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Affinity of Cherednik algebras on projective space

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We give sufficient conditions for the affinity of Etingof's sheaves of Cherednik algebras on projective space. To do this, we introduce the notion of pullback of modules under certain flat morphisms.

1. Introduction

1.1. In a seminal paper, Etingof and Ginzburg [2002] introduced the family of rational Cherednik algebras associated to a complex reflection group. Since their introduction, rational Cherednik algebras have been intensively studied, and found to be related to several other areas of mathematics. Their definition was vastly generalized in [Etingof 2004]. Given any smooth variety X and finite group W acting on X, Etingof defines a family of sheaves[1] of algebras $\mathscr{H}_{\omega,\mathbf{c}}(X, W)$ on X which are flat deformations of the skew group ring $\mathscr{D}_X \rtimes W$. Being sheaves of algebras, one would like to be able to use standard geometric techniques such as pullback and pushforward to study their representation theory. This paper is a small first step in developing these techniques. As motivation, we consider the question of affinity for these algebras when $X = \mathbb{P}(V)$.

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[1]Here, one must take the W-equivariant Zariski topology on X. See Section 2.1.

1.2. If V is a finite-dimensional vector space and W acts linearly on V, then there is an induced action of W on $\mathbb{P}(V)$. Thus, Etingof's construction gives us a sheaf of algebras $\mathcal{H}_{\omega,\mathbf{c}}(\mathbb{P}(V), W)$ on $\mathbb{P}(V)$. In trying to understand the representation theory of these algebras, one would like to know when they are affine, i.e., for which ω and $\mathbf{c}$ does the global sections functor give us an equivalence between the category of modules for $\mathcal{H}_{\omega,\mathbf{c}}(\mathbb{P}(V), W)$ and the category of modules for its global sections $\mathsf{H}_{\omega,\mathbf{c}}(\mathbb{P}(V), W)$. Our main result is an explicit combinatorial criterion on ω and $\mathbf{c}$ which guarantees that the corresponding Cherednik algebra is affine. We associate to ω, $\mathbf{c}$ and $\lambda \in \operatorname{Irr} W$ a pair of scalars a_λ, b_λ; see Section 5.5.

Theorem 1.2.1. *The sheaf of algebras $\mathcal{H}_{\omega,\mathbf{c}}(\mathbb{P}(V), W)$ is affine provided $a_\lambda \notin \mathbb{Z}_{\geq 0}$ and $b_\lambda \notin \mathbb{Z}_{>0}$ for all $\lambda \in \operatorname{Irr} W$.*

In order to prove this result, we introduce two key pieces of machinery. The first is the notion of pullback of $\mathcal{H}_{\omega,\mathbf{c}}$-modules under certain well-behaved maps (which we call melys). The second is to establish an equivalence between the category of (twisted) T-equivariant $\mathcal{H}_{\mathbf{c}}$-modules on a principal T-bundle $Y \to X$ and the category of modules for a Cherednik algebra $\mathcal{H}_{\omega,\mathbf{c}}$ on the base X of the bundle. With this machinery in place, the proof of the main result is essentially the same as for sheaves of twisted differential operators on $\mathbb{P}(V)$; see [Hotta et al. 2008, Theorem 1.6.5].

1.3. Being able to pull back D-modules is an extremely useful tool in studying the representation theory of sheaves of differential operators. Therefore, one would like to be able to do the same for Cherednik algebras. We show that this is possible, at least for some morphisms. A W-equivariant map $\varphi : Y \to X$ between smooth varieties is said to be *melys* if it is flat and, for all reflections (w, Z) in X, $\varphi^{-1}(Z)$ is contained in the fixed point set Y^w of w.

Theorem 1.3.1. *If $\varphi : Y \to X$ is melys, then pullback is an exact functor*

$$\varphi^* : \mathcal{H}_{\omega,\mathbf{c}}(X, W)\text{-}\mathsf{Mod} \longrightarrow \mathcal{H}_{\varphi^*\omega,\varphi^*\mathbf{c}}(Y, W)\text{-}\mathsf{Mod}.$$

The pullback functor is particularly well behaved when φ is étale. We define the melys site over X, a certain modification of the usual étale site over X. Using Theorem 1.3.1, we show that the Cherednik algebra forms a sheaf on this site.

One particularly rich source of melys morphisms is when $\pi : Y \to X$ is a principal T-bundle, where T is a torus acting on Y with the action commuting with the action of W. In this situation, one can perform quantum Hamiltonian reduction of the Cherednik algebra $\mathcal{H}_{\mathbf{c}}(Y, W)$ on Y to get a sheaf $\mathcal{H}_{\beta(\chi),\mathbf{c}}(X, W)$ of Cherednik algebras on X. As a consequence, one gets an equivalence between the category of (χ-twisted) T-equivariant $\mathcal{H}_{\mathbf{c}}(Y, W)$-modules on Y and the category of $\mathcal{H}_{\beta(\chi),\mathbf{c}}(X, W)$-modules on X.

Theorem 1.3.2. *Let $\chi \in \mathfrak{t}^*$. We have an isomorphism of sheaves of algebras on X*

$$\mathcal{H}_{\beta(\chi),\mathbf{c}}(X, W) \simeq (\pi_\bullet \mathcal{H}_{\mathbf{c}}(Y, W))^T / \langle \{t - \chi(t) \mid t \in \mathfrak{t}\} \rangle,$$

and the functor

$$(\mathcal{H}_{\mathbf{c}}(X, W), T, \chi)\text{-Mod} \longrightarrow \mathcal{H}_{\beta(\chi),\mathbf{c}}(Y, W)\text{-Mod}$$

given by $\mathcal{M} \mapsto (\pi_\bullet \mathcal{M})^T$ is an equivalence of categories, with quasi-inverse $\mathcal{N} \mapsto \pi^* \mathcal{N}$.

1.4. We also study a natural generalization of the Knizhnik–Zamolodchikov connection. The question of whether the Knizhnik–Zamolodchikov connection is flat is closely related to the issue of presenting the Cherednik algebra. In the appendix, we summarize for the reader unfamiliar with sheaves of twisted differential operators (TDOs) those basic properties that we require.

2. Sheaves of Cherednik algebras

In this section we introduce sheaves of Cherednik algebras on a smooth variety.

2.1. ***Conventions.*** Throughout, all our spaces will be equipped with the action of a finite group W. We do not assume that this action is effective. The morphisms $\varphi : Y \to X$ that we will consider will always be assumed to be W-equivariant. Since we wish to deal with objects such as $\mathcal{O}_X \rtimes W$, we work throughout with the *W-equivariant Zariski topology*: a subset $U \subset X$ is an open subset in this topology if and only if it is open in the Zariski topology and W-stable. Then, $\mathcal{O}_X \rtimes W$ becomes a sheaf on X. If $w \in W$, then X^w denotes the set of all points fixed under the automorphism w. The sheaf of vector fields (resp. one-forms) on a smooth variety X is denoted by Θ_X (resp. Ω^1_X).

2.2. Let X be a smooth, connected, quasiprojective variety over $\mathbb{C}$. Let Z be a smooth subvariety of X of codimension one. Locally, the ideal defining Z is principal, generated by one section, f_Z say. Then, the element

$$d \log f_Z := \frac{df_Z}{f_Z}$$

is a section of $\Omega^1_X(Z) = \Omega^1_X \otimes \mathcal{O}_X(Z)$. Contraction defines a pairing

$$\Theta_X \otimes \Omega^1_X(Z) \to \mathcal{O}_X(Z), \quad (\nu, \omega) \mapsto i_\nu(\omega).$$

Let $\Omega^{1,2}_X$ be the two-term subcomplex $\Omega^1_X \xrightarrow{d} (\Omega^2_X)^{\mathrm{cl}}$, concentrated in degrees 1 and 2, of the algebraic de Rham complex of X, where $(\Omega^2_X)^{\mathrm{cl}}$ denotes the subsheaf of closed forms in Ω^2_X. As noted in the appendix, sheaves of twisted differential operators on X are parametrized, up to isomorphism, by the second hypercohomology group $\mathbb{H}^2(X, \Omega_X^{1,2})$. Given $\omega \in \mathbb{H}^2(X, \Omega_X^{1,2})$, the corresponding sheaf of differential operators is denoted by $\mathscr{D}_X^{\omega}$.

2.3. ***Dunkl–Opdam operators.*** Let W be a finite group acting on X. Let $\mathcal{S}(X)$ be the set of pairs (w, Z) where $w \in W$ and Z is a connected component of X^w of codimension one. Any such Z is smooth. A pair (w, Z) in $\mathcal{S}(X)$ will be referred to as a *reflection* of (X, W). The group W acts on $\mathcal{S}(X)$, and we fix $\mathbf{c} : \mathcal{S}(X) \to \mathbb{C}$ to be a W-equivariant function, where W acts trivially on $\mathbb{C}$. A Picard algebroid $\mathscr{P}$ on X is said to be W-equivariant if there are isomorphisms $\psi_w : w^*(\mathscr{P}) \xrightarrow{\sim} \mathscr{P}$ of algebroids satisfying the usual cocycle condition such that the inclusion $\mathbb{O}_X \to \mathscr{P}$ and anchor map $\sigma : \mathscr{P} \to \Theta_X$ are W-equivariant. Since W acts rationally on $\mathbb{H}^2(X, \Omega_X^{1,2})$, each class $[\omega] \in \mathbb{H}^2(X, \Omega_X^{1,2})^W$ can be represented by an invariant 2-cocycle ω. The corresponding Picard algebroid $\mathscr{P}^{\omega}$ is W-equivariant. We fix one such W-equivariant Picard algebroid $\mathscr{P}^{\omega}$. Fix also an open affine, W-stable covering $\{U_i\}$ of X such that $\mathrm{Pic}(U_i) = 0$ for all i. Then, we can choose functions $f_{Z,i}$ defining $U_i \cap Z$. The union of all the Z is denoted by D. If $j : X - D \hookrightarrow X$ is the inclusion, then write $\mathscr{P}^{\omega}(D)$ for the sheaf $j_{\bullet}(\mathscr{P}^{\omega}|_{X-D})$.

Definition 2.3.1. For each $v \in \Gamma(U_i, \mathscr{P}^{\omega})$, the associated Dunkl–Opdam operator is

$$D_v = v + \sum_{(w,Z)\in\mathcal{S}(X)} \frac{2\mathbf{c}(w, Z)}{1 - \lambda_{w,Z}} i_{\sigma(v)}(d \log f_{Z,i})(w - 1), \tag{2.3.2}$$

where $\lambda_{w,Z}$ is the eigenvalue of w on each fiber of the conormal bundle of Z in X.

The operator D_v is a section of $\mathscr{P}^{\omega}(D) \rtimes W$ over U_i. The $\Gamma(U_i, \mathbb{O}_X \rtimes W)$-submodule of $\mathscr{P}^{\omega}(D) \rtimes W$ generated by $\Gamma(U_i, \mathbb{O}_X \rtimes W)$ and all the Dunkl–Opdam operators $\{D_v \mid v \in \Gamma(U_i, \mathscr{P}^{\omega})\}$ is denoted by $\Gamma(U_i, \mathscr{F}^1_{\omega,\mathbf{c}}(X, W))$. Though the definition of the Dunkl–Opdam operator D_v depends on the choice of functions $f_{Z,i}$, it is easy to see that the submodule $\Gamma(U_i, \mathscr{F}^1_{\omega,\mathbf{c}}(X, W))$ of $\Gamma(U_i, \mathscr{P}^{\omega}(D) \rtimes W)$ is independent of any choices. The modules $\Gamma(U_i, \mathscr{F}^1_{\omega,\mathbf{c}}(X, W))$ glue to form a sheaf $\mathscr{F}^1_{\omega,\mathbf{c}}(X, W)$ in the W-equivariant Zariski topology on X. As noted in the remark after Theorem 2.11 of [Etingof 2004], a calculation in each formal neighborhood of $x \in X$ shows that $[D_{v_1}, D_{v_2}] \in \mathscr{F}^1_{\omega,\mathbf{c}}(X, W)$ for all $v_1, v_2 \in \mathscr{P}^{\omega}$. However, there is *no* natural bracket on $\mathscr{F}^1_{\omega,\mathbf{c}}(X, W)$. The anchor map $\sigma : \mathscr{P}^{\omega}(D) \otimes W \to \Theta_X(D) \otimes W$ restricts to a map $\mathscr{F}^1_{\omega,\mathbf{c}}(X, W) \to \Theta_X \otimes W$ which fits into a short exact sequence

$$0 \longrightarrow \mathbb{O}_X \rtimes W \longrightarrow \mathscr{F}^1_{\omega,\mathbf{c}}(X, W) \xrightarrow{\sigma} \Theta_X \otimes W \longrightarrow 0. \tag{2.3.3}$$

Definition 2.3.4. We call the subsheaf of algebras of $j_{\bullet}(\mathscr{D}^{\omega}_{X-D} \rtimes W)$ generated by $\mathscr{F}^1_{\omega,\mathbf{c}}(X, W)$ the *sheaf of Cherednik algebras* associated to W, ω and $\mathbf{c}$. It is denoted $\mathscr{H}_{\omega,\mathbf{c}}(X, W)$.

The global sections of $\mathscr{H}_{\omega,\mathbf{c}}(X, W)$ are denoted $\mathsf{H}_{\omega,\mathbf{c}}(X, W)$.

2.4. There is a natural order filtration $\mathcal{F}^{\bullet}_{\omega,\mathbf{c}}(X, W)$ on $\mathcal{H}_{\omega,\mathbf{c}}(X, W)$, defined in one of two ways. Either one defines $\mathcal{F}^{\bullet}_{\omega,\mathbf{c}}(X, W)$ to be the restriction to $\mathcal{H}_{\omega,\mathbf{c}}(X, W)$ of the order filtration on $j_{\bullet}(\mathcal{D}^{\omega}_{X-D} \rtimes W)$, or, equivalently, one gives elements in $\mathcal{F}^{1}_{\omega,\mathbf{c}}(X, W)$ degree at most one, with $D \in \mathcal{F}^{1}_{\omega,\mathbf{c}}(X, W)$ having degree one if and only if $\sigma(D) \neq 0$, and then defines the filtration inductively by setting $\mathcal{F}^{i}_{\omega,\mathbf{c}}(X, W) = \mathcal{F}^{1}_{\omega,\mathbf{c}}(X, W)\mathcal{F}^{i-1}_{\omega,\mathbf{c}}(X, W)$. By definition, the filtration is exhaustive. Let $\pi : T^*X \to X$ be the projection map. Etingof [2004, Theorem 2.11] has shown that the algebras $\mathcal{H}_{\omega,\mathbf{c}}(X, W)$ are a flat deformation of $\mathcal{D}_X \rtimes W$. Equivalently, the PBW property holds for Cherednik algebras:

Theorem 2.4.1. *We have* $\operatorname{gr}_{\mathcal{F}} \mathcal{H}_{\omega,\mathbf{c}}(X, W) \simeq \pi_{\bullet}\mathcal{O}_{T^*X} \rtimes W$.

We note for later use that Theorem 2.4.1 implies that for any affine W-stable open set $U \subset X$, the algebra $\Gamma(U, \mathcal{H}_{\omega,\mathbf{c}}(X, W))$ has finite global dimension; its global dimension is bounded by $2 \dim X$.

2.5. Throughout, an $\mathcal{H}_{\omega,\mathbf{c}}(X, W)$-module will always mean an $\mathcal{H}_{\omega,\mathbf{c}}(X, W)$-module that is quasicoherent over $\mathcal{O}_X$. The category of all $\mathcal{H}_{\omega,\mathbf{c}}(X, W)$-modules is denoted by $\mathcal{H}_{\omega,\mathbf{c}}(X, W)$-Mod and the full subcategory of all modules coherent over $\mathcal{H}_{\omega,\mathbf{c}}(X, W)$ is denoted by $\mathcal{H}_{\omega,\mathbf{c}}(X, W)$-mod. A module $\mathcal{M} \in \mathcal{H}_{\omega,\mathbf{c}}(X, W)$-Mod is called *lisse* if it is coherent over $\mathcal{O}_X$.

3. Pullback of sheaves

In this section we show that modules for sheaves of Cherednik algebras can be pulled back under morphisms that are "melys" for the parameter **c**.

3.1. Let $\varphi : Y \to X$ be a W-equivariant morphism between smooth, connected, quasiprojective varieties. As explained in the appendix, given a Picard algebroid $\mathcal{P}^{\omega}_X$ on X, there is a φ-morphism $\mathcal{P}^{\varphi^*\omega}_Y \to \varphi^*\mathcal{P}^{\omega}_X$. This implies that the sheaf $\varphi^*\mathcal{D}^{\omega}_X$ is a left $\mathcal{D}^{\varphi^*\omega}_Y$-module. We give conditions on the map φ so that there exist a sheaf of Dunkl operators $\mathcal{F}^{1}_{\varphi^*\omega,\varphi^*\mathbf{c}}(Y, W)$ on Y and morphism of $\mathcal{O}_Y \rtimes W$-modules $\mathcal{F}^{1}_{\varphi^*\omega,\varphi^*\mathbf{c}}(Y, W) \to \varphi^*\mathcal{F}^{1}_{\omega,\mathbf{c}}(X, W)$. As a consequence $\varphi^*\mathcal{H}_{\omega,\mathbf{c}}(X, W)$ becomes a left $\mathcal{H}_{\varphi^*\omega,\varphi^*\mathbf{c}}(Y, W)$-module and we can pullback $\mathcal{H}_{\omega,\mathbf{c}}(X, W)$-modules to $\mathcal{H}_{\varphi^*\omega,\varphi^*\mathbf{c}}(Y, W)$-modules.

3.2. If the morphism φ is flat of relative dimension r, then there is a good notion of pullback of algebraic cycles, namely, $\varphi^* : C_k(X) \to C_{k+r}(Y)$, where $C_k(X)$ is the abelian group of k-dimensional algebraic cycles on X. See [Fulton 1998, Section 1.7]. The class in $C_k(X)$ of a k-dimensional subscheme Z of X is denoted by $[Z]$.

Lemma 3.2.1. *Let* $\varphi : Y \to X$ *be flat and* $(w, Z) \in \mathcal{S}(X)$. *Write* $\varphi^*[Z] = \sum_i n_i[Z_i]$, *where each* Z_i *is an irreducible subvariety of* Y. *Then,* w *permutes the* $[Z_i]$.

Moreover, if $\varphi^{-1}(Z)$ is set-theoretically contained in Y^w, then each irreducible component of $\varphi^{-1}(Z)$ is a connected component of Y^w of codimension one.

Proof. The first claim follows from the fact that, set-theoretically, $\varphi^{-1}(Z) = \bigcup_{n_i \neq 0} Z_i$. Since $\varphi^{-1}(Z)$ is a union of closed subvarieties of Y of codimension one and Y is assumed to be irreducible, it suffices for the second claim to show that $Y^w \neq Y$. Assume otherwise. Then, since φ is flat, $\varphi(Y^w) = \varphi(Y)$ is open in X, but is also contained in the closed subvariety X^w. Hence $X^w = X$. This contradicts the fact that Z is an irreducible component of X^w. □

3.3. Let $\mathcal{S}_{\mathbf{c}}(X)$ denote the set of all pairs $(w, Z) \in \mathcal{S}(X)$ such that $\mathbf{c}(w, Z) \neq 0$.

Definition 3.3.1. The morphism $\varphi : Y \to X$ is *melys* with respect to $\mathbf{c}$ if:

(1) φ is flat.

(2) For all $(w, Z) \in \mathcal{S}_{\mathbf{c}}(X)$, set-theoretically $\varphi^{-1}(Z) \subset Y^w$.

If φ is melys with respect to $\mathbf{c}$ then we define $\varphi^*\mathbf{c}$ on $\mathcal{S}(Y)$ by

$$(\varphi^*\mathbf{c})(w, Z') = \sum_{(w,Z)\in\mathcal{S}(X)} n_{Z,Z'}\, \mathbf{c}(w, Z),$$

where $\varphi^*[Z] = \sum_{Z'} n_{Z,Z'}[Z']$. Let $E = \bigcup_{\mathbf{c}(w,Z)\neq 0} Z$ and $D = \varphi^{-1}(E)$. Since φ is flat, each irreducible component of D has codimension one in X. Let $j : U := X - D \hookrightarrow X$ and $k : V = Y - E \hookrightarrow Y$; these are affine morphisms. For any quasicoherent sheaf $\mathcal{F}$ on X (resp. on Y), we denote by $\mathcal{F}(D)$ the sheaf $j_{\bullet}(\mathcal{F}|_U)$ (resp. by $\mathcal{F}(E)$ the sheaf $k_{\bullet}(\mathcal{F}|_V)$).

Lemma 3.3.2. *The sheaf $\varphi^*\mathcal{D}^{\omega}_Y(E) \rtimes W$ on X is a $\mathcal{D}^{\varphi^*\omega}_X(D) \rtimes W$-module, and there exists a morphism*

$$\gamma : \mathcal{D}^{\varphi^*\omega}_X(D) \rtimes W \longrightarrow \varphi^*\mathcal{D}^{\omega}_Y(E) \rtimes W$$

of $\mathcal{D}^{\varphi^*\omega}_X(D) \rtimes W$-modules.

Proof. The map φ restricts to a flat morphism $\Phi : U \to V$. By Lemma A.2.2, we have

$$\mathcal{P}^{\Phi^*\omega}_U \xrightarrow{\sim} \Phi^*\mathcal{P}^{\omega}_V \times_{\Phi^*\Theta_V} \Theta_U.$$

This induces a morphism $\gamma : \mathcal{D}^{\Phi^*\omega}_U \to \Phi^*\mathcal{D}^{\omega}_V$ of $\mathcal{D}^{\Phi^*\omega}_U$-modules. Since ω was chosen to be W-invariant, this extends to a morphism $\gamma : \mathcal{D}^{\Phi^*\omega}_U \rtimes W \to \Phi^*\mathcal{D}^{\omega}_V \rtimes W$ of $\mathcal{D}^{\Phi^*\omega}_U \rtimes W$-modules. Since $j_{\bullet}\mathcal{P}^{\Phi^*\omega}_U = \mathcal{P}^{\varphi^*\omega}_X(D)$, we have $j_{\bullet}(\mathcal{D}^{\Phi^*\omega}_U \rtimes W) =$

$\mathcal{D}_X^{\varphi^*\omega}(D) \rtimes W$. The diagram

$$\begin{tikzcd} U \arrow[r, hook, "j"] \arrow[d, "\Phi"'] & X \arrow[d, "\varphi"] \\ V \arrow[r, hook, "k"] & Y \end{tikzcd}$$

is Cartesian. Therefore, by flat base change, $j_\bullet \Phi^* \mathcal{P}_V^\omega \rtimes W = \varphi^* \mathcal{P}_Y^\omega(E) \rtimes W$ and hence $j_\bullet(\Phi^* \mathcal{D}_V^\omega \rtimes W) = \varphi^* \mathcal{D}_Y^\omega(E) \rtimes W$. □

3.4. By analogy with φ-morphisms (see Lemma A.2.2) we have:

Proposition 3.4.1. *There is a morphism*

$$\gamma : \mathcal{H}_{\varphi^*\omega,\varphi^*\mathbf{c}}(Y, W) \longrightarrow \varphi^* \mathcal{H}_{\omega,\mathbf{c}}(X, W)$$

of $\mathcal{H}_{\varphi^*\omega,\varphi^*\mathbf{c}}(Y, W)$-modules that induces an isomorphism of $\mathcal{O}_Y \rtimes W$-modules

$$\psi : \mathcal{F}^1_{\varphi^*\omega,\varphi^*\mathbf{c}}(Y, W) \xrightarrow{\sim} \varphi^* \mathcal{F}^1_{\omega,\mathbf{c}}(X, W) \times_{\varphi^*\Theta_X \otimes W} \Theta_Y \otimes W.$$

Proof. The algebra $\mathcal{H}_{\varphi^*\omega,\varphi^*\mathbf{c}}(Y, W)$ is a subalgebra of $\mathcal{D}_Y^{\varphi^*\omega}(E) \rtimes W$, whereas $\varphi^*\mathcal{H}_{\omega,\mathbf{c}}(X, W)$ is a subalgebra of $\varphi^*\mathcal{D}_X^\omega(D) \rtimes W$. Let $\gamma : \mathcal{H}_{\varphi^*\omega,\varphi^*\mathbf{c}}(Y, W) \to \varphi^*\mathcal{D}_X^\omega(D) \rtimes W$ be the restriction of the morphism $\gamma : \mathcal{D}_Y^{\varphi^*\omega}(E) \rtimes W \to \varphi^*\mathcal{D}_X^\omega(D) \rtimes W$ of Lemma 3.3.2. We claim that it suffices to show that the image of γ is contained in $\varphi^*\mathcal{H}_{\omega,\mathbf{c}}(X, W)$. Assuming this, the action of $\mathcal{H}_{\varphi^*\omega,\varphi^*\mathbf{c}}(Y, W)$ on $\varphi^*\mathcal{H}_{\omega,\mathbf{c}}(X, W)$ will just be the restriction of the action of $\mathcal{D}_Y^{\varphi^*\omega}(E) \rtimes W$ on $\varphi^*\mathcal{D}_X^\omega(D) \rtimes W$. Therefore, it is given by

$$a \cdot (g \otimes p) = \gamma([a, g]) \cdot (1 \otimes p) + g(\gamma(a) \cdot (1 \otimes p)),$$

where $a \in \mathcal{H}_{\varphi^*\omega,\varphi^*\mathbf{c}}(Y, W)$, $g \in \mathcal{O}_Y$ and $p \in \varphi^{-1}\mathcal{H}_{\omega,\mathbf{c}}(X, W)$. Here $[a, g]$ is thought of as an element of $\mathcal{H}_{\varphi^*\omega,\varphi^*\mathbf{c}}(Y, W)$. If $\gamma(a)$ is contained in $\varphi^*\mathcal{H}_{\omega,\mathbf{c}}(X, W)$ and $p \in \varphi^{-1}\mathcal{H}_{\omega,\mathbf{c}}(X, W)$, then $\gamma(a) \cdot (1 \otimes p)$ belongs to $\varphi^*\mathcal{H}_{\omega,\mathbf{c}}(X, W)$. Thus, it suffices to show that the image of γ is contained in $\varphi^*\mathcal{H}_{\omega,\mathbf{c}}(X, W)$ as claimed.

Since $\mathcal{H}_{\varphi^*\omega,\varphi^*\mathbf{c}}(Y, W)$ is generated as an algebra by $\mathcal{F}^1_{\varphi^*\omega,\varphi^*\mathbf{c}}(Y, W)$, it will suffice to show that the image of $\mathcal{F}^1_{\varphi^*\omega,\varphi^*\mathbf{c}}(Y, W)$ is contained in $\varphi^*\mathcal{F}^1_{\omega,\mathbf{c}}(X, W)$. This is a local calculation. Therefore, we may assume that both X and Y are affine and that the subvarieties Z of X with $(w, Z) \in \mathcal{S}_\mathbf{c}(X)$ are defined by the vanishing of functions f_Z. Let $p \in \mathcal{P}_Y^{\varphi^*\omega}$, and denote by D_p the associated Dunkl–Opdam operator given by (2.3.2). Let $\gamma(p) = \sum_i g^i \otimes q^i$ in $\varphi^*\mathcal{P}_X^\omega$. Then,

$$\gamma(D_p) = \sum_i g^i \otimes q^i + \sum_{(w,Z')} \frac{2(\varphi^*\mathbf{c})(w, Z')}{1 - \lambda_{w,Z'}} i_{\sigma_Y(p)}(d \log f_{Z'}) \otimes (w - 1).$$

If $\varphi^{-1}(Z) = Z'_1 \cup \cdots \cup Z'_l$ set-theoretically and $\varphi^*[Z] = \sum_{i=1}^l n_i [Z'_i]$, then $\varphi^* f_Z = u \prod_i f_{Z'_i}^{n_i}$, for some unit u, and scheme-theoretically $\varphi^{-1}(Z)$ is defined by the vanishing of the function $\prod_i f_{Z'_i}^{n_i}$. Therefore, by definition of the parameter $\varphi^*\mathbf{c}$,

$$\frac{2\mathbf{c}(w, Z)}{1 - \lambda_{w,Z}} \varphi^* d \log f_Z = \sum_{Z' \subset \varphi^{-1}(Z)} \frac{2(\varphi^*\mathbf{c})(w, Z')}{1 - \lambda_{w,Z'}} d \log f_{Z'} + h, \tag{3.4.2}$$

where $h \in \mathbb{O}_Y \rtimes W$. Hence, up to a term in $\varphi^* \mathbb{O}_X \rtimes W$,

$$\begin{aligned}
&\sum_{(w,Z')} \frac{2(\varphi^*\mathbf{c})(w, Z')}{1 - \lambda_{w,Z'}} i_{\sigma_Y(p)}(d \log f_{Z'}) \otimes (w-1) \\
&\qquad = \sum_{(w,Z)} \frac{2\mathbf{c}(w, Z)}{1 - \lambda_{w,Z}} i_{\sigma_Y(p)}(d \log \varphi^* f_Z) \otimes (w-1) \\
&\qquad = \sum_{(w,Z)} \frac{2\mathbf{c}(w, Z)}{1 - \lambda_{w,Z}} \frac{\sigma_Y(p)(\varphi^* f_Z)}{\varphi^* f_Z} \otimes (w-1) \\
&\qquad = \sum_{(w,Z)} \frac{2\mathbf{c}(w, Z)}{1 - \lambda_{w,Z}} \frac{1}{\varphi^* f_Z} \left(\sum_i g^i \varphi^*(\sigma_X(q^i)(f_Z)) \right) \otimes (w-1) \\
&\qquad = \sum_i g^i \otimes \left(\sum_{(w,Z)} \frac{2\mathbf{c}(w, Z)}{1 - \lambda_{w,Z}} \frac{\sigma_X(q^i)(f_Z)}{f_Z} (w-1) \right) \\
&\qquad = \sum_i g^i \otimes \left(\sum_{(w,Z)} \frac{2\mathbf{c}(w, Z)}{1 - \lambda_{w,Z}} i_{\sigma_X(q^i)}(d \log f_Z)(w-1) \right).
\end{aligned}$$

Thus, $\gamma(D_p) = \sum_i g^i \otimes D_{q^i}$, which lies in $\varphi^* \mathscr{F}^1_{\omega,\mathbf{c}}(X, W)$.

Finally, we show that the morphism γ induces the isomorphism ψ, as stated. Since φ is flat, pulling back the sequence (2.3.3) gives a short exact sequence

$$0 \longrightarrow \mathbb{O}_Y \rtimes W \longrightarrow \varphi^* \mathscr{F}^1_{\omega,\mathbf{c}}(X, W) \longrightarrow \varphi^* \Theta_X \otimes W \longrightarrow 0.$$

Using the fact that $\mathbb{O}_Y \rtimes W \times_{\varphi^*\Theta_X \otimes W} \Theta_Y \otimes W = \mathbb{O}_Y \rtimes W$, where $\mathbb{O}_Y \rtimes W \to \varphi^* \Theta_X \otimes W$ is the zero map, and the fact that $\varphi^* \Theta_X \otimes W \times_{\varphi^*\Theta_X \otimes W} \Theta_Y \otimes W = \Theta_Y \otimes W$, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{O}_Y \rtimes W \arrow[r] \arrow[d, equal] & \mathscr{F}^1_{\varphi^*\omega, \varphi^*\mathbf{c}}(Y, W) \arrow[r, "\sigma"] \arrow[d, "\psi"] & \Theta_Y \otimes W \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{O}_Y \rtimes W \arrow[r] & \varphi^* \mathscr{F}^1_{\omega,\mathbf{c}}(X, W) \times_{\varphi^*\Theta_X \otimes W} \Theta_Y \otimes W \arrow[r] & \Theta_Y \otimes W \arrow[r] & 0
\end{tikzcd}$$

By the five lemma, ψ is an isomorphism. $\square$

3.5. The morphism γ allows us to define an action of $\mathcal{H}_{\varphi^*\omega,\varphi^*\mathbf{c}}(Y, W)$ on $\varphi^*\mathcal{M}$ for any $\mathcal{H}_{\omega,\mathbf{c}}(X, W)$-module $\mathcal{M}$.

Corollary 3.5.1. *Assume that φ is melys with respect to* $\mathbf{c}$*. Then pullback is an exact functor*

$$\varphi^* : \mathcal{H}_{\omega,\mathbf{c}}(X, W)\text{-}\mathsf{Mod} \longrightarrow \mathcal{H}_{\varphi^*\omega,\varphi^*\mathbf{c}}(Y, W)\text{-}\mathsf{Mod}$$

extending the usual pullback $\varphi^* : \mathsf{QCoh}(X) \longrightarrow \mathsf{QCoh}(Y)$.

Proof. Proposition 3.4.1 implies that

$$\varphi^*\mathcal{M} = \varphi^*\mathcal{H}_{\omega,\mathbf{c}}(X, W) \otimes_{\varphi^{-1}\mathcal{H}_{\omega,\mathbf{c}}(X,W)} \varphi^{-1}\mathcal{M}$$

is naturally an $\mathcal{H}_{\varphi^*\omega,\varphi^*\mathbf{c}}(Y, W)$-module. Since φ is flat, pullback of quasicoherent $\mathcal{O}_X$-modules is an exact functor. □

It is clear by definition that φ^* maps $\mathcal{H}_{\omega,\mathbf{c}}(X, W)$-$\mathsf{mod}$ to $\mathcal{H}_{\varphi^*\omega,\varphi^*\mathbf{c}}(Y, W)$-$\mathsf{mod}$ and lisse $\mathcal{H}_{\omega,\mathbf{c}}(X, W)$-modules to lisse $\mathcal{H}_{\varphi^*\omega,\varphi^*\mathbf{c}}(Y, W)$-modules.

3.6. ***Étale morphisms.*** In this section we consider étale morphisms. Fix X, ω, W and $\mathbf{c}$ as above. Let $(X, \mathbf{c})_{\mathrm{mel}}$ be the full subcategory of Sch / X (schemes over X) consisting of all morphisms $Y \to X$ that are étale and melys with respect to $\mathbf{c}$. Then, one can easily check that $(X, \mathbf{c})_{\mathrm{mel}}$ is a site over X; see, e.g., [Milne 1980, Section II.1] for details on sites. We call $(X, \mathbf{c})_{\mathrm{mel}}$ the melys site over X. The following result is closely related to [Wilcox 2011, Proposition 2.3].

Proposition 3.6.1. *The sheaf* $\mathcal{H}_{\omega,\mathbf{c}}(X, W)$ *is a sheaf of algebras on the melys site* $(X, \mathbf{c})_{\mathrm{mel}}$.

Proof. Let $\varphi : Y \to X$ be an étale map, melys with respect to $\mathbf{c}$. We begin by showing that $\varphi^*\mathcal{H}_{\omega,\mathbf{c}}(X, W)$ is a sheaf of algebras and the morphism γ of Proposition 3.4.1 is an isomorphism of algebras.

As in Section 3.3, let $D = \bigcup Z$, $E = \varphi^{-1}(D)$, $U = X - D$ and $V = Y - E$. Since $\Phi : V \to U$ is étale, it is flat, and hence $\Phi^{-1}\mathcal{D}_U^\omega \rtimes W$ is a subsheaf of $\Phi^*\mathcal{D}_U^\omega \rtimes W$. As noted in Remark A.2.4, the natural map $\gamma : \mathcal{D}_V^{\Phi^*\omega} \rtimes W \to \Phi^*\mathcal{D}_U^\omega \rtimes W$ is an algebra isomorphism such that the restriction of γ^{-1} to $\Phi^{-1}\mathcal{D}_U^\omega \rtimes W$ is an algebra morphism $\Phi^{-1}\mathcal{D}_U^\omega \rtimes W \to \mathcal{D}_V^{\Phi^*\omega} \rtimes W$. Therefore, using flat base change as in the proof of Lemma 3.3.2, we get an algebra morphism $\gamma^{-1} : \varphi^{-1}\mathcal{D}_X^\omega(D) \rtimes W \to \mathcal{D}_Y^{\varphi^*\omega}(E) \rtimes W$. This morphism induces an algebra isomorphism

$$\gamma^{-1} : \varphi^*\mathcal{D}_X^\omega(D) \rtimes W \xrightarrow{\sim} \mathcal{D}_Y^{\varphi^*\omega}(E) \rtimes W,$$

where the multiplication in $\varphi^*\mathcal{D}_X^\omega(D) \rtimes W$ is given by

$$(g_1 \otimes q_1) \cdot (g_2 \otimes q_2) = (g_1 \otimes 1)u(q_1, g_2)(1 \otimes q_2),$$

with $u(q, g) := \gamma([\gamma^{-1}(q), g]) \in \Phi^*\mathcal{D}_X^\omega(D) \rtimes W$, for all $q, q_1, q_2 \in \varphi^{-1}\mathcal{D}_X^\omega(D) \rtimes W$

and all $g, g_1, g_2 \in \mathbb{O}_Y$. By Proposition 3.4.1, γ^{-1} restricts to an algebra morphism $\varphi^{-1}\mathscr{H}_{\omega,\mathbf{c}}(X, W) \to \mathscr{H}_{\varphi^*\omega,\varphi^*\mathbf{c}}(Y, W)$, inducing an isomorphism $\varphi^*\mathscr{H}_{\omega,\mathbf{c}}(X, W) \xrightarrow{\sim} \mathscr{H}_{\varphi^*\omega,\varphi^*\mathbf{c}}(Y, W)$. Let

$$\begin{tikzcd} Y_1 \arrow[rr, "\vartheta"] \arrow[dr, "\varphi_1"'] & & Y_2 \arrow[dl, "\varphi_2"] \\ & X & \end{tikzcd}$$

be a morphism in $(X, \mathbf{c})_{\mathrm{mel}}$. Then, Y_1 and Y_2 are smooth varieties and, by [Milne 1980, I, Corollary 3.6], ϑ is an étale morphism. Lemma 3.2.1 implies that it is also melys. Thus, the above computations show that $\mathscr{H}_{\omega,\mathbf{c}}(X, W)$ forms a presheaf on $(X, \mathbf{c})_{\mathrm{mel}}$.

To check that it is in fact a sheaf, it suffices to do so locally; see the proof of [Borho and Brylinski 1989, Proposition 0]. Therefore, we assume that X is affine and that we are given an étale, W-equivariant, affine covering $(i_\alpha : Y_\alpha \to X)$ of X; i.e., each Y_α is affine and the union of the images of the maps i_α cover X. Then we must prove that the sequence

$$0 \longrightarrow \mathsf{H}_{\omega,\mathbf{c}}(X, W) \longrightarrow \bigoplus_{\alpha} \mathsf{H}_{i_\alpha^*\omega, i_\alpha^*\mathbf{c}}(Y_\alpha, W) \longrightarrow \bigoplus_{\alpha,\beta} \mathsf{H}_{i_{\alpha,\beta}^*\omega, i_{\alpha,\beta}^*\mathbf{c}}(Y_\alpha \times_X Y_\beta, W)$$

is exact. Let $U, V_\alpha, \dots$ be the usual open subsets of $X, Y_\alpha, \dots$. Then, we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathsf{H}_{\omega,\mathbf{c}}(X, W) \arrow[r, "j"] \arrow[d, hook] & \bigoplus_{\alpha} \mathsf{H}_{i_\alpha^*\omega, i_\alpha^*\mathbf{c}}(Y_\alpha, W) \arrow[r, "k"] \arrow[d, hook] & \bigoplus_{\alpha,\beta} \mathsf{H}_{i_{\alpha,\beta}^*\omega, i_{\alpha,\beta}^*\mathbf{c}}(Y_\alpha \times_X Y_\beta, W) \arrow[d, hook] \\ 0 \arrow[r] & \Gamma(U, \mathscr{D}_U^\omega \rtimes W) \arrow[r] & \bigoplus_{\alpha} \Gamma\big(V_\alpha, \mathscr{D}_{V_\alpha}^{i_\alpha^*\omega} \rtimes W\big) \arrow[r] & \bigoplus_{\alpha,\beta} \Gamma\big(V_\alpha \times_U V_\beta, \mathscr{D}^{i_{\alpha,\beta}^*\omega} \rtimes W\big) \end{tikzcd}$$

The bottom row is exact because $\mathscr{D}_U^\omega \rtimes W$ is a sheaf on the melys site. Since the diagram commutes, j is injective and the image of j is contained in the kernel of k. Therefore, we just need to show that the image of j is exactly the kernel of k. The sequence on the bottom row is strictly filtered with respect to the order filtration and, as noted in Section 2.4, the Cherednik algebra inherits its natural filtration by restriction of the order filtration on $\mathscr{D}_U^\omega \rtimes W$. Therefore, the top row will be exact if and only if the corresponding sequence of associated graded objects is exact. But this sequence is also the associated graded of the analogous sequence for $\mathscr{D}_X \rtimes W$, which we know is exact. $\square$

3.7. ***The*** **KZ*-functor.*** Assume that W acts freely on the open sets $V \subset Y$ and $U \subset X$, and let $\omega = 0$. The proof of Proposition 3.4.1 makes it clear that pullback of

$\mathcal{H}_{\mathbf{c}}(X, W)$-modules is compatible with the KZ-functor. Denote by $\mathcal{H}_{\mathbf{c}}(X, W)$-Reg the full subcategory of $\mathcal{H}_{\mathbf{c}}(X, W)$-mod consisting of all lisse $\mathcal{H}_{\mathbf{c}}(X, W)$-modules whose restriction to U is an integrable connection with regular singularities. Let DR be the de Rham functor that maps integrable connections with regular singularities on U/W to representations of the fundamental group $\pi_1(U/W)$. The KZ-functor is defined by

$$\mathsf{KZ}_X(\mathcal{M}) = \mathsf{DR}\big([\rho_{\bullet}(\mathcal{M}|_U)]^W\big).$$

Then φ^* maps $\mathcal{H}_{\mathbf{c}}(X, W)$-Reg into $\mathcal{H}_{\varphi^*\mathbf{c}}(Y, W)$-Reg. Therefore, since the de Rham functor behaves well with respect to pullback [Hotta et al. 2008, Theorem 7.1.1], the following diagram commutes

$$\begin{tikzcd}
\mathcal{H}_{\mathbf{c}}(X, W)\text{-Reg} \arrow[r, "\varphi^*"] \arrow[d, "\mathsf{KZ}_X"'] & \mathcal{H}_{\varphi^*\mathbf{c}}(Y, W)\text{-Reg} \arrow[d, "\mathsf{KZ}_Y"] \\
\pi_1(U/W)\text{-mod} \arrow[r, "\Phi^*"] & \pi_1(V/W)\text{-mod}
\end{tikzcd}$$

The image of the KZ-functor is contained in the full subcategory of $\pi_1(U/W)$-mod consisting of all modules for a certain "Hecke" quotient of $\mathbb{C}\pi_1(U/W)$; see [Etingof 2004, Proposition 3.4].

3.8. ***Pushforward.*** It is also possible to define (derived) pushforward of modules under melys maps. Let $\varphi : Y \to X$ be melys with respect to $\mathbf{c}$, and denote by $\mathsf{Mod}\text{-}\mathcal{H}_{\omega,\mathbf{c}}(Y, W)$ the category of right $\mathcal{H}_{\omega,\mathbf{c}}(Y, W)$-modules. Then, the derived pushforward functor

$$\mathbb{R}\varphi_* : D^b(\mathsf{Mod}\text{-}\mathcal{H}_{\omega,\mathbf{c}}(Y, W)) \longrightarrow D^b(\mathsf{Mod}\text{-}\mathcal{H}_{\omega,\mathbf{c}}(X, W))$$

is given by

$$\mathbb{R}\varphi_*(\mathcal{M}) = \mathbb{R}\varphi_{\bullet}\big(\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{H}_{\omega,\mathbf{c}}(Y,W)} \varphi^*\mathcal{H}_{\omega,\mathbf{c}}(X, W)\big).$$

Let us justify the fact that the image of $\mathbb{R}\varphi_*$ is contained in $D^b(\mathsf{Mod}\text{-}\mathcal{H}_{\omega,\mathbf{c}}(X, W))$. First, as noted in Section 2.4, the PBW theorem implies that the sheaf $\mathcal{H}_{\omega,\mathbf{c}}(Y, W)$ has good homological properties. Since we have assumed that Y is quasiprojective, this implies that each $\mathcal{M} \in \mathsf{Mod}\text{-}\mathcal{H}_{\omega,\mathbf{c}}(Y, W)$ has a finite resolution by locally projective $\mathcal{H}_{\omega,\mathbf{c}}(Y, W)$-modules; see [Hotta et al. 2008, Section 1.4]. Hence, for $\mathcal{M} \in D^b(\mathsf{Mod}\text{-}\mathcal{H}_{\omega,\mathbf{c}}(Y, W))$, the complex $\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{H}_{\omega,\mathbf{c}}(Y,W)} \varphi^*\mathcal{H}_{\omega,\mathbf{c}}(X, W)$ belongs to $D^b(\mathsf{Mod}\text{-}\varphi^{-1}\mathcal{H}_{\omega,\mathbf{c}}(Y, W))$. That $\mathbb{R}\varphi_*(\mathcal{M})$ belongs to $D^b(\mathsf{Mod}\text{-}\mathcal{H}_{\omega,\mathbf{c}}(X, W))$ then follows, for instance, from [Hotta et al. 2008, Proposition 1.5.4].

We will also require pushforwards of left $\mathcal{H}_{\omega,\mathbf{c}}(Y, W)$-modules under open embeddings $j : Y \hookrightarrow X$. The following is standard; see, e.g., [Hotta et al. 2008, Proposition 1.5.4].

Lemma 3.8.1. *For $\mathcal{M} \in \mathcal{H}_{\omega,\mathbf{c}}(Y, W)$-Mod, the sheaves $\mathbb{R}^i j_\bullet(\mathcal{M})$, $i \geq 0$, belong to* $\mathcal{H}_{\omega,\mathbf{c}}(X, W)$-Mod.

It would be interesting to develop a notion of duality for Cherednik algebras, which would allow one to define pushforwards of left $\mathcal{H}_{\omega,\mathbf{c}}(Y, W)$-modules along arbitrary melys morphisms.

4. Twisted equivariant modules

In this section we define (twisted) G-equivariant $\mathcal{H}_{\omega,\mathbf{c}}(X, W)$-modules.

4.1. Let X be a smooth W-variety, and $\mathcal{H}_{\omega,\mathbf{c}}(X, W)$ a sheaf of Cherednik algebras on X. Assume that a *connected* algebraic group G also acts on X such that this action commutes with the action of W. Write $p, a : G \times X \longrightarrow X$ for the projection and action maps. Let $\mathcal{M}$ be an $\mathcal{H}_{\omega,\mathbf{c}}(X, W)$-module. Clearly, $p^*\mathcal{M}$ is an $\mathcal{H}_{\omega,\mathbf{c}}(G \times X, W) = \mathcal{D}_G \boxtimes \mathcal{H}_{\omega,\mathbf{c}}(X, W)$-module.

Lemma 4.1.1. *The action map a is melys for any $\mathbf{c}$, and therefore $a^*\mathcal{M}$ is an $\mathcal{H}_{\omega,\mathbf{c}}(G \times X, W)$-module.*

Proof. The action map a is smooth and hence flat. Let $(w, Z) \in \mathcal{S}(X)$. Since the action of G commutes with the action of W, X^w is G-stable. Moreover, the fact that G and Z are connected implies that Z itself is G-stable. Thus, $a^{-1}(Z) = G \times Z$ is contained in $(G \times X)^w = G \times X^w$. $\square$

4.2. The Lie algebra of G is denoted by $\mathfrak{g}$. Let $m : G \times G \to G$ the multiplication map and $s : X \to G \times X$ be defined by $s(x) = (e, x)$. Choose $\chi \in (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$, and let $\mathcal{O}_G^\chi$ be the $\mathcal{D}_G$-module $\mathcal{D}_G/\mathcal{D}_G\{v - \chi(v) \mid v \in \mathfrak{g}\}$, where we have identified $\mathfrak{g}$ with right-invariant vector fields on G. It is an irreducible integrable connection on G.

Definition 4.2.1. The module $\mathcal{M} \in \mathcal{H}_{\omega,\mathbf{c}}(X, W)$-Mod is called (G, χ)-monodromic if there exists an isomorphism $\theta : \mathcal{O}_G^\chi \boxtimes \mathcal{M} \xrightarrow{\sim} a^*\mathcal{M}$ of $\mathcal{H}_{\omega,\mathbf{c}}(G \times X, W)$-modules such that $s^*\theta = \mathrm{id}_{\mathcal{M}}$ and the diagram

$$\begin{tikzcd}
\mathcal{O}_G^\chi \boxtimes \mathcal{O}_G^\chi \boxtimes \mathcal{M} \arrow[rr, "\mathrm{id}_G \times \theta"] \arrow[d, "="'] & & \mathcal{O}_G^\chi \boxtimes a^*\mathcal{M} \arrow[d, "="] \\
(m \times \mathrm{id})^*(\mathcal{O}_G^\chi \boxtimes \mathcal{M}) \arrow[d, "(m \times \mathrm{id}_X)^*\theta"'] & & (\mathrm{id}_G \times a)^*(\mathcal{O}_G^\chi \boxtimes \mathcal{M}) \arrow[dl, "(\mathrm{id}_G \times a)^*\theta"] \\
(m \times \mathrm{id}_X)^* a^*\mathcal{M} \arrow[r, "="] & (\mathrm{id}_G \times a)^* a^*\mathcal{M} &
\end{tikzcd} \tag{4.2.2}$$

is commutative: $\mathcal{M}$ satisfies the *cocycle condition*.

We will denote the category of (G, χ)-monodromic $\mathcal{H}_{\omega,\mathbf{c}}(X, W)$-modules by $(\mathcal{H}_{\omega,\mathbf{c}}(X, W), G, \chi)$-Mod.

4.3. ***T-monodromic modules.*** Let T be a torus, i.e., a product of copies of the multiplicative group $\mathbb{C}^\times$. The Lie algebra of T is denoted by $\mathfrak{t}$. Let $\pi : Y \to X$ be a principal T-bundle, with X smooth. We assume that the finite group W acts on Y, the action commuting with the action of T. This implies that W also acts on X and that the map π is T-equivariant. Let $\mathcal{H}_{\mathbf{c}}(Y, W)$ be a sheaf of Cherednik[2] algebras on Y.

Lemma 4.3.1. *There is a morphism of Lie algebras $\mu_{\mathbf{c}} : \mathfrak{t} \to \mathcal{F}^1_{\mathbf{c}}(Y, W)$ such that the composite $\sigma \circ \mu_{\mathbf{c}}$ equals the usual moment map $\mu : \mathfrak{t} \to \Theta_Y \otimes W$.*

Proof. Since the action of T commutes with the action of W, the open set $V = Y - E$ is T-stable. Differentiating the action of T on U, there is a map $\mu' : \mathfrak{t} \to \mathcal{D}_Y(E) \rtimes W$. It is clear that $\sigma \circ \mu' = \mu$. Therefore, we just need to show that the image of μ' is contained in the subsheaf $\mathcal{F}^1_{\mathbf{c}}(Y, W)$. This is a local computation. Hence we may assume that $Y = X \times T$, in which case $\mathcal{H}_{\mathbf{c}}(Y, W) = \mathcal{H}_{\mathbf{c}}(X, W) \boxtimes \mathcal{D}_T$. Now the claim is clear. $\square$

The group T acts on $\mathcal{H}_{\mathbf{c}}(Y, W)$ and the map $\mu_{\mathbf{c}}$ is T-equivariant. Moreover, a local computation (using the fact that the bundle $Y \to X$ is locally trivial) shows that the image of $\mathfrak{t}$ is central in $(\pi_{\bullet}\mathcal{H}_{\mathbf{c}}(Y, W))^T$, and hence we may perform quantum Hamiltonian reduction. Recall that we define the map $\beta : \mathfrak{t}^* \to \mathbb{H}^2(X, \Omega^{1,2}_X)$ in (A.3.2).

Proposition 4.3.2. *Let $\chi \in \mathfrak{t}^*$. We have an isomorphism of sheaves of algebras on X*

$$\mathcal{H}_{\beta(\chi),\mathbf{c}}(X, W) \simeq (\pi_{\bullet}\mathcal{H}_{\mathbf{c}}(Y, W))^T / \langle\{\mu_{\mathbf{c}}(t) - \chi(t) \mid t \in \mathfrak{t}\}\rangle.$$

Proof. As in the proof of Proposition 3.4.1, let $D = \bigcup_{\mathbf{c}(w,Z)\neq 0} Z$, $U = X - D$, $E = \pi^{-1}(D)$ and $V = Y - E$. Then the restriction of π to V is a principal T-bundle $\Pi : V \to U$ and we have a Cartesian diagram

$$\begin{tikzcd} V \arrow[r, hook, "j"] \arrow[d, "\Pi"'] & Y \arrow[d, "\pi"] \\ U \arrow[r, hook, "k"] & X \end{tikzcd}$$

Proposition A.3.3 implies that there is an isomorphism

$$(\Pi_{\bullet}\mathcal{D}_V \rtimes W)^T / \langle\{\mu'(t) - \chi(t) \mid t \in \mathfrak{t}\}\rangle \xrightarrow{\sim} \mathcal{D}^{\beta(\chi)}_U \rtimes W. \tag{4.3.3}$$

[2]We assume, for simplicity, that the twist ω is zero. Presumably one can also deal with nontrivial twists.

Recall that $\mathscr{D}_X^{\beta(\chi)}(D) \rtimes W = k_{\bullet}(\mathscr{D}_U^{\beta(\chi)} \rtimes W)$. Since

$$k_{\bullet}(\Pi_{\bullet}\mathscr{D}_V \rtimes W)^T = (k_{\bullet}(\Pi_{\bullet}\mathscr{D}_V \rtimes W))^T = (\pi_{\bullet}(j_{\bullet}\mathscr{D}_V \rtimes W))^T$$

and $(\pi_{\bullet}\mathscr{H}_{\mathbf{c}}(Y, W))^T$ is a subalgebra of $(\pi_{\bullet} j_{\bullet}\mathscr{D}_V \rtimes W)^T$, we have a morphism of sheaves $\tau : (\pi_{\bullet}\mathscr{H}_{\mathbf{c}}(Y, W))^T \to \mathscr{D}_X^{\beta(\chi)}(D) \rtimes W$. The isomorphism (4.3.3) implies that $\langle\{\mu_{\mathbf{c}}(t) - \chi(t) \mid t \in \mathfrak{t}\}\rangle$ is contained in the kernel of τ. Therefore it suffices to show that $\langle\{\mu_{\mathbf{c}}(t) - \chi(t) \mid t \in \mathfrak{t}\}\rangle$ is precisely the kernel of τ and that the image of τ is $\mathscr{H}_{\beta(\chi),\mathbf{c}}(X, W)$. Both of these statements are local. Thus, we may assume without loss of generality that $Y = X \times T$. In this case, both statements reduce to the statement $\mathscr{D}(T)^T/\langle\{t - \chi(t) \mid t \in \mathfrak{t}\}\rangle \simeq \mathbb{C}$, which is clear. □

4.4. As for differential operators on principal T-bundles — see Section 2.5 of [Beilinson and Bernstein 1993]) — Proposition 4.3.2 implies an equivalence of categories:

Theorem 4.4.1. *The functor*

$$(\mathscr{H}_{\mathbf{c}}(X, W), T, \chi)\text{-Mod} \to \mathscr{H}_{\beta(\chi),\mathbf{c}}(Y, W)\text{-Mod}, \quad \mathscr{M} \mapsto (\pi_{\bullet}\mathscr{M})^T$$

is an equivalence of categories with quasi-inverse $\mathscr{N} \mapsto \pi^*\mathscr{N}$.

The above theorem can be extended in the obvious way to the category of weakly T-equivariant $\mathscr{H}_{\mathbf{c}}(X, W)$-modules with generalized central character $\overline{\chi} \in \mathfrak{t}^*/\mathbb{X}(T)$, as in [Beilinson and Bernstein 1993]. We leave the details to the interested reader.

5. Affinity of Cherednik algebras on projective space

In this section we prove the main result, which is a criterion for the affinity of Cherednik algebras on $\mathbb{P}(V)$.

5.1. Let V be a vector space and $W \subset \mathrm{GL}(V)$ a finite group. For each $(s, H) \in \mathscr{S}(V)$ and $(s, H^*) \in \mathscr{S}(V^*)$, we fix $\alpha_H \in V^*$ and $\alpha_H^\vee \in V$ such that $H = \operatorname{Ker}\alpha_H$ and $H^* = \operatorname{Ker}\alpha_H^\vee$, normalized so that $\alpha_H(\alpha_H^\vee) = 2$. Let $V^o = V - \{0\}$ and $\pi : V^o \to \mathbb{P}(V)$ be the quotient map. The map π is a principal T-bundle, where $T = \mathbb{C}^\times$ acts on V by dilations; i.e., $t \cdot v = t^{-1}v$ for $t \in T$ and $v \in V$. Since W acts on V it also acts on $\mathbb{P}(V)$. For each $s \in W$, $\operatorname{codim}\mathbb{P}(V)^s = 1$ if and only if s is a reflection, in which case $\mathbb{P}(V)^s = \mathbb{P}(H) \cup \mathbb{C} \cdot \alpha_H^\vee$.

Lemma 5.1.1. *We have $\mathbb{H}^2(\mathbb{P}(V), \Omega_{\mathbb{P}}^{1,2}) \simeq \mathbb{C}$, and the morphism β of* (A.3.2) *is an isomorphism.*

Proof. For each $n \in \mathbb{Z}$, let λ_n be the character of $\mathbb{C}^\times$ given by $t \mapsto t^n$. Then, $(\pi_{\bullet}\mathbb{O}_{V^o})^{\lambda_n} \simeq \mathbb{O}(n)$. This implies that β is injective. Therefore, it suffices to show that $\dim \mathbb{H}^2(\mathbb{P}(V), \Omega_{\mathbb{P}}^{1,2}) = 1$. Since $\mathbb{P}(V)$ can be covered by open sets isomorphic to $\mathbb{A}^{n-1}$, and $H_{\mathrm{DR}}^i(\mathbb{A}^{n-1}) = 0$ for $i \neq 0$, the algebraic de Rham complex is acyclic.

This implies that the map $d\mathcal{O}_{\mathbb{P}}[-1] \to \Omega^{1,2}_{\mathbb{P}}$ is a quasi-isomorphism. Therefore, the map $H^1(\mathbb{P}(V), d\mathcal{O}_{\mathbb{P}}) = \mathbb{H}^2(\mathbb{P}(V), d\mathcal{O}_{\mathbb{P}}[-1]) \to \mathbb{H}^2(\mathbb{P}(V), \Omega^{1,2}_{\mathbb{P}})$ is an isomorphism. The long exact sequence associated to the short exact sequence

$$0 \longrightarrow \mathbb{C}_{\mathbb{P}} \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow d\mathcal{O}_{\mathbb{P}} \longrightarrow 0$$

shows that $H^1(\mathbb{P}(V), d\mathcal{O}_{\mathbb{P}}) \simeq H^2(\mathbb{P}(V), \mathbb{C}_{\mathbb{P}})$ is one-dimensional. □

Lemma 5.1.1 implies the well-known fact that twisted differential operators on projective space are locally isomorphic, in the Zariski topology, to the usual differential operators. We identify $\mathbb{H}^2(\mathbb{P}(V), \Omega^{1,2}_{\mathbb{P}})$ with $\mathbb{C}$ so that if $\omega = n \in \mathbb{Z}$, then $\mathcal{D}^{\omega}_{\mathbb{P}(V)}$ acts on $\mathcal{O}(n)$. The action of W on $\mathbb{H}^2(\mathbb{P}(V), \Omega^{1,2}_{\mathbb{P}})$ is trivial; therefore the sheaf $\mathcal{D}^{\omega}_{\mathbb{P}(V)}$ is W-equivariant for all ω.

5.2. When $X = V$, the *rational Cherednik algebra* $\mathsf{H}_{\mathbf{c}}(V, W)$, as introduced by Etingof and Ginzburg, can be described as an algebra given by generators and relations. Namely, it is the quotient of the skew group algebra $T(V \oplus V^*) \rtimes W$ by the ideal generated by the relations

$$[x, x'] = 0, \quad [y, y'] = 0, \quad [y, x] = x(y) - \sum_{s \in \mathcal{S}} \mathbf{c}(s)\alpha_H(y)x(\alpha_H^{\vee})s \qquad (5.2.1)$$

for all $x, x' \in V^*$ and $y, y' \in V$. Let $x_1, \dots, x_n$ be a basis of V^* and $y_1, \dots, y_n \in V$ the dual basis. The Euler element is

$$\mathbf{h} = \sum_{i=1}^{n} x_i y_i - \sum_{s \in \mathcal{S}} \frac{2\mathbf{c}(s)}{1 - \lambda_s} s = \sum_{i=1}^{n} y_i x_i - n + \sum_{s \in \mathcal{S}} 2\mathbf{c}(s)\left(1 - \frac{1}{1 - \lambda_s}\right)s.$$

One can easily check that $[\mathbf{h}, x] = x$, $[\mathbf{h}, y] = -y$ and $[\mathbf{h}, w] = 0$ for all $x \in V^*$, $y \in V$ and $w \in W$. The element $\mathbf{h}$ defines an internal grading on $\mathsf{H}_{\mathbf{c}}(V, W)$, where $\deg(x) = 1$, $\deg(y) = -1$ and $\deg(w) = 0$. The m-th graded piece of $\mathsf{H}_{\mathbf{c}}(V, W)$ is denoted by $\mathsf{H}_{\mathbf{c}}(V, W)_m$.

5.3. ***Dunkl embedding.*** The open subset $U = V - D$ of V is the complement to the zero locus of $\prod_{s \in \mathcal{S}} \alpha_H$. For $y \in V$, thought of as a constant coefficient differential operator, the corresponding Dunkl operator D_y equals

$$\partial_y + \sum_{s \in \mathcal{S}} \frac{2\mathbf{c}(s)}{1 - \lambda_s} \frac{\alpha_H(y)}{\alpha_H}(s - 1) \in \Gamma(U, \mathcal{D}_U \rtimes W).$$

The presentation of $\mathsf{H}_{\mathbf{c}}(V, W)$ given above is identified with the Cherednik algebra, defined in terms of Dunkl operators, via the injective algebra homomorphism

$$\mathsf{H}_{\mathbf{c}}(V, W) \hookrightarrow \Gamma(U, \mathcal{D}_U \rtimes W), \quad w \mapsto w,\ x \mapsto x,\ y \mapsto D_y$$

for all $w \in W$, $x \in V^*$ and $y \in V$. The image of $\mathbf{h}$ under the Dunkl embedding is

$$\mathbf{h} = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} - \sum_{s \in \mathcal{S}} \frac{2\mathbf{c}(s)}{1-\lambda_s}. \tag{5.3.1}$$

5.4. ***The sheaf of Cherednik algebras on*** $\mathbb{P}(V)$. Set $\rho_{\mathbf{c}} = \sum_{s\in\mathcal{S}} 2\mathbf{c}(s)/(1-\lambda_s)$. As noted in Example 2.20 of [Etingof 2004], the global sections of $\mathcal{H}_{\omega,\mathbf{c}}(\mathbb{P}(V), W)$ are related to $\mathsf{H}_{\mathbf{c}}(V, W)$ as follows:

Lemma 5.4.1. *The space* $\mathsf{H}_{\omega,\mathbf{c}}(\mathbb{P}(V), W)$ *of global sections equals*

$$\mathsf{H}_{\mathbf{c}}(V, W)_0/(\mathbf{h} + \rho_{\mathbf{c}} - \omega).$$

Proof. By Proposition 4.3.2, we have a morphism

$$\mathsf{H}_{\mathbf{c}}(V, W)_0 = \mathsf{H}_{\mathbf{c}}(V, W)^T \to \mathsf{H}_{\mathbf{c}}(V^o, W)^T \to \mathsf{H}_{\omega,\mathbf{c}}(\mathbb{P}(V), W).$$

Equation (5.3.1) implies that the operator $\mathbf{h} + \rho_{\mathbf{c}} - \omega$ is in the kernel of this map because it is in the kernel of the composite

$$\mathsf{H}_{\mathbf{c}}(V, W)_0 \to \mathsf{H}_{\omega,\mathbf{c}}(\mathbb{P}(V), W) \to \mathcal{H}_{\omega,\mathbf{c}}(\mathbb{P}(V), W) \hookrightarrow \mathcal{D}^{\omega}_{\mathbb{P}(V)}(D) \rtimes W.$$

To prove that $\mathsf{H}_{\mathbf{c}}(V, W)_0/(\mathbf{h} + \rho_{\mathbf{c}} - \omega) \to \mathsf{H}_{\omega,\mathbf{c}}(\mathbb{P}(V), W)$ is an isomorphism, we consider the associated graded morphism. We have

$$\mathrm{gr}_{\mathcal{F}}\, \mathsf{H}_{\mathbf{c}}(V, W)_0 = \mathbb{C}[x_i y_j \mid i, j = 1, \dots, n] \rtimes W.$$

We claim that

$$\begin{aligned} \mathrm{gr}_{\mathcal{F}}\, \mathsf{H}_{\omega,\mathbf{c}}(\mathbb{P}(V), W) &= \Gamma(\mathbb{P}(V), \pi_{\bullet}\mathcal{O}_{T^*\mathbb{P}(V)} \rtimes W) \\ &= \left(\mathbb{C}[x_i y_j \mid i, j = 1, \dots, n] \Big/ \left(\sum_{i=1}^{n} x_i y_i\right)\right) \rtimes W. \end{aligned}$$

The second equality just follows from the usual description of $T^*\mathbb{P}(V)$ as the Hamiltonian reduction of $T^*V^o = V^o \times V^*$ with respect to the induced action of T. The first equality follows from Theorem 2.4.1, once one takes into account that the short exact sequences

$$0 \longrightarrow \mathcal{F}^{m-1}_{\omega,\mathbf{c}}(\mathbb{P}(V), W) \longrightarrow \mathcal{F}^{m}_{\omega,\mathbf{c}}(\mathbb{P}(V), W) \longrightarrow (\mathrm{Sym}^m\, \Theta_{\mathbb{P}(V)}) \otimes W \longrightarrow 0$$

imply by induction that $\mathbb{R}^i\Gamma(\mathcal{F}^m_{\omega,\mathbf{c}}(\mathbb{P}(V), W)) = 0$ for $i > 0$. Therefore, the filtered morphism $\mathsf{H}_{\mathbf{c}}(V, W)_0 \to \mathsf{H}_{\omega,\mathbf{c}}(\mathbb{P}(V), W)$ is surjective, and hence so too is $\mathsf{H}_{\mathbf{c}}(V, W)_0/(\mathbf{h} + \rho_{\mathbf{c}} - \omega) \to \mathsf{H}_{\omega,\mathbf{c}}(\mathbb{P}(V), W)$. On the other hand, the associated graded algebra of $\mathsf{H}_{\mathbf{c}}(V, W)_0/(\mathbf{h} + \rho_{\mathbf{c}} - \omega)$ is a quotient of the algebra

$$\left(\mathbb{C}[x_i y_j \mid i, j = 1, \dots, n]/\left(\sum_{i=1}^{n} x_i y_i\right)\right) \rtimes W. \qquad \square$$

5.5. Let Irr W be the set of all isomorphism classes of irreducible W-modules. The element

$$\mathbf{z} := \sum_{s\in\mathcal{S}} 2\mathbf{c}(s)\left(1 - \frac{1}{1-\lambda_s}\right)s = -\mathbf{z}_0 + \sum_{s\in\mathcal{S}} 2\mathbf{c}(s)s$$

belongs to the center of $\mathbb{C}W$. For each $\lambda \in \operatorname{Irr} W$, let c_λ be the scalar by which $\mathbf{z}$ acts on λ and d_λ the scalar by which $\mathbf{z}_0$ acts on λ. Set

$$a_\lambda := \rho_{\mathbf{c}} + c_\lambda - n - \omega, \quad b_\lambda := \rho_{\mathbf{c}} - d_\lambda - \omega.$$

The sheaf of algebras $\mathcal{H}_{\omega,\mathbf{c}}(\mathbb{P}(V), W)$ is said to be *affine* if the global sections functor Γ induces an equivalence of categories

$$\Gamma : \mathcal{H}_{\omega,\mathbf{c}}(\mathbb{P}(V), W)\text{-}\mathsf{Mod} \xrightarrow{\sim} \mathsf{H}_{\omega,\mathbf{c}}(\mathbb{P}(V), W)\text{-}\mathsf{Mod}.$$

Theorem 5.5.1. *Let a_λ and b_λ be as above.*

(1) *The functor Γ is exact, provided $a_\lambda \notin \mathbb{Z}_{\geq 0}$ for all $\lambda \in \operatorname{Irr} W$.*

(2) *The functor Γ is conservative, provided $b_\lambda \notin \mathbb{Z}_{>0}$ for all $\lambda \in \operatorname{Irr} W$.*

Hence, the sheaf of algebras $\mathcal{H}_{\omega,\mathbf{c}}(\mathbb{P}(V), W)$ is affine, provided $a_\lambda \notin \mathbb{Z}_{\geq 0}$ and $b_\lambda \notin \mathbb{Z}_{>0}$ for all $\lambda \in \operatorname{Irr} W$.

Our proof of Theorem 5.5.1 follows that of Theorem 1.6.5 in [Hotta et al. 2008].

Proof. The category of finitely generated $\mathsf{H}_c(V, W)$-modules supported on $\{0\} \subset V$ is denoted by $\mathcal{O}_-$. It is the category $\mathcal{O}$ for the rational Cherednik algebra as studied in [Ginzburg et al. 2003]. We use basic results from this article without reference. The element $\mathbf{h}$ acts locally finitely on modules in $\mathcal{O}_-$. The generalized eigenvalues of $\mathbf{h}$ on $M \in \mathcal{O}_-$ are the *weights* of M. Let $\Delta(\lambda)$, for $\lambda \in \operatorname{Irr} W$, denote the Verma modules in $\mathcal{O}_-$. It is isomorphic to $(\operatorname{Sym} V)\otimes\lambda$ as a $\operatorname{Sym} V \rtimes (\mathbb{C}W \otimes \mathbb{C}[\mathbf{h}])$-module. The weights of $\Delta(\lambda)$ are $c_\lambda - n - \mathbb{Z}_{\geq 0}$. If $M \in \mathcal{O}_-$, then there exist a projective module $P \in \mathcal{O}_-$ and a surjection $P \twoheadrightarrow M$. The fact that the module P has a Verma flag implies that the weights of M are contained in $\bigcup_{\lambda\in\operatorname{Irr} W} c_\lambda - n - \mathbb{Z}_{\geq 0}$. Therefore, zero is not a generalized eigenvalue of $\mathbf{h} + \rho_{\mathbf{c}} - \omega$ on M, provided $c_\lambda + \rho_{\mathbf{c}} - r - \omega - n \neq 0$ for all $r \in \mathbb{Z}_{\geq 0}$, i.e., provided $a_\lambda \notin \mathbb{Z}_{\geq 0}$.

Let $0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0$ be a short exact sequence in $\mathcal{H}_{\omega,\mathbf{c}}(\mathbb{P}(V), W)$-$\mathsf{mod}$. By Theorem 4.4.1, the terms of the sequence $0 \to \pi^*\mathcal{M}_1 \to \pi^*\mathcal{M}_2 \to \pi^*\mathcal{M}_3 \to 0$ belong to $(\mathcal{H}_{\mathbf{c}}(V^o, W), T, \omega)$-$\mathsf{mod}$. Moreover, the sequence is exact because π is smooth. Let $j : V^o \hookrightarrow V$. As noted in Lemma 3.8.1, the sheaves $\mathbb{R}^i j_\bullet(\pi^*\mathcal{M}_k)$ for $i \geq 0$ and $k = 1, 2, 3$ are $\mathsf{H}_{\omega,\mathbf{c}}(V, W)$-modules. The modules $\mathbb{R}^i j_\bullet(\pi^*\mathcal{M}_k)$ are supported on $\{0\}$ for all $i > 0$. Therefore, they belong to the ind-category Ind $\mathcal{O}_-$. The global sections $\Gamma(\mathbb{P}(V), \mathcal{M}_k)$ are the element of the $\Gamma(V, j_\bullet\pi^*\mathcal{M}_k)^T$. Therefore the long exact sequence

$$0 \longrightarrow \Gamma(V, j_\bullet\pi^*\mathcal{M}_1) \longrightarrow \Gamma(V, j_\bullet\pi^*\mathcal{M}_2) \\ \longrightarrow \Gamma(V, j_\bullet\pi^*\mathcal{M}_3) \longrightarrow \Gamma(V, \mathbb{R}^1 j_\bullet(\pi^*\mathcal{M}_1)) \longrightarrow \cdots$$

gives rise to

$$0 \longrightarrow \Gamma(\mathbb{P}(V), \mathcal{M}_1) \longrightarrow \Gamma(\mathbb{P}(V), \mathcal{M}_2) \\ \longrightarrow \Gamma(\mathbb{P}(V), \mathcal{M}_3) \longrightarrow \Gamma(V, \mathbb{R}^1 j_\bullet(\pi^*\mathcal{M}_1))^T \longrightarrow \cdots .$$

The space $\Gamma(V, \mathbb{R}^1 j_\bullet(\pi^*\mathcal{M}_1))^T$ can be identified with the space of generalized $\mathbf{h}$-eigenvectors in $\Gamma(V, \mathbb{R}^1 j_\bullet(\pi^*\mathcal{M}_1))$ with eigenvalue $\omega - \rho_{\mathbf{c}}$. But if $a_\lambda \notin \mathbb{Z}_{\geq 0}$ for all λ, then this space is necessarily zero. Hence the sequence $0 \to \Gamma(\mathbb{P}(V), \mathcal{M}_1) \to \Gamma(\mathbb{P}(V), \mathcal{M}_2) \to \Gamma(\mathbb{P}(V), \mathcal{M}_3) \to 0$ is exact.

Next we need to show if $b_\lambda \notin \mathbb{Z}_{>0}$ for all $\lambda \in \operatorname{Irr} W$, then Γ is conservative; i.e., $\Gamma(\mathbb{P}(V), \mathcal{M}) = 0$ implies that $\mathcal{M} = 0$. Assume that $\mathcal{M} \neq 0$. Since π is smooth and surjective, it is faithfully flat and $\pi^*\mathcal{M} = 0$ implies that $\mathcal{M} = 0$. Hence $\pi^*\mathcal{M} \neq 0$. Since $\pi^*\mathcal{M}$ is (T, ω)-monodromic, the Euler element $\mathbf{h}$ acts semisimply on $\Gamma(V, j_\bullet\pi^*\mathcal{M})$, hence it decomposes as

$$\Gamma(V, j_\bullet\pi^*\mathcal{M}) = \bigoplus_{\alpha\in\mathbb{Z}} \Gamma(V, j_\bullet\pi^*\mathcal{M})_{\alpha+\omega-\rho_{\mathbf{c}}}.$$

There is some $\alpha \in \mathbb{Z}$ for which $\Gamma(V, j_\bullet\pi^*\mathcal{M})_{\alpha+\omega-\rho_{\mathbf{c}}} \neq 0$. We first assume that $\alpha > 0$. Choose $0 \neq m \in \Gamma(V, j_\bullet\pi^*\mathcal{M})_{\alpha+\omega-\rho_{\mathbf{c}}}$. Since the space $\Gamma(V, j_\bullet\pi^*\mathcal{M})_{\alpha+\omega-\rho_{\mathbf{c}}}$ is a W-module, we may assume that m lies in some irreducible W-isotypic component (of type λ say) of $\Gamma(V, j_\bullet\pi^*\mathcal{M})_{\alpha+\omega-\rho_{\mathbf{c}}}$. We claim that there is some y such that $y \cdot m \neq 0$. Assume not; then $\mathbf{h} \cdot m = -d_\lambda m$. Hence $-d_\lambda = \alpha + \omega - \rho_{\mathbf{c}}$; i.e., $b_\lambda = \rho_{\mathbf{c}} - d_\lambda - \omega = \alpha \in \mathbb{Z}_{>0}$, contradicting our assumption on b_λ. Thus $y \cdot m \neq 0$. But $y \cdot m \in \Gamma(V, j_\bullet\pi^*\mathcal{M})_{\alpha-1+\omega-\rho_{\mathbf{c}}}$, so eventually we get a nonzero vector in $\Gamma(V, j_\bullet\pi^*\mathcal{M})_{\omega-\rho_{\mathbf{c}}}$ as required. Now, assume that $\alpha < 0$. If $m \in \Gamma(V^o, \pi^*\mathcal{M})_{\alpha+\omega-\rho_{\mathbf{c}}}$ is a nonzero section, then the support of m is not contained in $\{0\}$. On the other hand, if $x \cdot m = 0$ for all $x \in V^*$, then $\operatorname{Supp}(m) \subset \{0\}$ and hence $m = 0$. Hence $m \neq 0$ implies that there exists some $x \in V^*$ such that $x \cdot m \neq 0$. Repeating this argument, we eventually conclude that $\Gamma(V^o, \pi^*\mathcal{M})_{\omega-\rho_{\mathbf{c}}} \neq 0$. □

When W is trivial, Theorem 5.5.1 says that $\mathbb{P}(V)$ is $\mathcal{D}^\omega$-affine provided $\omega \notin \{-n, -n-1, \dots\}$, which equals the set of all $\omega \in \mathcal{A} \cup \mathcal{E}$ of [Van den Bergh 1991, Theorem 6.1.3].

Remark 5.5.2. The action of W on V induces an action of W on all the partial flag manifolds $\mathrm{GL}(V)/P$, where P is a parabolic of $\mathrm{GL}(V)$. However, one can check that there are reflections in $(\mathrm{GL}(V)/P, W)$ if and only if $\mathrm{GL}(V)/P = \mathbb{P}(V)$ or $\mathrm{GL}(V)/P$ is the Grassmannian of codimension-one subspaces in V.

5.6. ***Abelianization of W.*** In this section we assume that (V, W) is a complex reflection group. Pullback of melys morphisms can be used to relate the representation theory of $\mathsf{H}_{\mathbf{c}}(V, W)$ with that of $\mathsf{H}_{\mathbf{c}}(\Gamma)$, where Γ is a cyclic quotient of W. Let $\mathscr{A}$ denote the set of reflecting hyperplanes in V and, for each $H \in \mathscr{A}$, fix s_H a generator of the cyclic group $W_H = \{w \in W \mid w(H) = H\}$. Let $W_{\mathrm{ab}} = W/[W, W]$, and let $\chi_0, \dots, \chi_{k-1}$ denote the linear characters of W, where $k = |W_{\mathrm{ab}}|$. For each i and $H \in \mathscr{A}$ we let $a_{i,H}$ be the least positive integer such that $\chi_i(s_H) = (\det s_H)^{a_{i,H}}$. We write $\mathbb{N}(W_{\mathrm{ab}})$ for the free semigroup generated by $\chi_0, \dots, \chi_{k-1}$. Then there is an evaluation map $\mathbb{N}(W_{\mathrm{ab}}) \to \{\chi_0, \dots, \chi_{k-1}\}$ which sends $\underline{\chi} = \sum_{i=0}^{k-1} n_i \chi_i$ to $\mathrm{ev}(\underline{\chi}) = \prod_{i=0}^{k-1} \chi_i^{n_i}$. For each $\underline{\chi} = \sum_{i=0}^{k-1} n_i \chi_i$, define

$$m_H = \sum_{i=0}^{k-1} n_i a_{i,H} \quad \text{and} \quad f_{\underline{\chi}} = \prod_{H \in \mathscr{A}} \alpha_H^{m_H} \in \mathbb{C}[V].$$

Then it follows from Stanley's results [1977] on W-semi-invariants that

$$w \cdot f_{\underline{\chi}} = \mathrm{ev}(\underline{\chi})(w) f_{\underline{\chi}} \quad \text{for all } w \in W.$$

Fix $\underline{\chi} \in \mathbb{N}(W_{\mathrm{ab}})$. The one-dimensional space spanned by $f_{\underline{\chi}}$ in $\mathbb{C}[V]$ is denoted by $\mathfrak{t}^*$. Inclusion $\mathfrak{t}^* \hookrightarrow \mathbb{C}[V]$ defines a W-equivariant morphism $\varphi : V \to \mathfrak{t}$. It is melys for any parameter $\mathbf{c}$ associated to $(\mathfrak{t}, W)$. Define $\mathbf{c}' : \mathscr{S}(V) \to \mathbb{C}$ by $\mathbf{c}'(s, H) = m_H\, \mathbf{c}(s, \{0\})$ for all (s, H) such that $(s, \{0\}) \in \mathscr{S}(\mathfrak{t})$, and $\mathbf{c}'(s, H) = 0$ otherwise. Corollary 3.5.1 implies:

Proposition 5.6.1. *Pullback by φ defines an exact functor*

$$\mathsf{H}_{\mathbf{c}}(\mathfrak{t}, W)\text{-}\mathsf{Mod} \to \mathsf{H}_{\mathbf{c}'}(V, W)\text{-}\mathsf{Mod}.$$

One can check that (3.4.2) implies that φ^* maps a module $M \in \mathcal{O}_{\mathbf{c}}(\mathfrak{t}, W)$ to $\varphi^* M \in \mathcal{O}_{\mathbf{c}'}(V, W)$, since the term h of (3.4.2) will be zero in this case. Moreover, for any such M, we have $\text{GK-dim}(\varphi^* M) = \text{GK-dim}(M) + \dim V - 1$. Let Γ be the cyclic group $W/\operatorname{Ker}\mathrm{ev}(\underline{\chi})$. Representations of the rational Cherednik algebra $\mathsf{H}_{\mathbf{c}}(\mathfrak{t}, W)$ can be viewed as W-equivariant representations of $\mathsf{H}_{\mathbf{c}}(\mathfrak{t}, \Gamma)$; see [Chmutova 2005].

Remark 5.6.2. More generally, if $\mathfrak{t}^* \subset \mathbb{C}[V]$ is an irreducible W-module, then we get a morphism $\varphi : V \to \mathfrak{t}$. It seems likely that one can use the theory developed in [Bessis et al. 2002] to classify all $\mathfrak{t}$ such that φ is melys. However, there do not seem to be many examples where $\dim \mathfrak{t} > 1$.

6. A local presentation of the Cherednik algebra

In this section we give a local presentation of the sheaf of Cherednik algebras.

6.1. In this section only, we make the following assumptions:

- For each $(w, Z) \in \mathcal{S}(X)$, there exists a globally defined function f_Z such that $Z = V(f_Z)$.
- All Picard algebroids considered can be trivialized in the Zariski topology.

We fix a choice of functions f_Z.

6.2. ***The* KZ*-connection.*** Recall that $U = X - \bigcup_{(w,Z)} Z$, where the union is over all (w, Z) in $\mathcal{S}(X)$. Since we have fixed a choice of defining equations of the hypersurfaces Z, it is possible to write down a KZ-connection on U.

Definition 6.2.1. The Knizhnik–Zamolodchikov connection on U, with values in $\mathbb{O}_U \otimes \mathbb{C}W$, is defined to be

$$\omega_{X,\mathbf{c}} = \sum_{(w,Z)\in\mathcal{S}(X)} \frac{2\mathbf{c}(w, Z)}{1-\lambda_{w,Z}} (d \log f_Z) \otimes s.$$

The KZ-connection behaves well under melys morphisms:

Lemma 6.2.2. *Let $\varphi : Y \to X$ be a surjective morphism, melys for* $\mathbf{c}$. *Then*, $\varphi^* \omega_{Y,\mathbf{c}} = \omega_{X,\varphi^*\mathbf{c}}$.

Proof. The fact that φ is surjective implies that $\varphi^* f_Z$ is not a unit for all $(w, Z) \in \mathcal{S}_{\mathbf{c}}(X)$. Then, the lemma follows from (3.4.2), since the term h there can be chosen to be zero. □

6.3. Fix $\omega \in \mathbb{H}^2(X, \Omega_X^{1,2})^W$, trivializable in the Zariski topology. For $(w, Z) \in \mathcal{S}(X)$ and $\nu_1, \nu_2 \in \mathcal{P}^\omega$, define

$$\Xi_Z^w(\nu_1, \nu_2) := i_{\sigma(\nu_1)}(d \log f_Z)(w(\nu_2) - \nu_2) - i_{\sigma(\nu_2)}(d \log f_Z)(w(\nu_1) - \nu_1)$$

in $\mathcal{P}^\omega(D)$.

Lemma 6.3.1. *Let $(w, Z) \in \mathcal{S}(X)$, $g \in \mathbb{O}_X$ and $\nu_1, \nu_2 \in \mathcal{P}^\omega$. Then,*

$$i_{\sigma(\nu)}(d \log f_Z)(w(g) - g) \in \mathbb{O}_X \quad \textit{and} \quad \Xi_Z^w(\nu_1, \nu_2) \in \mathcal{P}^\omega.$$

Proof. If $g \in \mathbb{O}_X$ and $\nu \in \mathcal{P}^\omega$, then $i_{\sigma(\nu)}(d \log f_Z)(w(g)-g) \in \mathbb{O}_X$ because $w(g)-g \in I(Z)$. The second claim is that

$$\frac{\sigma(\nu_1)(f_Z)}{f_Z}(w(\nu_2) - \nu_2) - \frac{\sigma(\nu_2)(f_Z)}{f_Z}(w(\nu_1) - \nu_1) \in \mathcal{P}^\omega.$$

The statement is local and is clearly true in a neighborhood of any point of $X - Z$. Therefore, we may assume that we have fixed a point $x \in Z$. Choose a small, affine w-stable open subset U of X with coordinate system $x_1, \dots, x_n$ such that $w(x_1) = \zeta x_1$ and $w(x_i) = x_i$ for $i \neq 1$. Moreover, since we have assumed that the Picard algebroid $\mathcal{P}^\omega$ trivializes in the Zariski topology, we may assume that

$\mathcal{P}^{\omega}|_U = \mathbb{O}_U \oplus \Theta_U$. There exists some unit $u \in \Gamma(U, \mathbb{O}_X)$ such that $f_Z = u x_1$. The statement is clear if either v_1 or v_2 is in $\Gamma(U, \mathbb{O}_X)$. Thus, without loss of generality, $v_1, v_2 \in \Gamma(U, \Theta_X)$. Expanding,

$$\Xi_Z^w(v_1, v_2) = \frac{v_1(x_1)}{x_1}(w(v_2) - v_2) - \frac{v_2(x_1)}{x_1}(w(v_1) - v_1) + h$$

for some $h \in \Gamma(U, \Theta_X)$. There are $f_i, g_i \in \Gamma(U, \mathbb{O}_X)$ such that $v_1 = \sum_{i=1}^n f_i(\partial/\partial x_i)$ and $v_2 = \sum_{i=1}^n g_i(\partial/\partial x_i)$. We have

$$\begin{aligned}
\frac{v_1(x_1)}{x_1}(w(v_2) - v_2) &= \sum_{i,j=1}^n f_i x_1^{-1} \frac{\partial x_1}{\partial x_i}\left(w(g_j)\frac{\partial}{\partial w(x_j)} - g_j \frac{\partial}{\partial x_j}\right)\\
&= \sum_{j=1}^n f_1 x_1^{-1}\left(w(g_j)\frac{\partial}{\partial w(x_j)} - g_j\frac{\partial}{\partial x_j}\right)\\
&= \sum_{j=1}^n f_1 x_1^{-1}\left((w(g_j) - g_j)\frac{\partial}{\partial w(x_j)} + g_j\left(\frac{\partial}{\partial w(x_j)} - \frac{\partial}{\partial x_j}\right)\right)\\
&= f_1 g_1 x_1^{-1}(\zeta - 1)\frac{\partial}{\partial x_1} + \sum_{j=1}^n f_1 x_1^{-1}\left((w(g_j) - g_j)\frac{\partial}{\partial w(x_j)}\right).
\end{aligned}$$

Thus, if we define

$$h_1 = \sum_{j=1}^n f_1 x_1^{-1}\left((w(g_j) - g_j)\frac{\partial}{\partial w(x_j)}\right), \quad h_2 = \sum_{j=1}^n g_1 x_1^{-1}\left((w(f_j) - f_j)\frac{\partial}{\partial w(x_j)}\right),$$

which belong to $\Gamma(U, \mathcal{P}^{\omega})$, we have

$$\begin{aligned}
&\frac{v_1(x_1)}{x_1}(w(v_2) - v_2) - \frac{v_2(x_1)}{x_1}(w(v_1) - v_1)\\
&\qquad = f_1 g_1 x_1^{-1}(\zeta - 1)\frac{\partial}{\partial x_1} + h_1 - f_1 g_1 x_1^{-1}(\zeta - 1)\frac{\partial}{\partial x_1} - h_2 = h_1 - h_2,
\end{aligned}$$

which belongs to $\Gamma(U, \Theta_X)$. $\square$

6.4. We define the sheaf of algebras $\mathcal{U}_{\omega,\mathbf{c}}(X, W)$ to be the quotient of $T\mathcal{P}^{\omega} \rtimes W$ by the relations

$$v \otimes g - g \otimes v = \sigma(v)(g) + \sum_{(w,Z)} \frac{2\mathbf{c}(w, Z)}{1 - \lambda_{w,Z}} i_{\sigma(v)}(d \log f_Z)(w(g) - g)w, \quad (6.4.1)$$

$$v_1 \otimes v_2 - v_2 \otimes v_1 = [v_1, v_2] + \sum_{(w,Z)} \frac{2\mathbf{c}(w, Z)}{1 - \lambda_{w,Z}} \Xi_Z^w(v_1, v_2)w \quad (6.4.2)$$

for all $\nu, \nu_1, \nu_2 \in \mathscr{P}^{\omega}_X$ and $g \in \mathbb{O}_X$, and the relation[3] $1_{\mathscr{P}} = 1$.

Remark 6.4.3. When $X = V$ is a vector space and $\nu_1, \nu_2 \in V$ are constant coefficient vector fields, the right-hand side of (6.4.2) is zero and we get the usual relations of the rational Cherednik algebra.

Proposition 6.4.4. *The map $\nu \mapsto D_\nu$, $w \mapsto w$ for $\nu \in \mathscr{P}^\omega$ and $w \in W$ defines an isomorphism $\mathscr{U}_{\omega,\mathbf{c}}(X, W) \xrightarrow{\sim} \mathscr{H}_{\omega,\mathbf{c}}(X, W)$ if and only if the* KZ*-connection is flat.*

Proof. The proof is a direct calculation. It is straightforward to see that relation (6.4.1) always holds in $\mathscr{H}_{\omega,\mathbf{c}}(X, W)$. Therefore, we just need to check that relation (6.4.2) holds for Dunkl operators in $\mathscr{H}_{\omega,\mathbf{c}}(X, W)$ if and only if the KZ-connection is flat. Let $\nu_1, \nu_2 \in \mathscr{P}^\omega_X$, and D_{ν_1}, D_{ν_2} the corresponding Dunkl operators. We need to calculate the right-hand side of

$$[D_{\nu_1}, D_{\nu_2}] = \left[\nu_1 + \sum_{(w,Z)} \frac{2\mathbf{c}(w, Z)}{1-\lambda_{w,Z}} \frac{\sigma(\nu_1)(f_Z)}{f_Z}(w-1),\ \nu_2 + \sum_{(w,Z)} \frac{2\mathbf{c}(w, Z)}{1-\lambda_{w,Z}} \frac{\sigma(\nu_2)(f_Z)}{f_Z}(w-1)\right].$$

We have

$$\left[\frac{\sigma(\nu_1)(f_Z)}{f_Z}(w-1), \nu_2\right] = \frac{\sigma(\nu_2)\circ\sigma(\nu_1)(f_Z)}{f_Z}(w-1) - \frac{\sigma(\nu_1)(f_Z)\sigma(\nu_2)(f_Z)}{f_Z^2}(w-1) + \frac{\sigma(\nu_1)(f_Z)}{f_Z}(w(\nu_2)-\nu_2)w,$$

and hence

$$\sum_{(w,Z)} \frac{2\mathbf{c}(w, Z)}{1-\lambda_{w,Z}}\left(\left[\frac{\nu_1(f_Z)}{f_Z}(w-1), \nu_2\right] + \left[\nu_1, \frac{\nu_2(f_Z)}{f_Z}(w-1)\right]\right)$$

equals

$$\sum_{(w,Z)} \frac{2\mathbf{c}(w, Z)}{1-\lambda_{w,Z}}\left(\frac{[\nu_1,\nu_2](f_Z)}{f_Z}(w-1) + \frac{\nu_1(f_Z)}{f_Z}(w(\nu_2)-\nu_2)w - \frac{\nu_2(f_Z)}{f_Z}(w(\nu_1)-\nu_1)w\right).$$

Also,

$$\left[-\frac{\nu_1(f_Z)}{f_Z}w_1, \frac{\nu_2(f_{Z'})}{f_{Z'}}\right] + \left[\frac{\nu_1(f_Z)}{f_Z}, -\frac{\nu_2(f_{Z'})}{f_{Z'}}w_2\right] + \left[\frac{\nu_1(f_Z)}{f_Z}w_1, \frac{\nu_2(f_{Z'})}{f_{Z'}}w_2\right]$$

[3]Recall from Definition A.1.1 that $1_{\mathscr{P}}$ is defined to be the image of $1 \in \mathbb{O}_X$ under the map $i : \mathbb{O}_X \to \mathscr{P}$.

equals

$$-\frac{\nu_1(f_Z)}{f_Z}w_1\left(\frac{\nu_2(f_{Z'})}{f_{Z'}}\right)(w_1-1)+\frac{\nu_2(f_{Z'})}{f_{Z'}}w_2\left(\frac{\nu_1(f_Z)}{f_Z}\right)(w_2-1)$$
$$+\frac{\nu_1(f_Z)}{f_Z}w_1\left(\frac{\nu_2(f_{Z'})}{f_{Z'}}\right)w_1w_2-\frac{\nu_2(f_{Z'})}{f_{Z'}}w_2\left(\frac{\nu_1(f_Z)}{f_Z}\right)w_2w_1.$$

Combining the above equations, one sees that relation (6.4.2) holds for Dunkl operators in $\mathscr{H}_{\omega,\mathbf{c}}(X, W)$ if and only if

$$\sum_{\substack{(w_1,Z)\\(w_2,Z')}}\frac{4\mathbf{c}(w_1,Z)\mathbf{c}(w_2,Z')}{(1-\lambda_{w_1,Z})(1-\lambda_{w_2,Z'})}\left(\frac{\nu_2(f_Z)}{f_Z}\frac{\nu_1(f_{Z'})}{f_{Z'}}-\frac{\nu_1(f_Z)}{f_Z}\frac{\nu_2(f_{Z'})}{f_{Z'}}\right)w_1w_2=0.$$

Since the left-hand side equals

$$\left(\sum_{\substack{(w_1,Z)\\(w_2,Z')}}\frac{4\mathbf{c}(w_1,Z)\mathbf{c}(w_2,Z')}{(1-\lambda_{w_1,Z})(1-\lambda_{w_2,Z'})}(d\log f_Z\wedge d\log f_{Z'})\otimes w_1w_2\right)(\nu_2,\nu_1),$$

it will be zero for all ν_1, ν_2 if and only if the meromorphic two-form inside the bracket is zero. But this two-form is the curvature $\omega_{X,\mathbf{c}}\wedge\omega_{X,\mathbf{c}}$ of the KZ-connection. □

Proposition 6.4.4 implies that when the KZ-connection is flat, the algebra $\mathscr{U}_{\omega,\mathbf{c}}(X, W)$ is, up to isomorphism, independent of the choice of functions f_Z.

Appendix: TDOs

In the appendix we summarize the facts we need about twisted differential operators, following [Beilinson and Bernstein 1993] and [Kashiwara 1989].

A.1. ***Twisted differential operators.*** It is most natural to realize a sheaf of algebras of twisted differential operators as a quotient of the enveloping algebra of a Picard algebroid.

Definition A.1.1. An $\mathscr{O}_X$-module $\mathscr{L}$ is called a *Lie algebroid* if there exists a bracket $[-,-]:\mathscr{L}\otimes_{\mathbb{C}_X}\mathscr{L}\to\mathscr{L}$ and morphism of $\mathscr{O}_X$-modules $\sigma:\mathscr{L}\to\Theta_X$ (the anchor map) such that $(\mathscr{L},[-,-])$ is a sheaf of Lie algebras with the anchor map being a morphism of Lie algebras, and, for $l_1, l_2\in\mathscr{L}$ and $f\in\mathscr{O}_X$,

$$[l_1, fl_2]=f[l_1,l_2]+\sigma(l_1)(f)l_2.$$

If, moreover, there exists a map $i:\mathscr{O}_X\to\mathscr{L}$ of $\mathscr{O}_X$-modules such that the sequence

$$0\longrightarrow\mathscr{O}_X\longrightarrow\mathscr{L}\longrightarrow\Theta_X\longrightarrow 0$$

is exact and $i(1):=1_{\mathscr{L}}$ is central in $\mathscr{L}$, then $\mathscr{L}$ is called a *Picard algebroid*.

As in [Beilinson and Bernstein 1993], we denote by $\Omega_X^{1,2}$ the two-term subcomplex $\Omega_X^1 \xrightarrow{d} (\Omega_X^2)^{\mathrm{cl}}$, concentrated in degrees 1 and 2, of the algebraic de Rham complex of X.

Proposition A.1.2. *The Picard algebroids on X are parametrized up to isomorphism by $\mathbb{H}^2(X, \Omega_X^{1,2})$.*

Given $\omega \in \mathbb{H}^2(X, \Omega_X^{1,2})$, the corresponding Picard algebroid is denoted by $\mathcal{P}_X^\omega$. Associated to $\mathcal{P}_X^\omega$ is $\mathcal{D}_X^\omega$, the sheaf of differential operators on X with twist ω. It is the quotient of the enveloping algebra $\mathcal{U}(\mathcal{P}_X^\omega)$ of $\mathcal{P}_X^\omega$ by the ideal generated by $1_{\mathcal{P}_X^\omega} - 1$.

Definition A.1.3. A module for the Picard algebroid $\mathcal{P}$ is a quasicoherent $\mathcal{O}_X$-module $\mathcal{M}$ together with a map $- \cdot - : \mathcal{P} \otimes_{\mathbb{C}_X} \mathcal{M} \to \mathcal{M}$ such that $i(f) \cdot m = fm$ and $[p, q] \cdot m = p \cdot (q \cdot m) - q \cdot (p \cdot m)$ for all $p, q \in \mathcal{P}$, $m \in \mathcal{M}$ and $f \in \mathcal{O}_X$.

There is a natural equivalence between the category of $\mathcal{P}^\omega$-modules and the category of $\mathcal{D}^\omega$-modules.

A.2. *Functoriality.* We recall from Section 2.2 of [Beilinson and Bernstein 1993] the functoriality properties of Picard algebroids and twisted differential operators. Fix a morphism $\varphi : Y \to X$. Let $\mathcal{P}_X$ be a Picard algebroid on X and $\mathcal{P}_Y$ a Picard algebroid on Y.

Definition A.2.1. A φ-morphism $\gamma : \mathcal{P}_Y \to \mathcal{P}_X$ is an $\mathcal{O}_Y$-linear map $\gamma : \mathcal{P}_Y \to \varphi^*\mathcal{P}_X$ such that for any section $p \in \mathcal{P}_Y$ and $\gamma(p) = \sum_i g^i \otimes q^i$ with $g_i \in \mathcal{O}_Y$ and $q_i \in \varphi^{-1}\mathcal{P}_X$, we have

$$\gamma([p_1, p_2]) = \sum_{i,j} g_1^i g_2^j \otimes [q_1^i, q_2^j] + \sum_j \sigma(p_1)(g_2^j) \otimes q_2^j - \sum_i \sigma(p_2)(g_1^i) \otimes q_1^i$$

and $\sigma(n)(f^*g) = \sum_i g^i \varphi^*(\sigma(q^i)(g))$ for all $g \in \varphi^{-1}\mathcal{O}_X$.

The first fundamental theorem on differential forms [Matsumura 1989, Theorem 25.1] implies that there is a morphism of sheaves $\varphi^{-1}\Omega_X^1 \to \Omega_Y^1$. This extends to a morphism of complexes $\varphi^{-1}\Omega_X^{\bullet} \to \Omega_Y^{\bullet}$ and $\varphi^{-1}\Omega_X^{1,2} \to \Omega_Y^{1,2}$. By functoriality of hypercohomology, we get a map $\varphi^* : \mathbb{H}^2(X, \Omega_X^{1,2}) \to \mathbb{H}^2(Y, \Omega_Y^{1,2})$. For $\omega \in \mathbb{H}^2(X, \Omega_X^{1,2})$, let $\mathcal{P}_X^\omega$ be the corresponding Picard algebroid and $\mathcal{P}_Y$ the fiber product $\varphi^*\mathcal{P}_X^\omega \times_{\varphi^*\Theta_X} \Theta_Y$, where $\varphi^*\mathcal{P}_X^\omega \to \varphi^*\Theta_X$ is the anchor map and $\Theta_Y \to \varphi^*\Theta_X$ is $d\varphi$.

Lemma A.2.2. *The sheaf $\mathcal{P}_Y$ is a Picard algebroid, $\psi : \mathcal{P}_Y \to \varphi^*\mathcal{P}_X$ is a φ-morphism and we have an isomorphism of Picard algebroids $\mathcal{P}_Y \simeq \mathcal{P}_Y^{\varphi^*\omega}$.*

Thus, by definition, the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_Y \arrow[r] \arrow[d] & \mathcal{P}_Y^{\varphi^*\omega} \arrow[r, "\sigma_Y"] \arrow[d, "\psi"'] & \Theta_Y \arrow[r] \arrow[d] & 0 \\
 & \mathcal{O}_Y = \varphi^*\mathcal{O}_X \arrow[r] & \varphi^*\mathcal{P}_X^{\omega} \arrow[r, "\sigma_X"] & \varphi^*\Theta_X &
\end{tikzcd}$$

commutes. The projection $\mathcal{P}_Y^{\varphi^*\omega} \to \varphi^*\mathcal{P}_X^{\omega}$ extends to a morphism $\mathcal{D}_Y^{\varphi^*\omega} \to \varphi^*\mathcal{D}_X^{\omega}$, making $\varphi^*\mathcal{D}_X^{\omega}$ a left $\mathcal{D}_Y^{\varphi^*\omega}$-module. Let $\mathcal{M}$ be a left $\mathcal{D}_X^{\omega}$-module. Since $\varphi^*\mathcal{M} = \varphi^*\mathcal{D}_X^{\omega} \otimes_{\varphi^{-1}\mathcal{D}_X^{\omega}} \varphi^{-1}\mathcal{M}$, we have:

Proposition A.2.3. *For any $\mathcal{M} \in \mathcal{D}_X^{\omega}$-Mod, the sheaf $\varphi^*\mathcal{M}$ is a $\mathcal{D}_Y^{\varphi^*\omega}$-module.*

Remark A.2.4. If φ is étale, then $d\varphi : \Theta_Y \to \varphi^*\Theta_X$ is an isomorphism. Therefore, the projection $\mathcal{P}_Y^{\varphi^*\omega} \to \varphi^*\mathcal{P}_X^{\omega}$ is also an isomorphism and, in this case, the isomorphism $\gamma : \mathcal{D}_Y^{\varphi^*\omega} \to \varphi^*\mathcal{D}_X^{\omega}$ of left $\mathcal{D}_Y^{\varphi^*\omega}$-modules is actually an algebra isomorphism (in particular, $\varphi^*\mathcal{D}_X^{\omega}$ is a sheaf of algebras).

A.3. ***Monodromic $\mathcal{D}$-modules.*** Let T be a torus, i.e., a product of copies of the multiplicative group $\mathbb{C}^\times$. The Lie algebra of T is denoted by $\mathfrak{t}$. Let $\pi : Y \to X$ be a principal T-bundle, with X smooth. A common way of constructing sheaves of twisted differential operators on X is by quantum Hamiltonian reduction. Let $\mu : \mathfrak{t} \to \mathcal{D}_Y$ be the differential of the action of T on Y. Since $\mathcal{D}_Y$ is a T-equivariant sheaf, there is a stalkwise action of T on $\pi_\bullet\mathcal{D}_Y$. The map μ is T-equivariant and, since T acts trivially on $\mathfrak{t}$, μ descends to a map $\mathfrak{t} \to (\pi_\bullet\mathcal{D}_Y)^T$. The image of μ is central. Given a character $\chi : \mathfrak{t} \to \mathbb{C}$, let

$$\mathcal{D}_{X,\chi} := (\pi_\bullet\mathcal{D}_Y)^T / \langle\{\mu(t) - \chi(t) \mid t \in \mathfrak{t}\}\rangle. \tag{A.3.1}$$

Let $\mathbb{X}(T)$ be the lattice of characters of T. By differentiation, we may identify $\mathbb{X}(T)$ with a lattice in $\mathfrak{t}^*$ such that $\mathbb{X}(T) \otimes_{\mathbb{Z}} \mathbb{C} = \mathfrak{t}^*$. Given $\lambda \in \mathbb{X}(T)$, the sheaf of λ-semi-invariant sections $(\pi_\bullet\mathcal{O}_Y)^\lambda$ is a line bundle on X. Thus, we have a map $\mathbb{X}(T) \to H^1(X, \mathcal{O}_X^\times)$. Composing this with the map $\mathcal{O}_X^\times \xrightarrow{d\log} \operatorname{Ker}(d : \Omega_X^1 \to \Omega_X^2) \subset \Omega_X^{1,2}$ gives a map

$$\beta_{\mathbb{Z}} : \mathbb{X}(T) \longrightarrow H^1(X, \mathcal{O}_X^\times) \xrightarrow{d\log} \mathbb{H}^2(X, \Omega_X^{1,2})$$

of $\mathbb{Z}$-modules. Extending scalars, we get a map

$$\beta : \mathfrak{t}^* \to \mathbb{H}^2(X, \Omega_X^{1,2}). \tag{A.3.2}$$

Proposition A.3.3. *The sheaf of algebras $\mathcal{D}_{X,\chi}$ is a sheaf of twisted differential operators, isomorphic to $\mathcal{D}_X^{\beta(\chi)}$.*

Sketch of proof. Let $\lambda \in \mathbb{X}(T)$ and $\mathscr{L} := (\pi_\bullet \mathbb{O}_Y)^\lambda$ be the corresponding line bundle on X. If χ is the differential of λ, then (A.3.1) implies that $\mathfrak{D}_{X,\chi}$ acts on $\mathscr{L}$. As explained in [Beilinson and Bernstein 1993, Section 2.1.12], this implies that $\mathfrak{D}_{X,\chi} \simeq \mathfrak{D}_X^{\beta_{\mathbb{Z}}(\chi)}$. The fact that this extends to an isomorphism $\mathfrak{D}_{X,\chi} \simeq \mathfrak{D}_X^{\beta(\chi)}$ for all $\chi \in \mathfrak{t}^*$ follows from the Baer sum construction, as explained in [ibid., Section 2.1.3]. □

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