamma(k-1)}\right]\boldsymbol{E}_{s,k-s}(z, w).$$

Theorem 2.3. *Let $k \geq 6$ be even. The series $\boldsymbol{E}^*_{s,k-s}(z, w)$ has an analytic continuation to all $s, w \in \mathbb{C}$ and as a function of z is always in $S_k(\Gamma)$. For any f in $\mathcal{B}_k$, we have*

$$\langle \boldsymbol{E}^*_{s,k-s}(\,\cdot\,, w), f\rangle = L^*(f, s)L^*(f, w). \tag{2-13}$$

It follows directly from (2-13) and (2-1) that $\boldsymbol{E}^*_{s,k-s}(z, w)$ satisfies eight functional equations generated by

$$\boldsymbol{E}^*_{s,k-s}(z, w) = \boldsymbol{E}^*_{w,k-w}(z, s), \tag{2-14}$$

$$\boldsymbol{E}^*_{s,k-s}(z, w) = (-1)^{k/2}\boldsymbol{E}^*_{k-s,s}(z, w). \tag{2-15}$$

The next result shows how $\boldsymbol{E}^*_{s,k-s}$ is a generalization of the Rankin–Cohen bracket $[E_{k_1}, E_{k_2}]_n$.

Proposition 2.4. *For $n \in \mathbb{Z}_{\geq 1}$ and even $k_1, k_2 \geq 4$,*

$$n!\,[E_{k_1}, E_{k_2}]_n = \frac{2(-1)^{k_1/2}\pi^k\Gamma(k-1)}{(2\pi i)^n\zeta(k_1)\zeta(k_2)\Gamma(k_1)\Gamma(k_2)\Gamma(k-n-1)}\boldsymbol{E}^*_{k_1+n,k_2+n}(z, n+1).$$

Another way to understand these double Eisenstein series is through their connections to nonholomorphic Eisenstein series. Any smooth function transforming with weight k and with polynomial growth as $y \to \infty$ may be projected into S_k with respect to the Petersson scalar product. See [Diamantis and O'Sullivan 2010, §3.2] and the contained references. Denote this holomorphic projection by π_{hol}.

Proposition 2.5. *Let $k = k_1 + k_2 \geq 6$ for even $k_1, k_2 \geq 0$. Then for all $s, w \in \mathbb{C}$*

$$\boldsymbol{E}^*_{s,k-s}(z, w) = \pi_{\mathrm{hol}}\big[(-1)^{k_2/2}y^{-k/2}E^*_{k_1}(z, u)E^*_{k_2}(z, v)/(2\pi^{k/2})\big],$$

where

$$u = (s+w-k+1)/2 \quad \textit{and} \quad v = (-s+w+1)/2. \tag{2-16}$$

2C. ***Values of L-functions.*** For $f \in \mathcal{B}_k$, let K_f be the field obtained by adjoining to $\mathbb{Q}$ the Fourier coefficients of f. We will recall Zagier's proof of the next result in Section 8A.

Theorem 2.6 (Manin's periods theorem). *For each $f \in \mathcal{B}_k$ there exist real numbers $\omega_+(f)$, $\omega_-(f)$ such that*

$$L^*(f,s)/\omega_+(f),\ L^*(f,w)/\omega_-(f) \in K_f$$

for all s and w with $1 \leqslant s, w \leqslant k-1$ and s even and w odd.

Let $m \in \mathbb{Z}$ satisfy $m \leqslant 0$ or $m \geq k$. Then for these values outside the critical strip we have, according to [Kontsevich and Zagier 2001, §3.4] and the references therein,

$$L^*(f,m) \in \mathcal{P}[1/\pi],$$

where $\mathcal{P}$ is the ring of periods: complex numbers that may be expressed as an integral of an algebraic function over an algebraic domain. In contrast to the periods (1-4), we do not have much more precise information about the algebraic properties of the values $L^*(f,m)$. A special case of a theorem by Koblic [1975] shows, for example, that

$$L^*(f,m) \notin \mathbb{Z}\cdot L^*(f,1) + \mathbb{Z}\cdot L^*(f,2) + \cdots + \mathbb{Z}\cdot L^*(f,k-1).$$

Let $K(\boldsymbol{E}^*_{s,k-s}(\,\cdot\,,w))$ be the field obtained by adjoining to $\mathbb{Q}$ the Fourier coefficients of $\boldsymbol{E}^*_{s,k-s}(\,\cdot\,,w)$, and let $\omega_+(f)$ and $\omega_-(f)$ be as given in Theorem 2.6. Then we have:

Theorem 2.7. *For all $f \in \mathcal{B}_k$ and $s \in \mathbb{C}$,*

$$\begin{aligned} L^*(f,s)/\omega_+(f) &\in K(\boldsymbol{E}^*_{s,k-s}(\,\cdot\,,k-1))K_f,\\ L^*(f,s)/\omega_-(f) &\in K(\boldsymbol{E}^*_{k-2,2}(\,\cdot\,,s))K_f. \end{aligned}$$

The above theorem gives the link between Fourier coefficients of double Eisenstein series and arbitrary values of L-functions. We hope that this interesting connection will help shed light on $L^*(f,s)$ for s outside the set $\{1,2,\dots,k-1\}$. See the further discussion in Section 8B for the case when $s \in \mathbb{Z}$.

In Section 8C, we also prove results analogous to Theorem 2.7 for the completed L-function of f twisted by $e^{2\pi i m p/q}$ for $p/q \in \mathbb{Q}$:

$$L^*(f,s;p/q) := \frac{\Gamma(s)}{(2\pi)^s}\sum_{m=1}^{\infty}\frac{a_f(m)e^{2\pi i m p/q}}{m^s} = \int_0^\infty f(iy+p/q)y^{s-1}\,dy. \quad (2\text{-}17)$$

2D. ***Nonholomorphic double Eisenstein series.***

Definition 2.8. For $z \in \mathbb{H}$ and $w, s, s' \in \mathbb{C}$, we define the *nonholomorphic double Eisenstein series* as

$$\mathscr{E}(z, w; s, s') := \sum_{\substack{\gamma,\delta \in \Gamma_\infty\backslash\Gamma \\ \gamma\delta^{-1} \neq \Gamma_\infty}} \frac{\operatorname{Im}(\gamma z)^s \operatorname{Im}(\delta z)^{s'}}{|c_{\gamma\delta^{-1}}|^w}. \tag{2-18}$$

A simple comparison with (2-2) shows it is absolutely and uniformly convergent for $\operatorname{Re}(s), \operatorname{Re}(s') > 1$ and $\operatorname{Re}(w) > 0$. (This domain of convergence is improved in Proposition 4.3.) The most symmetric form of (2-18) is when $w = s + s'$. Define

$$\begin{aligned}\mathscr{E}^*(z; s, s') := 4\pi^{-s-s'}\Gamma(s)\Gamma(s')\zeta(3s+s')\zeta(s+3s')\mathscr{E}(z, s+s'; s, s') \\ + 2\theta(s)\theta(s')E(z, s+s').\end{aligned} \tag{2-19}$$

Theorem 2.9. *The completed double Eisenstein series $\mathscr{E}^*(z; s, s')$ has a meromorphic continuation to all $s, s' \in \mathbb{C}$ and satisfies the functional equations*

$$\mathscr{E}^*(z; s, s') = \mathscr{E}^*(z; s', s),$$
$$\mathscr{E}^*(z; s, s') = \mathscr{E}^*(z; 1-s, 1-s').$$

For any even Maass Hecke eigenform u_j,

$$\langle \mathscr{E}^*(z; s, s'), u_j \rangle = L^*(u_j, s+s'-1/2)L^*(u_j, s'-s+1/2).$$

3. Further background results and notation

We need to introduce two more families of modular forms.

Definition 3.1. For $z \in \mathbb{H}$, $k \geq 4$ in $2\mathbb{Z}$ and $m \in \mathbb{Z}_{\geq 0}$, the holomorphic *Poincaré series* is

$$P_k(z; m) := \sum_{\gamma \in \Gamma_\infty\backslash\Gamma} \frac{e^{2\pi i m \gamma z}}{j(\gamma, z)^k} = \frac{1}{2} \sum_{\gamma \in B\backslash\Gamma} \frac{e^{2\pi i m \gamma z}}{j(\gamma, z)^k}. \tag{3-1}$$

For $m \geq 1$, the series $P_k(z; m)$ span $S_k(\Gamma)$. The Eisenstein series $E_k(z) = P_k(z; 0)$ is not a cusp form but is in the space $M_k(\Gamma)$. The second family of modular forms is based on a series due to Cohen [1981].

Definition 3.2. The *generalized Cohen kernel* is given by

$$\mathscr{C}_k(z, s; p/q) := \frac{1}{2} \sum_{\gamma \in \Gamma} (\gamma z + p/q)^{-s} j(\gamma, z)^{-k} \tag{3-2}$$

for $p/q \in \mathbb{Q}$ and $s \in \mathbb{C}$ with $1 < \operatorname{Re}(s) < k-1$.

In [Diamantis and O'Sullivan 2010, §5], we studied $\mathscr{C}_k(z, s; p/q)$ (the factor $1/2$ is included to keep the notation consistent with that article, where $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$). We showed that, for each $s \in \mathbb{C}$ with $1 < \mathrm{Re}(s) < k-1$, $\mathscr{C}_k(z, s; p/q)$ converges to an element of $S_k(\Gamma)$ with a meromorphic continuation to all $s \in \mathbb{C}$. From Proposition 5.4 of the same work, we have

$$\langle \mathscr{C}_k(\cdot\,, s; p/q), f\rangle = 2^{2-k}\pi e^{-si\pi/2}\frac{\Gamma(k-1)}{\Gamma(s)\Gamma(k-s)}L^*(f, k-s; p/q), \tag{3-3}$$

which is a generalization of Cohen's lemma in [Kohnen and Zagier 1984, §1.2]. For simplicity, we write $\mathscr{C}_k(z, s)$ for $\mathscr{C}_k(z, s; 0)$. The twisted L-functions satisfy

$$\overline{L^*(f, s; p/q)} = L^*(f, \bar{s}; -p/q), \tag{3-4}$$

$$q^s L^*(f, s; p/q) = (-1)^{k/2} q^{k-s} L^*(f, k-s; -p'/q) \tag{3-5}$$

for $pp' \equiv 1 \bmod q$ as in [Kowalski et al. 2002, Appendix A.3].

Define $\mathcal{M}_n := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \,\middle|\, a, b, c, d \in \mathbb{Z},\ ad - bc = n \right\}$. Thus, $\mathcal{M}_1 = \Gamma$. For $k \in \mathbb{Z}$ and $g : \mathbb{H} \to \mathbb{C}$, set

$$(g|_k\gamma)(z) := \det(\gamma)^{k/2} g(\gamma z) j(\gamma, z)^{-k}$$

for all $\gamma \in \mathcal{M}_n$. The weight-k Hecke operator T_n acts on $g \in M_k$ by

$$(T_n g)(z) := n^{k/2-1} \sum_{\gamma \in \Gamma \backslash \mathcal{M}_n} (g|_k\gamma)(z) = n^{k-1} \sum_{\substack{ad=n \\ a,d>0}} d^{-k} \sum_{0 \leqslant b < d} g\left(\frac{az+b}{d}\right). \tag{3-6}$$

4. Basic properties of double Eisenstein series

We work more generally in this section with Γ a Fuchsian group of the first kind containing at least one cusp. Set

$$\varepsilon_\Gamma := \#\{\Gamma \cap \{-I\}\}. \tag{4-1}$$

Label the finite number of inequivalent cusps $\mathfrak{a}$, $\mathfrak{b}$, etc., and let $\Gamma_\mathfrak{a}$ be the subgroup of Γ fixing $\mathfrak{a}$. There exists a corresponding scaling matrix $\sigma_\mathfrak{a} \in \mathrm{SL}(2, \mathbb{R})$ such that $\sigma_\mathfrak{a}\infty = \mathfrak{a}$ and

$$\sigma_\mathfrak{a}{}^{-1}\Gamma_\mathfrak{a}\sigma_\mathfrak{a} = \begin{cases} B \cup -B & \text{if } -I \in \Gamma\ (\varepsilon_\Gamma = 1), \\ B & \text{if } -I \notin \Gamma\ (\varepsilon_\Gamma = 0). \end{cases}$$

Also set $\Gamma_\mathfrak{a}^* := \sigma_\mathfrak{a} B \sigma_\mathfrak{a}{}^{-1}$.

We recall some facts about $E_{k,\mathfrak{a}}(z, s)$, the nonholomorphic Eisenstein series associated to the cusp $\mathfrak{a}$; see for example [Iwaniec 2002, Chapter 3; Diamantis and O'Sullivan 2010, §2.1]. It is defined as

$$E_{k,\mathfrak{a}}(z, s) := \sum_{\gamma \in \Gamma_\mathfrak{a} \backslash \Gamma} \mathrm{Im}(\sigma_\mathfrak{a}{}^{-1}\gamma z)^s \left(\frac{j(\sigma_\mathfrak{a}{}^{-1}\gamma, z)}{|j(\sigma_\mathfrak{a}{}^{-1}\gamma, z)|} \right)^{-k}$$

and absolutely convergent for $\operatorname{Re}(s) > 1$. Put $E^*_{k,\mathfrak{a}}(z,s) := \theta_k(s)E_{k,\mathfrak{a}}(z,s)$ as in (2-5). Then we have the expansion

$$E^*_{0,\mathfrak{a}}(\sigma_{\mathfrak{b}}z,s) = \delta_{\mathfrak{ab}}\theta(s)y^s + \theta(1-s)Y_{\mathfrak{ab}}(s)y^{1-s} + \sum_{l\neq 0} Y_{\mathfrak{ab}}(l,s)W_s(lz), \tag{4-2}$$

and

$$E^*_{k,\mathfrak{a}}(\sigma_{\mathfrak{b}}z,s) = \delta_{\mathfrak{ab}}\theta_k(s)y^s + \theta_k(1-s)Y_{\mathfrak{ab}}(s)y^{1-s} + O(e^{-2\pi y}) \tag{4-3}$$

as $y \to \infty$ for all $k \in 2\mathbb{Z}$. Also, its functional equation is

$$E^*_{k,\mathfrak{a}}(z,1-s) = \sum_{\mathfrak{b}} Y_{\mathfrak{ab}}(1-s)E^*_{k,\mathfrak{b}}(z,s). \tag{4-4}$$

We gave the coefficients $Y_{\mathfrak{ab}}(s)$ and $Y_{\mathfrak{ab}}(l,s)$ explicitly in the case of $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ following (2-3), and in general, they involve series containing Kloosterman sums; see [Iwaniec 2002, (3.21) and (3.22)].

For the natural generalization of (2-10), we define the *double Eisenstein series associated to the cusp* $\mathfrak{a}$ as

$$\boldsymbol{E}_{s,k-s,\mathfrak{a}}(z,w) := \sum_{\substack{\gamma,\delta\in\Gamma^*_{\mathfrak{a}}\backslash\Gamma \\ c_{\sigma_{\mathfrak{a}}^{-1}\gamma\delta^{-1}\sigma_{\mathfrak{a}}}>0}} (c_{\sigma_{\mathfrak{a}}^{-1}\gamma\delta^{-1}\sigma_{\mathfrak{a}}})^{w-1}\left(\frac{j(\sigma_{\mathfrak{a}}^{-1}\gamma,z)}{j(\sigma_{\mathfrak{a}}^{-1}\delta,z)}\right)^{-s} j(\sigma_{\mathfrak{a}}^{-1}\delta,z)^{-k} \tag{4-5}$$

so that

$$\boldsymbol{E}_{s,k-s,\mathfrak{a}}(\sigma_{\mathfrak{a}}z,w) = j(\sigma_{\mathfrak{a}},z)^k \sum_{\substack{\gamma,\delta\in B\backslash\Gamma' \\ c_{\gamma\delta^{-1}}>0}} (c_{\gamma\delta^{-1}})^{w-1}\left(\frac{j(\gamma,z)}{j(\delta,z)}\right)^{-s} j(\delta,z)^{-k} \tag{4-6}$$

for $\Gamma' = \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}$, which is also a Fuchsian group of the first kind. To establish an initial domain of absolute convergence for (4-6), we consider

$$\sum_{\substack{\gamma,\delta\in B\backslash\Gamma' \\ c_{\gamma\delta^{-1}}>0}} \left|(c_{\gamma\delta^{-1}})^{w-1}\left(\frac{j(\gamma,z)}{j(\delta,z)}\right)^{-s} j(\delta,z)^{-k}\right|. \tag{4-7}$$

Recalling (2-11), we see that

$$|\rho^s| = |\rho|^\sigma e^{-t\arg(\rho)} \ll_t |\rho|^\sigma \quad \text{for } s = \sigma + it \in \mathbb{C}.$$

Therefore, with $r = \operatorname{Re}(w)$ and $\operatorname{Im}(\gamma z) = y|j(\gamma,z)|^{-2}$, we deduce that (4-7) is bounded by a constant depending on s times

$$y^{-k/2} \sum_{\substack{\gamma,\delta\in\Gamma_\infty\backslash\Gamma' \\ \gamma\delta^{-1}\neq\Gamma_\infty}} |c_{\gamma\delta^{-1}}|^{r-1} \operatorname{Im}(\gamma z)^{\sigma/2} \operatorname{Im}(\delta z)^{(k-\sigma)/2}. \tag{4-8}$$

Lemma 4.1. *There exists a constant $\kappa_\Gamma > 0$ so that for all $\gamma,\delta \in \Gamma$ with $c_{\gamma\delta^{-1}} > 0$*

$$\kappa_\Gamma \leqslant c_{\gamma\delta^{-1}} \leqslant \operatorname{Im}(\gamma z)^{-1/2}\operatorname{Im}(\delta z)^{-1/2}.$$

Proof. The existence of κ_Γ is described in [Iwaniec 2002, §2.5 and §2.6; Shimura 1971, Lemma 1.25]. Set $\varepsilon(\gamma, z) := j(\gamma, z)/|j(\gamma, z)| = e^{i \arg(j(\gamma,z))}$. It is easy to verify that, for all $\gamma, \delta \in \Gamma$ and $z \in \mathbb{H}$,

$$\begin{aligned} c_{\gamma\delta^{-1}} &= c_\gamma j(\delta, z) - c_\delta j(\gamma, z) \\ &= \left(\frac{j(\gamma, z) - \overline{j(\gamma, z)}}{2iy}\right) j(\delta, z) - \left(\frac{j(\delta, z) - \overline{j(\delta, z)}}{2iy}\right) j(\gamma, z) \\ &= (\varepsilon(\delta, z)^{-2} - \varepsilon(\gamma, z)^{-2}) j(\gamma, z) j(\delta, z)/(2iy). \end{aligned}$$

Therefore,

$$\begin{aligned} |c_{\gamma\delta^{-1}}| &= \left|\frac{\varepsilon(\gamma, z)}{\varepsilon(\delta, z)} - \frac{\varepsilon(\delta, z)}{\varepsilon(\gamma, z)}\right| \operatorname{Im}(\gamma z)^{-1/2} \operatorname{Im}(\delta z)^{-1/2}/2 \\ &= \left|\operatorname{Im}\left(\frac{\varepsilon(\gamma, z)}{\varepsilon(\delta, z)}\right)\right| \operatorname{Im}(\gamma z)^{-1/2} \operatorname{Im}(\delta z)^{-1/2} \\ &\leqslant \operatorname{Im}(\gamma z)^{-1/2} \operatorname{Im}(\delta z)^{-1/2}. \end{aligned}$$ □

It follows that for $r' = \max(r, 1)$ and $\gamma\delta^{-1} \notin \Gamma_\infty$

$$|c_{\gamma\delta^{-1}}|^{r-1} \ll \operatorname{Im}(\gamma z)^{(1-r')/2} \operatorname{Im}(\delta z)^{(1-r')/2} \tag{4-9}$$

for an implied constant depending on Γ and r. Combining (4-8) and (4-9) shows

$$\begin{aligned} \frac{\boldsymbol{E}_{s,k-s,\mathfrak{a}}(\sigma_\mathfrak{a} z, w)}{j(\sigma_\mathfrak{a}, z)^k} &\ll y^{-k/2} \sum_{\substack{\gamma,\delta\in\Gamma_\infty\backslash\Gamma' \\ \gamma\delta^{-1}\neq\Gamma_\infty}} \operatorname{Im}(\gamma z)^{(1-r'+\sigma)/2} \operatorname{Im}(\delta z)^{(1-r'+k-\sigma)/2} \qquad (4\text{-}10) \\ &= y^{-k/2}\left[E_\mathfrak{a}\left(\sigma_\mathfrak{a} z, \frac{1-r'+\sigma}{2}\right) E_\mathfrak{a}\left(\sigma_\mathfrak{a} z, \frac{1-r'+k-\sigma}{2}\right) - E_\mathfrak{a}\left(\sigma_\mathfrak{a} z, 1-r'+\frac{k}{2}\right)\right] \end{aligned}$$

on noting that $\operatorname{Im}(\gamma z) = \operatorname{Im}(\delta z)$ for $\gamma\delta^{-1} \in \Gamma_\infty$. Since $E_\mathfrak{a}(z, s)$ is absolutely convergent for $\sigma = \operatorname{Re}(s) > 1$, we have proved that the series $\boldsymbol{E}_{s,k-s,\mathfrak{a}}(\sigma_\mathfrak{a} z, w)$, defined in (4-6), is absolutely convergent for $2 < \sigma < k-2$ and $\operatorname{Re}(w) < \sigma - 1, k-1-\sigma$. This convergence is uniform for z in compact sets of $\mathbb{H}$ and for s and w in compact sets in $\mathbb{C}$ satisfying the above constraints.

We next verify that $\boldsymbol{E}_{s,k-s,\mathfrak{a}}(z, w)$ has weight k in the z variable. We have

$$f(z) \in M_k(\Gamma) \iff f(\sigma_\mathfrak{a} z) j(\sigma_\mathfrak{a}, z)^{-k} \in M_k(\sigma_\mathfrak{a}{}^{-1}\Gamma\sigma_\mathfrak{a}),$$

so with (4-6), we must prove that

$$g(z) := \sum_{\substack{\gamma,\delta\in B\backslash\Gamma' \\ c_{\gamma\delta^{-1}}>0}} (c_{\gamma\delta^{-1}})^{w-1} \left(\frac{j(\gamma, z)}{j(\delta, z)}\right)^{-s} j(\delta, z)^{-k}$$

is in $M_k(\Gamma')$. For all $\tau \in \Gamma'$,

$$\frac{g(\tau z)}{j(\tau,z)^k} = \sum_{\substack{\gamma,\delta\in B\backslash\Gamma' \\ c_{\gamma\delta^{-1}}>0}} (c_{\gamma\delta^{-1}})^{w-1}\left(\frac{j(\gamma,\tau z)}{j(\delta,\tau z)}\right)^{-s} j(\delta,\tau z)^{-k} j(\tau,z)^{-k}$$
$$= \sum_{\substack{\gamma,\delta\in B\backslash\Gamma' \\ c_{(\gamma\tau)(\delta\tau)^{-1}}>0}} (c_{(\gamma\tau)(\delta\tau)^{-1}})^{w-1}\left(\frac{j(\gamma\tau,z)}{j(\delta\tau,z)}\right)^{-s} j(\delta\tau,z)^{-k} = g(z)$$

as required.

We finally show that $\boldsymbol{E}_{s,k-s}$ is a cusp form. By (4-10), replacing z by $\sigma_{\mathfrak{a}}{}^{-1}\sigma_{\mathfrak{b}}z$ and using (4-3), for any cusp $\mathfrak{b}$ we obtain

$$\frac{\boldsymbol{E}_{s,k-s,\mathfrak{a}}(\sigma_{\mathfrak{b}}z,w)}{j(\sigma_{\mathfrak{b}},z)^k}$$
$$\ll y^{-k/2}\left[E_{\mathfrak{a}}\left(\sigma_{\mathfrak{b}}z,\frac{1-r'+\sigma}{2}\right)E_{\mathfrak{a}}\left(\sigma_{\mathfrak{b}}z,\frac{1-r'+k-\sigma}{2}\right)-E_{\mathfrak{a}}\left(\sigma_{\mathfrak{b}}z,1-r'+\frac{k}{2}\right)\right]$$
$$\ll y^{1+\sigma-k}+y^{1-\sigma}+y^{1+r'-k}+y^{r'-k}$$

and approaches 0 as $y\to\infty$. Thus, by a standard argument (see for example [Diamantis and O'Sullivan 2010, Proposition 5.3]), $\boldsymbol{E}_{s,k-s,\mathfrak{a}}(z,w)$ is a cusp form. Assembling these results, we have shown the following:

Proposition 4.2. *Let $z\in\mathbb{H}$ and $k\in\mathbb{Z}$, and let $s, w\in\mathbb{C}$ satisfy $2<\sigma<k-2$ and $\mathrm{Re}(w)<\sigma-1, k-1-\sigma$. For Γ a Fuchsian group of the first kind with cusp $\mathfrak{a}$, the series $\boldsymbol{E}_{s,k-s,\mathfrak{a}}(z,w)$ is absolutely and uniformly convergent for s, w and z in compact sets satisfying the above constraints. For each such s and w, we have $\boldsymbol{E}_{s,k-s,\mathfrak{a}}(z,w)\in S_k(\Gamma)$ as a function of z.*

The same techniques prove the next result for the nonholomorphic double Eisenstein series. Generalizing (2-18), we set

$$\mathscr{E}_{\mathfrak{a}}(\sigma_{\mathfrak{a}}z,w;s,s') := \sum_{\substack{\gamma,\delta\in\Gamma_\infty\backslash\sigma_{\mathfrak{a}}{}^{-1}\Gamma\sigma_{\mathfrak{a}} \\ \gamma\delta^{-1}\neq\Gamma_\infty}} \frac{\mathrm{Im}(\gamma z)^s\,\mathrm{Im}(\delta z)^{s'}}{|c_{\gamma\delta^{-1}}|^w}. \tag{4-11}$$

Proposition 4.3. *Let $z\in\mathbb{H}$ and $s, s', w\in\mathbb{C}$ with $\sigma=\mathrm{Re}(s)$ and $\sigma'=\mathrm{Re}(s')$. The series $\mathscr{E}_{\mathfrak{a}}(z,w;s,s')$ defined in (4-11) is absolutely and uniformly convergent for z, w, s and s' in compact sets satisfying*

$$\sigma,\sigma'>1 \quad \textit{and} \quad \mathrm{Re}(w)>2-2\sigma, 2-2\sigma'.$$

Unlike $\boldsymbol{E}_{s,k-s,\mathfrak{a}}(z,w)$, the series $\mathscr{E}_{\mathfrak{a}}(z,w;s,s')$ may have polynomial growth at cusps.

5. Further results on double Eisenstein series

5A. ***Analytic continuation: proof of Theorem 2.3.*** Our next task is to prove the meromorphic continuation of $\boldsymbol{E}_{s,k-s}(z,w)$ in s and w. For s and w in the initial domain of convergence, we begin with

$$\begin{aligned}
&\zeta(1-w+s)\zeta(1-w+k-s)\boldsymbol{E}_{s,k-s}(z,w)\\
&\quad=\sum_{u,v=1}^{\infty} u^{w-1-s}v^{w-1-k+s}\sum_{\substack{a,b,c,d\in\mathbb{Z}\\(a,b)=(c,d)=1\\ad-bc>0}}(ad-bc)^{w-1}\left(\frac{az+b}{cz+d}\right)^{-s}(cz+d)^{-k}\\
&\quad=\sum_{u,v=1}^{\infty}\sum_{\substack{a,b,c,d\in\mathbb{Z}\\(a,b)=(c,d)=1\\ad-bc>0}}(au\cdot dv-bu\cdot cv)^{w-1}\left(\frac{au\cdot z+bu}{cv\cdot z+dv}\right)^{-s}(cv\cdot z+dv)^{-k}\\
&\quad=\sum_{\substack{a,b,c,d\in\mathbb{Z}\\ad-bc>0}}(ad-bc)^{w-1}\left(\frac{az+b}{cz+d}\right)^{-s}(cz+d)^{-k} \qquad (5\text{-}1)\\
&\quad=\sum_{n=1}^{\infty}\frac{1}{n^{1-w}}\sum_{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathcal{M}_n}\left(\frac{az+b}{cz+d}\right)^{-s}(cz+d)^{-k}\\
&\quad=2\sum_{n=1}^{\infty}\frac{T_n\mathscr{C}_k(z,s)}{n^{k-w}}, \qquad (5\text{-}2)
\end{aligned}$$

recalling (3-2). With Proposition 4.2, we know $\boldsymbol{E}_{s,k-s}(z,w)\in S_k(\Gamma)$ so that

$$\begin{aligned}
&\boldsymbol{E}_{s,k-s}(z,w)=\sum_{f\in\mathscr{B}_k}\frac{\langle \boldsymbol{E}_{s,k-s}(\,\cdot\,,w),f\rangle}{\langle f,f\rangle}f(z) \implies\\
&\zeta(1-w+s)\zeta(1-w+k-s)\boldsymbol{E}_{s,k-s}(z,w)=2\sum_{n=1}^{\infty}\frac{1}{n^{k-w}}\sum_{f\in\mathscr{B}_k}\frac{\langle T_n\mathscr{C}_k(\,\cdot\,,s),f\rangle}{\langle f,f\rangle}f(z).
\end{aligned}$$

Then

$$\langle T_n\mathscr{C}_k(z,s),f\rangle=\langle\mathscr{C}_k(z,s),T_nf\rangle=a_f(n)\langle\mathscr{C}_k(z,s),f\rangle,$$

and with (3-3), we obtain

$$\begin{aligned}
&\zeta(1-w+s)\zeta(1-w+k-s)\boldsymbol{E}_{s,k-s}(z,w)\\
&\qquad=2^{3-w}\pi^{k+1-w}e^{-si\pi/2}\frac{\Gamma(k-1)}{\Gamma(s)\Gamma(k-s)\Gamma(k-w)}\\
&\qquad\qquad\times\sum_{f\in\mathscr{B}_k}L^*(f,k-s)L^*(f,k-w)\frac{f(z)}{\langle f,f\rangle}. \qquad (5\text{-}3)
\end{aligned}$$

Define the completed double Eisenstein series $\boldsymbol{E}^*$ with (2-12). Then (5-3) becomes

$$\boldsymbol{E}^*_{s,k-s}(z,w) = \sum_{f\in\mathscr{B}_k} L^*(f,s)L^*(f,w)\frac{f(z)}{\langle f,f\rangle}. \tag{5-4}$$

We also now see from (5-4) that $\boldsymbol{E}^*_{s,k-s}(z,w)$ has an analytic continuation to all s and w in $\mathbb{C}$ and satisfies (2-13) and the two functional equations (2-14) and (2-15). The dihedral group D_8 generated by (2-14) and (2-15) is described in [Diamantis and O'Sullivan 2010, §4.4]. □

5B. ***Twisted double Eisenstein series.*** In this section, we define the *twisted double Eisenstein series* by

$$\begin{aligned}\zeta(1-w+s)\zeta(1-w+k-s)&\boldsymbol{E}_{s,k-s}(z,w;\, p/q)\\ &:= \sum_{\substack{a,b,c,d\in\mathbb{Z}\\ ad-bc>0}} (ad-bc)^{w-1}\left(\frac{az+b}{cz+d}+\frac{p}{q}\right)^{-s}(cz+d)^{-k}\end{aligned} \tag{5-5}$$

for $p/q\in\mathbb{Q}$ with $q>0$ and establish its basic required properties. We remark that the above definition of $\boldsymbol{E}_{s,k-s}(z,w;\, p/q)$ comes from generalizing (5-1), but it is not clear how it can be extended to general Fuchsian groups.

Writing

$$\begin{aligned}(ad-bc)^{w-1}&\left(\frac{az+b}{cz+d}+\frac{p}{q}\right)^{-s}\\ &= q^{1-w+s}((aq+cp)d-(bq+dp)c)^{w-1}\left(\frac{(aq+cp)z+(bq+dp)}{cz+d}\right)^{-s},\end{aligned}$$

we see that (5-5) equals

$$q^{1-w+s}\sum_{\substack{a',b',c,d\in\mathbb{Z}\\ a'd-b'c>0}} (a'd-b'c)^{w-1}\left(\frac{a'z+b'}{cz+d}\right)^{-s}(cz+d)^{-k}$$

with $a'\equiv cp \bmod q$ and $b'\equiv dp \bmod q$. Hence, $\boldsymbol{E}_{s,k-s}(z,w;\, p/q)$ is a subseries of $\boldsymbol{E}_{s,k-s}(z,w)$ and, in the same domain of initial convergence, is an element of S_k.

The analog of (5-2) is

$$\zeta(1-w+s)\zeta(1-w+k-s)\boldsymbol{E}_{s,k-s}(z,w;\, p/q) = 2\sum_{n=1}^{\infty}\frac{T_n\mathscr{C}_k(z,s;\, p/q)}{n^{k-w}}. \tag{5-6}$$

Hence, with (3-3),

$$\begin{aligned}\zeta(1-w+s)\zeta(1-w+k-s)\boldsymbol{E}_{s,k-s}(z,w;\,p/q)\\ &= 2^{3-w}\pi^{k+1-w}e^{-si\pi/2}\frac{\Gamma(k-1)}{\Gamma(s)\Gamma(k-s)\Gamma(k-w)}\\ &\qquad\times\sum_{f\in\mathcal{B}_k}L^*(f,k-s;\,p/q)L^*(f,k-w)\frac{f(z)}{\langle f,f\rangle}.\end{aligned}\tag{5-7}$$

Define the completed double Eisenstein series $\boldsymbol{E}^*_{s,k-s}(z,w;\,p/q)$ with the same factor as (2-12), and we obtain

$$\langle \boldsymbol{E}^*_{s,k-s}(\,\cdot\,,w;\,p/q),f\rangle = L^*(f,k-s;\,p/q)L^*(f,k-w)\tag{5-8}$$

for any f in $\mathcal{B}_k$. Then (5-7) implies $\boldsymbol{E}^*_{s,k-s}(z,w;\,p/q)$ has an analytic continuation to all s and w in $\mathbb{C}$. It satisfies the two functional equations

$$\begin{aligned}\boldsymbol{E}^*_{s,k-s}(z,k-w;\,p/q) &= (-1)^{k/2}\boldsymbol{E}^*_{s,k-s}(z,w;\,p/q),\\ q^s\boldsymbol{E}^*_{k-s,s}(z,w;\,p/q) &= (-1)^{k/2}q^{k-s}\boldsymbol{E}^*_{s,k-s}(z,w;\,-p'/q)\end{aligned}$$

for $pp'\equiv 1 \bmod q$ using (2-1) and (3-5), respectively.

6. Applying the Rankin–Cohen bracket to Poincaré series

The main objective of this section is to show how double Eisenstein series arise naturally when the Rankin–Cohen bracket is applied to the usual Eisenstein series E_k. Proposition 2.4 will be a consequence of this. In fact, since there is no difficulty in extending these methods, we compute the Rankin–Cohen bracket of two arbitrary Poincaré series

$$[P_{k_1}(z;m_1),\,P_{k_2}(z;m_2)]_n$$

for $m_1,m_2\geq 0$. The result may be expressed in terms of the *double Poincaré series* defined below. In this way, the action of the Rankin–Cohen brackets on spaces of modular forms can be completely described. See also Corollary 6.5 at the end of this section.

Definition 6.1. Let $z\in\mathbb{H}$, $k_1,k_2\geq 3$ in $\mathbb{Z}$ and $m_1,m_2\in\mathbb{Z}_{\geq 0}$. For $w\in\mathbb{C}$ with $\mathrm{Re}(w)<k_1-1,k_2-1$, we define the *double Poincaré series*

$$\boldsymbol{P}_{k_1,k_2}(z,w;m_1,m_2) := \sum_{\substack{\gamma,\delta\in B\backslash\Gamma\\ c_{\gamma\delta^{-1}}>0}}(c_{\gamma\delta^{-1}})^{w-1}\frac{e^{2\pi i(m_1\gamma z+m_2\delta z)}}{j(\gamma,z)^{k_1}j(\delta,z)^{k_2}}.\tag{6-1}$$

The series (6-1) will vanish identically unless k_1 and k_2 have the same parity. Clearly, we have $\boldsymbol{E}_{k_1,k_2}(z,w)=\boldsymbol{P}_{k_1,k_2}(z,w;0,0)$. Since $|e^{2\pi i(m_1\gamma z+m_2\delta z)}|\leqslant 1$, it is a simple matter to verify that the work in Section 4 proves that $\boldsymbol{P}_{k_1,k_2}(z,w;m_1,m_2)$ converges absolutely and uniformly on compacta to a cusp form in $S_{k_1+k_2}(\Gamma)$.

For $l \in \mathbb{Z}_{\geq 0}$, it is convenient to set

$$Q_k(z, l; m) := \begin{cases} P_k(z; m) & \text{if } l = 0, \\ \frac{1}{2}\sum_{\gamma \in B\backslash\Gamma} \frac{e^{2\pi i m \gamma z}(c_\gamma)^l}{j(\gamma, z)^{k+l}} & \text{if } l \geq 1. \end{cases} \tag{6-2}$$

As in the proof of Proposition 4.2, Q_k is an absolutely convergent series for k even and at least 4. The next result may be verified by induction.

Lemma 6.2. *For every $j \in \mathbb{Z}_{\geq 0}$, we have the formulas*

$$\frac{d^j}{dz^j} E_k(z) = (-1)^j \frac{(k+j-1)!}{(k-1)!} Q_k(z, j; 0),$$

$$\frac{d^j}{dz^j} P_k(z; m) = \sum_{l=0}^{j} (-1)^{l+j} (2\pi i m)^l \frac{j!}{l!} \binom{k+j-1}{k+l-1} Q_{k+2l}(z, j-l; m) \quad \textit{for } m > 0.$$

Set

$$A_{k_1,k_2}(l, u)_n := \frac{(k_1+n-1)!\,(k_2+n-1)!}{l!\,u!\,(n-l-u)!\,(k_1+l-1)!\,(k_2+u-1)!}.$$

Proposition 6.3. *For $m_1, m_2 \in \mathbb{Z}_{\geq 1}$,*

$$\begin{aligned}[P_{k_1}(z; m_1), P_{k_2}(z; m_2)]_n = &\sum_{\substack{l,u \geq 0 \\ l+u \leqslant n}} A_{k_1,k_2}(l, u)_n (-2\pi i m_1)^l (2\pi i m_2)^u \\ &\times \boldsymbol{P}_{k_1+n+l-u,k_2+n-l+u}(z, n+1-l-u; m_1, m_2)/2 \\ &+ P_{k_1+k_2+2n}(z; m_1+m_2) \sum_{\substack{l,u \geq 0 \\ l+u = n}} A_{k_1,k_2}(l, u)_n (-2\pi i m_1)^l (2\pi i m_2)^u. \end{aligned}$$

Proof. With Lemma 6.2,

$$\begin{aligned} &[P_{k_1}(z; m_1), P_{k_2}(z; m_2)]_n \\ &\quad = \sum_{l=0}^{n} \sum_{u=0}^{n} (2\pi i m_1)^l (2\pi i m_2)^u \frac{(k_1+n-1)!\,(k_2+n-1)!}{l!\,u!\,(k_1+l-1)!\,(k_2+u-1)!} \\ &\qquad \times \sum_{r=l}^{n-u} (-1)^{n+l+u+r} \frac{Q_{k_1+2l}(z, r-l; m_1)\, Q_{k_2+2u}(z, n-r-u; m_2)}{(r-l)!\,(n-r-u)!}. \end{aligned} \tag{6-3}$$

The inner sum over r is

$$\begin{aligned} &\frac{(-1)^l}{4(n-l-u)!} \sum_{\gamma,\delta \in B\backslash\Gamma} \frac{e^{2\pi i (m_1 \gamma z + m_2 \delta z)}}{j(\gamma, z)^{k_1+2l} j(\delta, z)^{k_2+2u}} \\ &\qquad \times \sum_{r=l}^{n-u} \binom{n-l-u}{r-l} \left(\frac{c_\gamma}{j(\gamma, z)}\right)^{r-l} \left(\frac{-c_\delta}{j(\delta, z)}\right)^{n-r-u}, \end{aligned} \tag{6-4}$$

and, employing the binomial theorem, (6-4) reduces to

$$\frac{(-1)^l}{4(n-l-u)!}\sum_{\gamma,\delta\in B\backslash\Gamma}\frac{e^{2\pi i(m_1\gamma z+m_2\delta z)}}{j(\gamma,z)^{k_1+n+l-u}j(\delta,z)^{k_2+n-l+u}}(c_\gamma j(\delta,z)-c_\delta j(\gamma,z))^{n-l-u} \tag{6-5}$$

for $l+u<n$ and

$$\frac{(-1)^l}{4(n-l-u)!}\sum_{\gamma,\delta\in B\backslash\Gamma}\frac{e^{2\pi i(m_1\gamma z+m_2\delta z)}}{j(\gamma,z)^{k_1+n+l-u}j(\delta,z)^{k_2+n-l+u}} \tag{6-6}$$

for $l+u=n$. Noting that

$$c_\gamma j(\delta,z)-c_\delta j(\gamma,z)=\begin{vmatrix}c_\gamma & d_\gamma\\ c_\delta & d_\delta\end{vmatrix}=c_{\gamma\delta^{-1}}$$

means that (6-5) becomes

$$\frac{(-1)^l}{2(n-l-u)!}\boldsymbol{P}_{k_1+n+l-u,k_2+n-l+u}(z,n+1-l-u;m_1,m_2) \tag{6-7}$$

and (6-6) equals

$$\frac{(-1)^l}{(n-l-u)!}\biggl(\frac{\boldsymbol{P}_{k_1+n+l-u,k_2+n-l+u}(z,n+1-l-u;m_1,m_2)}{2} + P_{k_1+k_2+2n}(z;m_1+m_2)\biggr). \tag{6-8}$$

Putting (6-7) and (6-8) into (6-3) finishes the proof. □

In fact, Proposition 6.3 is also valid for m_1 or m_2 equaling 0 provided we agree that $(-2\pi i m_1)^l=1$ in the ambiguous case where $m_1=l=0$ and similarly that $(2\pi i m_2)^u=1$ when $m_2=u=0$. With this notational convention, the proof of the last proposition gives:

Corollary 6.4. *For $m>0$, we have*

$$\begin{aligned}[E_{k_1}(z),P_{k_2}(z;m)]_n &= \sum_{u=0}^{n}A_{k_1,k_2}(0,u)_n(2\pi i m)^u\\ &\times\frac{\boldsymbol{P}_{k_1+n-u,k_2+n+u}(z,n+1-u;0,m)}{2}+P_{k_1+k_2+2n}(z;m)\cdot A_{k_1,k_2}(0,n)_n(2\pi i m)^n,\\ [E_{k_1}(z),E_{k_2}(z)]_n &= A_{k_1,k_2}(0,0)_n\boldsymbol{E}_{k_1+n,k_2+n}(z,n+1)/2+E_{k_1+k_2}(z)\cdot\delta_{n,0}.\end{aligned} \tag{6-9}$$

Proposition 2.4 follows directly from (6-9). Combining Proposition 2.4 with Theorem 2.3 gives a new proof of Zagier's formula (1-2). His original proof in [1977, Proposition 6] employed Poincaré series.

Proof of Proposition 2.5. Let $F_{s,w}(z)=(-1)^{k_2/2}y^{-k/2}E^*_{k_1}(z,u)E^*_{k_2}(z,v)/(2\pi^{k/2})$ with $u=(s+w-k+1)/2$ and $v=(-s+w+1)/2$ as before in (2-16). Then

$F_{s,w}(z)$ has weight k and polynomial growth as $y \to \infty$. It is proved in [Diamantis and O'Sullivan 2010, Proposition 2.1] that

$$\langle F_{s,w}, f\rangle = L^*(f, s)L^*(f, w) \tag{6-10}$$

for all $f \in B_k$. Comparing (6-10) with (2-13) shows that

$$\boldsymbol{E}^*_{s,k-s}(\,\cdot\,, w) = \pi_{\mathrm{hol}}(F_{s,w}),$$

as required. □

A basic property of Rankin–Cohen brackets naturally emerges from Proposition 6.3 and Corollary 6.4.

Corollary 6.5. *For $g_1 \in M_{k_1}(\Gamma)$ and $g_2 \in M_{k_2}(\Gamma)$, we have $[g_1, g_2]_n \in S_{k_1+k_2+2n}(\Gamma)$ for $n > 0$.*

Proof. The space $M_{k_1}(\Gamma)$ is spanned by E_{k_1} and the Poincaré series $P_{k_1}(z; m)$ for $m \in \mathbb{Z}_{\geq 1}$. So we may write g_1, and similarly g_2, as a linear combination of Eisenstein and Poincaré series. Hence, $[g_1, g_2]_n$ is a linear combination of the Rankin–Cohen brackets appearing in Proposition 6.3 and Corollary 6.4. By these results, $[g_1, g_2]_n$ is a linear combination of double Poincaré and double Eisenstein series, which are in $S_{k_1+k_2+2n}(\Gamma)$ as we have already shown. □

It would be interesting to know if $\boldsymbol{P}_{k_1,k_2}(z, w; m_1, m_2)$ has a meromorphic continuation in w. As a corollary of work in the next section, we establish the continuation of $\boldsymbol{P}_{k_1,k_2}(z, w; 0, 0)$ to all $w \in \mathbb{C}$.

7. The Hecke action

The expression (5-2), giving $\boldsymbol{E}_{s,k-s}$ in terms of $\mathscr{C}_k$ acted upon by the Hecke operators, can be studied further and yields an interesting relation between $\boldsymbol{E}_{s,k-s}(z, w)$ and the generalized Cohen kernel $\mathscr{C}_k(z, s; p/q)$.

We have

$$\begin{aligned} T_n\mathscr{C}_k(z, s; p/q) &= n^{k-1} \sum_{\rho\in\Gamma\backslash\mathscr{M}_n} \mathscr{C}_k(\rho z, s; p/q)\cdot j(\rho, z)^{-k} \\ &= \tfrac{1}{2}n^{k-1} \sum_{\gamma\in\mathscr{M}_n} \left(\gamma z + \frac{p}{q}\right)^{-s} j(\gamma, z)^{-k}. \end{aligned}$$

To decompose $\mathscr{M}_n$ into left Γ-cosets, set

$$\mathscr{H} := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \,\middle|\, a, b, d \in \mathbb{Z}_{\geq 0}, ad = n, 0 \leqslant b < a \right\}$$

so that $\mathcal{M}_n = \bigcup_{\rho\in\mathcal{H}} \rho\Gamma$, a disjoint union. Hence,

$$\begin{aligned}
T_n\mathscr{E}_k(z, s; p/q) &= \tfrac{1}{2}n^{k-1}\sum_{\rho\in\mathcal{H}}\sum_{\gamma\in\Gamma}\left(\rho\gamma z+\frac{p}{q}\right)^{-s} j(\rho,\gamma z)^{-k} j(\gamma, z)^{-k}\\
&= \tfrac{1}{2}n^{k-1}\sum_{a|n}\left(\frac{n}{a}\right)^{-k}\left(\frac{a^2}{n}\right)^{-s}\sum_{0\leqslant b<a}\sum_{\gamma\in\Gamma}\left(\gamma z+\frac{b}{a}+\frac{n}{a^2}\frac{p}{q}\right)^{-s} j(\gamma, z)^{-k}\\
&= n^{s-1}\sum_{a|n}a^{k-2s}\sum_{0\leqslant b<a}\mathscr{E}_k\left(z, s; \frac{b}{a}+\frac{n}{a^2}\frac{p}{q}\right). \qquad (7\text{-}1)
\end{aligned}$$

Combining (7-1) in the case $p/q = 0$, with (5-2) we find

$$\begin{aligned}
&\frac{\zeta(1-w+s)\zeta(1-w+k-s)\boldsymbol{E}_{s,k-s}(z, w)}{2}\\
&\qquad = \sum_{n=1}^{\infty}\frac{T_n\mathscr{E}_k(z, s)}{n^{k-w}}\\
&\qquad = \sum_{n=1}^{\infty}n^{s+w-k-1}\sum_{a|n}a^{k-2s}\sum_{0\leqslant b<a}\mathscr{E}_k\left(z, s; \frac{b}{a}\right)\\
&\qquad = \sum_{a=1}^{\infty}a^{k-2s}\sum_{v=1}^{\infty}(av)^{s+w-k-1}\sum_{0\leqslant b<a}\mathscr{E}_k\left(z, s; \frac{b}{a}\right)\\
&\qquad = \zeta(k+1-s-w)\sum_{a=1}^{\infty}a^{w-s-1}\sum_{0\leqslant b<a}\mathscr{E}_k\left(z, s; \frac{b}{a}\right).
\end{aligned}$$

Consequently, for $2 < \sigma < k-2$ and $\mathrm{Re}(w) < \sigma - 1, k-1-\sigma$,

$$\zeta(1-w+s)\boldsymbol{E}_{s,k-s}(z, w) = 2\sum_{a=1}^{\infty}a^{w-s-1}\sum_{b=0}^{a-1}\mathscr{E}_k\left(z, s; \frac{b}{a}\right). \qquad (7\text{-}2)$$

Upon taking the inner product of both sides with $f \in \mathcal{B}_k$, by using (2-13) and (3-3) and then simplifying, we obtain

$$\begin{aligned}
&\frac{(2\pi)^{k-w}}{\Gamma(k-w)}L^*(f, s)L^*(f, w)\\
&\qquad = \zeta(k+1-s-w)\sum_{a=1}^{\infty}a^{w-s-1}\sum_{b=0}^{a-1}L^*\left(f, k-s; \frac{b}{a}\right). \qquad (7\text{-}3)
\end{aligned}$$

Since the eigenforms f in $\mathcal{B}_k$ span S_k, we may verify (7-2) by giving another proof of (7-3). Note that the right side of (7-3) equals

$$\zeta(k+1-s-w)\frac{\Gamma(k-s)}{(2\pi)^{k-s}}\sum_{a=1}^{\infty}a^{w-s-1}\sum_{b=0}^{a-1}\sum_{m=1}^{\infty}\frac{a_f(m)e^{2\pi imb/a}}{m^{k-s}}$$

$$=\zeta(k+1-s-w)\frac{\Gamma(k-s)}{(2\pi)^{k-s}}\sum_{m=1}^{\infty}\sum_{a|m}^{\infty}a^{w-s}\frac{a_f(m)}{m^{k-s}}$$

$$=\zeta(k+1-s-w)\frac{\Gamma(k-s)}{(2\pi)^{k-s}}\sum_{m=1}^{\infty}\frac{a_f(m)\sigma_{w-s}(m)}{m^{k-s}}.$$

The series

$$L(f\otimes E(\,\cdot\,,v),k-s):=\sum_{m=1}^{\infty}\frac{a_f(m)\sigma_{w-s}(m)}{m^{k-s}}$$

is a convolution L-series involving the Fourier coefficients of $f(z)$ and $E(z,v)$ for $2v=-s+w+1$ (as in (2-16)) and, recalling [Zagier 1977, (72)] or [Diamantis and O'Sullivan 2010, (2.11)],

$$\zeta(k+1-s-w)\frac{\Gamma(k-s)}{(2\pi)^{k-s}}L(f\otimes E(\,\cdot\,,v),k-s)=\frac{(2\pi)^{k-w}}{\Gamma(k-w)}L^*(f,k-s)L^*(f,k-w). \tag{7-4}$$

Applying the functional equation (2-1) confirms that the right side of (7-4) equals the left side of (7-3).

Looking to simplify (7-2) leads to the natural question, what are the relations between the $\mathscr{C}_k(z,s;\,p/q)$ for rational p/q in the interval [0, 1)? For example, it is a simple exercise with (3-3) and (3-5) to show that

$$q^{-s}\mathscr{C}_k(z,s;\,p/q)=e^{-si\pi}q^{-k+s}\mathscr{C}_k(z,k-s;\,-p'/q)$$

for $pp'\equiv 1 \bmod q$. With $s=k/2$ at the center of the critical strip, we get an even simpler relation:

$$\mathscr{C}_k(z,k/2;\,p/q)=(-1)^{k/2}\mathscr{C}_k(z,k/2;\,-p'/q). \tag{7-5}$$

A more interesting, but speculative, possibility would be to argue in the reverse direction in order to derive information about L-functions twisted by exponentials with *nonrational* exponents. Specifically, if we established, by other means, relations between the $\mathscr{C}_k(z,s;\,x)$ for $x\notin\mathbb{Q}$, then (7-2) and other results proven here might lead to relations for L-functions twisted by exponentials with nonrational exponents. That would be important because such L-functions play a prominent role in Kaczorowski and Perelli's program of classifying the Selberg class (see, e.g., [Kaczorowski and Perelli 1999]). Relations between these L-functions seem to be necessary for the extension of Kaczorowski and Perelli's classification to degree 2, to which L-functions of GL(2) cusp forms belong.

8. Periods of cusp forms

8A. ***Values of L-functions inside the critical strip.*** We first review Zagier's proof in [1977, §5] of Manin's periods theorem. This exhibits a general principle of proving algebraicity we will be using in the next sections.

For all $s, w \in \mathbb{C}$, it is convenient to define $H_{s,w} \in S_k$ by the conditions

$$\langle H_{s,w}, f\rangle = L^*(f, s)L^*(f, w) \quad \text{for all } f \in \mathcal{B}_k.$$

We need the following result:

Lemma 8.1. *For $g \in S_k$ with Fourier coefficients in the field K_g and $f \in \mathcal{B}_k$ with coefficients in K_f,*

$$\langle g, f\rangle/\langle f, f\rangle \in K_g K_f.$$

Proof. See the general result of Shimura [1976, Lemma 4]. It is also a simple extension of [Diamantis and O'Sullivan 2010, Lemma 4.3]. □

Let K_{critical} be the field obtained by adjoining to $\mathbb{Q}$ all the Fourier coefficients of

$$\bigl\{H_{s,k-1}, H_{k-2,w} \bigm| 1 \leqslant s, w \leqslant k-1,\ s \text{ even},\ w \text{ odd}\bigr\}.$$

Thus, with $f \in \mathcal{B}_k$ and employing Lemma 8.1,

$$L^*(f, k-1)L^*(f, k-2) = \langle H_{k-1,k-2}, f\rangle = c_f\langle f, f\rangle \tag{8-1}$$

for $c_f \in K_{\text{critical}}K_f$, and the left side of (8-1) is nonzero because the Euler product for $L^*(f, s)$ converges for $\operatorname{Re}(s) > k/2 + 1/2$. Set

$$\omega_+(f) := \frac{c_f\langle f, f\rangle}{L^*(f, k-1)} \quad \text{and} \quad \omega_-(f) := \frac{\langle f, f\rangle}{L^*(f, k-2)}. \tag{8-2}$$

Then $\omega_+(f)\omega_-(f) = \langle f, f\rangle$, and we have:

Lemma 8.2. *For each $f \in \mathcal{B}_k$,*

$$L^*(f, s)/\omega_+(f) \quad \textit{and} \quad L^*(f, w)/\omega_-(f) \in K_{\text{critical}}K_f$$

for all s and w with $1 \leqslant s, w \leqslant k-1$, s even and w odd.

Proof. For such s and w,

$$\frac{L^*(f, s)}{\omega_+(f)} = \frac{L^*(f, s)L^*(f, k-1)}{c_f\langle f, f\rangle} = \frac{\langle H_{s,k-1}, f\rangle}{c_f\langle f, f\rangle} = \frac{c'_f\langle f, f\rangle}{c_f\langle f, f\rangle} \in K_{\text{critical}}K_f,$$

$$\frac{L^*(f, w)}{\omega_-(f)} = \frac{L^*(f, w)L^*(f, k-2)}{c_f\langle f, f\rangle} = \frac{\langle H_{k-2,w}, f\rangle}{c_f\langle f, f\rangle} = \frac{c''_f\langle f, f\rangle}{c_f\langle f, f\rangle} \in K_{\text{critical}}K_f. \quad □$$

To deduce Manin's theorem from Lemma 8.2, we use Zagier's explicit expression for $H_{s,w}$. For $n \geq 0$, even $k_1, k_2 \geq 4$ and $k = k_1 + k_2 + 2n$, (1-2) implies

$$(-1)^{k_1/2} 2^{3-k} \frac{k_1 k_2}{B_{k_1} B_{k_2}} \binom{k-2}{n} H_{n+1,n+k_2} = \frac{[E_{k_1}, E_{k_2}]_n}{(2\pi i)^n}. \tag{8-3}$$

The Fourier coefficients of E_{k_1} and E_{k_2} are rational, and hence, the right side of (8-3) has rational coefficients. Then $H_{n+1,n+k_2}$ has Fourier coefficients in $\mathbb{Q}$ (and also for $k_1, k_2 = 2$ [Kohnen and Zagier 1984, p. 214]). It follows that $K_{\text{critical}} = \mathbb{Q}$ and Lemma 8.2 becomes Theorem 2.6, Manin's periods theorem.

8B. ***Arbitrary L-values.*** With the results of the last section, we may now give the proof of Theorem 2.7, restated here:

Theorem 8.3. *For all $f \in \mathcal{B}_k$ and $s \in \mathbb{C}$, with $\omega_+(f)$ and $\omega_-(f)$ as in Manin's theorem,*

$$L^*(f,s)/\omega_+(f) \in K(\boldsymbol{E}^*_{s,k-s}(\,\cdot\,, k-1))K_f,$$
$$L^*(f,s)/\omega_-(f) \in K(\boldsymbol{E}^*_{k-2,2}(\,\cdot\,, s))K_f.$$

Proof. By Theorem 2.3, we have $H_{s,w}(z) = \boldsymbol{E}^*_{s,k-s}(z, w)$ for all $s, w \in \mathbb{C}$. Thus, arguing as in Lemma 8.2 with $\boldsymbol{E}^*_{s,k-s}(\,\cdot\,, k-1) = H_{s,k-1}$ and $\boldsymbol{E}^*_{k-2,2}(\,\cdot\,, s) = H_{k-2,s}$ yields the theorem. □

We indicate briefly how the double Eisenstein series Fourier coefficients required to define $K(\boldsymbol{E}^*_{s,k-s}(\,\cdot\,, k-1))$ and $K(\boldsymbol{E}^*_{k-2,2}(\,\cdot\,, s))$ in Theorem 2.7 may be calculated when $s \in \mathbb{Z}$, using a slight extension of the methods in [Diamantis and O'Sullivan 2010, §3]. We wish to find the l-th Fourier coefficient, $a_{s,w}(l)$, of $H_{s,w}(z) = \boldsymbol{E}^*_{s,k-s}(z, w)$ for s even and w odd (and we assume $s, w \geq k/2 > 1$). With Proposition 2.5, this is $(-1)^{k_2/2}/(2\pi^{k/2})$ times the l-th Fourier coefficient of

$$\pi_{\text{hol}}\big[y^{-k/2} E^*_{k_1}(z,u) E^*_{k_2}(z,v)\big]$$

for $u = (s+w-k+1)/2$ and $v = (-s+w+1)/2$ both in $\mathbb{Z}$. Let

$$F(z) := y^{-k/2} E^*_{k_1}(z,u) E^*_{k_2}(z,v) - \frac{\theta_{k_1}(u)\theta_{k_2}(1-v)}{\theta_k(s+1-k/2)} y^{-k/2} E^*_k(z, s+1-k/2) - \frac{\theta_{k_1}(u)\theta_{k_2}(v)}{\theta_k(w+1-k/2)} y^{-k/2} E^*_k(z, w+1-k/2).$$

Then $\pi_{\text{hol}}(y^{-k/2} E^*_{k_1}(z,u) E^*_{k_2}(z,v)) = \pi_{\text{hol}}(F(z))$ because $\pi_{\text{hol}}(y^{-k/2} E^*_k(z,s)) = 0$ for every s. We have constructed F so that $F(z) \ll y^{-\varepsilon}$ as $y \to \infty$, and we may use [Diamantis and O'Sullivan 2010, Lemma 3.3] to obtain

$$a_{s,w}(l) = \frac{(-1)^{k_2/2}(4\pi l)^{k-1}}{(2\pi^{k/2})(k-2)!} \int_0^\infty F_l(y) e^{-2\pi l y} y^{k-2}\, dy$$

on writing $F(z)=\sum_{l\in\mathbb{Z}} e^{2\pi i l x} y^{-k/2} F_l(y)$. The functions $F_l(y)$ are sums involving the Fourier coefficients of $E^*_{k_1}(z,u)$ and $E^*_{k_2}(z,v)$ with $u,v\in\mathbb{Z}$. As shown in [Diamantis and O'Sullivan 2010, Theorem 3.1], these coefficients are simply expressed in terms of divisor functions, Bernoulli numbers and a combinatorial part. For s and w in the critical strip, this calculation yields an explicit finite formula for $a_{s,w}(l)$ in [Diamantis and O'Sullivan 2010, Theorem 1.3] (and another proof that $H_{s,w}$ in (8-3) has rational Fourier coefficients and that $K_{\text{critical}}=\mathbb{Q}$). For s and w outside the critical strip, we obtain infinite series representations for $a_{s,w}(l)$ but again involving nothing more complicated than divisor functions and Bernoulli numbers. Further details of this computation will appear in [O'Sullivan 2013].

8C. *Twisted periods.* There is an analog of Manin's periods theorem for twisted L-functions. Let $p/q\in\mathbb{Q}$, and let u be an integer with $1\leqslant u\leqslant k-1$. Manin shows in [1973, (13)] (see also [Lang 1976, Chapter 5]) that $i^u\int_0^{p/q} f(iy)y^{u-1}\,dy$ is an integral linear combination of periods $i^v\int_0^{\infty} f(iy)y^{v-1}\,dy$ for $v=1,\dots,k-1$. With (2-17), this proves

$$i^u q^{k-2} L^*(f,u;p/q)\in\mathbb{Z}\cdot iL^*(f,1)+\mathbb{Z}\cdot i^2L^*(f,2)+\cdots+\mathbb{Z}\cdot i^{k-1}L^*(f,k-1).$$

Therefore, Theorem 2.6 implies the next result.

Proposition 8.4. *For all $f\in\mathcal{B}_k$, $p/q\in\mathbb{Q}$ and integers u with $1\leqslant u\leqslant k-1$,*

$$L^*(f,u;p/q)\in K_f(i)\omega_+(f)+K_f(i)\omega_-(f).$$

Employing (5-8), a similar proof to that of Theorem 2.7 in the last section shows the following:

Proposition 8.5. *For all $f\in\mathcal{B}_k$, $p/q\in\mathbb{Q}$ and $s\in\mathbb{C}$ with $\omega_+(f)$ and $\omega_-(f)$ as in Manin's theorem,*

$$L^*(f,s;p/q)/\omega_+(f)\in K(\boldsymbol{E}^*_{k-s,s}(\cdot\,,1;p/q))K_f,$$
$$L^*(f,s;p/q)/\omega_-(f)\in K(\boldsymbol{E}^*_{k-s,s}(\cdot\,,2;p/q))K_f.$$

9. The nonholomorphic case

9A. *Background results and notation.* We will need a nonholomorphic analog of the Cohen kernel $\mathscr{C}_k(z,s)$.

Definition 9.1. With $z\in\mathbb{H}$ and $s,s'\in\mathbb{C}$, define the *nonholomorphic kernel* $\mathscr{K}$ as

$$\mathscr{K}(z;s,s'):=\frac{1}{2}\sum_{\gamma\in\Gamma}\frac{\operatorname{Im}(\gamma z)^{s+s'}}{|\gamma z|^{2s}}. \tag{9-1}$$

Following directly from the results in [Diamantis and O'Sullivan 2010, §5.2], it is absolutely convergent, uniformly on compacta, for $z\in\mathbb{H}$ and $\operatorname{Re}(s),\operatorname{Re}(s')>1/2$.

The kernel $\mathcal{K}(z; s, s')$ was introduced by Diaconu and Goldfeld [2007, (2.1)] (though they describe it there as a Poincaré series and their kernel is a product of Γ factors). Starting with the identity [Diaconu and Goldfeld 2007, Proposition 3.5]

$$\langle f \cdot \mathcal{K}(\,\cdot\,; s, s'), g\rangle \\ = \frac{\Gamma(s+s'+k-1)}{2^{s+s'+k-1}} \int_{-\infty}^{\infty} \frac{L^*(f, \alpha+i\beta)L^*(g, -s+s'+k-\alpha-i\beta)}{\Gamma(s+\alpha+i\beta)\Gamma(-s+s'+k-\alpha-i\beta)}\, d\beta$$

for f and g in $\mathcal{B}_k$, they provide a new method to establish estimates for the second moment of $L^*(f, s)$ along the critical line $\mathrm{Re}(s) = k/2$. They give similar results for $L^*(u_j, s)$, the L-function associated to a Maass form u_j as defined below.

The spectral decomposition of $\mathcal{K}(z; s, s')$ and its meromorphic continuation in the s and s' variables is shown in [Diaconu and Goldfeld 2007, §5]. We do the same; our treatment is slightly different, and we include it in Section 9B for completeness.

For $\Gamma = \mathrm{SL}(2, \mathbb{Z})$, the discrete spectrum of the Laplace operator $\Delta = -4y^2\partial_z\partial_{\bar{z}}$ is given by u_0, the constant eigenfunction, and u_j for $j \in \mathbb{Z}_{\geq 1}$ an orthogonal system of Maass cusp forms (see, e.g., [Iwaniec 2002, Chapters 4 and 7]) with Fourier expansions

$$u_j(z) = \sum_{n\neq 0} |n|^{-1/2} \nu_j(n) W_{s_j}(nz),$$

where u_j has eigenvalue $s_j(1-s_j)$ and by Weyl's law [Iwaniec 2002, (11.5)]

$$\#\{j \mid |\mathrm{Im}(s_j)| \leqslant T\} = T^2/12 + O(T \log T). \tag{9-2}$$

We may assume the u_j are Hecke eigenforms normalized to have $\nu_j(1) = 1$. Necessarily we have $\nu_j(n) \in \mathbb{R}$. Let ι be the antiholomorphic involution $(\iota u_j)(z) := u_j(-\bar{z})$. We may also assume each u_j is an eigenfunction of this operator, necessarily with eigenvalues ± 1. If $\iota u_j = u_j$, then $\nu_j(n) = \nu_j(-n)$ and u_j is called *even*. If $\iota u_j = -u_j$, then $\nu_j(n) = -\nu_j(-n)$ and u_j is *odd*.

The L-function associated to the Maass cusp form u_j is

$$L(u_j, s) = \sum_{n=1}^{\infty} \nu_j(n)/n^s,$$

convergent for $\mathrm{Re}(s) > 3/2$ since $\nu_j(n) \ll n^{1/2}$ by [Iwaniec 2002, (8.8)]. The completed L-function for an even form u_j is

$$L^*(u_j, s) := \pi^{-s}\Gamma\left(\frac{s+s_j-1/2}{2}\right)\Gamma\left(\frac{s-s_j+1/2}{2}\right)L(u_j, s), \tag{9-3}$$

and it satisfies

$$L^*(u_j, 1-s) = L^*(u_j, s) = \overline{L^*(u_j, \bar{s})}. \tag{9-4}$$

See [Bump 1997, p. 107] for (9-3), (9-4) and the analogous odd case.

To $E(z,s)$ (recall (2-3)) we associate the L-function

$$L(E(\,\cdot\,,s),w) := \sum_{m=1}^{\infty} \frac{\phi(m,s)}{m^w}.$$

The well-known identity $\sum_{m=1}^{\infty} \sigma_x(m)/m^w = \zeta(w)\zeta(w-x)$ implies

$$L(E(\,\cdot\,,s),w) = \frac{2\pi^s}{\Gamma(s)} \frac{\zeta(w+s-1/2)\zeta(w-s+1/2)}{\zeta(2s)}. \tag{9-5}$$

9B. ***The nonholomorphic kernel*** $\mathcal{H}$. Throughout this section, we use $s=\sigma+it$ and $s'=\sigma'+it'$. Recall $\mathcal{H}(z;s,s')$ defined in (9-1) for $\mathrm{Re}(s), \mathrm{Re}(s') > 1/2$. Our goal is to find the spectral decomposition of $\mathcal{H}(z;s,s')$ and prove its meromorphic continuation in s and s'. See [Diaconu and Goldfeld 2007, §5] and also [Iwaniec 2002, §7.4] for a similar decomposition and continuation of the automorphic Green function.

A routine verification (using [Jorgenson and O'Sullivan 2005, Lemma 9.2], for example) yields

$$\Delta\mathcal{H}(z;s,s') = (s+s')(1-s-s')\mathcal{H}(z;s,s') + 4ss'\mathcal{H}(z;s+1,s'+1). \tag{9-6}$$

Put

$$\xi_{\mathbb{Z}}(z,s) := \sum_{m\in\mathbb{Z}} \frac{1}{|z+m|^{2s}}.$$

Then

$$\mathcal{H}(z;s,s') = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma z)^{s+s'} \xi_{\mathbb{Z}}(\gamma z,s). \tag{9-7}$$

Use the Poisson summation formula as in [Iwaniec 2002, §3.4] or [Goldfeld 2006, Theorem 3.1.8] to see that

$$\xi_{\mathbb{Z}}(z,s) = \frac{\pi^{1/2}\Gamma(s-1/2)}{\Gamma(s)} y^{1-2s} + \frac{2\pi^s}{\Gamma(s)} y^{1/2-s} \sum_{m\neq 0} |m|^{s-1/2} K_{s-1/2}(2\pi|m|y) e^{2\pi i m x} \tag{9-8}$$

for $\mathrm{Re}(s) > 1/2$. Set

$$\xi^{\sharp}_{\mathbb{Z}}(z,s) := \sum_{m\neq 0} |m|^{s-1/2} K_{s-1/2}(2\pi|m|y) e^{2\pi i m x}. \tag{9-9}$$

Let $B_\rho := \{z\in\mathbb{C} \mid |z| \leqslant \rho\}$. Then with [Jorgenson and O'Sullivan 2008, Lemma 6.4],

$$\sqrt{y} K_{s-1/2}(2\pi y) \ll e^{-2\pi y}(y^{\rho+3} + y^{-\rho-3})$$

for all $s\in B_\rho$ and $\rho, y > 0$ with the implied constant depending only on ρ. Hence,

$$\xi^{\sharp}_{\mathbb{Z}}(z,s) \ll \sum_{m=1}^{\infty} e^{-2\pi m y}(m^{\rho+\sigma+2} y^{\rho+5/2} + m^{-\rho+\sigma-4} y^{-\rho-7/2}).$$

We also have [Jorgenson and O'Sullivan 2008, Lemma 6.2]

$$\sum_{m=1}^{\infty} m^{\rho} e^{-2m\pi y} \ll e^{-2\pi y}(1+y^{-\rho-1})$$

for all $y > 0$ with the implied constant depending only on $\rho \geq 0$. Therefore,

$$\xi_{\mathbb{Z}}^{\sharp}(z,s) \ll e^{-2\pi y}(y^{\rho+5/2}+y^{-\rho-9/2}). \tag{9-10}$$

Consider the weight-0 series

$$\mathcal{K}^{\sharp}(z;s,s') := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \operatorname{Im}(\gamma z)^{s'+1/2}\xi_{\mathbb{Z}}^{\sharp}(\gamma z,s). \tag{9-11}$$

With (9-10), we have

$$\mathcal{K}^{\sharp}(z;s,s') \ll \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} (\operatorname{Im}(\gamma z)^{\sigma'+\rho+3}+\operatorname{Im}(\gamma z)^{\sigma'-\rho-4})e^{-2\pi\operatorname{Im}(\gamma z)} \tag{9-12}$$

so that $\mathcal{K}^{\sharp}(z;s,s')$ is absolutely convergent for $\operatorname{Re}(s') > \rho+5$.

Proposition 9.2. *Let* $\rho > 0$ *and* $s, s' \in \mathbb{C}$ *satisfy* $\operatorname{Re}(s) > 1/2$, $\operatorname{Re}(s') > \rho+5$ *and* $s \in B_\rho$. *Then*

$$\mathcal{K}(z;s,s') = \frac{\pi^{1/2}\Gamma(s-1/2)}{\Gamma(s)}E(z,s'-s+1)+\frac{2\pi^s}{\Gamma(s)}\mathcal{K}^{\sharp}(z;s,s'), \tag{9-13}$$

and, for an implied constant depending only on s *and* s',

$$\mathcal{K}^{\sharp}(z;s,s') \ll y^{5+\rho-\sigma'} \quad \text{as } y\to\infty. \tag{9-14}$$

Proof. It is clear that (9-13) follows from (9-7), (9-8), (9-9) and (9-11) when s and s' are in the stated range. With (9-12) and employing (4-3), we deduce that as $y\to\infty$,

$$\begin{aligned}\mathcal{K}^{\sharp}(z;s,s') &\ll (y^{\sigma'+\rho+3}+y^{\sigma'-\rho-4})e^{-2\pi y}\\ &\qquad + \sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma\\ \gamma\neq\Gamma_\infty}} (\operatorname{Im}(\gamma z)^{\sigma'+\rho+3}+\operatorname{Im}(\gamma z)^{\sigma'-\rho-4})\\ &\ll e^{-\pi y}+y^{1-(\sigma'+\rho+3)}+y^{1-(\sigma'-\rho-4)}\\ &\ll y^{5+\rho-\sigma'}.\end{aligned}$$

□

Clearly, for $\operatorname{Re}(s') > \rho+5$, (9-13) gives the meromorphic continuation of $\mathcal{K}(z;s,s')$ to all $s \in B_\rho$. For these s and s', it follows from (9-14) that $\mathcal{K}^{\sharp}$, as a function of z, is bounded. Also use (9-6) and (9-13) to show that

$$\Delta\mathcal{K}^{\sharp}(z;s,s') = (s+s')(1-s-s')\mathcal{K}^{\sharp}(z;s,s')+4\pi s'\mathcal{K}^{\sharp}(z;s+1,s'+1),$$

and hence, $\Delta\mathcal{K}^\sharp$ is also bounded. Therefore, with [Iwaniec 2002, Theorems 4.7 and 7.3], $\mathcal{K}^\sharp$ has the spectral decomposition

$$\mathcal{K}^\sharp(z; s, s') = \sum_{j=0}^{\infty} \frac{\langle \mathcal{K}^\sharp(\,\cdot\,; s, s'), u_j\rangle}{\langle u_j, u_j\rangle} u_j(z) + \frac{1}{4\pi i} \int_{(1/2)} \langle \mathcal{K}^\sharp(\,\cdot\,; s, s'), E(\,\cdot\,, r)\rangle E(z, r)\, dr, \tag{9-15}$$

where the integral is from $1/2 - i\infty$ to $1/2 + i\infty$ and the convergence of (9-15) is pointwise absolute in z and uniform on compacta.

Lemma 9.3. *For $s \in B_\rho$ and $\mathrm{Re}(s') > \rho + 5$, we have*

$$\langle \mathcal{K}^\sharp(\,\cdot\,; s, s'), u_j\rangle = \frac{\pi^{1/2-s}}{4\Gamma(s')} L^*(u_j, s'-s+1/2)\Gamma\left(\frac{s'+s+s_j-1}{2}\right)\Gamma\left(\frac{s'+s-s_j}{2}\right)$$

when u_j is an even Maass cusp form. If u_j is odd or constant, then the inner product is zero.

Proof. Unfolding,

$$\begin{aligned}
\langle \mathcal{K}^\sharp(\,\cdot\,; s, s'), u_j\rangle
&= \int_{\Gamma\backslash\mathbb{H}} \mathcal{K}^\sharp(z; s, s')\overline{u_j(z)}\, d\mu(z) \\
&= \int_0^\infty \int_0^1 \left(\sum_{m\neq 0} y^{s'+1/2}|m|^{s-1/2} K_{s-1/2}(2\pi|m|y) e^{2\pi i m x}\right)\overline{u_j(z)}\, \frac{dx\, dy}{y^2} \\
&= 2\sum_{m\neq 0} \nu_j(m)|m|^{s-1/2} \int_0^\infty y^{s'} K_{s-1/2}(2\pi|m|y) K_{\overline{s_j}-1/2}(2\pi|m|y)\, \frac{dy}{y}.
\end{aligned}$$

Evaluating the integral [Iwaniec 2002, p. 205] yields

$$\langle \mathcal{K}^\sharp(\,\cdot\,; s, s'), u_j\rangle = \frac{L(u_j, s'-s+1/2)}{4\pi^{s'}\Gamma(s')} \prod \Gamma\left(\frac{s' \pm (s-1/2) \pm (\overline{s_j}-1/2)}{2}\right).$$

Using (9-3) and that $\overline{s_j} = 1 - s_j$ finishes the proof. □

In the same way, when $\mathrm{Re}(r) = 1/2$,

$$\langle \mathcal{K}^\sharp(\,\cdot\,; s, s'), E(\,\cdot\,, r)\rangle = \frac{L(\overline{E(\,\cdot\,, r)}, s'-s+1/2)}{4\pi^{s'}\Gamma(s')} \prod \Gamma\left(\frac{s' \pm (s-1/2) \pm (\bar{r}-1/2)}{2}\right).$$

Further, $\overline{E(z, r)} = E(z, \bar{r}) = E(z, 1-r)$, and with (9-5) we have shown the following:

Lemma 9.4. *For $s \in B_\rho$ and $\mathrm{Re}(s') > \rho + 5$,*

$$\langle \mathcal{K}^\sharp(\,\cdot\,; s, s'), E(\,\cdot\,, r)\rangle = \frac{\pi^{1/2-s}}{2\Gamma(s')\theta(1-r)}\Gamma\Big(\frac{s'+s-r}{2}\Big) \times \Gamma\Big(\frac{s'+s-1+r}{2}\Big)\theta\Big(\frac{s'-s+r}{2}\Big)\theta\Big(\frac{s'-s+1-r}{2}\Big).$$

Recall that $\theta(s) := \pi^{-s}\Gamma(s)\zeta(2s)$ as in (2-5). Let

$$\mathcal{K}_1(z; s, s') := \frac{\pi^{1/2}\Gamma(s-1/2)}{\Gamma(s)} E(z, s'-s+1),$$

$$\mathcal{K}_2(z; s, s') := \frac{\pi^{1/2}}{2\Gamma(s)\Gamma(s')} \sum_{\substack{j=1 \\ u_j \text{ even}}}^{\infty} L^*(u_j, s'-s+1/2)\Gamma\Big(\frac{s'+s+s_j-1}{2}\Big) \times \Gamma\Big(\frac{s'+s-s_j}{2}\Big)\frac{u_j(z)}{\langle u_j, u_j\rangle},$$

$$\mathcal{K}_3(z; s, s') := \frac{\pi^{1/2}}{\Gamma(s)\Gamma(s')}\frac{1}{4\pi i}\int_{(1/2)} \Gamma\Big(\frac{s'+s-r}{2}\Big)\Gamma\Big(\frac{s'+s-1+r}{2}\Big) \times \theta\Big(\frac{s'-s+r}{2}\Big)\theta\Big(\frac{s'-s+1-r}{2}\Big)\frac{E(z, r)}{\theta(1-r)}\,dr.$$

Assembling Proposition 9.2, (9-15) and Lemmas 9.3 and 9.4, we have proven the decomposition

$$\mathcal{K}(z; s, s') = \mathcal{K}_1(z; s, s') + \mathcal{K}_2(z; s, s') + \mathcal{K}_3(z; s, s') \tag{9-16}$$

for $s \in B_\rho$ and $\mathrm{Re}(s') > \rho + 5$. This agrees exactly with [Diaconu and Goldfeld 2007, (5.8)].

Clearly $\mathcal{K}_1(z; s, s')$ is a meromorphic function of s and s' in all of $\mathbb{C}$. The same is true for $\mathcal{K}_2(z; s, s')$ since the factors $L(u_j, s'-s+1/2)u_j(z)/\langle u_j, u_j\rangle$ have at most polynomial growth as $\mathrm{Im}(s_j) \to \infty$ while the Γ factors have exponential decay by Stirling's formula. See (9-2) and [Iwaniec 2002, §7 and §8] for the necessary bounds. The next result was first established in [Diaconu and Goldfeld 2007, §5].

Theorem 9.5. *The nonholomorphic kernel $\mathcal{K}(z; s, s')$ has a meromorphic continuation to all $s, s' \in \mathbb{C}$.*

Proof. As we have discussed, $\mathcal{K}_1(z; s, s')$ and $\mathcal{K}_2(z; s, s')$ are meromorphic functions of $s, s' \in \mathbb{C}$. The poles of $\Gamma(w)$ are at $w = 0, -1, -2, \ldots,$ and $\theta(w)$ has poles exactly at $w = 0, 1/2$ (with residues $-1/2$ and $1/2$, respectively). Therefore, the integral in $\mathcal{K}_3(z; s, s')$ is certainly an analytic function of s and s' for $\sigma' > \sigma + 1/2$ and $\sigma > 1/2$ since the Γ and θ factors have exponential decay as $|r| \to \infty$. Next, consider s fixed (with $\sigma > 1/2$) and s' varying. Consider a point r_0 with $\mathrm{Re}(r_0) = 1/2$.

Let $B(r_0)$ be a small disc centered at r_0 and $B(1-r_0)$ an identical disc at $1-r_0$. By deforming the path of integration to a new path C to the left of $B(r_0)$ and to the right of $B(1-r_0)$, we may, by Cauchy's theorem, analytically continue $\mathcal{K}_3(z; s, s')$ to s' with $s'-s \in B(r_0)$. Let C_1 be a clockwise contour around the left side of $B(r_0)$ and C_2 be a counterclockwise contour around the right side of $B(1-r_0)$ so that $C = (1/2)+C_1+C_2$. For $s'-s$ inside C_1 (and $1-(s'-s)$ inside C_2), we have

$$\pi^{-1/2}\Gamma(s)\Gamma(s')\cdot\mathcal{K}_3(z; s, s') = \frac{1}{4\pi i}\int_C * = \frac{1}{4\pi i}\int_{(1/2)} * + \frac{1}{4\pi i}\int_{C_1} * + \frac{1}{4\pi i}\int_{C_2} *,$$

where $*$ denotes the integrand in the definition of $\mathcal{K}_3$. Then

$$\begin{aligned}\frac{1}{4\pi i}\int_{C_1} * &= \frac{-2\pi i}{4\pi i}\biggl(\operatorname*{Res}_{r=s'-s}\theta\Bigl(\frac{s'-s+1-r}{2}\Bigr)\biggr)\\ &\qquad\times\Gamma(s)\Gamma(s'-1/2)\frac{\theta(s'-s)}{\theta(1-s'+s)}E(z, s'-s)\\ &= \tfrac{1}{2}\Gamma(s)\Gamma(s'-1/2)\frac{\theta(s'-s)}{\theta(1-s'+s)}E(z, s'-s)\\ &= \tfrac{1}{2}\Gamma(s)\Gamma(s'-1/2)E(z, s-s'+1).\end{aligned}$$

We get the same result for $(1/4\pi i)\int_{C_2}$, and for all s' with $\sigma-1/2 < \operatorname{Re}(s') < \sigma+1/2$, it follows that the continuation of $\mathcal{K}_3(z; s, s')$ is given by

$$\begin{aligned}&\pi^{-1/2}\Gamma(s)\Gamma(s')\cdot\mathcal{K}_3(z; s, s')\\ &\qquad = \Gamma(s)\Gamma(s'-1/2)E(z, s-s'+1) + \frac{1}{4\pi i}\int_{(1/2)} *. \qquad (9\text{-}17)\end{aligned}$$

Similarly, as s' crosses the line with real part $\sigma - 1/2$, the term

$$-\Gamma(s-1/2)\Gamma(s')E(z, s'-s+1)$$

must be added to the right side of (9-17). Thus, for all s' with $1/2 < \operatorname{Re}(s') < \sigma - 1/2$, the continuation of $\mathcal{K}(z; s, s')$ is

$$\mathcal{K}(z; s, s') = \frac{\pi^{1/2}\Gamma(s'-1/2)}{\Gamma(s')}E(z, s-s'+1) + \mathcal{K}_2(z; s, s') + \mathcal{K}_3(z; s, s'). \quad (9\text{-}18)$$

Clearly, with (9-17) and (9-18) we have demonstrated the meromorphic continuation of $\mathcal{K}(z; s, s')$ to all $s, s' \in \mathbb{C}$ with $\operatorname{Re}(s), \operatorname{Re}(s') > 1/2$. The continuation to all $s, s' \in \mathbb{C}$ follows in the same way with further terms in the expression for $\mathcal{K}(z; s, s')$ appearing from the residues of the poles of $\Gamma((s'+s-r)/2)\Gamma((s'+s-1+r)/2)$ as $\operatorname{Re}(s'+s) \to -\infty$. □

Proposition 9.6. *We have the functional equation*

$$\mathcal{K}(z; s, s') = \mathcal{K}(z; s', s). \qquad (9\text{-}19)$$

Proof. We may verify (9-19) by comparing (9-16) with (9-18) and using that $\mathcal{K}_2(z;s,s') = \mathcal{K}_2(z;s',s)$ by (9-4) and $\mathcal{K}_3(z;s,s') = \mathcal{K}_3(z;s',s)$ by (2-6). There is a second, easier proof: with $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, replace γ in (9-1) by $S\gamma$. $\square$

Proposition 9.7. *For all $s, s' \in \mathbb{C}$ and any even Maass Hecke eigenform u_j,*

$$\langle \mathcal{K}(\,\cdot\,;s,s'), u_j\rangle = \frac{\pi^{1/2}}{2\Gamma(s)\Gamma(s')}\Gamma\left(\frac{s'+s+s_j-1}{2}\right)\Gamma\left(\frac{s'+s-s_j}{2}\right)L^*\big(u_j, s'-s+\tfrac{1}{2}\big).$$

Proof. Since each u_j is orthogonal to Eisenstein series, we have by (9-16) (for $s \in B_\rho$ and $\mathrm{Re}(s') > \rho+5$) that

$$\langle \mathcal{K}(\,\cdot\,;s,s'), u_j\rangle = \langle \mathcal{K}_2(\,\cdot\,;s,s'), u_j\rangle.$$

The result follows, extending to all $s, s' \in \mathbb{C}$ by analytic continuation. $\square$

9C. *Nonholomorphic double Eisenstein series.* A similar argument to the proof of (5-2) shows that, for $\mathrm{Re}(s), \mathrm{Re}(s') > 1$ and $\mathrm{Re}(w) \geq 0$,

$$\zeta(w+2s)\zeta(w+2s')\mathcal{E}(z,w;s,s') = \frac{1}{2}\sum_{n=1}^{\infty}\frac{T_n\mathcal{K}(z;s,s')}{n^{w-1/2}}, \tag{9-20}$$

where, in this context [Goldfeld 2006, (3.12.3)], the appropriately normalized Hecke operator acts as

$$T_n\mathcal{K}(z) = \frac{1}{n^{1/2}}\sum_{\gamma\in\Gamma\backslash\mathcal{M}_n}\mathcal{K}(\gamma z).$$

For each Maass form, we have $T_n u_j = \nu_j(n)u_j$, and for the Eisenstein series, [Goldfeld 2006, Proposition 3.14.2] implies $T_nE(z,s) = n^{s-1/2}\sigma_{1-2s}(n)E(z,s)$. Therefore, as in (9-5),

$$\sum_{n=1}^{\infty}\frac{T_nE(z,s)}{n^{w-1/2}} = E(z,s)\sum_{n=1}^{\infty}\frac{\sigma_{1-2s}(n)}{n^{w-s}} = E(z,s)\zeta(w-s)\zeta(w+s-1).$$

Now choose any $\rho > 0$. For $s \in B_\rho$, $\mathrm{Re}(s) > 1$, $\mathrm{Re}(s') > \rho+5$ and $\mathrm{Re}(w) \geq 0$, we may apply T_n to both sides of (9-16) and obtain

$$\begin{aligned}
&\zeta(w+2s)\zeta(w+2s')\mathcal{E}(z,w;s,s')\\
&= \frac{\pi^{1/2}\Gamma(s-1/2)}{2\Gamma(s)}\zeta(s'-s+w)\zeta(s-s'+w-1)E(z,s'-s+1)\\
&+ \frac{\pi^{1/2}}{4\Gamma(s)\Gamma(s')}\sum_{\substack{j=1\\ u_j \text{ even}}}^{\infty} L^*(u_j,s'-s+1/2)\Gamma\left(\frac{s'+s+s_j-1}{2}\right)\Gamma\left(\frac{s'+s-s_j}{2}\right)\\
&\times L(u_j,w-1/2)\frac{u_j(z)}{\langle u_j,u_j\rangle} + \frac{\pi^{1/2}}{2\Gamma(s)\Gamma(s')}\frac{1}{4\pi i}\int_{(1/2)}\theta\left(\frac{s'-s+r}{2}\right)\theta\left(\frac{s'-s+1-r}{2}\right)\\
&\qquad\times\Gamma\left(\frac{s'+s-r}{2}\right)\Gamma\left(\frac{s'+s-1+r}{2}\right)\zeta(w-r)\zeta(w-1+r)\frac{E(z,r)}{\theta(1-r)}\,dr.
\end{aligned}\tag{9-21}$$

Put

$$\Omega(s,s';r) := \theta\Big(\frac{s'+s-r}{2}\Big)\theta\Big(\frac{s'+s-1+r}{2}\Big) \times \theta\Big(\frac{s'-s+r}{2}\Big)\theta\Big(\frac{s'-s+1-r}{2}\Big) \Big/ \theta(1-r).$$

Define the completed double Eisenstein series as in (2-19) and write

$$U(z;s,s') := \sum_{\substack{j=1 \\ u_j \text{ even}}}^{\infty} L^*(u_j, s+s'-1/2)L^*(u_j, s'-s+1/2)\frac{u_j(z)}{\langle u_j, u_j\rangle}.$$

As in the last section, Ω and U have exponential decay as $|r|, |\mathrm{Im}(s_j)| \to \infty$. Specializing (9-21) to $w = s + s'$, we have proved the next result.

Lemma 9.8. *For $s \in B_\rho$, $\mathrm{Re}(s) > 1$ and $\mathrm{Re}(s') > \rho + 5$,*

$$\mathscr{E}^*(z;s,s') = 2\theta(s)\theta(s')E(z;s+s') + 2\theta(1-s)\theta(s')E(z,s'-s+1) + U(z;s,s') + \frac{1}{2\pi i}\int_{(1/2)} \Omega(s,s';r)E(z,r)\,dr. \quad (9\text{-}22)$$

From this, we show the following:

Theorem 9.9. *The completed double Eisenstein series $\mathscr{E}^*(z;s,s')$ has a meromorphic continuation to all $s, s' \in \mathbb{C}$, and we have the functional equations*

$$\mathscr{E}^*(z;s,s') = \mathscr{E}^*(z;s',s), \quad (9\text{-}23)$$

$$\mathscr{E}^*(z;s,s') = \mathscr{E}^*(z;1-s,1-s'). \quad (9\text{-}24)$$

Proof. First note that (9-22) gives the meromorphic continuation of $\mathscr{E}^*(z;s,s')$ to all s and s' with $s \in B_\rho$ and $\mathrm{Re}(s') > \rho + 5$. As in the proof of Theorem 9.5, we see that the further continuation in s' is given by (9-22) along with residues that are picked up as the line of integration is crossed; for $s \in B_\rho$ fixed and $\mathrm{Re}(s') \to -\infty$, the continuation of $\mathscr{E}^*(z;s,s')$ is given by (9-22) plus each of the following:

$$\begin{aligned} 2\theta(s)\theta(1-s')E(z,s-s'+1) &\quad \text{when } \mathrm{Re}(s') < \sigma + 1/2, \\ -2\theta(1-s)\theta(s')E(z,s'-s+1) &\quad \text{when } \mathrm{Re}(s') < \sigma - 1/2, \\ 2\theta(1-s)\theta(1-s')E(z,2-s-s') &\quad \text{when } \mathrm{Re}(s') < -\sigma + 1/2, \\ -2\theta(s)\theta(s')E(z,s+s') &\quad \text{when } \mathrm{Re}(s') < -\sigma - 1/2. \end{aligned}$$

We have therefore shown the meromorphic continuation of $\mathscr{E}^*(z;s,s')$ to all $s \in B_\rho$ and $s' \in \mathbb{C}$. Hence, for all s' with $\mathrm{Re}(s') < -\rho - 4$, say, we have

$$\mathscr{E}^*(z;s,s') = 2\theta(1-s)\theta(1-s')E(z,2-s-s') + 2\theta(s)\theta(1-s')E(z,s-s'+1) + U(z;s,s') + \frac{1}{2\pi i}\int_{(1/2)} \Omega(s,s';r)E(z,r)\,dr. \quad (9\text{-}25)$$

The functional Equation (9-24) is a consequence of the easily checked symmetries $U(z;\, 1-s,\, 1-s') = U(z;\, s,\, s')$ and $\Omega(1-s,\, 1-s';\, r) = \Omega(s,\, s';\, r)$ and a comparison of (9-22) and (9-25). The Equation (9-23) has a similar proof or more simply follows from the definition (2-19). $\square$

Proposition 9.10. *For any even Maass Hecke eigenform u_j (as in Section 9A) and all $s, s' \in \mathbb{C}$,*

$$\langle \mathscr{E}^*(\,\cdot\,;\, s,\, s'),\, u_j\rangle = L^*(u_j,\, s+s'-1/2)L^*(u_j,\, s'-s+1/2).$$

Proof. As in Proposition 9.7, only $U(z;\, s,\, s')$ in (9-22) will contribute to the inner product. $\square$

With Theorem 9.9 and Proposition 9.10, we have proved Theorem 2.9.

10. Double Eisenstein series for general groups

We proved in Section 5A that for $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ the holomorphic double Eisenstein series $\boldsymbol{E}_{s,k-s}(z, w)$ may be continued to all s and w in $\mathbb{C}$ and satisfies a family of functional equations. That proof does not extend to groups where Hecke operators are not available. To show the continuation of $\boldsymbol{E}_{s,k-s,\mathfrak{a}}(z, w)$ for Γ an arbitrary Fuchsian group of the first kind, we first demonstrate a generalization of Proposition 2.5. Recall the definitions of u and v in (2-16) and ε_Γ in (4-1).

Theorem 10.1. *For s and w in the initial domain of convergence and even $k_1, k_2 \geq 0$ with $k = k_1 + k_2$, we have*

$$\begin{aligned}\boldsymbol{E}^*_{s,k-s,\mathfrak{a}}(z, w) \\ &= 2^{\varepsilon_\Gamma - 1}\pi_{\mathrm{hol}}\big[(-1)^{k_2/2} y^{-k/2} E^*_{k_1,\mathfrak{a}}(\,\cdot\,, 1-u) E^*_{k_2,\mathfrak{a}}(\,\cdot\,, 1-v)/(2\pi^{k/2})\big]. \quad (10\text{-}1)\end{aligned}$$

Proof. Let $g \in S_k(\Gamma)$, and set $\Gamma' = \sigma_\mathfrak{a}{}^{-1}\Gamma\sigma_\mathfrak{a}$. Then

$$\begin{aligned}\langle \boldsymbol{E}_{s,k-s,\mathfrak{a}}(\,\cdot\,, w), g\rangle &= \int_{\Gamma'\backslash\mathbb{H}} \mathrm{Im}(\sigma_\mathfrak{a} z)^k \bar{g}(\sigma_\mathfrak{a} z) \boldsymbol{E}_{s,k-s,\mathfrak{a}}(\sigma_\mathfrak{a} z, w)\, d\mu z \qquad (10\text{-}2)\\ &= \int_{\Gamma'\backslash\mathbb{H}} y^k \frac{\bar{g}(\sigma_\mathfrak{a} z)}{\bar{j}(\sigma_\mathfrak{a}, z)^k} \sum_{\delta \in B\backslash\Gamma'} j(\delta, z)^{-k} \Bigg[\sum_{\substack{\gamma \in B\backslash\Gamma' \\ c_{\gamma\delta^{-1}} > 0}} (c_{\gamma\delta^{-1}})^{w-1} \bigg(\frac{j(\gamma, z)}{j(\delta, z)}\bigg)^{-s}\Bigg] d\mu z.\end{aligned}$$

Since $g(\sigma_\mathfrak{a} z) j(\sigma_\mathfrak{a}, z)^{-k} \in S_k(\Gamma')$, we have

$$y^k \frac{\bar{g}(\sigma_\mathfrak{a} z)}{\bar{j}(\sigma_\mathfrak{a}, z)^k j(\delta, z)^k} = \mathrm{Im}(\delta z)^k \frac{\bar{g}(\sigma_\mathfrak{a}\delta z)}{\bar{j}(\sigma_\mathfrak{a}, \delta z)^k}.$$

Note also that $j(\gamma, z)/j(\delta, z) = j(\gamma\delta^{-1}, \delta z)$. Hence, (10-2) equals

$$2^{\varepsilon_\Gamma} \int_{\Gamma_\infty\backslash\mathbb{H}} y^k \frac{\bar{g}(\sigma_\mathfrak{a} z)}{\bar{j}(\sigma_\mathfrak{a}, z)^k} \Bigg[\sum_{\substack{\gamma \in B\backslash\Gamma' \\ c_\gamma > 0}} (c_\gamma)^{w-1} j(\gamma, z)^{-s}\Bigg] d\mu z. \qquad (10\text{-}3)$$

Writing

$$\sum_{\substack{\gamma\in B\backslash\Gamma' \\ c_\gamma>0}} (c_\gamma)^{w-1} j(\gamma,z)^{-s} = \sum_{\substack{\gamma\in B\backslash\Gamma'/B \\ c_\gamma>0}} (c_\gamma)^{w-1} \sum_{m\in\mathbb{Z}} j(\gamma,z+m)^{-s}$$

and using the Fourier expansion of g at $\mathfrak{a}$, $j(\sigma_{\mathfrak{a}},z)^{-k} g(\sigma_{\mathfrak{a}}z) = \sum_{n=1}^\infty a_{g,\mathfrak{a}}(n) e^{2\pi i n z}$, we get that (10-3) equals

$$2^{\varepsilon_\Gamma} \sum_{n=1}^\infty \overline{a_{g,\mathfrak{a}}}(n) \sum_{\substack{\gamma\in B\backslash\Gamma'/B \\ c_\gamma>0}} \frac{1}{(c_\gamma)^{s+1-w}} \int_0^\infty \int_{-\infty}^\infty y^{k-2} \frac{e^{-2\pi i n x - 2\pi n y}}{(x+d_\gamma/c_\gamma+iy)^s}\, dx\, dy$$
$$= 2^{\varepsilon_\Gamma} I_k(s) \sum_{n=1}^\infty \frac{\overline{a_{g,\mathfrak{a}}}(n)}{n^{k-s}} \sum_{\substack{\gamma\in B\backslash\Gamma'/B \\ c_\gamma>0}} \frac{e^{2\pi i n d_\gamma/c_\gamma}}{(c_\gamma)^{s+1-w}}$$

for

$$I_k(s) := \int_0^\infty \int_{-\infty}^\infty y^{k-2} \frac{e^{-2\pi i x - 2\pi y}}{(x+iy)^s}\, dx\, dy.$$

The inner integral over x may be evaluated with a formula of Laplace [Whittaker and Watson 1927, p. 246]:

$$\int_{-\infty}^\infty \frac{e^{-2\pi i x}}{(x+iy)^s}\, dx = e^{-2\pi y} \frac{(2\pi)^s}{\Gamma(s) e^{s i\pi/2}}$$

so that

$$I_k(s) = \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \frac{(2\pi)^s}{\Gamma(s) e^{s i\pi/2}}.$$

With (4-2) and, for example, [Iwaniec 2002, Chapter 3], we recognize

$$\sum_{\substack{\gamma\in B\backslash\Gamma'/B \\ c_\gamma>0}} \frac{e^{2\pi i n d_\gamma/c_\gamma}}{(c_\gamma)^{2s}} = \sum_{\substack{\gamma\in \Gamma_\infty\backslash\Gamma'/\Gamma_\infty \\ c_\gamma>0}} \frac{e^{2\pi i n d_\gamma/c_\gamma}}{(c_\gamma)^{2s}} = \frac{Y_{\mathfrak{a}\mathfrak{a}}(n,s)}{\zeta(2s) n^{s-1}}.$$

It follows that we have shown

$$\langle \boldsymbol{E}^*_{s,k-s,\mathfrak{a}}(\,\cdot\,,w), g\rangle = 2^{\varepsilon_\Gamma-1} \frac{\zeta(2-2u)\Gamma(k-s)\Gamma(k-w)}{(2\pi)^{2k-s-w}} \sum_{n=1}^\infty \frac{Y_{\mathfrak{a}\mathfrak{a}}(n,1-v)\overline{a_{g,\mathfrak{a}}}(n)}{n^{k-s-v}}.$$

Reasoning as in the proof of [Diamantis and O'Sullivan 2010, (2.10)], we also find, for all even $k_1, k_2 \geq 0$ with $k_1+k_2=k$,

$$\langle (-1)^{k_2/2} y^{-k/2} E^*_{k_1,\mathfrak{a}}(\,\cdot\,,1-u) E^*_{k_2,\mathfrak{b}}(\,\cdot\,,1-v)/(2\pi^{k/2}), g\rangle$$
$$= \frac{\zeta(2-2u)\Gamma(k-s)\Gamma(k-w)}{(2\pi)^{2k-s-w}} \sum_{n=1}^\infty \frac{Y_{\mathfrak{b}\mathfrak{a}}(n,1-v)\overline{a_{g,\mathfrak{a}}}(n)}{n^{k-s-v}}.$$

Since $\boldsymbol{E}^*_{s,k-s,\mathfrak{a}}(z,w) \in S_k(\Gamma)$ and $g\in S_k(\Gamma)$ is arbitrary, (10-1) follows. □

Corollary 10.2. *The double Eisenstein series* $\boldsymbol{E}^*_{s,k-s,\mathfrak{a}}(z, w)$ *has a meromorphic continuation to all* $s, w \in \mathbb{C}$ *and as a function of* z *is always in* $S_k(\Gamma)$. *It satisfies the functional equation*

$$\boldsymbol{E}^*_{k-s,s,\mathfrak{a}}(z, w) = (-1)^{k/2} \boldsymbol{E}^*_{s,k-s,\mathfrak{a}}(z, w). \tag{10-4}$$

Proof. Since $E^*_{k,\mathfrak{a}}(z, s)$ has a well-known continuation to all $s \in \mathbb{C}$, due to Selberg, the continuation of $\boldsymbol{E}^*_{s,k-s,\mathfrak{a}}(z, w)$ follows from (10-1). The change of variables $(s, w) \to (k - s, w)$ corresponds to $(u, v) \to (v, u)$, and so (10-4) is also a consequence of (10-1). □

If Γ has more than one cusp, then $\boldsymbol{E}^*_{s,k-s,\mathfrak{a}}(z, w)$ does not appear to possess a functional equation of the type (2-14) as $(s, w) \to (w, s)$. This corresponds on the right of (10-1) to $(u, v) \to (u, 1-v)$, and the functional equation for $E^*_{k_2,\mathfrak{a}}(\,\cdot\,, 1-v)$ involves a sum over cusps as in (4-4).

We remark that the functional Equation (10-4) also follows directly from (4-6) if $-I \in \Gamma$: replace γ and δ in the sum by $-\delta$ and γ, respectively.

Finally, it would be interesting to find the continuation in s and s' of the non-holomorphic double Eisenstein series $\mathscr{E}^*_{\mathfrak{a}}(z; s, s')$ for general groups. We expect that a similar decomposition to (9-22) should be true.

Acknowledgements

We thank Yuri Manin for his stimulating comments on an earlier version of this paper and the referee who provided the reference [Diaconu and Goldfeld 2007].

References

[Bruggeman et al. 2013] R. Bruggeman, J. Lewis, and D. Zagier, "Period functions for Maass wave forms and cohomology", preprint, 2013, Available at http://www.staff.science.uu.nl/~brugg103/notes/pfmwIIcoh130114.pdf. To appear in *Mem. Amer. Math. Soc.*

[Bump 1997] D. Bump, *Automorphic forms and representations*, Cambridge Studies in Advanced Mathematics **55**, Cambridge University Press, 1997. MR 97k:11080 Zbl 0868.11022

[Cohen 1981] H. Cohen, "Sur certaines sommes de séries liées aux périodes de formes modulaires", in *Journées de théorie analytique et élémentaire des nombres* (Limoges, 1980), Université de Limoges, 1981.

[Cohen et al. 1997] P. B. Cohen, Y. Manin, and D. Zagier, "Automorphic pseudodifferential operators", pp. 17–47 in *Algebraic aspects of integrable systems*, edited by A. S. Fokas and I. M. Gelfand, Progr. Nonlinear Differential Equations Appl. **26**, Birkhäuser, Boston, MA, 1997. MR 98e:11054 Zbl 1055.11514

[Deninger 1995] C. Deninger, "Higher order operations in Deligne cohomology", *Invent. Math.* **120**:2 (1995), 289–315. MR 96f:11085 Zbl 0847.55014

[Diaconu and Goldfeld 2007] A. Diaconu and D. Goldfeld, "Second moments of GL_2 automorphic L-functions", pp. 77–105 in *Analytic number theory*, edited by W. Duke and Y. Tschinkel, Clay Math. Proc. **7**, Amer. Math. Soc., Providence, RI, 2007. MR 2009e:11095 Zbl 1230.11058

[Diamantis and O'Sullivan 2010] N. Diamantis and C. O'Sullivan, "Kernels of L-functions of cusp forms", *Math. Ann.* **346**:4 (2010), 897–929. MR 2011d:11114 Zbl 05676435

[Gangl et al. 2006] H. Gangl, M. Kaneko, and D. Zagier, "Double zeta values and modular forms", pp. 71–106 in *Automorphic forms and zeta functions* (Tokyo, 2004), edited by S. Böcherer et al., World Sci. Publ., Hackensack, NJ, 2006. MR 2006m:11138 Zbl 1122.11057

[Goldfeld 2006] D. Goldfeld, *Automorphic forms and L-functions for the group* $\mathrm{GL}(n, \mathbb{R})$, Cambridge Studies in Advanced Mathematics **99**, Cambridge University Press, 2006. MR 2008d:11046 Zbl 1108.11039

[Iwaniec 2002] H. Iwaniec, *Spectral methods of automorphic forms*, 2nd ed., Graduate Studies in Mathematics **53**, American Mathematical Society, Providence, RI, 2002. MR 2003k:11085 Zbl 1006.11024

[Jorgenson and O'Sullivan 2005] J. Jorgenson and C. O'Sullivan, "Convolution Dirichlet series and a Kronecker limit formula for second-order Eisenstein series", *Nagoya Math. J.* **179** (2005), 47–102. MR 2006k:11080 Zbl 1098.11028

[Jorgenson and O'Sullivan 2008] J. Jorgenson and C. O'Sullivan, "Unipotent vector bundles and higher-order non-holomorphic Eisenstein series", *J. Théor. Nombres Bordeaux* **20**:1 (2008), 131–163. MR 2010g:11089 Zbl 1211.11064

[Kaczorowski and Perelli 1999] J. Kaczorowski and A. Perelli, "On the structure of the Selberg class, I: $0 \leqslant d \leqslant 1$", *Acta Math.* **182**:2 (1999), 207–241. MR 2000h:11097 Zbl 1126.11335

[Koblic 1975] N. I. Koblic, "Non-integrality of the periods of cusp forms outside the critical strip", *Funkcional. Anal. i Priložen.* **9**:3 (1975), 52–55. In Russian; translated in *Functional Anal. Appl.* **9**:3 (1976), 224–226. MR 53 #7948 Zbl 0343.10013

[Kohnen and Zagier 1984] W. Kohnen and D. Zagier, "Modular forms with rational periods", pp. 197–249 in *Modular forms* (Durham, 1983), edited by R. A. Rankin, Horwood, Chichester, 1984. MR 87h:11043 Zbl 0618.10019

[Kontsevich and Zagier 2001] M. Kontsevich and D. Zagier, "Periods", pp. 771–808 in *Mathematics unlimited: 2001 and beyond*, edited by B. Engquist and W. Schmid, Springer, Berlin, 2001. MR 2002i:11002 Zbl 1039.11002

[Kowalski et al. 2002] E. Kowalski, P. Michel, and J. VanderKam, "Rankin–Selberg L-functions in the level aspect", *Duke Math. J.* **114**:1 (2002), 123–191. MR 2004c:11070 Zbl 1035.11018

[Lang 1976] S. Lang, *Introduction to modular forms*, Grundlehren der Math. Wissenschaften **222**, Springer, Berlin, 1976. MR 55 #2751 Zbl 0344.10011

[Lewis and Zagier 2001] J. Lewis and D. Zagier, "Period functions for Maass wave forms, I", *Ann. of Math.* (2) **153**:1 (2001), 191–258. MR 2003d:11068 Zbl 1061.11021

[Manin 1973] J. I. Manin, "Periods of cusp forms, and p-adic Hecke series", *Mat. Sb.* (*N.S.*) **92(134)** (1973), 378–401, 503. In Russian; translated in *Math. USSR-Sb.***21**:3 (1973), 371–393. MR 49 #10638 Zbl 0293.14007

[Manin 2010] Y. I. Manin, "Remarks on modular symbols for Maass wave forms", *Algebra Number Theory* **4**:8 (2010), 1091–1114. MR 2012j:11103 Zbl 1229.11079

[Mühlenbruch 2006] T. Mühlenbruch, "Hecke operators on period functions for $\Gamma_0(n)$", *J. Number Theory* **118**:2 (2006), 208–235. MR 2007i:11062 Zbl 1122.11022

[O'Sullivan 2013] C. O'Sullivan, "Formulas for Eisenstein series", preprint, 2013.

[Rankin 1952] R. A. Rankin, "The scalar product of modular forms", *Proc. London Math. Soc.* (3) **2** (1952), 198–217. MR 14,139c Zbl 0049.33904

[Shimura 1971] G. Shimura, *Introduction to the arithmetic theory of automorphic functions*, Publications of the Mathematical Society of Japan **11**, Princeton University Press, 1971. MR 47 #3318 Zbl 0872.11023

[Shimura 1976] G. Shimura, "The special values of the zeta functions associated with cusp forms", *Comm. Pure Appl. Math.* **29**:6 (1976), 783–804. MR 55 #7925 Zbl 0348.10015

[Whittaker and Watson 1927] E. T. Whittaker and G. N. Watson, *A course of modern analysis*, 4th ed., Cambridge University Press, 1927. MR 31 #2375 Zbl 0951.30002

[Zagier 1977] D. Zagier, "Modular forms whose Fourier coefficients involve zeta-functions of quadratic fields", pp. 105–169 in *Modular functions of one variable, VI* (Bonn, 1976), edited by J.-P. Serre and D. B. Zagier, Lecture Notes in Math. **627**, Springer, Berlin, 1977. MR 58 #5525 Zbl 0372.10017

Communicated by Yuri Manin
Received 2012-05-30 Accepted 2012-12-21

diamant@mpim-bonn.mpg.de *School of Mathematical Sciences, University of Nottingham, University Park, Nottingham NG7 2RD, United Kingdom*

cosullivan@gc.cuny.edu *Department of Mathematics, The City University of New York Graduate Center, 365 Fifth Avenue, New York, NY 10016-4309, United States*

Algebra & Number Theory

msp.org/ant

See inside back cover or msp.org/ant for submission instructions.

The subscription price for 2013 is US $200/year for the electronic version, and $350/year (+$40, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscribers address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Algebra & Number Theory

Volume 7 No. 8 2013

1937-0652(2013)7:8;1-5