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The final log canonical model of $\overline{\mathcal{M}}_6$

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We describe the birational model of $\overline{\mathcal{M}}_6$ given by quadric hyperplane sections of the degree-5 del Pezzo surface. In the spirit of the genus-4 case treated by Fedorchuk, we show that it is the last nontrivial space in the log minimal model program for $\overline{\mathcal{M}}_6$. We also obtain a new upper bound for the moving slope of the moduli space.

1. Introduction

A general smooth curve C of genus 6 has five planar sextic models with four nodes in general linear position. Blowing up these four points and embedding the resulting surface in $\mathbb{P}^5$ via its complete anticanonical linear series, one finds that the canonical model of C is a quadric hyperplane section of a degree-5 del Pezzo surface S. As any four general points in $\mathbb{P}^2$ are projectively equivalent, this surface is unique up to isomorphism. Its automorphism group is finite and isomorphic to the symmetric group S_5 (see, e.g., [Shepherd-Barron 1989]). The surface S contains ten (-1)-curves, which are the four exceptional divisors of the blowup, together with the proper transforms of the six lines through pairs of the points. There are five ways of choosing four nonintersecting (-1)-curves on S, inducing five blowdown maps $S \to \mathbb{P}^2$, and restricting to the five g^2_6's on C. Residual to the latter are five g^1_4's, which can be seen in each planar model as the projection maps from the four nodes, together with the map that is induced on C by the linear system of conics passing through the nodes.

This description gives rise to a birational map

$$\varphi : \overline{\mathcal{M}}_6 \dashrightarrow X_6 := |-2K_S|/\mathrm{Aut}(S),$$

which is well defined and injective on the sublocus $(\mathcal{M}_6 \cup \Delta_0^{\mathrm{irr}}) \setminus \overline{\mathcal{GP}}_6$. Here Δ_0^{irr} denotes the locus of irreducible singular stable curves, and $\overline{\mathcal{GP}}_6$ is the closure of the Gieseker–Petri divisor of curves having fewer than five g^1_4's (or residually, g^2_6's). These have planar sextic models in which the nodes fail to be in general linear position, which forces the anticanonical image of the blown-up $\mathbb{P}^2$ to become

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singular. In the generic case, exactly three of the nodes become collinear, and the line through them is a (-2)-curve that gets contracted to an A_1 singularity. The class of the Gieseker–Petri divisor is computed in [Eisenbud and Harris 1987b] as

$$\left[\overline{\mathscr{GP}}_6\right] = 94\lambda - 12\delta_0 - 50\delta_1 - 78\delta_2 - 88\delta_3.$$

It is an extremal effective divisor of minimal slope on $\overline{\mathcal{M}}_6$ [Chang and Ran 1991].

The aim of this article is to study the birational model X_6, determine its place within the log minimal model program of $\overline{\mathcal{M}}_6$, and use it to derive an upper bound on the moving slope of this space. In order to do so, we will start in Section 2 by determining explicitly the way in which φ extends to the generic points of the divisors Δ_i for $i = 1, 2, 3$ and of $\overline{\mathscr{GP}}_6$. The divisors Δ_1 and Δ_2 are shown to be contracted by 1 and 4 dimensions as the low-genus components are replaced by a cusp and an A_5 singularity, respectively. The image of Δ_3 is at most one-dimensional, and $\overline{\mathscr{GP}}_6$ turns out to be contracted to a point. The curves parametrized by the latter two are shown to be mapped to the classes of certain nonreduced degree-10 curves on S.

In Section 3, we will then construct test families along which φ is defined and determine their intersection numbers with the standard generators of $\operatorname{Pic}(\overline{\mathcal{M}}_6)$ as well as with $\varphi^*\mathscr{O}_{X_6}(1)$. Having enough of those enables us in Section 4 to finally compute the class of the latter. This computation is then used to establish the upper bound $s'(\overline{\mathcal{M}}_6) \leq \frac{102}{13}$ for the moving slope of $\overline{\mathcal{M}}_6$ as well as to show that the log canonical model $\overline{\mathcal{M}}_6(\alpha)$ is isomorphic to X_6 for $\frac{16}{47} < \alpha \leq \frac{35}{102}$ and becomes trivial below this point.

2. Defining φ in codimension 1

In this section, we will see how φ is defined on the generic points of the codimension-one subloci of $\overline{\mathcal{M}}_6$ parametrizing curves whose canonical image does not lie on S. As mentioned in the introduction, these are the divisors Δ_i, $i = 1, 2, 3$, as well as $\overline{\mathscr{GP}}_6$, and they will turn out to constitute exactly the exceptional locus of φ.

Proposition 2.1. *A curve $C = C_1 \cup_p C_2 \in \Delta_1$ with p not a Weierstraß point on $C_2 \in \mathcal{M}_5$ is mapped to the class of a cuspidal curve whose pointed normalization is $(C_2,\ p)$. In particular, the map φ contracts Δ_1 by one dimension.*

Proof. This follows readily from the existence of a moduli space for pseudostable curves [Schubert 1991]. More concretely, let $\pi : \mathscr{C} \to B$ be a flat family of genus-6 curves whose general fiber is smooth and Gieseker–Petri general and with special fiber C. Then the twisted linear system $|\omega_\pi(C_1)|$ maps $\mathscr{C}$ to a flat family of curves in $|-2K_S|$. It restricts to $\mathscr{O}_{C_1}$ on C_1 and to $\omega_{C_2}(2p)$ on C_2, so it contracts C_1 and maps C_2 to a cuspidal curve of arithmetic genus 6, which lies on a smooth del Pezzo surface. □

Proposition 2.2. *Let $C = C_1 \cup_p C_2 \in \Delta_2$ be a curve such that*

- *the component $C_2 \in \mathcal{M}_4$ is Gieseker–Petri general and*
- *p is not a Weierstraß point on either component.*

Then C is mapped to the class of a curve consisting of C_2 together with a line that is 3-tangent to it at p. In particular, the map φ restricted to Δ_2 has 4-dimensional fibers.

Proof. Let $\mathscr{C} \to B$ be a flat family of genus-6 curves whose general fiber is smooth and Gieseker–Petri general and with special fiber C. Blow up the hyperelliptic conjugate $\tilde{p} \in C_1$ of p, and let $\pi : \mathscr{C}' \to B$ be the resulting family with central fiber C' and exceptional divisor R. Then the twisted line bundle $\mathscr{L} := \omega_\pi(2C_1)$ restricts to $\omega_{C_2}(3p)$, $\mathbb{O}_{C_1}$, and $\mathbb{O}_R(1)$ on the respective components of C'. By a detailed analysis, one can see that the family of linear systems $(\mathscr{L}, \pi_*\omega_\pi)$ restricts to $|\omega_{C_2}(3p)|$ on C_2 and maps R to the 3-tangent line at p while contracting C_1. A similar but harder analysis of this kind is carried out in Lemma 2.5 for the case of Δ_3, to which we refer.

In order to see that the central fiber lies on S as a section of $-2K_S$, it suffices to observe that a generic pointed curve $(C_2, p) \in \mathcal{M}_{4,1}$ has three quintic planar models with a flex at p. Each such model has two nodes, projecting from which gives the two g^1_3's. The 3-tangent line R at p meets C_2 at two other points, so $C_2 \cup R$ is a plane curve of degree 6 with four nodes (and an A_5 singularity). Blowing up the four nodes, which for generic (C_2, p) will be in general linear position, gives the claim.

For showing that the flat limit is unique, it suffices by [Fedorchuk 2012, Lemma 3.10] to show that, if C' is any curve in a small punctured neighborhood of $R \cup_p C_2$ inside $|-2K_S|$, then C is not the stable reduction of C' in any family in which it occurs as the central fiber. If C' is smooth, this is obviously satisfied, so assume it is still singular. If C' retains an A_5 singularity, then its genus-4 component must be different from (C_2, p) since C_2 has only a finite number of g^2_5's with a flex at p. Thus, its stable reduction cannot be isomorphic to C. If on the other hand the type of singularity changes, it can only become an A_k singularity with $1 \le k \le 4$. In case $k \le 3$, any irreducible component arising in the stable reduction has genus at most 1 while for $k = 4$ the stable tail is always a genus-2 curve attached at a Weierstraß point [Hassett 2000, Section 6]. Thus, the stable reduction cannot be isomorphic to C in these cases either. □

Proposition 2.3. *Let $C = C_1 \cup_p C_2 \in \Delta_3$ be a curve such that, on both components,*

- *p is not a Weierstraß point and*
- *p is not in the support of any odd theta characteristic (in particular, neither component is hyperelliptic).*

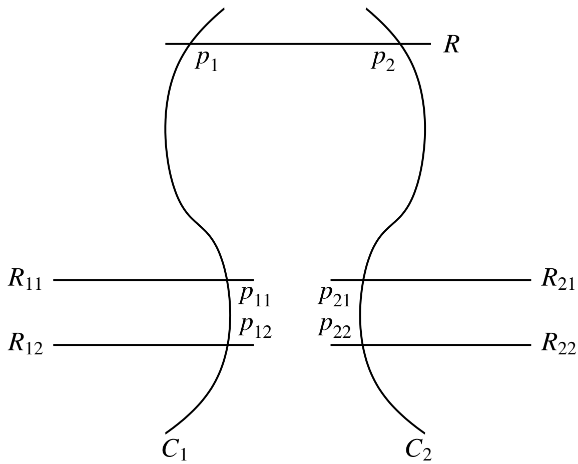

Figure 1. The central curve C'.

Then C is mapped to the class of two times a twisted cubic on S together with two (possibly reducible) conics meeting it tangentially. In particular, the image of Δ_3 under φ is at most 1-dimensional.

Proof. Let $\mathscr{C} \to B$ be a flat family of genus-6 curves whose general fiber is smooth and Gieseker–Petri general and with special fiber C. By assumption, the two base points of $|\omega_{C_i}(-2p)|$ are distinct from each other and from p for $i = 1, 2$. Blow up the total space $\mathscr{C}$ at p and at these four base points. Let $\pi : \mathscr{C}' \to B$ denote the resulting family with central fiber $C' = C_1 + C_2 + R + \sum R_{ij}$, where C_i are the proper transforms of the genus-3 components and R and R_{ij} are the exceptional divisors over p and the base points, respectively. For $i, j = 1, 2$, denote by p_{ij} the point of intersection of C_i with R_{ij} and by p_i the point of intersection of C_i with R (see Figure 1).

Consider the twisted sheaf $\mathscr{L} := \omega_\pi\big(3(C_1 + C_2) + \sum R_{ij}\big)$ on $\mathscr{C}'$. On the various components of C', it restricts to $\mathscr{O}_{C_i}$, $\mathscr{O}_R(6)$, and $\mathscr{O}_{R_{ij}}(1)$, respectively. The push-forward $\pi_*\mathscr{L}$ is not locally free (the central fiber has dimension 7 instead of 6), but it contains $\pi_*\omega_\pi$ as a locally free rank-6 subsheaf. The central fiber V of the image of this sheaf in $\pi_*\mathscr{L}$ is described in Lemma 2.5. The induced linear system $(\mathscr{L}|_{C'}, V)$ maps C' to the curve $C'' = R + 2R_1 + 2R_2 \subseteq \mathbb{P}^5$, which consists of the middle rational component R embedded as a degree-6 curve together with twice the tangent lines R_1 and R_2 at p_1 and p_2. The genus-3 components C_i are contracted to the points p_i. If one introduces coordinates $[x_0 : \cdots : x_5]$ in $\mathbb{P}^5$ corresponding to the basis of V given in Lemma 2.5, the image curve lies on the variety

$$\widetilde{S}_{2,3} = \bigcup_{[\lambda:\mu]\in\mathbb{P}^1} \overline{\varphi_1([\lambda : \mu])\varphi_2([\lambda : \mu])},$$

where

$$\varphi_1([\lambda:\mu]) := [\lambda^3 : 0 : \lambda^2\mu : \lambda\mu^2 : 0 : \mu^3],$$
$$\varphi_2([\lambda:\mu]) := [0 : \lambda^2 : 0 : 0 : \mu^2 : 0],$$

which is a projection of the rational normal scroll $S_{2,3} \subseteq \mathbb{P}^6$ from a point in the plane of the directrix. This surface is among the possible degenerations of the degree-5 del Pezzo surface investigated in [Coskun 2006, Proposition 3.2] and has the same Betti diagram. In equations, it is given by

$$\widetilde{S}_{2,3} = \left\{ \operatorname{rk} \begin{pmatrix} x_0 & x_1 & x_2 \\ x_3 & x_4 & x_5 \end{pmatrix} \leq 1 \right\} \cap \left\{ \operatorname{rk} \begin{pmatrix} x_0 & x_2 & x_3 \\ x_2 & x_3 & x_5 \end{pmatrix} \leq 1 \right\},$$

and C'' is a quadric section cut out for example by $x_1x_4 - x_0x_5$. When restricted to the directrix, the image of the projection is the line $\tilde{L} = \{x_0 = x_2 = x_3 = x_5 = 0\}$, which is the singular locus of $\widetilde{S}_{2,3}$. The two branch points q_i of this restriction are the intersection points of the double lines R_i with $\tilde{L}$.

The image of $\mathscr{C}'$ under the family of linear systems $(\mathscr{L}, \pi_*\omega_\pi)$ lies on a flat family of surfaces $\mathscr{S} \subseteq \mathbb{P}^5 \times B$ with general fiber S and special fiber $\widetilde{S}_{2,3}$. We will construct a birational modification of $\mathscr{S}$ whose central fiber is isomorphic to S. Let $\pi' : \mathscr{S}' \to B$ be the family obtained by blowing up $\tilde{L}$ and $S' \subseteq \mathscr{S}'$ the exceptional divisor. The proper transform of $\widetilde{S}_{2,3}$ in $\mathscr{S}'$ is $S_{2,3}$, and the intersection curve $L = S_{2,3} \cap S'$ is its directrix.

We want to show that $S' \cong S$. The ten (-1)-curves of the generic fiber cannot all specialize to points in the central limit since then the whole surface S would be contracted, contradicting flatness. Any exceptional curve that is not contracted must go to $\tilde{L}$ in the limit since it is the only curve on $\widetilde{S}_{2,3}$ having a normal sheaf of negative degree. By a chase around the intersection graph of the (-1)-curves on S, one can see that, if one of them is mapped dominantly to $\tilde{L}$, then at least four of them are. Since the graph is connected, the rest of them get mapped to points that lie on $\tilde{L}$. Using a base change ramified over 0 if necessary, we may assume that limits of noncontracted curves get separated in $\mathscr{S}'$ while the contracted ones are blown up to lines. Thus, there are ten distinct (-1)-curves on S', which by the list of possible limits in [Coskun 2006] forces it to be isomorphic to S (note that there are at most seven (-1)-curves on a singular degree-5 del Pezzo surface [Coray and Tsfasman 1988, Proposition 8.5]).

It remains to see what happens to the curve C'' in the process. Denote by $\psi : \mathscr{S}' \to \mathbb{P}^5 \times B$ the map induced by the family of linear systems $(\omega^\vee_{\pi'}(S_{2,3}), \pi'_*\omega^\vee_{\pi'})$. This restricts to $-K_{S'}$ on S' and to a subsystem of $|3F|$ on $S_{2,3}$. Thus, the map ψ contracts the latter and has degree 3 on L. This implies that $\mathscr{O}_{S'}(L) = \rho^*\mathscr{O}_{\mathbb{P}^2}(1)$ for one of the five maps $\rho : S' \to \mathbb{P}^2$, and there are exactly four exceptional curves $E_1, \dots, E_4 \subseteq S'$ that do not meet L. The blowdown fibration on S' is given

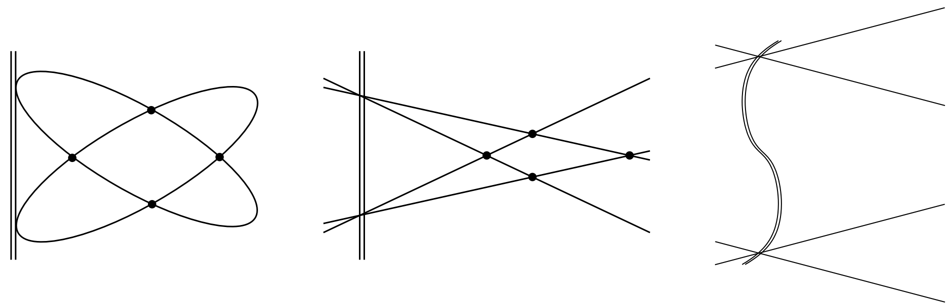

Figure 2. Two possibilities for the image of C under φ and the proper transform of the latter after blowing up the nodes.

by $\left|2L - \sum E_i\right|$, and it contains exactly 3 reducible conics. The flat pullback of C'' to $\mathscr{S}'$ contains the two conics in the fibration that meet L at the ramification points of the map $L \to \tilde{L}$, and the map ψ restricted to C'' contracts the two double lines R_i to the points q_i and maps R doubly onto L. Thus, the flat limit of C'' consists of twice the line L together with the two conics in the fibration which are tangent to L at the points q_i. Up to automorphisms, such a configuration has a 1-dimensional family of moduli, so the image of Δ_3 under φ is at most 1-dimensional. □

Remark 2.4. The image curve $\varphi(C)$ has two possible kinds of nonreduced planar singularities shown in Figure 2. The one on the left with local equation $y^2(y-x^2)=0$ appears in the proof of Proposition 2.3 in the curve C''. For the second one with equation $y^2(y^2 - x^2) = 0$, one can see directly using an appropriate family that it has the generic smooth genus 3 curve in its variety of stable tails. We will use this construction in the proof of Lemma 3.5.

Lemma 2.5. *Let $\mathscr{C}'$ and $\mathscr{L}$ be constructed as in the proof of Proposition 2.3, and let V be the central fiber of the image of $\pi_*\omega_\pi \hookrightarrow \pi_*\mathscr{L}$. Choose coordinates $[s:t]$ on each rational component such that on R_{1j} the coordinate t is centered at p_{1j}, on R_{2j} the coordinate s is centered at p_{2j} $(j = 1, 2)$, and on R the coordinate s is centered at p_1 and t at p_2. Then V is spanned by the following sections (on C_i the sections are constants and not listed in the table):*

R_{11}	R_{12}	R	R_{21}	R_{22}
0	0	s^6	t	t
0	0	s^5t	s	s
0	0	s^4t^2	0	0
0	0	s^2t^4	0	0
t	t	st^5	0	0
s	s	t^6	0	0

Proof. Let $\ell_R = (\mathcal{L}_R, V_R)$ be the R-aspect of the unique limit canonical series on the central fiber of $\mathcal{C}'$. By [Eisenbud and Harris 1987a, Theorem 2.2], we have that

$$\mathcal{L}_R = \omega_\pi\big(5(C_1 + C_2) + 4\sum R_{ij}\big)\big|_R = \mathcal{O}_R(10)$$

and ℓ_R has vanishing sequence $a^\ell_R(p_i) = (2, 3, 4, 6, 7, 8)$ at both p_i, so

$$V_R = s^2t^2\langle s^6, s^5t, s^4t^2, s^2t^4, st^5, t^6\rangle.$$

Since on R the inclusion $\mathcal{L}|_R \hookrightarrow \mathcal{L}_R$ restricts to $\mathcal{O}_R(6) \hookrightarrow \mathcal{O}_R(10)$, $\sigma \mapsto s^2t^2\sigma$, we have that $s^2t^2V|_R \subseteq V_R$. Since the dimensions match, the claim for the central column follows. By dimension considerations, it is clear that $\mathcal{L}$ must restrict to the complete linear series $|\mathcal{O}_{R_{ij}}(1)|$ on R_{ij}.

It remains to show that, if a section $\sigma \in V$ fulfills $\operatorname{ord}_{p_i}(\sigma|_R) \geq 2$, then $\sigma|_{R_{ij}} = 0$ for $j = 1, 2$. For this, let $\sigma_{C_i} \in H^0(C, \mathcal{O}_{\mathcal{C}'}(C_i)|_C)$ be the restriction of a generating section, and let $\varphi_i : H^0(C, \mathcal{L}(-C_i)|_C) \to H^0(C, \mathcal{L}|_C)$ be the map given by $\sigma \mapsto \sigma_{C_i} \cdot \sigma$. For a divisor D on $\mathcal{C}'$ and $k \in \mathbb{N}$, introduce the subspaces

$$\begin{aligned} V_{i,k}(D) &:= \big\{\sigma \in H^0(C, \mathcal{L} \otimes \mathcal{O}_{\mathcal{C}'}(D)|_C) \;\big|\; \operatorname{ord}_{p_i}(\sigma|_R) \geq k\big\}, \\ V_{i,k} &:= V_{i,k}(0). \end{aligned}$$

Since $\mathcal{L}|_{C_i} = \mathcal{O}_{C_i}$, we have that $\operatorname{im}(\varphi_i) = V_{i,1}$. Moreover, we certainly have the inclusion $\varphi_i(V_{i,1}(-C_i)) \subseteq V_{i,2}$ and

$$\begin{aligned} \operatorname{codim}(\varphi_i(V_{i,1}(-C_i)), V_{i,1}) &\leq \operatorname{codim}(V_{i,1}(-C_i), H^0(C, \mathcal{L}(-C_i)|_C)) \\ &\leq 1. \end{aligned}$$

From the description of the sections on R, it is apparent that $V_{i,2} \subsetneq V_{i,1}$, so we have in fact $\varphi_i(V_{i,1}(-C_i)) = V_{i,2}$. Thus, we get

$$\begin{aligned} V_{i,2} &= \varphi_i(V_{i,1}(-C_i)) \\ &= \varphi_i\big(\big\{\sigma \in H^0(C, \mathcal{L}(-C_i)|_C) \;\big|\; \sigma|_{R_{ij}} = 0 \text{ for } j = 1, 2\big\}\big) \\ &\subseteq \big\{\sigma \in H^0(C, \mathcal{L}|_C) \;\big|\; \sigma|_{R_{ij}} = 0 \text{ for } j = 1, 2\big\}. \end{aligned}$$ □

Proposition 2.6. *Let C be a smooth Gieseker–Petri special curve whose canonical image lies on a singular del Pezzo surface with a unique A_1 singularity but not passing through that singularity. Then φ maps C to a nonreduced degree-10 curve on S consisting of four times a line together with two times each of the three lines meeting it. In particular, φ contracts $\overline{\mathcal{GP}}_6$ to a point.*

Proof. This can be done by a geometric construction similar to [Fedorchuk 2012, Theorem 3.13]. Here we follow a simpler approach from [Jensen 2013]. A curve C as above has a planar sextic model with three collinear nodes, so the map $\mathcal{G}^1_4 \to \mathcal{M}_6$ is simply ramified over C. Thus, a neighborhood of the ramification point will map a (double cover of a) neighborhood of C to a family of $(4,4)$-curves on $\mathbb{P}^1 \times \mathbb{P}^1$. The image of the general fiber will be an irreducible curve with three nodes while the special fiber goes to four times the diagonal. Blowing up the nodes gives a flat family on S with central fiber as described. $\square$

Remark 2.7. A pencil of antibicanonical curves on a singular del Pezzo surface as above has slope $\frac{47}{6}$ like in the smooth case (for which see Lemma 3.1). This would seem to contradict the fact that φ contracts the Gieseker–Petri divisor, which has the same slope, to a point. However, any such pencil will contain a curve C having a node at the singular point. The normalization of such a curve is a trigonal curve of genus 5 since blowing up the node and blowing down four disjoint (-1)-curves gives a planar quintic model of C together with a line. Using this model, one can show that φ maps C to a configuration consisting of three times a line on S together with three lines and two conics meeting it. This arrangement obviously has moduli, so we deduce that φ is not defined on $\Delta_0^{\mathrm{trig}} := \{C \in \Delta_0 \mid C \text{ has a trigonal normalization}\}$, which is a component of $\Delta_0 \cap \overline{\mathcal{GP}}_6$.

3. Test families

In order to compute the class of $\varphi^*\mathcal{O}_{X_6}(1)$, we now construct some test families and record their intersection numbers with the standard generators of $\mathrm{Pic}(\overline{\mathcal{M}}_6)$ and with $\varphi^*\mathcal{O}_{X_6}(1)$. Those numbers not mentioned in the statements of the lemmas are implied to be 0.

Lemma 3.1. *A generic pencil T_1 of quadric hyperplane sections of S has intersection numbers*

$$T_1 \cdot \lambda = 6, \quad T_1 \cdot \delta_0 = 47, \quad T_1 \cdot \varphi^*\mathcal{O}_{X_6}(1) = 1.$$

Proof. Since all members of T_1 are irreducible, it suffices to show that $\varphi_*\lambda = \mathcal{O}_V(6)$ and $\varphi_*\delta = \mathcal{O}_V(47)$ on $V := |-2K_S| \cong \mathbb{P}^{15}$. This is completely parallel to the computation in [Fedorchuk 2012, Proposition 3.2]. If $Y := S \times V$ and $\mathcal{C} \subseteq Y$ denotes the universal curve, we have $\mathcal{O}_Y(\mathcal{C}) = \mathcal{O}_Y(-2K_S, 1)$, so by adjunction, $\omega_{\mathcal{C}/V} = \mathcal{O}_{\mathcal{C}}(-K_S, 1)$. Applying π_{2*} to the exact sequence

$$0 \to \mathcal{O}_Y(K_S, 0) \to \mathcal{O}_Y(-K_S, 1) \to \omega_{\mathcal{C}/V} \to 0,$$

we find that

$$\pi_{2*}\omega_{\mathcal{C}/V} \cong \pi_{2*}\mathcal{O}_Y(-K_S, 1) \cong H^0(S, -K_S) \otimes \mathcal{O}_V(1)$$

since $\pi_{2_*}\mathcal{O}_Y(K_S, 0) = R^1\pi_{2*}\mathcal{O}_Y(K_S, 0) = 0$ by Kodaira vanishing. Therefore, we obtain that $\varphi_*\lambda = \det \pi_{2*}\omega_{\mathcal{C}/V} = \mathcal{O}_V(6)$.

We also find that

$$\varphi_*\kappa = \pi_{2*}(\omega^2_{\mathcal{C}/V}) = \pi_{2*}((-2K_S, 1) \cdot (-K_S, 1)^2) = \mathcal{O}_V(25),$$

and from $\kappa = 12\lambda - \delta$, we deduce that $\varphi_*\delta = \mathbb{O}_V(47)$. $\square$

Lemma 3.2. *Let T_2 be the family obtained by attaching a fixed genus-5 curve to a base point of a general pencil of plane cubics. Then T_2 has intersection numbers*

$$T_2 \cdot \lambda = 1, \quad T_2 \cdot \delta_0 = 12, \quad T_2 \cdot \delta_1 = -1, \quad T_2 \cdot \varphi^* \mathbb{O}_{X_6}(1) = 0.$$

Proof. The first three intersection numbers are standard. By Proposition 2.1, φ is defined on T_2 and contracts it to a point. $\square$

Lemma 3.3. *There is a family T_3 of stable genus-6 curves having intersection numbers*

$$T_3 \cdot \lambda = 3, \quad T_3 \cdot \delta_0 = 30, \quad T_3 \cdot \delta_2 = -1, \quad T_3 \cdot \varphi^* \mathbb{O}_{X_6}(1) = 0.$$

Proof. In [Alper et al. 2011, Example 6.1], the authors construct for all $k \geq 2$ a complete family B_k of stable hyperelliptic curves of genus k with two marked points that are conjugate under the hyperelliptic involution. It is obtained by taking a double cover of the Hirzebruch surface $\mathbb{F}_1$ (considered as a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$), branched along $2k+2$ general sections of self-intersection 1. The markings are given as the preimage of the unique (-1)-curve, which does not meet the branch locus. The covering map to $\mathbb{F}_1$ restricts to the hyperelliptic g^1_2 on every fiber, and since the two markings are always distinct, they are never Weierstraß points.

From the family B_2, we construct our family T_3 by forgetting one marking and attaching at the other a fixed 1-pointed curve of genus 4. Then the first three intersection numbers directly carry over from the computation in [Alper et al. 2011, Example 6.1] (note that $T_3 \cdot \delta_2 = -B_2 \cdot \psi_1$). The last one follows by Proposition 2.2 since φ is defined on T_3 and contracts it to a point. $\square$

The following computation is used in the proof of Lemma 3.5:

Lemma 3.4. *Let X be a smooth threefold, $\mathscr{C} \subseteq X$ a surface with an ordinary k-fold point p, $\pi : \widetilde{X} \to X$ the blowup at p, and $\widetilde{\mathscr{C}}$ the proper transform of $\mathscr{C}$. Then $\chi(\mathbb{O}_{\widetilde{\mathscr{C}}}) = \chi(\mathbb{O}_{\mathscr{C}}) - \binom{k}{3}$.*

Proof. Let $E \subseteq \widetilde{X}$ be the exceptional divisor and $C = E \cap \widetilde{\mathscr{C}}$. By adjunction,

$$\begin{aligned} K_{\widetilde{\mathscr{C}}} &= (K_{\widetilde{X}} + \widetilde{\mathscr{C}})|_{\widetilde{\mathscr{C}}} \\ &= (\pi^* K_X + 2E + \pi^*\mathscr{C} - kE)|_{\widetilde{\mathscr{C}}} \\ &= \pi^* K_{\mathscr{C}} - (k-2)C, \end{aligned}$$

so Riemann–Roch for surfaces gives

$$\begin{aligned} \chi(\mathbb{O}_{\widetilde{\mathscr{C}}}) &= \chi(\mathbb{O}_{\widetilde{\mathscr{C}}}(-kC)) - kC^2 \\ &= \chi(\mathbb{O}_{\widetilde{\mathscr{C}}}(-kC)) + k^2. \end{aligned}$$

From the exact sequence

$$0 \to \mathcal{O}_X(-\mathscr{C}) \to \mathcal{O}_{\widetilde{X}}(-kE) \to \mathcal{O}_{\widetilde{\mathscr{C}}}(-kC) \to 0,$$

we get that

$$\chi(\mathcal{O}_{\widetilde{\mathscr{C}}}(-kC)) = \chi(\mathcal{O}_{\widetilde{X}}(-kE)) - \chi(\mathcal{O}_X) + \chi(\mathcal{O}_{\mathscr{C}}).$$

Finally, using induction on the exact sequence

$$0 \to \mathcal{O}_{\widetilde{X}}(-(i+1)E) \to \mathcal{O}_{\widetilde{X}}(-iE) \to \mathcal{O}_{\mathbb{P}^2}(i) \to 0$$

for $i = 0, \dots, k-1$, we conclude that

$$\chi(\mathcal{O}_{\widetilde{X}}(-kE)) = \chi(\mathcal{O}_X) - \sum_{i=0}^{k-1} \frac{i^2+3i+2}{2} = \chi(\mathcal{O}_X) - \frac{k^3+3k^2+2k}{6}.$$

Putting these three equations together gives the result. $\square$

Lemma 3.5. *There is a family T_4 of stable genus-6 curves having intersection numbers*

$$T_4 \cdot \lambda = 16, \quad T_4 \cdot \delta_0 = 118, \quad T_4 \cdot \delta_3 = 1, \quad T_4 \cdot \varphi^* \mathcal{O}_{X_6}(1) = 4.$$

Proof. Let X be the blowup of $\mathbb{P}^2 \times \mathbb{P}^1$ at four constant sections of the second projection, and let $\mathscr{C}, \mathscr{C}' \subseteq X$ denote the proper transforms of degree-4 families of plane sextic curves with assigned nodes at the blown-up points. Suppose $\mathscr{C}$ is chosen in such a way that it contains the curve pictured in Figure 2 on the right as a member and that the fourfold points of this fiber are also ordinary fourfold points of the total space while away from this special fiber the family is smooth and all singular fibers are irreducible nodal. The stable reduction of the special fiber is then a Δ_3-curve, which we furthermore assume to lie in the locus where the map φ is defined. The family $\mathscr{C}'$ is chosen generically so that all its members are irreducible stable curves.

Let $\pi : \widetilde{X} \to X$ be the blowup of X at the two fourfold points of $\mathscr{C}$; denote by $\widetilde{\mathscr{C}}$ the proper transform of $\mathscr{C}$ and by $E_1, E_2 \subseteq \widetilde{X}$ the exceptional divisors of π. Then $\widetilde{\mathscr{C}} = \pi^*\mathscr{C} - 4E_1 - 4E_2$ and $K_{\widetilde{X}} = \pi^* K_X + 2E_1 + 2E_2$, so

$$\begin{aligned} K^2_{\widetilde{\mathscr{C}}} &= (K_{\widetilde{X}} + \widetilde{\mathscr{C}})^2 \widetilde{\mathscr{C}} \\ &= (\pi^*(K_X + \mathscr{C}) - 2(E_1+E_2))^2 (\pi^*\mathscr{C} - 4(E_1+E_2)) \\ &= (K_X + \mathscr{C}')^2 \mathscr{C}' - 16(E_1^3 + E_2^3) = K^2_{\mathscr{C}'} - 32. \end{aligned}$$

By Lemma 3.4, we find that

$$\chi(\mathcal{O}_{\widetilde{\mathscr{C}}}) = \chi(\mathcal{O}_{\mathscr{C}}) - 2\binom{4}{3} = \chi(\mathcal{O}_{\mathscr{C}'}) - 8,$$

so $c_2(\widetilde{\mathscr{C}}) = c_2(\mathscr{C}') - 64$ by Noether's formula. If T_4 and T_4' denote the families in $\overline{\mathcal{M}}_6$ induced by $\widetilde{\mathscr{C}}$ and $\mathscr{C}'$, respectively, we find that $T_4 \cdot \lambda = T_4' \cdot \lambda - 8 = 4 \cdot 6 - 8 = 16$ (note that T_4' is numerically equivalent to $4T_1$, where T_1 is the pencil described in Lemma 3.1). Moreover, the difference in topological Euler characteristics between a general (smooth) fiber and the special (blown-up) fiber of $\widetilde{\mathscr{C}}$ is 6; thus, we find $T_4 \cdot \delta_0 = T_4' \cdot \delta_0 - 64 - 6 = 4 \cdot 47 - 70 = 118$. Finally, T_4 is constructed in such a way that $T_4 \cdot \delta_3 = 1$ and $T_4 \cdot \varphi^* \mathcal{O}_{X_6}(1) = 4$. □

Lemma 3.6. *There is a family T_5 of stable genus-6 curves having intersection numbers*

$$T_5 \cdot \lambda = 21, \quad T_5 \cdot \delta_0 = 164, \quad T_5 \cdot \varphi^* \mathcal{O}_{X_6}(1) = 10.$$

Proof. In order to construct T_5, we take a family of quadric hyperplane sections of a family of generically smooth anticanonically embedded del Pezzo surfaces with special fibers having A_1 singularities. More concretely, let $\widetilde{\mathscr{S}}$ be the blowup of $\mathbb{P}^2 \times \mathbb{P}^1$ along the four sections

$$\begin{aligned}
\Sigma_1 &= ([1:0:0], [\lambda:\mu]),\\
\Sigma_2 &= ([0:1:0], [\lambda:\mu]),\\
\Sigma_3 &= ([0:0:1], [\lambda:\mu]),\\
\Sigma_4 &= ([\lambda+\mu:\lambda:\mu], [\lambda:\mu]),
\end{aligned}$$

where $[\lambda : \mu] \in \mathbb{P}^1$ is the base parameter. We map $\widetilde{\mathscr{S}}$ into $\mathbb{P}^7 \times \mathbb{P}^1$ by taking a system of eight (3, 1)-forms that span the space of anticanonical forms in every fiber as given for example by

$$\begin{aligned}
f([x_0:x_1:x_2]) = \big[&x_0x_1(\lambda x_0 - (\lambda+\mu)x_1) &&: x_0^2(\mu x_1 - \lambda x_2)\\
&: x_0x_2(\mu x_0 - (\lambda+\mu)x_2) &&: x_0x_2(\mu x_1 - \lambda x_2)\\
&: x_0x_1(\mu x_1 - \lambda x_2) &&: x_1^2(\mu x_0 - (\lambda+\mu)x_2)\\
&: x_1x_2(\mu x_1 - \lambda x_2) &&: x_2^2(\lambda x_0 - (\lambda+\mu)x_1)\big].
\end{aligned}$$

This maps every fiber anticanonically into a 5-dimensional subspace of $\mathbb{P}^7$ that depends on $[\lambda : \mu] \in \mathbb{P}^1$. The image of the blown-up $\mathbb{P}^2$ is isomorphic to S except for the parameter values $[\lambda : \mu] = [1 : 0]$, $[0 : 1]$, and $[1 : -1]$, where three base points lie on a line that gets contracted to an A_1 singularity under the anticanonical embedding.

Denote the image of f by $\mathscr{S}$; let H_1 and H_2 be the generators of $\mathrm{Pic}(\mathbb{P}^7 \times \mathbb{P}^1)$ and $\widetilde{H}_1, \widetilde{H}_2, E_1, \ldots, E_4$ those of $\mathrm{Pic}(\widetilde{\mathscr{S}})$. Note that $f^*H_1 = 3\widetilde{H}_1 - \sum E_i + \widetilde{H}_2$ and $f^*H_2 = \widetilde{H}_2$. We claim that

$$\mathscr{S} \equiv 5H_1^5 + 9H_1^4 H_2 \in A^*(\mathbb{P}^7 \times \mathbb{P}^1).$$

Indeed, the first coefficient is just the degree in a fiber while the second one is computed as

$$\begin{aligned}\mathcal{S}\cdot H_1^3 &= \Big(3\widetilde{H}_1 - \sum_{i=1}^{4} E_i + \widetilde{H}_2\Big)^3 = 27\widetilde{H}_1^2\widetilde{H}_2 + 3\sum_{i=1}^{4}\widetilde{H}_2 E_i^2 - E_4^3 + 9\widetilde{H}_1 E_4^2\\ &= 27 - 12 + 3 - 9 = 9.\end{aligned}$$

Here we have used that $\widetilde{H}_2 E_i^2 = -1$ for $i = 1, \ldots, 4$ as it is just the self-intersection of the exceptional $\mathbb{P}^1$ in a fiber. Moreover, by the normal bundle exact sequence,

$$E_i^3 = K_{\mathbb{P}^2\times\mathbb{P}^1}\cdot\Sigma_i - \deg K_{\Sigma_i} = (-3\widetilde{H}_1 - 2\widetilde{H}_2)\widetilde{H}_1^2 + 2 = 0$$

for $i = 1, 2, 3$, and similarly,

$$E_4^3 = (-3\widetilde{H}_1 - 2\widetilde{H}_2)(\widetilde{H}_1^2 + \widetilde{H}_1\widetilde{H}_2) + 2 = -3.$$

Finally, $\widetilde{H}_1$ and $\widetilde{H}_2$ both restrict to the same thing on E_4 (namely the class of a fiber of the fibration $E_4 \to \Sigma_4$), so $\widetilde{H}_1 E_4^2 = \widetilde{H}_2 E_4^2 = -1$.

Let $\mathcal{C}$ be the family cut out on $\mathcal{S}$ by a generic hypersurface of bidegree (2, 2) so that $\mathcal{C} \equiv 10H_1^6 + 28H_1^5 H_2$. Since $K_{\widetilde{\mathcal{S}}} = \mathcal{O}_{\widetilde{\mathcal{S}}}(-3\widetilde{H}_1 + \sum E_i - 2\widetilde{H}_2)$, we find that $K_{\mathcal{S}} = \mathcal{O}_{\mathcal{S}}(-H_1 - H_2)$. Thus, $\omega_{\mathcal{S}/\mathbb{P}^1} = \mathcal{O}_{\mathcal{S}}(-H_1 + H_2)$, and by adjunction, $\omega_{\mathcal{C}/\mathbb{P}^1} = \mathcal{O}_{\mathcal{C}}(H_1 + 3H_2)$. If T_5 denotes the family induced in $\overline{\mathcal{M}}_6$ by $\mathcal{C}$, we then find that

$$T_5\cdot\kappa = \omega_{\mathcal{C}/\mathbb{P}^1}^2 = (H_1 + 3H_2)^2\cdot(10H_1^6 + 28H_1^5 H_2) = 88.$$

Next we note that $\mathcal{O}_{\mathcal{S}}(-\mathcal{C}) = 2K_{\mathcal{S}}$, so applying the Riemann–Roch theorem for threefolds to the short exact sequence $0 \to 2K_{\mathcal{S}} \to \mathcal{O}_{\mathcal{S}} \to \mathcal{O}_{\mathcal{C}} \to 0$, we get

$$\begin{aligned}\chi(\mathcal{O}_{\mathcal{C}}) &= \chi(\mathcal{O}_{\mathcal{S}}) - \chi(2K_{\mathcal{S}})\\ &= -\tfrac{1}{2}K_{\mathcal{S}}^3 + 4\chi(\mathcal{O}_{\mathcal{S}})\\ &= -\tfrac{1}{2}(-H_1 - H_2)^3(5H_1^5 + 9H_1^4 H_2) + 4\\ &= 16,\end{aligned}$$

where we used that $\chi(\mathcal{O}_{\mathcal{S}}) = 1$ because $\mathcal{S}$ is rational. Hence, if C denotes a generic fiber of $\mathcal{C}$, we get that $T_5\cdot\lambda = \chi(\mathcal{O}_{\mathcal{C}}) - (g(\mathbb{P}^1) - 1)(g(C) - 1) = 21$. Finally, by Mumford's relation, we obtain $T_5\cdot\delta_0 = 12\cdot 21 - 88 = 164$.

For computing $T_5\cdot\varphi^*\mathcal{O}_{X_6}(1)$, we note that we can also construct $\mathcal{S}$ as follows: blow up $\mathbb{P}^2\times\mathbb{P}^1$ at $[1:0:0]$, $[0:1:0]$, $[0:0:1]$, and $[1:1:1]$, embed it into $\mathbb{P}^7\times\mathbb{P}^1$ via

$$\begin{aligned}f'([x_0:x_1:x_2]) = \big[&x_0x_1(x_0-x_1) : x_0^2(x_1-x_2) : x_0x_2(x_0-x_2) : x_0x_2(x_1-x_2)\\ &: x_0x_1(x_1-x_2) : x_1^2(x_0-x_2) : x_1x_2(x_1-x_2) : x_2^2(x_0-x_1)\big],\end{aligned}$$

and take the proper transform of this constant family under the birational map $\psi : \mathbb{P}^7 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^7 \times \mathbb{P}^1$ given by

$$\begin{aligned}\psi([y_0 : \cdots : y_7]) = \big[&\lambda^2(\lambda+\mu)^2 y_0 : \lambda\mu(\lambda+\mu)^2 y_1 : \mu^2(\lambda+\mu)^2 y_2 : \lambda\mu^2(\lambda+\mu) y_3 \\ &: \lambda^2\mu(\lambda+\mu) y_4 : \lambda^2\mu(\lambda+\mu) y_5 : \lambda^2\mu^2 y_6 : \lambda\mu^2(\lambda+\mu) y_7\big].\end{aligned}$$

Denoting by $\mathcal{S}' \cong S \times \mathbb{P}^1$ the image of f', the intersection number $T_5 \cdot \varphi^*\mathcal{O}(1)$ is given by the number of curves in T_5 passing through a general fixed point of S. Since two general hyperplane sections cut out five general points on S, we compute that

$$T_5 \cdot \varphi^*\mathcal{O}_{X_6}(1) = \tfrac{1}{5}\mathcal{O}_{\mathcal{S}'}(H_1)^2 \cdot \psi^*\mathcal{O}_{\mathcal{S}}(\mathcal{C}) = \tfrac{1}{5} H_1^5 \cdot H_1^2 \cdot (2H_1 + 10H_2) = 10. \qquad \square$$

4. The moving slope of $\overline{\mathcal{M}}_6$

Proposition 4.1. *The moving slope of $\overline{\mathcal{M}}_6$ fulfills $\frac{47}{6} \leq s'(\overline{\mathcal{M}}_6) \leq \frac{102}{13}$.*

Proof. The lower bound is the slope of the effective cone of $\overline{\mathcal{M}}_6$ and was known before [Farkas 2010]. Using the test families T_1 through T_5 described in Section 3, we get that

$$\varphi^*\mathcal{O}_{X_6}(1) = 102\lambda - 13\delta_0 - 54\delta_1 - 84\delta_2 - 94\delta_3.$$

Since $\mathcal{O}_{X_6}(1)$ is ample on X_6 and φ is a rational contraction, this is a moving divisor on $\overline{\mathcal{M}}_6$, which gives the upper bound on the moving slope. $\square$

Remark 4.2. Note that $\frac{102}{13} \approx 7.846$ is strictly smaller than $\frac{65}{8} = 8.125$, which was the upper bound previously obtained in [Farkas 2010]. However, since our families T_4 and T_5 are not covering families for divisors contracted by φ, we cannot argue as in [Fedorchuk 2012, Corollary 3.7]. In particular, the actual moving slope may be lower than the upper bound given here.

Proposition 4.3. *The log canonical model $\overline{\mathcal{M}}_6(\alpha)$ is isomorphic to X_6 whenever $\frac{16}{47} < \alpha \leq \frac{35}{102}$. It is a point for $\alpha = \frac{16}{47}$, and empty for $\alpha < \frac{16}{47}$.*

Proof. This is completely analogous to [Fedorchuk 2012, Corollary 3.6]. Since

$$\begin{aligned}(K_{\overline{\mathcal{M}}_6} + \alpha\delta) &- \varphi^*\varphi_*(K_{\overline{\mathcal{M}}_6} + \alpha\delta) \\ &= (13\lambda - (2-\alpha)\delta) - \varphi^*\varphi_*(13\lambda - (2-\alpha)\delta) \\ &= \left(\tfrac{35}{2} - 51\alpha\right)\left[\overline{\mathcal{GP}}_6\right] + (9 - 11\alpha)\delta_1 + (19 - 29\alpha)\delta_2 + (34 - 96\alpha)\delta_3\end{aligned}$$

is an effective exceptional divisor for φ as long as $\alpha \leq \frac{35}{102}$, the upper bound follows. Moreover, $\varphi_*(13\lambda - (2-\alpha)\delta) = \mathcal{O}_{X_6}(47\alpha - 16)$, which gives the lower bound. $\square$

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