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Let P_d be a convex polygon with d vertices. The associated Wachspress surface W_d is a fundamental object in approximation theory, defined as the image of the rational map

$$\mathbb{P}^2 \xrightarrow{w_d} \mathbb{P}^{d-1},$$

determined by the Wachspress barycentric coordinates for P_d . We show w_d is a regular map on a blowup X_d of \mathbb{P}^2 and, if $d > 4$, is given by a very ample divisor on X_d so has a smooth image W_d . We determine generators for the ideal of W_d and prove that, in graded lex order, the initial ideal of I_{W_d} is given by a Stanley–Reisner ideal. As a consequence, we show that the associated surface is arithmetically Cohen–Macaulay and of Castelnuovo–Mumford regularity 2 and determine all the graded Betti numbers of I_{W_d} .

1. Introduction

Introduced by Möbius [1827], barycentric coordinates for triangles appear in a host of applications. Recent work in approximation theory has shown that it is also useful to define barycentric coordinates for a convex polygon P_d with $d \geq 4$ vertices (a d -gon). The idea is as follows. To deform a planar shape, first place the shape inside a control polygon. Then move the vertices of the control polygon, and use barycentric coordinates to extend this motion to the entire shape.

For a d -gon with $d \geq 4$, barycentric coordinates were defined by Wachspress [1975] in his work on finite elements; these coordinates are rational functions depending on the vertices $v(P_d)$ of P_d . Warren [2003] shows that Wachspress' coordinates are the unique rational barycentric coordinates of minimal degree. The Wachspress coordinates define a rational map w_d on \mathbb{P}^2 , whose value at a point $p \in P_d$ is the d -tuple of barycentric coordinates of p . The closure of the image of w_d is the Wachspress surface W_d , first defined and studied by Garcia-Puente and Sottile [2010] in their work on linear precision.

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In [Definition 1.3](#), we fix linear forms ℓ_i that are positive inside P_d and vanish on an edge. Let $A = \ell_1 \cdots \ell_d$, Z be the $\binom{d}{2}$ singular points of $\mathbb{V}(A)$, and $Y = Z \setminus v(P_d)$. We call Y the *external vertices* of P_d and show that w_d has basepoints only at Y . Let X_d be the blowup of \mathbb{P}^2 at Y . In [Section 2](#), we prove that W_d is the image of X_d , embedded by a certain divisor D_{d-2} on X_d . The global sections of D_{d-2} have a simple interpretation in terms of the edges $\mathbb{V}(\ell_i)$ of P_d : we prove that

$$H^0(\mathbb{C}_{X_d}(D_{d-2})) \quad \text{has basis} \quad \{\ell_3 \cdots \ell_d, \ell_1 \ell_4 \cdots \ell_d, \dots, \ell_2 \cdots \ell_{d-1}\}.$$

We show that D_{d-2} is very ample if $d > 4$; hence, $W_d \subseteq \mathbb{P}^{d-1}$ is a smooth surface.

1A. Statement of main results. For a d -gon P_d with $d \geq 4$:

- (1) We give explicit generators for $I_{W_d} \subseteq S = \mathbb{K}[x_1, \dots, x_d]$.
- (2) We determine $\text{in}_<(I_{W_d})$, where $<$ is graded lex order.
- (3) We prove $\text{in}_<(I_{W_d})$ is the Stanley–Reisner ideal of a graph Γ .
- (4) We prove that S/I_{W_d} is Cohen–Macaulay, and $\text{reg}(S/I_{W_d}) = 2$.
- (5) We determine the graded Betti numbers of S/I_{W_d} .

In [Section 1B](#), we give some quick background on geometric modeling, and in [Section 1C](#), we do the same for algebraic geometry (in particular, we define all the terms above). Our strategy runs as follows. In [Section 2](#), we study I_{W_d} by blowing up \mathbb{P}^2 at the external vertices. Define a divisor

$$D_{d-2} = (d-2)E_0 - \sum_{p \in Y} E_p$$

on X_d , where E_0 is the pullback of a line and E_p is the exceptional fiber over p . We show that D_{d-2} is very ample and that I_{W_d} is the ideal of the image of

$$X_d \rightarrow \mathbb{P}(H^0(D_{d-2})).$$

Riemann–Roch then yields the Hilbert polynomial of S/I_{W_d} .

In [Sections 3](#) and [4](#), we find distinguished sets of quadrics and cubics vanishing on W_d and use them to generate a subideal $I(d) \subseteq I_{W_d}$. In [Section 5](#), we tie everything together, showing that, in graded lex order, $I_\Gamma(d) \subseteq \text{in}_< I(d)$, where $I_\Gamma(d)$ is the Stanley–Reisner ideal of a certain graph. Using results on flat deformations and an analysis of associated primes, we prove

$$I_\Gamma(d) = \text{in}_<(I(d)).$$

The description in terms of the Stanley–Reisner ring yields the Hilbert series for $S/I_\Gamma(d)$. We prove that $S/I_\Gamma(d)$ is Cohen–Macaulay and has Castelnuovo–Mumford regularity 2, and it follows from uppersemicontinuity that the same is true for $S/I(d)$. The differentials on the quadratic generators of $I_\Gamma(d)$ turn out to

be easy to describe, and combining this with the regularity bound and knowledge of the Hilbert series yields the graded Betti numbers for $\text{in}_{\prec}(I(d))$.

Finally, we show that $I(d)$ has no linear syzygies on its quadratic generators, which allows us to prune the resolution of $\text{in}_{\prec}(I(d))$ to obtain the graded Betti numbers of $I(d)$. Comparing Hilbert polynomials shows that up to saturation

$$S/I(d) = S/I_{W_d}.$$

Since I_{W_d} is prime, it is saturated, and a short-exact-sequence argument shows that $S/I(d)$ is also saturated, concluding the proof.

1B. Geometric modeling background. Let P_d be a d -gon with vertices v_1, \dots, v_d and indices taken modulo d .

Definition 1.1. Functions $\{\beta_i : P_d \rightarrow \mathbb{R} \mid 1 \leq i \leq d\}$ are *barycentric coordinates* if, for all $p \in P_d$,

$$\beta_i(p) \geq 0, \quad p = \sum_{i=1}^d \beta_i(p)v_i, \quad \sum_{i=1}^d \beta_i(p) = 1.$$

Wachspress coordinates have a geometric description in terms of areas of subtriangles of the polygon. Let $A(a, b, c)$ denote the area of the triangle with vertices a , b , and c . For $1 \leq j \leq d$, set $\alpha_j := A(v_{j-1}, v_j, v_{j+1})$ and $A_j := A(p, v_j, v_{j+1})$.

Definition 1.2. For $1 \leq i \leq d$, the functions

$$\beta_i = \frac{b_i}{\sum_{j=1}^d b_j}, \quad \text{where } b_i = \alpha_i \prod_{j \neq i-1, i} A_j$$

are Wachspress barycentric coordinates for the d -gon P_d ; see [Figure 1](#).

We embed P_d in the plane $z = 1 \subseteq \mathbb{R}^3$ and form the cone with $\mathbf{0} \in \mathbb{R}^3$. Explicitly, to each vertex $v_i \in v(P_d)$, we associate the ray $\mathbf{v}_i := (v_i, 1) \in \mathbb{R}^3$. Let \mathbf{P}_d denote the cone generated by the rays \mathbf{v}_i , and $v(\mathbf{P}_d) := \{\mathbf{v}_i \mid v_i \in v(P_d)\}$. The cone over

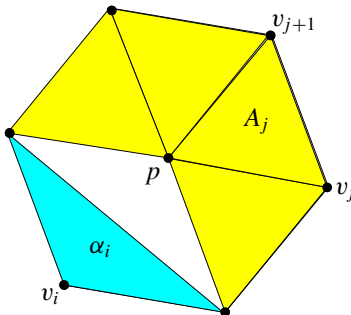


Figure 1. Wachspress coordinates for a polygon.

the edge $[v_i, v_{i+1}]$ corresponds to a facet of P_d with normal vector $\mathbf{n}_i := \mathbf{v}_i \times \mathbf{v}_{i+1}$. We redefine α_j and A_j to be the determinants $|\mathbf{v}_{j-1}\mathbf{v}_j\mathbf{v}_{j+1}|$ and $|\mathbf{v}_j\mathbf{v}_{j+1}\mathbf{p}|$, where $\mathbf{p} = (x, y, z)$. This scales the b_i by a factor of 2 so leaves the β_i unchanged, save for homogenizing the A_j with respect to z , and allows us to define Wachspress coordinates for nonconvex polygons, although Property 1 of barycentric coordinates fails when P_d is nonconvex.

Definition 1.3. $\ell_j := A_j = \mathbf{n}_j \cdot \mathbf{p} = |\mathbf{v}_j\mathbf{v}_{j+1}\mathbf{p}|.$

The ℓ_j are homogeneous linear forms in (x, y, z) and vanish on the cone over the edge $[v_j, v_{j+1}]$. We use [Theorem 1.6](#) below, but Warren’s proof does not require convexity. Our results hold over an arbitrary field \mathbb{K} as long as no three of the lines $\mathbb{V}(\ell_i) \subseteq \mathbb{P}^2$ meet at a point. For the first condition of [Definition 1.1](#) to make sense, \mathbb{K} should be an ordered field.

Definition 1.4. The *dual cone* to P_d is the cone spanned by the normals $\mathbf{n}_1, \dots, \mathbf{n}_d$ and is denoted P_d^* .

Triangulating P_d yields a triangulation of P_d , and the volume of the parallelepiped S spanned by vertices $\{\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k, \mathbf{0}\}$ is $a_S = |\mathbf{v}_i\mathbf{v}_j\mathbf{v}_k|$.

Definition 1.5. Let C be a cone defined by a polygon P_d and $T(C)$ a triangulation of C obtained from a triangulation of P_d as above. The *adjoint* of C is

$$\mathcal{A}_{T(C)}(\mathbf{p}) = \sum_{S \in T(C)} a_S \prod_{\mathbf{v} \in \mathbf{v}(P_d) \setminus \mathbf{v}(S)} (\mathbf{v} \cdot \mathbf{p}) \in \mathbb{K}[x, y, z]_{d-3}.$$

Theorem 1.6 [[Warren 1996](#)]. $\mathcal{A}_{T(C)}(\mathbf{p})$ is independent of the triangulation $T(C)$.

1C. Algebraic geometry background. Next, we review some background in algebraic geometry, referring to [[Eisenbud 1995](#); [Hartshorne 1977](#); [Schenck 2003](#)] for more detail. Homogenizing the numerators of Wachspress coordinates yields our main object of study:

Definition 1.7. The Wachspress map defined by a polygon P_d is the rational map $\mathbb{P}^2 \xrightarrow{w_d} \mathbb{P}^{d-1}$ given on the open set $U_{z \neq 0} \subseteq \mathbb{P}^2$ by $(x, y) \mapsto (b_1(x, y), \dots, b_d(x, y))$. The Wachspress variety W_d is the closure of the image of w_d .

The polynomial ring $S = \mathbb{K}[x_1, \dots, x_d]$ is a graded ring: it has a direct-sum decomposition into homogeneous pieces. A finitely generated graded S -module N admits a similar decomposition; if $s \in S_p$ and $n \in N_q$, then $s \cdot n \in N_{p+q}$. In particular, each N_q is a $(S_0 = \mathbb{K})$ -vector space.

Definition 1.8. For a finitely generated graded S -module N , the Hilbert series $\text{HS}(N, t) = \sum \dim_{\mathbb{K}} N_q t^q$.

Definition 1.9. A free resolution for an S -module N is an exact sequence

$$\mathbb{F} : \cdots \rightarrow F_i \xrightarrow{d_i} F_{i-1} \rightarrow \cdots \rightarrow F_0 \rightarrow N \rightarrow 0,$$

where the F_i are free S -modules.

If N is graded, then the F_i are also graded, so letting $S(-m)$ denote a rank-1 free module generated in degree m , we may write $F_i = \bigoplus_j S(-j)^{a_{i,j}}$. By the Hilbert syzygy theorem [Eisenbud 1995], a finitely generated, graded S -module N has a free resolution of length at most d with all the F_i of finite rank.

Definition 1.10. For a finitely generated graded S -module N , a free resolution is minimal if, for each i , $\text{Im}(d_i) \subseteq \mathfrak{m}F_{i-1}$, where $\mathfrak{m} = \langle x_1, \dots, x_d \rangle$. The graded Betti numbers of N are the $a_{i,j}$ that appear in a minimal free resolution, and the Castelnuovo–Mumford regularity of N is $\max_{i,j} \{a_{i,j} - i\}$.

While the differentials d_i that appear in a minimal free resolution of N are not unique, the ranks and degrees of the free modules that appear are unique. The graded Betti numbers are displayed in a *Betti table*. Reading this table right and down, starting at $(0, 0)$, the entry $b_{ij} := a_{i,i+j}$, and the regularity of N is the index of the bottommost nonzero row in the Betti table for N .

Example 1.11. In Examples 2.9 and 3.11 of [Garcia-Puente and Sottile 2010], it is shown that I_{W_6} is generated by three quadrics and one cubic. The variety $\mathbb{V}(\ell_1 \cdots \ell_6)$ of the edges of P_6 has $\binom{6}{2} = 15$ singular points, of which six are vertices of P_6 , and S/I_{W_6} has Betti table

total	1	4	6	3
0	1	–	–	–
1	–	3	–	–
2	–	1	6	3

For example, $b_{1,2} = a_{1,3} = 1$ reflects that I_{W_6} has a cubic generator, and S/I_{W_6} has regularity 2. The Hilbert series can be read off the Betti table:

$$\text{HS}(S/I_{W_6}, t) = \frac{1 - 3t^2 - t^3 + 6t^4 - 3t^5}{(1 - t)^6} = \frac{1 + 3t + 3t^2}{(1 - t)^3}.$$

Theorem 5.11 gives a complete description of the Betti table of S/I_{W_d} .

2. $H^0(D_{d-2})$ and the Wachspress surface

2A. Background on blowups of \mathbb{P}^2 . Fix points $p_1, \dots, p_k \in \mathbb{P}^2$, and let

$$X \xrightarrow{\pi} \mathbb{P}^2 \tag{1}$$

be the blowup of \mathbb{P}^2 at these points. Then $\text{Pic}(X)$ is generated by the exceptional curves E_i over the points p_i and the proper transform E_0 of a line in \mathbb{P}^2 . A classical

geometric problem asks for a relationship between numerical properties of a divisor $D_m = mE_0 - \sum a_i E_i$ on X and the geometry of

$$X \xrightarrow{\phi} \mathbb{P}(H^0(D_m)^\vee).$$

First, we discuss some basics. Let m and a_i be nonnegative, let I_{p_i} denote the ideal of a point p_i , and define

$$J = \bigcap_{i=1}^k I_{p_i}^{a_i} \subseteq \mathbb{K}[x, y, z] = R. \tag{2}$$

Then $H^0(D_m)$ is isomorphic to the m -th graded piece J_m of J (see [Harbourne 2002]). Davis and Geramita [1988] show that, if $\gamma(J)$ denotes the smallest degree t such that J_t defines J scheme theoretically, then D_m is very ample if $m > \gamma(J)$, and if $m = \gamma(J)$, then D_m is very ample if and only if J does not contain m collinear points, counted with multiplicity. Note that $\gamma(J) \leq \text{reg}(J)$.

2B. Wachspress surfaces. For a polygon P_d , fix defining linear forms ℓ_i as in Definition 1.3 and let $A := \ell_1 \cdots \ell_d$; the edges of P_d are defined by the $\mathbb{V}(\ell_i)$. Let Z denote the $\binom{d}{2}$ singular points of $\mathbb{V}(A)$ and $Y = Z \setminus \nu(P_d)$. Finally, X_d will be the blowup of \mathbb{P}^2 at Y . We study the divisor

$$D_{d-2} = (d - 2)E_0 - \sum_{p \in Y} E_p$$

on X_d . First, we present some preliminaries.

Definition 2.1. Let L be the ideal in $R = \mathbb{K}[x, y, z]$ given by

$$L = \langle \ell_3 \cdots \ell_d, \ell_1 \ell_4 \cdots \ell_d, \dots, \ell_2 \cdots \ell_{d-1} \rangle = \langle A/\ell_1 \ell_2, A/\ell_2 \ell_3, \dots, A/\ell_d \ell_1 \rangle,$$

where $A = \prod_{i=1}^d \ell_i$.

For any variety V , we use I_V to denote the ideal of polynomials vanishing on V .

Lemma 2.2. *The ideals L and I_Y are equal up to saturation at $\langle x, y, z \rangle$.*

Proof. Being equal up to saturation at $\langle x, y, z \rangle$ means that the localizations at any associated prime except $\langle x, y, z \rangle$ are equal. The ideal I_p of a point p is a prime ideal. Recall that the localization of a ring T at a prime ideal \mathfrak{p} is a new ring $T_{\mathfrak{p}}$ whose elements are of the form f/g with $f, g \in T$ and $g \notin \mathfrak{p}$. Localize R at I_p , where $p \in Y$. Then in R_{I_p} , ℓ_i is a unit if $p \notin \mathbb{V}(\ell_i)$. Without loss of generality, suppose forms ℓ_1 and ℓ_2 vanish on p (note that all points of Y are intersections of exactly two lines) and the remaining forms do not. Thus, $L_{I_p} = \langle \ell_1, \ell_2 \rangle = (I_Y)_{I_p}$. \square

The ideal L is not saturated.

Lemma 2.3. *I_Y is generated by one form F of degree $d - 3$ and $d - 3$ forms of degree $d - 2$. Hence, a basis for L_{d-2} consists of $F \cdot x, F \cdot y, F \cdot z$, and the $d - 3$ forms.*

Proof. First, note that I_Y cannot contain any form of degree $d - 4$ since Y contains d sets of $d - 3$ collinear points. So the smallest degree of a minimal generator for I_Y is $d - 3$. Since Y consists of $\binom{d-1}{2} - 1$ distinct points and the space of forms of degree $d - 3$ has dimension $\binom{d-1}{2}$, there is at least one form F of degree $d - 3$ in I_Y . We claim that it is unique. To see this, first note that no ℓ_i can divide F : by symmetry, if one ℓ_i divides F , they all must, which is impossible for degree reasons. Now suppose G is a second form of degree $d - 3$ in I_Y . Let $p \in v(P_d)$ and $\mathbb{V}(\ell_i)$ be a line corresponding to an edge containing p . $F(p)$ must be nonzero since if not $\mathbb{V}(F)$ would contain $d - 2$ collinear points of $\mathbb{V}(\ell_i)$, forcing $\mathbb{V}(F)$ to contain $\mathbb{V}(\ell_i)$, a contradiction. This also holds for G . But in this case, $F(p)G - G(p)F$ is a polynomial of degree $d - 3$ vanishing at $d - 2$ collinear points, again a contradiction. So F is unique (up to scaling), which shows that the Hilbert function satisfies

$$\text{HF}(R/L, d - 3) = |Y|,$$

so $\text{HF}(R/L, t) = |Y|$ for all $t \geq d - 3$ (see [Schenck 2003]). As the polynomials $A/\ell_i \ell_{i+1}$ are linearly independent and there are the correct number, L_{d-2} must be the degree- $(d - 2)$ component of I_Y . \square

Theorem 2.4. *The minimal free resolution of R/L is*

$$0 \rightarrow R(-d) \xrightarrow{d_3} R(-d+1)^d \xrightarrow{d_2} R(-d+2)^d \xrightarrow{\begin{bmatrix} A & A & \dots & A \\ \ell_1 \ell_2 & \ell_2 \ell_3 & \dots & \ell_d \ell_1 \end{bmatrix}} R \rightarrow R/L \rightarrow 0,$$

$$\text{where } d_2 = \begin{bmatrix} \ell_1 & 0 & \dots & \dots & 0 & 0 & m_1 \\ -\ell_3 & \ell_2 & 0 & \dots & \vdots & \vdots & m_2 \\ 0 & -\ell_4 & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ell_{d-2} & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & -\ell_d & \ell_{d-1} & \vdots \\ 0 & \dots & \dots & 0 & 0 & -\ell_1 & m_d \end{bmatrix}$$

and the m_i are linear forms.

Proof. By Lemma 2.3, the generators of I_Y are known. Since I_Y is saturated, the Hilbert–Burch theorem implies that the free resolution of R/I_Y has the form

$$0 \rightarrow R(-d + 1)^{d-3} \rightarrow R(-d + 3) \oplus R(-d + 2)^{d-3} \rightarrow R \rightarrow R/I_Y \rightarrow 0.$$

Writing I_Y as $\langle f_1, \dots, f_{d-3}, F \rangle$ and L as $\langle f_1, \dots, f_{d-3}, xF, yF, zF \rangle$, the task is to understand the syzygies on L given the description above of the syzygies on I_Y . From the Hilbert–Burch resolution, any minimal syzygy on I_Y is of the form

$$\sum g_i f_i + qF = 0,$$

where g_i are linear and q is a quadric (or zero). Since

$$qF = g_1xF + g_2yF + g_3zF \quad \text{with } g_i \text{ linear,}$$

all $d - 3$ syzygies on I_Y lift to give linear syzygies on L . Furthermore, we obtain three linear syzygies on $\{xF, yF, zF\}$ from the three Koszul syzygies on $\{x, y, z\}$. It is clear from the construction that these d linear syzygies are linearly independent. Since $\text{HF}(R/L, d - 1) = |Y|$, this means we have determined all the linear first syzygies. Furthermore, the three Koszul first syzygies on $\{xF, yF, zF\}$ generate a linear second syzygy, so the complex given above is a subcomplex of the minimal free resolution. A check shows that the Buchsbaum–Eisenbud criterion [1973] holds, so the complex above is actually exact and hence a free resolution. The differential d_2 above involves the canonical generators $A/\ell_i\ell_{i+1}$ rather than a set involving $\{xF, yF, zF\}$. Since the $d - 1$ linear syzygies appearing in the first $d - 1$ columns of d_2 are linearly independent, they agree up to a change of basis; the last column of d_2 is a vector of linear forms determined by the change of basis. \square

Theorem 2.5.

(i) $H^0(D_{d-2}) \simeq \text{Span}_{\mathbb{K}}\{A/\ell_1\ell_2, A/\ell_2\ell_3, \dots\}$.

(ii) $H^1(D_{d-2}) = 0 = H^2(D_{d-2})$.

Proof. The remark following Equation (2) shows that $H^0(D_{d-2}) \simeq L_{d-2}$. Since $K = -3E_0 + \sum_{p \in Y} E_p$ (see [Hartshorne 1977]), by Serre duality,

$$H^2(D_{d-2}) \simeq H^0\left((-d - 1)E_0 + \sum_{p \in Y} E_p\right),$$

which is clearly zero. Using that X_d is rational, it follows from Riemann–Roch that

$$h^0(D_{d-2}) - h^1(D_{d-2}) = \frac{D_{d-2}^2 - D_{d-2} \cdot K}{2} + 1.$$

The intersection pairing on X_d is given by $E_i^2 = 1$ if $i = 0$ and -1 if $i \neq 0$, and

$$E_i \cdot E_j = 0 \quad \text{if } i \neq j.$$

Thus,

$$D_{d-2}^2 = (d - 2)^2 - |Y| \quad \text{and} \quad -D_{d-2}K = 3(d - 2) - |Y|, \quad (3)$$

yielding

$$h^0(D_{d-2}) - h^1(D_{d-2}) = \frac{d^2 - d - 2 - 2|Y|}{2} + 1 = d. \quad (4)$$

Thus, $h^0(D_{d-2}) - h^1(D_{d-2}) = d$. Now apply the remark following Equation (2). \square

Corollary 2.6. *If $d > 4$, D_{d-2} is very ample, so the image of X_d in \mathbb{P}^{d-1} is smooth.*

Proof. By [Theorem 2.4](#), the ideal L is $d - 2$ regular. Furthermore, the set Y contains d sets of $d - 3$ collinear points but no set of $d - 2$ collinear points if $d > 4$. The result follows from the Davis–Geramita criterion. \square

Theorem 2.7. $W_4 \simeq \mathbb{P}^1 \times \mathbb{P}^1$, and $X_4 \rightarrow W_4$ is an isomorphism away from the (-1) curve $E_0 - E_1 - E_2$, which is contracted to a smooth point.

Proof. The surface X_4 is \mathbb{P}^2 blown up at two points, which is toric, and isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at a point. By Proposition 6.12 of [\[Cox et al. 2011\]](#), D_2 is basepoint free. Since $D_2^2 = 2$, W_4 is an irreducible quadric surface in \mathbb{P}^3 . As $D_2 \cdot (E_0 - E_1 - E_2) = 0$, the result follows. \square

Replacing D_{d-2} with tD_{d-2} , a computation as in Equations (3) and (4) and Serre vanishing shows that the Hilbert polynomial $\text{HP}(S/I_{W_d}, t)$ is equal to

$$\frac{((d-2)^2 - |Y|)t^2 + (3(d-2) - |Y|)t}{2} + 1 = \frac{d^2 - 5d + 8}{4}t^2 - \frac{d^2 - 9d + 12}{4}t + 1. \quad (5)$$

3. The Wachspress quadrics

In this section, we construct a set of quadrics that vanish on W_d . These quadrics are polynomials that are expressed as a scalar product with a fixed vector τ . The vector τ defines a linear projection $\mathbb{P}^{d-1} \dashrightarrow \mathbb{P}^2$, also denoted by τ , given by

$$\mathbf{x} \mapsto \sum_{i=1}^d x_i \mathbf{v}_i,$$

where $\mathbf{x} = [x_1 : \dots : x_d] \in \mathbb{P}^{d-1}$. By the second property of barycentric coordinates, the composition $\tau \circ w_d : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is the identity map on \mathbb{P}^2 . Since $\mathbf{v}_i \in \mathbb{K}^3$, the vector τ is a triple of linear forms $(\tau_1, \tau_2, \tau_3) \in S^3$. The linear subspace \mathcal{C} of \mathbb{P}^{d-1} where the projection is undefined is the *center of projection*, and $I_{\mathcal{C}} = \langle \tau_1, \tau_2, \tau_3 \rangle$.

3A. Diagonal monomials. A *diagonal monomial* is a monomial $x_i x_j \in S_2$ such that $j \notin \{i - 1, i, i + 1\}$. We write \mathcal{D} for the subspace of S_2 spanned by the diagonal monomials; identifying x_i with the vertex v_i , a diagonal monomial is a diagonal in P_d ; see [Figure 2](#).

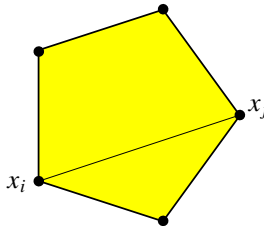


Figure 2. A diagonal monomial.

Lemma 3.1. *Any quadric that vanishes on W_d is a linear combination of elements of \mathcal{D} .*

Proof. Let Q be a polynomial in $(I_{W_d})_2$. Then $Q(w_d) = Q(b_1, \dots, b_d) = 0$. On the edge $[v_k, v_{k+1}]$, all the b_i vanish except b_k and b_{k+1} . Thus, on this edge, the expression $Q(w_d) = 0$ is

$$c_1 b_k^2 + c_2 b_k b_{k+1} + c_3 b_{k+1}^2 = 0 \tag{6}$$

for some constants c_1, c_2 , and c_3 in \mathbb{K} . Recall that $b_i(v_j) = 0$ if $i \neq j$ and $b_i(v_i) \neq 0$ for each i . Evaluating (6) at v_k and v_{k+1} , we conclude $c_1 = c_3 = 0$. At an interior point of edge $[v_k, v_{k+1}]$, neither b_k nor b_{k+1} vanishes. This implies that $c_2 = 0$. A similar calculation on each edge shows that all coefficients of nondiagonal terms in Q are zero. \square

3B. The map to $(I_\ell)_2$. We define a surjective map onto $(I_\ell)_2$ and use the map to calculate the dimension of the vector space of polynomials in $(I_\ell)_2$ that are supported on diagonal monomials. Let S_1^3 denote the space of triples of linear forms on \mathbb{P}^{d-1} . Define the map $\Psi : S_1^3 \rightarrow (I_\ell)_2$ by $F \mapsto F \cdot \tau$, where \cdot is the scalar product.

Lemma 3.2. *The kernel of Ψ is three-dimensional.*

Proof. Since I_ℓ is a complete intersection, the kernel is generated by the three Koszul syzygies on the τ_i . \square

Next we determine conditions on F so that $\Psi(F) \in \mathcal{D}$. If $\mathbf{u}_i \in \mathbb{K}^3$ for $i = 1, \dots, d$, then

$$F = \sum_{i=1}^d x_i \mathbf{u}_i$$

is an element of S_1^3 . Viewing the projection τ as an element of S_1^3 , we have

$$\Psi(F) = F \cdot \tau = \left(\sum_{i=1}^d x_i \mathbf{u}_i \right) \cdot \left(\sum_{i=1}^d x_i \mathbf{v}_i \right) = \sum_{i,j=1}^d (\mathbf{u}_i \cdot \mathbf{v}_j + \mathbf{u}_j \cdot \mathbf{v}_i) x_i x_j. \tag{7}$$

If $\Psi(F) \in \mathcal{D}$, then the coefficients of nondiagonal monomials must vanish:

$$\mathbf{u}_i \cdot \mathbf{v}_i = 0 \quad \text{and} \quad \mathbf{u}_i \cdot \mathbf{v}_{i+1} + \mathbf{u}_{i+1} \cdot \mathbf{v}_i = 0 \quad \text{for all } i. \tag{8}$$

Lemma 3.3. *The dimension of the vector space $\mathcal{D} \cap (I_\ell)_2$ is $d - 3$.*

Proof. We show the conditions in (8) give $2d$ independent conditions on the $3d$ -dimensional vector space S_1^3 , and the solution space is $\Psi^{-1}(\mathcal{D} \cap (I_\ell)_2)$; thus,

$\dim(\Psi^{-1}(\mathfrak{D} \cap (I_{\mathcal{E}})_2)) = d$. The conditions are represented by the matrix equation

$$\begin{pmatrix} \mathbf{v}_1 \cdot \mathbf{u}_1 \\ \vdots \\ \mathbf{v}_d \cdot \mathbf{u}_d \\ \mathbf{v}_1 \cdot \mathbf{u}_2 + \mathbf{v}_2 \cdot \mathbf{u}_1 \\ \vdots \\ \mathbf{v}_d \cdot \mathbf{u}_1 + \mathbf{v}_1 \cdot \mathbf{u}_d \end{pmatrix} = \overbrace{\begin{pmatrix} \mathbf{v}_1^T & 0 & \cdots & 0 \\ 0 & \mathbf{v}_2^T & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \mathbf{v}_d^T \\ \mathbf{v}_2^T & \mathbf{v}_1^T & & 0 \\ 0 & & \ddots & \\ \mathbf{v}_d^T & & & \mathbf{v}_1^T \end{pmatrix}}^M \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \vdots \\ \mathbf{u}_d \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix},$$

where the \mathbf{v}_i and \mathbf{u}_i are column vectors and the superscript T indicates transpose. The matrix M in the middle is a $2d \times 3d$ matrix, and the proof will be complete if the rows are shown to be independent. Denote the rows of M by $r_1, \dots, r_d, r_{d+1}, \dots, r_{2d}$, and let $c_1 r_1 + \cdots + c_d r_d + c_{d+1} r_{d+1} + \cdots + c_{2d} r_{2d}$ be a dependence relation among them. The first three columns of M give the dependence relation $c_1 \mathbf{v}_1 + c_{d+1} \mathbf{v}_2 + c_{2d} \mathbf{v}_d = 0$. Since \mathbf{v}_d , \mathbf{v}_1 , and \mathbf{v}_2 define adjacent rays of a polyhedral cone, they must be independent, so c_1 , c_{d+1} , and c_{2d} must be zero. Repeating the process at each triple \mathbf{v}_{i-1} , \mathbf{v}_i , and \mathbf{v}_{i+1} shows the rest of the c_i 's vanish. Since the restriction $\Psi : \Psi^{-1}(\mathfrak{D} \cap (I_{\mathcal{E}})_2) \rightarrow \mathfrak{D} \cap (I_{\mathcal{E}})_2$ remains surjective, we find $\dim(\mathfrak{D} \cap (I_{\mathcal{E}})_2) = \dim(\Psi^{-1}(\mathfrak{D} \cap (I_{\mathcal{E}})_2)) - \dim(\ker(\Psi)) = d - 3$. \square

3C. Wachspress quadrics. We now compute the dimension and a generating set for $(I_{W_d})_2$.

Definition 3.4. Let $\gamma(i)$ denote the set $\{1, \dots, d\} \setminus \{i-1, i\}$, $\gamma(i, j) = \gamma(i) \cap \gamma(j)$, and $\gamma(i, j, k) = \gamma(i) \cap \gamma(j) \cap \gamma(k)$.

The image of a diagonal monomial $x_i x_j$ under the pullback map $w_d^* : S \rightarrow R$ is

$$b_i b_j = \alpha_i \alpha_j \prod_{k \in \gamma(i)} \ell_k \prod_{m \in \gamma(j)} \ell_m = \alpha_i \alpha_j \prod_{k=1}^d \ell_k \prod_{m \in \gamma(i, j)} \ell_m,$$

and each diagonal monomial has a common factor $A = \prod_{k=1}^d \ell_k$. To find the quadratic relations among Wachspress coordinates, it suffices to find linear relations among products $\prod_{m \in \gamma(i, j)} \ell_m \in R_{d-4}$ for diagonal pairs i and j . Define the map $\phi : \mathfrak{D} \rightarrow R_{d-4}$ by $x_i x_j \mapsto b_i b_j / A$, and extend by linearity; this is w_d^* restricted to \mathfrak{D} and divided by A . By Lemma 3.1, it follows that $(I_{W_d})_2 = \ker(\phi) \subseteq \mathfrak{D}$.

Lemma 3.5. *The dimension of $(I_{W_d})_2$ is $d - 3$.*

Proof. We will show $\phi : \mathfrak{D} \rightarrow R_{d-4}$ is surjective with $\dim(\ker \phi) = d - 3$. To see this, note that there are $d - 3$ diagonal monomials that have x_1 as a factor. We show

	$x_{2,4}$	\cdots	$x_{2,d}$	$x_{3,5}$	\cdots	$x_{3,d}$	\cdots	$x_{d-3,d-1}$	$x_{d-3,d}$	$x_{d-2,d}$
$p_{1,3}$	*									
\vdots			\ddots							
$p_{1,d-1}$			*							
$p_{2,4}$	*			*						
\vdots					\ddots					
$p_{2,d-1}$			*			*				
\vdots							\ddots			
$p_{(d-4)(d-2)}$								*		
$p_{(d-4)(d-1)}$								*	*	
$p_{(d-3)(d-1)}$								*	*	*

Table 1. Values of images of diagonal monomials at external vertices.

that the images of the remaining

$$d(d-3)/2 - (d-3) = (d-3)(d-2)/2 = \dim(R_{d-4})$$

diagonal monomials are independent. Let $p_{s,t} = \ell_s \cap \ell_t$ and $x_{p,q} = x_p x_q$. In Table 1, a star, *, represents a nonzero number and a blank space is zero. The (i, j) entry in the table represents the value of the image of the diagonal monomial in column j at the external vertex in row i . The external vertices not lying on ℓ_d are arranged down the rows with their indices in lexicographic order.

Since Table 1 is lower triangular, the images are independent. We have found $\dim(R_{d-4})$ independent images, and hence, ϕ is surjective. This is a map from a vector space of dimension $d(d-3)/2$ to one of dimension $(d-2)(d-3)/2$. The map is surjective, so the kernel has dimension $d(d-3)/2 - (d-2)(d-3)/2 = d-3$. \square

There is a generating set for $(I_{W_d})_2$ where each generator is a scalar product with the vector τ . The other vectors in these scalar products are

$$\Lambda_k = \frac{x_{k+1}}{\alpha_{k+1}} \mathbf{n}_{k+1} - \frac{x_k}{\alpha_k} \mathbf{n}_{k-1} \in S_1^3.$$

Lemma 3.6. *The vectors $\{\Lambda_1, \dots, \Lambda_d\}$ form a basis for the space $\Psi^{-1}(\mathcal{D} \cap (I_{\mathcal{Q}})_2)$.*

Proof. Suppose that $\sum_{k=1}^d c_k \Lambda_k = 0$ is a linear dependence relation among the Λ_k . The coefficient of a variable x_k is

$$\frac{1}{\alpha_k} (c_{k-1} \mathbf{n}_k - c_k \mathbf{n}_{k-1}).$$

By the dependence relation, this must be zero, which implies that \mathbf{n}_{k-1} and \mathbf{n}_k are scalar multiples. This is impossible since they are normal vectors of adjacent facets

of a polyhedral cone. Hence, $c_{k-1} = c_k = 0$ for all k , which shows that the Λ_k are independent.

In the proof of [Lemma 3.3](#), we showed that $\dim(\Psi^{-1}(\mathcal{D} \cap (I_{\mathcal{C}})_2)) = d$, and we have just shown $\dim(\langle \Lambda_k \mid k = 1, \dots, d \rangle) = d$. To prove the result, it suffices to show $\langle \Lambda_k \mid k = 1, \dots, d \rangle \subseteq \Psi^{-1}(\mathcal{D} \cap (I_{\mathcal{C}})_2)$. The conditions of [\(8\)](#) are required for $\Lambda_k \in S_1^3$ to lie in $\Psi^{-1}(\mathcal{D} \cap (I_{\mathcal{C}})_2)$. We show these conditions are satisfied for each Λ_k .

Let $\mathbf{u}_i = 0$ if $i \neq k, k+1$, $\mathbf{u}_k = -\mathbf{n}_{k-1}/\alpha_k$, and $\mathbf{u}_{k+1} = \mathbf{n}_{k+1}/\alpha_{k+1}$ for each fixed k . Then

$$\Lambda_k = \frac{x_{k+1}}{\alpha_{k+1}} \mathbf{n}_{k+1} - \frac{x_k}{\alpha_k} \mathbf{n}_{k-1} = \sum_{i=1}^d \mathbf{u}_i x_i.$$

Since $\mathbf{n}_{k-1} \cdot \mathbf{v}_k = 0$, $\mathbf{n}_{k+1} \cdot \mathbf{v}_{k+1} = 0$, and $\mathbf{u}_i = 0$ for $i \neq k, k+1$, we have that $\mathbf{u}_i \cdot \mathbf{v}_i = 0$ for each $i = 1, \dots, d$. The expression $\mathbf{u}_i \cdot \mathbf{v}_{i+1} + \mathbf{u}_{i+1} \cdot \mathbf{v}_i$ is zero for all $i \neq k-1, k, k+1$ simply because $\mathbf{u}_i = 0$ for $i \neq k, k+1$. We have

$$\begin{aligned} \mathbf{u}_k \cdot \mathbf{v}_{k+1} + \mathbf{u}_{k+1} \cdot \mathbf{v}_k &= -\frac{\mathbf{n}_{k-1}}{\alpha_k} \cdot \mathbf{v}_{k+1} + \frac{\mathbf{n}_{k+1}}{\alpha_{k+1}} \cdot \mathbf{v}_k \\ &= -\frac{\mathbf{v}_{k-1} \times \mathbf{v}_k \cdot \mathbf{v}_{k+1}}{\alpha_k} + \frac{\mathbf{v}_{k+1} \times \mathbf{v}_{k+2} \cdot \mathbf{v}_k}{\alpha_{k+1}} \\ &= -\frac{|\mathbf{v}_{k-1} \mathbf{v}_k \mathbf{v}_{k+1}|}{\alpha_k} + \frac{|\mathbf{v}_{k+1} \mathbf{v}_{k+2} \mathbf{v}_k|}{\alpha_{k+1}} = 0 \end{aligned}$$

as $\alpha_j = |\mathbf{v}_{j-1} \mathbf{v}_j \mathbf{v}_{j+1}|$. It is easy to show that the expression $\mathbf{u}_i \cdot \mathbf{v}_{i+1} + \mathbf{u}_{i+1} \cdot \mathbf{v}_i$ is zero for $i = k \pm 1$. Thus, the \mathbf{u}_i satisfy the conditions in [\(8\)](#), so $\Lambda_k \in \Psi^{-1}(\mathcal{D} \cap (I_{\mathcal{C}})_2)$. \square

Theorem 3.7 (Wachspress quadrics). *The Wachspress quadrics $(I_{W_d})_2$ are those elements of S_2 that are diagonally supported and vanish on \mathcal{C} . The quadrics $Q_k = \Lambda_k \cdot \tau$ for $k = 1, \dots, d$ span $(I_{W_d})_2$.*

Proof. Let \mathbf{p} be the vector (x, y, z) . By definition of Wachspress coordinates,

$$\tau(w_d(\mathbf{p})) = \sum_{i=1}^d b_i(\mathbf{p}) \mathbf{v}_i = \mathbf{p} \sum_{i=1}^d b_i(\mathbf{p}).$$

We have

$$\begin{aligned} \Lambda_k(w_d(\mathbf{p})) &= \frac{b_{k+1}(\mathbf{p})}{\alpha_{k+1}} \mathbf{n}_{k+1} - \frac{b_k(\mathbf{p})}{\alpha_k} \mathbf{n}_{k-1} \\ &= \left(\prod_{j \neq k, k+1} \ell_j \right) \mathbf{n}_{k+1} - \left(\prod_{j \neq k-1, k} \ell_j \right) \mathbf{n}_{k-1} \\ &= \left(\prod_{j \neq k-1, k, k+1} \ell_j \right) (\ell_{k-1} \mathbf{n}_{k+1} - \ell_{k+1} \mathbf{n}_{k-1}) \\ &= H[\mathbf{n}_{k+1}(\mathbf{n}_{k-1} \cdot \mathbf{p}) - \mathbf{n}_{k-1}(\mathbf{n}_{k+1} \cdot \mathbf{p})], \end{aligned}$$

where $H = \prod_{j \neq k-1, k, k+1} \ell_j$. Set $\bar{H} := H \sum_{i=1}^d b_i(\mathbf{p})$. Then we have

$$\begin{aligned} Q_k(w_d(\mathbf{p})) &= \tau(w_d(\mathbf{p})) \cdot \Lambda_k(w_d(\mathbf{p})) \\ &= \bar{H} \mathbf{p} \cdot [\mathbf{n}_{k+1}(\mathbf{n}_{k-1} \cdot \mathbf{p}) - \mathbf{n}_{k-1}(\mathbf{n}_{k+1} \cdot \mathbf{p})] \\ &= \bar{H}[(\mathbf{n}_{k+1} \cdot \mathbf{p})(\mathbf{n}_{k-1} \cdot \mathbf{p}) - (\mathbf{n}_{k-1} \cdot \mathbf{p})(\mathbf{n}_{k+1} \cdot \mathbf{p})] = 0. \end{aligned}$$

We have just shown that $Q_k \in (I_{W_d})_2$. By Lemma 3.6, $\Psi^{-1}(\mathcal{D} \cap (I_{\ell})_2)$ is spanned by the Λ_k . Observe that $\langle Q_1, \dots, Q_d \rangle = \Psi(\langle \Lambda_k \rangle) = \mathcal{D} \cap (I_{\ell})_2$. Thus, $\dim(\langle Q_1, \dots, Q_d \rangle) = d - 3$, and by Lemma 3.5, $\dim((I_{W_d})_2) = d - 3$. Therefore, since $\langle Q_1, \dots, Q_d \rangle \subseteq (I_{W_d})_2$, we have $\langle Q_1, \dots, Q_d \rangle = (I_{W_d})_2 = \mathcal{D} \cap (I_{\ell})_2$. \square

Corollary 3.8. *The quadrics $\{\Lambda_2 \cdot \tau, \dots, \Lambda_{d-2} \cdot \tau\}$ are a basis for the quadrics in I_{W_d} , and in graded lex order, $\{x_1x_3, \dots, x_1x_{d-1}\}$ is a basis for $\text{in}_{<}(I_{W_d})_2$.*

Proof. Expanding the expression for $\Lambda_i \cdot \tau$ yields

$$\Lambda_i \cdot \tau = x_1x_{i+1} \left(\frac{\mathbf{v}_1 \cdot \mathbf{n}_{i+1}}{\alpha_{i+1}} \right) - x_1x_i \left(\frac{\mathbf{v}_1 \cdot \mathbf{n}_{i-1}}{\alpha_i} \right) + \zeta_i,$$

where $\zeta_i \in \mathbb{K}[x_2, \dots, x_d]$. Since $\mathbf{n}_i = \mathbf{v}_i \times \mathbf{v}_{i+1}$,

$$\Lambda_2 \cdot \tau = x_1x_3 \left(\frac{\mathbf{v}_1 \cdot \mathbf{n}_3}{\alpha_3} \right) + \zeta_2.$$

Since no three of the lines $\mathbb{V}(l_i)$ are concurrent, $\mathbf{v}_i \cdot \mathbf{n}_j$ is nonzero unless $j \in \{i, i + 1\}$, so we may use the lead term of $\Lambda_2 \cdot \tau$ to reduce $\Lambda_3 \cdot \tau$ to $x_1x_4 + f(x_2, \dots, x_d)$. Repeating the process proves that

$$\{x_1x_3, \dots, x_1x_{d-1}\} \subseteq \text{in}_{<}(I_{W_d})_2.$$

By Lemma 3.5, $(I_{W_d})_2$ has dimension $d - 3$, which concludes the proof. \square

Corollary 3.9. *There are no linear first syzygies on $(I_{W_d})_2$.*

Proof. By Corollary 3.8, we may assume that a basis for $(I_{W_d})_2$ has the form

$$\begin{aligned} &x_1x_3 + \zeta_3(x_2, \dots, x_d), \\ &x_1x_4 + \zeta_4(x_2, \dots, x_d), \\ &x_1x_5 + \zeta_5(x_2, \dots, x_d), \\ &\vdots \\ &x_1x_{d-1} + \zeta_{d-1}(x_2, \dots, x_d). \end{aligned}$$

Since the ζ_i do not involve x_1 , this implies that any linear first syzygy on $(I_{W_d})_2$ must be a linear combination of the Koszul syzygies on $\{x_3, \dots, x_{d-1}\}$. Now change the term order to graded lex with $x_i > x_{i+1} > \dots > x_d > x_1 > x_2 > \dots > x_{i-1}$. In

this order, arguing as in the proof of [Corollary 3.8](#) shows that we may assume a basis for $(I_{W_d})_2$ has the form

$$\begin{aligned} & x_i x_{i+2} + \zeta_{i+2}(x_1, \dots, \widehat{x}_i, \dots, x_d), \\ & x_i x_{i+3} + \zeta_{i+3}(x_1, \dots, \widehat{x}_i, \dots, x_d), \\ & x_i x_{i+4} + \zeta_{i+4}(x_1, \dots, \widehat{x}_i, \dots, x_d), \\ & \quad \vdots \\ & x_i x_{i-2} + \zeta_{i-2}(x_1, \dots, \widehat{x}_i, \dots, x_d). \end{aligned}$$

Hence, any linear first syzygy on $(I_{W_d})_2$ must be a combination of Koszul syzygies on $x_{i+2}, x_{i+3}, \dots, x_{i-2}$. Iterating this process for the term orders above shows there can be no linear first syzygies on $(I_{W_d})_2$. \square

3D. Decomposition of $\mathbb{V}((I_{W_d})_2)$. We now prove that $\mathbb{V}((I_{W_d})_2) = \mathcal{C} \cup W_d$. The results in [Sections 4](#) and [5](#) are independent of this fact.

Lemma 3.10. *For any i, j , and k , we have*

$$|\mathbf{n}_i \mathbf{n}_j \mathbf{n}_k| = |\mathbf{v}_j \mathbf{v}_k \mathbf{v}_{k+1}| \cdot |\mathbf{v}_i \mathbf{v}_{i+1} \mathbf{v}_{j+1}| - |\mathbf{v}_{j+1} \mathbf{v}_k \mathbf{v}_{k+1}| \cdot |\mathbf{v}_i \mathbf{v}_{i+1} \mathbf{v}_j|.$$

Proof. Apply the formulas $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$ and $|\mathbf{abc}| = \mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$:

$$\begin{aligned} |\mathbf{n}_i \mathbf{n}_j \mathbf{n}_k| &= \mathbf{n}_i \times \mathbf{n}_j \cdot \mathbf{n}_k = (\mathbf{n}_i \times (\mathbf{v}_j \times \mathbf{v}_{j+1})) \cdot \mathbf{n}_k \\ &= [\mathbf{v}_j (\mathbf{n}_i \cdot \mathbf{v}_{j+1}) - \mathbf{v}_{j+1} (\mathbf{n}_i \cdot \mathbf{v}_j)] \cdot \mathbf{n}_k \\ &= (\mathbf{v}_j \cdot \mathbf{n}_k) (\mathbf{n}_i \cdot \mathbf{v}_{j+1}) - (\mathbf{v}_{j+1} \cdot \mathbf{n}_k) (\mathbf{n}_i \cdot \mathbf{v}_j) \\ &= |\mathbf{v}_j \mathbf{v}_k \mathbf{v}_{k+1}| \cdot |\mathbf{v}_i \mathbf{v}_{i+1} \mathbf{v}_{j+1}| - |\mathbf{v}_{j+1} \mathbf{v}_k \mathbf{v}_{k+1}| \cdot |\mathbf{v}_i \mathbf{v}_{i+1} \mathbf{v}_j|. \end{aligned} \quad \square$$

Corollary 3.11. *We have $|\mathbf{n}_i \mathbf{n}_j \mathbf{n}_{j+1}| = \alpha_{j+1} |\mathbf{v}_i \mathbf{v}_{i+1} \mathbf{v}_{j+1}|$.*

Proof. This follows from [Lemma 3.10](#) and the definition of α_{j+1} . \square

Corollary 3.12. *We have $|\mathbf{n}_{i-1} \mathbf{n}_i \mathbf{n}_{i+1}| = \alpha_i \alpha_{i+1}$.*

Proof. This follows from [Lemma 3.10](#) and the definition of α_i and α_{i+1} . \square

Lemma 3.13. *Let $\mathbf{x} = [x_1 : \dots : x_d] \in \mathbb{V}((I_{W_d})_2) \setminus \mathcal{C}$. If $\tau(\mathbf{x})$ is a base point $p_{ij} = \mathbf{n}_i \times \mathbf{n}_j$, then \mathbf{x} lies on the exceptional line \widehat{p}_{ij} over p_{ij} .*

Proof. Since indices are cyclic, we assume that $i = 1$. Thus, $\tau(\mathbf{x}) = p_{1,j} = \mathbf{n}_1 \times \mathbf{n}_j$ for some $j \notin \{d, 1, 2\}$. The relation $Q_1(\mathbf{x}) = \Lambda_1 \cdot \tau(\mathbf{x}) = \Lambda_1 \cdot (\mathbf{n}_1 \times \mathbf{n}_j) = 0$ yields

$$L(1) := x_2 \mathbf{n}_2 \cdot p_{1,j} - x_1 \mathbf{n}_d \cdot p_{1,j} = 0. \quad (9)$$

The relation $Q_j(\mathbf{x}) = 0$ implies

$$L(j) := x_{j+1} |\mathbf{n}_{j+1} \mathbf{n}_1 \mathbf{n}_j| - x_j |\mathbf{n}_2 \mathbf{n}_1 \mathbf{n}_j| = 0. \quad (10)$$

Also,

$$Q_2(\mathbf{x}) = (x_3\mathbf{n}_3 - x_2\mathbf{n}_1) \cdot \mathbf{n}_1 \times \mathbf{n}_j = x_3|\mathbf{n}_3\mathbf{n}_1\mathbf{n}_j| = 0,$$

implying $x_3 = 0$ since $|\mathbf{n}_3\mathbf{n}_1\mathbf{n}_j| \neq 0$ if $j \neq 3$. Assume $x_k = 0$ for $3 \leq k < j - 1$. Note that

$$Q_k(\mathbf{x}) = (x_{k+1}\mathbf{n}_{k+1} - x_k\mathbf{n}_{k-1}) \cdot \mathbf{n}_1 \times \mathbf{n}_j = x_{k+1}|\mathbf{n}_{k+1}\mathbf{n}_1\mathbf{n}_j| = 0;$$

hence, $x_{k+1} = 0$ since $|\mathbf{n}_{k+1}\mathbf{n}_1\mathbf{n}_j| \neq 0$ and by induction $x_k = 0$ for $3 \leq k \leq j - 1$. An analogous argument shows that $x_k = 0$ for $j + 2 \leq k \leq d$. Hence, \mathbf{x} lies on the line $\mathbb{V}(L(1), L(j), x_k \mid k \notin \{1, 2, j, j + 1\})$, which is the exceptional line $\hat{p}_{1,j}$. \square

Theorem 3.14. *The subset $\mathbb{V}(\langle(I_{W_d})_2\rangle) \setminus \mathcal{C}$ is contained in W_d . It follows that the variety $\mathbb{V}(\langle(I_{W_d})_2\rangle)$ has irreducible decomposition $W_d \cup \mathcal{C}$.*

Proof. Let $\mathbf{x} = [x_1 : \dots : x_d] \in \mathbb{V}(\langle(I_{W_d})_2\rangle) \setminus \mathcal{C}$. The Wachspress quadrics give the relations

$$x_{r+1}\mathbf{n}_{r+1} \cdot \boldsymbol{\tau} = x_r\mathbf{n}_{r-1} \cdot \boldsymbol{\tau} \tag{11}$$

for each $r = 1, \dots, d$. By [Theorem 1.6](#), the adjoint is independent of triangulation,

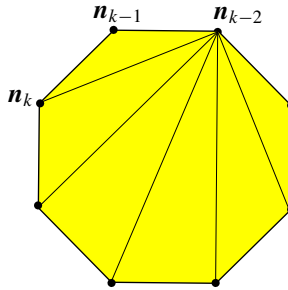


Figure 3. Triangulation used for adjoint.

so we use \mathcal{A} to denote the adjoint, specifying the triangulation if necessary. We now show, for each $k \in \{1, \dots, d\}$, $b_k(\boldsymbol{\tau}(\mathbf{x})) = \mathcal{A}(\boldsymbol{\tau}(\mathbf{x}))x_k$, where the triangulation above is used for the adjoint \mathcal{A} . It follows from the uniqueness of Wachspress coordinates that the denominator $\sum_{i=1}^d b_i$ of β_i is the adjoint of \mathbf{P}_d^* , so it follows that

$$w_d(\boldsymbol{\tau}(\mathbf{x})) = \mathcal{A}(\boldsymbol{\tau}(\mathbf{x}))\mathbf{x}. \tag{12}$$

Provided $\mathcal{A}(\boldsymbol{\tau}(\mathbf{x})) \neq 0$, the result follows since $w_d(\boldsymbol{\tau}(\mathbf{x})) \in \mathbb{P}^{d-1}$ is a nonzero scalar multiple of \mathbf{x} ; hence, \mathbf{x} is in the image of the Wachspress map and thus lies on W_d . If $\mathbf{x} \in \mathbb{V}(\langle(I_{W_d})_2\rangle) \setminus \mathcal{C}$ and $\mathcal{A}(\boldsymbol{\tau}(\mathbf{x})) = 0$, then by [\(12\)](#) $w_d(\boldsymbol{\tau}(\mathbf{x})) = 0$, and hence, $\boldsymbol{\tau}(\mathbf{x})$ is a basepoint of w_d . Thus, $\boldsymbol{\tau}(\mathbf{x}) = \mathbf{n}_i \times \mathbf{n}_j$ for some diagonal pair (i, j) . By [Lemma 3.13](#),

\mathbf{x} lies on an exceptional line and hence lies on W_d . To prove the claim, note that since all indices are cyclic it suffices to assume $k = 3$. Let $|\mathbf{n}_i \mathbf{n}_j \mathbf{n}_k| = |\mathbf{n}_{ijk}|$ and

$$\mathbf{n}_{i_1, \dots, i_m} \cdot \tau := \prod_{j=1}^m (\mathbf{n}_{i_j} \cdot \tau).$$

This is the product of m linear forms in S , and with this notation,

$$b_3(\tau) = \mathbf{n}_{1,4,5, \dots, d} \cdot \tau.$$

For each $r \in \{3, \dots, d\}$, define

$$\sigma_r := (\mathbf{n}_{4, \dots, r} \cdot \tau) \mathbf{n}_1 \cdot \left[\sum_{i=3}^r \mathbf{v}_i (\mathbf{n}_{r+1, \dots, d} \cdot \tau) x_i + \sum_{i=r+1}^d \mathbf{v}_i (\mathbf{n}_{r-1, \dots, i-2} \cdot \tau) (\mathbf{n}_{i+1, \dots, d} \cdot \tau) x_r \right],$$

where we set $\mathbf{n}_{i, \dots, j} \cdot \tau = 1$ if $j < i$. We show $x_3 \mathcal{A}(\tau(\mathbf{x})) = \sigma_3 = \sigma_d = b_3(\tau(\mathbf{x}))$. First, we show $\sigma_3 = x_3 \mathcal{A}(\tau)$: to see this, note that

$$x_3 \mathcal{A}(\tau) = |\mathbf{n}_{123}| (\mathbf{n}_{4, \dots, d} \cdot \tau) x_3 + \sum_{i=4}^d |\mathbf{n}_{1, i-1, i}| (\mathbf{n}_{2, \dots, i-2} \cdot \tau) (\mathbf{n}_{i+1, \dots, d} \cdot \tau) x_3, \quad (13)$$

where we express the adjoint \mathcal{A} using the triangulation in [Figure 3](#). Applying the scalar triple product to $|\mathbf{n}_{123}|$ and $|\mathbf{n}_{1, i-1, i}|$ in the expression (13) yields

$$\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3) (\mathbf{n}_{4, \dots, d} \cdot \tau) x_3 + \sum_{i=4}^d \mathbf{n}_1 \cdot (\mathbf{n}_{i-1} \times \mathbf{n}_i) (\mathbf{n}_{2, \dots, i-2} \cdot \tau) (\mathbf{n}_{i+1, \dots, d} \cdot \tau) x_3. \quad (14)$$

Factoring an \mathbf{n}_1 and noting that $\mathbf{n}_i \times \mathbf{n}_{i+1} = \mathbf{v}_{i+1}$, (14) becomes

$$\mathbf{n}_1 \cdot \left[\mathbf{v}_3 (\mathbf{n}_{4, \dots, d} \cdot \tau) x_3 + \sum_{i=4}^d \mathbf{v}_i (\mathbf{n}_{2, \dots, i-2} \cdot \tau) (\mathbf{n}_{i+1, \dots, d} \cdot \tau) x_3 \right] = \sigma_3.$$

Now we show $\sigma_d = b_3(\tau)$. Since $\mathbf{n}_{d+1, \dots, d} \cdot \tau = 1$,

$$\sigma_d = (\mathbf{n}_{4, \dots, d} \cdot \tau) \mathbf{n}_1 \cdot \left(\sum_{i=3}^d \mathbf{v}_i (\mathbf{n}_{d+1, \dots, d} \cdot \tau) x_i \right) = (\mathbf{n}_{4, \dots, d} \cdot \tau) \mathbf{n}_1 \cdot \left(\sum_{i=3}^d \mathbf{v}_i x_i \right). \quad (15)$$

Observing that $\mathbf{n}_1 \cdot \sum_{i=1}^2 x_i \mathbf{v}_i = 0$, we see that (15) is

$$(\mathbf{n}_{4, \dots, d} \cdot \tau) (\mathbf{n}_1 \cdot \tau) = \mathbf{n}_{1,4, \dots, d} \cdot \tau = b_3(\tau).$$

We now claim that for $r \in \{3, \dots, d - 1\}$ we have $\sigma_r = \sigma_{r+1}$. Indeed,

$$\begin{aligned} \sigma_r &= (\mathbf{n}_{4, \dots, r} \cdot \tau) \mathbf{n}_1 \cdot \left[\sum_{i=3}^r \mathbf{v}_i(\mathbf{n}_{r+1, \dots, d} \cdot \tau) x_i \right. \\ &\quad \left. + \sum_{i=r+1}^d \mathbf{v}_i(\mathbf{n}_{r, \dots, i-2} \cdot \tau)(\mathbf{n}_{i+1, \dots, d} \cdot \tau)(\mathbf{n}_{r-1} \cdot \tau) x_r \right] \\ &= (\mathbf{n}_{4, \dots, r} \cdot \tau) \mathbf{n}_1 \cdot \left[\sum_{i=3}^r \mathbf{v}_i(\mathbf{n}_{r+1, \dots, d} \cdot \tau) x_i \right. \\ &\quad \left. + \sum_{i=r+1}^d \mathbf{v}_i(\mathbf{n}_{r, \dots, i-2} \cdot \tau)(\mathbf{n}_{i+1, \dots, d} \cdot \tau)(\mathbf{n}_{r+1} \cdot \tau) x_{r+1} \right], \end{aligned}$$

where we have applied (11) to the last term. Factoring out $\mathbf{n}_{r+1} \cdot \tau$ yields

$$(\mathbf{n}_{4, \dots, r+1} \cdot \tau) \mathbf{n}_1 \cdot \left[\sum_{i=3}^r \mathbf{v}_i(\mathbf{n}_{r+2, \dots, d} \cdot \tau) x_i + \sum_{i=r+1}^d \mathbf{v}_i(\mathbf{n}_{r, \dots, i-2} \cdot \tau)(\mathbf{n}_{i+1, \dots, d} \cdot \tau) x_{r+1} \right].$$

Lastly, since the expressions in both summations agree at the index $i = r + 1$, we can shift the indices of summation,

$$(\mathbf{n}_{4, \dots, r+1} \cdot \tau) \mathbf{n}_1 \cdot \left[\sum_{i=3}^{r+1} \mathbf{v}_i(\mathbf{n}_{r+2, \dots, d} \cdot \tau) x_i + \sum_{i=r+2}^d \mathbf{v}_i(\mathbf{n}_{r, \dots, i-2} \cdot \tau)(\mathbf{n}_{i+1, \dots, d} \cdot \tau) x_{r+1} \right],$$

which is precisely σ_{r+1} , proving the claim. The claim shows that $\sigma_3 = \sigma_d$; hence, (12) holds, and so \mathbf{x} lies in W_d if $\mathcal{A}(\tau(\mathbf{x})) \neq 0$. \square

4. The Wachspress cubics

Theorem 3.14 shows that the Wachspress quadrics do not suffice to cut out the Wachspress variety W_d . We now construct cubics, the *Wachspress cubics*, that lie in I_{W_d} and do not arise from the Wachspress quadrics. These cubics are determinants of 3×3 matrices of linear forms. The key to showing that they are in I_{W_d} is to write them as a difference of adjoints $\mathcal{A}_{T_1(C)} - \mathcal{A}_{T_2(C)}$, where $T_1(C)$ and $T_2(C)$ are two different triangulations of a subcone C of the dual cone P_d^* . By Theorem 1.6, the difference is zero, so the cubic is in I_{W_d} .

4A. Construction of Wachspress cubics. As in Lemma 3.6, let

$$\Lambda_r = \frac{x_{r+1}}{\alpha_{r+1}} \mathbf{n}_{r+1} - \frac{x_r}{\alpha_r} \mathbf{n}_{r-1}.$$

Theorem 4.1. *If $i \neq j \neq k \neq i$, then $w_{i,j,k} := |\Lambda_i, \Lambda_j, \Lambda_k| \in I_{W_d}$.*

Proof. We break the proof into two parts. First, suppose no pair of (i, j, k) corresponds to an edge of P_d . We call such an (i, j, k) a T -triple. A direct

calculation shows that, if (i, j, k) is a T -triple, then evaluating the monomial $x_i x_j x_k$ at Wachspress coordinates yields

$$x_i x_j x_k(w_d) = b_i b_j b_k = A^2 \prod_{m \in \gamma(i, j, k)} \ell_m, \tag{16}$$

where $\gamma(i, j, k)$ is as in [Definition 3.4](#). Since there are no T -triples if $d < 6$, we may assume $d \geq 6$. Changing variables by replacing x_i with x_i/α_i , we may ignore the constants α_i . Using the definition of the Λ 's, observe that

$$\begin{aligned} w_{i, j, k} &= |\mathbf{n}_{i+1} \mathbf{n}_{j+1} \mathbf{n}_{k+1} | x_{i+1} x_{j+1} x_{k+1} - |\mathbf{n}_{i+1} \mathbf{n}_{j+1} \mathbf{n}_{k-1} | x_{i+1} x_{j+1} x_k \\ &\quad - |\mathbf{n}_{i+1} \mathbf{n}_{j-1} \mathbf{n}_{k+1} | x_{i+1} x_j x_{k+1} + |\mathbf{n}_{i+1} \mathbf{n}_{j-1} \mathbf{n}_{k-1} | x_{i+1} x_j x_k \\ &\quad - |\mathbf{n}_{i-1} \mathbf{n}_{j+1} \mathbf{n}_{k+1} | x_i x_{j+1} x_{k+1} + |\mathbf{n}_{i-1} \mathbf{n}_{j+1} \mathbf{n}_{k-1} | x_i x_{j+1} x_k \\ &\quad + |\mathbf{n}_{i-1} \mathbf{n}_{j-1} \mathbf{n}_{k+1} | x_i x_j x_{k+1} - |\mathbf{n}_{i-1} \mathbf{n}_{j-1} \mathbf{n}_{k-1} | x_i x_j x_k. \end{aligned} \tag{17}$$

There are several situations to consider, depending on various possibilities for interactions among the indices. Interactions may occur if $i + 1 = j - 1$ or $j + 1 = k - 1$ or $k + 1 = i - 1$, so there are four cases:

1. All three hold.
2. Two hold.
3. One holds.
4. None hold.

Case 1. The indices (i, j, k) satisfy Case 1 if and only if $d = 6$. For $d = 6$, there are only two T -triples: $(1, 3, 5)$ and $(2, 4, 6)$. We show that $w_{1,3,5}$ vanishes on Wachspress coordinates; the case of $w_{2,4,6}$ is similar. All but two of the determinants in [Equation \(17\)](#) vanish, leaving

$$w_{1,3,5} = |\Lambda_1, \Lambda_3, \Lambda_5| = |\mathbf{n}_2 \mathbf{n}_4 \mathbf{n}_6 | x_2 x_4 x_6 - |\mathbf{n}_6 \mathbf{n}_2 \mathbf{n}_4 | x_1 x_3 x_5. \tag{18}$$

Notice that the coefficients are equal, and we conclude by showing that

$$x_1 x_3 x_5 - x_2 x_4 x_6$$

vanishes on Wachspress coordinates. The monomials $x_1 x_3 x_5$ and $x_2 x_4 x_6$ evaluated at Wachspress coordinates are $b_1 b_3 b_5$ and $b_2 b_4 b_6$, respectively. Both of these are equal to A^2 , so $x_1 x_3 x_5 - x_2 x_4 x_6$ vanishes on Wachspress coordinates.

Case 2. We can assume without loss of generality $i + 1 \neq j - 1$, $j + 1 = k - 1$, and $k + 1 = i - 1$. Four coefficients vanish in [\(17\)](#), yielding

$$\begin{aligned} w_{i, j, k} &= |\mathbf{n}_{i+1} \mathbf{n}_{j+1} \mathbf{n}_{i-1} | x_{i+1} x_{j+1} x_{i-1} \\ &\quad - |\mathbf{n}_{i+1} \mathbf{n}_{j-1} \mathbf{n}_{i-1} | x_{i+1} x_j x_{i-1} \\ &\quad + |\mathbf{n}_{i+1} \mathbf{n}_{j-1} \mathbf{n}_{j+1} | x_{i+1} x_j x_{i-2} \\ &\quad - |\mathbf{n}_{i-1} \mathbf{n}_{j-1} \mathbf{n}_{j+1} | x_i x_j x_{i-2}. \end{aligned}$$

Evaluating this at Wachspress coordinates yields

$$\begin{aligned}
 w_{i,j,k} \circ w_d &= |\mathbf{n}_{i+1}\mathbf{n}_{j+1}\mathbf{n}_{i-1}| \prod_{m \in \gamma(i+1,j+1,i-1)} \ell_m + |\mathbf{n}_{i+1}\mathbf{n}_{j-1}\mathbf{n}_{i-1}| \prod_{m \in \gamma(i+1,j,i-1)} \ell_m \\
 &\quad - |\mathbf{n}_{i+1}\mathbf{n}_{j-1}\mathbf{n}_{j+1}| \prod_{m \in \gamma(i+1,j,i-1)} \ell_m - |\mathbf{n}_{i-1}\mathbf{n}_{j-1}\mathbf{n}_{j+1}| \prod_{m \in \gamma(i,j,i-1)} \ell_m \\
 &= A^2 \left(\prod_{m \in \gamma(i-1,i+1,j+1,j)} \ell_m \right) (|\mathbf{n}_{i+1}\mathbf{n}_{j+1}\mathbf{n}_{i-1}| \ell_{j-1} - |\mathbf{n}_{i+1}\mathbf{n}_{j-1}\mathbf{n}_{i-1}| \ell_{j+1} \\
 &\quad + |\mathbf{n}_{i+1}\mathbf{n}_{j-1}\mathbf{n}_{j+1}| \ell_{i-1} - |\mathbf{n}_{i-1}\mathbf{n}_{j-1}\mathbf{n}_{j+1}| \ell_{i+1}) \\
 &= A^2 \left(\prod_{m \in \gamma(i-1,i+1,j+1,j)} \ell_m \right) [(|\mathbf{n}_{i+1}\mathbf{n}_{j+1}\mathbf{n}_{i-1}| \ell_{j-1} + |\mathbf{n}_{i-1}\mathbf{n}_{j+1}\mathbf{n}_{j-1}| \ell_{i+1}) \\
 &\quad - (|\mathbf{n}_{i+1}\mathbf{n}_{j-1}\mathbf{n}_{i-1}| \ell_{j+1} + |\mathbf{n}_{i+1}\mathbf{n}_{j+1}\mathbf{n}_{j-1}| \ell_{i-1})],
 \end{aligned}$$

where

$$A = \prod_{i=1}^d \ell_i.$$

The last factor is the difference of two adjoints with respect to the triangulations of the quadrilateral in Figure 4. The vanishing can be seen directly: write $\mathbf{n}_1, \dots, \mathbf{n}_4$ for $\mathbf{n}_{i-1}, \mathbf{n}_{i+1}, \mathbf{n}_{j-1}$, and \mathbf{n}_{j+1} . Then the last factor is

$$|\mathbf{n}_2\mathbf{n}_3\mathbf{n}_4| \ell_1 - |\mathbf{n}_1\mathbf{n}_3\mathbf{n}_4| \ell_2 + |\mathbf{n}_1\mathbf{n}_2\mathbf{n}_4| \ell_3 - |\mathbf{n}_1\mathbf{n}_2\mathbf{n}_3| \ell_4.$$

Applying $\frac{d}{dx}$ to this shows the x coefficient is

$$|\mathbf{n}_2\mathbf{n}_3\mathbf{n}_4| \mathbf{n}_{11} - |\mathbf{n}_1\mathbf{n}_3\mathbf{n}_4| \mathbf{n}_{21} + |\mathbf{n}_1\mathbf{n}_2\mathbf{n}_4| \mathbf{n}_{31} - |\mathbf{n}_1\mathbf{n}_2\mathbf{n}_3| \mathbf{n}_{41}.$$

This is the determinant of the matrix of the \mathbf{n}_i with a repeat row for the x coordinates \mathbf{n}_{i1} , so it vanishes. Reason similarly for the y and z coefficients.

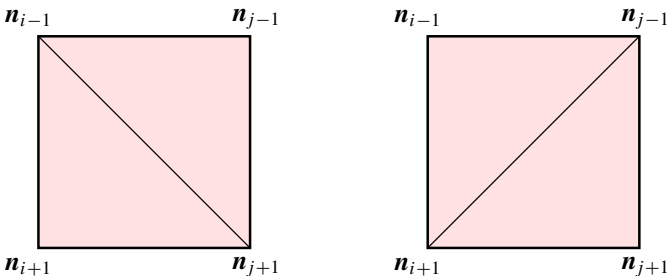


Figure 4. Case 2 triangulation.

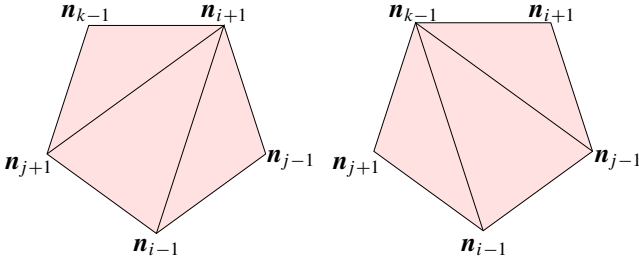


Figure 5. Case 3 triangulation.

Case 3. Assume without loss of generality $i + 1 \neq j - 1$, $j + 1 \neq k - 1$, and $k + 1 = i - 1$. In this case, two coefficients vanish in (17), and after evaluating at Wachspress coordinates, we obtain

$$\begin{aligned}
 & w_{i,j,k} \circ w_d \\
 &= |\mathbf{n}_{i+1}\mathbf{n}_{j+1}\mathbf{n}_{i-1}| \prod_{m \in \gamma(i+1, j+1, k+1)} \ell_m - |\mathbf{n}_{i+1}\mathbf{n}_{j+1}\mathbf{n}_{k-1}| \prod_{m \in \gamma(i+1, j+1, k)} \ell_m \\
 &\quad - |\mathbf{n}_{i+1}\mathbf{n}_{j-1}\mathbf{n}_{i-1}| \prod_{m \in \gamma(i+1, j, k+1)} \ell_m + |\mathbf{n}_{i+1}\mathbf{n}_{j-1}\mathbf{n}_{k-1}| \prod_{m \in \gamma(i+1, j, k)} \ell_m \\
 &\quad + |\mathbf{n}_{i-1}\mathbf{n}_{j+1}\mathbf{n}_{k-1}| \prod_{m \in \gamma(i, j+1, k)} \ell_m - |\mathbf{n}_{i-1}\mathbf{n}_{j-1}\mathbf{n}_{k-1}| \prod_{m \in \gamma(i, j, k)} \ell_m \\
 &= A^2 \left(\prod_{\substack{m \in \gamma(i, j, k, \\ i+1, j+1, k+1)}} \ell_m \right) (|\mathbf{n}_{i+1}\mathbf{n}_{j+1}\mathbf{n}_{i-1}| \ell_{j-1} \ell_{k-1} - |\mathbf{n}_{i+1}\mathbf{n}_{j+1}\mathbf{n}_{k-1}| \ell_{i-1} \ell_{j-1} \\
 &\quad - |\mathbf{n}_{i+1}\mathbf{n}_{j-1}\mathbf{n}_{i-1}| \ell_{j+1} \ell_{k-1} + |\mathbf{n}_{i+1}\mathbf{n}_{j-1}\mathbf{n}_{k-1}| \ell_{j+1} \ell_{i-1} \\
 &\quad + |\mathbf{n}_{i-1}\mathbf{n}_{j+1}\mathbf{n}_{k-1}| \ell_{i+1} \ell_{j-1} - |\mathbf{n}_{i-1}\mathbf{n}_{j-1}\mathbf{n}_{k-1}| \ell_{i+1} \ell_{j+1}).
 \end{aligned}$$

The last factor is the difference of adjoints with respect to the triangulations of the pentagon in Figure 5.

Case 4. In this case, evaluation at Wachspress coordinates yields

$$\begin{aligned}
 & w_{i,j,k} \circ w_d = |\mathbf{n}_{i+1}\mathbf{n}_{j+1}\mathbf{n}_{k+1}| \prod_{m \in \gamma(i+1, j+1, k+1)} \ell_m - |\mathbf{n}_{i+1}\mathbf{n}_{j+1}\mathbf{n}_{k-1}| \prod_{m \in \gamma(i+1, j+1, k)} \ell_m \\
 &\quad - |\mathbf{n}_{i+1}\mathbf{n}_{j-1}\mathbf{n}_{k+1}| \prod_{m \in \gamma(i+1, j, k+1)} \ell_m + |\mathbf{n}_{i+1}\mathbf{n}_{j-1}\mathbf{n}_{k-1}| \prod_{m \in \gamma(i+1, j, k)} \ell_m \\
 &\quad - |\mathbf{n}_{i-1}\mathbf{n}_{j+1}\mathbf{n}_{k+1}| \prod_{m \in \gamma(i, j+1, k+1)} \ell_m + |\mathbf{n}_{i-1}\mathbf{n}_{j+1}\mathbf{n}_{k-1}| \prod_{m \in \gamma(i, j+1, k)} \ell_m \\
 &\quad + |\mathbf{n}_{i-1}\mathbf{n}_{j-1}\mathbf{n}_{k+1}| \prod_{m \in \gamma(i, j, k+1)} \ell_m - |\mathbf{n}_{i-1}\mathbf{n}_{j-1}\mathbf{n}_{k-1}| \prod_{m \in \gamma(i, j, k)} \ell_m
 \end{aligned}$$

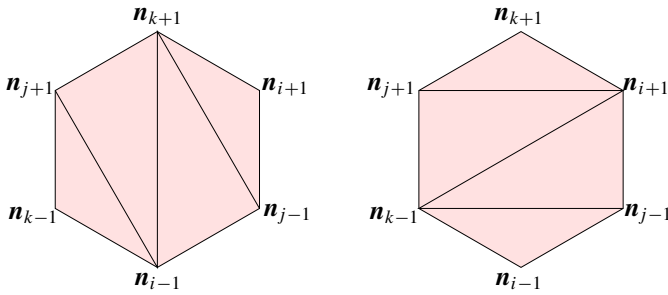


Figure 6. Case 4 triangulation.

$$\begin{aligned}
 &= A^2 \left(\prod_{\substack{m \in \gamma(i, j, k, \\ i+1, j+1, k+1)}} \ell_m \right) \\
 &\quad \times (|n_{i+1}n_{j+1}n_{k+1}| \ell_{i-1} \ell_{j-1} \ell_{k-1} - |n_{i+1}n_{j+1}n_{k-1}| \ell_{i-1} \ell_{j-1} \ell_{k+1} \\
 &\quad - |n_{i+1}n_{j-1}n_{k+1}| \ell_{i-1} \ell_{j+1} \ell_{k-1} + |n_{i+1}n_{j-1}n_{k-1}| \ell_{i-1} \ell_{j+1} \ell_{k+1} \\
 &\quad - |n_{i-1}n_{j+1}n_{k+1}| \ell_{i+1} \ell_{j-1} \ell_{k-1} + |n_{i-1}n_{j+1}n_{k-1}| \ell_{j+1} \ell_{i-1} \ell_{k+1} \\
 &\quad + |n_{i-1}n_{j-1}n_{k+1}| \ell_{i+1} \ell_{j+1} \ell_{k-1} - |n_{i-1}n_{j-1}n_{k-1}| \ell_{i+1} \ell_{j+1} \ell_{k+1}).
 \end{aligned}$$

The last factor is the difference of adjoints expressed using the triangulations of the hexagon in Figure 6. This completes the analysis when (i, j, k) is a T -triple.

Next, we consider the situation when (i, j, k) contains a pair of consecutive indices. Suppose first that there are exactly two consecutive vertices; without loss of generality, we assume the indices are $(2, 3, i)$ with $i > 4$. We have

$$\begin{aligned}
 w_{2,3,i} := |\Lambda_2 \Lambda_3 \Lambda_i| &= |n_2 n_4 n_{i+1}| x_3 x_4 x_{i+1} - |n_3 n_4 n_{i-1}| x_3 x_4 x_i \\
 &\quad - |n_3 n_2 n_{i+1}| x_3 x_3 x_{i+1} + |n_3 n_2 n_{i-1}| x_3 x_3 x_i \\
 &\quad - |n_1 n_4 n_{i+1}| x_2 x_4 x_{i+1} + |n_1 n_4 n_{i-1}| x_2 x_4 x_i \\
 &\quad + |n_1 n_2 n_{i+1}| x_2 x_3 x_{i+1} - |n_1 n_2 n_{i-1}| x_2 x_3 x_i.
 \end{aligned}$$

We show that $w_{2,3,i} \circ w_d$ is a multiple of the difference between two expressions of the adjoint polynomial of a polygon with respect to two different triangulations. After evaluation at w_d , each monomial has a common factor of $A \prod_{j \neq 2,3} \ell_j$. Thus, we can express

$$\frac{w_{2,3,i}(w_d)}{A \prod_{j \neq 2,3} \ell_j}$$

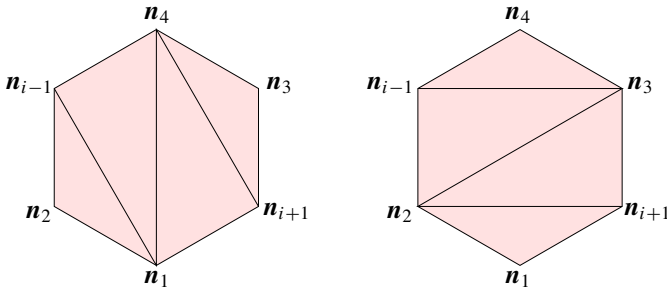


Figure 7. Triangulations for the non- T -triples.

as

$$\begin{aligned} \frac{w_{2,3,i}(w_d)}{A \prod_{j \neq 2,3} \ell_j} &= |n_2 n_4 n_{i+1}| \prod_{j \neq 3,4,i+1} \ell_j - |n_3 n_4 n_{i-1}| \prod_{j \neq 3,4,i-1} \ell_j \\ &\quad - |n_3 n_2 n_{i+1}| \prod_{j \neq 2,3,i+1} \ell_j + |n_3 n_2 n_{i-1}| \prod_{j \neq 2,3,i-1} \ell_j \\ &\quad - |n_1 n_4 n_{i+1}| \prod_{j \neq 1,4,i+1} \ell_j + |n_1 n_4 n_{i-1}| \prod_{j \neq 1,4,i-1} \ell_j \\ &\quad + |n_1 n_2 n_{i+1}| \prod_{j \neq 1,2,i+1} \ell_j - |n_1 n_2 n_{i-1}| \prod_{j \neq 1,2,i-1} \ell_j \\ &= \left(\prod_{j \in \gamma(2,4,i,i+1)} \ell_j \right) (|n_2 n_4 n_{i+1}| \ell_1 \ell_3 \ell_{i-1} - |n_3 n_4 n_{i-1}| \ell_1 \ell_2 \ell_{i+1} \\ &\quad - |n_3 n_2 n_{i+1}| \ell_1 \ell_4 \ell_{i-1} + |n_3 n_2 n_{i-1}| \ell_1 \ell_4 \ell_{i+1} \\ &\quad - |n_1 n_4 n_{i+1}| \ell_2 \ell_3 \ell_{i-1} + |n_1 n_4 n_{i-1}| \ell_2 \ell_3 \ell_{i+1} \\ &\quad + |n_1 n_2 n_{i+1}| \ell_3 \ell_4 \ell_{i-1} - |n_1 n_2 n_{i-1}| \ell_3 \ell_4 \ell_{i+1}). \end{aligned}$$

The factor in parentheses is the difference of the adjoints computed with respect to the triangulations of the polygon in [Figure 7](#).

Finally, for the case where the three vertices are consecutive, assume without loss of generality the triple is $(2, 3, 4)$, and proceed as above. In this case, the triangulations that arise are those that appear in [Figure 5](#). \square

Definition 4.2. $I(d)$ is the ideal generated by the Wachspress quadrics appearing in [Corollary 3.8](#) and the Wachspress cubics appearing in [Theorem 4.1](#).

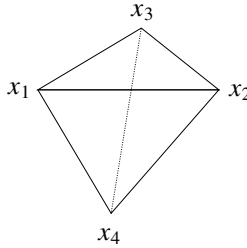
5. Gröbner basis, Stanley–Reisner ring, and free resolution

In this section, we determine the initial ideal of $I(d)$ in graded lex order and prove $I(d) = I_{W_d}$. First, we present some preliminaries.

5A. Simplicial complexes and combinatorial commutative algebra. An abstract n -simplex is a set consisting of all subsets of an $(n + 1)$ -element ground set. Typically a simplex is viewed as a geometric object; for example, a 2-simplex on the set $\{a, b, c\}$ can be visualized as a triangle with the subset $\{a, b, c\}$ corresponding to the whole triangle, $\{a, b\}$ an edge, and $\{a\}$ a vertex. For this reason, elements of the ground set are called the vertices.

Definition 5.1 [Ziegler 1995]. A simplicial complex Δ on a vertex set V is a collection of subsets σ of V such that, if $\sigma \in \Delta$ and $\tau \subset \sigma$, then $\tau \in \Delta$. If $|\sigma| = i + 1$, then σ is called an i -face. Let $f_i(\Delta)$ denote the number of i -faces of Δ , and define $\dim(\Delta) = \max\{i \mid f_i(\Delta) \neq 0\}$. If $\dim(\Delta) = n - 1$, we define $f_\Delta(t) = \sum_{i=0}^n f_{i-1}t^{n-i}$. The ordered list of coefficients of $f_\Delta(t)$ is the f -vector of Δ , and the coefficients of $h_\Delta(t) := f_\Delta(t - 1)$ are the h -vector of Δ .

Example 5.2. Consider the 1-skeleton of a tetrahedron with vertices x_1, x_2, x_3, x_4 , as in the figure.



The corresponding simplicial complex Δ consists of all vertices and edges, so $\Delta = \{\emptyset, \{x_i\}, \{x_i, x_j\} \mid 1 \leq i \leq 4 \text{ and } i < j \leq 4\}$. Thus, $f(\Delta) = (1, 4, 6)$ and $h(\Delta) = (1, 2, 3)$; the empty face gives $f_{-1}(\Delta) = 1$.

A simplicial complex Δ can be used to define a commutative ring, known as the Stanley–Reisner ring. This construction allows us to use tools of commutative algebra to prove results about the topology or combinatorics of Δ .

Definition 5.3. Let Δ be a simplicial complex on vertices $\{x_1, \dots, x_n\}$. The Stanley–Reisner ideal I_Δ is

$$I_\Delta = \langle x_{i_1} \cdots x_{i_j} \mid \{x_{i_1}, \dots, x_{i_j}\} \text{ is not a face of } \Delta \rangle \subseteq \mathbb{K}[x_1, \dots, x_n],$$

and the Stanley–Reisner ring is $\mathbb{K}[x_1, \dots, x_k]/I_\Delta$.

In Example 5.2, since Δ has no 2-faces,

$$I_\Delta = \langle x_1x_2x_3, x_1x_2x_4, x_1x_3x_4, x_2x_3x_4 \rangle = \bigcap_{1 \leq i < j \leq 4} \langle x_i, x_j \rangle.$$

Definition 5.4. A prime ideal P is associated to a graded S -module N if P is the annihilator of some $n \in N$, and $\text{Ass}(N)$ is the set of all associated primes of N .

Definition 5.5. Let $\text{codim}(N) = \min\{\text{codim}(P) \mid P \in \text{Ass}(N)\}$ for a finitely generated graded S -module N . The projective dimension $\text{pdim}(N)$ is the length of a minimal free resolution of N ; N is Cohen–Macaulay if $\text{codim}(N) = \text{pdim}(N)$. S/I is arithmetically Cohen–Macaulay if it is Cohen–Macaulay as an S -module.

5B. Application to Wachspress surfaces.

Definition 5.6. Define $I_\Gamma(d) \subseteq \mathbb{K}[x_1, \dots, x_d]$ as

$$I_\Gamma(d) = \langle x_1x_3, \dots, x_1x_{d-1} \rangle + K_{2,d-1},$$

where $K_{2,d-1}$ consists of all square-free cubic monomials in x_2, \dots, x_{d-1} .

Theorem 5.7. *The quotient $S/I_\Gamma(d)$ is arithmetically Cohen–Macaulay, of Castelnuovo–Mumford regularity two, and has Hilbert series*

$$\text{HS}(S/I_\Gamma(d), t) = \frac{1 + (d - 3)t + \binom{d-3}{2}t^2}{(1 - t)^3}.$$

Proof. The ideal $I_\Gamma(d)$ is the Stanley–Reisner ideal of a one-dimensional simplicial complex Γ consisting of a complete graph on vertices $\{x_2, \dots, x_{d-1}\}$ with a single additional edge $\overline{x_1x_2}$ attached. All connected graphs are shellable, so since shellable implies Cohen–Macaulay (see [Miller and Sturmfels 2005]), $S/I_\Gamma(d)$ is Cohen–Macaulay. Since $I_\Gamma(d)$ contains no terms involving x_d , if $S' = \mathbb{K}[x_1, \dots, x_{d-1}]$, then

$$S/I_\Gamma(d) \simeq S'/I_\Gamma(d) \otimes \mathbb{K}[x_d].$$

The Hilbert series of a Stanley–Reisner ring has numerator equal to the h -vector of the associated simplicial complex (see [Schenck 2003]), which in this case is a graph on $d - 1$ vertices with $\binom{d-2}{2} + 1$ edges. Converting $f(\Gamma) = (1, d - 1, \binom{d-2}{2} + 1)$ to $h(\Gamma)$ yields the Hilbert series of $S'/I_\Gamma(d)$. The Hilbert series of a graph has denominator $(1 - t)^2$, and tensoring with $\mathbb{K}[x_d]$ contributes a factor of $1/(1 - t)$, yielding the result. □

Theorem 5.8. *In graded lex order, $\text{in}_< I(d) = I_\Gamma(d)$.*

Proof. First, note that

$$I_\Gamma(d) \subseteq \text{in}_< I(d),$$

which follows from Corollary 3.8 and Theorem 4.1, combined with the observation that, in graded lex order, $\text{in}(|\Lambda_i \Lambda_j \Lambda_k|) = x_i x_j x_k$ if $i < j < k$ as long as $k \neq d$. Since $I(d) \subseteq I_{W_d}$, there is a surjection

$$S/I(d) \twoheadrightarrow S/I_{W_d};$$

hence, $\text{HP}(S/I(d), t) \geq \text{HP}(S/I_{W_d}, t)$. Since

$$\text{HP}(S/I(d), t) = \text{HP}(S/\text{in}_< I(d), t)$$

and

$$I_\Gamma(d) \subseteq \text{in}_< I(d),$$

we have

$$\text{HP}(S/I_\Gamma(d), t) \geq \text{HP}(S/\text{in}_< I(d), t) = \text{HP}(S/I(d), t) \geq \text{HP}(S/I_{W_d}, t).$$

The Hilbert polynomial $\text{HP}(S/I_{W_d}, t)$ is given by Equation (5). The Hilbert series of $S/I_\Gamma(d)$ is given by Theorem 5.7, from which we can extract the Hilbert polynomial:

$$\text{HP}(S/I_\Gamma(d), t) = \binom{d-3}{2} \binom{t}{2} + (d-3) \binom{t+1}{2} + \binom{t+2}{2},$$

and a check shows this agrees with Equation (5). Since $I_\Gamma(d) \subseteq \text{in}_< I(d)$, equality of the Hilbert polynomials implies that in high degree (i.e., up to saturation)

$$I_\Gamma(d) = \text{in}_< I(d) \quad \text{and} \quad I(d) = I_{W_d}.$$

Consider the short exact sequence

$$0 \rightarrow \text{in}_< I(d)/I_\Gamma(d) \rightarrow S/I_\Gamma(d) \rightarrow S/\text{in}_< I(d) \rightarrow 0.$$

By Lemma 3.6 of [Eisenbud 1995],

$$\text{Ass}(\text{in}_< I(d)/I_\Gamma(d)) \subseteq \text{Ass}(S/I_\Gamma(d)). \tag{19}$$

Since $\text{HP}(S/I_\Gamma(d), t) = \text{HP}(S/\text{in}_< I(d), t)$, the module $\text{in}_< I(d)/I_\Gamma(d)$ must vanish in high degree so is supported at \mathfrak{m} , which is of codimension d . But $I_\Gamma(d)$ is a radical ideal supported in codimension $d - 3$, so it follows from Equation (19) that $\text{in}_< I(d)/I_\Gamma(d)$ must vanish. \square

Corollary 5.9. *The ideal $I(d)$ is the ideal of the image of*

$$X_d \rightarrow \mathbb{P}(H^0(D_{d-2})).$$

In particular, $I(d) = I_{W_d}$, and $S/I(d)$ is arithmetically Cohen–Macaulay.

Proof. By the results of Sections 2 and 3, $I(d) \subseteq I_{W_d}$, and the proof of Theorem 5.8 showed that they are equal up to saturation. Hence, $I_{W_d}/I(d)$ is supported at \mathfrak{m} . Consider the short exact sequence

$$0 \rightarrow I_{W_d}/I(d) \rightarrow S/I(d) \rightarrow S/(I_{W_d}) \rightarrow 0.$$

Since $S/I_\Gamma(d) = S/\text{in}_< I(d)$ is arithmetically Cohen–Macaulay of codimension $d - 3$, by uppersemicontinuity [Herzog 2005], so is $S/I(d)$, so $I_{W_d}/I(d) = 0$. \square

Corollary 5.10. *The quotient S/I_{W_d} has regularity 2.*

Proof. Since $S/I(d)$ is Cohen–Macaulay, reducing modulo a linear regular sequence of length 3 yields an Artinian ring with the same regularity, which is equal to the socle degree [Eisenbud 2005]. By Theorems 5.7 and 5.8, this is 2, so the regularity of S/I_{W_d} is 2. \square

Theorem 5.11. *The nonzero graded Betti numbers of the minimal free resolution of $S/I(d)$ are given by $b_{12} = d - 3$ and for $i \geq 3$ by*

$$b_{i-2,i} = \binom{d-3}{i} - (d-3)\binom{d-3}{i-1} + \binom{d-3}{2}\binom{d-3}{i-2}.$$

Proof. By [Corollary 5.10](#), there are only two rows in the Betti table of $S/I(d)$. By [Corollary 3.9](#), the top row is empty, save for the quadratic generators at the first step. Thus, the entire Betti diagram may be obtained from the Hilbert series, which is given in [Theorem 5.7](#), and the result follows. \square

We are at work on generalizing the results here to higher dimensions.

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