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Decompositions of commutative monoid congruences and binomial ideals

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Primary decomposition of commutative monoid congruences is insensitive to certain features of primary decomposition in commutative rings. These features are captured by the more refined theory of *mesoprimary decomposition* of congruences, introduced here complete with witnesses and associated prime objects. The combinatorial theory of mesoprimary decomposition lifts to arbitrary binomial ideals in monoid algebras. The resulting *binomial mesoprimary decomposition* is a new type of intersection decomposition for binomial ideals that enjoys computational efficiency and independence from ground field hypotheses. Binomial primary decompositions are easily recovered from mesoprimary decomposition.

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1. Introduction

Overview. Primary decomposition of ideals and modules has been a mainstay of commutative algebra since Emmy Noether's unification of scattered results [1921]. A formally analogous theory for congruences on commutative monoids made its first appearance in [Drbohlav 1963], and subsequently the topic of decompositions has similarly played a central role in commutative semigroup theory [Grillet 2001]. Our first goal is to demonstrate that the formal analogy in the setting of finitely generated monoids and congruences — the *combinatorial setting* — fails to capture the essence of primary decomposition in noetherian rings and modules. We justify this claim, and rectify it, by exhibiting a more sensitive theory of *mesoprimary decomposition* of congruences, complete with witnesses, associated prime objects, and other facets of control afforded in parallel with primary decomposition in rings. We then proceed beyond formal analogy by lifting our witnessed theory of mesoprimary decomposition to the *arithmetic setting* of binomial ideals in semigroup rings, at the interface of commutative ring theory with finitely generated monoids.

Mesoprimary decomposition of binomial ideals is not binomial primary decomposition, but a new type of intersection decomposition for binomial ideals, with numerous advantages over ordinary primary decomposition, such as combinatorial clarity, independence from properties of the ground field, and computational efficiency. Nevertheless, binomial primary decomposition is easily recovered from mesoprimary decomposition. In essence, by lifting mesoprimary decomposition of congruences, binomial mesoprimary decomposition distills the coefficient-free combinatorics inherent in primary decomposition of binomial ideals and isolates the precise manner in which coefficients subsequently determine the primary components. The subtlety of coefficient arithmetic causes the lifting procedure to fail verbatim translation, particularly where redundancy is involved. Part of our study therefore contrasts the slightly different notions of witness and associatedness in the combinatorial and arithmetic settings.

General motivation. The need for natural decompositions in the monoid and binomial contexts has become increasingly important in recent years, in view of appearances and applications in numerous areas. Some of these directly involve commutative monoids, such as schemes over $\mathbb{F}_1$ [Connes and Consani 2011; Deitmar 2005], where monoids form the foundation just as rings do for usual schemes. Another instance is the arrival of misère quotients in combinatorial game theory, where monoids provide data structures for recording and computing winning strategies [Plambeck 2005; Plambeck and Siegel 2008] (see also [Miller 2011] for an algebraic introduction). At the same time, binomial ideals interact with other parts of mathematics and the sciences, motivating research into applicable descriptions of their decompositions. For example, dynamics of mass-action kinetics, where steady states in detailed-balanced cases are described by vanishing of binomial trajectories, arise from stoichiometric exponential growth and decay [Adleman et al. 2014]; binomial decompositions in mass-action kinetics can identify which species persist or become extinct [Shiu and Sturmfels 2010]. In algebraic statistics, decompositions of binomial ideals give insight into how a set of conditional independence statements among random variables can be realized [Drton et al. 2009; Herzog et al. 2010]. More generally, the connectivity of lattice point walks in polyhedra can be analyzed using decompositions of binomial ideals [Diaconis et al. 1998; Kahle et al. 2014b]. These applications rely on decompositions of *unital* ideals — generated by monomials and differences of monomials — into unital ideals; these are mesoprimary decompositions. The algebra, geometry, and combinatorics of binomial primary decomposition interact with systems of differential equations of hypergeometric type [Gelfand et al. 1987; Gelfand et al. 1989], whose solutions are eigenfunctions for binomial differential operators encoding the infinitesimal action of an algebraic torus. In fact, it was in the hypergeometric framework that the combinatorics of binomial primary decomposition had its origin [Matusevich et al. 2005; Dickenstein et al. 2010a; 2010b], providing tight control over series solutions. In the meantime, mesoprimary decomposition serves as an improved method for presenting and visualizing binomial primary decomposition in algorithmic output [Kahle 2012]. Beyond that, the methods here have already found a theoretical application to combinatorial game theory [Guo and Miller 2011; Miller 2013].

Conventions. Unless otherwise stated, Q denotes a finitely generated (equivalently, noetherian) commutative monoid, and $\Bbbk$ denotes an arbitrary field.

Gathering primary components rationally. Staring at output of binomial primary decomposition algorithms intimates that certain primary components belong together.

Example 1.1. During investigations of presentations of misère quotients of combinatorial games (culminating in the definition of lattice games [Guo et al. 2009; Guo and Miller 2011]), Macaulay2 [Grayson and Stillman] produced long lists of primary binomial ideals. In one instance, eight of the components were

$$\begin{array}{ll}\langle e-1,\, d-1,\, b-1,\, a-1,\, c^3\rangle, & \langle e-1,\, d-1,\, b-1,\, a+1,\, c^3\rangle,\\ \langle e-1,\, d+1,\, b-1,\, a-1,\, c^3\rangle, & \langle e-1,\, d+1,\, b-1,\, a+1,\, c^3\rangle,\\ \langle e+1,\, d-1,\, b+1,\, a-1,\, c^3\rangle, & \langle e+1,\, d-1,\, b+1,\, a+1,\, c^3\rangle,\\ \langle e+1,\, d+1,\, b+1,\, a-1,\, c^3\rangle, & \langle e+1,\, d+1,\, b+1,\, a+1,\, c^3\rangle.\end{array}$$

The urge to gather these eight into one piece (a piece of eight?), namely their intersection

$$\langle b-e,\, e^2-1,\, d^2-1,\, a^2-1,\, c^3\rangle,$$

is irresistible. (Who would rather sift through the big list?) And it would have become more so had the exponents in the single gathered component been larger integers, for then the coefficients in the long list of primary ideals would not even have been rational numbers, though the intersection would still have been rational.

An arbitrary binomial prime ideal $I_{\rho,P}$ in a finitely generated monoid algebra $\Bbbk[Q]$ is determined by a monoid prime ideal $P \subset Q$ and a character $\rho : K \to \Bbbk^*$ defined on a subgroup of the unit group $G_P \subseteq Q_P$ in the localization of Q along P (Definition 3.9 and Theorem 11.14). A binomial ideal $I \subseteq \Bbbk[Q]$ might possess many associated primes sharing the same P and K, differing only in the character ρ. *Mesoprimary ideals* (Definition 10.4; see also Propositions 12.10 and 15.1) are data structures for keeping track of primary components for such groups of associated binomial primes. The term "group" here is used in the ordinary nonmathematical sense, but it is appropriate mathematically: the primary components of a mesoprimary ideal over an algebraically closed field are indexed by the characters of a finite abelian group, namely the quotient $\mathrm{sat}(K)/K$ of the saturation of K in G_P (Propositions 11.9 and 15.4). Gathering primary components into mesoprimary ideals saves space just as writing the presentation for a finite abelian group instead of listing every one of its characters does.

The situation is not typically as simple as in Example 1.1. Indeed, upon inspecting a binomial primary decomposition, it can be difficult to determine which mesoprimary ideals ought to occur, and which mesoprimary ideal each primary component ought to contribute to. Furthermore, some primary components of a mesoprimary ideal can be absent, even if the mesoprimary ideal clearly ought to appear.

Example 1.2. If $\mathrm{char}(\Bbbk) \neq 2$, the ideal $I = \langle y - x^2 y,\, y^2 - xy^2,\, y^3\rangle \subseteq \Bbbk[x, y]$ has primary decomposition $I = \langle y\rangle \cap \langle 1+x,\, y^2\rangle \cap \langle 1-x,\, y^3\rangle$. The ideal I is unital, being generated by differences of monomials, so the component $\langle 1+x,\, y^2\rangle$ feels out of place. Yet there are no obvious components to gather. What's missing is a "phantom" component $\langle 1-x,\, y^2\rangle$, hidden by $\langle 1-x,\, y^3\rangle$. Gathering yields $\langle 1+x,\, y^2\rangle \cap \langle 1-x,\, y^2\rangle = \langle 1-x^2,\, y^2\rangle$. If $\mathrm{char}(\Bbbk) = 2$, then $I = \langle y\rangle \cap \langle 1-x^2,\, y^2 - xy^2,\, y^3\rangle$ is a primary decomposition of I. While this decomposition is forced to be unital, it feels not fine enough. Indeed, $1-x^2$ and $1-x$ look like they should contribute two associated objects, and in all but a single characteristic they do. Independent of the characteristic the mesoprimary decomposition splits the second component: $I = \langle y\rangle \cap \langle 1-x^2,\, y^2\rangle \cap \langle 1-x,\, y^3\rangle$.

A *mesoprimary decomposition* of a binomial ideal I is an expression of I as an intersection of *mesoprimary components* (Definition 12.14), each of which is a mesoprimary ideal. Mesoprimary decompositions of binomial ideals always exist (Definition 13.1 and Theorem 13.2) in a form that realizes our initial intent (Theorems 15.6 and 15.9). However, an arbitrary intersection of mesoprimary ideals is not a mesoprimary decomposition, even if the intersection is a binomial ideal; exigent additional conditions must be met regarding the interaction of the combinatorics and the arithmetic of the mesoprimary components, as compared with that of I (Remark 13.6). In summary, mesoprimary decomposition gathers primary components so that:

(1) The decomposition into binomial ideals requires no hypotheses on the field $\Bbbk$.

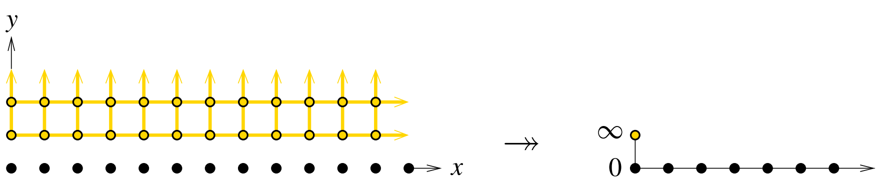
(2) Specifying one mesoprimary component takes the place of individually listing all primary components arising from saturated extensions of a fixed character.

(3) The combinatorics of the components and their associated prime objects reflects accurately and faithfully the combinatorics of the decomposed binomial ideal.

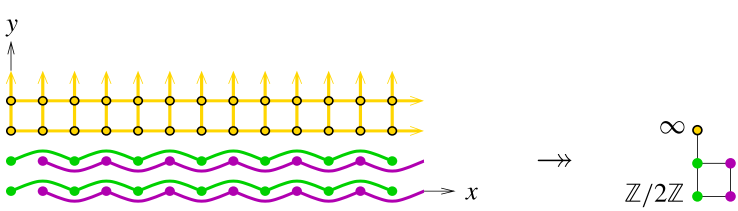
Congruences: binomial combinatorics. The simple (and not new) idea of binomial combinatorics is that a binomial ideal $I \subseteq \Bbbk[Q]$ determines an equivalence relation $\sim$ on Q that sets $u \sim v$ if I contains a two-term binomial $\mathbf{t}^u - \lambda \mathbf{t}^v$ (Definition 2.15). The quotient $\bar{Q} = Q/\sim$ modulo this relation is a monoid.

Example 1.3. The following ideals induce the depicted congruences on $\mathbb{N}^2$ and quotient monoids. The congruence classes are the connected components of the graphs drawn in the left-hand pictures. Each element labeled 0 is the identity of the quotient monoid. Each element labeled ∞ in the right-hand picture is *nil* (Definition 2.9 and Remark 2.10) in the quotient monoid; its congruence class comprises all monomials in the given binomial ideal. In items (2) and (4), the groups labeling the rows indicate how the group in the bottom row acts on the higher rows. In all four items, every element outside of the bottom row of the quotient monoid is *nilpotent* (Definition 2.9).

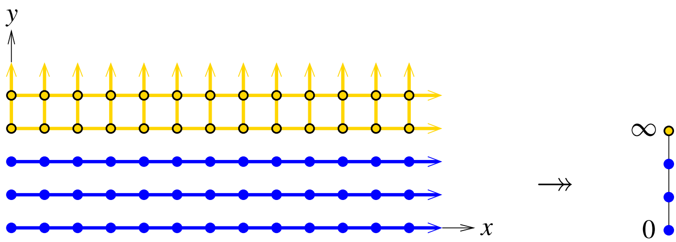
(1) For the ideal $\langle y \rangle \subset \Bbbk[x, y]$, the quotient monoid is $\mathbb{N} \cup \infty$:



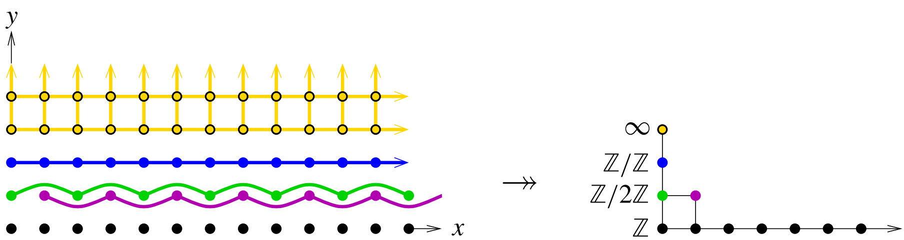
(2) For the ideal $\langle 1 - x^2, y^2 \rangle \subset \Bbbk[x, y]$, the quotient monoid is a copy of the group $\mathbb{Z}/2\mathbb{Z}$ (the bottom row), a free module over $\mathbb{Z}/2\mathbb{Z}$ (the middle row), and a nil:



(3) For the ideal $\langle 1-x,\, y^3\rangle \subset \Bbbk[x,\, y]$, the quotient monoid is the quotient $\mathbb{N}/(3+\mathbb{N})$ of the natural numbers modulo the *Rees congruence* of the ideal $3+\mathbb{N}$, which makes all elements of the ideal equivalent and leaves the other elements of $\mathbb{N}$ alone:



(4) For the ideal $\langle y-x^2y,\, y^2-xy^2,\, y^3\rangle \subset \Bbbk[x,\, y]$, the quotient monoid is a disjoint union of the group $\mathbb{Z}$ and three $\mathbb{Z}$-modules:



We examined (the literature on) monoid congruences on the premise that an appropriate decomposition theory for them should lift, either directly or analogously, to the desired mesoprimary theory for binomial ideals. However, although we found rich decomposition theories for commutative semigroups [Grillet 2001], the expected analogue of binomial primary decomposition was absent.

The most promising development we encountered along these lines is Grillet's discovery of conditions guaranteeing that a commutative semigroup can be realized as a subsemigroup of the multiplicative semigroup of a primary ring — that is, a ring with just one associated prime [Grillet 1975]. That work covers ground anticipating — in a more general setting — the characterization of primary binomial ideals over algebraically closed fields of characteristic zero [Dickenstein et al. 2010b].

The closest monoid relative in the literature to primary decomposition in rings seems to be primary decomposition of congruences [Drbohlav 1963] (see [Gilmer 1984] for a treatment in the context of semigroup rings). However, one of our motivating discoveries is that primary decomposition of congruences, being much closer to a shadow of cellular binomial decomposition (see Theorem 10.6), falls short of serving as a rubric for either primary or mesoprimary decomposition of binomial ideals. Indeed, congruences that are *prime*, meaning that quotients modulo them are cancellative except perhaps for a nil (Definition 2.12.4), fail to be irreducible

(Example 2.22). Furthermore, congruences that are *primary*, meaning that every element in the quotient is either nilpotent or cancellative (Definition 2.12.1), admit further decompositions into pieces that are visibly more “homogeneous”, in a manner more analogous to primary decomposition in the presence of embedded primes than to irreducible decomposition of primary ideals.

Example 1.4. All of the congruences depicted in Example 1.3 are primary, but the first three are visibly more homogeneous: in each one, the nonnil rows all look the same. In fact, the fourth congruence is the *common refinement* (Section 3) of the first three. This is equivalent, given that all of the ideals (and their intersection) are binomial, to saying that the fourth binomial ideal equals the intersection of the first three, since the ideals in question are all unital and contain monomials; see Remark 2.16 and Theorem 9.12. This intersection is the mesoprimary decomposition from Example 1.2.

Primary binomial ideals in characteristic zero induce *primitive* congruences (Definition 2.12 and Theorem 10.6), but congruences usually do not admit expressions as intersections (common refinements) of primitive congruences. The reason stems from the same phenomenon that requires one to assume, for binomial primary decomposition, that the base field is algebraically closed: decompositions of ideals generated by binomials — even unital ones — usually require nontrivial roots of unity. Viewed another way, the arithmetic part of binomial primary decomposition has a combinatorial ramification: intersecting multiple primary ideals inducing the same primitive congruence results in a single mesoprimary ideal whose associated prime congruence has finite index in the primitive one (Proposition 15.4). In essence, primary congruences on Q are too coarse to reflect binomial primary decomposition in $\Bbbk[Q]$ accurately, and primitive congruences on Q are too fine, requiring additional arithmetic data from $\Bbbk$ to resolve otherwise indistinguishable associated primes in $\Bbbk[Q]$.

An additional layer of complication arises from the fact that primary binomial ideals in positive characteristic need not induce nicely filtered congruences (Example 10.8). The reason for this failure is not under our control: the ideal $\langle (x-1)^p, y(x-1), y^2 \rangle$ happens to be primary, the ideal $\langle x^p - 1, y(x-1), y^2 \rangle$ happens to be binomial, and — accidentally, one may conclude — they coincide in characteristic p. This highlights that even the “binomiality” of a ring-theoretic construction can depend on the characteristic, and consequently no study of binomial ideals can skirt the resulting distinctions.

The true monoid congruence analogue of primary decomposition in rings is a suitable compromise, developed (in Sections 2–8) as *mesoprimary decomposition for congruences* (Definition 8.1 and Theorem 8.3). The type of homogeneity mentioned before Example 1.4, discovered by Grillet [1975] (Remark 2.13(4)), characterizes mesoprimary congruences (Corollary 6.7 and Remark 6.8). These are also distinguished (Theorem 6.1) as those with just one *associated prime congruence* (Definitions 2.12.4 and 5.2), a notion new to monoid theory. For comparison, a congruence is primary precisely when it has just one *associated prime ideal* (Definition 4.7 and Corollary 4.21).

The development of binomial mesoprimary decomposition in the latter half of the paper (Sections 9–16) mirrors the first half directly. Arithmetic existence statements build on combinatorial ones by exhibiting lifts of statements or requirements concerning elements equivalent under congruences to statements or requirements concerning binomials with nonzero coefficients.

It is worth warning the reader at this juncture of the inevitable clash of terminology in translating between combinatorics and arithmetic; see the table in Section 10, which in particular explains the source of our term *mesoprimary* to mean "between the two occurrences of 'primary'". To aid readers coming from commutative ring theory, the basic notions from semigroup theory are reviewed from scratch (Sections 2 and 3). For readers interested primarily in monoids, we complete the entire combinatorial theory in Section 8, before starting the arithmetic theory in Section 9.

Witnessed associated objects. In ordinary primary decomposition, a witness is an element whose annihilator is (an associated) prime. Our *witnesses* also have *associated* prime objects (Definitions 4.7, 5.2, and 12.1). Continuing the parallel, our notions of associatedness are defined by local combinatorial or algebraic conditions but equivalently characterized by the consistent appearance of prime objects in every primary decomposition (Theorems 4.20 and 15.11). The local conditions defining witnesses incorporate the combinatorial quiddity of having prime annihilator in ordinary ring theory.

The proof of concept for mesoprimary decomposition as a mode to connect the combinatorial and arithmetic settings lies in a fundamental discovery: there is a combinatorially defined set of witnesses that captures decompositions of both a binomial ideal and its induced congruence. To yield finite decompositions, however, not all witnesses are to be believed. The *key witnesses* for congruences (Definition 4.7) and *essential witnesses* for binomial ideals (Definition 12.1) yield finitely many components whose intersections suffice. These key and essential decompositions can generally fail to be minimal in ways that even retain symmetry. In the cellular binomial ideal case, we demonstrate a systematic reduction to *character witnesses* (Definition 16.3) that should have an extension to general binomial ideals. The dichotomy between key and essential witnesses demands care, as do other subtle distinctions between the combinatorial and arithmetic aspects of the theory, since they necessitate occasional slight weakenings, or failures of the combinatorics to lift; see Remarks 12.20 and 12.21, for instance.

2. Taxonomy of congruences on monoids

Fix a *commutative semigroup* Q: a set with an associative, commutative binary operation (usually denoted by $+$ here). Assume that Q has an identity, usually denoted by 0 here, so Q is a *monoid*. An *ideal* $T \subseteq Q$ is a subset such that $T+Q \subseteq T$, and T is *prime* if $t+s \in T$ implies $t \in T$ or $s \in T$. The ideal generated by elements $q_1, \ldots, q_s$ is written $\langle q_1, \ldots, q_s \rangle$. A *congruence* $\sim$ on Q is an equivalence relation that is additively closed: $a \sim b \Longrightarrow a+c \sim b+c$ for all $a, b, c \in Q$. The quotient $Q/{\sim}$ by any congruence is a monoid. The minimal relation satisfying this definition is equality itself, called the *identity congruence*. The congruence that equates all pairs of elements in Q, and has trivial quotient, is the *universal congruence*. For any ideal $T \subseteq Q$, under the *Rees congruence* $\sim_T$ all elements of T form one class, while all elements outside of T are singletons.

Definition 2.1. A *module* over a commutative monoid Q is a nonempty set T with an *action* of Q, which means a map $Q \times T \to T$, written $(q, t) \mapsto q+t$, that satisfies

- $0+t = t$ for all $t \in T$, and
- $(q+q')+t = q+(q'+t)$,

the latter meaning that the action respects addition. A *congruence* on a module is an equivalence relation that is preserved by the action. A *module homomorphism* over a given monoid is a set map that respects the actions. For any element $q \in Q$, the *addition morphism* $\phi_q : Q \to \langle q \rangle$ is the module morphism defined by $p \mapsto p+q$. The *kernel* $\ker(\phi)$ of a module homomorphism $\phi : T_1 \to T_2$ is the congruence on T_1 under which $t \sim s \Longleftrightarrow \phi(t) = \phi(s)$.

Remark 2.2. For general semigroups Grillet [2007] defines an *act* as a set with an action of a semigroup that satisfies only the second bullet in Definition 2.1, even if the semigroup was a monoid to start with. To every semigroup S a formal identity element e can be adjoined (even if S is already a monoid) to form the monoid $S \cup \{e\}$. Upon this operation an S-act turns into an $(S \cup \{e\})$-module as it automatically satisfies the first item in Definition 2.1.

Remark 2.3. A subsemigroup of a monoid may have an identity, and in that case it may or may not be the identity of the monoid. To the contrary, a submonoid is required to have the same identity as its ambient monoid. In this sense a subsemigroup of a monoid can be a monoid without being a submonoid.

Definition 2.4. A *subgroup of a monoid* is a subsemigroup that is a group.

Definition 2.5. *Green's preorder* on a monoid is the divisibility preorder $p \preceq q \Longleftrightarrow \langle p \rangle \supseteq \langle q \rangle$. *Green's relation* on a monoid is $p \sim q \Longleftrightarrow \langle p \rangle = \langle q \rangle$.

Lemma 2.6 [Grillet 2001, Proposition I.4.1]. *The quotient of a commutative monoid modulo Green's relation is partially ordered by divisibility.*

Remark 2.7. Green's relation measures the extent to which group-like behavior occurs in a monoid. Idempotents and nontrivial units are obstructions to partially ordering a monoid by divisibility. In particular, a monoid with trivial unit group is partially ordered if Green's relation is trivial. Note that our divisibility preorder is the opposite direction compared to Grillet's, to be compatible with divisibility of monomials.

The following observation, which relies crucially on the noetherian hypothesis, is applied in the proof of Proposition 7.9.

Lemma 2.8. *Fix a noetherian commutative monoid Q. If $p \in Q$ and the Green's class of w satisfies $[w] = [p+w]$, then the map $[w] \to [p+w]$ of Green's classes induced by adding p is bijective.*

Proof. Suppose that $v \in [w] = [p+w]$. For surjectivity, first note that $v \in p+\langle w\rangle$, because $v \in \langle v\rangle = \langle p+w\rangle = p+\langle w\rangle$. Consequently $v \in p+[w]$ because $[v] = [w]$ is the (unique) minimal element in the poset of Green's classes with representatives in $\langle w\rangle$ (that is, $[v]$ can't lie in $p+[u]$ if $[u] \succ [w]$).

Since the sets in question can be infinite, injectivity requires additional reasoning. Suppose that $v \in [w]$ satisfies $p+w = p+v$. By surjectivity, for $k \in \mathbb{N}$ choose $w_k,\, w'_k \in [w]$ so that $k\cdot p+w_k = w$ and $k\cdot p+w'_k = v$. If $\sim_k$ is the kernel congruence of addition by $k\cdot p$, then $\sim_k$ refines $\sim_\ell$ whenever $k \leq \ell$. The noetherian property implies that the chain of kernel congruences stabilizes: $\sim_k = \sim_{k+1}$ for $k \gg 0$. But $w_k \sim_{k+1} w'_k$ for all k because $p+w = p+v$, whence $w_k \sim_k w'_k$ for $k \gg 0$ by stability. For $k \gg 0$, then, $w = k\cdot p+w_k \sim k\cdot p+w'_k = v$. □

Definition 2.9. A nonidentity element ∞ in a monoid Q is *nil* if $q+\infty = \infty$ for all $q \in Q$. An element $q \in Q$ is

- *nilpotent* if one of its multiples nq is nil for some nonnegative integer $n \in \mathbb{N}$,
- *cancellative* if addition by it is injective: $q+a = q+b \Longrightarrow a = b$ in Q,
- *partly cancellative* if $q+a = q+b \neq \infty \Longrightarrow a = b$ for all cancellative $a, b \in Q$.

A set S of elements in a monoid is *torsion-free* if $na = nb \Longrightarrow a = b$ for all $n \in \mathbb{N}$ whenever $a, b \in S$. An *affine semigroup* is a monoid isomorphic to a finitely generated submonoid of a free abelian group. A *nilmonoid* is a monoid whose nonidentity elements are all nilpotent.

Remark 2.10. In the literature a nil is often called a zero instead; but when we work with monoid algebras, we need to distinguish the nil monomial $\boldsymbol{t}^\infty$ from the zero element 0 of the algebra (see Section 9 for ramifications of this distinction), and we need to identify the identity monomial $\boldsymbol{t}^0$ with the unit element 1 of the algebra.

Remark 2.11. The condition $a + c = b + c'$ for cancellative c, c' means that a and b are off by a unit in the localization Q' of Q obtained by inverting all of its cancellative elements. Note that the natural map $Q \to Q'$ is injective.

Definition 2.12. Fix a commutative monoid Q, a congruence $\sim$, and use a bar to denote passage to the quotient $\overline{Q} = Q/{\sim}$. The congruence $\sim$ is

(1) *primary* if every element of $\overline{Q}$ is either nilpotent or cancellative,

(2) *mesoprimary* if it is primary and every element of $\overline{Q}$ is partly cancellative,

(3) *primitive* if it is mesoprimary and the cancellative subset of $\overline{Q}$ is torsion-free,

(4) *prime* if every element of $\overline{Q}$ is either nil or cancellative,

(5) *toric* if the nonnil elements of $\overline{Q}$ form an affine semigroup.

Remark 2.13. The notions just defined are nearly or exactly the same as concepts that have appeared in the literature on monoids.

(1) Our definitions of prime and primary congruences agree with those in the literature [Gilmer 1984, §5]. In the case of prime congruences, where the nonnil elements of $\overline{Q}$ form a cancellative monoid, this is easy. In the case of primary congruences, for $q \in Q$ the condition Gilmer expresses as $q+a \sim q+b$ for all $a, b \in Q$ is equivalent to the class $\bar{q}$ being a nil in $\overline{Q} = Q/{\sim}$, so q lies in the nil class, and the condition that Gilmer expresses by saying that q lies in the radical of the nil class is equivalent to $\bar{q}$ being nilpotent in $\overline{Q}$.

(2) Our definition of affine semigroup differs slightly from [Grillet 2001, §II.7]: Grillet requires the unit group to be trivial, whereas we do not. Equivalently, our affine semigroups are the finitely generated, cancellative, torsion-free commutative monoids, while Grillet additionally requires affine semigroups to be reduced (that is, to have trivial unit group).

(3) A congruence on Q is primary if and only if $\overline{Q}$ is a *subelementary* monoid, by definition [Grillet 2001, §VI.2.2].

(4) A congruence on Q is mesoprimary if and only if the subelementary monoid $\overline{Q}'$, obtained from the monoid $\overline{Q}$ in the previous item by inverting its cancellative elements, is *homogeneous* [Grillet 2001, §VI.5.3]; this is Corollary 6.7, below.

Lemma 2.14. *For monoid congruences,*

- *toric* $\Longrightarrow$ *prime* $\Longrightarrow$ *mesoprimary* $\Longrightarrow$ *primary, and*
- *toric* $\Longrightarrow$ *primitive* $\Longrightarrow$ *mesoprimary* $\Longrightarrow$ *primary.*

Proof. The only implication that is not immediate from the definitions is that prime implies mesoprimary. For this, assume $\sim$ is a prime congruence and $\bar{q}+\bar{a}=\bar{q}+\bar{b}$ in $\overline{Q}$ with neither side being nil. Then $\bar{q}$ is not nil, whence $\bar{a}=\bar{b}$ by cancellativity. $\square$

Definition 2.15. The *semigroup algebra* $\Bbbk[Q] = \bigoplus_{q\in Q} \Bbbk\cdot \mathbf{t}^q$ is the direct sum with multiplication $\mathbf{t}^p\mathbf{t}^q = \mathbf{t}^{p+q}$. Any congruence $\sim$ on Q induces a grading of $\Bbbk[Q]$ by $\bar{Q} = Q/\sim$ in which the *monomial* $\mathbf{t}^q$ has degree $\bar{q} \in \bar{Q}$ whenever $q \mapsto \bar{q}$ under the quotient map $Q \to \bar{Q}$. A *binomial ideal* $I \subseteq \Bbbk[Q]$ is an ideal generated by *binomials* $\mathbf{t}^p - \lambda\mathbf{t}^q$, where $\lambda \in \Bbbk$ is a scalar possibly equal to $0 \in \Bbbk$. A binomial ideal is *unital* if all coefficients λ are equal to either 0 or 1. The ideal I *induces* the congruence $\sim_I$ in which $p \sim_I q$ whenever $\mathbf{t}^p - \lambda\mathbf{t}^q \in I$ for some unit $\lambda \in \Bbbk^*$.

Remark 2.16. Giving a congruence on Q is the same as giving a unital ideal in $\Bbbk[Q]$ that is generated by unital binomials $\mathbf{t}^p - \mathbf{t}^q$ (and no monomials). In particular, every congruence is induced by some binomial ideal. That said, other binomial ideals can induce the same congruence as the canonical unital ideal, by rescaling the variables or via Theorem 9.12, for instance.

Example 2.17 (some congruences from unital ideals).

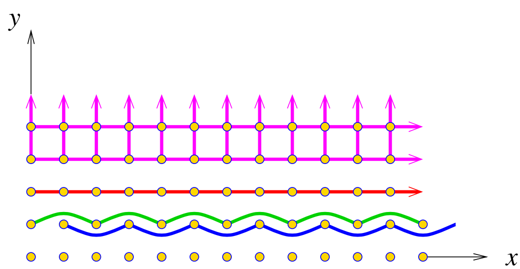
(1) The prime ideal $\langle x - y\rangle \subset \Bbbk[x, y]$ induces a toric congruence such that $\overline{\mathbb{N}^2} \cong \mathbb{N}$.

(2) The ideal $\langle x^2 - y^2\rangle \subset \Bbbk[x, y]$ induces a prime congruence with $\overline{\mathbb{N}^2}$ isomorphic to the submonoid $Q \subseteq G = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ generated by $(1, 0)$ and $(1, 1)$. The monoid is not torsion-free, since $x^2 = y^2$ but $x \neq y$ in $\Bbbk[Q]$. Therefore the congruence on $\mathbb{N}^2$ is not toric, since Q generates G as a group.

(3) The ideal $\langle x^2 - x\rangle \subset \Bbbk[x]$ induces the same toric congruence on $\mathbb{N}$ as the prime ideal $\langle x\rangle$ does, but $\langle x^2 - x\rangle$ is not primary (in fact, not even cellular; see Definition 10.4). Nevertheless $\sim_{\langle x^2-1\rangle} = \sim_{\langle x\rangle}$ is irreducible according to Definition 2.21.

(4) The $\langle x, y\rangle$-primary ideal $\langle x^2, x - y\rangle$ induces the primitive congruence on $\mathbb{N}^2$ with $\overline{\mathbb{N}^2} \cong \{0, x, \infty\} =: Q$. The monoid algebra $\Bbbk[Q]$ has a presentation $\Bbbk[x, y]/J$, where $J = \langle x - y, x - x^2\rangle = \langle x - 1, y - 1\rangle \cap \langle x, y\rangle$ induces the same congruence.

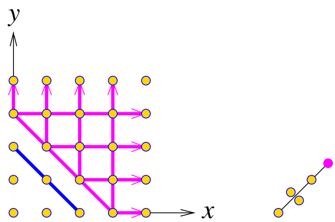
(5) The binomial ideal $\langle y - x^2 y, y^2 - x y^2, y^3\rangle$ induces a primary congruence whose classes are depicted as connected components of the graph in the following figure.



This congruence exhibits the distinction between primary and mesoprimary congruences: for a primary congruence, no injectivity is required of addition by a nilpotent element. In the picture, this means that translating two dots in different classes upward by one unit can force them into the same nonnil class. To make the congruence mesoprimary, homogenize the bottom three rows by replacing any two of them with the third; after that, upward translation on two dots keeps them in separate classes unless both land in the nil class. This replacement procedure also exhibits the distinction between mesoprimary and primitive congruences: it results in a primitive congruence only if the bottom row or the third row is preserved; preserving the second row yields torsion in the cancellative part of $\bar{Q}$.

The following example demonstrates the partly cancellative property.

Example 2.18. Partly cancellative elements can still merge congruence classes. For instance, consider the congruence on $\mathbb{N}^2$ induced by $I = \langle x^2 - xy, xy - y^2, x^3, y^3 \rangle \subseteq \Bbbk[x, y]$. In the figure



the congruence on $\mathbb{N}^2$ appears at left, and the quotient $\overline{\mathbb{N}^2}$ appears at right. The quotient is the monoid $\mathbb{N}$ with two copies of 1 modulo the Rees congruence of $\langle 3 \rangle$ (declare all elements in $\langle 3 \rangle$ congruent). The two copies of 1 become identified upon addition by either: $1 + 1 = 1 + 1' = 1' + 1' = 2$. Nonetheless, both 1 and $1'$ are partly cancellative and the congruence is mesoprimary.

The next result will be applied in the proofs of Theorems 8.4 and 10.6. The conclusion says that Q/F is a nilmonoid whose Green's preorder is an order (i.e., is antisymmetric). Equivalently, it says that Q/F is *naturally partially ordered*, or a *holoid* [Grillet 2001, §V.2.2].

Lemma 2.19. *Fix a monoid Q whose identity congruence is primary, so the non-nilpotent elements of Q constitute a cancellative submonoid $F \subseteq Q$. The quotient monoid Q/F defined by the congruence*

$$p \sim q \iff p + f = q + g \text{ for some } f, g \in F$$

is a nilmonoid partially ordered by divisibility. If Q is finitely generated, Q/F is finite.

Proof. This is more or less [Grillet 2001, Proposition VI.3.3], but the proof is simple. Every nonidentity element of Q/F is nilpotent by definition, so when Q is finitely generated, Q/F is finite. The rest follows because every nilmonoid is partially ordered by divisibility; this is easy, and can be found in [Grillet 2001, Proposition IV.3.1]. □

Remark 2.20. It is a crucial assumption for Lemma 2.19 that every element is nilpotent or cancellative, excluding idempotents. If every cancellative element is a unit, e.g., after localizing at the nilpotent ideal (see Section 4), then Q/F equals Q modulo Green's relation.

Concluding this section, we comment on the notion of irreducibility for congruences, which is, despite the close connection between binomial ideals and their congruences, quite different from irreducibility for ideals.

Definition 2.21. A congruence is *irreducible* if it cannot be expressed as the common refinement of two congruences neither of which equals the given one.

The theories of irreducible and primary decomposition for congruences in commutative monoids are not as nice as for (binomial) ideals in rings. The following example might come as a nasty surprise (it did to us). Quotients by irreducible congruences are characterized in [Grillet 2001, Theorem VI.5.3].

Example 2.22. The identity congruence on $\mathbb{N}^2$ is reducible: it is the common refinement of the congruences induced by $\langle x-1\rangle$ and $\langle y-1\rangle$. Ring-theoretically, this is due to the fact that $\langle x-1\rangle \cap \langle y-1\rangle$ does not contain binomials.

Example 2.22 demonstrates the sad reality that prime congruences need not be irreducible. In a wider sense, unrestricted primary or irreducible decomposition of congruences decomposes them into components that are too fine to provide nuanced information about their combinatorics. The theory of mesoprimary decomposition, with its well-founded notions of associatedness for prime ideals and prime congruences, is our remedy.

3. Primary decomposition and localization in monoids

We review the notion of primary decomposition for congruences on finitely generated commutative monoids, which traces back to Drbohlav [1963]. This decomposition is only a coarse approximation of mesoprimary decomposition, a central goal of this paper. In general, a *decomposition* of a congruence is an expression of it as a common refinement of congruences. The notion of refinement here is standard: formally, an equivalence relation on Q is a reflexive, symmetric, transitive subset of $Q \times Q$; one relation $\sim$ *refines* another relation $\approx$ if $\approx$ contains $\sim$ (we also say $\approx$ *coarsens* $\sim$); and the *common refinement* of a family of equivalence relations is their intersection in $Q \times Q$.

Remark 3.1. Every congruence in this setting admits a *primary decomposition*: an expression as the common refinement of finitely many primary congruences

[Gilmer 1984, Theorem 5.7]. Similarly to the case of rings, this follows from the existence of irreducible decomposition using a noetherian induction argument. Any decomposition theory that is finer than primary decomposition — that is, any theory that further decomposes each primary component — yields a greater number of congruences, each of which is coarser than some primary component.

Remark 3.2. The preimage under any monoid homomorphism of a prime ideal is prime. Since $\mathbb{N}^n$ has only finitely many prime ideals and a finitely generated commutative monoid Q has a presentation $\mathbb{N}^n \twoheadrightarrow Q$, it follows that Q has only finitely many prime ideals. Precisely one of these is the *maximal ideal* of Q.

Convention 3.3. To avoid tedious case distinctions in the following, we consider the empty set as an ideal of any monoid, and in fact we declare it to be a prime ideal (its complement is, after all, a submonoid). The empty set considered as an ideal will be denoted by $\varnothing \subset Q$; this symbol is never used for any other purpose in this paper.

Definition 3.4. The *nilpotent ideal* of a congruence $\sim$ on Q is the ideal of Q consisting of all elements with nilpotent image in $Q/{\sim}$. If P is the nilpotent ideal of a primary congruence $\sim$, then $\sim$ is *P-primary*.

Lemma 3.5. *If $\sim$ is a primary congruence, then the nilpotent ideal is prime. If $Q/{\sim}$ is cancellative, then $\sim$ is $\varnothing$-primary.* □

Remark 3.6. If $q_1, \dots, q_n$ generate Q, then a primary congruence defines a partition of $\{q_1, \dots, q_n\}$ into generators with cancellative and nilpotent images, respectively. In this case the nilpotent ideal is generated by the generators q_i with nilpotent images.

Proposition 3.7. *The common refinement of finitely many P-primary congruences is P-primary.*

Proof. It suffices by induction to show this for two P-primary congruences $\sim_1$ and $\sim_2$. Reducing modulo their intersection, we can assume that the intersection is the identity congruence on Q. Denote by Q_1 and Q_2 the quotients modulo $\sim_1$ and $\sim_2$, respectively. By assumption $P \subset Q$ is the nilpotent ideal of both $\sim_1$ and $\sim_2$. We claim that if $p \in P$ then p is nilpotent already in Q. Indeed, a sufficiently high multiple of p is congruent to nil under both $\sim_1$ and $\sim_2$, and since their intersection is trivial this can only happen if that multiple is nil. On the other hand, if $p \notin P$, then it must be cancellative: if there exist $a, b \in Q$ with $a+p=b+p$, then $a \sim_1 b$ and $a \sim_2 b$ both hold — whence $a = b$, in fact — since p is cancellative modulo $\sim_1$ and $\sim_2$. □

Remark 3.8. Albeit in different language, [Gilmer 1984, Theorem 5.6.2] contains a variant of the statement of Proposition 3.7.

Passing from the theory surrounding P-primary congruences to that for general congruences is best accomplished by localizing.

Definition 3.9. The *localization* T_P of a Q-module T at a prime ideal $P \subset Q$ is the set of formal differences $t - q$ for $t \in T$ and $q \notin P$, with $t - q$ and $t' - q'$ identified when $w + q' + t = w + q + t'$ for some $w \in Q \setminus P$. Conventions for this are as follows:

- The localization Q_P of Q itself is naturally a monoid, and T_P is a Q_P-module.
- The image of P in Q_P is the maximal ideal P_P of Q_P.
- Any given congruence $\sim$ on Q induces a congruence on Q_P, also denoted $\sim$.
- If $\bar{Q} = Q/\sim$ then we write $\bar{Q}_P = Q_P/\sim$.
- The *unit group at* P is the subgroup $G_P = Q_P \setminus P_P$.

Example 3.10. Localizing Q at the empty prime ideal yields the universal group $Q_\varnothing$. When Q has a nil, $Q_\varnothing$ is trivial. In fact, the universal group $Q_\varnothing$ is trivial precisely when Q has a nil. (Proof: If $Q_\varnothing$ is trivial, then q becomes equal to 0 after inverting every element of Q. Thus there is an element $x_q \in Q$ such that $x_q + q = x_q$. As Q is generated by a finite set $S \subseteq Q$, the sum of the elements x_s for $s \in S$ exists, and it is nil in Q.)

By definition, the group of units of Q_P acts on itself and also on the set $\bar{Q}_P$ of equivalence classes modulo any congruence on Q_P. Here and in what follows, we often think of the quotient $\bar{Q}$ explicitly as a set of congruence classes in Q. Thus $\bar{Q}_P$ is a set of congruence classes in Q_P. We record this fact for future reference.

Lemma 3.11. *Let $P \subset Q$ be a prime ideal. Given any congruence on Q, the unit group of Q_P acts on the quotient $\bar{Q}_P$ modulo the induced congruence on Q_P.* □

In analogy with what happens over rings, primary decomposition of congruences behaves well under localization.

Theorem 3.12. *Primary decomposition of congruences commutes with localization: if $\sim = \sim_1 \cap \cdots \cap \sim_r$ is a primary decomposition of the congruence $\sim$ on Q, and $P \subset Q$ is a prime ideal, then each of the congruences induced by $\sim_1, \ldots, \sim_r$ on Q_P is primary or universal, and their common refinement is the congruence induced by $\sim$ on Q_P.*

Proof. If some element of Q lies outside of P but becomes nilpotent in $Q/\sim_j$, then $\sim_j$ induces the universal congruence on Q_P, so assume no such element exists. Suppose that $q - u \in Q_P$. Our assumption means that u has cancellative image in $Q/\sim_j$. It follows that $q - u \in Q_P$ becomes cancellative in $Q_P/\sim_j$ as long as $q - u$ does not become nilpotent in $Q_P/\sim_j$. Therefore $\sim_j$ induces a primary congruence on Q_P. The rest of the proof is covered by the following lemma. □

Lemma 3.13. *Localization commutes with finite common refinement of congruences: if* $\sim = \sim_1 \cap \cdots \cap \sim_r$ *as congruences on* Q, *and* $P \subset Q$ *is a prime ideal, then the induced congruences on the localization* Q_P *still satisfy* $\sim = \sim_1 \cap \cdots \cap \sim_r$.

Proof. For the duration of this proof, a dot denotes passage to Q_P, so $\dot\sim$ is the congruence on Q_P induced by $\sim$ on Q. If $v \mathrel{\dot\sim_j} w$ in Q_P for all j, then for each j there is an element $u_j \in Q \setminus P$ with $u_j + v \sim_j u_j + w$. Summing these elements u_j yields an element $u = u_1 + \cdots + u_r$ such that $u + v \sim_j u + w$ for all j, whence $u + v \sim u + w$ by definition of $\sim$ as the common refinement. Therefore $v \mathrel{\dot\sim} w$. This logic easily reverses to show that $v \mathrel{\dot\sim} w \implies v \mathrel{\dot\sim_j} w$ for all j. We conclude that $\dot\sim = \dot\sim_1 \cap \cdots \cap \dot\sim_r$, as desired. □

4. Witnesses and associated prime ideals of congruences

Our aim in this section is to show that primary decompositions of congruences in finitely generated commutative monoids have well-defined associated prime ideals. These, and their witnesses, reflect the combinatorial features of a given congruence more accurately than does primary decomposition alone.

Definition 4.1. For any ideal $T \subseteq Q$, the *annihilator modulo* T is the common refinement $\operatorname{ann}(T) = \bigcap_{t \in T} \ker(\phi_t)$ of the kernels of the addition morphisms ϕ_t for $t \in T$.

Remark 4.2. If $q_1 + v = q_2$ then $\ker(\phi_{q_1})$ refines $\ker(\phi_{q_2})$. Therefore, in the definition of $\operatorname{ann}(T)$, it suffices to intersect only over generators of T. Equivalently, if T is generated by $t_1, \ldots, t_r$, then $\operatorname{ann}(T) = \ker(\phi_{t_1} \oplus \cdots \oplus \phi_{t_r} : Q \to T^{\oplus r})$. If $T = \varnothing$ is the empty ideal, then $\operatorname{ann}(T)$ is the universal congruence (that has just one class).

Example 4.3. To explain the "annihilator" terminology, let Q be a monoid with nil ∞ and write $\Bbbk[Q]^- := \Bbbk[Q]/\langle \mathbf{t}^\infty \rangle$. If $T \subseteq Q$ is a monoid ideal, then $\operatorname{ann}(T)$ is the congruence induced by the binomials (and the monomials) in the ideal $(0 : \Bbbk\{T\}) = \{f \in \Bbbk[Q] \mid f\,\Bbbk\{T\} = 0 \text{ in } \Bbbk[Q]^-\}$.

Definition 4.4. Fix a prime ideal $P \subset Q$ with $P_P \subset Q_P$ minimally generated by $p_1, \ldots, p_r$. The *P-covers* of $q \in Q$ are the elements $q + p_i \in Q_P$ for $i = 1, \ldots, r$. The *cover morphisms at* P are the morphisms $\phi_i : Q_P \to \langle p_i \rangle_P$ defined via $q \mapsto q + p_i$; if P is the maximal ideal, then the ϕ_i are called simply *the cover morphisms* of Q.

Remark 4.5. The set of P-cover morphisms depends on the choice of generators $p_1, \ldots, p_r$ and may be infinite if, for example, Q_P has a lot of units. However, modulo Green's relation on Q_P there is a unique finite minimal generating set of any ideal, and every minimal generating set for P_P maps bijectively to it.

Lemma 4.6. *For a fixed prime P, the set of kernels of P-cover morphisms is finite.*

Proof. Two cover morphisms ϕ_p and $\phi_{p'}$ for elements p, p' that are Green's-equivalent in Q_P have the same kernel, because if $p \in \langle p' \rangle$ then there exists an element u such that $p = p' + u$, and thus the kernel of $\phi_{p'}$ refines the kernel of ϕ_p and vice versa. □

Next comes the first main new definition of the paper (note that the concept of mesoprimary congruence in Definition 2.12 is equivalent to a notion already available in the literature; cf. Remark 2.13(4), whose details can best be seen in action in the proofs of Proposition 7.9 and Theorem 8.4.

Definition 4.7. Let $\sim$ be a congruence on Q and $P \subset Q$ a prime ideal. Consider the localized quotient $\overline{Q}_P$. For each $q \in Q$ let $\overline{q}$ be its image in $\overline{Q}_P$. An element $\overline{q}$ is *exclusively maximal* in a subset $S \subseteq \overline{Q}_P$ if $\overline{q}$ is the unique maximal element of S under Green's preorder. An element $w \in Q$ is a:

(1) *witness for P* if the class of $\overline{w}$ is nonsingleton under the kernel of each cover morphism (i.e., the class $\overline{p} + \overline{w}$ is nonsingleton for all $p \in P$) and in each of its nonsingleton kernel classes, $\overline{w}$ is not exclusively maximal.

(2) *key witness for P* if the class of $\overline{w}$ is nonsingleton under the intersection of the kernels of all cover morphisms (i.e., if the class of $\overline{w}$ is nonsingleton under $\operatorname{ann}(\overline{P}_P)$) and $\overline{w}$ is not exclusively maximal in the nonsingleton class;

The ideal P is an *associated prime ideal of* $\sim$ if the annihilator modulo $\overline{P}_P \subset \overline{Q}_P$ is not the identity congruence.

Convention 4.8. A (key) witness is a (key) witness for some prime ideal P. When we speak of the set of (key) witnesses for a given congruence we mean the set of pairs (w, P) where $w \in Q$ is a (key) witness for a prime ideal $P \subset Q$. If the congruence $\sim$ is not clear from context, a (key) witness may be called a (key) $\sim$-witness.

Lemma 4.9. *A prime ideal $P \subset Q$ is associated to a congruence $\sim$ on Q if and only if Q has a key witness for P.*

Proof. Once the annihilator $\operatorname{ann}(\overline{P}_P)$ does not equal the identity congruence, it has a class of size 2 or more; at least one element therein avoids being exclusively maximal. □

Definition 4.10. Fix the notation of Definition 4.7.

(1) An *aide*[1] for a witness w and a generator $p \in P$ is an element $w' \in Q$ whose image $\overline{w}' \in \overline{Q}_P$ is (i) distinct from $\overline{w}$, but (ii) congruent to $\overline{w}$ in the kernel of

[1]The English word "aide" is fortuitously a transliteration of the Hebrew word for "witness". In talmudic courts, a pair of witnesses was required for any conviction.

the cover morphism ϕ_p, and (iii) maximal (under Green's preorder) in the set $\{\overline{w}, \overline{w}'\}$.

(2) A *key aide* for a key witness w is an element $w' \in Q$ whose image $\overline{w}' \in \overline{Q}_P$ is (i) distinct from $\overline{w}$, but (ii) congruent to $\overline{w}$ in the intersection of the kernels of all cover morphisms, and (iii) maximal (under Green's preorder) in the set $\{\overline{w}, \overline{w}'\}$.

Lemma 4.11. *Every witness for P and generator $p \in P$ has an aide. Every key witness has a key aide.*

Proof. In each case, there is a nonsingleton class containing $\overline{w} \in \overline{Q}_P$, so there exists an element $\overline{w}' \neq \overline{w}$ in this class. The point is to choose $\overline{w}'$ so that it does not precede $\overline{w}$ under Green's preorder and so that $\overline{w}'$ lies in the image of the composite morphism $Q \to \overline{Q} \to \overline{Q}_P$. The existence of $\overline{w}'$ not preceding $\overline{w}$ is a consequence of $\overline{w}$ not being exclusively maximal. Now use that every element of $\overline{Q}_P$ is off from the image of $\overline{Q}$ by an element of P, and that $Q \to \overline{Q}$ is surjective. □

Remark 4.12. Every key witness is a witness, because any key aide is an aide for all generators of P.

Remark 4.13. An aide w' for a witness w and $p \in P$ can be a witness but need not be:

- Adding $\overline{p}$ could join $\overline{w}$ to $\overline{w}'$ while some other element of P fails to join $\overline{w}$ to $\overline{w}'$.
- $\overline{w}'$ can be exclusively maximal in its class under the kernel of the cover morphism.

Similarly, a key aide can be a witness (and hence a key witness) but need not be; however, in the key case only the second circumstance (i.e., exclusive maximality) can occur.

In the set of (key) witnesses for a congruence, a single $w \in Q$ can occur multiple times for different P. For instance, this happens when $\varnothing$ is associated.

Example 4.14. The condition for an element to be a witness for the empty prime ideal $\varnothing$ is vacuous: there are no cover morphisms. Furthermore, the congruence $\operatorname{ann}(\varnothing)$ in the definition of key witness is an empty intersection of congruences, so it is the universal congruence on $\overline{Q}_\varnothing$. Thus the empty ideal is associated to a congruence if and only if the universal group $\overline{Q}_\varnothing$ of the quotient modulo that congruence is nontrivial, and that occurs precisely when $\overline{Q}$ has no nil (see Example 3.10). Every $q \in Q$ is a (key) witness in this case but at the same time $\overline{Q}_\varnothing$ has only one class under Green's relation.

The following series of examples demonstrates various features of associatedness of prime ideals and their witnesses.

Example 4.15. As usual it will be convenient to describe congruences on $\mathbb{N}^n$ by unital binomial ideals in polynomial rings. We use $e_x, e_y, \ldots$ to denote the generators of $\mathbb{N}^n$ corresponding to variables $x, y, \ldots$ in the polynomial ring $\Bbbk[\mathbb{N}^n]$, but we write the addition morphisms as $\phi_x, \phi_y, \ldots$ instead of $\phi_{e_x}, \phi_{e_y}, \ldots$, for simplicity.

(1) Let $\sim$ be the congruence on $\mathbb{N}^2$ induced by the binomial ideal $\langle x^2 - xy, xy - y^2\rangle$ of $\Bbbk[x, y]$. The set of associated prime ideals in $\mathbb{N}^2$ consists of the empty ideal $\varnothing$ and the maximal ideal $P = \langle e_x, e_y\rangle$. Localization at the maximal ideal does nothing and there are only two cover morphisms, given by adding e_x and e_y, respectively. To establish that P is associated, note that e_x and e_y themselves are key witnesses for P, congruent under $\mathrm{ann}(P)$, and serve as aides for one another. Indeed, $\mathrm{ann}(P)$, the intersection of the two kernels, contains the pair (e_x, e_y) since $e_x + e_x \sim e_y + e_x$ and also $e_x + e_y \sim e_y + e_y$. The identity $0 \in \mathbb{N}^2$ is not a witness for P. Neither $\langle e_x\rangle$ nor $\langle e_y\rangle$ is associated since adjoining inverses to either turns the quotient $\mathbb{N}^2/\sim$ into a cancellative monoid. In this case all kernels of addition morphisms are trivial. Finally, localizing at the empty prime ideal amounts to considering the induced congruence on $\mathbb{Z}^2$, which is induced by the binomial ideal $\langle x - y\rangle \subset \Bbbk[x^{\pm}, y^{\pm}]$. Since the quotient is nontrivial, $\varnothing$ is associated too. Every element of $\mathbb{N}^2$ is a witness for $\varnothing$, but taken together they form only one Green's class in $\mathbb{Z}^2$.

(2) Let $\sim$ be the congruence on $\mathbb{N}^3$ induced by $\langle x^2 - xy, y^2 - xy, x(z - 1)\rangle \subset \Bbbk[x, y, z]$. The associated prime ideals are $\langle e_x, e_y\rangle$ and $\varnothing$. The argument for $\varnothing$ is the same as in item (1). The localization of $\sim$ at $\langle e_x, e_y\rangle$ is induced by the same ideal, considered in $\Bbbk[x, y, z^{\pm}]$. This says that e_z is cancellative, i.e., that the addition morphism $\phi_z : q \mapsto q + e_z$ is injective. The set of key witnesses is invariant under the ϕ_z-action. It consists of $e_y + ke_z$ and $e_x + ke_z$ for $k \in \mathbb{N}$. The translates of e_y all become equivalent when adding e_x or e_y. Any translates of e_x are witnesses since they are each joined to a translate of e_y. No e_z-translate of 0 is a witness, though. Again, all witnesses are key.

(3) Let $\sim$ be the congruence on $\mathbb{N}^4$ induced by

$$\langle x^2 - xy, y^2 - xy, x(z - 1), y(w - 1)\rangle \subset \Bbbk[x, y, z, w].$$

The associated prime ideals are again $\varnothing$ and $P = \langle e_x, e_y\rangle$. The set of witnesses for P is determined as follows. The element $0 \in \mathbb{N}^4$ is a witness that is not key. The kernel congruences of ϕ_x and ϕ_y are generated by $\{(0, e_z), (e_x, e_y)\}$ and $\{(0, e_w), (e_x, e_y)\}$ in $\mathbb{N}^4 \times \mathbb{N}^4$, respectively. This shows the witness property and also, because their common refinement leaves it singleton similarly to Example 2.22, that 0 is not key. In contrast, e_x and e_y are key witnesses because $\phi_x(e_x) = \phi_x(e_y)$ and likewise for ϕ_y. A mesoprimary decomposition (Theorem 13.2) of the binomial ideal defining $\sim$ has components corresponding to all three witnesses, while a mesoprimary decomposition of the congruence $\sim$ itself needs components only for the two key witnesses (Theorem 8.4). Why the extra binomial component? The common refinement of the congruences induced by $\langle z-1, x^2, y\rangle$ and $\langle w-1, x, y^2\rangle$ leaves the class of 0 singleton, but the intersection of the ideals is merely free of binomials, rather than being altogether zero.

This next example demonstrates how the monoid prime ideal P matters in the definition of a (key) witness for P, and how the same element can be a witness for different P.

Example 4.16. Fix the congruence $\sim$ induced on $\mathbb{N}^4$ by the unital binomial ideal $\langle x(z-1), x(w-1), y(z-1), y^2\rangle \subset \Bbbk[x, y, z, w]$. The associated prime ideals of $\sim$ are $\langle e_x, e_y\rangle$ and $\langle e_y\rangle$. Consider the addition morphisms ϕ_x and ϕ_y. The key witnesses for $\langle e_y\rangle$ are $e_y + ke_x$ and all their translates in the e_z and e_w directions. No element in the ideal $\langle e_x\rangle$ can be a witness for a monoid prime containing e_x because ϕ_x acts injectively on that ideal. Indeed, the witnesses for $\langle e_x, e_y\rangle$ are $0 \in \mathbb{N}^4$ together with all its translates in the e_z direction, and e_y together with its translates in the e_z and e_w directions.

The final example on witnesses demonstrates the prohibition on exclusive maximality, which in particular bars ∞ and idempotents from being witnesses. See Remark 7.10 for a deeper explanation of the ban on exclusive maximality.

Example 4.17. Let $P = \langle e_x, e_y\rangle$ be the maximal ideal of $\mathbb{N}^2$.

(1) Under the Rees congruence induced by the monomial ideal $\langle x^2, y^2\rangle$, the element $e_x + e_y$ is joined to nil under both cover morphisms. Only $e_x + e_y$ is a P-witness, and is in fact a key witness. In contrast, ∞ is a key aide but not a witness, and hence certainly not a key witness.

(2) Under the congruence induced by the unital binomial ideal $\langle y, x^2 - x\rangle$, both cover morphisms join the identity 0 to e_x. However, only the identity is a witness, because e_x lies in the ideal that 0 generates.

Lemma 4.18. *If P is maximal among the prime ideals associated to the components in a primary decomposition, then* $\operatorname{ann}(P)$ *refines all P'-primary components with $P' \subsetneq P$.*

Proof. Fix a P'-primary component $\approx$ with $P' \subsetneq P$, and choose $p \in P \setminus P'$, so that $\bar{p} \in Q/{\approx}$ is cancellative. By definition, if $a, b \in Q$ are congruent modulo $\operatorname{ann}(P)$ then $a+p$ and $b+p$ are congruent modulo the original congruence, so $a+p \approx b+p$, and therefore $a \approx b$ by the cancellative property of $\bar{p}$. Thus $\operatorname{ann}(P)$ refines $\approx$. □

Lemma 4.19. *For all primes $P \not\supseteq P'$, the congruence on Q_P induced by any P'-primary congruence on Q is universal on Q_P.*

Proof. Localization adjoins an inverse for a nilpotent element. □

Despite the oddities in Example 2.22, primary decomposition of congruences is combinatorially well-behaved: the associated prime ideals of a congruence reflect which components are necessary in every primary decomposition.

Theorem 4.20. *A prime $P \subset Q$ is associated to a congruence $\sim$ on Q if and only if every primary decomposition of $\sim$ has a P-primary component. Moreover, if P is not associated to $\sim$, then every P-primary component in every primary decomposition of $\sim$ is* ***redundant****: omitting it leaves another primary decomposition of $\sim$.*

Proof. Suppose that a primary decomposition with no P-primary component is given. Working modulo $\sim$, assume that the congruence to be decomposed is the identity congruence on Q. After localizing along P, the induced congruences on Q_P form a primary decomposition of the identity congruence there by Theorem 3.12, with all P'-primary components for $P' \not\subseteq P$ being universal and thus redundant by Lemma 4.19. That is to say, we can assume that P is the maximal monoid prime ideal of Q. Since the primary decomposition has no P-primary component, Lemma 4.18 implies that $\operatorname{ann}(P)$ refines all primary components, and thus it refines their intersection. Thus $\operatorname{ann}(P)$ is trivial and P is not associated.

To prove the rest of the statement, it suffices to show that P is an associated prime of $\sim$ if some primary decomposition of $\sim$ has a P-primary component $\sim_P$ that is *irredundant* in the sense that omitting $\sim_P$ yields a coarser congruence than $\sim$. Write $\approx$ for the (not necessarily primary) common refinement of all other congruences in the decomposition. Thus $\sim_P \cap \approx$ is a nontrivial decomposition of the identity congruence. Choose $a \neq b \in Q$ with $a \approx b$ but $a \not\sim_P b$. Let $T = \{t \in Q \mid t+a \sim_P t+b\}$. Since $\sim_P$ is P-primary, the radical of T is P. Modulo Green's relation on Q_P, find a maximal element $\hat{t}$ not in the image of T. If $t \in Q$ maps to $\hat{t}$ then the images of $t+a$ and $t+b$ in Q_P are joined under each cover morphism. Therefore their class is nonsingleton under $\operatorname{ann}(P)$, so one of them is a key witness for P. □

Theorem 4.20 implies a natural characterization of primary congruences.

Corollary 4.21. *A congruence is primary if and only if it has exactly one associated prime ideal.*

Remark 4.22. Via the Rees congruence construction, primary decomposition of congruences is a refinement of primary decomposition of ideals in monoids. There is an extensive literature on the second type of decomposition surveyed in [Anderson and Johnson 1984]. Our definitions are aligned with those in the literature: the Rees congruence of a monoid ideal is primary if and only if that monoid ideal is primary. In this case its unique associated monoid prime ideal is the unique associated monoid prime ideal of the congruence.

5. Associated prime congruences

Each primary congruence on a finitely generated commutative monoid Q has a unique associated prime ideal. One of the most basic insights in this paper is that a single primary congruence can have several associated prime congruences. The first definition says how a congruence looks near a given $q \in Q$.

Definition 5.1. Fix a prime ideal $P \subseteq Q$, a congruence $\sim$ on Q, and an element $q \in Q$. The *P-prime congruence* of $\sim$ *at* q is the kernel of the morphism $Q \to (\langle \bar{q} \rangle / \langle \bar{q} + P \rangle)_P$ induced by the quotient $Q \to Q/{\sim} = \bar{Q}$, addition $\phi_{\bar{q}} : \bar{Q} \to \langle \bar{q} \rangle$, and localization at P.

Definition 5.2. A prime congruence $\approx$ on Q is *associated* to an arbitrary congruence $\sim$ if $\approx$ equals the P-prime congruence of $\sim$ at a key witness for P.

Remark 5.3. The definition implies that the associated prime P of $\approx$ is associated to $\sim$ too. If P is clear from the context, such as after $\approx$ is fixed, then we also speak of a key witness for P simply as a *key witness*.

Lemma 5.4. *If $p, q \in Q$ are equivalent under Green's relation, that is, if $\langle p \rangle = \langle q \rangle$, then their P-prime congruences agree for each P.*

Proof. The same argument as for Lemma 4.6 applies. □

Example 5.5. In the situation of Example 4.16, the associated prime congruences are induced by the ideals $\langle x, y \rangle$, $\langle x, y, z-1 \rangle$, and $\langle y, z-1, w-1 \rangle$. The first two correspond to witnesses for $\langle e_x, e_y \rangle$, while the third corresponds to all of the witnesses for $\langle e_y \rangle$.

The following and Lemma 2.19 are the central finiteness results, reflected in all of the following development, particularly Theorem 8.4.

Theorem 5.6. *Fix a congruence $\sim$ on a finitely generated commutative monoid Q. For each of the finitely many primes P of Q, the key $\sim$-witnesses for P generate only finitely many Green's classes in the localization Q_P along P. Consequently, each congruence on Q has only finitely many associated prime congruences.*

Proof. Since the definition of key witness for P is already local, it suffices to treat the case where P is the maximal ideal of Q. Form a relation on Q by joining every key witness w to a key aide a. This relation is a congruence by the definitions of key witness and key aide. The claim about Green's classes holds because Q is noetherian. To prove the consequence for associated prime congruences, use Lemma 5.4. □

Example 5.7. The congruence in Example 2.18 is primary with respect to the maximal ideal. The (key) witnesses are e_x, e_y, and also $2e_x$, $e_x + e_y$, and $2e_y$, since their class gets joined to nil under ϕ_x and ϕ_y. Although the witnesses look combinatorially different, the only associated prime congruence is the identity congruence on the monoid $\{0, \infty\}$. This is forced, as the identity is the only cancellative element in $\overline{Q}$.

If on Q the identity congruence is primary, then the assignment of witnesses to their P-prime congruences is order-preserving. It would be interesting to understand which posets of witnesses and associated prime congruences can occur (Problem 17.4).

6. Characterization of mesoprimary congruences

In parallel with the theory of ordinary primary ideals in commutative rings, the mesoprimary condition admits a characterization in terms of associated prime congruences. Definition 2.12 was made with this proposition in mind.

Theorem 6.1. *A congruence is mesoprimary if and only if it has exactly one associated prime congruence.*

Proof. Fix a P-primary congruence $\sim$ on Q. If $\sim$ is mesoprimary and w is not nil, then the P-prime congruence of $\sim$ at w coincides with the P-prime congruence of $\sim$ at the identity because $\overline{w}$ is partly cancellative. The uniqueness of the associated prime congruence follows from the special case where w is a key witness.

On the other hand, assume $\sim$ has a unique associated prime congruence. Then $\sim$ is primary by Corollary 4.21. Replacing Q with $\overline{q}$, assume $\sim$ is the identity congruence on Q. Suppose that a and b are distinct elements whose images in Q_P satisfy $a+u=b$ for some unit $u \in Q_P$. Using the partial order from Lemma 2.19, let $w \in Q$ be any element such that $w+a \neq w+b$ and the image of w modulo the cancellative elements $F \subseteq Q$ is maximal with this property. Let w' be any maximal nonnil element whose image in the poset Q/F is comparable to w but not below. The choices of w and w' make them both key witnesses: w' has ∞ as an aide, and w is verified directly to be a key witness since $p+w = p+(w+a-b)$ in Q_P for all $p \in P$. Replacing w' with $w'+c$ for some cancellative element c if necessary, assume that $w' = w+q$ for some $q \in Q$. Uniqueness of the associated prime congruence, combined with the relation $\phi_{w'} = \phi_q \circ \phi_w$ among addition morphisms, implies that $w'+a \neq w'+b$. By maximality of w' in Q/F, the relation $v+a = v+b$ can only hold for v such that $v+a = \infty$. Thus $\sim$ is partly cancellative. □

Remark 6.2. A primary congruence has only one associated monoid prime ideal by Corollary 4.21. Theorem 6.1 makes precise the notion that further decomposition along the associated prime congruences is natural, as is visible already in Example 1.3.

Quotients by mesoprimary congruences can be described fairly explicitly in terms related to the action in Lemma 3.11. Making this description into a precise alternative characterization of mesoprimary congruences requires some specialized notions involving monoid actions.

Definition 6.3. The action of a monoid F on an F-module T is *semifree* if

- $t \mapsto f+t$ is an injection $T \hookrightarrow T$ for all $f \in F$, and
- $f \mapsto f+t$ is an injection $F \hookrightarrow T$ for all $t \in T$.

Remark 6.4. The letter "F" stands for "face": in practice, the monoid F is often a face of an affine semigroup, and thinking of it that way is good for intuition.

Lemma 6.5. *An action of a cancellative monoid F on an F-module T is semifree if and only if the localization map $T \hookrightarrow T_\varnothing$ is injective and the universal group $F_\varnothing$ acts freely on $T_\varnothing$.*

Proof. The cancellative condition means that the natural map $F \hookrightarrow F_\varnothing$ is injective. Using this fact, the "if" direction is elementary, and omitted. In the other direction, the semifree case, the first injectivity condition guarantees that $t-f=t'-f' \iff f'+t=f+t'$. In particular, $t-0=t'-0 \iff t=t'$, so the natural map $T \hookrightarrow T_\varnothing$ is injective. The second injectivity condition guarantees that the action of $F_\varnothing$ is free: $(f-f')+(t-w)=t-w \iff (f+t)-(f'+w)=t-w \iff (w+f)+t=(f'+w)+t$, and by the second injectivity condition this occurs if and only if $f+w=f'+w$, which is equivalent to $f=f'$ because F is cancellative. □

In contrast to group actions, monoid actions need not define equivalence relations, because the relation $t \sim f+t$ can fail to be symmetric. The relation is already reflexive and transitive, however, precisely by the two axioms for monoid actions.

Definition 6.6. An *orbit* of a monoid action of F on T is an equivalence class under the symmetrization of the relation $\{(s,t) \mid f+s=t \text{ for some } f \in F\} \subseteq T \times T$.

Combinatorially, from an F-module T, one can construct a directed graph with vertex set T and an edge from s to t if $t=f+s$ for some $f \in F$. Then an orbit is a connected component of the underlying undirected graph.

Corollary 6.7. *A congruence $\sim$ on a finitely generated commutative monoid Q is mesoprimary if and only if the set F of nonnilpotent elements in $\overline{Q}=Q/{\sim}$ is a cancellative monoid that acts semifreely on $\overline{Q} \setminus \{\infty\}$ with finitely many orbits.*

Proof. Whether we assume the mesoprimary condition on $\sim$ or the condition on the nonnilpotent elements in $\overline{Q}$, we can in each case deduce that $\sim$ is P-primary for some prime $P \subset Q$. The image of $Q \setminus P$ in $\overline{Q}$ is the cancellative submonoid F by definition, which has finitely many orbits by Lemma 2.19. The only feature of the corollary's statement that distinguishes mesoprimary congruences from general primary ones is semifreeness, which we claim is equivalent to uniqueness of the associated prime congruence in Theorem 6.1. Indeed, F acts semifreely if and only if the P-prime congruences at all nonnil elements of $\overline{Q}$ coincide. Those coincidences certainly imply that the P-prime congruences at all witnesses coincide, in which case $\sim$ is mesoprimary. On the other hand, if $\sim$ is mesoprimary, then the P-prime congruences at all key witnesses coincide. They all coincide with the P-prime congruence at the identity, or else there would be two key witnesses, one sharing its P-prime congruence with the identity and the other not. Since the image in Q/F of every nonnil element of Q lies between the identity and a key witness, the P-prime congruence of every nonnil element is forced to agree with the one shared by the identity and the key witnesses. □

Remark 6.8. As the proof of Corollary 6.7 shows, one interpretation of the structure theorem in the statement is that a P-primary congruence has the same P-prime congruence at every nonnil element as soon as it has the same P-prime congruence at every key witness, and that is what it means to be mesoprimary.

Proposition 6.9. *Given a finite set of congruences on Q, all of which are mesoprimary with the same associated prime congruence, their common refinement is also mesoprimary with the same associated prime congruence.*

Proof. Let $\sim$ be the common refinement of finitely many P-mesoprimary congruences. Then $\sim$ is P-primary by Proposition 3.7. Applying Theorem 6.1, it suffices to show that the P-prime congruence of $\sim$ at any element $q \in Q$ that lies outside the nil class of $\sim$ is the same as the P-prime congruence of $\sim$ at the identity.

Lemma 3.13 implies that we may assume P is the maximal ideal of Q with unit group $G = Q \setminus P$. Under each of the given mesoprimary congruences, Corollary 6.7 (in the guise of Remark 6.8) implies that the class of q is either nil or its intersection with the orbit $G+q$ equals $K+q$, where $K \subseteq G$ is the subgroup that stabilizes (fixes as a set, but not necessarily pointwise) the class of the identity under each of the mesoprimary congruences. Since the nil class contains $K+q$ once it contains q, the class of q under $\sim$ is either nil or its intersection with the orbit $G+q$ equals $K+q$. Having excluded nil by our choice of q, the intersection must be $K+q$. Thus the P-prime congruence at q under $\sim$ coincides with the P-prime congruence at q under (every) one of the mesoprimary congruences modulo which q is not nil.

In particular, letting q be the identity shows that K is the intersection of the identity class of $\sim$ with G. Consequently, the P-prime congruence of $\sim$ at q coincides with the P-prime congruence of $\sim$ at the identity, as desired. □

7. Coprincipal congruences

In commutative rings, irreducible decomposition underlies primary decomposition. Analogously, coprincipal decomposition underlies mesoprimary decomposition of commutative monoid congruences (but see the remarks and examples after Theorem 8.4).

Definition 7.1. A *peak* of a monoid Q is a nonnil element $q \in Q$ such that $q+a=\infty$ for all nonunit $a \in Q$. The *cogenerators* of a P-primary congruence on Q are the elements of Q whose images in $\bar{Q}_P$ are peaks.

Definition 7.2. A congruence $\sim$ on Q is *coprincipal* if it is P-mesoprimary for some monoid prime P and additionally the quotient of $\bar{Q}_P$ modulo its Green's relation has precisely one peak.

Example 7.3. The congruence in Example 2.18 is coprincipal. It is P-mesoprimary for $P = Q \setminus \{0\}$ and its unique peak is the class of 2.

Definition 7.4. Fix a congruence on Q with quotient $\bar{Q}$. The *order ideal* $Q^P_{\preceq q}$ *cogenerated by* $q \in Q$ *at a prime ideal* $P \subset Q$ consists of those $a \in Q$ whose image precedes that of q in the partially ordered quotient of $\bar{Q}_P$ modulo its Green's relation (Lemma 2.6).

Example 7.5. Let $\sim$ be the congruence on $\mathbb{N}$ induced by the binomial ideal $\langle x^3 - x^6\rangle \subset \Bbbk[x]$. Set $P = \langle e\rangle$, where $e = e_x$ is the generator of $\mathbb{N}$.

(1) The order ideal $\mathbb{N}^P_{\preceq e}$ consists of e itself and $0 \in \mathbb{N}$.

(2) Including $2e$ yields the order ideal $\mathbb{N}^P_{\preceq 2e} = \{0, e, 2e\}$.

(3) The order ideals $\mathbb{N}^P_{\preceq q}$ for $q = me$ with $m \geq 3$ all coincide with $\mathbb{N}$ itself. Thus, in general, order ideals $Q^P_{\preceq q} \subseteq Q$ need not be finite, although their images in $\bar{Q}_P$ modulo Green's relation always are.

(4) The order ideals $\mathbb{N}^{\varnothing}_{\preceq q}$ for $q \in \mathbb{N}$ all coincide with $\mathbb{N}$ itself.

Example 7.6. Let $\sim$ be the identity congruence on $Q = \mathbb{N}^3$, and set $P = \langle e, f\rangle$, where e, f are two of the three generators of $\mathbb{N}^3$, the third being g. The order ideal $Q^P_{\preceq e+f+2g}$ consists of the lattice points on the nonnegative g-axis together with their translates by e, f, and $e+f$. The answer would have been the same had $e+f+2g$ been replaced by $e+f$, or $e+f+g$, or $e+f+mg$ for any $m \in \mathbb{N}$.

Definition 7.7. Fix a congruence $\sim$. The congruence *cogenerated by* q *along* P is the coarsening $\sim^P_q$ of $\sim$ obtained by first joining any pair of elements in $Q \setminus Q^P_{\preceq q}$ and also joining any pair $(a, b) \in Q$ such that

(i) the images $\bar{a}$ and $\bar{b}$ in $\bar{Q}_P$ differ by a unit in $\bar{Q}_P$, and

(ii) $\bar{c}+\bar{a} = \bar{c}+\bar{b} = \bar{q} \in \bar{Q}_P$ for some $c \in Q_P$.

Example 7.8. The congruence $\sim^P_q$ in Definition 7.7 need not be primary, and hence it need not be coprincipal. Essentially, the prime P has to be small enough to foster the mesoprimary condition. In Example 4.17.2, the congruence cogenerated by $q = e_x$ along $P' = \{e_x, \infty\}$ is not primary. However, along $P = \{\infty\}$, localization inverts more, causing e_x to be joined with 0, resulting in a primary — and hence coprincipal — congruence.

Proposition 7.9. *Fix a congruence* $\sim$ *and a witness* w *for a prime* P. *All elements of* P *are nilpotent modulo the congruence* $\sim_w^P$, *whose nil class is* $Q \setminus Q_{\preceq w}^P$.

Proof. Given an aide w' for w and a generator p of P, one of two things must happen, and in both cases $p+w$ is nil modulo $\sim_w^P$. Write $[q]$ for the Green's class of $\overline{q} \in \overline{Q}_P$.

(1) $[w] \neq [w']$. In this case, either $[w] < [w']$ or $[w]$ and $[w']$ are incomparable, but these both imply that w' maps to nil modulo the coprincipal congruence $\sim_w^P$, so $p+w = p+w'$ is nil modulo $\sim_w^P$.

(2) $[w] = [w']$; that is, their images lie in the same Green's class. In this case, $[p+w] > [w]$ by Lemma 2.8, since addition by p joins $[w]$ to $[w']$.

Since p is an arbitrary generator of P, it follows that $P+w$ is nil modulo $\sim_w^P$. This implies that every element of P is nilpotent modulo $\sim_w^P$, as follows. There are only finitely many Green's classes beneath $[w]$, so the Green's classes of multiples of any given nonunit element $a \in Q_P/\sim_w^P$ are not all distinct: there must be repeats. Suppose $[\alpha \cdot a] = [\beta \cdot a]$ for some positive integers $\alpha < \beta$. Every nonnil element of $Q_P/\sim_w^P$ precedes $\overline{w}$ in Green's preorder. Therefore, if neither $\alpha \cdot a$ nor $\beta \cdot a$ is nil, then there is some $c \in Q$ such that $[\alpha \cdot a] + c = [w]$, whence

$$[w] = [\beta \cdot a] + c = (\beta - \alpha) \cdot a + [\alpha \cdot a] + c = (\beta - \alpha) \cdot a + [w] \subseteq P + [w]$$

is nil modulo $\sim_w^P$, contradicting the choice of w.

The statement about the nil class holds because $Q \setminus Q_{\preceq w}^P$ is an ideal of Q (so its image is nil) that does not contain w itself (so the image of w is not made nil by the first relations in Definition 7.7) or any element in $Q_{\preceq w}^P$ (so none of the relations defined by (i) and (ii) in Definition 7.7 make w or any other element of $Q_{\preceq w}^P$ nil). □

Remark 7.10. Proposition 7.9 can fail if w is merely an aide — even a key aide. The not-exclusively-maximal property of a witness guarantees existence of an aide that can be set congruent to nil modulo the coprincipal congruence without forcing w to be nil as well. In Example 4.17.2, for instance, there is no way to define a coprincipal congruence cogenerated by e_x in such a way that e_x is nilpotent without it being nil.

Theorem 7.11. *Given a congruence* $\sim$, *the congruence* $\sim_w^P$ *cogenerated by any witness* w *for* P *is coprincipal, with associated prime ideal* P.

Proof. Every nonunit in the localization $Q_P/\sim_w^P$ of the quotient monoid $Q/\sim_w^P$ along P is nilpotent by Proposition 7.9. The statement about the nil class in that same proposition implies that the Green's class of w is the unique peak. The localization morphism $Q/\sim_w^P \to Q_P/\sim_w^P$ is injective by condition (i) in Definition 7.7. Condition (ii) there forces the P-prime congruence at the identity to equal the P-prime congruence at w, which consequently forces the P-prime congruences at all nonnil elements to coincide, since they lie between the P-prime congruences at the identity and at w. Therefore the action of the unit group of $Q_P/\sim_w^P$ on its nonnilpotent elements is free. The proof is complete by Corollary 6.7, using the characterization of semifreeness in Lemma 6.5. □

Definition 7.12. If w is a witness for an associated P-prime congruence of $\sim$, then the congruence $\sim_w^P$ is the *coprincipal component* of $\sim$ cogenerated by w along P. If the prime ideal P is clear from context, e.g., if w is already specified to be a witness for P, then we simply speak of the coprincipal component cogenerated by w.

Example 7.13. Consider the congruence on $\mathbb{N}^2$ induced by $I = \langle x^3 - x^2, y^3 - y^2 \rangle$. The quotient $Q = \mathbb{N}^2/\sim_I$ has nine elements, with the class of $2e_x + 2e_y$ being nil. The quotient also has two idempotents, namely the classes of $2e_x$ and $2e_y$. Neither of the congruences cogenerated by $q = e_x + 2e_y$ and $q = e_y + 2e_x$ along $P = \langle e_x, e_y \rangle$ is primary; however, these elements are not P-witnesses. In fact, there are no P-witnesses: the maximal ideal is not associated. In contrast, the coprincipal components for the witnesses $(2e_2, \langle e_1 \rangle)$ and $(2e_1, \langle e_2 \rangle)$ are mesoprimary, as per Theorem 7.11.

Example 7.14. In the setting of Example 7.5, the coprincipal component of $\sim$ cogenerated by any $q \in \mathbb{N}$ along $\varnothing$ is induced by the binomial ideal $\langle 1 - x^3 \rangle$. The component cogenerated by the key witness $2e$ along $\langle e \rangle$ is induced by the binomial ideal $\langle x^3 \rangle$.

Proposition 7.15. *Given any witness w for an associated P-prime congruence of $\sim$, the coprincipal component of $\sim$ cogenerated by w along P is refined by $\sim$.*

Proof. Starting from $\sim$ the coprincipal component is formed by identifying additional pairs of elements. □

Proposition 7.16. *Any mesoprimary congruence $\sim$ equals the common refinement of the coprincipal components of $\sim$ cogenerated by the cogenerators of $\sim$.*

Proof. Fix a P-mesoprimary congruence $\sim$. By Proposition 7.15 each coprincipal component at a cogenerator coarsens $\sim$. On the other hand, suppose that $q \not\sim q'$. Let $\overline{q}$ and $\overline{q}'$ denote their images in the localized quotient $\overline{q}_P$. By mesoprimariness, $\overline{q} \neq \overline{q}'$. Modulo Green's relation on $\overline{q}_P$, every element precedes a peak. If exactly one of $\overline{q}$ and $\overline{q}'$ precedes some peak $\overline{w}$, then modulo $\sim_w^P$ exactly one of q and q' maps to nil, so they are incongruent. If no such peak exists, then $\overline{q}$ and $\overline{q}'$ both precede some peak $\overline{w}$. For $\overline{q}$ and $\overline{q}'$ to be joined by $\sim_w^P$ they must differ by a unit and satisfy $\overline{q} + \overline{c} = \overline{q}' + \overline{c} = \overline{w}$ for some $c \in Q_P$, all by Definition 7.7. However, since $\sim$ is mesoprimary, $\overline{q} + \overline{c} = \overline{q}' + \overline{c}$ implies that both sides are nil. Consequently, $q \not\sim_w^P q'$. □

8. Mesoprimary decompositions of congruences

Definition 8.1. Fix a congruence $\sim$ on a finitely generated commutative monoid Q.

(1) An expression of $\sim$ as the common refinement of finitely many mesoprimary congruences is a *mesoprimary decomposition* if, for each mesoprimary congruence $\approx$ that appears in the decomposition with associated prime ideal $P \subset Q$, the P-prime congruences of $\sim$ and $\approx$ at every cogenerator of $\approx$ coincide.

(2) Each mesoprimary congruence that appears is a *mesoprimary component* of $\sim$.

(3) If every cogenerator of every P-mesoprimary component $\approx$ is a key $\sim$-witness for P, then the decomposition is a *key* mesoprimary decomposition.

Example 8.2. According to Definition 8.1, the decomposition in Example 2.22 is not a mesoprimary decomposition because the intersectands are not components of the identity congruence: the combinatorics at the witnesses for the mesoprimary congruences in the decomposition do not agree with the combinatorics of the identity congruence. More precisely, the $\varnothing$-prime congruence at each element of $\mathbb{N}^2$ is the identity congruence, not the congruence induced by $\langle x-1\rangle$ or $\langle y-1\rangle$.

Theorem 8.3. *Every congruence on a finitely generated commutative monoid admits a key mesoprimary decomposition.*

Proof. Two examples are the decompositions in Theorem 8.4 and Corollary 8.11, by Remark 8.5 and finiteness of the set of Green's classes of witnesses in Theorem 5.6. □

In the remainder of this section, Convention 4.8 leads to some simplification of terminology. The first statement to benefit is our first main decomposition theorem (the other being Corollary 8.11), which generalizes to arbitrary monoid congruences the notion of irreducible decomposition for monoid ideals; see Examples 8.6 and 8.7.

Theorem 8.4. *Every congruence on a finitely generated commutative monoid is the common refinement of the coprincipal congruences cogenerated by its key witnesses.*

Proof. Fix a congruence $\sim$ on Q. Proposition 7.15 implies that the intersection of all of the coprincipal congruences for witnesses is refined by $\sim$. On the other hand, suppose that $q \not\sim q'$ for two elements $q, q' \in Q$. The proof is done once we find a prime $P \subset Q$ and a key witness $w \in Q$ whose coprincipal congruence $\sim_w^P$ on Q fails to join q to q'.

Let $T = \{t \in Q \mid t+q \sim t+q'\}$ be the ideal of elements joining q to q'. Fix a prime ideal P minimal among primes of Q containing T. The images $\hat{q}$ and $\hat{q}'$ of q and q' in the localization Q_P remain incongruent because P contains T. In contrast, every element in the localized image T_P joins $\hat{q}$ to $\hat{q}'$; that is, $\hat{t}+\hat{q} \sim \hat{t}+\hat{q}'$ for all $\hat{t} \in T_P$. Since the maximal ideal P_P of Q_P is minimal over T_P by minimality of P over T, there is a maximal Green's class among those represented by the elements

$\{\hat{t} \in Q_P \mid \hat{t}+\hat{q} \not\sim \hat{t}+\hat{q}'\}$. If the image of t lies in such a maximal Green's class, then in Q at least one of the elements $w = t+q$ and $w' = t+q'$ — namely one whose image in $\bar{Q}_P$ is not strictly greater than the other under Green's preorder — is a key witness by definition. Assuming, by symmetry, that w is a key witness, the localization of the congruence $\sim_w^P$ satisfies $\hat{q} \not\sim_w^P \hat{q}'$, so $q \not\sim_w^P q'$ before localization. □

Remark 8.5. In Theorem 8.4 it makes no difference whether one uses all the key witnesses or just one per Green's class. This follows instantly from the definition of a coprincipal component; indeed, for a given Green's class of key witnesses, the coprincipal components are all equal — not just equivalent, but literally the same congruence.

Example 8.6. For a monomial ideal in an affine semigroup ring, the coprincipal decomposition of the Rees congruence afforded by Theorem 8.4 arises equivalently from the Rees congruences of the components in the unique irredundant irreducible decomposition into monomial ideals [Miller 2002, Theorem 2.4]; see also [Miller and Sturmfels 2005, Corollary 11.5 and Proposition 11.41].

Example 8.7. Unlike the case in Example 8.6, the decomposition in Theorem 8.4 can be redundant in general. This happens for the congruence in Example 4.15(1). The decomposition produced by Theorem 8.4 has three mesoprimary components: $\sim_w^P$ for $P = \langle e_x, e_y \rangle$ and $w \in \{(e_x, 0), (0, e_y)\}$ arise from joining e_y and e_x, respectively, to nil. A third component $\sim^\varnothing$ arises for $P = \varnothing$ (with any element as a witness) and is induced by $\langle x - y \rangle$. The decomposition into three congruences is redundant: the given congruence is already the common refinement of $\sim^\varnothing$ and either of $\sim_w^P$, the point being that once $\sim^\varnothing$ is given, one only needs to separate $(1, 0)$ from $(0, 1)$. That said, the points $(1, 0)$ and $(0, 1)$ represent distinct Green's classes of key witnesses for the associated prime congruence induced by the binomial ideal $\langle x, y \rangle$. There is simply no way of constructing an irredundant coprincipal decomposition without breaking the symmetry: no systematic method of eliminating one of the redundant components in this example would have a way to choose between them.

Remark 8.8. A coprincipal congruence can have more than one Green's class of key witnesses, such as Example 2.18. In any such case the mesoprimary decomposition from Theorem 8.4 produces more than one coprincipal component. By Proposition 7.16, however, it is guaranteed that the original congruence appears as the component for the Green's class of the unique peak, and thus all other components are redundant. This phenomenon prevents arbitrary coprincipal congruences from accurately reflecting the combinatorics of irreducible decomposition of binomial ideals. One irreducible decomposition of the coprincipal ideal $I = \langle x^2 - xy, xy - y^2, x^3 \rangle$ from Example 2.18 is $I = \langle x - y, x^3 \rangle \cap \langle x^2, y \rangle \cap \langle x, y^2 \rangle$, as can be seen by applying [Vasconcelos 1998, Proposition 3.1.7].

Remark 8.9. Any irreducible congruence is mesoprimary: if a congruence is not mesoprimary then it has at least two associated prime congruences by Theorem 6.1, and then it is reducible by mesoprimary decomposition. However, irreducible decompositions of congruences do not, in general, reflect the combinatorics of congruences in a manner that is witnessed combinatorially by the congruence itself.

Lemma 8.10. *Every cogenerator of the common refinement of a finite set of P-mesoprimary congruences is a cogenerator of one of the given mesoprimary congruences.*

Proof. If w is a cogenerator of the common refinement $\sim$, then w is not nil modulo $\sim$, so w is not nil modulo (at least) one of the given mesoprimary congruences. On the other hand, $p+w$ is nil modulo $\sim$ for all $p \in P$, whence $P+w$ is nil modulo each one of the given mesoprimary congruences. Therefore w is a cogenerator of each of the given mesoprimary congruences modulo which it is not nil. □

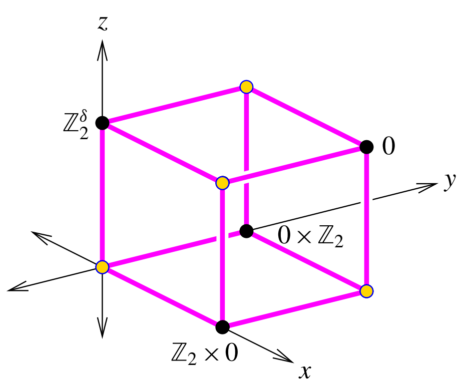
Combining Theorem 8.4 with Proposition 6.9 and Lemma 8.10 yields the next result, culminating our study of commutative monoid congruence decompositions.

Corollary 8.11. *Every congruence on a finitely generated commutative monoid admits a key mesoprimary decomposition with one component per associated prime congruence.*

Example 8.12. In general, the set of key witnesses is properly contained in the set of witnesses. Example 4.15(3) shows one way this can happen. Exploiting the weirdness of irreducible decomposition of the identity congruence is not necessary: consider the primary congruence induced by the (cellular) binomial ideal

$$I = \langle a^2-1,\, b^2-1,\, x(b-1),\, y(a-1),\, z(a-b),\, x^2,\, y^2,\, z^2\rangle.$$

The geometry of the quotient is shown here, where $\mathbb{Z}_2^\delta$ is the diagonal copy of $\mathbb{Z}_2$ in $\mathbb{Z}_2 \times \mathbb{Z}_2$, i.e., the copy generated by (1, 1):



The solid dots indicate key witnesses and are labeled with quotients of $\overline{Q}$ modulo the corresponding stabilizers, under the action from Lemma 3.11. The origin is not a key witness because the common refinement of the three kernels of the addition morphisms is trivial. According to Theorem 8.4, a coprincipal mesoprimary decomposition of $\sim_I$ is induced by the following decomposition of I into unital binomial ideals:

$$I = \langle a-1, b-1, z^2, y^2, x^2\rangle \cap \langle a^2-1, b-1, z, y, x^2\rangle \\ \cap \langle a-1, b^2-1, z, x, y^2\rangle \cap \langle ab-1, a-b, y, x, z^2\rangle.$$

The heart of the remainder of this paper — the ring-theoretic part — is to make the corresponding decomposition of arbitrary (nonunital) binomial ideals precise. For reference, the primary decomposition of I is

$$I = \langle a-1, b-1, z^2, y^2, x^2\rangle \cap \langle a+1, b-1, z, y, x^2\rangle \\ \cap \langle a-1, b+1, z, x, y^2\rangle \cap \langle a+1, b+1, y, x, z^2\rangle.$$

9. Augmentation ideals, kernels, and nils

One of our goals is to compare the combinatorics of congruences on a commutative monoid Q in purely monoid-theoretic settings with their ring-theoretic counterparts. It is therefore important to note that various binomial ideals $I \subset \Bbbk[Q]$ can induce the same congruence on Q. One way for this to happen is an arithmetic way, via binomials involving the same monomials but different sets of coefficients; this occurs for binomial primes $I_{\rho,P}$ whose characters share their domain of definition (see Section 12).

Example 9.1. Let $\mathrm{char}(\Bbbk) \neq 2$. In the polynomial ring $\Bbbk[x, y, z]$, both of the ideals $I = \langle x(z-1), y(z-1), z^2-1, x^2, xy, y^2\rangle$ and $I' = \langle x(z-1), y(z+1), z^2-1, x^2, y^2\rangle$ induce the same congruence; note that I' contains $\langle xy\rangle$, so the only difference between these two ideals is the character on $\mathbb{Z} = \{0\} \times \{0\} \times \mathbb{Z} \subseteq \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ induced by the monomials $y, zy, z^2y, \ldots$ due to the generator $y(z+1)$ instead of $y(z-1)$.

Another way, demonstrated in parts (3) and (4) of Example 2.17, is combinatorial: when Q has a nil ∞, the binomial ideal $\langle \mathbf{t}^\infty\rangle$ induces the same (trivial) congruence on Q as the zero ideal $\langle 0\rangle \subseteq \Bbbk[Q]$. Nils are the only way for this to occur.

Lemma 9.2. *Fix a binomial ideal $I \subseteq \Bbbk[Q]$ whose congruence $\sim_I$ is trivial* (*every class is a singleton*). *Then $I = 0$ or $I = \langle \mathbf{t}^\infty\rangle$ for a nil $\infty \in Q$.*

Proof. If $I \neq 0$ then I must be a monomial ideal with a unique monomial, or else the congruence $\sim_I$ has a class of size at least 2. Hence the result follows because a monoid can have at most one nil. □

Definition 9.3. If $\infty \in Q$ is a nil, then the *truncated algebra* is $\Bbbk[Q]^- := \Bbbk[Q]/\langle \mathbf{t}^\infty\rangle$. By convention, if Q has no nil, then we set $\Bbbk[Q]^- := \Bbbk[Q]$.

Remark 9.4. Truncated algebras arise naturally from monoid algebras because of differences in the way quotients of monoids and monoid algebras by ideals are formed. If $F \subseteq Q$ is a monoid ideal and $\sim_F$ its Rees congruence, the quotient $\Bbbk[Q] \to \Bbbk[Q]/M_F$ modulo the monomial ideal $M_F = \langle \boldsymbol{t}^f \mid f \in F \rangle$ equals $\Bbbk[Q/\sim_F]^-$ rather than $\Bbbk[Q/\sim_F]$ itself. We shall see that if Q has a nil, then $\Bbbk[Q]$ and $\Bbbk[Q]^-$ reflect certain aspects of the algebra of Q to varying degrees of accuracy.

More generally, if the congruence induced by a (not necessarily unital) binomial ideal I results in a quotient $Q/\sim_I$ that has a nil, then throwing in monomials from the nil class results in an ideal that determines the same congruence.

Proposition 9.5. *Fix a binomial ideal $I \subseteq \Bbbk[Q]$. The only binomial ideals containing I that determine the same congruence $\sim_I$ are I itself and, if $\bar{Q} = Q/\sim_I$ has a nil ∞, the ideal $I + \langle \boldsymbol{t}^q \mid \bar{q} = \infty \rangle$, where the bar denotes passage from $q \in Q$ to its image $\bar{q} \in \bar{Q}$.*

Proof. Under the grading of the quotient algebra $\Bbbk[Q]/I$ by $\bar{Q} = Q/\sim_I$, the dimension of the graded piece $(\Bbbk[Q]/I)_{\bar{q}}$ as a vector space over $\Bbbk$ is either 0 or 1, depending on whether I contains a monomial in the corresponding class. Since the (exponents on) monomials in I form a single class, the dimension can only be 0 for at most one $\bar{q}$, and $\bar{q}$ must be a nil in $\bar{Q}$. Now note that $\Bbbk[Q]/I$ is close enough to the monoid algebra $\Bbbk[\bar{Q}]$ for the argument from Lemma 9.2 to work, and lift the result from $\Bbbk[Q]/I$ to $\Bbbk[Q]$. □

The two binomial ideals in Proposition 9.5 are unequal precisely when I contains no monomials, and in this case it is trivial to form the second ideal by inserting monomials. In special circumstances, it is possible to reverse this procedure. To this end, we wish to examine the transition from $\Bbbk[Q]$ to the truncated algebra $\Bbbk[Q]^-$ (when Q has a nil) in terms of primary decomposition of binomial ideals. This naturally leads to the following concept refining that of a nil.

Definition 9.6. A *kernel* of a commutative monoid Q is a nonempty ideal contained in all nonempty ideals of Q. (Such an ideal might not exist.)

Example 9.7. A nil is the same thing as a kernel of cardinality 1.

The existence of a nil in Q, or a finite kernel more generally, is reflected by a certain kind of maximal ideal of $\Bbbk[Q]$ being an associated prime of $\Bbbk[Q]$.

Definition 9.8. Fix a commutative monoid Q, and write $\Bbbk^* = \Bbbk \setminus \{0\}$. The *unital augmentation ideal* in the monoid algebra $\Bbbk[Q]$ is the ideal

$$I^1_{\text{aug}} := \langle \boldsymbol{t}^q - 1 \mid q \in Q \rangle$$

generated by all monomial differences. More generally, an *augmentation ideal* for a given binomial ideal $I \subseteq \Bbbk[Q]$ is a proper ideal of the form

$$I_{\text{aug}} := \langle \boldsymbol{t}^q - \lambda_q \mid q \in Q, \lambda_q \in \Bbbk^* \rangle \subseteq \Bbbk[Q]$$

such that $I \cap I_{\text{aug}}$ is a binomial ideal.

Example 9.9. The ideal $I = \langle x^2 \rangle \subset \Bbbk[x, y]$ induces a primary congruence (a Rees congruence) identifying all monomials in I. A compatible augmentation ideal is $I_{\text{aug}} = \langle x-1, y-1 \rangle$, which satisfies $I \cap I_{\text{aug}} = \langle x^2-x^3, yx^2-x^2 \rangle$. This intersection induces the same congruence $\sim$ as I does. Note that $\Bbbk[x, y]/(I \cap I_{\text{aug}}) \cong \Bbbk[\mathbb{N}^2/{\sim}]$ is isomorphic to the semigroup algebra of $\mathbb{N}^2/{\sim}$ while $\Bbbk[\mathbb{N}^2]/I \cong \Bbbk[\mathbb{N}^2/{\sim}]^-$ is the truncated algebra.

Lemma 9.10. *Given an augmentation ideal I_{aug} as in Definition 9.8, the association $q \mapsto \lambda_q$ constitutes a monoid homomorphism $\phi : Q \to \Bbbk^*$.*

Proof. The maximal ideals of $\Bbbk[Q]$ with residue field $\Bbbk$ are in bijection with the monoid homomorphisms $Q \to \Bbbk$; Definition 9.8 guarantees that the image lies in $\Bbbk^*$. □

Proposition 9.11. *Fix a monoid algebra $\Bbbk[Q]$ over a field $\Bbbk$, with Q finitely generated. An augmentation ideal is associated to $\Bbbk[Q]$ if and only if Q has a finite kernel, and in that case the unital augmentation ideal is associated to $\Bbbk[Q]$.*

Proof. If Q has a finite kernel K, then I^1_{aug} is the annihilator of the sum $f = \sum_{k \in K} \boldsymbol{t}^k$. Indeed, $q + K \subseteq K$ is an ideal of $Q \Longrightarrow q + K = K$ for all $q \in Q \Longrightarrow \boldsymbol{t}^q f = f$ for all $q \in Q \Longrightarrow (\boldsymbol{t}^q - 1) f = 0$ for all $q \in Q \Longrightarrow I^1_{\text{aug}} \subseteq \operatorname{ann}(f)$; but I^1_{aug} is a maximal ideal.

Now suppose that an augmentation ideal I_{aug} is associated to $\Bbbk[Q]$. The homomorphism $q \mapsto \lambda_q$ in Lemma 9.10 induces an automorphism of $\Bbbk[Q]$ that rescales the monomials by $\boldsymbol{t}^q \mapsto \lambda_q \boldsymbol{t}^q$. This automorphism takes I_{aug} to I^1_{aug}. Therefore, we may as well assume $I_{\text{aug}} = I^1_{\text{aug}}$ is the unital augmentation ideal. Let $K \subseteq Q$ be a nonempty subset such that $f = \sum_{k \in K} \mu_k \boldsymbol{t}^k$ is annihilated by I^1_{aug}, where $\mu_k \in \Bbbk^*$ for all $k \in K$. It suffices to show that K is a kernel of Q. But $\boldsymbol{t}^q f = f$ for all $q \in Q$ implies that $q + K = K$ for all $q \in Q$, which implies both that K is an ideal of Q (since $q + K \subseteq K$ for all q) and also that K is contained in every ideal of Q (since $K + q \supseteq K$). □

Theorem 9.12. *If $I_\ell \supset \cdots \supset I_0$ is a chain of distinct binomial ideals in $\Bbbk[Q]$ inducing the same congruence on Q, then $\ell \leq 1$. Moreover, if $\ell = 1$ then I_1 contains monomials and I_0 does not: $I_0 = I_1 \cap I_{\text{aug}}$ for an augmentation ideal I_{aug} compatible with I_1.*

Proof. The first sentence follows from Proposition 9.5, as does the statement about monomials when $\ell = 1$. It remains to show that $I_0 = I_1 \cap I_{\text{aug}}$ if $\ell = 1$. Set $I = I_0$. Under the grading of the quotient algebra $\Bbbk[Q]/I$ by $\bar{Q} = Q/{\sim_I}$, the dimension of the graded piece $(\Bbbk[Q]/I)_{\bar{q}}$ as a vector space over $\Bbbk$ is 1 for all $\bar{q} \in \bar{Q}$. Let $\overline{\infty} \in \bar{Q}$ be the nil, which exists because it is the class of all exponents on monomials in I_1. Fix a nonzero element $\boldsymbol{t}^{\overline{\infty}} \in \Bbbk[Q]/I$ of degree $\overline{\infty}$. Then $\boldsymbol{t}^q \boldsymbol{t}^{\overline{\infty}} = \lambda_q \boldsymbol{t}^{\overline{\infty}}$ for each

$q \in Q$. Set $I_{\mathrm{aug}} = \langle \mathbf{t}^q - \lambda_q \mid q \in Q \rangle$. Then $I_{\mathrm{aug}} \supseteq I$ by construction, but $I_{\mathrm{aug}} \not\supseteq I_1$, since I_1 contains monomials and I_{aug} does not. Therefore $I_1 \supsetneq I_1 \cap I_{\mathrm{aug}} \supseteq I$, whence $I_1 \cap I_{\mathrm{aug}} = I$, because $I_1/I = \langle \mathbf{t}^{\overline{\infty}} \rangle \subseteq \Bbbk[Q]/I$ has dimension 1 as a vector space over $\Bbbk$ by Proposition 9.5. □

Example 9.13. The ideal $I = \langle x^2 - xy, xy - 2y^2 \rangle \subseteq \Bbbk[x, y]$ contains monomials even when $\mathrm{char}(\Bbbk) \neq 2$, because I contains both of $x^2y - xy^2$ and $x^2y - 2xy^2$, so x^2y and xy^2 lie in I. However, Theorem 9.12 implies that there is no augmentation ideal compatible with I. Indeed, every binomial ideal I' contained in I and inducing the same congruence necessarily contains a binomial of the form $x^2 - \lambda xy$ and one of the form $xy - \mu y^2$, so I' contains both $x^2 - xy$ and $xy - 2y^2$ (and therefore $I' = I$) since $xy \notin I$.

10. Taxonomy of binomial ideals in monoid algebras

The concepts of primary, mesoprimary, primitive, prime, and toric congruence from Definition 2.12 have precise analogues for binomial ideals in monoid algebras. As a small measure to aid the reader with conflicting usages of the terms “primary” and “prime”, long since immovably set in the literature, the items in the following definition are listed in the order corresponding exactly to Definition 2.12, as Theorem 10.6 makes precise; for quick reference, consult the following table.

...congruence on Q	...binomial ideal in $\Bbbk[Q]$
primary	cellular
mesoprimary	mesoprimary
primitive	primary
prime	mesoprime
toric	prime

This table explains our choice of terminology: “mesoprimary” sits between the two occurrences of “primary”, being stronger than one and weaker than the other.

Our choice to work over fields that need not be algebraically closed forces us to consider slight generalizations of group algebras.

Definition 10.1. A *twisted group algebra* over a field $\Bbbk$ is a $\Bbbk$-algebra that is graded by a group G and, after tensoring with the algebraic closure $\overline{\Bbbk}$, is isomorphic to the group algebra $\overline{\Bbbk}[G]$ via a G-graded isomorphism. A *monomial homomorphism* from a monoid algebra to a twisted group algebra takes each monomial to a homogeneous element (possibly 0).

Example 10.2. The ring $R = \mathbb{Q}[x]/\langle x^3 - 2 \rangle$ is not isomorphic to the group algebra $\mathbb{Q}[G]$ for $G = \mathbb{Z}/3\mathbb{Z}$ over $\mathbb{Q}$, because no element of R is a cube root of 2. On the other hand, the element $y = x\sqrt[3]{2} \in R_{\mathbb{C}} := R \otimes_{\mathbb{Q}} \mathbb{C}$ generates $R_{\mathbb{C}}$, yielding the presentation $R_{\mathbb{C}} = \mathbb{C}[y]/\langle y^3 - 1\rangle \cong \mathbb{C}[G]$. Therefore R is a nontrivial twisted group algebra for the group $G = \mathbb{Z}/3\mathbb{Z}$ over the rational numbers $\mathbb{Q}$.

Generalizing the manipulations in Example 10.2 yields the following.

Proposition 10.3. *The twisted group algebras R over $\Bbbk$ (for a finitely generated group G) are precisely the quotients of Laurent polynomial rings over $\Bbbk$ by binomial ideals.*

Proof. Let R be a twisted group algebra. Every G-graded piece of R has dimension $\dim_{\Bbbk}(R_g) = 1$ for all $g \in G$, because this is true after tensoring with $\overline{\Bbbk}$ by definition. Thus R admits a binomial presentation $R \cong \Bbbk[\mathbb{N}^n]/I$ [Eisenbud and Sturmfels 1996, Proposition 1.11]. Every monomial $\boldsymbol{x}^u \in \Bbbk[\mathbb{N}^n]$ becomes invertible in R because every such monomial becomes invertible in $R\overline{\Bbbk} := R \otimes_{\Bbbk} \overline{\Bbbk}$. Therefore $R \cong \Bbbk[\mathbb{Z}^n]/I$ is a binomial quotient of a Laurent polynomial ring. On the other hand, the characterization of Laurent binomial ideals I [Eisenbud and Sturmfels 1996, Theorem 2.1] (or see Lemma 11.10, below) implies that there is a unique sublattice $L \subseteq \mathbb{Z}^n$ and character $\sigma : L \to \Bbbk$ such that $I = \langle \boldsymbol{x}^q - \sigma(q) \mid q \in L\rangle$. Over $\Bbbk$, not much more can be said, in general; but over $\overline{\Bbbk}$, the fact that $\overline{\Bbbk}^*$ is an injective abelian group implies that σ extends to a character $\rho : \mathbb{Z}^n \to \overline{\Bbbk}^*$. If y_i is the image in $R\overline{\Bbbk}$ of $\rho(x_i)x_i \in \overline{\Bbbk}[\mathbb{Z}^n]$, then naturally $R\overline{\Bbbk} = \overline{\Bbbk}[y_1, \ldots, y_n] = \overline{\Bbbk}[G]$ for $G = \mathbb{Z}^n/L$. □

Definition 10.4. A binomial ideal $I \subset \Bbbk[Q]$ in the monoid algebra for a monoid Q is

(1) *cellular* if every monomial $\boldsymbol{t}^q \in \Bbbk[Q]/I$ is either nilpotent or a nonzerodivisor;

(2) *mesoprimary* if it is maximal among the proper binomial ideals inducing a given mesoprimary congruence (as per Theorem 9.12);

(3) *primary* if the quotient $\Bbbk[Q]/I$ has precisely one associated prime ideal;

(4) *mesoprime* if I is the kernel of a monomial homomorphism from $\Bbbk[Q]$ to a twisted group algebra over $\Bbbk$;

(5) *prime* if $\Bbbk[Q]/I$ is an integral domain: $fg = 0$ in $\Bbbk[Q]/I$ implies $f = 0$ or $g = 0$.

Remark 10.5. The maximality for a mesoprimary ideal $I \subseteq \Bbbk[Q]$ amounts to stipulating that the nil class of $\sim_I$ consists of elements $q \in Q$ with $\boldsymbol{t}^q \in I$, the alternative being that none of these monomials lie in I but differences of scalar multiples thereof do.

Theorem 10.6. *For $\alpha \in \{1, 2, 4\}$, a binomial ideal I satisfies condition (α) of Definition 10.4 if and only if its induced congruence satisfies condition (α) of Definition 2.12 and I is maximal among proper ideals inducing that congruence. For $\alpha = 5$, the same holds if $\Bbbk$ is algebraically closed. For $\alpha = 3$, the condition in Definition 2.12 implies the one in Definition 10.4 in general, and the converse holds if $\Bbbk$ is algebraically closed of characteristic* 0.

Proof. Fix a binomial ideal I and use notation as in Definition 2.12 for $\sim = \sim_I$. We first assume that I satisfies Definition 10.4(α) and show that I satisfies Definition 2.12(α).

(1) If a monomial $\boldsymbol{t}^q \in \Bbbk[Q]/I$ is nilpotent or a nonzerodivisor then the image $\bar{q} \in \bar{Q}$ of q is nilpotent or cancellative, respectively.

(2) By definition.

(3) Pick a presentation $\mathbb{N}^n \twoheadrightarrow Q$. The kernel of the induced surjection $\Bbbk[\mathbb{N}^n] \twoheadrightarrow \Bbbk[Q]$ is a binomial ideal [Gilmer 1984, §7], so the preimage of I in $\Bbbk[\mathbb{N}^n]$ is a primary binomial ideal $I' \subseteq \Bbbk[\mathbb{N}^n]$ such that $\mathbb{N}^n/{\sim_{I'}} = \bar{Q}$. Replacing I by I' if necessary, we therefore may as well assume $Q = \mathbb{N}^n$, since the definitions of primitive congruence and primary ideal depend only on the quotients $\overline{\mathbb{N}^n} = \bar{Q}$ and $\Bbbk[\mathbb{N}^n]/I' = \Bbbk[Q]/I$.

Each binomial prime in $\Bbbk[\mathbb{N}^n] = \Bbbk[x_1, \dots, x_n]$ can be expressed as a sum $\mathfrak{p}_b + \mathfrak{m}_J \subseteq \Bbbk[\mathbb{N}^n]$ of its "binomial portion" $\mathfrak{p}_b$, which is a prime binomial ideal containing no monomials, and a monomial prime $\mathfrak{m}_J := \langle x_i \mid i \notin J \rangle$, which is generated by the variables whose indices are *not* contained in $J \subseteq \{1, \dots, n\}$ [Eisenbud and Sturmfels 1996, Corollary 2.6]; this deduction relies on the algebraically closed hypothesis. Rescaling the variables of $\Bbbk[\mathbb{N}^n]$ if necessary, we can assume that the unique associated prime $\mathfrak{p} = \mathfrak{p}_b + \mathfrak{m}_J$ of $\Bbbk[\mathbb{N}^n]/I$ is *unital*—that is, $\mathfrak{p}_b$ is a unital ideal. Given that $\Bbbk$ is algebraically closed of characteristic 0, the $\mathfrak{p}$-primary condition on I implies that it contains $\mathfrak{p}_b$ [Eisenbud and Sturmfels 1996, Theorem 7.1′]. Therefore, replacing $\Bbbk[\mathbb{N}^n]$ by $\Bbbk[\mathbb{N}^n]/\mathfrak{p}_b$ and I by $I/\mathfrak{p}_b$, we assume that Q is an affine semigroup and $\mathfrak{p}$ is generated by monomials. The desired result now follows from [Dickenstein et al. 2010b, Theorem 2.15 and Proposition 2.13] or [Miller 2011, Theorem 2.23], the latter being an equivalent statement that directly implies the characterization of mesoprimary congruences in Corollary 6.7.

(4) If $\bar{q}$ is not nil then $\boldsymbol{t}^q \in \Bbbk[Q]$ lies outside of I, so $\boldsymbol{t}^q$ maps to a nonzero monomial in the twisted group algebra, whence $\bar{q}$ is cancellative because G is cancellative.

(5) When I is a monomial prime in an affine semigroup ring, the result is obvious. But prime $\Longrightarrow$ primary, so the reduction to that case in part (3) applies.

Moreover, since $I = \mathfrak{p}$ contains $\mathfrak{p}_b$ already, the characteristic 0 hypothesis is superfluous.

For this half of the theorem, it remains to explain, for $\alpha \neq 2$, why I is maximal among ideals inducing $\sim$. For that, it suffices by Theorem 9.12 to show that I contains a monomial if $\overline{Q}$ has a nil ∞. For part (1) (the cellular case), if $\overline{q} = \infty$, then by definition of nil there is for each $r \in \mathbb{N}$ a binomial $\boldsymbol{t}^q - \lambda_r \boldsymbol{t}^{rq} \in I$ for some $\lambda_r \in \Bbbk^*$, so $\boldsymbol{t}^q(1 - \lambda_r \boldsymbol{t}^{(r-1)q}) \in I$, whence $\boldsymbol{t}^q$ is a zerodivisor modulo I and thus nilpotent modulo I—say $\boldsymbol{t}^{rq} \in I$; then $\boldsymbol{t}^q - \lambda_r \boldsymbol{t}^{rq} \in I \implies \boldsymbol{t}^q \in I$. For part (3) (the primary case), Theorem 9.12 implies that I has at least two associated primes—one or more arising from an augmentation ideal—if maximality fails. For part (4) (the mesoprime case), any monomial $\boldsymbol{t}^q$ with $\overline{q} = \infty$ must lie in I because a group has no nil. For part (5) (the prime case), the maximality is a special case of part (1), because prime $\implies$ cellular for binomial ideals.

Next, assuming that I is maximal among the binomial ideals inducing a congruence $\sim$ on Q satisfying Definition 2.12(α), we prove that I satisfies Definition 10.4(α). As a matter of notation, write $\bar{\boldsymbol{t}}^q$ for the image of $\boldsymbol{t}^q$ in $\Bbbk[Q]/I$. In all cases, if $q \in Q$ is an element whose image $\overline{q} \in \overline{Q}$ is nil, then $\bar{\boldsymbol{t}}^q = 0$ by Theorem 9.12, using the maximality property of I. Consequently, if $q \in Q$ is nilpotent, then $\bar{\boldsymbol{t}}^q$ is nilpotent in $\Bbbk[Q]/I$.

(1) By the previous paragraph, if $q \in Q$, then either the monomial $\bar{\boldsymbol{t}}^q$ is nilpotent or $\overline{q}$ is cancellative. In the latter case, multiplication by $\bar{\boldsymbol{t}}^q$ is injective on $\Bbbk[Q]/I$ because $\Bbbk[Q]/I$ is $\overline{Q}$-graded and addition by $\overline{q}$ is injective on $\overline{Q}$.

(2) By definition.

(3) The quotient $\overline{Q}$ satisfies the condition of Corollary 6.7 in which the cancellative monoid $F \subseteq \overline{Q}$ is an affine semigroup. Each orbit is a finite union of translates $\overline{q} + F$ because $\overline{Q}$ itself is generated by F and finitely many nilpotent elements. The proof now proceeds as in [Dickenstein et al. 2010b, Proposition 2.13]: owing to the partial order on the set of orbits afforded by Lemma 2.19, the $\overline{Q}/F$-grading on $\Bbbk[Q]/I$ induces a filtration by $\Bbbk[Q]$-submodules with associated graded module

$$\operatorname{gr}(\Bbbk[Q]/I) \cong \bigoplus_{F\text{-orbits } T} \Bbbk\{T\},$$

where $\Bbbk\{T\}$ is the vector space over $\Bbbk$ with basis T. The isomorphism above is as $\Bbbk[F]$-modules, or equivalently, as $\Bbbk[Q]$-modules annihilated by the kernel $\mathfrak{p}_F$ of the surjection $\Bbbk[Q] \twoheadrightarrow \Bbbk[F]$, with the $\Bbbk[F]$-module structure on $\Bbbk\{T\}$ induced by the F-action on T. Since $\Bbbk\{T\}$ is torsion-free as a $\Bbbk[F]$-module, the direct sum over T has only one associated prime, namely $\mathfrak{p}_F$, whence $\Bbbk[Q]/I$ does too.

(4) Set $\bar{Q}' = \bar{Q} \setminus \{\infty\}$ if $\bar{Q}$ has a nil, and $\bar{Q}' = \bar{Q}$ otherwise. By maximality of I, the quotient $\Bbbk[Q]/I$ is $\bar{Q}'$-graded. By part (1), every nonzero monomial $\bar{t}^q \in \Bbbk[Q]/I$ is a nonzerodivisor. Therefore $\Bbbk[Q]/I$ injects into its localization R obtained by inverting the nonzero monomials. Any presentation $\mathbb{Z}^n \twoheadrightarrow G$ for the universal group G of Q results in a presentation $\Bbbk[\mathbb{Z}^n] \twoheadrightarrow \Bbbk[G] \twoheadrightarrow \Bbbk[G]/I = R$ whose kernel is a binomial ideal. Thus R is a twisted group algebra over $\Bbbk$ by Proposition 10.3.

(5) The argument for part (4) works in this case, too, but now $\bar{Q}'$ is an affine semigroup, so that $\bar{\Bbbk} \otimes_{\Bbbk} R$, and hence also $\Bbbk[Q]/I$, are integral domains. □

Corollary 10.7. *For binomial ideals in* $\Bbbk[Q]$, *over an arbitrary field except where noted,*

- *prime* $\Longrightarrow$ *mesoprime* $\Longrightarrow$ *mesoprimary* $\Longrightarrow$ *cellular; and*
- *prime* $\Longrightarrow$ *primary* $\Longrightarrow$ *mesoprimary* $\Longrightarrow$ *cellular (we only claim the second implication when* $\Bbbk$ *is algebraically closed of characteristic* 0*).*

Proof. Use Theorem 10.6: if I is maximal among binomial ideals inducing a congruence from Definition 2.12, then it is maximal among binomial ideals inducing any of the weaker congruences from Lemma 2.14. This proves every implication except for prime $\Longrightarrow$ mesoprime, which a priori requires $\Bbbk$ to be algebraically closed, if Theorem 10.6 is being applied. But in fact the implication holds in general, even though the quotient by a prime binomial ideal I need not be an affine semigroup ring if $\Bbbk$ is not algebraically closed. This is a consequence of the stronger statement in Theorem 11.14, below. □

Example 10.8. In general a primary ideal need not be mesoprimary. For instance, $\langle 1 - x^p, y - xy, y^2\rangle$ is primary in characteristic p, but the congruence it induces has two associated prime congruences regardless of the characteristic.

Remark 10.9. The given proof of the implication Definition 10.4(3) $\Longrightarrow$ Definition 2.12(3) fails in characteristic p, even if the field $\Bbbk$ is algebraically closed, because primary binomial ideals in characteristic p do not necessarily contain the binomial part of their associated prime [Eisenbud and Sturmfels 1996, Theorem 7.1′].

Theorem 10.6 implies the following result, which reflects the table on page 1332 homogeneously across all of its rows, and shows that all of the richness in Definition 10.4 is already exhibited by *unital ideals*: those generated by monomials and unital binomials.

Corollary 10.10. *A congruence satisfies a part of Definition 2.12 if and only if the kernel of the surjection* $\Bbbk[Q] \twoheadrightarrow \Bbbk[\bar{Q}]^-$ *satisfies the corresponding part of Definition 10.4.* □

11. Monomial localization, characters, and mesoprimes

For arithmetic reasons, intersections of binomial ideals need not reflect their combinatorics completely accurately. The simplest example is $\langle x^2-1\rangle = \langle x-1\rangle \cap \langle x+1\rangle$, whose congruence fails to equal the common refinement of the congruences induced by $\langle x-1\rangle$ and $\langle x+1\rangle$. Precise statements about relations between combinatorics and arithmetic use characters on subgroups of the unit groups of localizations of Q.

Localizations of monoids at their prime ideals corresponds to inverting monomials in their monoid algebras.

Definition 11.1. For a prime ideal $P \subset Q$, the corresponding monomial ideal in $\Bbbk[Q]$ is $\mathfrak{m}_P = \langle \boldsymbol{t}^p \mid p \in P\rangle$.

Remark 11.2. When P is maximal, $\mathfrak{m}_P$ is the maximal proper Q-graded ideal in the monoid algebra $\Bbbk[Q]$, but it need not be maximal in the set of all proper ideals of $\Bbbk[Q]$.

Definition 11.3. The *monomial localization* $\Bbbk[Q]_P$ *of* $\Bbbk[Q]$ *along* P is the monoid algebra of the localization Q_P, arising by adjoining inverses to all monomials outside of $\mathfrak{m}_P$. The *monomial localization* of any $\Bbbk[Q]$-module M *along* P is $M_P = M \otimes_{\Bbbk[Q]} \Bbbk[Q]_P$.

Localization behaves well upon passing between algebra and combinatorics; it forms part of the justification for characterizing algebraic notions, such as the concept of I-witness in the next section, in combinatorial terms.

Lemma 11.4. *If $I \subseteq \Bbbk[Q]$ is a binomial ideal inducing the congruence $\sim$ on Q with quotient $\bar{Q}$, then for any monoid prime $P \subset Q$, the quotient of Q_P modulo the congruence induced by I_P is the monoid localization $\bar{Q}_P$ from Definition 3.9.*

Proof. Immediate from the definitions. □

Definition 11.5. For any group L, a *character* is a homomorphism $\rho : L \to \Bbbk^*$. A character $\rho' : L' \to \Bbbk^*$ *extends* ρ if $L \subseteq L'$ is a subgroup and $\rho'(\ell) = \rho(\ell)$ for $\ell \in L$. The extension is *finite* if L'/L is finite.

Convention 11.6. The domain L is part of the data of a character $\rho : L \to \Bbbk^*$; that is, we simply speak of the character ρ, and write L_ρ if it is necessary to specify L.

Definition 11.7. Fix a subgroup $K \subseteq G_P$ of the unit group G_P at P. For any character $\rho : K \to \Bbbk^*$, the *P-mesoprime of* ρ is the preimage $I_{\rho,P}$ in $\Bbbk[Q]$ of the ideal

$$(I_{\rho,P})_P := \langle \boldsymbol{t}^u - \rho(u-v)\boldsymbol{t}^v \mid u-v \in K\rangle + \mathfrak{m}_P \subseteq \Bbbk[Q]_P.$$

Viewing P as implicit in the definition of ρ, the symbol I_ρ refers to the preimage in $\Bbbk[Q]$ of the ideal $\langle \boldsymbol{t}^u - \rho(u-v)\boldsymbol{t}^v \mid u-v \in K\rangle \subseteq \Bbbk[Q]_P$.

Definition 11.8. A subgroup $L \subseteq G$ of an abelian group is *saturated* in G if there is no subgroup of G in which L is properly contained with finite index. The *saturation* $\mathrm{sat}(L)$ of L is the intersection of all saturated subgroups of G that contain L. For any prime number $p \in \mathbb{N}$, the largest subgroup of $\mathrm{sat}(L)$ whose quotient modulo L has order

- a power of p is denoted $\mathrm{sat}_p(L)$,
- coprime to p is denoted $\mathrm{sat}'_p(L)$.

For $p = 0$, set $\mathrm{sat}_p(L) = L$ and $\mathrm{sat}'_p(L) = \mathrm{sat}(L)$.

The following implies, in particular, that the set of saturations of a character is finite. The statement is actually a slight generalization of [Eisenbud and Sturmfels 1996, Corollary 2.2], in that the domain L of ρ is allowed to be a subgroup of an arbitrary finitely generated abelian unit group G_P, and $I_{\rho,P}$ is not an arbitrary ideal in a finitely generated group algebra, but rather an ideal containing the maximal monomial ideal in an arbitrary finitely generated monoid algebra. However, the generalization follows from the original by working modulo the maximal monomial ideal and lifting to any presentation of G_P, taking note that all of the characters in question are trivial on the kernel of the presentation.

Proposition 11.9 [Eisenbud and Sturmfels 1996, Corollary 2.2]. *Fix an algebraically closed field* $\Bbbk$ *of characteristic* $p \geq 0$. *Let* $\rho : L \to \Bbbk^*$ *be a character on a subgroup* $L \subseteq G_P$, *and write* g *for the order of* $\mathrm{sat}'_p(L)/L$. *There are* g *distinct characters* $\rho_1, \ldots, \rho_g$ *on* $\mathrm{sat}'_p(L)$ *that extend* ρ. *For each* ρ_j *there is a unique character* ρ'_j *on* $\mathrm{sat}(L)$ *extending* ρ_j. *There is a unique character* ρ' *that extends* ρ *and is defined on* $\mathrm{sat}_p(L)$. *Moreover*,

(1) $\sqrt{I_{\rho,P}} = I_{\rho',P}$,

(2) $\mathrm{Ass}(S/I_{\rho,P}) = \{I_{\rho'_j,P} \mid j = 1, \ldots, g\}$, *and*

(3) $I_{\rho,P} = \bigcap_{j=1}^{g} I_{\rho_j,P}$.

The following lemma is a variant of [Dickenstein et al. 2010a, Lemma 2.9] and [Eisenbud and Sturmfels 1996, Theorem 2.1].

Lemma 11.10. *If* $\Bbbk[\Phi]$ *is the group algebra of a finitely generated abelian group* Φ, *then for any proper binomial ideal* $I \subset \Bbbk[\Phi]$ *there is a subgroup* $L \subseteq \Phi$ *and a character* $\rho : L \to \Bbbk^*$ *such that* $I = I_\rho$.

Proof. The binomial ideal is of the form $\langle 1 - \lambda_u \boldsymbol{t}^u \mid u \in \mathcal{U}\rangle$ for some finite $\mathcal{U} \subseteq \Phi$. First off, $\mathcal{U}$ is a subgroup of Φ since $1 - \lambda\mu\boldsymbol{t}^{u+v} = \mu\boldsymbol{t}^v(1 - \lambda\boldsymbol{t}^u) + (1 - \mu\boldsymbol{t}^v)$ for all $\lambda, \mu \in \Bbbk$, including $\lambda = \lambda_u$ and $\mu = \lambda_v$. The set $\mathcal{U}$ is closed under inverses because $(1 - \lambda\boldsymbol{t}^u)/\lambda\boldsymbol{t}^u = -(1 - \boldsymbol{t}^{-u}/\lambda)$ when $\lambda \neq 0$, and $I \neq \Bbbk[\Phi] \Longrightarrow \lambda_u \neq 0$. The very same arguments show that the map $\rho : \mathcal{U} \to \Bbbk^*$ defined by $u \mapsto \lambda_u$ is a homomorphism. □

Definition 11.11. Fix a binomial ideal $I \subseteq \Bbbk[Q]$.

(1) The *stabilizer* of an element $q \in Q$ along a prime ideal $P \subset Q$ is the subgroup $K_q^P \subseteq G_P$ (sometimes denoted by K_q if P is clear from context) fixing the class of $q \in Q_P$ under the action from Lemma 3.11 for the congruence $\sim_I$.

(2) For $\mathbf{t}^q \notin I_P$, the *character (of I_P) at q* is the homomorphism $\rho = \rho_q^P : K_q^P \to \Bbbk^*$ such that the $\Bbbk[G_P]$-module map $\Bbbk[G_P] \to \Bbbk[Q_P]/I_P$ taking $1 \mapsto \mathbf{t}^q$ has kernel I_ρ.

(3) The ideal $I_q^P := I_{\rho,P} \subseteq \Bbbk[Q]$ is the *P-mesoprime of I at q*.

Remark 11.12. The homomorphism $\Bbbk[G_P] \to \Bbbk[Q_P]/I_P$ in Definition 11.11(2) has kernel of the form I_ρ by Lemma 11.10. Indeed, the kernel is a priori the binomial ideal $(I_P : \mathbf{t}^q) \cap \Bbbk[G_P]$, which is not the unit ideal in $\Bbbk[G_P]$ because $\mathbf{t}^q$ lies outside of I_P.

Saturations of subgroups (Definition 11.8) are more or less combinatorial in nature. Saturations of characters, on the other hand, are more subtle, because arithmetic properties of the target field $\Bbbk$ can enter.

Definition 11.13. Fix a subgroup L of an abelian group G. A character $\rho : L \to \Bbbk^*$ is

- *saturated* if the subgroup L is saturated, and
- *arithmetically saturated* if ρ has no finite proper extensions.

A *saturation* of ρ is an extension of ρ to $\mathrm{sat}(L)$.

The importance of saturated characters has been demonstrated in Proposition 11.9, which required the algebraically closed hypothesis. Without it, the arithmetically saturated condition holds sway, and equivalence of primality with saturation can break.

Theorem 11.14. *If a binomial ideal in $\Bbbk[Q]$ over an arbitrary field $\Bbbk$ is prime then it is a mesoprime $I_{\rho,P}$ for an arithmetically saturated character ρ. The converse holds if $\Bbbk$ is algebraically closed, and it can fail if not.*

Proof. Suppose that $\Bbbk[Q]/I$ is a domain. The ideal of monoid elements $p \in Q$ such that $\mathbf{t}^p \in I$ is a monoid prime P. Replacing Q with the monoid $Q \setminus P$ and I with its image in $\Bbbk[Q \setminus P] = \Bbbk[Q]/\langle \mathbf{t}^p \mid p \in P\rangle$, it suffices to prove that $I = I_\rho$ for an arithmetically saturated character when Q is cancellative and I contains no monomials. Since $\Bbbk[Q]$ injects into its localization $\Bbbk[Q]_\varnothing = \Bbbk[\Phi]$ for the universal group $\Phi = Q_\varnothing$, and I contains no monomials, Lemma 11.10 implies the existence of a subgroup $L \subseteq \Phi$ and a character $\rho : L \to \Bbbk^*$ such that $I = I_\rho$. It remains to show that I_ρ is not prime if ρ is not arithmetically saturated. Suppose $\sigma : K \to \Bbbk^*$ properly extends ρ to a subgroup $K \subseteq \mathrm{sat}(L)$. Then $I_\sigma \supsetneq I_\rho$. By restricting σ to a subgroup of K that still properly contains L, we can assume that $|K/L| > 1$ and one of the following occurs:

- $\Bbbk$ has positive characteristic p and $|K/L|$ is a power of p.
- $\Bbbk$ has positive characteristic p and $|K/L|$ is relatively prime to p.
- $\Bbbk$ has characteristic 0.

Proposition 11.9 implies that in the first case, the extension $\bar{I}_\sigma$ of I_σ to $\bar{\Bbbk}$ has the same radical as the extension $\bar{I}_\rho$, in which case I_ρ itself is not a radical ideal. In the remaining two cases, Proposition 11.9 implies that $\bar{I}_\rho = \bar{I}_\sigma \cap \bar{J}$, with no associated prime of either intersectand containing an associated prime of the other. It follows that $I_\rho = I_\sigma \cap J$, where I_σ and $J := (I_\rho \mid I_\sigma)$ both properly contain I_ρ, so I_ρ is not prime.

The $\Bbbk = \bar{\Bbbk}$ converse is implicit in Proposition 11.9, and anyway follows easily from [Eisenbud and Sturmfels 1996, Theorem 2.1]. Example 11.15 demonstrates failure of the general converse. □

Example 11.15. The ideal $I_\rho \subset \mathbb{Q}[x]$ for the character $\rho : 4\mathbb{Z} \to \mathbb{Q}^*$ defined by $\rho(4) = -4$ is $\langle x^4 + 4\rangle$. This ideal is not prime because it factors as $\langle x^4 + 4\rangle = \langle x^2 - 2x + 2\rangle \cap \langle x^2 + 2x + 2\rangle$. Nonetheless, ρ is arithmetically saturated because $x^4 + 4$ has no binomial factors of degree 2.

Example 11.16. The ideal $\langle x^3 - 2\rangle$ in Example 10.2 is prime (by Eisenstein's criterion, for example). Therefore the character $\rho : 3\mathbb{Z} \to \mathbb{Q}^*$ sending $3 \mapsto 2$ is arithmetically saturated, viewing $3\mathbb{Z}$ as a subgroup of $\mathbb{Z}$: any proper extension of ρ to a character $\mathbb{Z} \to \mathbb{Q}^*$ would require a cube root of 2.

12. Coprincipal and mesoprimary components of binomial ideals

Definition 12.1. Fix a binomial ideal $I \subseteq \Bbbk[Q]$ inducing a congruence $\sim$ on Q.

(1) An element $w \in Q$ is an *I-witness* for a monoid prime P if it is a $\sim$-witness for P or if $P = \varnothing$ is the empty monoid ideal and I contains no monomials.

(2) An element $w \in Q$ is an *essential* I-witness for a monoid prime P if w is a key $\sim_I$-witness or some polynomial annihilated by $\mathfrak{m}_P$ in $\Bbbk[Q_P]/I_P$ (Definitions 11.1 and 11.3) has $\mathbf{t}^w$ minimal (under Green's preorder) among its nonzero monomials.

(3) If $I_{\rho,P}$ is the P-mesoprime of I (Definition 11.11) at some I-witness w for P, then w is an *I-witness for $I_{\rho,P}$*.

(4) $I_{\rho,P}$ is an *associated mesoprime* of I if there is an essential I-witness for $I_{\rho,P}$.

Lemma 12.2. *Every essential I-witness for P is an I-witness for P.*

Proof. Assume that $f \in \Bbbk[Q]$ such that $\mathfrak{m}_P f \subseteq I_P$. Let $m = \lambda \mathbf{t}^w$ be a term of f (that is, a nonzero constant times a monomial) minimal under Green's preorder on Q_P restricted to the terms of f. Fix a nonunit monoid element $p \in Q_P$.

Since $\boldsymbol{t}^p f \in I_P$, the term $\boldsymbol{t}^p m$ must equal, modulo I_P, some sum of terms whose monomials $\boldsymbol{t}^{p+a}$ have $\boldsymbol{t}^a$ appearing with nonzero coefficient in f. It follows that $\boldsymbol{t}^p m$ shares its $\bar{Q}_P$-graded degree with at least one of these monomials $\boldsymbol{t}^{p+a}$, where $\bar{Q} = Q/{\sim_I}$. Thus w is a witness by minimality of m: at least one of the elements a is an aide for w and p. □

Example 12.3. If $I = \langle y - x^2 y,\, y^2 - xy^2,\, y^3\rangle$ is the binomial ideal from Example 2.17(5) then $I_{\rho,P} = \langle x^2 - \lambda,\, y\rangle$ for $P = \langle e_y\rangle$ and $\rho : \langle(2,0)\rangle \to \Bbbk^*$ defined by $\rho(2,0) = \lambda$ induces the associated prime congruence of $\sim_I$ for any $\lambda \in \Bbbk^*$. The monomial $x^a y \in \Bbbk[x, y]$ is a witness for any $a \in \mathbb{N}$, and it lies in one of two possible essential witness classes, depending on the parity of a; see the figure in Example 2.17. However, only $\lambda = 1$ gives the associated mesoprime itself, as opposed to merely inducing its congruence.

Lemma 12.4. *Every binomial ideal in $\Bbbk[Q]$ has only finitely many essential witnesses.*

Proof. Theorem 5.6 takes care of key witnesses, so it is enough to treat witnesses arising from annihilation by $\mathfrak{m}_P$. As Q has finitely many prime ideals, it suffices to bound the number of essential witnesses for a fixed prime ideal P. By definition, $\mathfrak{m}_P$ annihilates the $\Bbbk[Q_P]$-submodule of $\Bbbk[Q_P]/I_P$ consisting of polynomials giving rise to essential I-witnesses. Hence the $\Bbbk[Q_P]$-submodule in question is finitely generated over $\Bbbk[G_P] = \Bbbk[Q_P]/\mathfrak{m}_P$, so only finitely G_P-orbits of (exponents on) monomials are involved. □

Remark 12.5. All associated mesoprimes of a unital binomial ideal (generated by differences of monomials with unit coefficients) are unital.

Remark 12.6. When I contains no monomials, every monomial is an essential I-witness for the empty monoid ideal $\varnothing \subset Q$. The condition that $I_{\rho,\varnothing}$ be an associated mesoprime of I for some (unique) character ρ is similar to the condition that $\varnothing$ be associated to the congruence $\sim$ induced by I, but it is not equivalent. These conditions differ only when I is minimal and not maximal among binomial ideals inducing $\sim$ (see Theorem 9.12) — that is, when $\sim$ has a nil class but I nonetheless contains no monomials — in which case I has an associated mesoprime $I_{\rho,\varnothing}$ but $\varnothing$ is not associated to $\sim$.

Lemma 12.7. *If w is an I-witness for $I_{\rho,P}$, then the localization along P of the P-mesoprime I_w^P of I at w satisfies $(I_w^P)_P = (I_{\rho,P})_P = (I_P : \boldsymbol{t}^w) + \mathfrak{m}_P$.*

Proof. The first equality is by Definition 11.11. For the second, use Theorem 7.11, which implies that I and $I_P + \boldsymbol{t}^w \mathfrak{m}_P$ have the same P-mesoprime at w. It follows that the natural isomorphism $\Bbbk[G_P] \to \Bbbk[Q_P]/\mathfrak{m}_P$ induced by the inclusion $\Bbbk[G_P] \to \Bbbk[Q_P]$ descends to an isomorphism

$$\Bbbk[G_P]/(I_P : \boldsymbol{t}^w) \cap \Bbbk[G_P] \to \Bbbk[Q_P]/((I_P : \boldsymbol{t}^w) + \mathfrak{m}_P).$$

Now apply Remark 11.12. □

Remark 12.8. If $Q = \mathbb{N}^n$ and I is unital, then all information about associated mesoprimes is contained in the set of *associated lattices* $L \subset \mathbb{Z}^J$, each of which comes with an *associated subset* $J \subseteq \{1, \ldots, n\}$. Indeed, a prime ideal P of $\mathbb{N}^n$ is the complement of a face $\mathbb{N}^J$ of $\mathbb{N}^n$, and specifying a prime congruence on $\mathbb{N}^n$ amounts to choosing such a face along with a lattice $L \subset \mathbb{Z}^J$. To see why, first observe that localization along P inverts the face, turning $\mathbb{N}^n$ into $\mathbb{Z}^J \times \mathbb{N}^{\bar{J}} = G_P \times \mathbb{N}^{\bar{J}}$. Subsequently passing to the quotient by a given prime congruence, the complement of the face maps to nil, and the subgroup L is the stabilizer of any class under the action of $\mathbb{Z}^J = G_P$ on the quotient. We were led to associated lattices (before the more general associated prime congruences) in part by [Eisenbud and Sturmfels 1996, Theorem 8.1]. Although that theorem only covers cellular cases, the upshot is that a collection of associated lattice ideals contributes associated primes.

Remark 12.9. When the domain K of a character $\rho : K \to \Bbbk^*$ is a saturated subgroup of G_P, the ideal $I_{\rho,P}$ is often an associated prime of a binomial ideal I without being an associated mesoprime of I. The reason is that the congruences induced by associated P-mesoprimes are immediately visible in the congruence induced by I_P, whereas the associated primes of I usually induce coarser congruences (larger congruence classes) than those visible. The quintessential example to consider is the lattice ideal I for an unsaturated sublattice of $\mathbb{Z}^n$: the lattice ideal for the saturation is an associated prime of I, but the unique associated mesoprime of I is I itself.

Proposition 12.10. *A binomial ideal $I \subseteq \Bbbk[Q]$ is mesoprimary if and only if I has exactly one associated mesoprime.*

Proof. If I is mesoprimary then it is cellular by Corollary 10.7 and the congruence $\sim_I$ is mesoprimary by Definition 10.4. If w is any witness (essential or not) for the unique associated prime congruence and $I' = (I : \mathbf{t}^w)$ is the annihilator of the image of $\mathbf{t}^w$ in $\Bbbk[Q]/I$, then multiplication by $\mathbf{t}^w$ induces an isomorphism $I_P + \mathfrak{m}_P \to I'_P + \mathfrak{m}_P$, so every associated mesoprime of I is equal to $I + \mathfrak{m}_P$.

On the other hand, assume that I has only one associated mesoprime, and that its associated monoid prime is $P \subset Q$. The congruence $\sim$ induced by I is mesoprimary by Lemma 4.9 and Theorem 6.1. Either I contains a monomial, in which case it is already maximal among ideals inducing its congruence by Theorem 9.12, or else I contains no monomials, in which case the unique associated monoid prime ideal is $P = \varnothing$ by definition. When $P = \varnothing$, if I is not maximal then $\sim$ has a witness for some monoid prime ideal other than $\varnothing$ by Remark 12.6, as $\sim$ has an associated monoid prime but $\varnothing$ is not one of them. Thus uniqueness of the associated mesoprime implies maximality. □

Remark 12.11. Building on Remark 6.8, Proposition 12.10 says that the character of I_P is the same at every nonzero monomial as soon as it is the same at every essential witness monomial, and that is what it means to be a mesoprimary ideal.

Definition 12.12. Given a monoid prime $P \subset Q$, a mesoprimary binomial ideal in $\Bbbk[Q]$ is P*-mesoprimary* if the associated prime ideal of its induced congruence is P.

The principal use of the following definition, which builds on the notion of order ideal from Definition 7.4, concerns the case where the set $\boldsymbol{w}$ consists of a single witness. The more general case arises during the construction of mesoprimary decompositions with as few components as possible (Corollary 13.5).

Definition 12.13. Fix a binomial ideal $I \subseteq \Bbbk[Q]$, a prime $P \subset Q$, and a finite subset $\boldsymbol{w} \subseteq Q$. The *monomial ideal* $M^P_{\boldsymbol{w}}(I) \subseteq \Bbbk[Q]$ *cogenerated by* $\boldsymbol{w}$ *along* P is generated by the monomials $\boldsymbol{t}^u \in \Bbbk[Q]$ such that u lies outside of the order ideal $Q^P_{\preceq w}$ cogenerated by w at P (Definition 7.4) under the congruence $\sim_I$ for all $w \in \boldsymbol{w}$.

Definition 12.14. Fix a binomial ideal $I \subseteq \Bbbk[Q]$ and a finite set $\boldsymbol{w} \subseteq Q$ such that the P-mesoprime I^P_w of I at w is $I_{\rho,P}$ for all $w \in \boldsymbol{w}$. The P*-mesoprimary component of* I *cogenerated by* $\boldsymbol{w}$ is the preimage $W^P_{\boldsymbol{w}}(I)$ in $\Bbbk[Q]$ of the ideal $I_P + I_\rho + M^P_{\boldsymbol{w}}(I) \subseteq \Bbbk[Q]_P$.

Remark 12.15. Comparing to Definition 7.7, adding $M^P_{\boldsymbol{w}}(I)$ in Definition 12.14 joins all pairs of elements in $Q \setminus Q^P_{\preceq q}$, while adding I_ρ joins the pairs $(a, b) \in Q$ satisfying conditions (i) and (ii) in Definition 7.7.

Definition 12.16. A *cogenerator* of a mesoprimary binomial ideal $I \subseteq \Bbbk[Q]$, or of the quotient $\Bbbk[Q]/I$, is a monoid element that is a cogenerator of the induced congruence. A *monomial cogenerator* is a monomial in $\Bbbk[Q]$ whose exponent is a cogenerator.

The nomenclature in Definition 12.14 is justified by the following result, which arithmetizes the combination of Theorem 7.11 and Lemma 8.10.

Proposition 12.17. *If* $\boldsymbol{w}$ *consists of* I*-witnesses for* P*, then the ideal* $W^P_{\boldsymbol{w}}(I)$ *in Definition 12.14 is mesoprimary with associated mesoprime* $I_{\rho,P}$*. Moreover, if* I *induces* $\sim$ *on* Q*, then* $W^P_{\boldsymbol{w}}(I)$ *induces the common refinement of the coprincipal components* $\sim^P_w$ *cogenerated by the elements in* $\boldsymbol{w}$ *along* P*. Every cogenerator of* $W^P_{\boldsymbol{w}}(I)$ *lies in* $\boldsymbol{w}$*.*

Proof. The claim has little content if $P = \varnothing$, as then $I_{\rho,P} = I_\rho = I_P$, so assume $P \neq \varnothing$. Since $W^P_{\boldsymbol{w}}(I)$ contains monomials by definition, it suffices by Theorem 9.12 to verify that $W^P_{\boldsymbol{w}}(I)$ induces the common refinement $\approx$ of coprincipal congruences in question, given that $\approx$ is mesoprimary by Theorem 7.11 and Proposition 6.9.

By construction (specifically, Definition 7.7; see also Remark 12.15), the mesoprimary congruence $\approx$ refines the congruence $\approx'$ induced by $W^P_{\boldsymbol{w}}(I)$: the monomial ideal $M_{\boldsymbol{w}}^P(I)$ sets all elements outside of the order ideal equivalent to one another, and the generators of I_ρ carry out the remaining required identifications. The harder direction is showing that no more relations are introduced.

Since $W_{\boldsymbol{w}}^P(I)$ is obtained from an extension to the localization $\Bbbk[Q]_P$ along P, we may as well assume that $Q = Q_P$, so P is the maximal ideal of Q. The congruences induced by I and I_ρ each individually refine the congruence $\approx$ (not to be confused with $\approx'$ here); for I this is by Theorem 8.4, and for I_ρ this is by Corollary 6.7 (see also Remark 6.8). Therefore both I and I_ρ are ideals graded by $Q/\!\approx$. We deduce that $W_{\boldsymbol{w}}^P(I)$ is graded by $Q/\!\approx$ as well, since $M_{\boldsymbol{w}}^P(I)$ is a monomial ideal and hence is automatically graded by $Q/\!\approx$. Consequently, each nonnil congruence class of $\approx'$ is contained in some congruence class of $\approx$.

It remains to treat the nil class of $\approx'$. Assuming $a \in Q$ with $\boldsymbol{t}^a \notin M_{\boldsymbol{w}}^P(I)$, it suffices to show $\boldsymbol{t}^a \notin W_{\boldsymbol{w}}^P(I)$. Choose $w \in \boldsymbol{w}$ with a in the order ideal $Q_{\preceq w}^P = Q_{\preceq w}^P(\sim)$, which can be done by definition of $M_{\boldsymbol{w}}^P(I)$. Next pick $u \in Q$ such that the images of $u+a$ and w in $Q/\!\approx$ are Green's-equivalent to one another; this is possible by definition of the order ideal $Q_{\preceq w}^P$. Use a double bar to denote passage from Q to $Q/\!\approx$, so $\overline{\overline{q}} \in Q/\!\approx$ is the image of q for any $q \in Q$. The choice of the character ρ was made precisely so that the graded piece $(I)_{\overline{\overline{q}}}$ of the ideal I contains the graded piece $(I_\rho)_{\overline{\overline{q}}}$ whenever $\overline{\overline{q}}$ is Green's-equivalent to $\overline{\overline{w}}$ in $Q/\!\approx$. This means that I_ρ adds no new relations to I in degree $\overline{\overline{q}}$. Since $M_{\boldsymbol{w}}^P(I)$ adds no new relations to I in degree $\overline{\overline{q}}$ by definition, $W_{\boldsymbol{w}}^P(I)_{\overline{\overline{q}}} = (I)_{\overline{\overline{q}}}$ for $q = u+a$. The class of $u+a$ is not nil in $Q/\!\sim$ because the character of I_P at $u+a$ is ρ. Hence $\boldsymbol{t}^a \notin W_{\boldsymbol{w}}^P(I)$.

The final claim of the Proposition follows from Lemma 8.10. □

Definition 12.18. A binomial ideal is *coprincipal* if it is mesoprimary and its induced congruence is coprincipal. A *coprincipal component* $W_q^P(I)$ *of* I *cogenerated by* q *at* P is a P-mesoprimary component $W_{\{q\}}^P(I)$ cogenerated by a single element q.

Corollary 12.19. *If* $I \subseteq \Bbbk[Q]$ *is a binomial ideal and* w *is an* I*-witness for* P*, then the coprincipal component of* I *cogenerated by* w *at* P *is a coprincipal binomial ideal.*

Proof. Immediate from Proposition 12.17 and the definitions. □

Remark 12.20. It would be superb if intersecting any pair of mesoprimary ideals with the same associated mesoprime resulted in another mesoprimary ideal. More precisely, a direct binomial ideal analogue of Proposition 6.9 would be desirable. Unfortunately, the binomial analogue is false in general: in $\Bbbk[x, y]$, the intersection of the mesoprimary ideals $\langle x-2y\rangle + \langle x, y\rangle^3$ and $\langle x-y\rangle + \langle x, y\rangle^3$ is not mesoprimary when $\mathrm{char}(\Bbbk) \neq 2$; it is not even a binomial ideal. Heuristically, if I_1 and I_2 are mesoprimary ideals in $\Bbbk[Q_P]$ with associated mesoprime $I_{\rho,P}$, then in each of I_1 and I_2 there are "vertical" binomials from I_ρ, whose coefficients are dictated by the character ρ, and "horizontal" binomials conglomerating the vertical fibers, with more arbitrary coefficients. (The vertical and horizontal directions in Examples 1.3 and 2.17 are reversed for aesthetic reasons; the usage here makes sense in Examples 4.15, 4.16, 8.12, 9.1, and 17.5.) When the horizontal coefficients from I_1 and I_2 conflict, the intersection need not be binomial.

That said, the analogue of Proposition 6.9 is true once control is granted over binomiality, and that comes for free when I_1 and I_2 both arise from a single ideal via sets of witnesses as in Proposition 12.17. In that sense, the binomial analogue of Proposition 6.9 is "true enough" for the relevant aspects of the theory of mesoprimary decomposition to succeed, namely Corollary 13.5.

Remark 12.21. The existence of a mesoprimary ideal inducing a given congruence is automatic by Remark 2.16. However, the question becomes more subtle when a given associated mesoprime other than the unital one is desired. Roughly speaking, we do not know how to construct mesoprimary ideals with given associated mesoprimes de novo, although by Proposition 12.17 we do know how to construct mesoprimary ideals given the foundation of a binomial ideal to start from. More precisely, fix a monoid prime $P \subset Q$, a P-mesoprimary congruence $\approx$ on Q, and a character $\rho : K \to \Bbbk^*$ on the stabilizer K of some element that is not nil in the localization of $Q/{\approx}$ along P. It would be convenient to say that there exists a mesoprimary ideal J inducing $\approx$ with associated mesoprime $I_{\rho,P}$, but it is not clear to us whether this should be true. What guarantees existence in the cases we care about, namely Proposition 12.17, is the I-witnessed nature of $\approx$: each I-witness prefers a particular character over all others — the one it sees by virtue of it being an I-witness — and that is the only one required for the theory of mesoprimary decomposition.

In a different light, the problem is one of automorphisms. The associated mesoprime of any unital P-mesoprimary ideal I is $I_{1,P}$ for the trivial character. Suppose, for simplicity, that the ground field $\Bbbk$ is algebraically closed. Then, for any mesoprime $I_{\rho,P}$, there is an automorphism of $\Bbbk[Q]$ taking $I_{1,P}$ to $I_{\rho,P}$; this amounts to the feasibility of extending the character $\rho : K \to \Bbbk^*$ to the entire group G_P of units of Q_P. To transform I into a mesoprimary ideal with associated mesoprime $I_{\rho,P}$, however, the character must be extended appropriately to all of Q_P, not just to G_P. It is not clear to us whether issues of horizontal coefficients (see Remark 12.20) can intervene, particularly when the inclusion of G_P into Q_P fails to split.

Remark 12.22. Independent of the existence question, it is not clear how to describe the class of mesoprimary ideals inducing a given congruence and with a given associated mesoprime. Certainly, a solution to the problem in Remark 12.21 need not be unique. For instance in the nilpotent situation, the one-parameter family $\langle x - \lambda y, x^2, xy, y^2\rangle$ (for $\lambda \neq 0$) consists of mesoprimary ideals over the associated mesoprime $\langle x, y\rangle$, all inducing the same congruence.

13. Mesoprimary decomposition of binomial ideals

This section makes precise the sense in which mesoprimary decomposition of congruences lifts to a parallel combinatorial theory for binomial ideals in monoid algebras.

Definition 13.1. Fix a binomial ideal $I \subseteq \Bbbk[Q]$ in a finitely generated commutative monoid algebra over a field $\Bbbk$.

(1) An expression of I as an intersection of finitely many mesoprimary ideals is a *mesoprimary decomposition* if, for each prime $P \subset Q$ and P-mesoprimary intersectand J, the P-mesoprimes of I and J at every cogenerator of J coincide.

(2) The decomposition is a *combinatorial* mesoprimary decomposition if every cogenerator of every component J in the decomposition is an essential I-witness.

Theorem 13.2. *Fix a finitely generated commutative monoid Q and a field $\Bbbk$. Every binomial ideal in the algebra $\Bbbk[Q]$ admits a combinatorial mesoprimary decomposition.*

Proof. Examples include those in Theorem 13.3 and Corollary 13.5, below, where the finiteness of the intersection in Theorem 13.3 is Lemma 12.4. □

The use of all essential witnesses and not merely key witnesses in the next result stems from the element f in the proof, which can have more than two terms. See also Example 16.6, which shows that nonkey witnesses can be necessary for the intersection of the corresponding coprincipal components to be a binomial ideal. On the other hand, the restriction to essential witnesses instead of all witnesses ensures finiteness of the number of intersectands, according to Lemma 12.4.

Theorem 13.3. *Fix a finitely generated commutative monoid Q and a field $\Bbbk$. Every binomial ideal in the monoid algebra $\Bbbk[Q]$ is the intersection of the coprincipal components cogenerated by its essential witnesses.*

Proof. Pick an element f outside of I. The goal is to show that f lies outside of the coprincipal component of I cogenerated by some essential witness. First assume that f lies in the monomial localization $I_{P'}$ along every nonmaximal prime P'. Thus f is annihilated, modulo I, by some power of the maximal monomial ideal $\mathfrak{m}_P \subseteq \Bbbk[Q]$. Replacing f by a monomial multiple of f, assume that f is annihilated, modulo I, by the entire maximal monomial ideal; that is, assume $\mathfrak{m}_P f \subseteq I$. By Definition 12.1, some essential I-witness w for P is the exponent on a monomial $\boldsymbol{t}^w$ with nonzero coefficient in f. Minimality of w ensures that all terms of f other than $\boldsymbol{t}^w$ itself vanish modulo $W_w^P(I)$, whence $f \notin W_w^P(I)$.

The argument just completed proves, in particular, the case where Q has only one prime ideal. Now assume that Q has more than one prime ideal. By the argument already given, assume the image of f under monomial localization along some nonmaximal monoid prime P lies outside of I_P. Induction on the number of prime ideals of Q implies that the localized image of f lies outside of some P-coprincipal component of I_P. By Definition 12.14, a P-coprincipal component of I_P is the localization along P of a P-coprincipal component of I. Lemma 13.4 implies that f lies outside of that P-coprincipal component before localization, as desired. □

Lemma 13.4. *If I is a P-mesoprimary ideal, then localization along a monoid prime is either injective or* 0 *on $\Bbbk[Q]/I$, with injectivity precisely when the prime contains P.*

Proof. By Definition 2.12, any P-mesoprimary congruence on Q is P-primary, whence the quotient $\bar{Q}$ either injects into its localization along the given prime (if the prime contains P) or else $\bar{Q}$ becomes trivial upon localization (if some element of P — which is nilpotent in $\bar{Q}$ — is inverted). Lemma 11.4 implies that the result for congruences lifts to binomial ideals. □

Using Theorem 13.3 and Proposition 12.17, one can find a mesoprimary decomposition that minimizes the number of components by intersecting all coprincipal components for a given associated mesoprime.

Corollary 13.5. *Fix a finitely generated commutative monoid Q and a field $\Bbbk$. Every binomial ideal in the monoid algebra $\Bbbk[Q]$ admits a combinatorial mesoprimary decomposition with one component per associated mesoprime.*

Remark 13.6. The existence of any mesoprimary decomposition — let alone a combinatorial one as in Theorem 13.2 — is much stronger than mere existence of a decomposition as an intersection of mesoprimary ideals, essentially because of the phenomenon in Remark 12.9. The strength is particularly visible when the field $\Bbbk$ is algebraically closed of characteristic 0. In that case, every binomial primary decomposition of I expresses I as an intersection of mesoprimary ideals by Corollary 10.7, but a mesoprime must honor stringent combinatorial conditions to be an associated mesoprime of I, and a mesoprimary ideal for an associated mesoprime must honor stringent combinatorial conditions to be an intersectand in a mesoprimary decomposition of I. The difference between ordinary and combinatorial mesoprimary decompositions is a relatively slight distinction among potential cogenerator locations: in the ordinary case, I is merely required to possess the correct characters at the cogenerators of the intersectands, whereas in the combinatorial case only certain intrinsically defined elements possessing the correct characters from I are allowed as cogenerators of components.

14. Binomial localization

Upon localization of a binomial quotient $\Bbbk[Q]/I$ at a binomial prime, some monomials become units and others are annihilated. The units are easy: if the prime is $I_{\sigma,P}$, then the monomials outside of $\mathfrak{m}_P$ become units. The question of which monomials die is much more subtle. There are two potential reasons that a monomial gets killed upon ordinary localization (Theorem 14.9): a combinatorial one and an arithmetic one. Combinatorially, a monomial dies if its class under $\sim_I$ points into P (Definition 14.1); arithmetically, a monomial dies if the character of I_P at it is incommensurate with ρ (Definition 14.6). These annihilations result from the inversion of two different types of binomials: in the combinatorial case the inverted binomials have one monomial outside of $\mathfrak{m}_P$, and in the arithmetic case the inverted binomials lie along the unit group G_P locally at P. The relevant monomials die because locally each becomes a binomial unit multiple of a binomial in I; see the proof of Theorem 14.9.

Definition 14.1. Given a prime $P \subset Q$, and a congruence $\sim$ on Q, the congruence class of $q \in Q$ *points into* P if $q + p \sim q$ in the localization Q_P for some $p \in P$.

Lemma 14.2. *Given a prime $P \subset Q$ and a congruence $\sim$ on Q, the set of elements in Q whose class points into P is an ideal of Q.*

Proof. If $q + p \sim q$ then $u + q + p \sim u + q$ by additivity of $\sim$. □

Definition 14.3. The *P-infinite ideal* $M^P_\infty(\sim) \subseteq Q$ for a prime $P \subset Q$ and congruence $\sim$ on Q is generated by the elements of Q whose classes point into P. If $\sim = \sim_I$ is induced by a binomial ideal $I \subseteq \Bbbk[Q]$, then $M^P_\infty(I) \subseteq \Bbbk[Q]$ is the corresponding *P-infinite monomial ideal*.

Remark 14.4. The terminology involving infinity stems from [Dickenstein et al. 2010b, Lemma 2.10], which concerns binomial localization at a monomial prime of an affine semigroup ring: when the ambient monoid Q is an affine semigroup, a class that points into P is infinite. The focus on monomial primes in affine semigroup rings arises there because the field is algebraically closed of characteristic 0 and the ideals to be localized are $I_{\rho,P}$-primary (and hence contain I_ρ), so the binomial localization procedure can be carried out in the affine semigroup ring $\Bbbk[Q]/I_\rho$. Definitions 14.1 and 14.3 lift the picture from $(I+I_\rho)/I_\rho \subseteq \Bbbk[Q]/I_\rho$ to $I+I_\rho \subseteq \Bbbk[Q]$ itself; but see Remark 14.7.

Lemma 14.5. *Let R be a set of characters on subgroups of the unit group G_P of Q_P. Given a binomial ideal $I \subseteq \Bbbk[Q]$, the set*

$$\{q \in Q \mid \textit{the character } \rho^P_q \textit{ of } I_P \textit{ at } q \textit{ is not a restriction of every character from } R\}$$

is an ideal of Q.

Proof. The character ρ^P_{p+q} of I_P at $p+q$ is an extension of ρ^P_q. □

Definition 14.6. Given a binomial ideal $I \subseteq \Bbbk[Q]$ and a mesoprime $I_{\rho,P}$, the *incommensurate ideal* of I at ρ is the ideal $M^P_\rho(I) \subseteq \Bbbk[Q]$ spanned over $\Bbbk$ by all monomials $\mathbf{t}^q$ such that the character of I_P at q is not a restriction of ρ.

Remark 14.7. The condition for a monomial to lie in the incommensurate ideal is phrased arithmetically, but in reality many monomials in it are there for combinatorial reasons: if the domain of the character of I_P at q fails to be contained in the (saturation of) the domain of ρ — that is, if the stabilizer of the class of q in $Q/{\sim_I}$ is too big — then q has no hope of being commensurate with ρ. This type of combinatorial obstruction to commensurability also contributes infinite classes in [Dickenstein et al. 2010b, Lemma 2.10].

Definition 14.8. The *binomial localization* of $I \subseteq \Bbbk[Q]$ at a binomial prime $I_{\sigma,P}$ is the sum $I + M_\infty^P(I) + M_\sigma^P(I) \subseteq \Bbbk[Q]$ of I plus its P-infinite and incommensurate ideals.

The point of this section is to compare the previous definition with ordinary (inhomogeneous) localization of a $\Bbbk[Q]$-module at a binomial prime $I_{\sigma,P}$, obtained by inverting all elements of $\Bbbk[Q]$ outside of $I_{\sigma,P}$.

Theorem 14.9. *Given a binomial ideal $I \subseteq \Bbbk[Q]$ over an arbitrary field $\Bbbk$, the kernel of the localization homomorphism from $\Bbbk[Q]$ to the ordinary localization of $\Bbbk[Q]/I$ at a binomial prime $I_{\sigma,P}$ contains the binomial localization of I at $I_{\sigma,P}$.*

Proof. First suppose that the class of $q \in Q$ points into P. Pick $p \in P$ such that $q + p \sim q$. This congruence means that there is a binomial $\mathbf{t}^q - \lambda \mathbf{t}^{q+p} = \mathbf{t}^q(1 - \lambda \mathbf{t}^p)$ in I. But $1 - \lambda \mathbf{t}^p$ lies outside of $I_{\sigma,P}$ because its image modulo $\mathfrak{m}_P$ is already 1. Therefore $1 - \lambda \mathbf{t}^p$ is a unit in the ordinary localization of $\Bbbk[Q]/I$ at $I_{\sigma,P}$, so $\mathbf{t}^q$ is 0 there.

Next suppose that $\mathbf{t}^q \in M_\sigma^P(I)$. By definition, there is a binomial $1 - \lambda \mathbf{t}^g$ for some $g \in G_P$ such that $\lambda \neq \sigma(g)$ and $\mathbf{t}^q(1 - \lambda \mathbf{t}^g) \in I_P$. The element $1 - \lambda \mathbf{t}^g$ lies outside of $I_{\sigma,P}$ by definition. Therefore the argument in the previous paragraph works in this case, too. We conclude that the binomial localization of I is contained in the kernel. □

Remark 14.10. How is Theorem 14.9 to be applied? While the binomial localization I' of I at $I_{\sigma,P}$ might not coincide with the kernel of ordinary localization at $I_{\sigma,P}$, it is always the case, by Theorem 14.9, that I and I' have the same ordinary localization at $I_{\sigma,P}$. Therefore, for the purpose of detecting $I_{\sigma,P}$-primary components, I' is just as good as I was in the first place. But the combinatorics of I' might be much simplified, thereby clarifying the role of $I_{\sigma,P}$ in the primary decomposition of I. See the proof of Theorem 15.11 for a quintessential example.

15. Primary decomposition of binomial ideals

Passing from mesoprimary and coprincipal ideals and decompositions to primary ideals and decompositions requires a minimal amount of knowledge concerning primary decomposition of mesoprimary ideals themselves. To speak about binomial primary decomposition of binomial ideals in $\Bbbk[Q]$, we are forced to assume, in appropriate locations, that $\Bbbk$ is algebraically closed (Example 11.15); we write $\Bbbk = \overline{\Bbbk}$ in that case. Doing so guarantees that each binomial ideal $I \subset \Bbbk[Q]$ has binomial associated primes [Eisenbud and Sturmfels 1996, Theorem 6.1]. However, most of this section works for an arbitrary ground field, so we are explicit about our hypotheses in this section. One reason is that the characterization of binomial prime ideals (Theorem 11.14) does not rely on properties of $\Bbbk$: every binomial prime can be expressed as a sum $\mathfrak{p} + \mathfrak{m}_P$ in which $P \subset Q$ is a monoid prime ideal and $\mathfrak{p}$ is a binomial ideal (unique and prime modulo $\mathfrak{m}_P$, but not necessarily in $\Bbbk[Q]$) that contains no monomials.

Proposition 15.1. *Fix an arbitrary field $\Bbbk$. If $I \subset \Bbbk[Q]$ is mesoprimary with associated mesoprime $I_{\rho,P}$, and the localized quotient monoid $\overline{Q}_P = Q_P/\!\sim_I$ has unit group G, then* (i) *localizing along P induces an injection $\Bbbk[Q]/I \hookrightarrow (\Bbbk[Q]/I)_P$, and* (ii) *$(\Bbbk[Q]/I)_P$ has finitely many nonzero $(\overline{Q}_P/G)$-graded pieces, all isomorphic to $(\Bbbk[Q]/I_{\rho,P})_P$. Conditions* (i) *and* (ii) *characterize mesoprimary ideals I with associated mesoprime $I_{\rho,P}$.*

Proof. The monomials outside of $\mathfrak{m}_P$ are nonzerodivisors on the quotient modulo any P-mesoprimary ideal by definition; hence the injection (i). Claim (ii) and the statement about characterizing mesoprimary ideals follow from Proposition 12.10 (see also Definition 11.11, Remark 11.12, and Lemma 12.7). □

Corollary 15.2. *Fix an arbitrary field $\Bbbk$. If $I \subset \Bbbk[Q]$ is mesoprimary, then the associated primes of I are exactly the minimal primes of its unique associated mesoprime. In particular, I is primary if it is mesoprimary and its associated mesoprime is prime.*

Proof. The partial order on the monoid $\overline{Q}_P/G$ afforded by Lemma 2.19 induces a filtration of $(\Bbbk[Q]/I)_P$ by $\Bbbk[Q]_P$-submodules whose associated graded module is free of finite rank — in fact isomorphic to $(\Bbbk[Q]/I)_P$ itself — as a module over $(\Bbbk[Q]/I_{\rho,P})_P$. □

Remark 15.3. Corollary 15.2 says that, although one expects to derive information about associated primes of I from the characters at its witnesses, when I is mesoprimary the appropriate characters appear at the identity $1 \in \Bbbk[Q]$. This is another manifestation of semifreeness (Remark 6.8), detailed in the present case in Proposition 15.1.

Primary decomposition of mesoprimary ideals reduces to that of mesoprimes.

Proposition 15.4. *Fix $\Bbbk = \overline{\Bbbk}$. Any mesoprimary ideal $I \subset \Bbbk[Q]$ with associated mesoprime $I_{\rho,P}$ has unique minimal primary decomposition $I = \bigcap_\sigma (I + I_\sigma)$, if $I_{\rho,P} = \bigcap_\sigma I_{\sigma,P}$ is the unique minimal primary decomposition of $I_{\rho,P}$ from Proposition 11.9.*

Proof. Adding the binomials I_σ to the mesoprimary ideal I coarsens its congruence to another mesoprimary one, so each ideal $I+I_\sigma$ is mesoprimary, and hence primary by Corollary 15.2. The intersection $J = \bigcap_\sigma (I+I_\sigma)$ obviously contains I, and we need that $J \subseteq I$, or equivalently that $I_\rho = \bigcap_\sigma I_\sigma$ maps to 0 in the quotient $\Bbbk[Q]/I$. This is a consequence of Proposition 15.1, completing the proof. □

Remark 15.5. If I is coprincipal in Proposition 15.4, then every primary component there is a coprincipal ideal. Indeed, the partially ordered monoid of Green's classes that is used to detect (or construct) coprincipal ideals is the same for I and for $I+I_\sigma$.

The remainder of this section outlines the main consequences of mesoprimary decomposition for primary decomposition.

Theorem 15.6. *Fix a binomial ideal $I \subseteq \bar{\Bbbk}[Q]$ over an algebraically closed field $\bar{\Bbbk}$. Refining any mesoprimary decomposition of I by canonical primary decomposition of its components yields a binomial primary decomposition of I. In characteristic* 0, *each primary component in this decomposition induces a primitive congruence on Q.*

Proof. Proposition 15.4 implies binomiality of the primary decomposition. For the final claim, it suffices to prove that every component $I+I_\sigma$ in Proposition 15.4 induces a primitive congruence in characteristic 0. But since σ is a saturation of ρ, the quotient of Q_P modulo the congruence induced by $I+I_\sigma$ is exactly the quotient of $Q_P/{\sim_I}$ by the torsion subgroup of its unit group. □

Remark 15.7. No choices are necessary to construct the coprincipal decomposition in Theorem 13.3 or the combinatorial mesoprimary decomposition in Corollary 13.5, and hence no choices are necessary to construct the primary decomposition in Theorem 15.6: these decompositions are all canonically recovered from essentially combinatorial data — a set of witnesses and monoid primes, plus the congruence induced by the binomial ideal — just as in the monomial case. Canonicality in the binomial context, however, comes at the price of nonminimality. Some redundancy can be eliminated using Section 16, but without arbitrary, unmotivated (and often symmetry-breaking) choices, redundancy can stubbornly persist. The reason is that the redundancy is already inherent in the combinatorics; that is, it happens at the level of monoids, congruences, and witnesses, before coefficients enter the picture. Note that by "canonical" we mean in the sense of "determined without extra data or requirements". In contrast, Ortiz [1959] uses the adjective "canonical" to refer to primary decompositions that minimize a certain index of nilpotency. Regardless of the name, Ojeda [2011] proves that the components in Ortiz's "canonical" decompositions are binomial when the original ideal is binomial, but these decompositions generally differ from the ones here, which rely solely on intrinsic data.

Remark 15.8. In positive characteristic p, primary binomial ideals need not be mesoprimary. This feature of mesoprimary decomposition reflects its freedom from characteristic. For instance, according to Hasse's local-to-global principle the ideal $\langle x^p - 1,\, y(x-1),\, y^2\rangle$ has no business being primary: in all but one characteristic it has two or more associated objects that accidentally coincide in characteristic p.

When the base field $\Bbbk$ is not algebraically closed, the binomial ideal I need not possess a binomial primary decomposition over $\Bbbk$ (see Example 11.15, for instance), but it does have one over the algebraic closure $\overline{\Bbbk}$. One of our original motivations for seeking a theory of mesoprimary decomposition was to gather primary components in such a way that Galois automorphisms (of $\overline{\Bbbk}$ over $\Bbbk$) permute them. In particular, if two primes are Galois translates of one another, then we wanted their corresponding primary components to look combinatorially the same.

Theorem 15.9. *If the ideal I in Theorem 15.6 is defined over a subfield $\Bbbk$ of its algebraic closure $\overline{\Bbbk}$, then the primary decomposition there is fixed by the Galois group* $\mathrm{Gal}(\overline{\Bbbk}/\Bbbk)$. *More precisely, if $\pi \in \mathrm{Gal}(\overline{\Bbbk}/\Bbbk)$ is a Galois automorphism and C is one of the primary components of I from Theorem 15.6, then $\pi(C)$ is another one of them.*

Proof. The Galois group fixes every mesoprimary component of I, and the primary decomposition of a mesoprimary ideal (Proposition 15.4) is canonical. □

Our final result on the primary-to-mesoprimary correspondence shows that, for general binomial ideals, every associated prime is detected by an associated mesoprime. For cellular binomial ideals, the relationship between associated mesoprimes and associated primes is even more perfectly precise. The cellular case of the following result over an algebraically closed field is [Eisenbud and Sturmfels 1996, Theorem 8.1] and its converse; the latter was stated and used without proof after [Eisenbud and Sturmfels 1996, Algorithm 9.5]. First, a matter of notation.

Definition 15.10. Fix a cellular binomial ideal $I \subset \Bbbk[Q]$. If $P \subset Q$ is the prime ideal of exponents on monomials that are nilpotent modulo I, then I is *P-cellular*.

Theorem 15.11. *Fix a binomial ideal $I \subseteq \Bbbk[Q]$ over an arbitrary field $\Bbbk$.*

(1) *Each associated prime of I is minimal over some associated mesoprime of I.*

(2) *If I is cellular, then the binomial converse holds: every binomial prime that is minimal over an associated mesoprime of I is an associated prime of I.*

Proof. For part (1), apply Corollary 15.2 to the components of I under any mesoprimary decomposition from Theorem 13.2.

For the cellular converse, suppose that I is P-cellular, and that a binomial prime $I_{\sigma,P}$ is minimal over some associated mesoprime $I_{\rho,P}$ of I. The submodule of $\Bbbk[Q]/I$ generated by a witness for $I_{\rho,P}$ is isomorphic to a quotient $\Bbbk[Q]/I'$, for a binomial ideal I' all of whose witness characters are extensions of ρ. After subsequently binomially localizing at $I_{\sigma,P}$, the only surviving characters are restrictions of σ, and hence sit between σ and ρ. In particular, this is true for the character at any given monomial $\boldsymbol{t}^q$ such that q is a cogenerator of the induced congruence. Such a monomial generates a mesoprime submodule with $I_{\sigma,P}$ among its associated primes by Corollary 15.2. Therefore $I_{\sigma,P}$ is associated to I', and hence to I by Theorem 14.9; see Remark 14.10. □

Example 15.12. Given an associated prime of I as in Theorem 5.11(1), the associated mesoprime guaranteed by the theorem need not be unique. This phenomenon is illustrated by Example 2.17.5. The binomial prime $\langle x-1, y\rangle$ for the trivial character on the x-axis $\mathbb{N}\times\{0\}$ is associated to I and has two possible choices of associated mesoprime, namely $\langle x-1, y\rangle$ and $\langle x^2-1, y\rangle$. Combinatorially, the row of dots at height 1 consists of two classes, each being the nonnegative points in a coset of an unsaturated lattice, while the row of dots at height 2 comprise just one class, the nonnegative points in a coset of the saturation. In general, when the group of units G_P acts, there could be a whole G_P-orbit of classes corresponding to an unsaturated subgroup K, and a higher G_P-orbit with an associated subgroup anything between K and its saturation.

Example 15.13. Unmixed (cellular) binomial ideals need not be mesoprimary. Consider the cellular binomial ideal $\langle x^2-1, y(x-1), y^2\rangle \subset \Bbbk[x, y]$. It is not mesoprimary, but because its associated primes are $\langle x-1, y\rangle$ and $\langle x+1, y\rangle$, it is unmixed (even primary if $\mathrm{char}(\Bbbk)=2$). Consequently, the unmixed decompositions of [Eisenbud and Sturmfels 1996, Corollary 8.2] and [Ojeda Martínez de Castilla and Piedra-Sánchez 2000, Algorithm A4] do not decompose this ideal and thus do not lead to mesoprimary — let alone coprincipal — decompositions, even in cellular cases.

16. Character witnesses and false witnesses

The set of I-witnesses in the arithmetic setting of a binomial ideal I in a monoid algebra can be redundant in a manner that parallels the redundancy of witnesses in the combinatorial setting of monoid congruences. In the combinatorial setting, some of the redundancy is naturally eliminated by restricting to key witnesses; in the arithmetic setting here, character witnesses (Definition 16.3) play an analogous role. For cellular binomial ideals this is Theorem 16.9. Lifting to the general (i.e., noncellular) case is possible but would take us too far afield to be included here.

Definition 16.1. Fix a binomial ideal $I \subset \Bbbk[Q]$, an element $q \in Q$, and a monoid prime ideal $P \subset Q$. A *P-cover extension at q* is an extension of the character $\rho_q^P : K_q \to \Bbbk^*$ of I_P at q to the character $\rho_{p+q}^P : K_{p+q} \to \Bbbk^*$ at a P-cover $p+q$ of q (Definitions 4.4 and 11.11).

There can be many — even infinitely many — choices of minimal generating sets for P (Remark 4.5), but just as in Lemma 4.6, there are not too many P-cover extensions.

Lemma 16.2. *In the situation of Definition 16.1, the set of P-cover extensions at q is finite, in the sense that only finitely many stabilizers K_{p+q} occur, and only finitely many characters defined on each stabilizer occur among the characters ρ^P_{p+q}.*

Proof. Let $\bar{Q}$ be the quotient of Q modulo the congruence determined by I. If the images of p and p' are Green's-equivalent in $\bar{Q}$, then the stabilizers K_{p+q} and $K_{p'+q}$ coincide, as do the extensions to ρ^P_{p+q} and $\rho^P_{p'+q}$. Now apply Remark 4.5. □

Definition 16.3. Fix a prime $P \subset Q$, a P-cellular binomial ideal $I \subset \Bbbk[Q]$, and $w \in Q$.

(1) The *testimony* of w at P is the set $T_P(w)$ of P-cover extension characters.

(2) The testimony $T_P(w)$ is *suspicious* if the intersection of the corresponding mesoprimes equals the P-mesoprime I^P_w (Definition 11.7); that is, if $I^P_w = \bigcap_{\rho \in T_P(w)} I_{\rho,P}$.

(3) A *false witness* is an I-witness w for P that is not maximal (under Green's preorder) among I-witnesses for P and whose testimony at P is suspicious.

(4) An I-witness that is not false is a *character witness*.

Remark 16.4. For algebraically closed $\Bbbk = \bar{\Bbbk}$, Definition 16.3(4) becomes transparent, as follows. Minimal primary decompositions of mesoprimes $I_{\rho,P}$ (Proposition 11.9) are easy and canonical in that case: every saturated finite extension of ρ appears exactly once. A finite intersection of mesoprimes $I_{\sigma,P}$, each containing $I_{\rho,P}$, equals $I_{\rho,P}$ when, among all of the saturated finite extensions of the characters σ, every saturated finite extension of ρ appears at least once. A character witness for P with associated mesoprime $I_{\rho,P}$ is a witness in possession of a new character (a saturated finite extension) not present in its testimony. By the same token, a witness is false if it has no new characters to mention: the set of characters in its testimony is suspiciously complete.

The relation between the different types of witnesses from monoid land (key witnesses) and binomial land (character witnesses) is not as strong as one may hope. For example, a key witness can be a false witness (Example 16.5), and a character witness might not be a key witness (Example 16.6). It is also possible for a nonkey witness to be a false witness (Example 16.7). All of these examples are cellular binomial ideals.

Example 16.5. Consider the ideal $I' = \langle x(z-1),\, y(z+1),\, z^2-1,\, x^2,\, y^2\rangle$ from Example 9.1 and let P be the monoid prime of $\mathbb{N}^3$ such that $\mathfrak{m}_P = \langle x, y\rangle$. Then $0 \in \mathbb{N}^3$ is a key I'-witness for P that is a false I'-witness: the P-mesoprimes at the

P-covers of 0 are $\langle z-1\rangle$ and $\langle z+1\rangle$, whose characters form the complete set of saturated finite extensions of the character for $\langle z^2-1\rangle$. The testimony is suspicious because $\langle z-1\rangle \cap \langle z+1\rangle = \langle z^2-1\rangle$. In contrast, $0 \in \mathbb{N}^3$ is a character I-witness for P, where the ideal $I = \langle x(z-1),\, y(z-1),\, z^2-1,\, x^2,\, xy,\, y^2\rangle$ induces the same congruence as I'.

Example 16.6. In Definition 16.3, the intersection of the mesoprimes is the analogue of intersecting the kernels of the cover morphisms in Definition 4.7. The necessity of allowing all (nonkey) witnesses as potential character witnesses stems from the phenomenon in Example 2.22 (the common refinement of the congruences induced by $\langle x-1\rangle$ and $\langle y-1\rangle$ is trivial whereas the intersection of these ideals not) but is better illustrated by $I = \langle x^2-xy,\, y^2-xy,\, x(z-1),\, y(w-1),\, x^3\rangle \subset \Bbbk[x, y, z, w]$, which throws an extra generator x^3 into the ideal from Example 4.15(3). In contrast with that example, the extra monomial causes I to be cellular: the primary congruence it induces has associated monoid prime $P = \langle e_x, e_y\rangle$. But the P-prime congruence at the character I-witness $0 \in \mathbb{N}^4$ remains trivial, being the common refinement of the congruences induced by $\langle z-1\rangle$ and $\langle w-1\rangle$. This trivial P-prime congruence at 0 indicates a total lack of binomials in the $\bar{Q}$-degree 0 part of the intersection $\langle z-1,\, x^2,\, y\rangle \cap \langle w-1,\, x,\, y^2\rangle$, but this lack is accompanied by nonbinomial elements. An additional intersectand, namely the prime ideal $\langle x, y\rangle$ itself, is required to enforce binomiality.

In terms of Definition 16.3, the testimony consists entirely of saturated but infinite extensions of the character of I_P at $0 \in \mathbb{N}^4$. Therefore no saturated finite extensions occur, in the sense of Remark 16.4, making $0 \in \mathbb{N}^4$ a rather strong character I-witness, even though it is not a key witness for the congruence induced by I.

Example 16.7. Nonkey witnesses can be false witnesses. In Example 8.12 the origin is a false witness because $\langle a^2-1,\, b-1\rangle \cap \langle a-1,\, b^2-1\rangle \cap \langle ab-1,\, a-b\rangle = \langle a^2-1,\, b^2-1\rangle$ exhibits suspicious testimony.

Definition 16.8. Fix a cellular binomial ideal $I \subseteq \Bbbk[Q]$ in a finitely generated commutative monoid algebra over a field $\Bbbk$. A mesoprimary decomposition of I is *characteristic* if every cogenerator for every mesoprimary component is a character I-witness.

Theorem 16.9. *Fix I, a cellular binomial ideal. I admits a characteristic mesoprimary decomposition. In fact, I is the intersection of the coprincipal ideals cogenerated by its character witnesses. More generally, if I is expressed as an intersection of coprincipal components of I, then any component cogenerated by a false witness is redundant.*

In particular, the components for false witnesses can be thrown out (with their testimony) from the coprincipal decomposition in Theorem 13.3 for a cellular binomial ideal.

Proof. P-cellular ideals have only finitely many Green's classes of witnesses for P, because their induced congruences have only finitely many Green's classes to begin with by Lemma 2.19. Therefore the intersection over character witnesses is finite.

Express I as an intersection of mesoprimary components of I cogenerated by single witnesses, one of which is $W = W_w^P(I)$, cogenerated by a false witness w. Given an element $f \notin W$, we need f to lie outside of the intersection I' of the other components. It suffices to show that f lies outside at least one of the other components. To that end, there is no harm in localizing along P, because by Lemma 13.4 if f lies outside of a coprincipal component after localizing then it does so before localizing. Henceforth, therefore, assume P is the maximal monoid ideal. Furthermore, if f' is a monomial multiple of f that remains outside of W, then concluding that $f' \notin I'$ is enough. Therefore, replacing f by a monomial multiple of f, assume f is annihilated, modulo W, by the entire maximal monomial ideal. Write $f = f_{\preceq w} + f_{\not\preceq w}$, where $f_{\preceq w}$ is the sum of the terms of f whose exponents lie in $w + G$ for $G = Q \setminus P$, the unit group of Q.

The first goal is to show that $f \in I' \Longrightarrow f_{\preceq w} \in I'$. Let v be any I-witness and set $W' = W_v^P(I)$. When $w \not\prec v$ in Green's preorder, it is automatic that $f_{\preceq w} \in W'$, for then all monomials with exponents in $w + G$ lie in W'. Therefore assume $w \prec v$ and $f \in W'$. The relation $w \prec v$ implies that w is not nil modulo the congruence $\sim_{W'}$ induced by W', and consequently no term of $f_{\preceq w}$ has an exponent that is congruent under $\sim_{W'}$ to the exponent on a term of $f_{\not\preceq w}$. Therefore $f \in W' \Longrightarrow f_{\preceq w} \in W'$.

We have reduced to showing that $f \notin W \Longrightarrow f \notin I'$ when $f = f_{\preceq w}$, so assume $f = f_{\preceq w} \notin W$. For each generator $p \in P$, let $\sigma_p \in T_P(w)$ be the corresponding P-cover extension character. The crucial observation is that, since f is a sum over $w + G$,

$$\boldsymbol{t}^p f \in I \iff f \in W + I_{\sigma_p}.$$

This equivalence holds by tracing through all of the definitions; it implies that $\boldsymbol{t}^p f \in I$ for all generators $p \in P$ precisely when

$$f \in \bigcap_p (W + I_{\sigma_p}) = W + \bigcap_p I_{\sigma_p} = W + I_w^P = W,$$

where the first displayed equality is a consequence of Proposition 15.1. Since $f \notin W$, it follows that there is some generator $p \in P$ such that $\boldsymbol{t}^p f \notin I$. But $\boldsymbol{t}^p f \in W$ by construction, so $\boldsymbol{t}^p f$ lies outside of some other coprincipal component of I, and hence so does f itself, as desired. □

Where did cellularity enter the proof of the preceding proposition? Beyond finiteness of witnesses, the conclusion that no term of $f_{\preceq w}$ has an exponent congruent under $\sim_{W'}$ to the exponent on a term of $f_{\not\preceq w}$ would be false if W' were allowed to be a coprincipal component for a monoid prime strictly contained in P; see the next example.

Example 16.10. Let

$$I = \langle x^2 - x\dot{x},\, x\dot{x} - \dot{x}^2,\, x^3,\, x^2y,\, z^2 - 1,\, x^2(z-1),\, y(z+1),\, y(x-\dot{x})\rangle \subseteq \Bbbk[x, \dot{x}, y, z].$$

(The variables x and $\dot{x}$ correspond to x and y in Example 2.18.) Then $\dot{x}$ is a false I-witness monomial for the monoid prime P corresponding to $\mathfrak{m}_P = \langle x, \dot{x}, y\rangle$: the character at $\dot{x}$ is $z^2 - 1$, while at $x\dot{x}$ it is $z - 1$ and at $y\dot{x}$ it is $z + 1$. Omitting the coprincipal component $\langle x^2, x\dot{x}, \dot{x}^2, y, z^2 - 1\rangle$ of I cogenerated by $\dot{x}$ from the coprincipal decomposition of I in Theorem 13.3 leaves $\langle x^3, x^2 - x\dot{x}, x\dot{x} - \dot{x}^2, y, z - 1\rangle \cap \langle x^2, x - \dot{x}, z + 1\rangle$, which is not a binomial ideal. The element $f = x - \dot{x}$ has a monomial $\dot{x} = \boldsymbol{t}^w$ whose exponent w is congruent to the exponent of $x = \boldsymbol{t}^q$ under $\sim_{W'}$ for $W' = \langle x^2, x - \dot{x}, z + 1\rangle$ even though q and w are incomparable. W' is P'-mesoprimary for $\mathfrak{m}_{P'} = \langle x, \dot{x}\rangle \subsetneq \mathfrak{m}_P$.

Remark 16.11. One reason Theorem 16.9 restricts to the cellular case is the automatic finiteness for witnesses. In contrast, in Section 12 the notion of essential witness does the job by Lemma 12.4. In general, even modulo Green's equivalence the set of I-witnesses can be infinite. For example, infiniteness causes Proposition 12.17 to fail when $I = \langle x^2y - y^2x\rangle$ if one uses all I-witnesses for $\mathfrak{m}_P = \langle x, y\rangle$. The sets of essential and character witnesses do not coincide, because of the false key witnesses in Example 16.5, but it is possible that every character I-witness could be an essential I-witness.

Question 16.12. Are there redundant character witnesses? key witnesses?

17. Open problems

Beyond Question 16.12, the results of this paper raise other problems implicitly in the remarks, and still others that constitute future research directions beyond the scope of this paper. We collect some of these problems here.

17.1. *Intersections of binomial ideals.*

Problem 17.1. Characterize when an intersection of binomial ideals is binomial.

Problem 17.1 was originally posed by Eisenbud and Sturmfels [1996, Problem 4.9], who answered it in the reduced situation [1996, Theorem 4.1]. In our language, that theorem contains information about the associated prime ideals of the congruence induced by a radical binomial ideal. It is possible that the general case could reduce to the radical case, by considering what the congruence or the P-prime characters induced by the intersection could possibly look like at each monoid element. This type of consideration underlies the definition of character witness (Definition 16.3), where nonbinomiality at specific monoid elements would otherwise occur, without specifically throwing in additional binomials, because of incompatibility of congruences or characters arising from covers.

As a stepping stone to a full answer to Problem 17.1, one might consider [Eisenbud and Sturmfels 1996, Problem 6.6]: does intersecting the minimal primary components of a binomial ideal result in another binomial ideal?

17.2. ***Choices of vertical coefficients.*** Remarks 12.21 and 12.22 raise the following.

Problem 17.2. Characterize the mesoprimary ideals that induce a fixed mesoprimary congruence with a fixed associated mesoprime. In particular, decide when the set of such mesoprimary ideals is nonempty.

17.3. ***Primary binomial ideals in positive characteristic.*** Lack of knowledge concerning the combinatorics of primary binomial ideals in positive characteristic is an obstacle in our investigations. In particular we do not know exactly which primary binomial ideals are mesoprimary.

Problem 17.3. Characterize primary binomial ideals with nontrivial mesoprimary decompositions.

17.4. ***Posets of mesoprimes.***

Problem 17.4. Characterize the posets of associated prime congruences of primary congruences.

The problem could have been stated for arbitrary congruences, but then every finite poset would be possible, because every finite poset occurs as the set of associated primes of a monomial ideal (this is a good exercise, but it follows from [Miller 1998]). Problem 17.4 is equivalent to characterizing posets of associated mesoprimes of unital cellular binomial ideals. Such posets always possess a unique minimal element, represented by the identity element of the finite partially ordered monoid of Green's classes in Lemma 2.19. When devising examples for the present paper, we often used a technique to "place" associated mesoprimes at desired locations, illustrated as follows.

Example 17.5. Let $\Delta \subsetneq \Gamma$ be simplicial complexes on $\{1, \dots, n\}$ and consider the polynomial ring in $2n$ variables $S = \Bbbk[x_1, \dots, x_n, y_1, \dots, y_n]$. For any $A \in \Gamma \setminus \Delta$ write $x_A := \prod_{i \in A} x_i$. The poset of associated mesoprimes of the cellular binomial ideal

$$I_{\Gamma\setminus\Delta} = \sum_{A \in \Gamma\setminus\Delta} I_A + \langle x_i^2 \mid i = 1, \dots, n\rangle \subset S \quad \text{for} \quad I_A = \langle x_A(y_i - 1) \mid i \in A\rangle$$

is isomorphic to $(\Gamma \setminus \Delta) \cup \{\varnothing\}$.

Remark 17.6. The construction in the previous example is fairly general, and one might hope that complete generality is possible. In practice this problem will be about understanding what happens to the partial order on $\mathbb{N}^n$ when passing to a quotient and under the order-preserving map assigning to a witness its associated prime congruence.

Remark 17.7. Definition 5.2 requires associated prime congruences to appear at key witnesses. Allowing arbitrary witnesses yields an a priori different notion of associated prime congruence: although the P-prime congruence at an arbitrary witness for P agrees with the P-prime congruence at some key witness, the key witness might be for a monoid prime smaller than P. This phenomenon does not occur for primary congruences, however, as they have only one associated monoid prime. Thus Problem 17.4 would have the same answer if Definition 5.2 had allowed arbitrary witnesses.

Nonetheless, this line of thinking indicates that care must be taken in lifting Problem 17.4 to the arithmetic setting, where Definition 12.1 requires associated mesoprimes to appear at arbitrary witnesses, not at a subset of all witnesses. For instance, a P-mesoprime can be associated to an ideal even though it only appears at a false witness; this occurs in both Example 16.5 and Example 16.7. This idiosyncrasy in the definition of associated mesoprime motivates a new definition.

Definition 17.8. An associated mesoprime of a binomial ideal I is *truly associated* if it is the P-mesoprime of I at a character I-witness for P.

Problem 17.9. Characterize the posets of associated mesoprimes of cellular binomial ideals. Do the same for posets of truly associated mesoprimes.

Remark 17.10. The family of posets referred to in (either version of) Problem 17.9 contains the family of posets in Problem 17.4 by Remark 17.7 applied to the case of unital binomial ideals.

17.5. ***Mesoprimary decomposition of modules.*** Grillet [2007] shows how subdirect decompositions of semigroups induce subdirect decompositions of sets acted on by semigroups; see Remark 2.2. In a similar vein, mesoprimary decomposition ought to extend to finitely generated monoid actions.

Problem 17.11. Generalize mesoprimary decomposition of congruences to Q-modules.

The generalization ought to parallel the manner in which ordinary primary decomposition of ideals in rings extends to primary decomposition of modules over rings. In the arithmetic setting of mesoprimary decomposition, however, even the first step of the extension requires thought.

Question 17.12. What is a binomial module over a commutative monoid algebra?

A good theory of such modules should yield the desired generalization.

Problem 17.13. Extend mesoprimary decomposition to binomial $\Bbbk[Q]$-modules.

17.6. ***Homological invariants of binomial rings.*** The combinatorics of the free commutative monoid $\mathbb{N}^n$ gives rise to formulas and constructions for all sorts of homological invariants involving monomial ideals — Betti numbers, Bass numbers, free resolutions, local cohomology, and so on — due to the $\mathbb{N}^n$-grading; see [Miller and Sturmfels 2005]. Gradings by more general affine semigroups yield formulas and constructions for local cohomology over affine semigroup rings (with maximal support [Ishida 1988] as well as with more arbitrary monomial support [Helm and Miller 2003; 2005]), and Betti numbers for toric ideals [Stanley 1996, Theorem I.7.9], etc. Having identified the combinatorics controlling decompositions of binomial ideals, the way is open to generalize monomial homological algebra.

Question 17.14. Do there exist combinatorial (monoid-theoretic) formulas for local cohomology, Tor, and Ext involving binomial quotients of polynomial rings?

Remark 17.15. In contrast, it is unclear to us whether combinatorial formulas for local cohomology with binomial support should exist, partly because of ill-behaved characteristic dependence; see [Iyengar et al. 2007, Example 21.31].

As soon as there is some control over Betti tables, Boij–Söderberg theory [Fløystad 2012] enters the picture. There one decomposes the Betti table $\beta(M)$ of a module M over a polynomial ring S as a rational linear combination of certain *pure tables* π_d:

$$\beta(M) = \sum a_d \pi_d.$$

Question 17.16. What combinatorics, if any, explains the coefficients a_d of $S/I_{\rho,P}$ as an S-module when $I_{\rho,P}$ is a mesoprime?

Even the special case of Boij–Söderberg theory for toric ideals is currently open.

17.7. ***Test sets in integer programming.*** Let $A \in \mathbb{Z}^{d\times n}$ be an integer matrix. An *integer program* is an optimization problem that seeks, for a given cost vector $\omega \in \mathbb{R}^n$, to maximize $\omega \cdot u$ over the integer points in the polyhedron $\mathcal{F}_b = \{u \in \mathbb{N}^n \mid Au = b\}$ for $b \in \mathbb{N}A := A(\mathbb{N}^n) \subseteq \mathbb{Z}^d$. A solution to this problem is the computation of a *test set*: a set $\mathcal{B}$ of differences between points in $\mathcal{F}_b$ such that for any candidate solution u to the optimization problem, its optimality can be tested by comparing it to $u+v$ for $v \in \mathcal{B}$. Computing a Gröbner basis of the toric ideal $I_A = \langle \mathbf{x}^u - \mathbf{x}^v \mid u, v \in \mathbb{N}^n \text{ and } Au = Av\rangle$ provides a simultaneous test set for all right-hand sides b, but this procedure may be computationally prohibitive. The hope behind the following problem is that for many b a test set is significantly simpler than a Gröbner basis.

Problem 17.17. Fix a finite set $\mathcal{B} \subset \ker_{\mathbb{Z}} A$.

(1) Characterize the multidegrees $b \in \mathbb{N}A$ for which $\mathcal{B}$ is a test set.

(2) Quantify the failure of $\mathcal{B}$ to be a test set in large fibers $\mathcal{F}_b$.

Intuition for the second problem comes from the geometry of mesoprimary components, or better yet, coprincipal components: their thicknesses in various directions should provide bounds on how close an integer point in $\mathscr{F}_b$ can get to optimality using $\mathscr{B}$. Indeed, starting at some $u \in \mathbb{Z}^n$ and successively progressing to the (local) optimum achieved by moving along vectors in $\mathscr{B}$ is equivalent to normal form reduction of $\boldsymbol{x}^u$ using binomials in the ideal $I_{\mathscr{B}} = \langle \boldsymbol{x}^u - \boldsymbol{x}^v \mid u - v \in \mathscr{B} \rangle$. Classes for the congruence induced by $I_{\mathscr{B}}$ can be thought of, roughly speaking, as polyhedra of the form $\mathscr{F}_b$ with bits (the "skerries" from [Dickenstein et al. 2010b, Section 1.1]) eaten away from the boundary; mesoprimary decomposition controls the missing boundary bits.

Diaconis, Eisenbud, and Sturmfels suggested — though not in the presence of a cost vector — to systematically study lattice walks with step set $\mathscr{B}$ using primary decomposition of $I_{\mathscr{B}}$ [Diaconis et al. 1998]. Given the unsuitability of primary decomposition for combinatorial purposes, the method should be updated to work with mesoprimary decompositions. This is especially true in the presence of unsaturated lattices among the minimal primes of $I_{\mathscr{B}}$, in which case the combinatorial flavor of the problem becomes clouded in the arithmetic (rather than combinatorics) of binomial primary decomposition.

A first step toward Problem 17.17 was developed in [Kahle et al. 2014b]. There the authors study only the connectivity of $\mathscr{F}_b$ as a function of the position of b in the cone $\mathbb{Q}_+ A$. Additionally all ideals there are radical, and consequently the subtleties of mesoprimary decomposition play no role.

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Locally analytic representations and sheaves on the Bruhat–Tits building

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Let L be a finite field extension of $\mathbb{Q}_p$ and let G be the group of L-rational points of a split connected reductive group over L. We view G as a locally L-analytic group with Lie algebra $\mathfrak{g}$. The purpose of this work is to propose a construction which extends the localization of smooth G-representations of P. Schneider and U. Stuhler to the case of locally analytic G-representations. We define a functor from admissible locally analytic G-representations with prescribed infinitesimal character to a category of equivariant sheaves on the Bruhat–Tits building of G. For smooth representations, the corresponding sheaves are closely related to the sheaves of Schneider and Stuhler. The functor is also compatible, in a certain sense, with the localization of $\mathfrak{g}$-modules on the flag variety by A. Beilinson and J. Bernstein.

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1. Introduction

Let L be a finite field extension of the field $\mathbb{Q}_p$ of p-adic numbers. Let $\boldsymbol{G}$ be a connected split reductive group over L and $\boldsymbol{B} \subset \boldsymbol{G}$ a Borel subgroup defined over L. Let $\boldsymbol{T} \subset \boldsymbol{G}$ be a maximal torus contained in $\boldsymbol{B}$. Let $G := \boldsymbol{G}(L)$, $T := \boldsymbol{T}(L)$ denote the groups of rational points, viewed as locally L-analytic groups. Let $\mathfrak{g}$ and $\mathfrak{t}$ be the corresponding Lie algebras.

The purpose of this work is to propose a construction which extends the localization theory for smooth G-representations of P. Schneider and U. Stuhler [1997] to the case of locally analytic G-representations. In more concrete terms, we define an exact functor from admissible locally analytic G-representations with prescribed infinitesimal character to a category of equivariant sheaves on the Bruhat–Tits building of G. The functor is also compatible, in a certain sense, with the localization theory for $\mathfrak{g}$-modules on the flag variety of $\boldsymbol{G}$ by A. Beilinson and J. Bernstein [1981], and J.-L. Brylinski and M. Kashiwara [1980; 1981].

To give more details, let $\mathscr{B}$ be the (semisimple) Bruhat–Tits building of G. The torus $\boldsymbol{T}$ determines an apartment A in $\mathscr{B}$. We fix a fundamental chamber $\mathscr{C} \subset A$ and a special vertex $x_0 \in \overline{\mathscr{C}}$, which will be used as an origin for the affine space A. In [Schneider and Stuhler 1997] the authors consider, for any facet $F \subset \mathscr{B}$, a well-behaved filtration

$$P_F \supset U_F^{(0)} \supset U_F^{(1)} \supset \cdots$$

of the pointwise stabilizer P_F of F in G by open pro-p subgroups $U_F^{(e)}$. For any point $z \in \mathscr{B}$, one sets $U_z^{(e)} := U_F^{(e)}$, where F is the unique facet containing z. It forms a fundamental system of neighborhoods of $1 \in P_z$, where P_z is the stabilizer of z. Let from now on $e \geq 0$ be a fixed number (called a *level* [loc. cit.]).

Using the groups $U_z^{(e)}$, Schneider and Stuhler [1997, Section IV] defined an exact functor

$$V \mapsto \underset{\approx}{V}$$

from smooth complex G-representations to sheaves of complex vector spaces on $\mathscr{B}$. The stalk of the sheaf $\underset{\approx}{V}$ at a point z is given by the coinvariants $V_{U_z^{(e)}}$ and the restriction of $\underset{\approx}{V}$ to a facet $F \subset \mathscr{B}$ equals the constant sheaf with fiber $V_{U_F^{(e)}}$. The functor $V \mapsto \underset{\approx}{V}$ has particularly good properties when restricted to the subcategory of representations generated by their $U_{x_0}^{(e)}$-fixed vectors. It is a major tool in the proof of the Zelevinsky conjecture [loc. cit.].

From now on we fix a complete discretely valued field extension K of L. The functor $V \mapsto \underset{\approx}{V}$ can be defined in exactly the same way for smooth G-representations on K-vector spaces, and produces sheaves of K-vector spaces on $\mathscr{B}$. The naive extension of the functor $V \mapsto \underset{\approx}{V}$ to locally analytic representations, by taking coinvariants as above, does not have good properties. For instance, applying this procedure to an irreducible finite-dimensional algebraic representation, which is not the trivial representation, produces the zero sheaf. Moreover, if we aim at a picture which is related to the localization theory of $\mathfrak{g}$-modules, then localizing an irreducible algebraic representation should give a line bundle.

We consider the variety of Borel subgroups

$$X = \boldsymbol{G}/\boldsymbol{B}$$

of $\boldsymbol{G}$. We let $\mathcal{O}_X$ be its structure sheaf and $\mathcal{D}_X$ be its sheaf of differential operators. Deriving the left regular action of $\boldsymbol{G}$ on X yields an algebra homomorphism

$$\alpha : \underline{U}(\mathfrak{g}) \to \mathcal{D}_X$$

where the source refers to the constant sheaf on X with fiber equal to the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. Let $Z(\mathfrak{g})$ be the center of the ring $U(\mathfrak{g})$.

The torus $\boldsymbol{T}$ determines a root system. Let ρ be half the sum over the positive roots with respect to $\boldsymbol{B}$. For any algebraic character $\chi - \rho$ of the torus $\boldsymbol{T}$ we have the sheaf $\mathcal{D}_\chi$ of differential endomorphisms of the line bundle on X associated with $\chi - \rho$. Any trivialization of the line bundle induces a local isomorphism between $\mathcal{D}_\chi$ and $\mathcal{D}_X$, and we have $\mathcal{D}_\rho = \mathcal{D}_X$. More generally, if $\chi - \rho$ is an arbitrary character of $\mathfrak{t}$ there is a sheaf of so-called *twisted* differential operators $\mathcal{D}_\chi$ on X. As in the former case, it comes equipped with a morphism $\mathcal{O}_X \hookrightarrow \mathcal{D}_\chi$ which is locally isomorphic to the canonical morphism $\mathcal{O}_X \hookrightarrow \mathcal{D}_X$. Moreover, there is an algebra homomorphism $\underline{U}(\mathfrak{g}) \to \mathcal{D}_\chi$ locally isomorphic to α. The sheaf $\mathcal{D}_\chi$ was first introduced in [Beilinson and Bernstein 1981] as a certain quotient sheaf of the skew tensor product algebra $\mathcal{O}_X \# U(\mathfrak{g})$, where we use # to indicate that the multiplication on the tensor product $\mathcal{O}_X \otimes U(\mathfrak{g})$ involves the action of $U(\mathfrak{g})$ on $\mathcal{O}_X$.

Let χ be a character of $\mathfrak{t}$. Let θ be the character of $Z(\mathfrak{g})$ associated with χ via the classical Harish-Chandra homomorphism. The above map factors via a homomorphism

$$\underline{U}(\mathfrak{g})_\theta \to \mathcal{D}_\chi$$

where $U(\mathfrak{g})_\theta = U(\mathfrak{g}) \otimes_{Z(\mathfrak{g}),\theta} L$. If χ is dominant and regular, a version of the localization theorem due to Beilinson and Bernstein asserts that the functor

$$\Delta_\chi : M \mapsto \mathcal{D}_\chi \otimes_{\underline{U}(\mathfrak{g})_\theta} \underline{M}$$

is an equivalence of categories between the (left) $U(\mathfrak{g})_\theta$-modules and the (left) $\mathcal{D}_\chi$-modules which are quasicoherent as $\mathcal{O}_X$-modules. The underlined objects refer to the associated constant sheaves on X. We remark that a seminal application of this theorem (or rather its complex version) leads to a proof of the Kazhdan–Lusztig multiplicity conjecture [Beilinson and Bernstein 1981; Brylinski and Kashiwara 1980; 1981].

The starting point of our work is a result of V. Berkovich [Berkovich 1990; Rémy et al. 2010] according to which the building $\mathscr{B}$ may be viewed as a locally closed subspace

$$\mathscr{B} \hookrightarrow X^{\mathrm{an}}$$

of the Berkovich analytification X^{an} of X. This makes it possible to "compare" sheaves on $\mathscr{B}$ and X^{an} in various ways. Most of what has been said above about the scheme X extends to the analytic space X^{an}. In particular, there is an analytic version $\mathcal{D}_{\chi}^{\mathrm{an}}$ of $\mathcal{D}_{\chi}$ and an analytic version $\Delta_{\chi}(\,\cdot\,)^{\mathrm{an}}$ of the functor Δ_{χ} (Section 6).

For technical reasons we have to assume at some point in this paper that $L = \mathbb{Q}_p$, with $p > 2$ an odd prime. (However, we have no doubts that our results eventually extend to general L and p). To describe our proposed locally analytic "localization functor" under this assumption we let $D(G)$ be the algebra of K-valued locally analytic distributions on G. It naturally contains $U(\mathfrak{g})$. Recall that the category of admissible locally analytic G-representations over K (in the sense of Schneider and J. Teitelbaum [2003]) is antiequivalent to a full abelian subcategory of the (left) $D(G)$-modules, the so-called coadmissible modules. A similar result holds over any compact open subgroup $U_z^{(e)}$.

From now on we fix a central character

$$\theta : Z(\mathfrak{g}_K) \to K$$

and a toral character $\chi \in \mathfrak{t}_K^*$ associated to θ via the classical Harish-Chandra homomorphism. For example, the character $\chi = \rho$ corresponds to the trivial infinitesimal character θ_0 with $\ker\theta_0 = Z(\mathfrak{g}_K) \cap U(\mathfrak{g}_K)\mathfrak{g}_K$. The ring $Z(\mathfrak{g}_K)$ lies in the center of the ring $D(G)$ [Schneider and Teitelbaum 2002, Proposition 3.7], so that we may consider the central reduction

$$D(G)_\theta := D(G) \otimes_{Z(\mathfrak{g}_K),\theta} K.$$

We propose to study the abelian category of (left) $D(G)_\theta$-modules which are coadmissible over $D(G)$. As remarked above it is antiequivalent to the category of admissible locally analytic G-representations over K with infinitesimal character θ. We emphasize that *any* topologically irreducible admissible locally analytic G-representation admits, up to a finite extension of K, an infinitesimal character [Dospinescu and Schraen 2013, Corollary 3.10].

To start with, consider a point $z \in \mathscr{B}$. The group $U_z^{(e)}$ carries a natural p-valuation in the sense of M. Lazard [1965, III.2.1]. According to the general locally analytic theory [Schneider and Teitelbaum 2003, Section 4], this induces a family of norms $\|\cdot\|_r$ on the distribution algebra $D(U_z^{(e)})$ for $r \in [r_0, 1)$, where $r_0 := p^{-1}$. We let $D_r(U_z^{(e)})$ be the corresponding completion of $D(U_z^{(e)})$ and $D_r(U_z^{(e)})_\theta$ its central reduction. In Section 8.2 we introduce sheaves of distribution algebras $\underline{D}_r$ and $\underline{D}_{r,\theta}$

on $\mathscr{B}$ with stalks

$$(\underline{D}_r)_z = D_r(U_z^{(e)}), \quad (\underline{D}_{r,\theta})_z = D_r(U_z^{(e)})_\theta$$

for all points $z \in \mathscr{B}$. The inclusions $U(\mathfrak{g}) \subset D_r(U_z^{(e)})$ sheafify to a morphism $\underline{U}(\mathfrak{g}_K)_\theta \to \underline{D}_{r,\theta}$. Similarly, for any coadmissible $D(G)_\theta$-module M we consider a $\underline{D}_{r,\theta}$-module $\underline{M}_r$ on $\mathscr{B}$ having stalks

$$(\underline{M}_r)_z = D_r(U_z^{(e)})_\theta \otimes_{D(U_z^{(e)})_\theta} M$$

for all points $z \in \mathscr{B}$. The formation of $\underline{M}_r$ is functorial in M. The sheaves $\underline{D}_{r,\theta}$, $\underline{M}_r$ are constructible and will formally replace the constant sheaves appearing in the definition of the functors Δ_χ, $\Delta_\chi^{\mathrm{an}}$.[1]

Consider the restriction of the structure sheaf of X^{an} to $\mathscr{B}$, i.e.,

$$\mathcal{O}_{\mathscr{B}} = \mathcal{O}_{X^{\mathrm{an}}}|_{\mathscr{B}}.$$

We then define a sheaf of noncommutative rings $\mathscr{D}_{r,\chi}$ on $\mathscr{B}$ which is also a module over $\mathcal{O}_{\mathscr{B}}$ and which is vaguely reminiscent of a "sheaf of twisted differential operators". It has a natural G-equivariant structure. It depends on the level e, but, following the usage of Schneider and Stuhler [1997, Section IV.1], we suppress this in our notation. More importantly, it depends on the "radius" r, which is genuine to the locally analytic situation and is related to a choice of completed distribution algebra $D_r(U_z^{(e)})$ at each point $z \in \mathscr{B}$. Completely analogous to constructing $\mathcal{D}_\chi$ out of the skew tensor product algebra $\mathcal{O}_X \# U(\mathfrak{g}_K)$ (cf. [Beilinson and Bernstein 1981]) we obtain the sheaf $\mathscr{D}_{r,\chi}$ out of a skew tensor product algebra of the form $\mathcal{O}_{\mathscr{B}} \# \underline{D}_r$.

To describe the sheaf $\mathscr{D}_{r,\chi}$ we observe first that, for any point $z \in \mathscr{B}$, the inclusion $U_z^{(e)} \subset P_z$ implies that there is a locally analytic $U_z^{(e)}$-action on the analytic stalk $\mathcal{O}_{\mathscr{B},z}$. We therefore have the corresponding skew group ring $\mathcal{O}_{\mathscr{B},z} \# U_z^{(e)}$ as well as the skew enveloping algebra $\mathcal{O}_{\mathscr{B},z} \# U(\mathfrak{g})$, familiar objects from noncommutative ring theory [McConnell and Robson 1987]. In Section 3 and in Sections 6.3 and 6.4, we explain how the completed tensor product

$$\mathcal{O}_{\mathscr{B},z} \hat{\otimes}_L D_r(U_z^{(e)})$$

can be endowed with a unique structure of a topological K-algebra such that the $\mathcal{O}_{\mathscr{B},z}$-linear maps

$$\mathcal{O}_{\mathscr{B},z} \# U_z^{(e)} \to \mathcal{O}_{\mathscr{B},z} \hat{\otimes}_L D_r(U_z^{(e)}), \quad \mathcal{O}_{\mathscr{B},z} \# U(\mathfrak{g}) \to \mathcal{O}_{\mathscr{B},z} \hat{\otimes}_L D_r(U_z^{(e)}), \quad (1.1.2)$$

[1] We assume from now on that e is sufficiently large (later in the paper we require $e > e_{\mathrm{st}}$, where e_{st} is defined in Lemma 6.2.6) and that the radius r is equal to $\sqrt[p^m]{1/p}$ for some $m \geq 0$; see Lemma 7.4.7.

induced by $U_z^{(e)} \subset D(U_z^{(e)})^\times$ and $U(\mathfrak{g}) \subset D(U_z^{(e)})$ respectively, become ring homomorphisms. To emphasize this skew multiplication we denote the target of the two maps in (1.1.2) by $\mathcal{O}_{\mathscr{B},z}\# D_r(U_z^{(e)})$, keeping in mind that there is a *completed* tensor product involved. This process leads to a sheaf of K-algebras $\mathcal{O}_{\mathscr{B}}\#\underline{D}_r$ on $\mathscr{B}$ with stalks

$$(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)_z = \mathcal{O}_{\mathscr{B},z}\# D_r(U_z^{(e)})$$

at points $z \in \mathscr{B}$. It comes equipped with a morphism $\mathcal{O}_{\mathscr{B}}\# U(\mathfrak{g}) \to \mathcal{O}_{\mathscr{B}}\#\underline{D}_r$ giving back the second map in (1.1.2) at a point $z \in \mathscr{B}$.

To generalize the formalism of *twisting* to this new situation we proceed similarly to [Beilinson and Bernstein 1981]. Let $\mathcal{T}_{X^{\mathrm{an}}}$ be the tangent sheaf of X^{an} and let $\alpha^{\mathrm{an}} : \mathfrak{g} \to \mathcal{T}_{X^{\mathrm{an}}}$ be the analytification of the map $\alpha|_{\mathfrak{g}}$. There is the sheaf of L-Lie algebras

$$\mathfrak{b}^{\circ,\mathrm{an}} := \ker(\mathcal{O}_{X^{\mathrm{an}}} \otimes_L \mathfrak{g} \xrightarrow{\alpha^{\mathrm{an}}} \mathcal{T}_{X^{\mathrm{an}}}).$$

The inclusion $\boldsymbol{T} \subset \boldsymbol{B}$ induces an isomorphism of Lie algebras

$$\mathcal{O}_{X^{\mathrm{an}}} \otimes_L \mathfrak{t} \xrightarrow{\sim} \mathfrak{b}^{\circ,\mathrm{an}}/[\mathfrak{b}^{\circ,\mathrm{an}}, \mathfrak{b}^{\circ,\mathrm{an}}].$$

We have thus an obvious $\mathcal{O}_{X^{\mathrm{an}}}$-linear extension of the character $\chi - \rho$ of $\mathfrak{t}_K$ to $\mathfrak{b}^{\circ,\mathrm{an}} \otimes_L K$. Its kernel, restricted to the building $\mathscr{B}$, generates a two-sided ideal $\mathscr{I}_\chi^{\mathrm{an}}$ in $\mathcal{O}_{\mathscr{B}}\#\underline{D}_r$ and we set

$$\mathscr{D}_{r,\chi} := \big(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r\big)/\mathscr{I}_\chi^{\mathrm{an}}.$$

Let $\mathcal{D}_{\mathscr{B},\chi}^{\mathrm{an}}$ denote the restriction of $\mathcal{D}_\chi^{\mathrm{an}}$ to the building $\mathscr{B}$. The sheaf $\mathscr{D}_{r,\chi}$ comes with an algebra homomorphism $\mathcal{D}_{\mathscr{B},\chi}^{\mathrm{an}} \to \mathscr{D}_{r,\chi}$ induced from the inclusion $\mathcal{O}_{\mathscr{B}}\# U(\mathfrak{g}_K) \to \mathcal{O}_{\mathscr{B}}\#\underline{D}_r$. Most importantly, the canonical morphism $\underline{D}_r \to \mathcal{O}_{\mathscr{B}}\#\underline{D}_r$ induces a canonical morphism $\underline{D}_{r,\theta} \to \mathscr{D}_{r,\chi}$ making the diagram

$$\begin{array}{ccc} \underline{U}(\mathfrak{g}_K)_\theta & \longrightarrow & \mathcal{D}_{\mathscr{B},\chi}^{\mathrm{an}} \\ \downarrow & & \downarrow \\ \underline{D}_{r,\theta} & \longrightarrow & \mathscr{D}_{r,\chi} \end{array}$$

commutative. In this situation we prove that

$$M \mapsto \mathscr{L}_{r,\chi}(M) := \mathscr{D}_{r,\chi} \otimes_{\underline{D}_{r,\theta}} \underline{M}_r$$

is an exact covariant functor from coadmissible $D(G)_\theta$ modules into G-equivariant (left) $\mathscr{D}_{r,\chi}$-modules. The stalk of the sheaf $\mathscr{L}_{r,\chi}(M)$ at a point $z \in \mathscr{B}$ with residue field $\kappa(z)$ equals the $(\chi-\rho)$-coinvariants of the $\mathfrak{t}_K$-module

$$(\kappa(z)\hat{\otimes}_L \underline{M}_{r,z})/\mathfrak{n}_{\pi(z)}(\kappa(z)\hat{\otimes}_L \underline{M}_{r,z})$$

as it should [Beilinson and Bernstein 1981]. Here, $\mathfrak{n}_{\pi(z)}$ equals the nilpotent radical of the Borel subalgebra of $\kappa(z) \otimes_L \mathfrak{g}$ defined by the point $\pi(z) \in X$, where $\pi : X^{\mathrm{an}} \to X$ is the canonical map. We tentatively call $\mathscr{L}_{r,\chi}$ a locally analytic "localization functor". We suppress the dependence of $\mathscr{L}_{r,\chi}$ on the level e in our notation.

We prove the following compatibilities with the Schneider–Stuhler and the Beilinson–Bernstein localizations. Suppose first that the coadmissible module M is associated to a *smooth* G-representation V. Since $\mathfrak{g}M = 0$ it has infinitesimal character $\theta = \theta_0$ and the natural choice of twisting is therefore $\chi = \rho$. We establish a canonical isomorphism (Theorem 9.2.5) of $\mathcal{O}_{\mathscr{B}}$-modules

$$\mathscr{L}_{r_0,\rho}(M) \xrightarrow{\sim} \mathcal{O}_{\mathscr{B}} \otimes_L \underset{\approx}{\check{V}}$$

where $\check{V}$ is the smooth dual of V and $\underset{\approx}{\check{V}}$ the sheaf associated to $\check{V}$ by Schneider–Stuhler. The isomorphism is natural in M.

Secondly, suppose the coadmissible module M is associated to a *finite dimensional algebraic* G-representation. The functor $\Delta_\chi(\cdot)^{\mathrm{an}}$ may be applied to its underlying $\mathfrak{g}$-module and gives a $\mathcal{D}^{\mathrm{an}}_\chi$-module on X^{an} and then, via restriction, a $\mathcal{D}^{\mathrm{an}}_{\mathscr{B},\chi}$-module $\Delta_\chi(M)^{\mathrm{an}}_{\mathscr{B}}$ on $\mathscr{B}$. We prove (Theorem 10.1.1) that there is a number $r(M) \in [r_0, 1)$ which is intrinsic to M and a canonical isomorphism of $\mathcal{D}^{\mathrm{an}}_{\mathscr{B},\chi}$-modules

$$\mathscr{L}_{r,\chi}(M) \xrightarrow{\sim} \Delta_\chi(M)^{\mathrm{an}}_{\mathscr{B}}$$

for $r \geq r(M)$. The isomorphism is natural in M.

As a class of examples we finally investigate the localizations of locally analytic representations in the image of the functor $\mathcal{F}^G_B$ introduced by S. Orlik and investigated in [Orlik and Strauch 2010a]. The image of $\mathcal{F}^G_B$ comprises a wide class of interesting representations and contains all principal series representations as well as all locally algebraic representations (e.g., tensor products of smooth with algebraic representations).

This paper is the first of a series of papers whose aim is to develop a localization theory for locally analytic representations. Here we only make a first step in this direction, focusing on the building and merging the theory of Schneider and Stuhler with the theory of Beilinson and Bernstein, resp. Brylinski and Kashiwara. One approach to get a more complete picture would be to extend the construction given here to a compactification $\overline{\mathscr{B}}$ of the building. The compactification which one would take here is, of course, the closure of $\mathscr{B}$ in X^{an}. Moreover, for intended applications like functorial resolutions and the computation of Ext groups, one has to develop a "homological theory", in analogy to [Schneider and Stuhler 1997, Section II]. However, the sheaves produced in this way (using a compactification) would still have too many global sections. For instance, the space of global sections would be a module for the ring of meromorphic functions on X^{an} with poles outside $\overline{\mathscr{B}}$, and this is a very large ring. The aim would be to produce sheaves whose global sections give back the $D(G)$-module one started with. In [Patel et al. 2013] we explore an approach (in the case of GL(2)) which is based on the use of (a family of) semistable formal models $\mathfrak{X}$ of X^{an}, and we replace $\mathcal{O}_{\mathscr{B}}$ by the pull-back of $\mathcal{O}_{\mathfrak{X}} \otimes L$ via the specialization map $X^{\text{an}} \to \mathfrak{X}$, and the rôle of $\mathscr{D}_{r,\chi}$ is played by arithmetic logarithmic differential operators. In this regard we want to mention related works by C. Noot-Huyghe [2009], and K. Ardakov and S. Wadsley [2013]. While Noot-Huyghe studies localizations of arithmetic $\mathscr{D}$-modules on smooth formal models of X, Ardakov and Wadsley define and study localizations of representations of Iwasawa algebras on smooth models. Our present paper is in some sense complementary to these papers, as our focus is on noncompact groups.

Despite the many aspects (like compactifications, homological theory, relation with formal models) that still have to be explored, given the many technical details that one has to take care of we thought it worthwhile to give an account of the constructions as developed up to this point.

Notation. Let p be an odd prime. Let $L/\mathbb{Q}_p$ be a finite extension and $K \subseteq \mathbb{C}_p$ a complete discretely valued extension of L. The absolute value $|\cdot|$ on $\mathbb{C}_p$ is normalized by $|p| = p^{-1}$. Let $o_L \subset L$ be the ring of integers and $\varpi_L \in o_L$ a uniformizer. We denote by v_L always the normalized p-adic valuation on L, i.e., $v_L(\varpi) = 1$. Let n and $e(L/\mathbb{Q}_p)$ be the degree and the ramification index of the extension $L/\mathbb{Q}_p$ respectively. Similarly, $o_K \subset K$ denotes the integers in K and $\varpi_K \in o_K$ a uniformizer. Let $k := o_K/(\varpi_K)$ denote the residue field of K.

The letter $\boldsymbol{G}$ always denotes a connected reductive linear algebraic group over L which is split over L and $G = \boldsymbol{G}(L)$ denotes its group of rational points.

2. Distribution algebras and locally analytic representations

For notions and notation from nonarchimedean functional analysis we refer to [Schneider 2002]. If not indicated otherwise, topological tensor products of locally convex vector spaces are always taken with respect to the projective tensor product topology.

2.1. ***Distribution algebras.*** In this section we recall some definitions and results about algebras of distributions attached to locally analytic groups [Schneider and Teitelbaum 2002; 2003]. We consider a locally L-analytic group H and denote by $C^{\text{an}}(H, K)$ the locally convex K-vector space of locally L-analytic functions on H as defined in [Schneider and Teitelbaum 2002]. The strong dual

$$D(H, K) := C^{\text{an}}(H, K)'_b$$

is the algebra of K-valued locally analytic distributions on H where the multiplication is given by the usual convolution product. This multiplication is separately continuous. However, if H is compact, then $D(H, K)$ is a K-Fréchet algebra. The algebra $D(H, K)$ comes equipped with a continuous K-algebra homomorphism

$$\Delta : D(H, K) \to D(H, K) \hat{\otimes}_{K,\iota} D(H, K)$$

which has all the usual properties of a comultiplication [Schneider and Teitelbaum 2005, Section 3 and Appendix]. Here ι refers to the (complete) inductive tensor product.[2] If H is compact, then $D(H, K)$ is a Fréchet space and the inductive and projective tensor product topology on the right-hand side coincide [Schneider 2002, 17.6]. Of course, $\Delta(\delta_h) = \delta_h \otimes \delta_h$ for $h \in H$.

The universal enveloping algebra $U(\mathfrak{h})$ of the Lie algebra $\mathfrak{h} := \mathrm{Lie}(H)$ of H acts naturally on $C^{\mathrm{an}}(H, K)$. On elements $\mathfrak{x} \in \mathfrak{h}$ this action is given by

$$(\mathfrak{x} f)(h) = \frac{d}{dt}\big(t \mapsto f(\exp_H(-t\mathfrak{x})h)\big)|_{t=0},$$

where $\exp_H : \mathfrak{h} \dashrightarrow H$ denotes the exponential map of H, defined in a small neighborhood of 0 in $\mathfrak{h}$. This gives rise to an embedding of $U(\mathfrak{h})_K := U(\mathfrak{h}) \otimes_L K$ into $D(H, K)$ via

$$U(\mathfrak{h})_K \hookrightarrow D(H, K), \quad \mathfrak{x} \mapsto (f \mapsto (\dot{\mathfrak{x}} f)(1)).$$

Here $\mathfrak{x} \mapsto \dot{\mathfrak{x}}$ is the unique antiautomorphism of the K-algebra $U(\mathfrak{h})_K$ that induces multiplication by -1 on $\mathfrak{h}$. The comultiplication Δ restricted to $U(\mathfrak{g})_K$ gives the usual comultiplication of the Hopf algebra $U(\mathfrak{g})_K$, i.e., $\Delta(\mathfrak{x}) = \mathfrak{x} \otimes 1 + 1 \otimes \mathfrak{x}$ for all $\mathfrak{x} \in \mathfrak{h}$.

2.2. *Norms and completions of distribution algebras.*

2.2.1. *p-valuations.* Let H be a compact locally $\mathbb{Q}_p$-analytic group. Recall (see [Lazard 1965]) that a *p-valuation* ω on H is a real-valued function $\omega : H \setminus \{1\} \to (1/(p-1), \infty) \subset \mathbb{R}$ satisfying

(i) $\omega(gh^{-1}) \geq \min\,(\omega(g), \omega(h))$,

(ii) $\omega(g^{-1}h^{-1}gh) \geq \omega(g) + \omega(h)$,

(iii) $\omega(g^p) = \omega(g) + 1$,

for all $g, h \in H$. As usual one puts $\omega(1) := \infty$ and interprets the above inequalities in the obvious sense if a term $\omega(1)$ occurs.

Let ω be a p-valuation on H. It follows from [loc. cit., III.3.1.3/7/9] that the topology on H is defined by ω [loc. cit., II.1.1.5] and H is a pro-p group. Moreover,

[2] This is the only exception to our general convention to only consider the projective tensor product topology.

there is a topological generating system $h_1, \ldots, h_d$ of H such that the map

$$\mathbb{Z}_p^d \to H, \quad (a_1, \ldots, a_d) \mapsto h_1^{a_1} \cdots h_d^{a_d}$$

is well-defined and a homeomorphism. Moreover,

$$\omega(h_1^{a_1} \cdots h_d^{a_d}) = \min\{\omega(h_i) + v_p(a_i) \mid i = 1, \ldots, d\},$$

where v_p denotes the p-adic valuation on $\mathbb{Z}_p$. The sequence $(h_1, \ldots, h_d)$ is called a *p-basis* (or an *ordered basis*; see [Schneider and Teitelbaum 2003, Section 4]) of the p-valued group (H, ω).

Finally, a p-valued group (H, ω) is called *p-saturated* if any $g \in H$ such that $\omega(g) > p/(p-1)$ is a p-th power in H.

2.2.2. *The canonical p-valuation on uniform pro-p groups.* We recall some definitions and results about pro-p groups [Dixon et al. 1999, Chapters 3 and 4] in the case $p \neq 2$. In this subsection H will be a pro-p group which is equipped with its topology of a profinite group. Then H is called *powerful* if H/H^p is abelian. Here, H^p is the closure of the subgroup generated by the p-th powers of its elements. If H is topologically finitely generated one can show that the subgroup H^p is open and hence automatically closed. The lower p-series $(P_i(H))_{i \geq 1}$ of an arbitrary pro-p group H is defined inductively by

$$P_1(H) := H, \quad P_{i+1}(H) := \overline{P_i(H)^p [P_i(H), H]}.$$

If H is topologically finitely generated, then the groups $P_i(H)$ are all open in H and form a fundamental system of neighborhoods of 1 [loc. cit., Proposition 1.16]. A pro-p group H is called *uniform* if it is topologically finitely generated, powerful and its lower p-series satisfies $(H : P_2(H)) = (P_i(H) : P_{i+1}(H))$ for all $i \geq 1$. If H is a topologically finitely generated powerful pro-p group then $P_i(H)$ is a uniform pro-p group for all sufficiently large i [loc. cit., 4.2]. Moreover, any compact $\mathbb{Q}_p$-analytic group contains an open normal uniform pro-p subgroup [loc. cit., 8.34]. According to [loc. cit., Theorem 9.10], any uniform pro-p group H determines a powerful $\mathbb{Z}_p$-Lie algebra $\mathcal{L}(H)$.[3] Now let H be a uniform pro-p group. It carries a distinguished p-valuation ω^{can} which is associated to the lower p-series and which we call the *canonical p-valuation*. For $h \neq 1$, it is defined by $\omega^{\text{can}}(h) = \max\{i \geq 1 : h \in P_i(H)\}$.

2.2.3. *Norms arising from p-valuations.* In this section we let H be a compact $\mathbb{Q}_p$-analytic group endowed with a p-valuation ω that has rational values. For convenience of the reader we briefly recall [Schneider and Teitelbaum 2003, Section 4] the construction of a suitable family of submultiplicative norms $\|\cdot\|_r$, $r \in [1/p, 1)$ on the algebra $D(H, K)$.

[3]The adjective *powerful* refers here to the property $[\mathcal{L}(H), \mathcal{L}(H)] \subseteq p\mathcal{L}(H)$.

Let $h_1, \dots, h_d$ be an ordered basis for (H, ω). The homeomorphism $\psi : \mathbb{Z}_p^d \simeq H$ given by $(a_1, \dots, a_d) \mapsto h_1^{a_1} \cdots h_d^{a_d}$ is a global chart for the $\mathbb{Q}_p$-analytic manifold H. By functoriality of $C^{\mathrm{an}}(\,\cdot\,, K)$ it induces an isomorphism

$$\psi^* : C^{\mathrm{an}}(H, K) \xrightarrow{\sim} C^{\mathrm{an}}(\mathbb{Z}_p^d, K)$$

of topological K-vector spaces. Using Mahler expansions [Lazard 1965, III.1.2.4] we may express elements of $C(\mathbb{Z}_p^d, K)$, the space of continuous K-valued functions on $\mathbb{Z}_p^d$, as series $f(x) = \sum_{\alpha \in \mathbb{N}_0^d} c_\alpha \binom{x}{\alpha}$, where $c_\alpha \in K$ and $\binom{x}{\alpha} = \binom{x_1}{\alpha_1} \cdots \binom{x_d}{\alpha_d}$ for $x = (x_1, \dots, x_d) \in \mathbb{Z}_p^d$ and multi-indices $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$. Further, we have $|c_\alpha| \to 0$ as $|\alpha| = \alpha_1 + \cdots + \alpha_d \to \infty$. A continuous function $f \in C(\mathbb{Z}_p^d, K)$ is locally analytic if and only if $|c_\alpha| r^{|\alpha|} \to 0$ for some real number $r > 1$ [loc. cit., III.1.3.9].

Put $b_i := h_i - 1 \in \mathbb{Z}[H]$ and $\boldsymbol{b}^\alpha := b_1^{\alpha_1} \cdots b_d^{\alpha_d}$ for $\alpha \in \mathbb{N}_0^d$. Identifying group elements with Dirac distributions induces a K-algebra embedding $K[H] \hookrightarrow D(H, K)$, $h \mapsto \delta_h$. In the light of the dual isomorphism $\psi_* : D(\mathbb{Z}_p^d, K) \xrightarrow{\sim} D(H, K)$ we see that any $\delta \in D(H, K)$ has a unique convergent expansion $\delta = \sum_{\alpha \in \mathbb{N}_0^d} d_\alpha \boldsymbol{b}^\alpha$ with $d_\alpha \in K$ such that the set $\{|d_\alpha| r^{|\alpha|}\}_\alpha$ is bounded for all $0 < r < 1$. Conversely, any such series is convergent in $D(H, K)$. By construction the value $\delta(f) \in K$ of such a series on a function $f \in C^{\mathrm{an}}(H, K)$ equals $\delta(f) = \sum_\alpha d_\alpha c_\alpha$, where c_α are the Mahler coefficients of $\psi^*(f)$.

To take the original p-valuation ω into account we define $\tau\alpha := \sum_i \omega(h_i)\alpha_i$ for $\alpha \in \mathbb{N}_0^d$. The family of norms $\|\cdot\|_r$, $0 < r < 1$, on $D(H, K)$ defined on a series δ as above via $\|\delta\|_r := \sup_\alpha |d_\alpha| r^{\tau\alpha}$ defines the Fréchet topology on $D(H, K)$. Let $D_r(H, K)$ denote the norm completion of $D(H, K)$ with respect to $\|\cdot\|_r$. Thus we obtain

$$D_r(H, K) = \left\{ \sum_{\alpha \in \mathbb{N}_0^d} d_\alpha \boldsymbol{b}^\alpha \mid d_\alpha \in K,\ \lim_{|\alpha| \to \infty} |d_\alpha| r^{\tau\alpha} = 0 \right\}.$$

There is an obvious norm-decreasing linear map $D_{r'}(H, K) \to D_r(H, K)$ whenever $r \le r'$.

The norms $\|\cdot\|_r$ belonging to the subfamily $\frac{1}{p} \le r < 1$ are submultiplicative [loc. cit., Proposition 4.2] and do not depend on the choice of ordered basis [loc. cit., before Theorem 4.11]. In particular, each $D_r(H, K)$ is a K-Banach algebra in this case. If we equip the projective limit $\varprojlim_r D_r(H, K)$ with the projective limit topology the natural map

$$D(H, K) \xrightarrow{\sim} \varprojlim_r D_r(H, K)$$

is an isomorphism of topological K-algebras. Finally, it is easy to see that the comultiplication Δ completes to continuous "comultiplications"

$$\Delta_r : D_r(H, K) \to D_r(H, K) \hat{\otimes}_K D_r(H, K)$$

for any r in the above range. We make two final remarks in case H is a uniform pro-p group and ω is its canonical p-valuation; see Section 2.2.2. In this case each group $P_m(H), m \geq 0$ is a uniform pro-p group.

(i) For $r = \frac{1}{p}$ there is a canonical isomorphism between $D_{1/p}(H, \mathbb{Q}_p)$ and the p-adic completion (with p inverted) of the universal enveloping algebra of the $\mathbb{Z}_p$-Lie algebra $\frac{1}{p}\mathcal{L}(H)$ [Ardakov and Wadsley 2013, Theorem 10.4, Remark 10.5(c)].

(ii) Let

$$r_m := \sqrt[p^m]{1/p}$$

for $m \geq 0$. In particular, $r_0 = 1/p$. Since $P_{m+1}(H)$ is uniform pro-p we may consider the corresponding $\|\cdot\|_{r_0}$- norm on its distribution algebra $D(P_{m+1}(H))$. In this situation the ring extension $D(P_{m+1}(H)) \subset D(H)$ completes in the $\|\cdot\|_{r_m}$-norm topology on $D(H)$ to a ring extension

$$D_{r_0}(P_{m+1}(H)) \subset D_{r_m}(H)$$

and $D_{r_m}(H)$ is a finite and free (left or right) module over $D_{r_0}(P_{m+1}(H)))$ with basis given by any system of coset representatives for the finite group $H/P_{m+1}(H)$ [Schmidt 2013, Lemma 5.11].

2.3. ***Coadmissible modules.*** We keep all notations from the preceding section but suppose that the p-valuation ω on H satisfies additionally

(HYP) (H, ω) is p-saturated and the ordered basis $h_1, \dots, h_d$ of H satisfies $\omega(h_i) + \omega(h_j) > p/(p-1)$ for any $1 \leq i \neq j \leq d$.

Remark. This implies that H is a uniform pro-p group. Conversely, the canonical p-valuation on a uniform pro-p group (p arbitrary) satisfies (HYP). For both statements we refer to [Schneider and Teitelbaum 2003, Remark before Lemma 4.4] and [Schmidt 2008, Proposition 2.1].

Suppose in the following $r \in (p^{-1}, 1)$ and $r \in p^{\mathbb{Q}}$. In this case the norm $\|\cdot\|_r$ on $D_r(H, K)$ is multiplicative and $D_r(H, K)$ is a (left and right) noetherian integral domain [Schneider and Teitelbaum 2003, Theorem 4.5]. For two numbers $r \leq r'$ in the given range the ring homomorphism

$$D_{r'}(H, K) \to D_r(H, K)$$

makes the target a flat (left or right) module over the source [Schneider and Teitelbaum 2003, Theorem 4.9]. The above isomorphism $D(H, K) \xrightarrow{\sim} \varprojlim_r D_r(H, K)$ realizes therefore a *Fréchet–Stein structure* on $D(H, K)$ in the sense of [loc. cit., Section 3]. The latter allows one to define a well-behaved abelian full subcategory $\mathcal{C}_H$

of the (left) $D(H, K)$-modules, the so-called *coadmissible modules*. By definition, an abstract (left) $D(H, K)$-module M is coadmissible if for all r in the given range

(i) $M_r := D_r(H, K) \otimes_{D(H,K)} M$ is finitely generated over $D_r(H, K)$,

(ii) the natural map $M \xrightarrow{\sim} \varprojlim_r M_r$ is an isomorphism.

The projective system $\{M_r\}_r$ is sometimes called the *coherent sheaf* associated to M. To give an example, any finitely presented $D(H, K)$-module is coadmissible.

More generally, for any compact locally L-analytic group H the ring $D(H, K)$ has the structure of a Fréchet–Stein algebra [loc. cit., Theorem 5.1]. In particular, we may define the notion of a coadmissible module over $D(H, K)$ for any compact L-analytic group in a similar manner. For a general locally L-analytic group G, a $D(G, K)$-module M is coadmissible if it is coadmissible as a $D(H, K)$-module for every compact open subgroup $H \subset G$. It follows from [loc. cit.] that it is sufficient to check this for a single compact open subgroup.

2.4. ***Locally analytic representations.*** A topological abelian group M which is a (left) module over a topological ring R is a *separately continuous* (left) module, if the map $R \times M \to M$ giving the action is separately continuous. Any separately continuous bilinear map between Fréchet spaces is jointly continuous [Bourbaki 1987, III.30, Corollary 1].

After this preliminary remark, we recall some facts about locally analytic representations. A K-vector space V which equals a locally convex inductive limit $V = \varinjlim_{n\in\mathbb{N}} V_n$ over a countable system of K-Banach spaces V_n, where the transition maps $V_n \to V_{n+1}$ are injective compact linear maps is called a vector space *of compact type*. We recall that such a space is Hausdorff, complete, bornological and reflexive [Schneider and Teitelbaum 2002, Theorem 1.1]. Its strong dual V'_b is a nuclear Fréchet space satisfying $V'_b = \varprojlim_n (V_n)'_b$. We will make frequent use of the following property of such spaces.

Proposition 2.4.1. *Let W be a K-Banach space. The continuous linear map*

$$\pi : \varinjlim V_n \hat{\otimes}_K W \to (\varinjlim V_n) \hat{\otimes}_K W$$

is bijective and the source of π is Hausdorff. Here, the target of π equals the Hausdorff completion of the projective tensor product $(\varinjlim V_n) \otimes_K W$.

Proof. The first assertion follows from [Schneider and Teitelbaum 2002, Proposition 1.5] together with [Schneider 2002, Corollary 18.8]. Since the target of π is Hausdorff, the second assertion follows from [Schneider 2002, 4.6]. □

Now let H be a locally L-analytic group, V a Hausdorff locally convex K-vector space and $\rho : H \to \mathrm{GL}(V)$ a homomorphism. Then V (or the pair (V, ρ)) is called a *locally analytic representation of H* if the topological K-vector space V is barrelled, each $h \in H$ acts K-linearly and continuously on V, and the orbit maps $\rho_v : H \to V$, $h \mapsto \rho(h)(v)$ are locally analytic maps for all $v \in V$ [Schneider and Teitelbaum 2002, Section 3]. If V is of compact type, then the contragredient H-action on its strong dual V'_b extends to a separately continuous left $D(H, K)$-module on a nuclear Fréchet space.

In this way the functor $V \mapsto V'_b$ induces an antiequivalence of categories between locally analytic H-representations on K-vector spaces of compact type (with continuous linear H-maps as morphisms) and separately continuous $D(H, K)$-modules on nuclear Fréchet spaces (with continuous $D(H, K)$-module maps as morphisms).

A locally analytic H-representation V is said to be admissible if V'_b is a coadmissible $D(H, K)$-module. The above functor restricts to an antiequivalence between the corresponding categories of admissible locally analytic representations and coadmissible $D(H, K)$-modules.

3. Completed skew group rings

In this section we will describe a general method of completing certain skew group rings. We recall our general convention that in this paper we only consider the completed tensor product of locally convex vector spaces with respect to the projective tensor product topology.[4]

3.1. ***Preliminaries.*** Let H be a compact locally L-analytic group and let A be a locally convex L-algebra equipped with a locally analytic H-representation $\rho : H \to \mathrm{GL}(A)$ by automorphisms of L-algebras. The H-action on A extends to $D(H, L)$ and makes A a separately continuous $D(H, L)$-module [Schneider and Teitelbaum 2002, Proposition 3.2]. On the other hand, $D(H, L)$ is a topological module over itself via left multiplication. The completion $A \hat{\otimes}_L D(H, L)$ is thus a separately continuous $D(H, L) \hat{\otimes}_L D(H, L)$-module. We view it as a separately continuous $D(H, L)$-module by restricting scalars via the comultiplication Δ. This allows us to define the L-bilinear map

$$(A \otimes_L D(H, L)) \times (A \hat{\otimes}_L D(H, L)) \to A \hat{\otimes}_L D(H, L)$$

given by $\left(\sum_i f_i \otimes \delta_i, b\right) \mapsto \sum_i f_i \cdot \delta_i(b)$. We consider the product topology on the source. In view of the separate continuity of all operations involved together with [Schneider 2002, Lemma 17.1] this map is separately continuous. Since the target is complete it extends in a bilinear and separately continuous manner to the completion of the source. In other words, $A \hat{\otimes}_L D(H, L)$ becomes a separately continuous L-algebra. Of course, $A \hat{\otimes}_L D(H, K)$ is then a separately continuous K-algebra. To emphasize its skew multiplication we denote it in the following

[4] The only exception occurred in Section 2.1.

by
$$A \,\#_L\, D(H, K)$$
or even by $A\#D(H, K)$. This should not cause confusion. However, one has to keep in mind that there is a *completed* tensor product involved. If A is a Fréchet algebra, then the multiplication on $A\#D(H, K)$ is jointly continuous, i.e., $A\#D(H, K)$ is a topological algebra in the usual sense.

3.2. *Skew group rings, skew enveloping algebras and their completions.*

3.2.1. Using the action ρ we may form the abstract skew group ring $A\#H$ [McConnell and Robson 1987, 1.5.4]. We remind the reader that it equals the free left A-module with elements of H as a basis and with multiplication defined by $(ag)\cdot(bh) := a(\rho(g)(b))gh$ for any $a, b \in A$ and $g, h \in H$. Each element of $A\#H$ has a unique expression as $\sum_{h\in H} a_h h$ with $a_h = 0$ for all but finitely many $h \in H$. Evidently, $A\#H$ contains H as a subgroup of its group of units and A as a subring. Furthermore, the inclusion $L[H] \subseteq D(H, L)$ gives rise to an A-linear map
$$A\#H = A \otimes_L L[H] \to A\#D(H, L). \tag{3.2.2}$$

On the other hand, let $\mathfrak{h} := \mathrm{Lie}(H)$. Differentiating the locally analytic action ρ gives a homomorphism of L-Lie algebras $\alpha : \mathfrak{h} \to \mathrm{Der}_L(A)$ into the L-derivations of the algebra A making the diagram
$$\begin{array}{ccc} U(\mathfrak{h}) & \xrightarrow{\ \alpha\ } & \mathrm{End}_L(A) \\ \big\downarrow \subseteq & & \big\downarrow Id \\ D(H, L) & \xrightarrow{\ \rho\ } & \mathrm{End}_L(A) \end{array}$$
commutative [Schneider and Teitelbaum 2002, 3.1]. We may therefore form the *skew enveloping algebra* $A\#U(\mathfrak{h})$ [McConnell and Robson 1987, 1.7.10]. We recall that this is an L-algebra whose underlying L-vector space equals the tensor product $A \otimes_L U(\mathfrak{h})$. The multiplication is defined by
$$(f_1 \otimes \mathfrak{x}_1)\cdot(f_2 \otimes \mathfrak{x}_2) = (f_1\alpha(\mathfrak{x}_1)(f_2)) \otimes \mathfrak{x}_2 + (f_1 f_2) \otimes (\mathfrak{x}_1\mathfrak{x}_2),$$
for $f_i \otimes \mathfrak{x}_i \in A \otimes_L \mathfrak{h}$. Also, the inclusion $U(\mathfrak{h}) \subseteq D(H, L)$ induces an A-linear map
$$A\#U(\mathfrak{h}) \to A\#D(H, L). \tag{3.2.3}$$

Proposition 3.2.4. *The A-linear maps (3.2.2) and (3.2.3) are L-algebra homomorphisms. The first of these maps has dense image.*

Proof. The first statement follows from the identities

(i) $(1 \hat{\otimes} \delta_g)\cdot(f \hat{\otimes} 1) = (\rho(g)(f)) \hat{\otimes} \delta_g$ for any $g \in H$, $f \in A$,

(ii) $(1 \hat{\otimes} \mathfrak{x})\cdot(f \hat{\otimes} 1) = (\alpha(\mathfrak{x})(f)) \hat{\otimes} 1 + f \hat{\otimes} \mathfrak{x}$ for any $\mathfrak{x} \in \mathfrak{h}$, $f \in A$

in $A \hat{\otimes}_L D(H, L)$. In turn these identities follow from $\Delta(\delta_g) = \delta_g \hat{\otimes} \delta_g$ and $\Delta(\mathfrak{x}) = \mathfrak{x} \hat{\otimes} 1 + 1 \hat{\otimes} \mathfrak{x}$. The final statement follows from [Schneider and Teitelbaum 2002, Lemma 3.1]. □

3.2.5. In this paragraph we assume that $L = \mathbb{Q}_p$ and that the compact locally $\mathbb{Q}_p$-analytic group H is endowed with a p-valuation ω. Recall from Section 2.2.3 that $r_0 := p^{-1}$. Consider the norm completion $D_r(H, L)$ for some arbitrary but fixed $r \in [r_0, 1)$. Let us assume for a moment that the natural map $D(H, L) \to D_r(H, L)$ satisfies the following hypothesis:

($\star$) The separately continuous $D(H, L)$-module structure of A extends to a separately continuous $D_r(H, L)$-module structure.

If we replace in the above discussion the comultiplication Δ by its completion Δ_r we obtain in an entirely analogous manner a completion $A \hat{\otimes}_L D_r(H, K)$ of the skew group ring $A\#H$, base changed to K. It satisfies *mutatis mutandis* the statement of the preceding proposition. As before we will often abbreviate it by $A\#D_r(H, K)$.

4. Sheaves on the Bruhat–Tits building and smooth representations

4.1. *Filtrations of stabilizer subgroups.*

4.1.1. Let $\boldsymbol{T}$ be a maximal L-split torus in $\boldsymbol{G}$. Let $X^*(\boldsymbol{T})$ resp. $X_*(\boldsymbol{T})$ be the group of algebraic characters resp. cocharacters of $\boldsymbol{T}$. Let $\Phi = \Phi(\boldsymbol{G}, \boldsymbol{T}) \subset X^*(\boldsymbol{T})$ denote the root system determined by the adjoint action of $\boldsymbol{T}$ on the Lie algebra of $\boldsymbol{G}$. Let W denote the corresponding Weyl group. For each $\alpha \in \Phi$ we have the unipotent root subgroup $\boldsymbol{U}_\alpha \subseteq \boldsymbol{G}$. Since $\boldsymbol{G}$ is split the choice of a *Chevalley basis* determines a system of L-group isomorphisms

$$x_\alpha : \mathbb{G}_a \xrightarrow{\sim} \boldsymbol{U}_\alpha$$

for each $\alpha \in \Phi$ (an *épinglage*) satisfying Chevalley's commutation relations [1955, p. 27]. Let $X_*(\boldsymbol{C})$ denote the group of L-algebraic cocharacters of the connected center $\boldsymbol{C}$ of $\boldsymbol{G}$.

We denote by G, T, U_α the groups of L-rational points of $\boldsymbol{G}, \boldsymbol{T}, \boldsymbol{U}_\alpha (\alpha \in \Phi)$ respectively. Recall the normalized p-adic valuation v_L on L, i.e., $v_L(\varpi) = 1$. For $\alpha \in \Phi$ we denote by $(U_{\alpha,r})_{r \in \mathbb{R}}$ the filtration of U_α arising from the valuation v_L on L via the isomorphism x_α. It is an exhaustive and separated discrete filtration by subgroups. Put $U_{\alpha,\infty} := \{1\}$.

4.1.2. Let $\mathscr{B} = \mathscr{B}(G)$ be the semisimple Bruhat–Tits building of G. The torus $\boldsymbol{T}$ determines an apartment A in $\mathscr{B}$. Recall that a point z in the Coxeter complex A is called *special* if for any direction of wall there is a wall of A actually passing through z [Bruhat and Tits 1972, 1.3.7]. As in [Cartier 1979, 3.5] we choose once and for all a special vertex x_0 in A and a chamber $\mathscr{C} \subset A$ containing it. We use the point x_0 to identify the affine space A with the real vector space

$$A = (X_*(\boldsymbol{T})/X_*(\boldsymbol{C})) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Each root $\alpha \in \Phi$ induces therefore a linear form $\alpha : A \to \mathbb{R}$ in an obvious way. For any nonempty subset $\Omega \subseteq A$ we let $f_\Omega : \Phi \to \mathbb{R} \cup \{\infty\}$, $\alpha \mapsto -\inf_{x\in\Omega} \alpha(x)$. It is a *concave* function in the sense of [Bruhat and Tits 1972, 6.4.1–5]. We emphasize that the concept of a concave function is developed in [loc. cit.] more generally for functions taking values in the set

$$\tilde{\mathbb{R}} := \mathbb{R} \cup \{r+ : r \in \mathbb{R}\} \cup \{\infty\}.$$

The latter naturally has the structure of a totally ordered commutative monoid extending the total order and the addition on $\mathbb{R}$. For any $\alpha \in \Phi$ and $r \in \mathbb{R}$ we define

$$U_{\alpha,r+} := \bigcup_{s\in\mathbb{R},\, s>r} U_{\alpha,s}.$$

For any concave function $f : \Phi \to \tilde{\mathbb{R}}$ we then have the group

$$U_f := \text{subgroup of } G \text{ generated by all } U_{\alpha,f(\alpha)} \text{ for } \alpha \in \Phi. \tag{4.1.3}$$

4.1.4. For each nonempty subset $\Omega \subseteq \mathscr{B}$ we let

$$P_\Omega := \{g \in G : gz = z \text{ for any } z \in \Omega\}$$

be its pointwise stabilizer in G. For any facet $F \subseteq \mathscr{B}$ we will recall from [Schneider and Stuhler 1997, I.2] a certain decreasing filtration of P_F by open normal pro-p subgroups which will be most important for all that follows in this article. To do this we first consider a facet F in the apartment A. For $\alpha \in \Phi$ we put $f_F^*(\alpha) := f_F(\alpha)+$ if $\alpha|_F$ is constant and $f_F^*(\alpha) := f_F(\alpha)$ otherwise. This yields a concave function $f_F^* : \Phi \to \tilde{\mathbb{R}}$. With f_F^* also the functions $f_F^* + e$, for any integer $e \geq 0$, are concave. Hence there is the descending sequence of subgroups

$$U_{f_F^*} \supseteq U_{f_F^*+1} \supseteq U_{f_F^*+2} \supseteq \cdots$$

4.1.5. On the other hand we let $\mathfrak{T} := \operatorname{Spec}(o_L[X^*(\boldsymbol{T})])$ and

$$T^{(e)} := \ker(\mathfrak{T}(o_L) \to \mathfrak{T}(o_L/\varpi_L^{e+1} o_L))$$

for any $e \geq 0$ (see [Schneider and Stuhler 1997, proof of Proposition I.2.6]) and finally define

$$U_F^{(e)} := U_{f_F^*+e} \cdot T^{(e)}$$

for each $e \geq 0$ [loc. cit., p. 21]. This definition is extended to *any* facet F in $\mathscr{B}$ by putting $U_F^{(e)} := gU_{F'}^{(e)}g^{-1}$ if $F = gF'$ with $g \in G$ and F' a facet in A. We thus obtain a filtration

$$P_F \supseteq U_F^{(0)} \supseteq U_F^{(1)} \supseteq \cdots$$

of the pointwise stabilizer P_F by normal subgroups. As in [loc. cit.] we define, for any point $z \in \mathscr{B}$,

$$U_z^{(e)} := U_F^{(e)}$$

where F is the unique facet of $\mathscr{B}$ that contains z. The group $U_z^{(e)}$ fixes the point z. By construction we have

$$U_{gz}^{(e)} = gU_z^{(e)}g^{-1} \tag{4.1.6}$$

for any $z \in \mathscr{B}$ and any $g \in G$.

Remark. We emphasize that the definition of the groups $\{U_F^{(e)}\}_{F \subset \mathscr{B}, e \geq 0}$ depends on the choice of the special vertex x_0 as an origin for A. We also remark that the very same groups appear in the work of Moy and Prasad on unrefined minimal types [Moy and Prasad 1994; Vignéras 1997].

We will make use of the following basic properties of the groups $U_F^{(e)}$. To formulate them let

$$\Phi = \Phi^+ \cup \Phi^-$$

be any fixed decomposition of Φ into positive and negative roots.

Proposition 4.1.7 [Schneider and Stuhler 1997, Propositions I.2.7 and I.2.11 and Corollary I.2.9]. (i) *Let $F \subset A$ be a facet. For any $e \geq 0$ the product map induces a bijection*

$$\Big(\prod_{\alpha \in \Phi^-} U_{f_F^* + e} \cap U_\alpha\Big) \times T^{(e)} \times \Big(\prod_{\alpha \in \Phi^+} U_{f_F^* + e} \cap U_\alpha\Big) \xrightarrow{\sim} U_F^{(e)}$$

whatever ordering of the factors of the left hand side we choose. Moreover, we have

$$U_{f_F^* + e} \cap U_\alpha = U_{\alpha, f_F^*(\alpha) + e}$$

for any $\alpha \in \Phi$.

(ii) *For any facet $F \subset \mathscr{B}$ the $U_F^{(e)}$ for $e \geq 0$ form a fundamental system of compact open neighborhoods of* 1 *in G,*

(iii) *$U_{F'}^{(e)} \subseteq U_F^{(e)}$ for any two facets F, F' in $\mathscr{B}$ such that $F' \subseteq \bar{F}$.*

4.1.8. As an example and in view of later applications we give a more concrete description of the groups $\{U_{x_0}^{(e)}\}_{e\geq 0}$. The stabilizer $P_{\{x_0\}}$ in G of the vertex x_0 is a special, good, maximal compact open subgroup of G [Cartier 1979, 3.5]. We let $\mathfrak{G}$ be the connected reductive o_L-group scheme with generic fiber $\boldsymbol{G}$ associated with the special vertex x_0 [Tits 1979, 3.4; Bruhat and Tits 1984, 4.6.22]. Its group of o_L-valued points $\mathfrak{G}(o_L)$ can be identified with $P_{\{x_0\}}$. For $e \geq 0$ we therefore have in $P_{\{x_0\}}$ the normal subgroup $\mathfrak{G}(\varpi^e) := \ker(\mathfrak{G}(o_L) \to \mathfrak{G}(o_L/\varpi^e o_L))$.

Now the concave function $f_{\{x_0\}}$ vanishes identically whence $f^*_{\{x_0\}}$ has constant value $0+$. Thus,

$$U_{\alpha, f^*_{\{x_0\}}(\alpha)+e} = \bigcup_{s>0} \{a \in L : v_L(a) \geq e+s\} = \varpi^{e+1} o_L$$

for any $e \geq 0$. By Proposition 4.1.7(i) and the definition of $T^{(e)}$ we therefore have a canonical isomorphism $U_{x_0}^{(e)} \xrightarrow{\sim} \mathfrak{G}(\varpi^{e+1})$ for any $e \geq 0$.

4.2. ***The Schneider–Stuhler construction.*** We now review the construction of a certain "localization" functor constructed by P. Schneider and U. Stuhler [1997, IV.1]. In fact, there will be a functor for each "level" $e \geq 0$. Following [loc. cit.], we will suppress this dependence in our notation. In [Schneider and Stuhler 1997] only complex representations are considered. However, all results remain true over our characteristic zero field K [Vignéras 1997].

4.2.1. Recall that a *smooth* representation V of G is a K-vector space V together with a linear action of G such that the stabilizer of each vector is open in G. A morphism between two such representations is simply a K-linear G-equivariant map.

Now let us fix an integer $e \geq 0$ and let V be a smooth representation. For any subgroup $U \subseteq G$ we have the K-vector space

$$V_U := \text{maximal quotient of } V \text{ on which the } U\text{-action is trivial}$$

of U-coinvariants of V. For any open subset $\Omega \subseteq \mathscr{B}$ we let

$$\underset{\approx}{V}(\Omega) := K\text{-vector space of all maps } s : \Omega \to \dot{\bigcup_{z\in\Omega}} V_{U_z^{(e)}} \text{ such that}$$

- $s(z) \in V_{U_z^{(e)}}$ for all $z \in \Omega$,
- there is an open covering $\Omega = \bigcup_{i\in I} \Omega_i$ and vectors $v_i \in V$ with

$$s(z) = \text{class of } v_i \in V_{U_z^{(e)}}$$

  for any $z \in \Omega_i$ and $i \in I$.

We summarize some properties of this construction in the following proposition. Recall that a sheaf on a polysimplicial space is called *constructible* if its restriction to a given geometric polysimplex is a constant sheaf [Kashiwara and Schapira 1990, 8.1].

Proposition 4.2.2. (i) *The correspondence* $\Omega \mapsto \underset{\approx}{V}(\Omega)$ *is a sheaf of* K*-vector spaces.*

(ii) *For any* $z \in \mathscr{B}$ *the stalk of the sheaf* $\underset{\approx}{V}$ *at* z *equals* $(\underset{\approx}{V})_z = V_{U_z^{(e)}}$.

(iii) $\underset{\approx}{V}$ *is a constructible sheaf whose restriction to any facet* F *of* $\mathscr{B}$ *is constant with value* $V_{U_F^{(e)}}$.

(iv) *The correspondence* $V \mapsto \underset{\approx}{V}$ *is an exact functor from smooth* G*-representations to sheaves of* K*-vector spaces on* $\mathscr{B}$.

Proof. Part (i) follows from the local nature of the preceding definition. Part (ii) and (iii) are [Schneider and Stuhler 1997, Lemma IV.1.1]. Part (iv) follows from (ii) because of $char(K) = 0$. □

We recall that the smooth representation V is called *admissible* if the H-invariants V^H form a finite dimensional K-vector space for any compact open subgroup H of G. In this situation the natural projection map $V \to V_H$ induces a linear isomorphism $V^H \xrightarrow{\sim} V_H$. For an admissible representation V we may therefore deduce from Proposition 4.2.2(ii) that the stalks of $\underset{\approx}{V}$ are finite dimensional K-vector spaces. We emphasize again that the functor $V \mapsto \underset{\approx}{V}$ depends on the level $e \geq 0$.

4.3. ***p-valuations on certain stabilizer subgroups.*** We keep the notations from the preceding paragraph and define certain p-valuations on the groups $U_F^{(e)}$. However, for the rest of this section we assume $L = \mathbb{Q}_p$.

Lemma 4.3.1. *Let* F *be a facet in* $\mathscr{B}$ *and* $e, e' \geq 0$. *The commutator group* $(U_F^{(e)}, U_F^{(e')})$ *satisfies*

$$(U_F^{(e)}, U_F^{(e')}) \subseteq U_F^{(e+e')}.$$

Proof. Choosing a facet F' in A and an element $g \in G$ such that $F' = gF$ we may assume that F lies in A. Define a function $h_F : \Phi \cup \{0\} \to \tilde{\mathbb{R}}$ via $h_F|_\Phi := f_F^*$ and $h_F(0) := 0+$. Then $g := h_F + e$ and $f := h_F + e'$ are concave functions in the sense of [Bruhat and Tits 1972, Definition 6.4.3]. Consider the function $h : \Phi \cup \{0\} \to \tilde{\mathbb{R}} \cup \{-\infty\}$ defined as

$$h(a) := \inf\Big\{\sum_i f(a_i) + \sum_j g(b_j)\Big\}$$

where the infimum is taken over the set of pairs of finite nonempty sets (a_i) and (b_j) of elements in $\Phi \cup \{0\}$ such that $a = \sum_i a_i + \sum_j b_j$. Using that the functions f and g are concave one finds

$$h_F(a) + e + e' \leq h(a)$$

for any $a \in \Phi \cup \{0\}$. By [loc. cit., Proposition 6.4.44], the function h is therefore concave and has the property $(U_f, U_g) \subseteq U_h \subseteq U_{h_F+e+e'}$. Here, the groups involved are defined completely analogous to (4.1.3) (see [loc. cit., Definition 6.4.42]). It remains to observe that $U_{h_F+a} = U_F^{(a)}$ for any integer $a \geq 0$ [Schneider and Stuhler 1997, p. 21]. □

Let l be the rank of the torus $\boldsymbol{T}$. By construction of $\mathfrak{T}$ any trivialization $\boldsymbol{T} \simeq (\mathbb{G}_m)^l$ yields an identification $\mathfrak{T} \simeq (\mathbb{G}_{m/o_L})^l$ which makes the structure of the topological groups $T^{(e)}$, $e \geq 0$ explicit. Moreover, we assume in the following $e \geq 2$. For each $g \in U_F^{(e)} \setminus \{1\}$ let

$$\omega_F^{(e)}(g) := \sup\{n \geq 0 : g \in U_F^{(n)}\}.$$

The following corollary is essentially due to H. Frommer [2003, 1.3, proof of Proposition 6]. For sake of completeness we include a proof.

Corollary 4.3.2. *The function*

$$\omega_F^{(e)} : U_F^{(e)} \setminus \{1\} \to (1/(p-1), \infty) \subset \mathbb{R}$$

is a p-valuation on $U_F^{(e)}$ for $e \geq 2$.

Proof. The first axiom (i) is obvious and (ii) follows from the lemma. Let $g \in U_F^{(e)}$ with $n := \omega_F^{(e)}(g)$. We claim $\omega_F^{(e)}(g^p) = n+1$. The root space decomposition (Proposition 4.1.7),

$$m : \Big(\prod_{\alpha\in\Phi^-} U_{\alpha, f_F^*(\alpha)+n}\Big) \times T^{(n)} \times \Big(\prod_{\alpha\in\Phi^+} U_{\alpha, f_F^*(\alpha)+n}\Big) \xrightarrow{\sim} U_F^{(n)},$$

is in an obvious sense compatible with variation of the level n. If $g \in T^{(n)}$ the claim is immediate. The same is true if $g \in U_{\alpha, f_F^*(\alpha)+n}$ for some $\alpha \in \Phi$: indeed the filtration of U_α is induced by the p-adic valuation on $\mathbb{Q}_p$ via $x_\alpha : \mathbb{Q}_p \simeq U_\alpha$. In general let $m(h_1, \ldots, h_d) = g$. By what we have just said there is $1 \leq i \leq d$ such that $\omega^{(e)}(h_i^p) = n+1$ and $\omega^{(e)}(h_j^p) \geq n+1$ for all $j \neq i$. Furthermore, $h_1^p \cdots h_d^p g' = g^p$, where $g' \in (U_F^{(n)}, U_F^{(n)}) \subseteq U_F^{(2n)}$. Since $n \geq 2$ we have $2n \geq n+2$ and hence $g^p \in U_F^{(n+1)}$. If $g^p \in U_F^{(n+2)}$ then $h_1^p \cdots h_d^p = g^p g'^{-1} \in U_F^{(n+2)}$, which contradicts the existence of h_i. Hence $\omega^{(e)}(g^p) = n+1$, which verifies axiom (iii). □

4.3.3. For a given root $\alpha \in \Phi$ let u_α be a topological generator for the group $U_{\alpha, f_F^*(\alpha)+e}$. Let $t_1, \ldots, t_l$ be topological generators for the group $T^{(e)}$. In the light of the decomposition of Proposition 4.1.7(i) it is easy to see that the set

$$\{u_\alpha\}_{\alpha\in\Phi^-} \cup \{t_i\}_{i=1,\ldots,l} \cup \{u_\alpha\}_{\alpha\in\Phi^+}$$

arranged in the order suggested in Frommer's proof is an ordered basis for the p-valued group $(U_F^{(e)}, \omega_F^{(e)})$. Of course, $\omega_F^{(e)}(h) = e$ for any element h of this ordered basis.

For technical reasons we will work in the following with the slightly simpler p-valuations

$$\mathring{\omega}_F^{(e)} := \omega_F^{(e)} - (e-1)$$

satisfying $\mathring{\omega}_F^{(e)}(h) = 1$ for any element h of the above ordered basis. If $z \in \mathscr{B}$ lies in the facet $F \subset \mathscr{B}$ we write $\mathring{\omega}_z^{(e)}$ for $\mathring{\omega}_F^{(e)}$.

Remark 4.3.4. The tangent map at $1 \in H := U_F^{(2)}$ corresponding to the p-power map equals multiplication by p and thus, is an isomorphism. It follows from Proposition 4.1.7(ii) that there is $e(F) \geq 2$ such that for any $e \geq e(F)$ any element $g \in U_F^{(e+1)}$ is a p-th power h^p with $h \in H$. Since H is p-valued, axiom (iii) implies $h \in U_F^{(e)}$. This means that $(U_F^{(e)}, \mathring{\omega}_F^{(e)})$ is p-saturated. For $e \geq e(F)$ the group $U_F^{(e)}$ is therefore a uniform pro-p group (apply remark before Lemma 4.4 in [Schneider and Teitelbaum 2003] to $\mathring{\omega}_F^{(e)}$ and use $p \neq 2$). Since any facet in $\mathscr{B}$ is conjugate to a facet in $\mathscr{C}$ we deduce from (4.1.6) that there is a number $e_{\mathrm{uni}} \geq 2$ such that all the groups $U_F^{(e)}$ for $F \subset \mathscr{B}$ are uniform pro-p groups whenever $e \geq e_{\mathrm{uni}}$. In this situation, Proposition A1 of [Huber et al. 2011] asserts that the subgroups

$$U_F^{(e)} \supset U_F^{(e+1)} \supset U_F^{(e+2)} \supset \cdots$$

form the lower p-series of the group $U_F^{(e)}$.

For technical reasons that will become apparent in Section 7.4 we include the following additional property into the definition of e_{uni}. Let $\overline{\mathscr{C}}$ be the closure of the fundamental chamber $\mathscr{C} \subset A$ and let $x_0 \in A$ be the chosen origin; see Section 4.1.2. We choose once and for all $e_{\mathrm{uni}} \geq 2$ such that, for $e \geq e_{\mathrm{uni}}$, all groups $U_F^{(e)}$ are uniform pro-p groups and such that $U_z^{(e)} \subseteq U_{wx_0}^{(0)}$ for all points $z \in \overline{\mathscr{C}}$ and all $w \in W$.

We may apply the discussion of Section 2.1 to $(U_F^{(e)}, \mathring{\omega}_F^{(e)})$ and the above ordered basis to obtain a family of norms $\|\cdot\|_r$, $r \in [1/p, 1)$ on $D(U_F^{(e)}, K)$ with completions $D_r(U_F^{(e)}, K)$ being K-Banach algebras. For facets F, F' in $\mathscr{B}$ such that $F' \subseteq \overline{F}$ we shall need a certain "gluing" lemma for these algebras.

Lemma 4.3.5. *Let F, F' be two facets in $\mathscr{B}$ such that $F' \subseteq \overline{F}$. The inclusion $U_{F'}^{(e)} \subseteq U_F^{(e)}$ extends to a norm-decreasing algebra homomorphism*

$$\sigma_r^{F'F} : D_r(U_{F'}^{(e)}, K) \to D_r(U_F^{(e)}, K).$$

Moreover, (i) $\sigma_r^{FF} = \mathrm{id}$ *and* (ii) $\sigma_r^{F'F} \circ \sigma_r^{F''F'} = \sigma_r^{F''F}$ *if F'' is a third facet in $\mathscr{B}$ with $F'' \subseteq \overline{F'}$.*

Finally, $\sigma_r^{F'F}$ restricted to $\mathrm{Lie}(U_{F'}^{(e)})$ *equals the map* $\mathrm{Lie}(U_{F'}^{(e)}) \simeq \mathrm{Lie}(U_F^{(e)}) \subset D_r(U_F^{(e)}, K)$ *where the first arrow is the canonical Lie algebra isomorphism from* [Bourbaki 1972, III Section 3.8].

Proof. By functoriality [Kohlhaase 2007, 1.1] of $D(\cdot, K)$ we obtain an algebra homomorphism

$$\sigma : D(U_{F'}^{(e)}, K) \to D(U_F^{(e)}, K).$$

Let $h_1', \ldots, h_d'$ and $h_1, \ldots, h_d$ be the ordered bases of $U_{F'}^{(e)}$ and $U_F^{(e)}$ respectively. Let $b_i' = h_i' - 1 \in \mathbb{Z}[U_{F'}^{(e)}]$ and $\boldsymbol{b}'^m := b_1'^{\,m_1} \cdots b_d'^{\,m_d}$ for $m \in \mathbb{N}_0^d$. Given an element

$$\lambda = \sum_{m \in \mathbb{N}_0^d} d_m \boldsymbol{b}'^m \in D(U_{F'}^{(e)}, K)$$

we have $\|\lambda\|_r = \sup_m |d_m| \, \|b_i'\|_r$. Because

$$\|\sigma(\lambda)\|_r \leq \sup_m |d_m| (\|\sigma(b_1')\|_r)^{m_1} \cdots (\|\sigma(b_d')\|_r)^{m_d})$$

it therefore suffices to prove $\|\sigma(b_i')\|_r \leq \|b_i'\|_r$ for any i. If h_i' belongs to the toral part of the ordered basis of $U_{F'}^{(e)}$ then we may assume $\sigma(b_i') = b_i'$ and we are done. Let therefore $\alpha \in \Phi$ and consider the corresponding elements h_α' and h_α in the ordered bases of $U_{F'}^{(e)}$ and $U_F^{(e)}$ respectively. By the root space decomposition we have

$$U_{\alpha, f_{F'}^*(\alpha)+e} \subseteq U_{\alpha, f_F^*(\alpha)+e} = (h_\alpha)^{\mathbb{Z}_p}.$$

Let therefore $a \in \mathbb{Z}_p$ such that $h_\alpha' = (h_\alpha)^a$. Since a change of ordered basis does not affect the norms in question (see Section 2.2.3) we may assume $a = p^s$ for some natural number $s \geq 0$. Then

$$h_\alpha' - 1 = (h_\alpha + 1 - 1)^{p^s} - 1 = \sum_{k=1,\ldots,p^s} \binom{p^s}{k} (h_\alpha - 1)^k$$

and therefore

$$\|\sigma(h_\alpha' - 1)\|_r \leq \max_{k=1,\ldots,p^s} \left| \binom{p^s}{k} \right| \, \|(h_\alpha - 1)\|_r^k = \max_{k=1,\ldots,p^s} \left| \binom{p^s}{k} \right| r^k \leq r = \|h_\alpha' - 1\|_r$$

which shows the claim and the existence of $\sigma_r^{FF'}$. Properties (i) and (ii) in the second paragraph of the statement follow from functoriality of $D(\cdot, K)$ by passing to completions. Since $U_{F'}^{(e)} \subseteq U_F^{(e)}$ is an open immersion of Lie groups the final statement is clear. □

5. Sheaves on the flag variety and Lie algebra representations

5.1. *Differential operators on the flag variety.*

5.1.1. Let X denote the variety of Borel subgroups of $\boldsymbol{G}$. It is a smooth and projective L-variety. Let $\mathcal{O}_X$ be its structure sheaf. Let $\mathfrak{g}$ be the Lie algebra of $\boldsymbol{G}$.

Differentiating the natural (left) action of $\boldsymbol{G}$ on X yields a homomorphism of Lie algebras

$$\alpha : \mathfrak{g} \to \Gamma(X, \mathcal{T}_X)$$

into the global sections of the tangent sheaf $\mathcal{T}_X = \mathcal{D}er_L(\mathcal{O}_X)$ of X [Demazure and Gabriel 1970, II Section 4.4.4]. In the following we identify an abelian group (algebra, module etc.) with the corresponding constant sheaf on X. This should not cause any confusion. Letting

$$\mathfrak{g}^\circ := \mathcal{O}_X \otimes_L \mathfrak{g}$$

the map α extends to a morphism of $\mathcal{O}_X$-modules $\alpha^\circ : \mathfrak{g}^\circ \to \mathcal{T}_X$. Defining $[\mathfrak{x}, f] := \alpha(\mathfrak{x})(f)$ for $\mathfrak{x} \in \mathfrak{g}$ and a local section f of $\mathcal{O}_X$ makes $\mathfrak{g}^\circ$ a sheaf of L-Lie algebras.[5] Then α° is a morphism of L-Lie algebras. We let $\mathfrak{b}^\circ := \ker \alpha^\circ$, a subalgebra of $\mathfrak{g}^\circ$, and $\mathfrak{n}^\circ := [\mathfrak{b}^\circ, \mathfrak{b}^\circ]$ its derived algebra. At a point $x \in X$ with residue field $\kappa(x)$ the reduced stalks of the sheaves $\mathfrak{b}^\circ$ and $\mathfrak{n}^\circ$ equal the Borel subalgebra $\mathfrak{b}_x$ of $\kappa(x) \otimes_L \mathfrak{g}$ defined by x and its nilpotent radical $\mathfrak{n}_x \subset \mathfrak{b}_x$ respectively. Let $\mathfrak{h}$ denote the abstract Cartan algebra of $\mathfrak{g}$ [Miličić 1993a, Section 2]. We view the $\mathcal{O}_X$-module $\mathcal{O}_X \otimes_L \mathfrak{h}$ as an abelian L-Lie algebra. By definition of $\mathfrak{h}$ there is a canonical isomorphism of $\mathcal{O}_X$-modules and L-Lie algebras

$$\mathfrak{b}^\circ/\mathfrak{n}^\circ \xrightarrow{\simeq} \mathcal{O}_X \otimes_L \mathfrak{h}. \tag{5.1.2}$$

Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. The enveloping algebra of the Lie algebra $\mathfrak{g}^\circ$ has the underlying $\mathcal{O}_X$-module $\mathcal{O}_X \otimes_L U(\mathfrak{g})$. Its L-algebra of local sections over an open affine $V \subseteq X$ is the skew enveloping algebra $\mathcal{O}_X(V)\#U(\mathfrak{g})$ relative to $\alpha : \mathfrak{g} \to \mathrm{Der}_L(\mathcal{O}_X(V))$ (in the sense of sec 3). To emphasize this skew multiplication we follow [Bezrukavnikov et al. 2008, 3.1.3] and denote the enveloping algebra of $\mathfrak{g}^\circ$ by

$$\mathcal{O}_X\#U(\mathfrak{g}).$$

5.1.3. To bring in the torus $\boldsymbol{T}$ we choose a Borel subgroup $\boldsymbol{B} \subset \boldsymbol{G}$ defined over L containing $\boldsymbol{T}$. Let $\boldsymbol{N} \subset \boldsymbol{B}$ be the unipotent radical of $\boldsymbol{B}$ and let $\boldsymbol{N}^-$ be the unipotent radical of the Borel subgroup opposite to $\boldsymbol{B}$. We denote by

$$q : \boldsymbol{G} \to \boldsymbol{G}/\boldsymbol{B} = X$$

the canonical projection. Let $\mathfrak{b}$ be the Lie algebra of $\boldsymbol{B}$ and $\mathfrak{n} \subset \mathfrak{b}$ its nilpotent radical. If $\mathfrak{t}$ denotes the Lie algebra of $\boldsymbol{T}$ the map $\mathfrak{t} \subset \mathfrak{b} \to \mathfrak{b}/\mathfrak{n} \simeq \mathfrak{h}$ induces an isomorphism $\mathfrak{t} \simeq \mathfrak{h}$ of L-Lie algebras. We will once and for all identify these two

[5] Following [Beilinson and Bernstein 1981] we call such a sheaf simply a Lie algebra over X in the sequel. This abuse of language should not cause confusion.

Lie algebras via this isomorphism. Consequently, (5.1.2) yields a morphism of $\mathcal{O}_X$-modules and L-Lie algebras

$$\mathfrak{b}^\circ \to \mathfrak{b}^\circ/\mathfrak{n}^\circ \xrightarrow{\sim} \mathcal{O}_X \otimes_L \mathfrak{t}.$$

Given a linear form $\lambda \in \mathfrak{t}^*$ it extends $\mathcal{O}_X$-linearly to the target of this morphism and may then be pulled-back to $\mathfrak{b}^\circ$. This gives a $\mathcal{O}_X$-linear morphism $\lambda^\circ : \mathfrak{b}^\circ \to \mathcal{O}_X$.

5.1.4. Let $\rho := \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$. Given $\chi \in \mathfrak{t}^*$ we put $\lambda := \chi - \rho$. Denote by $\mathcal{I}_\chi$ the right ideal sheaf of $\mathcal{O}_X\#U(\mathfrak{g})$ generated by $\ker\lambda^\circ$, i.e., by the expressions

$$\xi - \lambda^\circ(\xi)$$

with ξ a local section of $\mathfrak{b}^\circ \subset \mathfrak{g}^\circ \subset \mathcal{O}_X\#U(\mathfrak{g})$. It is a two-sided ideal and we let

$$\mathcal{D}_\chi := (\mathcal{O}_X\#U(\mathfrak{g}))/\mathcal{I}_\chi$$

be the quotient sheaf. This is a sheaf of noncommutative L-algebras on X endowed with a natural algebra morphism $U(\mathfrak{g}) \to \Gamma(X, \mathcal{D}_\chi)$ induced by $\mathfrak{x} \mapsto 1 \otimes \mathfrak{x}$ for $\mathfrak{x} \in U(\mathfrak{g})$. On the other hand $\mathcal{D}_\chi$ is an $\mathcal{O}_X$-module through the (injective) L-algebra morphism $\mathcal{O}_X \to \mathcal{D}_\chi$ induced by $f \mapsto f \otimes 1$. This allows to define the full subcategory $\mathcal{M}_{qc}(\mathcal{D}_\chi)$ of the (left) $\mathcal{D}_\chi$-modules consisting of modules which are quasicoherent as $\mathcal{O}_X$-modules. It is abelian.

5.1.5. For future reference we briefly discuss a refinement of the above construction of the sheaf $\mathcal{D}_\chi$. The right ideal of $\mathcal{O}_X\#U(\mathfrak{g})$ generated by $\mathfrak{n}^\circ$ is a two-sided ideal and, following [Miličić 1993a, Section 3] we let

$$\mathcal{D}_\mathfrak{t} := (\mathcal{O}_X\#U(\mathfrak{g}))/\mathfrak{n}^\circ(\mathcal{O}_X\#U(\mathfrak{g}))$$

be the quotient sheaf. We have the open subscheme $U_1 := q(\mathbf{N}^-)$ of X. Choose a representative $\dot{w} \in G$ for every $w \in W$ with $\dot{1} = 1$. The translates $U_w := \dot{w}U_1$ for all $w \in W$ form a Zariski covering of X. Let $\mathfrak{n}^-$ be the Lie algebra of $\mathbf{N}^-$ and put $\mathfrak{n}^{-,w} := \mathrm{Ad}(\dot{w})(\mathfrak{n}^-)$ for any $w \in W$.

For any $w \in W$ there are obvious canonical maps from $\mathcal{O}_X(U_w)$, $U(\mathfrak{n}^{-,w})$ and $U(\mathfrak{t})$ to $\mathcal{O}_X(U_w)\#U(\mathfrak{g})$ and therefore to $\mathcal{D}_\mathfrak{t}(U_w)$. According to [Miličić 1993b, Lemma C.1.3] they induce a K-algebra isomorphism

$$(\mathcal{O}_X(U_w)\#U(\mathfrak{n}^{-,w})) \otimes_L U(\mathfrak{t}) \xrightarrow{\sim} \mathcal{D}_\mathfrak{t}(U_w). \tag{5.1.6}$$

Note that $\mathbf{N}^- \cong \mathbb{A}_L^{|\Phi^-|}$ implies that the skew enveloping algebra $\mathcal{O}_X(U_w)\#U(\mathfrak{n}^{-,w})$ is equal to the usual algebra of differential operators $\mathcal{D}_X(U_w)$ on the translated affine space $U_w = \dot{w}U_1$.

The above discussion implies that the canonical homomorphism

$$U(\mathfrak{t}) \to \mathcal{O}_X\#U(\mathfrak{g}), \mathfrak{x} \mapsto 1 \otimes \mathfrak{x}$$

induces a central embedding $U(\mathfrak{t}) \hookrightarrow \mathcal{D}_{\mathfrak{t}}$. In particular, the sheaf $(\ker\lambda)\mathcal{D}_{\mathfrak{t}}$ is a two-sided ideal in $\mathcal{D}_{\mathfrak{t}}$. According to the discussion before Theorem 3.2 in [Miličić 1993a, p. 138], the canonical map $\mathcal{D}_{\mathfrak{t}} \to \mathcal{D}_{\chi}$ coming from $\mathfrak{n}^{\circ} \subset \ker\lambda^{\circ}$ induces

$$\mathcal{D}_{\mathfrak{t}} \otimes_{U(\mathfrak{t})} L_{\lambda} = \mathcal{D}_{\mathfrak{t}}/(\ker\lambda)\mathcal{D}_{\mathfrak{t}} \xrightarrow{\sim} \mathcal{D}_{\chi},$$

an isomorphism of sheaves of K-algebras.

Remark. According to the above we may view the formation of the sheaf $\mathcal{D}_{\chi}$ as a two-step process. In a first step on constructs the sheaf $\mathcal{D}_{\mathfrak{t}}$ whose sections over the Weyl translates of the big cell U_1 are explicitly computable. Secondly, one performs a central reduction $\mathcal{D}_{\mathfrak{t}} \otimes_{U(\mathfrak{t})} L_{\lambda}$ via the chosen character $\lambda = \chi - \rho$. This point of view will be useful in later investigations.

5.2. *The Beilinson–Bernstein localization theorem.*

5.2.1. We recall some notions related to the classical *Harish-Chandra isomorphism*. To begin with let $S(\mathfrak{t})$ be the symmetric algebra of $\mathfrak{t}$ and let $S(\mathfrak{t})^W$ be the subalgebra of Weyl invariants. Let $Z(\mathfrak{g})$ be the center of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. The classical Harish-Chandra map is an algebra isomorphism $Z(\mathfrak{g}) \xrightarrow{\sim} S(\mathfrak{t})^W$ relating central characters and highest weights of irreducible highest weight $\mathfrak{g}$-modules in a meaningful way [Dixmier 1996, 7.4]. Given a linear form $\chi \in \mathfrak{t}^*$ we let

$$\sigma(\chi) : Z(\mathfrak{g}) \to L$$

denote the central character associated with χ via the Harish-Chandra map. Recall that $\chi \in \mathfrak{t}^*$ is called *dominant* if $\chi(\check{\alpha}) \notin \{-1, -2, \dots\}$ for any coroot $\check{\alpha}$ with $\alpha \in \Phi^+$. It is called *regular* if $w(\chi) \neq \chi$ for any $w \in W$ with $w \neq 1$.

Let $\theta := \sigma(\chi)$ and put $U(\mathfrak{g})_{\theta} := U(\mathfrak{g}) \otimes_{Z(\mathfrak{g}),\theta} L$ for the corresponding central reduction.

Theorem 5.2.2 [Beilinson and Bernstein 1981].

(i) *The algebra morphism $U(\mathfrak{g}) \to \Gamma(X, \mathcal{D}_{\chi})$ induces an isomorphism $U(\mathfrak{g})_{\theta} \simeq \Gamma(X, \mathcal{D}_{\chi})$.*

(ii) *If χ is dominant and regular the functor $M \mapsto \mathcal{D}_{\chi} \otimes_{U(\mathfrak{g})_{\theta}} M$ is an equivalence of categories between the (left) $U(\mathfrak{g})_{\theta}$-modules and $\mathcal{M}_{qc}(\mathcal{D}_{\chi})$.*

(iii) *Let M be a $U(\mathfrak{g})_{\theta}$-module. The reduced stalk of the sheaf $\mathcal{D}_{\chi} \otimes_{U(\mathfrak{g})_{\theta}} M$ at a point $x \in X$ equals the λ-coinvariants of the $\mathfrak{h}$-module $(\kappa(x) \otimes_L M)/\mathfrak{n}_x(\kappa(x) \otimes_L M)$.*

Remarks. (i) In [Beilinson and Bernstein 1981] the theorem is proved under the assumption that the base field is algebraically closed. However, all proofs in that paper go through over an arbitrary characteristic zero field in the case where the Lie algebra $\mathfrak{g}$ is split over the base field. In the following, this is the only case we shall require.

(ii) If $\lambda := \chi - \rho \in X^*(\boldsymbol{T}) \subset \mathfrak{t}^*$ and if $\mathcal{O}(\lambda)$ denotes the associated invertible sheaf on X then $\mathcal{D}_\chi$ can be identified with the sheaf of differential endomorphisms of $\mathcal{O}(\lambda)$ [Miličić 1993a, p. 138]. It is therefore a *twisted sheaf of differential operators* on X in the sense of [Beilinson and Bernstein 1981, Section 1]. In particular, if $\chi = \rho$ the map α° induces an isomorphism $\mathcal{D}_\rho \xrightarrow{\sim} \mathcal{D}_X$ with the usual sheaf of differential operators on X [Grothendieck 1967, Section 16.8]. In this case, $\mathcal{M}_{qc}(\mathcal{D}_\chi)$ equals therefore the usual category of algebraic D-modules on X in the sense of [Borel et al. 1987].

6. Berkovich analytifications

6.1. *Differential operators on the analytic flag variety.*

6.1.1. For the theory of Berkovich analytic spaces we refer to [Berkovich 1990; 1993]. We keep the notations introduced in the preceding section. In particular, X denotes the variety of Borel subgroups of $\boldsymbol{G}$. Being a scheme of finite type over L we have an associated Berkovich analytic space X^{an} over L [loc. cit., Theorem 3.4.1]. In the preceding section we recalled a part of the algebraic Beilinson–Bernstein localization theory over X. It admits the following "analytification" over X^{an}.

By construction X^{an} comes equipped with a canonical morphism

$$\pi : X^{\text{an}} \to X$$

of locally ringed spaces. Let π^* be the associated inverse image functor from $\mathcal{O}_X$-modules to $\mathcal{O}_{X^{\text{an}}}$-modules. Here $\mathcal{O}_{X^{\text{an}}}$ denotes the structure sheaf of the locally ringed space X^{an}. As with any morphism of locally ringed spaces we have the sheaf

$$\mathcal{T}_{X^{\text{an}}} := \mathcal{D}er_L(\mathcal{O}_{X^{\text{an}}})$$

of L-derivations of $\mathcal{O}_{X^{\text{an}}}$ [Grothendieck 1967, 16.5.4]. By definition $\Gamma(X^{\text{an}}, \mathcal{T}_{X^{\text{an}}}) = \text{Der}_L(\mathcal{O}_{X^{\text{an}}})$. Since X^{an} is smooth over L the results of [Berkovich 1993, 3.3 and 3.5] imply that the stalk of this sheaf at a point $x \in X^{\text{an}}$ equals $\mathcal{T}_{X^{\text{an}},x} = \text{Der}_L(\mathcal{O}_{X^{\text{an}},x})$.

Let $\boldsymbol{G}^{\text{an}}$ denote the analytic space associated to the variety $\boldsymbol{G}$ and let $\pi_{\boldsymbol{G}} : \boldsymbol{G}^{\text{an}} \to \boldsymbol{G}$ be the canonical morphism. The space $\boldsymbol{G}^{\text{an}}$ is a group object in the category of L-analytic spaces (a *L-analytic group* in the terminology of [Berkovich 1990, 5.1]). The unit sections of $\boldsymbol{G}$ and $\boldsymbol{G}^{\text{an}}$ correspond via $\pi_{\boldsymbol{G}}$, which allows us to canonically identify the Lie algebra of $\boldsymbol{G}^{\text{an}}$ with $\mathfrak{g}$ [loc. cit., Theorem 3.4.1(ii)]. By functoriality the group $\boldsymbol{G}^{\text{an}}$ acts on X^{an}. The following result is proved as in the scheme case.

Lemma 6.1.2. *The group action induces a Lie algebra homomorphism*

$$\mathfrak{g} \to \Gamma(X^{\text{an}}, \mathcal{T}_{X^{\text{an}}}).$$

We define

$$\mathfrak{g}^{\circ,\mathrm{an}} := \mathcal{O}_{X^{\mathrm{an}}} \otimes_L \mathfrak{g} = \pi^*(\mathfrak{g}^\circ).$$

The preceding lemma allows on the one hand, to define a structure of L-Lie algebra on $\mathfrak{g}^{\circ,\mathrm{an}}$. The respective enveloping algebra will be denoted by $\mathcal{O}_{X^{\mathrm{an}}} \# U(\mathfrak{g})$. On the other hand, the map from the lemma extends to a $\mathcal{O}_{X^{\mathrm{an}}}$-linear morphism of L-Lie algebras

$$\alpha^{\circ,\mathrm{an}} : \mathfrak{g}^{\circ,an} \to \mathcal{T}_{X^{\mathrm{an}}}. \tag{6.1.3}$$

As in the algebraic case we put $\mathfrak{b}^{\circ,\mathrm{an}} := \ker \alpha^{\circ,\mathrm{an}}$ and $\mathfrak{n}^{\circ,\mathrm{an}} := [\mathfrak{b}^{\circ,\mathrm{an}}, \mathfrak{b}^{\circ,\mathrm{an}}]$. Again, we obtain a morphism $\mathfrak{b}^{\circ,\mathrm{an}} \to \mathcal{O}_{X^{\mathrm{an}}} \otimes_L \mathfrak{t}$. Given $\chi \in \mathfrak{t}^*$ and $\lambda := \chi - \rho$ we denote by $\mathcal{I}^{\mathrm{an}}$ resp. $\mathcal{I}^{\mathrm{an}}_\chi$ the right ideal sheaf of $\mathcal{O}_{X^{\mathrm{an}}} \# U(\mathfrak{g})$ generated by $\mathfrak{n}^{\circ,an}$ resp. $\ker \lambda^{\circ,\mathrm{an}}$ where $\lambda^{\circ,\mathrm{an}}$ equals the $\mathcal{O}_{X^{\mathrm{an}}}$-linear form of $\mathfrak{b}^{\circ,\mathrm{an}}$ induced by λ. These are two-sided ideals. We let

$$\mathcal{D}^{\mathrm{an}}_{\mathfrak{t}} := (\mathcal{O}_{X^{\mathrm{an}}} \# U(\mathfrak{g}))\,/\mathcal{I}^{\mathrm{an}} \quad \text{and} \quad \mathcal{D}^{\mathrm{an}}_\chi := (\mathcal{O}_{X^{\mathrm{an}}} \# U(\mathfrak{g}))\,/\mathcal{I}^{\mathrm{an}}_\chi$$

be the quotient sheaves. We view $\mathcal{D}^{\mathrm{an}}_\chi$ as a sheaf of twisted differential operators on X^{an}.

All these constructions are compatible with their algebraic counterparts via the functor π^*. For example, using the fact that $\pi^*(\mathcal{T}_X) = \mathcal{T}_{X^{\mathrm{an}}}$ it follows from the above proof that $\alpha^{\circ,\mathrm{an}} = \pi^*(\alpha^\circ)$. Moreover, all that has been said in Section 5 on the relation between the sheaves $\mathcal{D}_{\mathfrak{t}}$ and $\mathcal{D}_\chi$ remains true for its analytifications. In particular, $\mathcal{D}^{\mathrm{an}}_\chi$ is a central reduction of $\mathcal{D}^{\mathrm{an}}_{\mathfrak{t}}$ via the character $\lambda : U(\mathfrak{t}) \to L$:

$$\mathcal{D}^{\mathrm{an}}_{\mathfrak{t}}/(\ker \lambda)\mathcal{D}^{\mathrm{an}}_{\mathfrak{t}} \xrightarrow{\sim} \mathcal{D}^{\mathrm{an}}_\chi. \tag{6.1.4}$$

6.2. *The Berkovich embedding and analytic stalks.* Recall our chosen Borel subgroup $\boldsymbol{B} \subset \boldsymbol{G}$ containing $\boldsymbol{T}$ and the quotient morphism $q : \boldsymbol{G} \to \boldsymbol{G}/\boldsymbol{B} = X$. We will make heavy use of the following result of V. Berkovich which was taken up and generalized in a conceptual way by B. Rémy, A. Thuillier and A. Werner. Let $\eta \in X$ be the generic point of X.

Theorem 6.2.1. *There exists a G-equivariant injective map*

$$\vartheta_{\boldsymbol{B}} : \mathscr{B} \to X^{\mathrm{an}}$$

which is a homeomorphism onto its image. The latter is a locally closed subspace of X^{an} contained in the preimage $\pi^{-1}(\eta)$ of the generic point of X.

Proof. This is [Berkovich 1990, 5.5.1]. We sketch the construction in the language of [Rémy et al. 2010]. The map is constructed in three steps. First one attaches to any point $z \in \mathscr{B}$ an L-affinoid subgroup $\boldsymbol{G}_z$ of $\boldsymbol{G}^{\mathrm{an}}$ whose rational points coincide with the stabilizer of z in G. In a second step one attaches to $\boldsymbol{G}_z$ the unique point in its Shilov boundary (the *sup-norm* on $\boldsymbol{G}_z$) which defines a map $\vartheta : \mathscr{B} \to \boldsymbol{G}^{\mathrm{an}}$.

In a final step one composes this map with the analytification of the orbit map $\boldsymbol{G} \to X$, $g \mapsto g.\boldsymbol{B}$. The last assertion follows from the next lemma. □

Lemma 6.2.2. *Let $z \in \mathscr{B}$. The local rings $\mathcal{O}_{X^{\mathrm{an}},\vartheta_{\boldsymbol{B}}(z)}$ and $\mathcal{O}_{X,\pi(\vartheta_{\boldsymbol{B}}(z))}$ are fields. In particular, $\pi(\vartheta_{\boldsymbol{B}}(z))) = \eta$, the generic point of X.*

Proof. This is a direct consequence of [Rémy et al. 2010, Corollary 2.18] and the sentence immediately following that corollary. □

Since X^{an} is a compact Hausdorff space by [Berkovich 1990, 3.4.8], the closure of the image of $\vartheta_{\boldsymbol{B}}$ in X^{an} is a compactification of $\mathscr{B}$ [loc. cit., Remark 3.31]. It is called the *Berkovich compactification* of $\mathscr{B}$ of type $\varnothing$ [loc. cit., Definition 3.30]. We will in the following often identify $\mathscr{B}$ with its image under $\vartheta_{\boldsymbol{B}}$ and hence, view $\mathscr{B}$ as a locally closed subspace of X^{an}.

6.2.3. By [Berkovich 1993, 1.5] the space X^{an} is a *good* analytic space (in the sense of [loc. cit., Remark 1.2.16] which means that any point of X^{an} lies in the topological interior of an affinoid domain. In particular, given $x \in X^{\mathrm{an}}$ the stalk $\mathcal{O}_{X^{\mathrm{an}},x}$ may be written as

$$\mathcal{O}_{X^{\mathrm{an}},x} = \varinjlim_{x \in V} \mathcal{A}_V$$

where the inductive limit ranges over the affinoid neighborhoods V of x and where $\mathcal{A}_V$ denotes the associated affinoid algebra. As usual a subset of neighborhoods of x will be called *cofinal* if it is cofinal in the directed partially ordered set of all neighborhoods of x. If V is an affinoid neighborhood of x, the corresponding affinoid algebra $\mathcal{A}_V$ carries its Banach topology. We endow $\mathcal{O}_{X^{\mathrm{an}},x}$ with the locally convex final topology [Schneider 2002, Section 5.E] arising from the above inductive limit. This topology makes $\mathcal{O}_{X^{\mathrm{an}},x}$ a topological L-algebra. We need another, rather technical, property of this topology.

Lemma 6.2.4. *Let $x \in X^{\mathrm{an}}$. There is a sequence $V_1 \supset V_2 \supset V_3 \supset \dots$ of irreducible reduced strictly affinoid neighborhoods of x which is cofinal and has the property: the homomorphism of affinoid algebras $\mathcal{A}_{V_i} \to \mathcal{A}_{V_{i+1}}$ associated with the inclusion $V_{i+1} \subset V_i$ is flat and an injective compact linear map between Banach spaces. In particular, the stalk $\mathcal{O}_{X^{\mathrm{an}},x}$ is a vector space of compact type.*

Proof. Being an analytification of a variety over L, the analytic space X^{an} is closed (in the sense of [Berkovich 1990, p. 49]); cf. [loc. cit., 3.4.1]. Since the valuation on L is nontrivial, X^{an} is strictly k-analytic [loc. cit., Proposition 3.1.2]. Let V be a strictly affinoid neighborhood of x in X^{an} so that x lies in the topological interior of V. In the following we will use basic results on the relative interior $\mathrm{Int}(Y/Z)$ of an analytic morphism $Y \to Z$ [loc. cit., 2.5, 3.1]. As usual we write $\mathrm{Int}(Y)$ in case of the structure morphism $Y \to \mathscr{M}(L)$. Since X^{an} is closed we have by definition $\mathrm{Int}(X^{\mathrm{an}}) = X^{\mathrm{an}}$. Moreover, Proposition 3.1.3(ii) of [loc. cit.]

implies $\mathrm{Int}(V) = \mathrm{Int}(V/X^{\mathrm{an}})$. By part (i) of the same proposition the topological interior of V is equal to $\mathrm{Int}(V/X^{\mathrm{an}})$ and, thus, $x \in \mathrm{Int}(V)$. Now the residue field of L being finite, there is a countable basis $\{W_n\}_{n\in\mathbb{N}}$ of neighborhoods of x (see discussion after [loc. cit., 3.2.8]) which consists of strictly affinoid subdomains (even Laurent domains) of V [loc. cit., Proposition 3.2.9]. By smoothness of X^{an} the local ring $\mathcal{O}_{X^{\mathrm{an}},x}$ is noetherian regular and hence an integral domain. We may therefore assume that all W_n are reduced and irreducible [loc. cit., last sentence of 2.3]. Consider $V_1 := W_{n_1}$ for some $n_1 \in \mathbb{N}$. As we have just seen $x \in \mathrm{Int}(V_1)$. Since $\mathrm{Int}(V_1)$ is an open neighborhood of x there is $n_2 > n_1$ such that $W_{n_2} \subseteq \mathrm{Int}(V_1)$. We put $V_2 := W_{n_2}$ and repeat the above argument with V_1 replaced by V_2. In this way we find a cofinal sequence $V_1 \supset V_2 \supset V_3 \ldots$ of strictly irreducible reduced affinoid neighborhoods of x with the property $\mathrm{Int}(V_i) \supseteq V_{i+1}$ for all $i \geq 1$. According to [loc. cit., Proposition 2.5.9], the bounded homomorphism of L-Banach algebras $\mathcal{A}_{V_i} \to \mathcal{A}_{V_{i+1}}$ associated with the inclusion $V_{i+1} \subset V_i$ is inner with respect to L (in the sense of [loc. cit., Definition 2.5.1]. The arguments in [Emerton 2011, Proposition 2.1.16] now show that $\mathcal{A}_{V_i} \to \mathcal{A}_{V_{i+1}}$ is a compact linear map between Banach spaces. Finally, this latter map is injective because V_i is irreducible and V_{i+1} contains a nonempty open subset of V_i. It is also flat since, by construction, V_{i+1} is an affinoid subdomain of V_i [Berkovich 1990, Proposition 2.2.4(ii)]. □

6.2.5. In this paragraph and the next lemma we assume $L = \mathbb{Q}_p$. Consider for a given $z \in \mathscr{B}$ the group $U_z^{(e)} \subset G$; see Section 4.1.5. For $e \geq e_{\mathrm{uni}}$ the group $U_z^{(e)}$ is uniform pro-p; see 4.3.4. As such, it has a $\mathbb{Z}_p$-Lie algebra $\mathcal{L}(U_z^{(e)})$, which is powerful, and the exponential map $\exp_{U_z^{(e)}} : \mathcal{L}(U_z^{(e)}) \to U_z^{(e)}$ is well-defined and a bijection [Dixon et al. 1999, Section 9.4]. Using the Baker–Campbell–Hausdorff series one can then associate to the lattice $\mathcal{L}(U_z^{(e)})$ a $\mathbb{Q}_p$-analytic affinoid subgroup $\mathbb{U}_z^{(e)} \subset \boldsymbol{G}^{\mathrm{an}}$, which has the property that $\mathbb{U}_z^{(e)}(\mathbb{Q}_p) = U_z^{(e)}$.[6] ($U_z^{(e)}$ is a good analytic open subgroup of G in the sense of [Emerton 2011, Section 5.2].) Let $V \subset X^{\mathrm{an}}$ be an affinoid domain. We say that $U_z^{(e)}$ acts *analytically* on V, if there is an action of the affinoid group $\mathbb{U}_z^{(e)}$ on V compatible with the action of $\mathbb{U}_z^{(e)}$ on X^{an}, i.e., if there is a commutative diagram of group operations

$$\begin{array}{ccc} \mathbb{U}_z^{(e)} \times V & \longrightarrow & V \\ \big\downarrow & & \big\downarrow \\ \boldsymbol{G}^{\mathrm{an}} \times X^{\mathrm{an}} & \longrightarrow & X^{\mathrm{an}} \end{array}$$

[6] Only here do we use that $L = \mathbb{Q}_p$. For general L it would be necessary to show that $U_z^{(e)}$ is actually an L-uniform pro-p group; see [Orlik and Strauch 2010b, 2.2.5]. This can be done, but we do not work here in this generality.

where the vertical maps are inclusions (and the products on the left are taken in the category of L-analytic spaces).

Lemma 6.2.6. *There exists a number $e_{\mathrm{st}} \geq e_{\mathrm{uni}}$ with the following property. For any point $z \in \mathscr{B}$, viewed as a point in X^{an}, there is a fundamental system of strictly affinoid neighborhoods $\{V_n\}_{n\geq 0}$ of z with the properties as in Lemma 6.2.4, and such that for all $n \geq 0$ and $e \geq e_{\mathrm{st}}$ the group $U_z^{(e)}$ acts analytically on V_n.*

The proof, which is lengthy, is given in an appendix, in order not to interrupt the discussion at this point.

6.3. *A structure sheaf on the building.*

6.3.1. To be able to compare the localization of Schneider–Stuhler and Beilinson–Bernstein we equip the topological space $\mathscr{B}$ with a sheaf of commutative and topological L-algebras. Recall that a subset $V \subset X^{\mathrm{an}}$ is called a *special domain* if it is a finite union of affinoid domains, and to any special domain V there is associated an L-Banach algebra $\mathcal{A}_V$ [Berkovich 1990, 2.2.6]. The sheaf $\mathcal{O}_{X^{\mathrm{an}}}$ is naturally a sheaf of locally convex algebras as follows: given an open subset $U \subset X^{\mathrm{an}}$ we have

$$\mathcal{O}_{X^{\mathrm{an}}}(U) = \varprojlim_{V \subset U} \mathcal{A}_V\,,$$

where the limit is taken over all special domains (or affinoid domains) of X^{an} contained in U. Here, $\mathcal{A}_V$ is the L-Banach algebra corresponding to V and the projective limit is equipped with the projective limit topology. Because the residue field of L is finite, X^{an} has a countable basis of open subsets [Berkovich 1990, 3.2.9]. Therefore, one can cover U with a countable set of special domains and $\mathcal{O}_{X^{\mathrm{an}}}(U)$ is thus a countable projective limit of Banach algebras, hence a Fréchet algebra over L.

We then consider the exact functor $\vartheta_{\boldsymbol{B}}^{-1}$ from abelian sheaves on X^{an} to abelian sheaves on $\mathscr{B}$ given by restriction along $\vartheta_{\boldsymbol{B}} : \mathscr{B} \hookrightarrow X^{\mathrm{an}}$. Let

$$\mathcal{O}_{\mathscr{B}} := \vartheta_{\boldsymbol{B}}^{-1}(\mathcal{O}_{X^{\mathrm{an}}}).$$

For any subset $C \subset X^{\mathrm{an}}$ we can consider $\mathcal{O}_{X^{\mathrm{an}}}(C)$, the vector space of sections of $\mathcal{O}_{X^{\mathrm{an}}}$ over C, i.e., the global sections of the restriction of the sheaf $\mathcal{O}_{X^{\mathrm{an}}}$ to C.

Proposition 6.3.2. *For any subset $C \subset X^{\mathrm{an}}$ we have*

$$\mathcal{O}_{X^{\mathrm{an}}}(C) = \varinjlim_{C \subset U} \mathcal{O}_{X^{\mathrm{an}}}(U)$$

where U runs through all open neighborhoods of C in X^{an}.

Proof. As was pointed out in 6.3.1, the compact Hausdorff topological space X^{an} has a countable basis of open subsets. By Urysohn's metrization theorem, it is therefore metrizable, and we may apply [Godement 1958, II.3.3, Corollary 1]. □

In particular, given an open set $\Omega \subseteq \mathscr{B}$ we have

$$\mathcal{O}_{\mathscr{B}}(\Omega) = \varinjlim_{\Omega \subset U} \mathcal{O}_{X^{\mathrm{an}}}(U)$$

where U runs through the open neighborhoods of Ω in X^{an}. Using the locally convex inductive limit topology on the right-hand side, the sheaf $\mathcal{O}_{\mathscr{B}}$ becomes in this way a sheaf of locally convex algebras. We point out that the stalk $\mathcal{O}_{\mathscr{B},z} = \mathcal{O}_{X^{\mathrm{an}},z}$ for any point $z \in \mathscr{B}$ is in fact a field; see Lemma 6.2.2. We summarize some properties of $\vartheta_{\boldsymbol{B}}^{-1}$:

(1) $\vartheta_{\boldsymbol{B}}^{-1}$ preserves (commutative) rings, L-algebras, L-Lie algebras and G-equivariance.

(2) $\vartheta_{\boldsymbol{B}}^{-1}$ maps $\mathcal{O}_{X^{\mathrm{an}}}$-modules into $\mathcal{O}_{\mathscr{B}}$-modules.

(3) $\vartheta_{\boldsymbol{B}}^{-1}$ induces a Lie algebra homomorphism $\mathrm{Der}_L(\mathcal{O}_{X^{\mathrm{an}}}) \to \mathrm{Der}_L(\mathcal{O}_{\mathscr{B}})$.

(4) $\mathcal{O}_{\mathscr{B}}$ is a sheaf of *locally convex* L-algebras. For every $z \in \mathscr{B}$ the stalk $\mathcal{O}_{\mathscr{B},z}$ is of compact type with a defining system $\mathcal{A}_{V_n}$ of Banach algebras, where $(V_n)_n$ is a fundamental system of affinoid neighborhoods as in Lemma 6.2.6.

Composing the map $\mathfrak{g} \to \mathrm{Der}_L(\mathcal{O}_{X^{\mathrm{an}}})$ from Lemma 6.1.2 with (3) yields a Lie algebra homomorphism $\mathfrak{g} \to \mathrm{Der}_L(\mathcal{O}_{\mathscr{B}})$ and the associated skew enveloping algebra $\mathcal{O}_{\mathscr{B}}\#U(\mathfrak{g})$. By (1),(2) we have the L-Lie algebras and $\mathcal{O}_{\mathscr{B}}$-modules $\mathfrak{n}_{\mathscr{B}}^{\circ,\mathrm{an}} := \vartheta_{\boldsymbol{B}}^{-1}(\mathfrak{n}^{\circ,\mathrm{an}})$ and $\mathfrak{b}_{\mathscr{B}}^{\circ,\mathrm{an}} := \vartheta_{\boldsymbol{B}}^{-1}(\mathfrak{b}^{\circ,\mathrm{an}})$. Similarly,

$$\lambda_{\mathscr{B}}^{\circ,\mathrm{an}} := \vartheta_{\boldsymbol{B}}^{-1}(\lambda^{\circ,\mathrm{an}}) : \mathfrak{b}_{\mathscr{B}}^{\circ,\mathrm{an}} \to \mathcal{O}_{\mathscr{B}}$$

is a morphism of L-Lie algebras and $\mathcal{O}_{\mathscr{B}}$-modules. Let $\mathcal{I}_{\mathscr{B},\mathfrak{t}}^{\mathrm{an}}$ resp. $\mathcal{I}_{\mathscr{B},\chi}^{\mathrm{an}}$ be the right ideal sheaf of $\mathcal{O}_{\mathscr{B}}\#U(\mathfrak{g})$ generated by $\mathfrak{n}_{\mathscr{B}}^{\circ,\mathrm{an}}$ resp. $\ker(\lambda_{\mathscr{B}}^{\circ,\mathrm{an}})$. One checks that these are two-sided ideals. We let

$$\mathcal{D}_{\mathscr{B},\mathfrak{t}}^{\mathrm{an}} := (\mathcal{O}_{\mathscr{B}}\#U(\mathfrak{g}))/\mathcal{I}_{\mathscr{B},\mathfrak{t}}^{\mathrm{an}} \quad \text{and} \quad \mathcal{D}_{\mathscr{B},\chi}^{\mathrm{an}} := (\mathcal{O}_{\mathscr{B}}\#U(\mathfrak{g}))/\mathcal{I}_{\mathscr{B},\chi}^{\mathrm{an}}.$$

Note that by exactness of $\vartheta_{\boldsymbol{B}}^{-1}$ we have

$$\mathcal{D}_{\mathscr{B},\mathfrak{t}}^{\mathrm{an}} = \vartheta_{\boldsymbol{B}}^{-1}(\mathcal{D}_{\mathfrak{t}}^{\mathrm{an}}) \quad \text{and} \quad \mathcal{D}_{\mathscr{B},\chi}^{\mathrm{an}} = \vartheta_{\boldsymbol{B}}^{-1}(\mathcal{D}_{\chi}^{\mathrm{an}}).$$

6.3.3. The sheaf $\mathcal{D}_{\mathscr{B},\chi}^{\mathrm{an}}$ of twisted differential operators on $\mathscr{B}$ is formed with respect to the Lie algebra action of $\mathfrak{g}$ on the ambient space $\mathscr{B} \subset X^{\mathrm{an}}$. In an attempt to keep track of the whole analytic G-action on X^{an} we will produce in the following a natural injective morphism of sheaves of algebras

$$\mathcal{D}_{\mathscr{B},\chi}^{\mathrm{an}} \to \mathscr{D}_{r,\chi}$$

with target a sheaf of what we tentatively call *twisted distribution operators* on $\mathscr{B}$. Actually, there will be one such sheaf for each “radius” $r \in [r_0, 1)$ in $p^{\mathbb{Q}}$ and each sufficiently large "level" $e > 0$. Again, following [Schneider and Stuhler 1997] we suppress the dependence on the level in our notation.

6.4. ***Mahler series and completed skew group rings.***

6.4.1. Suppose for a moment that $\mathcal{A}$ is an arbitrary L-Banach algebra. Since $\mathbb{Q}_p \subset \mathcal{A}$ the completely valued $\mathbb{Z}_p$-module $(\mathcal{A}, |\cdot|)$ is saturated in the sense of [Lazard 1965, I.2.2.10]. Consequently, we have the theory of Mahler expansions over $\mathcal{A}$ at our disposal [loc. cit., III.1.2.4 and III.1.3.9]. In this situation we prove a version of the well-known relation between decay of Mahler coefficients and overconvergence.

Proposition 6.4.2. *Let $f = \sum_{\alpha \in \mathbb{N}_0^d} a_\alpha x^\alpha$ be a d-variable power series over $\mathcal{A}$ converging on the disc $|x_i| \le R$ for some $R > 1$. Let $c > 0$ be a constant such that $|a_\alpha| \le cR^{-|\alpha|}$ for all α. Let*

$$f(\cdot) = \sum_{\alpha \in \mathbb{N}_0^d} c_\alpha \binom{\cdot}{\alpha},$$

$c_\alpha \in \mathcal{A}$, be the Mahler series expansion of f. Then $|c_\alpha| \le cs^{|\alpha|}$ for all α, where $s = \tilde{r}R^{-1}$ with $\tilde{r} = p^{-1/(p-1)}$.

Proof. We prove the lemma in case $d = 1$. The general case follows along the same lines but with more notation. We define the following series of polynomials over $\mathbb{Z}$

$$(x)_0 = 1, \dots, (x)_k = x(x-1)\cdots(x-k+1)$$

for $k \ge 1$. The $\mathbb{Z}$-module $\mathbb{Z}[x]$ has the $\mathbb{Z}$-bases $\{x^k\}_{k\ge 0}$ and $\{(x)_k\}_{k\ge 0}$ and the transition matrices are unipotent upper triangular. We may therefore write

$$x^n = \sum_{k=0,\dots,n} s(n,k)(x)_k \tag{6.4.3}$$

with $s(n,k) \in \mathbb{Z}$. Put $b_k := c_k/k!$. Then

$$\sum_{k\ge 0} c_k \binom{x}{k} = \sum_{k \ge 0} b_\kappa (x)_k$$

is a uniform limit of continuous functions (even polynomials) on $\mathbb{Z}_p$ [Robert 2000, Theorem VI.4.7]. We now proceed as in (the proof of) [Washington 1997, Proposition 5.8]. Fix $i \ge 1$ and write

$$\sum_{n \le i} a_n x^n = \sum_{k \le i} b_{k,i}(x)_k$$

as polynomials over $\mathcal{A}$ with some elements $b_{k,i} \in \mathcal{A}$. Inserting (6.4.3) and comparing coefficients yields $b_{k,i} = \sum_{k \le n \le i} a_n s(n,k)$ and consequently,

$$|b_{k,i}| \le \max_{k \le n \le i} |a_n| \le \max_{k \le n \le i} (cR^{-n}) \le cR^{-k}$$

since $R^{-1} < 1$. It follows that, for $j \geq i$, we have

$$|b_{k,j} - b_{k,i}| = \left| \sum_{n=i+1}^{j} a_n s(n,k) \right| \leq R^{-(i+1)}.$$

We easily deduce from this that $\{b_{k,i}\}_{i \geq 0}$ is a Cauchy sequence in the Banach space $\mathcal{A}$. Let $\tilde{b}_k$ be its limit. Clearly, $|\tilde{b}_k| \leq cR^{-k}$. Put $\tilde{c}_k := k!\,\tilde{b}_k$. Since $|k!| \leq \tilde{r}^k$ we obtain $|\tilde{c}_k| \leq c(\tilde{r}R^{-1})^k = cs^k$ for all k. By definition of $\tilde{b}_k$ the series of polynomials

$$\sum_{k \geq 0} \tilde{c}_k \binom{x}{k} = \sum_{k \geq 0} \tilde{b}_\kappa (x)_k$$

converges pointwise to the limit

$$\lim_{i \to \infty} \sum_{k \leq i} b_{k,i}(x)_k = \lim_{i \to \infty} \sum_{n \leq i} a_n x^n = f(x).$$

By [Robert 2000, IV.2.3, p. 173] this convergence is uniform and so uniqueness of Mahler expansions implies $\tilde{c}_k = c_k$ for all k. This proves the lemma. □

Corollary 6.4.4. *Let $L = \mathbb{Q}_p$, $z \in \mathscr{B}$, and $e > e_{\mathrm{uni}}$.*

(i) *Consider an affinoid domain V of X^{an} on which $U_z^{(e-1)}$ acts analytically in the sense of Section 6.2.5, and let $\mathcal{A}_V$ be the corresponding Banach algebra.*

- (a) *For any p-basis $(h_1, \dots, h_d)$ of $U_z^{(e-1)}$* (see Section 2.2), *and for any $f \in \mathcal{A}_V$ the orbit map $U^{(e-1)} \to \mathcal{A}_V$, $h = h_1^{x_1} \cdots h_d^{x_d} \mapsto h.f$, can be expanded as a strictly convergent power series $\sum_{\nu \in \mathbb{N}^d} f_\nu x_1^{\nu_1} \cdots x_d^{\nu_d}$ with $f_\nu \in \mathcal{A}_V$ and $|f_\nu|_V \to 0$ as $|\nu| \to \infty$. ($|\cdot|_V$ denotes the supremum norm on $\mathcal{A}_V$.)*
- (b) *The representation $\rho : U_z^{(e)} \to \mathrm{GL}(\mathcal{A}_V)$, $(\rho(h).f)(w) = f(h^{-1}.w)$, satisfies the assumption ($\star$) of Section 3.2.5 for any $r \in [r_0, 1)$. In particular, the ring*

$$\mathcal{A}_V \# D_r(U_z^{(e)}, K)$$

exists for all $r \in [r_0, 1)$.

(ii) *More generally, let $V = V_1 \cup \ldots \cup V_m$ be a special domain of X^{an}* (see 6.3.1), *where V_i is affinoid for $1 \leq i \leq m$, and suppose that $U^{(e-1)}$ acts analytically on each V_i, $1 \leq i \leq m$. Then the representation $\rho : U_z^{(e)} \to \mathrm{GL}(\mathcal{A}_V)$, $(\rho(h).f)(w) = f(h^{-1}.w)$, satisfies the assumption ($\star$) of Section 3.2.5 for any $r \in [r_0, 1)$. In particular, the ring*

$$\mathcal{A}_V \# D_r(U_z^{(e)}, K)$$

exists for all $r \in [r_0, 1)$.

Proof. (i)(a) To simplify notation put $U = U_z^{(e-1)}$ and $\mathbb{U} = \mathbb{U}_z^{(e-1)}$; see Section 6.2.5. Let $\mathcal{A}_{\mathbb{U}}$ be the affinoid algebra of $\mathbb{U}$. The p-basis gives rise to an isomorphism

$\mathcal{A}_{\mathbb{U}} \simeq \mathbb{Q}_p\langle x_1, \dots, x_d\rangle$, where the latter denotes strictly convergent power series. The action of $\mathbb{U}$ on V corresponds to a morphism of affinoid algebras

$$\mathcal{A}_V \to \mathcal{A}_{\mathbb{U}} \hat{\otimes}_L \mathcal{A}_V \simeq \mathcal{A}_V\langle x_1, \dots, x_d\rangle.$$

On the right we have the algebra of strictly convergent power series over $\mathcal{A}_V$. This proves the first assertion.

(i)(b) By 4.3.4 we have that $U_z^{(e)}$ is the second member of the lower p-series of $U_z^{(e-1)}$. Therefore, if $(h_1, \dots, h_d)$ is the p-basis for $U_z^{(e-1)}$ used in (i), it follows that $(h_1^p, \dots, h_d^p)$ is a p-basis for $U_z^{(e)}$. Denote by $(y_1, \dots, y_d)$ the coordinates on $U_z^{(e)}$ corresponding to this p-basis. Then, applying (i) to the group $U_z^{(e)}$, we find

$$\rho(h).f = \sum_{\nu\in\mathbb{N}^d} f_\nu y_1^{\nu_1} \cdots y_d^{\nu_d} \tag{6.4.5}$$

when $h = (h_1^p)^{y_1} \cdots (h_d^p)^{y_d} \in U_z^{(e)}$ and $f \in \mathcal{A}_V$. Therefore, the right-hand side of (6.4.5) converges on the disc $|y_i| \le p$. Next consider the Mahler expansion

$$\rho\big((h_1^p)^{y_1} \cdots (h_d^p)^{y_d}\big).f = \sum_{\alpha\in\mathbb{N}^d} c_{f,\alpha}\binom{y}{\alpha}.$$

By Proposition 6.4.2 we have $|c_{f,\alpha}|_V \le cs^{|\alpha|}$ with some $c > 0$ and $s = r_1 p^{-1} < p^{-1} = r_0$.

Write $\delta \in D_r(U_z^{(e)}, L)$ as a series $\delta = \sum_{\alpha\in\mathbb{N}^d} d_\alpha \boldsymbol{b}^\alpha$ with $\boldsymbol{b}^\alpha = (h_1^p - 1)^{\alpha_1} \cdots \cdot (h_d^p - 1)^{\alpha_d}$ and $d_\alpha \in L$ such that $|d_\alpha| r^{|\alpha|} \to 0$. Since $s < r_0 \le r$ and $|c_{f,\alpha}|_V \le cs^{|\alpha|}$ the sum

$$\delta.f = \delta\big(h \mapsto \rho(h).f\big) = \sum_{\alpha\in\mathbb{N}^d} d_\alpha c_{f,\alpha} \tag{6.4.6}$$

converges in the Banach space $\mathcal{A}_V$. The map $(\delta, f) \mapsto \delta.f$ makes $\mathcal{A}_V$ a topological module over $D_r(U_z^{(e)}, L)$ in a way compatible with the map $D(U_z^{(e)}) \to D_r(U_z^{(e)})$. The last assertion is contained in Section 3.2.5.

(ii) As X is separated, X^{an} is Hausdorff and therefore separated [Berkovich 1990, 3.4.8 and 3.1.5]. This implies that the intersection of any two affinoid domains in X^{an} is again an affinoid domain [loc. cit., 3.1.6]. In this case,

$$\mathcal{A}_V = \ker\bigg(\prod_{i=1}^m \mathcal{A}_{V_i} \rightrightarrows \prod_{i,j} \mathcal{A}_{V_i\cap V_j}\bigg);$$

see [loc. cit., 2.2.6 and 3.3]. We can now apply the assertions in (i)(b) to each factor $\mathcal{A}_{V_i}$ and $\mathcal{A}_{V_i\cap V_j}$, and to the corresponding products, and deduce statement (ii). □

6.4.7. Later on we will sometimes need to consider the action of $U_z^{(e)}$ on rings of the form $\mathcal{A}_V$, where V is a special domain in X^{an} as in Corollary 6.4.4(ii). In order to conveniently refer to this situation, we will say that $U_z^{(e)}$ acts *analytically* on a special domain V, if one can write $V = V_1 \cup \ldots \cup V_m$ as finite union of affinoid domains $V_i \subset X^{\mathrm{an}}$, $1 \le i \le m$, with the property that $U_z^{(e)}$ acts analytically on each V_i, $1 \le i \le m$.

Until the end of this section we will assume $L = \mathbb{Q}_p$, $e > e_{\mathrm{st}}$ and $r \in [r_0, 1)$.

Proposition 6.4.8. *Let $z \in \mathscr{B}$, and let $(V_n)_n$ be a descending sequence of affinoid neighborhoods of z as in Lemma 6.2.6. Then the stalk $\mathcal{O}_{\mathscr{B},z}$ is equal to the inductive limit of the Banach algebras $\mathcal{A}_{V_n}$, the completed skew group rings $\mathcal{A}_{V_n} \# D_r(U_z^{(e)}, K)$ and $\mathcal{O}_{\mathscr{B},z} \# D_r(U_z^{(e)}, K)$ exist, and the natural map*

$$\varinjlim_n \big(\mathcal{A}_{V_n} \# D_r(U_z^{(e)}, K)\big) \to \mathcal{O}_{\mathscr{B},z} \# D_r(U_z^{(e)}, K)$$

is an isomorphism of K-algebras.

Proof. By Lemmas 6.2.4 and 6.2.6, the stalk $\mathcal{O}_{\mathscr{B},z}$ is the inductive limit of the Banach algebras $\mathcal{A}_{V_n}$, and the transition maps $\mathcal{A}_{V_n} \to \mathcal{A}_{V_{n+1}}$ are compact and injective. By Proposition 2.4.1, the natural map

$$\varinjlim \big(\mathcal{A}_{V_n} \hat{\otimes}_L D_r(U_z^{(e)}, K)\big) \to \big(\varinjlim \mathcal{A}_{V_n}\big) \hat{\otimes}_L D_r(U_z^{(e)}, K) = \mathcal{O}_{\mathscr{B},z} \hat{\otimes}_L D_r(U_z^{(e)}, K) \quad (6.4.9)$$

is an isomorphism of vector spaces. By Corollary 6.4.4(i)(b) the ring $\mathcal{A}_{V_n}$ is a $D_r(U_z^{(e)}, L)$-module for every n and $r \in [r_0, 1)$, and the transition maps are homomorphisms of $D_r(U_z^{(e)}, L)$-modules. This shows that $\mathcal{O}_{\mathscr{B},z}$ is naturally a $D_r(U_z^{(e)}, L)$-module, hence $\mathcal{O}_{\mathscr{B},z} \# D_r(U_z^{(e)}, L)$ and $\mathcal{O}_{\mathscr{B},z} \# D_r(U_z^{(e)}, K)$ exist. The natural map $\mathcal{A}_{V_n} \# D_r(U_z^{(e)}, K) \to \mathcal{O}_{\mathscr{B},z} \# D_r(U_z^{(e)}, K)$ is a ring homomorphism, and the map (6.4.9) is an isomorphism of K-algebras. □

The following corollary is immediate and recorded only for future reference.

Corollary 6.4.10. *Let V be a neighborhood of z which is a special subset of X^{an}. Suppose $U^{(e-1)}$ acts analytically on V in the sense of Section 6.4.7. Let ι_z be the natural map $\mathcal{A}_V \to \mathcal{O}_{\mathscr{B},z}$ sending a function to its germ at z. The map $\iota_z \hat{\otimes} \mathrm{id}$ is an algebra homomorphism*

$$\mathcal{A}_V \# D_r(U_z^{(e)}, K) \to \mathcal{O}_{\mathscr{B},z} \# D_r(U_z^{(e)}, K).$$

Corollary 6.4.11. *Let V be a neighborhood of z which is a special subset of X^{an}. Suppose $U^{(e-1)}$ acts analytically on V in the sense of Section 6.4.7. Then the inclusions*

$$L[U_z^{(e)}] \subseteq D_r(U_z^{(e)}, K)$$

and $U(\mathfrak{g})_K \subseteq D_r(U_z^{(e)}, K)$ *induce algebra homomorphisms*

(i) $\mathcal{A}_V \# U_z^{(e)} = \mathcal{A}_V \otimes_L L[U_z^{(e)}] \to \mathcal{A}_V \# D_r(U_z^{(e)}, K)$,

(ii) $\mathcal{A}_V \# U(\mathfrak{g})_K \to \mathcal{A}_V \# D_r(U_z^{(e)}, K)$.

If V runs through a sequence of affinoid neighborhoods of z as in Lemma 6.2.6 these maps assemble to algebra homomorphisms

(i) $\mathcal{O}_{\mathscr{B},z} \# U_z^{(e)} = \mathcal{O}_{\mathscr{B},z} \otimes_L L[U_z^{(e)}] \to \mathcal{O}_{\mathscr{B},z} \# D_r(U_z^{(e)}, K)$,

(ii) $\mathcal{O}_{\mathscr{B},z} \# U(\mathfrak{g})_K \to \mathcal{O}_{\mathscr{B},z} \# D_r(U_z^{(e)}, K)$.

Proof. Consider the case of $\mathcal{A}_V$. The existence of the map (i) follows from Proposition 3.2.4. The same is true for the map (ii) once we convince ourselves that there is a commutative diagram of algebra homomorphisms

$$\begin{array}{ccc} U(\mathfrak{g}) & \xrightarrow{\alpha^{\circ,\mathrm{an}}} & \mathrm{End}_L(\mathcal{A}_V) \\ \big\downarrow \subseteq & & \big\downarrow Id \\ D_r(U_z^{(e)}, L) & \longrightarrow & \mathrm{End}_L(\mathcal{A}_V) \end{array}$$

where the upper horizontal arrow is derived from (6.1.3) and the lower horizontal arrow describes the $D_r(U_z^{(e)}, L)$-module structure of $\mathcal{A}_V$ as given by Corollary 6.4.4. Restricting the lower horizontal arrow to $\mathfrak{g}$ amounts to differentiating the analytic $U_z^{(e)}$-action on $\mathcal{A}_V$. This action comes from the algebraic action of $\boldsymbol{G}$ on X. The diagram commutes by the remark following Lemma 6.1.2. Having settled the case $\mathcal{A}_V$ the case of $\mathcal{O}_{\mathscr{B},z}$ now follows by passage to the inductive limit. □

As a result of this discussion we have associated to each point $z \in \mathscr{B} \subset X^{\mathrm{an}}$ the (noncommutative) K-algebra $\mathcal{O}_{\mathscr{B},z} \# D_r(U_z^{(e)}, K)$. As we have seen, it comes together with an injective algebra homomorphism

$$\mathcal{O}_{\mathscr{B},z} \# U(\mathfrak{g})_K \hookrightarrow \mathcal{O}_{\mathscr{B},z} \# D_r(U_z^{(e)}, K). \tag{6.4.12}$$

In the next section we will sheafify this situation and obtain a sheaf of noncommutative K-algebras $\mathcal{O}_{\mathscr{B}} \# D_r$ on $\mathscr{B}$ together with an injective morphism of sheaves of algebras

$$\mathcal{O}_{\mathscr{B}} \# U(\mathfrak{g})_K \hookrightarrow \mathcal{O}_{\mathscr{B}} \# D_r$$

inducing the map (6.4.12) at all points $z \in \mathscr{B}$. To do this we shall need a simple "gluing property" of the algebras $\mathcal{O}_{\mathscr{B},z} \# D_r(U_z^{(e)}, K)$.

Lemma 6.4.13. *Let F, F' be facets in $\mathscr{B}$ such that $F' \subseteq \overline{F}$ and let*

$$\sigma_r^{F'F} : D_r(U_{F'}^{(e)}, K) \to D_r(U_F^{(e)}, K)$$

be the corresponding algebra homomorphism. Suppose V and V' are two special domains in X^{an} on which $U_F^{(e-1)}$ and $U_{F'}^{(e-1)}$, respectively, act analytically (see Section 6.4.7). *If $V \subset V'$ the map $\mathrm{res}_V^{V'} \hat{\otimes} \sigma_r^{F'F}$ is a continuous algebra homomorphism,*

$$\mathrm{res}_V^{V'} \hat{\otimes} \sigma_r^{F'F} : \mathcal{A}_{V'} \# D_r(U_{F'}^{(e)}, K) \to \mathcal{A}_V \# D_r(U_F^{(e)}, K).$$

Proof. Since the map $\sigma_r^{F'F}$ is induced from the inclusion $U_{F'}^{(e)} \subseteq U_F^{(e)}$ there is a commutative diagram

$$\begin{array}{ccc} D_r(U_{F'}^{(e)}, L) \times \mathcal{A}_{V'} & \longrightarrow & \mathcal{A}_{V'} \\ \downarrow{\scriptstyle \sigma_r^{F'F} \times \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} \\ D_r(U_F^{(e)}, L) \times \mathcal{A}_V & \longrightarrow & \mathcal{A}_V \end{array}$$

where the horizontal arrows describe the module structures of $\mathcal{A}_{V'}$ and $\mathcal{A}_V$ over $D_r(U_{F'}^{(e)}, L)$ and $D_r(U_F^{(e)}, L)$ respectively; see Corollary 6.4.4. The assertion follows now from the construction of the skew multiplication of the source and target of $\mathrm{res}_V^{V'} \hat{\otimes} \sigma_r^{F'F}$ (see Section 3). □

6.4.14. For any subset $C \subset X^{\mathrm{an}}$ we have by Proposition 6.3.2

$$\mathcal{O}_{X^{\mathrm{an}}}(C) = \varinjlim_U \mathcal{O}_{X^{\mathrm{an}}}(U),$$

where U runs over all open neighborhoods of C in X^{an}. Obviously, if C is contained in $\mathscr{B}$ we have $\mathcal{O}_{\mathscr{B}}(C) = \mathcal{O}_{X^{\mathrm{an}}}(C)$. We recall that the *star* of a facet F' in $\mathscr{B}$ is the subset of $\mathscr{B}$ defined by

$$\mathrm{St}(F) := \text{ union of all facets } F' \subseteq \mathscr{B} \text{ such that } F \subseteq \overline{F'}.$$

These stars form a locally finite open covering of $\mathscr{B}$.

Proposition 6.4.15. *Let $F \subset \mathscr{B}$ be a facet, and let $C \subset \mathrm{St}(F)$ be a compact set.*

(i) *There is a countable fundamental system of neighborhoods $V_1 \supset V_2 \supset \cdots$ of C in X^{an} with the following properties*:

- *For all i the neighborhood V_i is a special subdomain on which $U_F^{(e)}$ acts analytically.*
- *For all $i < j$ the induced map $\mathcal{A}_{V_i} \to \mathcal{A}_{V_j}$ is compact and injective.*

(ii) *Let $(V_i)_i$ be as in* (i). *Then the rings $\mathcal{A}_{V_i} \# D_r(U_F^{(e)})$ exist for all i and the maps*

$$\mathcal{A}_{V_i} \# D_r(U_F^{(e)}) \to \mathcal{A}_{V_{i+1}} \# D_r(U_F^{(e)})$$

induced by the restriction maps $\mathcal{A}_{V_i} \to \mathcal{A}_{V_{i+1}}$ are homomorphisms of K-algebras.

(iii) *Let* $(V_i)_i$ *be as in* (i). *Then the maps*

$$\mathcal{A}_{V_i} \hat{\otimes}_L D_r(U_F^{(e)}) \to \mathcal{O}_{\mathscr{B}}(C) \hat{\otimes}_L D_r(U_F^{(e)})$$

induced by the canonical maps $\mathcal{A}_{V_i} \to \mathcal{O}_{\mathscr{B}}(C)$ *induce an isomorphism of vector spaces*

$$\varinjlim_i \big(\mathcal{A}_{V_i} \hat{\otimes}_L D_r(U_F^{(e)})\big) \to \mathcal{O}_{\mathscr{B}}(C) \hat{\otimes}_L D_r(U_F^{(e)}). \tag{6.4.16}$$

(*iv*) *The left-hand side of* (6.4.16) *carries a unique structure of a* K*-algebra*, *such that the canonical maps*

$$\mathcal{A}_{V_i} \hat{\otimes}_L D_r(U_F^{(e)}) \to \varinjlim_i \big(\mathcal{A}_{V_i} \hat{\otimes}_L D_r(U_F^{(e)})\big)$$

become K*-algebra homomorphisms. Consequently*, *via transport of structure*, *we give* $\mathcal{O}_{\mathscr{B}}(C) \hat{\otimes}_L D_r(U_F^{(e)})$ *the unique* K*-algebra structure*, *henceforth denoted by* $\mathcal{O}_{\mathscr{B}}(C)\# D_r(U_F^{(e)})$, *such that* (6.4.16) *becomes an isomorphism of* K*-algebras.*

Proof. (i) By [Berkovich 1990, 3.2.9] there is a countable fundamental system of open neighborhoods $W_1 \supset W_2 \supset \cdots$ of C in X^{an}. We are going to find inductively the special domain $V_i \subset W_i$. To begin, use Lemma 6.2.6 to find for each $x \in C$ an affinoid neighborhood $W_{1,x} \subset W_1$ on which $U_F^{(e)}$ acts analytically. Clearly, we may furthermore assume that every $W_{1,x}$ is connected. Denote by $\mathrm{Int}(W_{1,x})$ the topological interior of $W_{1,x}$. As C is compact it is contained in a finite union $\mathrm{Int}(W_{1,x_1}) \cup \ldots \cup \mathrm{Int}(W_{1,x_{m_1}})$. Put

$$V_1 = W_{1,x_1} \cup \ldots \cup W_{1,x_{m_1}}.$$

Now suppose we have found a special domain $V_i = W_{i,z_1} \cup \ldots \cup W_{i,z_{m_i}}$ contained in W_i with the property that for all $1 \le j \le m_i$

- W_{i,z_j} is a connected affinoid neighborhood of z_i on which $U_F^{(e)}$ acts analytically, for all $1 \le j \le m_i$,
- C is contained in the union of the $\mathrm{Int}(W_{i,z_j})$, for $1 \le j \le m_i$.

For a given $z' \in C$ choose $j(z') \in \{1, \ldots, m_i\}$ such that z' is contained in $\mathrm{Int}(W_{i,z_{j(z')}})$. Use again Lemma 6.2.6 to find a connected affinoid neighborhood $W_{i+1,z'}$ of z' contained in $W_{i+1} \cap \mathrm{Int}(W_{i,z_{j(z')}})$ on which $U_F^{(e)}$ acts analytically. Put $z = z_{j(z')}$. As $W_{i,z}$ is connected, it is irreducible [loc. cit., 3.1.8], and so is $Spec(\mathcal{A}_{W_{i,z}})$. The ring $\mathcal{A}_{W_{i,z}}$ is hence an integral domain, and the restriction map

$$\mathcal{A}_{W_{i,z}} \to \mathcal{A}_{W_{i+1,z'}} \tag{6.4.17}$$

is thus injective. Moreover, since $W_{i+1,z'}$ is contained in $\mathrm{Int}(W_{i,z})$, the map (6.4.17) is inner, by [loc. cit., 2.5.9]. The arguments in [Emerton 2011, 2.1.16] then show that (6.4.17) is a compact map of Banach spaces. Now choose $z'_1, \ldots, z'_{m_{i+1}}$ such that C is contained in the union of the $\mathrm{Int}(W_{i+1,z'_j})$, for $1 \le j \le m_{i+1}$, and let V_{i+1} be the union of the W_{i+1,z'_j}, for $1 \le j \le m_{i+1}$. Recall that

$$\mathcal{A}_{V_i} = \ker\Bigl(\prod_{j=1}^{m_i} \mathcal{A}_{W_{i,z_j}} \rightrightarrows \prod_{j,j'} \mathcal{A}_{W_{i,z_j} \cap W_{i,z_{j'}}}\Bigr);$$

see [Berkovich 1990, 2.2.6 and 3.3]. Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{A}_{V_i} & \longrightarrow & \prod_{j=1}^{m_i} \mathcal{A}_{W_{i,z_j}} \\ \downarrow & & \downarrow \\ \mathcal{A}_{V_{i+1}} & \longrightarrow & \prod_{j=1}^{m_{i+1}} \mathcal{A}_{W_{i+1,z'_j}} \end{array}$$

The horizontal arrows are the obvious inclusions. The vertical arrow on the right is the one induced by the maps (6.4.17), and is thus injective and compact. The canonical vertical arrow on the left is thus injective and compact too [Schneider 2002, 16.7 (ii)]. This proves the first assertion.

(ii) This follows immediately from the second part of Corollary 6.4.4, together with Lemma 6.4.13.

(iii) Because $(V_i)_i$ is a fundamental system of neighborhoods of C we have, by Proposition 6.3.2, $\mathcal{O}_{\mathscr{B}}(C) = \varinjlim_i \mathcal{A}_{V_i}$. The assertion now follows from (i) and Proposition 2.4.1.

(iv) Using (ii) we see that the right-hand side of (6.4.16) has a canonical K-algebra structure. The remaining assertions are now clear. □

7. A sheaf of "distribution operators" on the building

In this section we assume throughout $L = \mathbb{Q}_p$, $e > e_{\mathrm{st}}$ (compare Lemma 6.2.6), and $r \in [r_0, 1)$. Because $e - 1 \ge e_{\mathrm{st}} \ge e_{\mathrm{uni}}$ (see again Lemma 6.2.6), all groups $U_z^{(e-1)}$ are uniform pro-p groups; see Remark 4.3.4. We will work from now on exclusively over the coefficient field K. To ease notation we will therefore drop this coefficient field from the notation when working with distribution algebras. We thus write $D(G) = D(G, K)$, $D_r(U_F^{(e)}) = D_r(U_F^{(e)}, K)$ etc.

Recall that the sheaf of (twisted) differential operators $\mathcal{D}_\chi$ on X may be constructed from the skew tensor product $\mathcal{O}_X \# U(\mathfrak{g})$; see Section 5. In a similar way we are going to construct a sheaf of "distribution operators" on $\mathscr{B}$ starting from a twisted tensor product $\mathcal{O}_{\mathscr{B}} \# \underline{D}_r$. Here, $\underline{D}_r$ replaces the constant sheaf $\underline{U(\mathfrak{g})}$ and equals a sheaf of distribution algebras on $\mathscr{B}$. It will be a constructible sheaf with respect to the usual polysimplicial structure of $\mathscr{B}$. Recall from Section 4.2 that a sheaf on a polysimplicial space is called *constructible* if its restriction to a given geometric polysimplex is a constant sheaf.

7.1. ***A constructible sheaf of distribution algebras.*** Given an open set $\Omega \subset \mathscr{B}$ we have for any $z \in \Omega$ the natural map

$$\iota_z : \mathcal{O}_{\mathscr{B}}(\Omega) \to \mathcal{O}_{\mathscr{B},z}, \qquad f \mapsto \text{germ of } f \text{ at } z.$$

Definition 7.1.1. For an open subset $\Omega \subseteq \mathscr{B}$ let

$$\underline{D}_r(\Omega) := K\text{-vector space of all maps } s : \Omega \to \dot{\bigcup_{z \in \Omega}} D_r(U_z^{(e)}) \text{ such that}$$

(1) $s(z) \in D_r(U_z^{(e)})$ for all $z \in \Omega$, and

(2) for each facet $F \subseteq \mathscr{B}$ there exists a finite open covering $\Omega \cap \mathrm{St}(F) = \bigcup_{i \in I} \Omega_i$ with the property: for each i with $\Omega_i \cap F \neq \varnothing$ there is an element $s_i \in D_r(U_F^{(e)})$ satisfying the following conditions:

  (a) $s(z) = s_i$ for any $z \in \Omega_i \cap F$.

  (b) $s(z') = \sigma_r^{FF'}(s_i)$ for any $z' \in \Omega_i$. Here, F' is the unique facet in $\mathrm{St}(F)$ that contains z'.

From (a) it is easy to see that the restriction of $\underline{D}_r$ to a facet F is the constant sheaf with value $D_r(U_F^{(e)})$. Hence $\underline{D}_r$ is constructible. Furthermore, if $\Omega' \subseteq \Omega$ is an open subset there is the obvious restriction map $\underline{D}_r(\Omega) \to \underline{D}_r(\Omega')$. The proof of the following result is implicitly contained in the proofs of Lemmas 7.2.2 and 7.2.3 below.

Lemma 7.1.2. *With pointwise multiplication $\underline{D}_r$ is a sheaf of K-algebras. For $z \in \mathscr{B}$ one has $(\underline{D}_r)_z = D_r(U_z^{(e)})$.*

7.2. ***Sheaves of completed skew group rings.***

Definition 7.2.1. For an open subset $\Omega \subseteq \mathscr{B}$ let

$$(\mathcal{O}_{\mathscr{B}} \# \underline{D}_r)(\Omega) := K\text{-vector space of all maps } s : \Omega \to \dot{\bigcup_{z \in \Omega}} \mathcal{O}_{\mathscr{B},z} \# D_r(U_z^{(e)}) \text{ such that}$$

(1) $s(z) \in \mathcal{O}_{\mathscr{B},z} \# D_r(U_z^{(e)})$ for all $z \in \Omega$, and

(2) for each facet $F \subseteq \mathscr{B}$ there exists a finite open covering $\Omega \cap \mathrm{St}(F) = \bigcup_{i \in I} \Omega_i$ with the property that, for each i with $\Omega_i \cap F \neq \varnothing$, there exists

$$s_i \in \mathcal{O}_{\mathscr{B}}(\Omega_i) \hat{\otimes}_L D_r(U_F^{(e)})$$

satisfying the following conditions:

  (a) $s(z) = (\iota_z \hat{\otimes} \mathrm{id})(s_i)$ for any $z \in \Omega_i \cap F$.

  (b) $s(z') = (\iota_{z'} \hat{\otimes} \sigma_r^{FF'})(s_i)$ for any $z' \in \Omega_i$. Here, F' is the unique facet in $\mathrm{St}(F)$ that contains z'.

Consider a map $s:\Omega\to\dot{\bigcup}_{z\in\Omega}\mathcal{O}_{\mathscr{B},z}\# D_r(U_z^{(e)})$ satisfying (1). It will be convenient to call an open covering $\Omega\cap\mathrm{St}(F)=\bigcup_{i\in I}\Omega_i$ together with the elements s_i such that (a) and (b) hold a *datum* for s with respect to the facet F. Any open covering of $\Omega\cap\mathrm{St}(F)$ which is a refinement of the covering $\{\Omega_i\}_{i\in I}$, together with the same set of elements s_i is again a datum for s with respect to F.

Suppose $\Omega'\subseteq\Omega$ is an open subset and let $s\in(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)(\Omega)$. Let $F\subseteq\mathscr{B}$ be a facet. Given a corresponding datum $\{\Omega_i\}_{i\in I}$ for s put $\Omega_i':=\Omega'\cap\Omega_i$. Together with the elements s_i, in case $\Omega_i'\cap F\neq\varnothing$, we obtain a datum for the function $s|_{\Omega'}$. It follows that $(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)$ is a presheaf of K-vector spaces on $\mathscr{B}$.

In the following it will be convenient to define $\mathcal{F}(\Omega)$ as the K-vector space of all maps

$$s:\Omega\to\dot{\bigcup_{z\in\Omega}}\mathcal{O}_{\mathscr{B},z}\#_L D_r(U_z^{(e)})$$

satisfying condition (1) in Definition 7.2.1. It is clear that pointwise multiplication makes $\mathcal{F}$ a sheaf of K-algebras on $\mathscr{B}$ such that $(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)$ is a subpresheaf of K-vector spaces.

Lemma 7.2.2. *The induced multiplication makes $(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)\subseteq\mathcal{F}$ an inclusion of sheaves of K-algebras.*

Proof. Take an open subset $\Omega\subseteq\mathscr{B}$. We first show that for $s,s'\in(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)(\Omega)$ we have $ss'\in(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)(\Omega)$, i.e., that $(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)(\Omega)$ is a subalgebra of $\mathcal{F}(\Omega)$.

To do this let $F\subseteq\mathscr{B}$ be a facet. Let $\{\Omega_i\}_{i\in I}$ and $\{\Omega_j'\}_{j\in J}$ be corresponding data for s and s' respectively. Passing to $\{\Omega_{ij}\}_{ij}$ with $\Omega_{ij}=\Omega_i\cap\Omega_j'$ and refining the coverings if necessary, we may assume: there exists one datum $\{\Omega_i\}_{i\in I}$ for both s,s' and each Ω_i is contained in a compact subset of $\mathrm{St}(F)$. We will produce a datum for ss' by passing to a suitable open covering of Ω_i whenever Ω_i and F intersect. To this end, let us fix such an $i\in I$. We choose connected compact subsets $C\subset\mathrm{St}(F)$ whose open interiors C° form a covering of Ω_i. We have the K-algebra $\mathcal{O}_{\mathscr{B}}(C)\# D_r(U_F^{(e)})$ from Proposition 6.4.15. We apply the base change $(\cdot)\hat{\otimes}_L D_r(U_F^{(e)})$ to the restriction map $\mathcal{O}_{\mathscr{B}}(\Omega)\to\mathcal{O}_{\mathscr{B}}(C)$ and consider the image of s_i and s_i' in $\mathcal{O}_{\mathscr{B}}(C)\# D_r(U_F^{(e)})$. Let $s_is_i'\in\mathcal{O}_{\mathscr{B}}(C)\# D_r(U_F^{(e)})$ be their product. We apply the base change $(\cdot)\hat{\otimes}_L D_r(U_F^{(e)})$ to the restriction map $\mathcal{O}_{\mathscr{B}}(C)\to\mathcal{O}_{\mathscr{B}}(C^\circ)$ and consider the image of s_is_i' in $\mathcal{O}_{\mathscr{B}}(C^\circ)\hat{\otimes}_L D_r(U_F^{(e)})$. We denote this image again by s_is_i'. According to the definition of the product on $\mathcal{O}_{\mathscr{B}}(C)\# D_r(U_F^{(e)})$ in Proposition 6.4.15 we find, for any $z\in C^\circ\cap F$, that

$$(ss')(z)=s(z)s'(z)=(\iota_z\hat{\otimes}\,\mathrm{id})(s_i)\cdot(\iota_z\hat{\otimes}\,\mathrm{id})(s_i')=(\iota_z\hat{\otimes}\,\mathrm{id})(s_is_i')$$

using Corollary 6.4.10 and we find, for any $z'\in C^\circ$, that

$$(ss')(z')=s(z')s'(z')=(\iota_z\hat{\otimes}\,\sigma_r^{F'F})(s_i)\cdot(\iota_z\hat{\otimes}\,\sigma_r^{F'F})(s_i')=(\iota_z\hat{\otimes}\,\sigma_r^{F'F})(s_is_i')$$

using Lemma 6.4.13 (F' is the unique facet in $\mathrm{St}(F)$ that contains z'). This shows that, if we replace each such Ω_i by the open covering given by the corresponding C° and invoke the corresponding sections $s_i s_i' \in \mathcal{O}_{\mathscr{B}}(C^\circ)\hat{\otimes}_L D_r(U_F^{(e)})$, we will have a datum for ss' relative to F. Consequently, $ss' \in (\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)(\Omega)$ and hence, $(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)(\Omega)$ is a subalgebra of $\mathcal{F}(\Omega)$. If $\Omega' \subseteq \Omega$ is an open subset the restriction map $(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)(\Omega) \to (\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)(\Omega')$ is obviously multiplicative. Thus, $(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)$ is a presheaf of K-algebras.

Let us show that $(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)$ is in fact a sheaf. Since $(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r) \subseteq \mathcal{F}$ is a sub-presheaf and $\mathcal{F}$ is a sheaf it suffices to prove the following: if

$$\Omega = \bigcup_{j\in J} U_j$$

is an open covering of an open subset $\Omega \subseteq \mathscr{B}$ and if $s_j \in (\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)(U_j)$ are local sections with $s_j|_{U_j\cap U_i} = s_i|_{U_i\cap U_j}$ for all $i, j \in J$ then the unique section $s \in \mathcal{F}(\Omega)$ with $s|_{U_j} = s_j$ for all $j \in J$ lies in $(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)(\Omega)$. To do this let $F \subseteq \mathscr{B}$ be a facet. Consider for each $j \in J$ a datum $\{U_{ji}\}_{i\in I}$ for s_j. In particular, $U_j \cap \mathrm{St}(F) = \bigcup_{i\in I} U_{ji}$ and there are distinguished elements

$$s_{ji} \in \mathcal{O}_{\mathscr{B}}(U_{ji}) \,\hat{\otimes}_L\, D_r(U_F^{(e)})$$

whenever $U_{ji} \cap F \neq \varnothing$ intersect. Then $\Omega \cap \mathrm{St}(F) = \bigcup_{ji} U_{ji}$ (together with the elements s_{ji} whenever $U_{ji} \cap F \neq \varnothing$) is a datum for s. Indeed, given $z \in U_{ji} \cap F$ one has $s(z) = s_j(z) = (\iota_z \,\hat{\otimes}\, \mathrm{id})(s_{ji})$ which shows condition (a) in Definition 7.2.1. Moreover, if $z' \in U_{ji}$ one has $s(z') = s_j(z') = (\iota_z \,\hat{\otimes}\, \sigma_r^{F'F})(s_{ji})$, which shows (b). Together this means $s \in (\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)(\Omega)$. □

The next lemma shows that the stalks of the sheaf $(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)$ are as expected.

Lemma 7.2.3. *The canonical map* $(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)_z \xrightarrow{\sim} \mathcal{O}_{\mathscr{B},z}\#D_r(U_z^{(e)})$ *is an isomorphism of K-algebras for any $z \in \mathscr{B}$.*

Proof. There is the K-algebra homomorphism

$$(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)_z \to \mathcal{O}_{\mathscr{B},z}\#D_r(U_z^{(e)}), \qquad \text{germ of } s \text{ at } z \mapsto s(z).$$

Let us show that this map is injective. Let $[s]$ be the germ of a local section $s \in (\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)(\Omega)$ over some open subset $\Omega \subseteq \mathscr{B}$ with the property $s(z) = 0$. Let F be the unique facet of $\mathscr{B}$ that contains z and let $\{\Omega_i\}_{i\in I}$ be a corresponding datum for s. According to Lemma 6.2.4 we may write the stalk

$$\mathcal{O}_{\mathscr{B},z} = \varinjlim_{V} \mathcal{A}_V$$

as a compact inductive limit of integral affinoid algebras with injective transition maps.

Let us abbreviate $\mathcal{E} := D_r(U_F^{(e)})$. If $W \subseteq V$ is an inclusion of affinoids occurring in the above inductive limit, then [Emerton 2011, Corollary 1.1.27] implies that the base changed map

$$\mathcal{A}_V \hat{\otimes}_L \mathcal{E} \to \mathcal{A}_W \hat{\otimes}_L \mathcal{E}$$

remains injective. Let $i_0 \in I$ such that $z \in \Omega_{i_0} \cap F$ and consider the map

$$\iota_z \hat{\otimes} \mathrm{id} : \mathcal{O}_{\mathscr{B}}(\Omega_{i_0}) \hat{\otimes}_L \mathcal{E} \to \mathcal{O}_{\mathscr{B},z} \hat{\otimes}_L \mathcal{E} \simeq \varinjlim_V (\mathcal{A}_V \hat{\otimes}_L \mathcal{E}).$$

The last isomorphism here is due to Proposition 2.4.1. Let V be an affinoid in the inductive limit on the right-hand side such that $\mathcal{A}_V \hat{\otimes}_L \mathcal{E}$ contains the image of s_{i_0} under $\iota_z \hat{\otimes} \mathrm{id}$. Choose an open subset $U \subseteq X^{\mathrm{an}}$ in V containing z and replace Ω_{i_0} by the intersection $\mathscr{B} \cap U$. Then replace s_{i_0} by its restriction to this intersection, in other words, s_{i_0} lies now in the image of the map

$$\mathcal{A}_V \hat{\otimes}_L \mathcal{E} \to \mathcal{O}_{\mathscr{B}}(\Omega_{i_0}) \hat{\otimes}_L \mathcal{E}.$$

By our discussion above, the natural map from $\mathcal{A}_V \hat{\otimes}_L \mathcal{E}$ into $\mathcal{O}_{\mathscr{B},z} \hat{\otimes}_L \mathcal{E}$ is injective and lifts the map $\iota_z \hat{\otimes} \mathrm{id}$. We may therefore deduce from

$$0 = s(z) = (\iota_z \hat{\otimes} \mathrm{id})(s_{i_0})$$

that $s_{i_0} = 0$. Given $z' \in \Omega_{i_0}$ let F' be the unique facet of $\mathrm{St}(F)$ containing z'. Then $s(z') = (\iota_z \hat{\otimes} \sigma_r^{F'F})(s_{i_0}) = 0$ according to condition (b) and, consequently, $s|_{\Omega_{i_0}} = 0$. Since Ω_{i_0} is an open neighborhood of z this shows $[s] = 0$ and proves injectivity.

Let us now show that our map is surjective. Let $t \in \mathcal{O}_{\mathscr{B},z} \# D_r(U_z^{(e)})$ be an element in the target. Since the stalk $\mathcal{O}_{\mathscr{B},z}$ is an inductive limit with compact and injective transition maps and since $D_r(U_z^{(e)})$ is a Banach space, Proposition 2.4.1 implies that there is an open neighborhood Ω' of z and an element $\tilde{s} \in \mathcal{O}_{\mathscr{B}}(\Omega') \hat{\otimes}_L D_r(U_z^{(e)})$ such that $(\iota_z \hat{\otimes} \mathrm{id})(\tilde{s}) = t$. Let $F \subseteq \mathscr{B}$ be a facet containing z and define

$$\Omega := \Omega' \cap \mathrm{St}(F), \quad s := (\mathrm{res}_\Omega^W \hat{\otimes} \mathrm{id})(\tilde{s}) \in \mathcal{O}_{\mathscr{B}}(\Omega) \hat{\otimes}_L D_r(U_z^{(e)}).$$

Since $\mathrm{St}(F)$ is an open neighborhood of z and contains only finitely many facets of $\mathscr{B}$ we may pass to a smaller Ω' (and hence Ω) and therefore assume: any $F' \in (\mathscr{B} \setminus \mathrm{St}(F))$ satisfies $F' \cap \Omega = \varnothing$. For any $z' \in \Omega$ let $s(z') := (\iota_{z'} \hat{\otimes} \sigma_r^{FF'})(s)$ where F' denotes the facet in $\mathrm{St}(F)$ containing z'. This defines a function

$$s : \Omega \to \dot{\bigcup_{z' \in \Omega}} \mathcal{O}_{\mathscr{B},z} \# D_r(U_{z'}^{(e)})$$

satisfying condition (1) of Definition 7.2.1. According to Lemma 4.3.5 one has $\sigma_r^{FF} = \mathrm{id}$ whence

$$s(z) = (\iota_z \hat{\otimes} \mathrm{id})(s) = (\iota_z \hat{\otimes} \mathrm{id})(\tilde{s}) = t.$$

Thus, the germ of s at z will be a preimage of t once we have shown that $s \in (\mathcal{O}_{\mathscr{B}} \# \underline{D}_r)(\Omega)$. To do this consider an arbitrary facet $F' \subset \mathscr{B}$ together with the covering of $\Omega \cap \mathrm{St}(F')$ consisting of the single element

$$\Omega_0 := \Omega \cap \mathrm{St}(F').$$

Suppose $\Omega_0 \cap F' \neq \varnothing$. We have to exhibit an element $s_0 \in \mathcal{O}_{\mathscr{B}}(\Omega_0) \hat{\otimes}_L D_r(U_{F'}^{(e)})$ satisfying conditions (a) and (b) in Definition 7.2.1. Since $F' \in \mathrm{St}(F)$ we may define $s_0 := (\mathrm{id} \hat{\otimes} \sigma_r^{FF'})(s)$. For any $z' \in \Omega \cap F'$ we compute

$$s(z') = (\iota_{z'} \hat{\otimes} \sigma_r^{FF'})(s) = (\iota_{z'} \hat{\otimes} \mathrm{id})(\mathrm{id} \hat{\otimes} \sigma_r^{FF'})(s) = (\iota_{z'} \hat{\otimes} \mathrm{id})(s_0)$$

which shows (a). Moreover, for any $z' \in \Omega_0$ we compute

$$s(z') = (\iota_{z'} \hat{\otimes} \sigma_r^{FF''})(s) = (\iota_{z'} \hat{\otimes} \sigma_r^{F'F''})(\iota_{z'} \hat{\otimes} \sigma_r^{FF'})(s) = (\iota_{z'} \hat{\otimes} \sigma_r^{F'F''})(s_0)$$

by Lemma 4.3.5. Here F'' denote the facet of $\mathrm{St}(F')$ that contains z'. This shows (b) and completes the proof. □

Corollary 7.2.4. *The $\mathcal{O}_{\mathscr{B},z}$-module structure on $(\mathcal{O}_{\mathscr{B}} \# \underline{D}_r)_z$ for any $z \in \mathscr{B}$ sheafifies to a $\mathcal{O}_{\mathscr{B}}$-module structure on $(\mathcal{O}_{\mathscr{B}} \# \underline{D}_r)$ (compatible with scalar multiplication by L).*

Proof. As with any sheaf [Godement 1958, II.1.2] we may regard $(\mathcal{O}_{\mathscr{B}} \# \underline{D}_r)$ as the sheaf of *continuous* sections of its étale space

$$\begin{array}{c} \dot{\bigcup}_{z \in \mathscr{B}} (\mathcal{O}_{\mathscr{B}} \# \underline{D}_r)_z \\ \big\downarrow \\ \mathscr{B} \end{array}$$

and the same applies to the sheaf $\mathcal{O}_{\mathscr{B}}$. By the preceding proposition we have $(\mathcal{O}_{\mathscr{B}} \# \underline{D}_r)_z = \mathcal{O}_{\mathscr{B},z} \# D_r(U_z^{(e)})$ for any $z \in \mathscr{B}$. Let $\Omega \subseteq \mathscr{B}$ be an open subset, $s \in (\mathcal{O}_{\mathscr{B}} \# \underline{D}_r)$, $f \in \mathcal{O}_{\mathscr{B}}(\Omega)$. For $z \in \Omega$ put $(f \cdot s)(z) := f(z) \cdot s(z)$. This visibly defines an element $f \cdot s \in \mathcal{F}(\Omega)$. The "$\mathcal{O}_{\mathscr{B}}$-linearity" in conditions (a) and (b) proves $f \cdot s \in (\mathcal{O}_{\mathscr{B}} \# \underline{D}_r)$. It follows that $(\mathcal{O}_{\mathscr{B}} \# \underline{D}_r)$ is an $\mathcal{O}_{\mathscr{B}}$-module in the prescribed way. □

Proposition 7.2.5. *The natural map*

$$D_r(U_z^{(e)}) \to (\mathcal{O}_{\mathscr{B}} \# \underline{D}_r)_z, \quad \delta \mapsto 1 \hat{\otimes} \delta,$$

sheafifies to a morphism of sheaves of K-algebras $\underline{D}_r \to \mathcal{O}_{\mathscr{B}} \# \underline{D}_r$.

Proof. This is easy to see. □

Recall from (6.4.12) that we have for any $z \in \mathscr{B}$ a canonical K-algebra homomorphism

$$\mathcal{O}_{\mathscr{B},z}\# U(\mathfrak{g})_K \to \mathcal{O}_{\mathscr{B},z}\# D_r(U_z^{(e)}).$$

Proposition 7.2.6. *The homomorphisms (6.4.12) sheafify into a morphism*

$$\mathcal{O}_{\mathscr{B}}\# U(\mathfrak{g})_K \to \mathcal{O}_{\mathscr{B}}\# \underline{D}_r$$

of sheaves of K-algebras. This morphism is $\mathcal{O}_{\mathscr{B}}$-linear.

Proof. We view $\mathcal{O}_{\mathscr{B}}\# U(\mathfrak{g})_K$ as the sheaf of continuous sections of its étale space

$$\begin{array}{c} \dot{\bigcup}_{z\in\mathscr{B}}\mathcal{O}_{\mathscr{B},z}\,\#_L\, U(\mathfrak{g})_K \\ \Big\downarrow \\ \mathscr{B} \end{array}$$

Composing such a section with (6.4.12) defines a morphism $i : \mathcal{O}_{\mathscr{B}}\# U(\mathfrak{g})_K \to \mathcal{F}$ of sheaves of K-algebras and we will prove that its image lies in the subsheaf $(\mathcal{O}_{\mathscr{B}}\# \underline{D}_r)$. To do this let $\Omega \subseteq \mathscr{B}$ be an open subset and $s \in \mathcal{O}_{\mathscr{B}}\# U(\mathfrak{g})_K(\Omega)$ a local section.

Let $F \subseteq \mathscr{B}$ be a facet. Consider the covering of $\Omega \cap \mathrm{St}(F)$ consisting of the single element $\Omega_0 := \Omega \cap \mathrm{St}(F)$. In case $\Omega_0 \cap F \neq \varnothing$ let s_0 be the image of $\tilde{s}$ under the map

$$\mathcal{O}_{\mathscr{B}}(\Omega) \otimes_L U(\mathfrak{g})_K \to \mathcal{O}_{\mathscr{B}}(\Omega)\,\hat{\otimes}_L\, D_r(U_F^{(e)})$$

induced by $U(\mathfrak{g})_K \subseteq D_r(U_F^{(e)})$. For any $z \in \Omega_0 \cap F$ we obviously have $i(s)(z) = (\iota_z \hat{\otimes} \mathrm{id})(s_0)$, which shows condition (a). For any $z \in \Omega_0$ we find $i(s)(z) = (\iota_z \hat{\otimes} \mathrm{id})(s_0) = (\iota_z \hat{\otimes} \sigma_r^{FF'})(s_0)$ by the last statement of Lemma 4.3.5. Here F' denotes the facet containing z. This shows (b). In the light of the definitions it is clear that the resulting morphism $\mathcal{O}_{\mathscr{B}}\# U(\mathfrak{g})_K \to \mathcal{O}_{\mathscr{B}}\# \underline{D}_r$ is $\mathcal{O}_{\mathscr{B}}$-linear. □

7.3. *Infinitesimal characters.* We will write $\mathfrak{g}_K := \mathfrak{g} \otimes_{\mathbb{Q}_p} K$, $\mathfrak{t}_K := \mathfrak{t} \otimes_{\mathbb{Q}_p} K$ etc.

7.3.1. According to [Schneider and Teitelbaum 2002, Proposition 3.7] the ring $Z(\mathfrak{g}_K)$ lies in the center of the ring $D(G)$. In the following we fix a central character

$$\theta : Z(\mathfrak{g}_K) \to K$$

and we let

$$D(G)_\theta := D(G) \otimes_{Z(\mathfrak{g}_K),\theta} K$$

be the corresponding central reduction of $D(G)$. A (left) $D(G)_\theta$-module M is called *coadmissible* if it is coadmissible as $D(G)$-module via the natural map $D(G) \to D(G)_\theta$, $\delta \mapsto \delta \hat{\otimes} 1$. In the following we will study the abelian category of coadmissible $D(G)_\theta$-modules. As explained in the beginning, this category is antiequivalent to the category of admissible locally analytic G-representations over K which have infinitesimal character θ.

Example. Let $\lambda_0 : D(T) \to K$ denote the character of $D(T)$ induced by the augmentation map $K[T] \to K$. The restriction of λ_0 to the Lie algebra $\mathfrak{t}_K \subset D(T)$ vanishes identically whence $\chi = \rho$. Let $\theta_0 : Z(\mathfrak{g}_K) \to K$ be the infinitesimal character associated to ρ via the Harish-Chandra homomorphism. Then $\ker \theta_0 = Z(\mathfrak{g}_K) \cap U(\mathfrak{g}_K)\mathfrak{g}_K$.

Remark. K. Ardakov and S. Wadsley [2013, Section 8] have established a version of Quillen's lemma for p-adically completed universal enveloping algebras. It implies that any topologically irreducible admissible locally analytic G-representation admits, up to a finite extension of K, a central character and an infinitesimal character [Dospinescu and Schraen 2013].

7.3.2. To investigate the local situation let F be a facet in $\mathscr{B}$. We have

$$Z(\mathfrak{g}_K) \subseteq D(U_F^{(e)}) \cap Z(D(G)) \subseteq Z(D(U_F^{(e)})),$$

again according to [Schneider and Teitelbaum 2002, Proposition 3.7]. We let

$$D_r(U_F^{(e)})_\theta := D_r(U_F^{(e)}) \otimes_{Z(\mathfrak{g})_K,\theta} K$$

be the corresponding central reduction of $D_r(U_F^{(e)})$.

Let F, F' be two facets in $\mathscr{B}$ such that $F' \subseteq \overline{F}$ and consider the homomorphism $\sigma_r^{F'F}$. According to the last statement of Lemma 4.3.5 it factors by continuity into a homomorphism

$$\sigma_r^{F'F} : D_r(U_{F'}^{(e)})_\theta \to D_r(U_F^{(e)})_\theta.$$

We may therefore define a sheaf of K-algebras $\underline{D}_{r,\theta}$ in complete analogy with the sheaf $\underline{D}_r$ by replacing each $D_r(U_z^{(e)})$ and each $D_r(U_F^{(e)})$ by their central reductions. In particular, $(\underline{D}_{r,\theta})_z = D_r(U_z^{(e)})_\theta$ for any $z \in \mathscr{B}$ and there is an obvious quotient morphism

$$\underline{D}_r \to \underline{D}_{r,\theta}.$$

7.4. ***Twisting.*** We now bring in a toral character

$$\chi : \mathfrak{t}_K \to K$$

such that $\sigma(\chi) = \theta$. We consider the two-sided ideals $\mathcal{I}_{\mathscr{B},\mathfrak{t}}^{\mathrm{an}}$ and $\mathcal{I}_{\mathscr{B},\chi}^{\mathrm{an}}$ of $\mathcal{O}_{\mathscr{B}} \# U(\mathfrak{g}_K)$. Denote the right ideal in $\mathcal{O}_{\mathscr{B}} \# \underline{D}_r$ generated by the image of the first resp. second under the morphism

$$\mathcal{O}_{\mathscr{B}} \# U(\mathfrak{g}_K) \to \mathcal{O}_{\mathscr{B}} \# \underline{D}_r$$

by $\mathscr{I}_{\mathfrak{t}}^{\mathrm{an}}$ resp. $\mathscr{I}_{\chi}^{\mathrm{an}}$.

Recall that we assume $e > e_{\mathrm{st}}$ throughout this section and that this implies $e > e_{\mathrm{uni}}$.

Proposition 7.4.1. *Let $z \in \mathscr{B}$ a point and let V be a strictly affinoid neighborhood of z on which $U_z^{(e-1)}$ acts analytically. Then the ring $\mathcal{A}_V \# D_{r_0}(U_z^{(e)})$ is noetherian.*

Proof. As $e > e_{\mathrm{uni}}$ the group $U_z^{(e-1)}$ is a uniform pro-p group. Let

$$\mathfrak{h}_{\mathbb{Z}_p} = \mathcal{L}(U_z^{(e-1)}) \subset \mathfrak{g}$$

be the $\mathbb{Z}_p$-Lie algebra of $U_z^{(e-1)}$ [Dixon et al. 1999, Section 9.4]. We consider the bijective exponential map $\exp : \mathfrak{h}_{\mathbb{Z}_p} \to U_z^{(e-1)}$ which is used to define the affinoid analytic subgroup $\mathbb{U}_z^{(e-1)} \subset \boldsymbol{G}^{\mathrm{an}}$; see Section 6.2.5. This exponential map gives then rise to an exponential map of affinoid analytic spaces $\exp : \mathbb{B} \otimes_{\mathbb{Z}_p} \mathfrak{h}_{\mathbb{Z}_p} \to \mathbb{U}_z^{(e-1)}$, where $\mathbb{B}$ is the closed unit disc over $\mathbb{Q}_p$ and $\mathbb{B} \otimes_{\mathbb{Z}_p} \mathfrak{h}_{\mathbb{Z}_p}$ is the strictly $\mathbb{Q}_p$-analytic space whose affinoid algebra is

$$\mathrm{Sym}_{\mathbb{Z}_p}(\mathfrak{h}_{\mathbb{Z}_p}^{\vee})^{\wedge} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Here, $\mathfrak{h}_{\mathbb{Z}_p}^{\vee} = \mathrm{Hom}_{\mathbb{Z}_p}(\mathfrak{h}_{\mathbb{Z}_p}, \mathbb{Z}_p)$ and $(\,\cdot\,)^{\wedge}$ means the p-adic completion.

The affinoid algebra $\mathcal{A}_V$ is a $\mathfrak{g}$-module. As a first step we want to show that the subring $A \subset \mathcal{A}_V$ of power-bounded elements is stable under the action of $\mathfrak{h}_{\mathbb{Z}_p}$. Because $U_z^{(e-1)}$ acts analytically on V we have for any $\mathfrak{x} \in \mathfrak{h}_{\mathbb{Z}_p}$ and $f \in A$

$$\exp(t\mathfrak{x}).f = \sum_{n \geq 0} \left(\frac{\mathfrak{x}^n}{n!}.f \right) t^n,$$

where the right-hand side is a convergent power series in $t \in \mathbb{B}$. If we evaluate this identity at a point $z' \in V$ we get

$$f\left(\exp(-t\mathfrak{x}).z'\right) = \sum_{n} \left(\frac{\mathfrak{x}^n}{n!}.f \right)(z') t^n, \tag{7.4.2}$$

which holds for all $t \in \mathbb{B}$. The left-hand side of (7.4.2) is bounded by 1 in absolute value for all $t \in \mathbb{B}$, and so is the right-hand side. But this means that all coefficients $(\frac{1}{n!}\mathfrak{x}^n.f)(z') \in \mathcal{A}_V$ on the right-hand side of (7.4.2) must be bounded by one in absolute value, and, in particular, the coefficient $(\mathfrak{x}.f)(z')$. This shows that the supremum norm of $\mathfrak{x}.f$ on V is bounded by 1, i.e., that we have $\mathfrak{x}.f \in A$.

We let $U(\mathfrak{h}_{\mathbb{Z}_p})$ be the universal enveloping algebra over $\mathbb{Z}_p$ of $\mathfrak{h}_{\mathbb{Z}_p}$. As we have seen above, the ring A is a $U(\mathfrak{h}_{\mathbb{Z}_p})$-module, and we can consider the skew enveloping algebra $A\#U(\mathfrak{h}_{\mathbb{Z}_p}) := A \otimes_{\mathbb{Z}_p} U(\mathfrak{h}_{\mathbb{Z}_p})$. We denote its p-adic completion by

$$R_A := A\#\hat{U}(\mathfrak{h}_{\mathbb{Z}_p}).$$

In a manner completely analogous to Section 3, this becomes a p-adically complete topological $\mathbb{Z}_p$-algebra. Its mod p-reduction is equal to

$$gr_0(R_A) := \bar{A}\#U(\mathfrak{h}_{\mathbb{F}_p})$$

where $\bar{A} = A/pA$ and $\mathfrak{h}_{\mathbb{F}_p} := \mathfrak{h}_{\mathbb{Z}_p} \otimes \mathbb{F}_p$. The vector space underlying $gr_0(R_A)$ equals $\bar{A} \otimes_{\mathbb{F}_p} U(\mathfrak{h}_{\mathbb{F}_p})$. The second factor in this tensor product has its PBW-filtration. It induces a positive $\mathbb{Z}$-filtration on $gr_0(R_A)$ with $\bar{A}$ concentrated in degree zero. Let deg be the degree function of this filtration. If $f \in \bar{A}$, $\mathfrak{x} \in \mathfrak{h}_{\mathbb{F}_p}$ we have $[f, \mathfrak{x}] = \mathfrak{x}(f)$ from which it follows that $gr_0(R_A)$ is a $\mathbb{Z}$-filtered ring. Moreover,

$$\deg\,[f, \mathfrak{x}] < \deg\,\mathfrak{x}$$

which means that the associated graded ring

$$\mathrm{Gr}(R_A) := gr\ gr_0(R_A) = \bar{A} \otimes_{\mathbb{F}_p} S(\mathfrak{h}_{\mathbb{F}_p})$$

is commutative and therefore a polynomial ring over $\bar{A}$. Since $\bar{A}$ is noetherian, so is $\mathrm{Gr}(R_A)$. By [Schneider and Teitelbaum 2003, Proposition 1.1] the ring $gr_0(R_A)$ is noetherian. Now R_A is complete with respect to the p-adic topology and the graded ring associated with the p-adic filtration equals

$$gr(R_A) = (gr_0 R_A)[Z, Z^{-1}],$$

the Laurent polynomials over $gr_0(R_A)$ in one variable Z (e.g., [Ardakov and Wadsley 2013, Lemma 3.1]). It is noetherian, since $gr_0(R_A)$ is noetherian. Another application of [Schneider and Teitelbaum 2003, Proposition 1.2] now yields that R_A is noetherian. The embedding $\mathfrak{h}_{\mathbb{Z}_p} \subset \mathfrak{g} \subset D(U_z^{(e)})$ induces a ring isomorphism

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{U}(\mathfrak{h}_{\mathbb{Z}_p}) \xrightarrow{\sim} D_{r_0}(U_z^{(e)}, \mathbb{Q}_p);$$

see [Schmidt 2013, Proposition 6.3]. We have $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} A = \mathcal{A}_V$ and thus a ring isomorphism

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} R_A \simeq \mathcal{A}_V \# D_{r_0}(U_z^{(e)}, \mathbb{Q}_p).$$

Therefore, the right-hand side is noetherian. Base change from $\mathbb{Q}_p$ to K finally yields the assertion of the lemma. □

Lemma 7.4.3. *For all $m \geq 0$ the inclusion $U_z^{(e+m)} \subseteq U_z^{(e)}$ induces a finite free ring homomorphism*

$$D_{r_0}(U_z^{(e+m)}) \to D_{r_m}(U_z^{(e)}),$$

which is an isometry between Banach algebras. A basis for this free extension is given by any choice of system of coset representatives for the finite group $U_z^{(e)}/U_z^{(e+m)}$.

Proof. Since $e \geq e_{\mathrm{uni}}$ each group $U_z^{(e)}$ is a uniform pro-p group with lower p-series given by the subgroups $U_z^{(e+m)}$ for $m \geq 0$. The claim follows therefore from the discussion at the end of Section 2.2. □

Keep the assumptions of the preceding proposition and lemma. Put

$$\mathcal{E} := \mathcal{A}_V \# D_{r_0}(U_z^{(e+m)}) \quad \text{and} \quad \mathcal{E}' := \mathcal{A}_V \# D_{r_m}(U_z^{(e)}).$$

Consider the subsheaves $\mathfrak{n}^{\circ,\mathrm{an}}$ and $\ker \lambda^{\circ,\mathrm{an}}$ of the sheaf $\mathcal{O}_{X^{\mathrm{an}}} \# U(\mathfrak{g})$ on X^{an}. Let $\mathcal{K}$ be the vector space of sections[7] over the affinoid $V \subset X^{\mathrm{an}}$ of one of these subsheaves. Put

$$\mathcal{F} := \mathcal{E}/\mathcal{K}\mathcal{E} \quad \text{and} \quad \mathcal{F}' := \mathcal{E}'/\mathcal{K}\mathcal{E}'.$$

The ring homomorphism of the preceding lemma induces a ring homomorphism $\phi : \mathcal{E} \to \mathcal{E}'$ and a linear homomorphism $\mathcal{F} \to \mathcal{F}'$. The latter fits into a homomorphism

$$\mathcal{F} \otimes_{D_{r_0}(U_z^{(e+m)})} D_{r_m}(U_z^{(e)}) \to \mathcal{F}' \tag{7.4.4}$$

of $(\mathcal{F}, D_{r_m}(U_z^{(e)}))$-bimodules.

Lemma 7.4.5. *The rings $\mathcal{E}$ and $\mathcal{E}'$ are noetherian. The homomorphism (7.4.4) is an isomorphism.*

Proof. Since $\mathcal{A}_V$ is a noetherian ring, so is the ring $\mathcal{A}_V \# U(\mathfrak{g})$ [McConnell and Robson 1987, 1.7.14]. Choose generators $x_1, \ldots, x_s$ for the right $\mathcal{A}_V \# U(\mathfrak{g})$-ideal generated by the vector space $\mathcal{K}$. These generators determine a free presentation

$$\bigoplus_{i=1,\ldots,s} \mathcal{E} \to \mathcal{E} \to \mathcal{F} \to 0$$

of the right $\mathcal{E}$-module $\mathcal{F}$. The bijectivity of the natural map

$$\mathcal{E} \otimes_{D_{r_0}(U_z^{(e+m)})} D_{r_m}(U_z^{(e)}) \to \mathcal{E}',\ f \otimes h \mapsto \phi(f) \cdot h$$

of $(\mathcal{E}, D_{r_m}(U_z^{(e)}))$-bimodules can be checked on the level of vector spaces. It follows there from functoriality of $\mathcal{A}_V \hat{\otimes}_L (\cdot)$ applied to the obvious bijective linear map

$$D_{r_0}(U_z^{(e+m)}) \otimes_{D_{r_0}(U_z^{(e+m)})} D_{r_m}(U_z^{(e)}) \xrightarrow{\sim} D_{r_m}(U_z^{(e)}).$$

Using 7.4.3 we conclude that $\mathcal{E}'$ is a finite free left $\mathcal{E}$-module. By 7.4.1, the ring $\mathcal{E}$ is noetherian and so $\mathcal{E}'$ is left noetherian. A similar argument shows that $\mathcal{E}'$ is right noetherian. Finally, $\mathcal{E}$ is a right $D_{r_0}(U_z^{(e+m)})$-module in the obvious way. Applying the functor $(\cdot) \otimes_{D_{r_0}(U_z^{(e+m)})} D_{r_m}(U_z^{(e)})$ to the above presentation yields the isomorphism

$$\mathcal{F} \otimes_{D_{r_0}(U_z^{(e+m)})} D_{r_m}(U_z^{(e)}) \simeq \mathcal{E}'/\mathcal{K}\mathcal{E}' = \mathcal{F}'. \qquad \square$$

[7] Of course, V is not an open subset of the topological space X^{an}. However, all sheaves in fact extend to sheaves with respect to the Grothendieck topology on the analytic space X^{an}. This technical point is of minor importance.

We keep our assumptions: $z \in \mathscr{B}$ is a point and V is a strictly affinoid neighborhood of z on which $U_z^{(e-1)}$ acts analytically. Let $Y_i \subseteq V$ be finitely many affinoid domains such that the completed skew group ring $\mathcal{A}_{Y_i} \# D_{r_0}(U_z^{(e)})$ exist. If $Y = \cap_i Y_i$ and $Y' = \bigcup_i Y_i$, then one may verify that the skew group rings $\mathcal{A}_Y \# D_{r_0}(U_z^{(e)})$ and $\mathcal{A}_{Y'} \# D_{r_0}(U_z^{(e)})$ exist as well. Indeed, since V is separated, the case of Y is straightforward and the case of Y' follows from considering a short exact sequence as in the proof of Corollary 6.4.4(ii). For any affinoid domain $Y \subseteq V$ such that the skew group ring $\mathcal{A}_Y \# D_{r_0}(U_z^{(e)})$ exists, we have inside this ring the right ideal generated by $\mathfrak{n}^{\circ,\mathrm{an}}(Y)$. Such affinoid domains Y together with finite coverings form a G-topology on V and we may consider the sheaf $\mathcal{I}_{\mathfrak{t}}^{\mathrm{an},z}$ associated to the presheaf

$$Y \mapsto \mathfrak{n}^{\circ,\mathrm{an}}(Y).\mathcal{A}_Y \# D_{r_0}(U_z^{(e)})$$

on V. It follows from Lemma 6.2.6 that we have for its stalk at z, that

$$(\mathcal{I}_{\mathfrak{t}}^{\mathrm{an},z})_z = \mathfrak{n}_z^{\circ,\mathrm{an}}.\mathcal{O}_{\mathscr{B},z} \# D_{r_0}(U_z^{(e)}) = \mathscr{I}_{\mathfrak{t},z}^{\mathrm{an}}$$

with the ideal sheaf $\mathscr{I}_{\mathfrak{t}}^{\mathrm{an}} \subseteq \mathcal{O}_{\mathscr{B}} \# \underline{D}_{r_0}$. There is an analogous sheaf $\mathcal{I}_{\chi}^{\mathrm{an},z}$ on V defined by replacing $\mathfrak{n}^{\circ}$ with $\ker \lambda^{\circ}$.

Finally, fix once and for all a neighborhood basis of z consisting of strict affinoids $V' \subset V$ on which $U_z^{(e-1)}$ acts analytically (Lemma 6.2.6). We give $\mathcal{O}_{\mathscr{B},z} \# D_r(U_z^{(e)})$ the inductive limit topology from the isomorphism (6.4.9) via transport of structure.

Corollary 7.4.6. *Let $r = r_m$ for some $m \geq 0$ and keep the previous assumptions and notations.*

(1) *The isomorphism (5.1.6) induces an isometric isomorphism of Banach spaces*

$$(\mathcal{A}_V \# D_{r_0}(U_z^{(e,-)})) \hat{\otimes}_L D_{r_0}(U_z^{(e,t)}) \xrightarrow{\sim} (\mathcal{A}_V \# D_{r_0}(U_z^{(e)}))/\mathcal{I}_{\mathfrak{t}}^{\mathrm{an},z}(V).$$

Here $U_z^{(e,-)}$ and $U_z^{(e,t)}$ are respectively the negative and toral parts in the root space decomposition of the group $U_z^{(e)}$ appearing in Proposition 4.1.7.

(2) *The isomorphism* (1) *induces an isometric isomorphism of Banach spaces*

$$\mathcal{A}_V \# D_{r_0}(U_z^{(e,-)}) \xrightarrow{\sim} (\mathcal{A}_V \# D_{r_0}(U_z^{(e)}))/\mathcal{I}_{\chi}^{\mathrm{an},z}(V).$$

(3) *The ideals $\mathscr{I}_{\mathfrak{t},z}^{\mathrm{an}}$ and $\mathscr{I}_{\chi,z}^{\mathrm{an}}$ are closed in $\mathcal{O}_{\mathscr{B},z} \# D_r(U_z^{(e)})$* (*in the inductive limit topology*).

Proof. Let us first assume $K = \mathbb{Q}_p$. Ad (1): We begin by introducing certain integral structures in the situation of 5.1.5: let $\mathfrak{B} \subset \mathfrak{G}$ be a Borel subgroup scheme over $\mathbb{Z}_p$ with generic fiber $\boldsymbol{B}$ and containing the Néron model $\mathfrak{T} := \mathrm{Spec}(\mathbb{Z}_p[X^*(\boldsymbol{T})])$ of $\boldsymbol{T}$. Denote the unipotent radical of $\mathfrak{B}$ by $\mathfrak{N}$. Let $\mathfrak{X} := \mathfrak{G}/\mathfrak{B}$. The group scheme $\mathfrak{G}$ acts on $\mathfrak{X}$ by left translations and we have a derived action of its Lie algebra $\mathfrak{g}_{\mathbb{Z}_p}$ on $\mathfrak{X}$. Let $\mathfrak{N}^-$ be the unipotent radical of the Borel subgroup scheme opposite to

$\mathfrak{B}$. We denote by $\mathfrak{U}_1 = q(\mathfrak{N}^-)$ the image of $\mathfrak{N}^-$ under the natural projection map $q : \mathfrak{G} \to \mathfrak{X}$. For each $w \in W$, we fix a representative $\dot{w}$ in $\mathfrak{G}(\mathbb{Z}_p)$ with $\dot{1} = 1$ and put $\mathfrak{U}_w := \dot{w}\mathfrak{U}_1$. The $\mathfrak{U}_w$, $w \in W$ form a Zariski covering of $\mathfrak{X}$ and each $\mathfrak{U}_w$ has generic fiber U_w.

If U_w^{an} denotes the rigid analytification of U_w and $U_w^{\mathrm{an},0}$ denotes the Raynaud generic fiber of $\mathfrak{U}_w$, then there is a natural morphism

$$U_w^{\mathrm{an},0} \hookrightarrow U_w^{\mathrm{an}}$$

identifying the source with an affinoid subdomain in the target. Let $\mathcal{B}_w$ be the affinoid algebra of $U_w^{\mathrm{an},0}$. Put

$$V_w := V \cap U_w^{\mathrm{an},0}.$$

The V_w, $w \in W$ form a finite admissible affinoid covering of V. Let $V_{ww'} := V_w \cap V_{w'}$ for $w, w' \in W$ and denote by $\mathcal{A}_w$ and $\mathcal{A}_{ww'}$ the affinoid algebras corresponding to V_w and $V_{ww'}$ respectively. We have a commutative diagram of restriction maps

$$\begin{array}{ccc} \mathcal{B}_w & \longrightarrow & \mathcal{B}_{ww'} \\ \downarrow & & \downarrow \\ \mathcal{A}_w & \longrightarrow & \mathcal{A}_{ww'} \end{array} .$$

After these preliminary remarks we establish the isomorphism in (1) in several steps. Note that G-equivariance reduces us to proving the statement in the case where z is contained in the closure $\overline{\mathscr{C}}$ of the fundamental chamber $\mathscr{C}$. So let $z \in \overline{\mathscr{C}}$ in the following. We follow the notation of the proof of Proposition 7.4.1 and denote by $\mathcal{L}(\,\cdot\,)$ the $\mathbb{Z}_p$-Lie algebra of a uniform pro-p group. In particular,

$$\mathfrak{h}_{\mathbb{Z}_p} := \mathcal{L}(U_z^{(e-1)}), \quad \mathfrak{h}_{\mathbb{Z}_p}^- := \mathcal{L}(U_z^{(e-1,-)}).$$

Let $\hat{U}(\mathfrak{h}_{\mathbb{Z}_p})$ and $\hat{U}(\mathfrak{h}_{\mathbb{Z}_p}^-)$ denote the p-adic completions of the universal enveloping algebras of these Lie algebras. Note that

$$\hat{U}(\mathfrak{h}_{\mathbb{Z}_p}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = D_{r_0}(U_z^{(e)}), \quad \hat{U}(\mathfrak{h}_{\mathbb{Z}_p}^-) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = D_{r_0}(U_z^{(e,-)}).$$

Since $e > e_{\mathrm{uni}}$ we have

$$U_z^{(e-1)} \subseteq U_{wx_0}^{(0)}$$

for all $w \in W$; see Remark 4.3.4. We fix in the following two elements $w, w' \in W$. Note that $U_{wx_0}^{(0)}$ acts analytically on the affinoid domain $U_w^{\mathrm{an},0}$ and therefore so does $U_z^{(e-1)}$. Hence, $U_z^{(e-1)}$ acts analytically on V_w and $V_{ww'}$. From the proof of Proposition 7.4.1 we know that the induced action of the Lie algebra $\mathfrak{h}_{\mathbb{Z}_p}$ stabilizes the subrings $\mathcal{A}_V^\circ$, $\mathcal{A}_w^\circ$ and $\mathcal{A}_{ww'}^\circ$ of power-bounded elements in $\mathcal{A}_V$, $\mathcal{A}_w$ and $\mathcal{A}_{ww'}$ respectively. To simplify notation, we denote in the following by $\mathcal{A}^\circ$ one of the rings $\mathcal{A}_V^\circ$, $\mathcal{A}_w^\circ$ or $\mathcal{A}_{ww'}^\circ$ and let $\mathcal{A} := \mathcal{A}^\circ \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. The affinoid space $\mathscr{M}(\mathcal{A})$ equals therefore one of the affinoid domains V, V_w or $V_{ww'}$ of V. As we have just seen, the space of sections $\mathcal{I}_{\mathfrak{t}}^{\mathrm{an},z}(\mathscr{M}(\mathcal{A}))$ is defined.

The root space decomposition of $U_z^{(e-1)}$ (Proposition 4.1.7) induces a decomposition

$$\mathfrak{h}_{\mathbb{Z}_p} = \mathfrak{h}_{\mathbb{Z}_p}^- \oplus \mathfrak{h}_{\mathbb{Z}_p}^t \oplus \mathfrak{h}_{\mathbb{Z}_p}^+$$

which upon tensoring with $\mathbb{Q}_p$ gives the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{t} \oplus \mathfrak{n}^+$ of the reductive Lie algebra $\mathfrak{g}$. Let

$$\iota : U(\mathfrak{h}_{\mathbb{Z}_p}^-) \otimes_{\mathbb{Z}_p} U(\mathfrak{h}_{\mathbb{Z}_p}^t) \hookrightarrow U(\mathfrak{h}_{\mathbb{Z}_p})$$

be the linear PBW-map induced from this decomposition and form the linear map

$$f : (\mathcal{A}^\circ \# U(\mathfrak{h}_{\mathbb{Z}_p}^-)) \otimes_{\mathbb{Z}_p} U(\mathfrak{h}_{\mathbb{Z}_p}^t) \to (\mathcal{A}^\circ \# U(\mathfrak{h}_{\mathbb{Z}_p})),\ (f \otimes \mathfrak{x}) \otimes \mathfrak{y} \mapsto f \otimes \iota(\mathfrak{x} \otimes \mathfrak{y}).$$

Let $\hat{f}$ be the p-adic completion of the map f. Composing $\hat{f} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ with the natural projection map

$$\mathcal{A}\# D_{r_0}(U_z^{(e)}) \to (\mathcal{A}\# D_{r_0}(U_z^{(e)}))/\mathcal{I}_{\mathfrak{t}}^{\mathrm{an},z}(\mathscr{M}(\mathcal{A}))$$

yields a linear map

$$\psi_{\mathcal{A}} : (\mathcal{A}\# D_{r_0}(U_z^{(e,-)}))\hat{\otimes}_{\mathbb{Q}_p} D_{r_0}(U_z^{(e,t)}) \to (\mathcal{A}\# D_{r_0}(U_z^{(e)}))/\mathcal{I}_{\mathfrak{t}}^{\mathrm{an},z}(\mathscr{M}(\mathcal{A})).$$

According to Proposition 7.4.1, the Banach algebra $\mathcal{A}\# D_{r_0}(U_z^{(e)})$ is noetherian, and hence, its right ideal $\mathcal{I}_{\mathfrak{t}}^{\mathrm{an},z}(\mathscr{M}(\mathcal{A}))$ is closed. If we endow the target of $\psi_{\mathcal{A}}$ with the quotient norm, then $\psi_{\mathcal{A}}$ becomes a norm-decreasing linear map between Banach spaces. In the case where $\mathcal{A} = \mathcal{A}_V$, i.e., $\mathscr{M}(\mathcal{A}) = V$, we denote this map by ψ_V. We claim that ψ_V is our searched for isomorphism appearing in (1). To prove this we will show, as a first step, that the map $\psi_{\mathcal{A}}$ is an isometric isomorphism of Banach spaces for $\mathcal{A}$ equal to one of the rings $\mathcal{A}_w$ or $\mathcal{A}_{ww'}$. We will do this with the help of auxiliary isomorphisms coming from [Ardakov and Wadsley 2013] and involving the "congruence group" $U_{wx_0}^{(0)}$. So suppose that $\mathcal{A}$ is either $\mathcal{A}_w$ or $\mathcal{A}_{ww'}$, so that $\mathscr{M}(\mathcal{A}) \subseteq U_w^{\mathrm{an},0}$. We may apply essentially the same construction above to the algebra $\mathcal{A}$ and the group $U_{wx_0}^{(0)}$ and obtain from [loc. cit.] an isometric isomorphism of Banach spaces

$$\psi_{\mathcal{A}}^0 : (\mathcal{A}\# D_{r_0}(U_{wx_0}^{(0,-)}))\hat{\otimes}_L D_{r_0}(U_{wx_0}^{(0,t)}) \to (\mathcal{A}\# D_{r_0}(U_{wx_0}^{(0)}))/J_{\mathcal{A}},$$

$J_{\mathcal{A}}$ being the right ideal induced by $\mathfrak{n}^{\circ,\mathrm{an}}(\mathscr{M}(\mathcal{A}))$. Let us explain this isomorphism in more detail. Since $U_{x_0}^{(0)} = \mathfrak{G}(p)$, we have $\mathcal{L}(U_{wx_0}^{(0)}) = p\mathfrak{g}_{\mathbb{Z}_p}^w$ as $\mathbb{Z}_p$-Lie algebras where $\mathfrak{g}_{\mathbb{Z}_p}^w := \mathrm{Ad}(\dot{w})(\mathfrak{g}_{\mathbb{Z}_p})$. In particular, $D_{r_0}(U_{wx_0}^{(0)})$ equals the p-adic completion (with p inverted) of the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{Z}_p}^w)_1 := U(p\mathfrak{g}_{\mathbb{Z}_p}^w)$. Our sheaf $\mathcal{D}_{\mathfrak{t}}$ on X, as introduced in 5.1.5, equals the pull-back along the inclusion map $X \subset \mathfrak{X}$ of the relative enveloping algebra

$$\tilde{\mathcal{D}} := \xi_*(\mathcal{D}_{\tilde{\mathfrak{X}}})^{\mathfrak{H}}$$

of the locally trivial $\mathfrak{H}$-torsor

$$\xi : \tilde{\mathfrak{X}} \to \mathfrak{X}$$

appearing in [loc. cit., 4.7]. Here, $\tilde{\mathfrak{X}}$ denotes the homogeneous space $\mathfrak{G}/\mathfrak{N}$ with ring of crystalline level zero differential operators $\mathcal{D}_{\tilde{\mathfrak{X}}}$, the symbol $\mathfrak{H}$ denotes the abstract Cartan group $\mathfrak{B}/\mathfrak{N}$ and ξ equals the map $g\mathfrak{N} \mapsto g\mathfrak{B}$. In this situation, the map $\psi^0_{\mathcal{A}}$ and its properties follow from [loc. cit., Lemma 6.4(a)] and its proof like this: in the notation of [loc. cit.] choose $n = 1$ and put $U := \mathfrak{U}_w$ or $\mathfrak{U}_w \cap \mathfrak{U}_{w'}$ depending on $\mathcal{A} = \mathcal{A}_w$ or $\mathcal{A}_{ww'}$ respectively. Since U trivializes the torsor ξ, we have the isomorphism

$$(\mathcal{D}_1)|_U \otimes_{\mathbb{Z}_p} U(\mathfrak{t}_{\mathbb{Z}_p})_1 \simeq (\tilde{\mathcal{D}}_1)|_U$$

of sheaves of $\mathbb{Z}_p$-algebras. Here, $\mathcal{D}$ denotes the sheaf of "crystalline level zero" differential operators on $\mathfrak{X}$, the subscript $(\,\cdot\,)_1$ refers to the first deformation functor of [loc. cit.] and we have identified the Lie algebra of the $\mathbb{Z}_p$-group scheme $\mathfrak{H}$ with $\mathfrak{t}_{\mathbb{Z}_p} := \mathrm{Lie}(\mathfrak{T})$ via the morphism $\mathfrak{T} \subset \mathfrak{B} \to \mathfrak{H}$. Let $\mathfrak{n}_{\mathbb{Z}_p} := \mathrm{Lie}(\mathfrak{N}^-)$. Then

$$U(\mathfrak{t}_{\mathbb{Z}_p})_1 = U(p\mathfrak{t}_{\mathbb{Z}_p}) = U(\mathcal{L}(U^{(0,t)}_{x_0}))$$

and

$$U(\mathfrak{n}^w_{\mathbb{Z}_p})_1 = U(p\mathfrak{n}^w_{\mathbb{Z}_p}) = U(\mathcal{L}(U^{(0,-)}_{wx_0}))$$

as $\mathbb{Z}_p$-subalgebras inside the $\mathbb{Q}_p$-algebra $U(\mathfrak{g})$. The above isomorphism of sheaves extends to an isomorphism

$$(\widehat{(\mathcal{D}_1)})|_U \hat{\otimes}_{\mathbb{Z}_p} \widehat{U(\mathfrak{t}_{\mathbb{Z}_p})}_1 \simeq (\widehat{(\tilde{\mathcal{D}}_1)})|_U$$

involving the p-adic completions of the former sheaves. We may view these sheaves as sheaves on the formal scheme $Spf\,\mathcal{A}^\circ$. Taking global sections and subsequent inversion of p yields an isometric isomorphism of Banach spaces

$$(\mathcal{A}\# D_{r_0}(U^{(0,-)}_{wx_0}))\hat{\otimes}_{\mathbb{Q}_p} D_{r_0}(U^{(0,t)}_{wx_0}) \to (\mathcal{A}\# D_{r_0}(U^{(0)}_{wx_0}))/J_{\mathcal{A}}$$

with, as already indicated, $J_{\mathcal{A}}$ equal to the right ideal induced by $\mathfrak{n}^{\circ,\mathrm{an}}(\mathscr{M}(\mathcal{A}))$. This is our promised isomorphism $\psi^0_{\mathcal{A}}$. Note that there is a canonical map

$$\mathcal{I}^{\mathrm{an},z}_{\mathfrak{t}}(\mathscr{M}(\mathcal{A})) \to J_{\mathcal{A}}$$

induced from the inclusion $D_{r_0}(U^{(e)}_z) \to D_{r_0}(U^{(0)}_{wx_0})$.

The maps $\psi_{\mathcal{A}}$ and $\psi^0_{\mathcal{A}}$ fit into the diagram

$$\begin{array}{ccc}
(\mathcal{A}\#U(\mathfrak{n}^-))\otimes_{\mathbb{Q}_p}U(\mathfrak{t}) & \xrightarrow{\simeq} & (\mathcal{A}\#U(\mathfrak{g}))/\mathcal{I}^{\mathrm{an}}_{\mathfrak{t}}(\mathscr{M}(\mathcal{A})) \\
\downarrow & & \downarrow \\
(\mathcal{A}\#D_{r_0}(U_z^{(e,-)}))\hat{\otimes}_{\mathbb{Q}_p}D_{r_0}(U_z^{(e,t)}) & \xrightarrow[\psi_{\mathcal{A}}]{} & (\mathcal{A}\#D_{r_0}(U_z^{(e)}))/\mathcal{I}^{\mathrm{an},z}_{\mathfrak{t}}(\mathscr{M}(\mathcal{A})) \\
\downarrow & & \downarrow \\
(\mathcal{A}\#D_{r_0}(U_{wx_0}^{(0,-)}))\hat{\otimes}_{\mathbb{Q}_p}D_{r_0}(U_{wx_0}^{(0,t)}) & \xrightarrow[\psi^0_{\mathcal{A}}]{\simeq} & (\mathcal{A}\#D_{r_0}(U_{wx_0}^{(0)}))/J_{\mathcal{A}}
\end{array} \qquad (\dagger)$$

where the top horizontal arrow is the pull-back along the canonical morphism

$$\mathscr{M}(\mathcal{A})\subset U_w^{\mathrm{an}}\to U_w$$

of the isomorphism (5.1.6) and therefore an algebra isomorphism itself. The bottom vertical arrows are induced by the inclusion $U_z^{(e)}\subseteq U_{wx_0}^{(0)}$.

The group $U_z^{(e,-)}$ is an open subgroup of the p-adic group $\boldsymbol{N}^-(\mathbb{Q}_p)$. Applying $D_{r_0}(\cdot)$ yields therefore a completion of $U(\mathfrak{n}^-)$. Similarly, applying $D_{r_0}(\cdot)$ to the groups $U_z^{(e,t)}$ and $U_z^{(e)}$ yields a completion of $U(\mathfrak{t})$ and $U(\mathfrak{g})$ respectively. These completions define the top vertical arrows. Unwinding the definitions of all the maps involved shows that the diagram commutes. The bottom left vertical arrow obviously is injective. Since $\psi^0_{\mathcal{A}}$ is bijective, the commutativity of the lower square implies $\psi_{\mathcal{A}}$ to be injective. For its surjectivity, consider the inverse map of the upper horizontal isomorphism. It is induced from the PBW-projection $\mathfrak{g}\to\mathfrak{n}^-\oplus\mathfrak{t}$. The corresponding projection map arising from the triangular decomposition of $\mathfrak{h}_{\mathbb{Z}_p}$ completes to a linear map

$$(\mathcal{A}\#D_{r_0}(U_z^{(e)}))\to(\mathcal{A}\#D_{r_0}(U_z^{(e,-)}))\hat{\otimes}_{\mathbb{Q}_p}D_{r_0}(U_z^{(e,t)}),$$

which factors through the target of $\psi_{\mathcal{A}}$ and gives a section to $\psi_{\mathcal{A}}$. All in all, we have shown that $\psi_{\mathcal{A}}$ is an isometric isomorphism between Banach spaces. This completes our first step.

In a second step, we show that the map ψ_V is an isometric isomorphism of Banach spaces. We put $\psi_{V_w}:=\psi_{\mathcal{A}}$ in case $\mathcal{A}=\mathcal{A}_w$ and $\psi_{V_{ww'}}:=\psi_{\mathcal{A}}$ in case $\mathcal{A}=\mathcal{A}_{ww'}$. Abbreviate

$$D:=D_{r_0}(U_z^{(e)}),\quad D^t:=D_{r_0}(U_z^{(e,t)}),\quad D^-:=D_{r_0}(U_z^{(e,-)}).$$

The covering $V=\bigcup_{w\in W}V_w$ gives rise to the exact restriction sequence

$$0\to\mathcal{A}_V\to\bigoplus_w\mathcal{A}_w\to\bigoplus_{w<w'}\mathcal{A}_{ww'}$$

where we have ordered the elements of W in some arbitrary way. It induces two complexes, namely

$$0 \to (\mathcal{A}_V \# D^-)\hat{\otimes} D^t \to \bigoplus_w (\mathcal{A}_w \# D^-)\hat{\otimes} D^t \to \bigoplus_{w<w'} (\mathcal{A}_{ww'} \# D^-)\hat{\otimes} D^t \qquad \text{(I)}$$

and

$$0 \to (\mathcal{A}_V \# D)/\mathcal{I}_{\mathfrak{t}}^{\mathrm{an},z}(V) \to \bigoplus_w (\mathcal{A}_w \# D)/\mathcal{I}_{\mathfrak{t}}^{\mathrm{an},z}(V_w) \to \bigoplus_{w<w'} (\mathcal{A}_{ww'} \# D)/\mathcal{I}_{\mathfrak{t}}^{\mathrm{an},z}(V_{ww'}). \qquad \text{(II)}$$

The maps ψ_V, ψ_{V_w} and $\psi_{V_{ww'}}$ induce a morphism between (I) and (II). We claim that (I) is exact. Indeed, exactness may be shown on the level of vector spaces. So let d be the rank of the finitely generated free $\mathbb{Z}_p$-module $\mathfrak{h}^-_{\mathbb{Z}_p} \oplus \mathfrak{h}^t_{\mathbb{Z}_p}$. By construction, the Banach space $D^- \hat{\otimes} D^t$ is isomorphic to the Banach space underlying the Tate algebra of the d-dimensional closed unit disc B. Our assertion follows now from the sheaf property of $\mathcal{O}_{V\times_K B}$ applied to the admissible covering $V \times_K B = \bigcup_w V_w \times_K B$. Since the maps ψ_{V_w} are all isomorphisms by our first step, it follows that ψ_V is injective. To establish its surjectivity, observe that the second arrow in (II),

$$(\mathcal{A}_V \# D)/\mathcal{I}_{\mathfrak{t}}^{\mathrm{an},z}(V) \to \bigoplus_w (\mathcal{A}_w \# D)/\mathcal{I}_{\mathfrak{t}}^{\mathrm{an},z}(V_w),$$

is injective. This follows from an easy diagram chase in

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathcal{I}_{\mathfrak{t}}^{\mathrm{an},z}(V) & \longrightarrow & \bigoplus_w \mathcal{I}_{\mathfrak{t}}^{\mathrm{an},z}(V_w) & \longrightarrow & \bigoplus_{w<w'} \mathcal{I}_{\mathfrak{t}}^{\mathrm{an},z}(V_{ww'}) \;, \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & (\mathcal{A}_V \# D) & \longrightarrow & \bigoplus_w (\mathcal{A}_{V_w} \# D) & \longrightarrow & \bigoplus_{w<w'} (\mathcal{A}_{V_{ww'}} \# D) \end{array}$$

where we have exact rows and injective vertical maps. Since all the maps ψ_{V_w} and $\psi_{V_{ww'}}$ are isomorphisms by our first step and since (II) is a complex, this implies the surjectivity of ψ_V. Then ψ_V must be an isometric isomorphism of Banach spaces. This completes the proof of (1). Treating the ideal $\mathcal{I}_{\chi}^{\mathrm{an},z}$ in the same way gives (2).

Ad (3): By G-equivariance we may assume that z is contained in the closure of the fundamental chamber $\mathscr{C}$. Recall our fixed choice of neighborhood basis consisting of strict affinoids $V \subset X^{\mathrm{an}}$ on which $U_z^{(e-1)}$ acts analytically. Since the isomorphism in (1) is compatible with the restriction maps $\mathcal{A}_V \to \mathcal{A}_{V'}$ associated to an inclusion $V' \subset V$ we see that Proposition 2.4.1 applies to the locally convex inductive limit

$$\varinjlim_V \mathcal{A}_V \# D_{r_0}(U_z^{(e)})/\mathcal{I}_{\mathfrak{t}}^{\mathrm{an},z}(V).$$

The limit is therefore Hausdorff. Moreover, the isomorphism (7.4.4) appearing in Lemma 7.4.5 is also compatible with the map $\mathcal{A}_V \to \mathcal{A}_{V'}$. Since a finite direct sum of Hausdorff spaces is again Hausdorff, Lemma 7.4.3 implies that the locally convex inductive limit

$$\varinjlim_V \mathcal{A}_V \# D_{r_m}(U_z^{(e)})/\mathcal{I}_{\mathfrak{t},m}^{\mathrm{an},z}(V)$$

is Hausdorff. Here, the sheaf $\mathcal{I}_{\mathfrak{t},m}^{\mathrm{an},z}$ is defined by replacing in the definition of $\mathcal{I}_{\mathfrak{t}}^{\mathrm{an},z}$ the ring $D_{r_0}(U_z^{(e)})$ by its subring $D_{r_m}(U_z^{(e)})$. We may now finish the proof of (3). Proposition 2.4.1 gives a (topological) linear isomorphism

$$\mathcal{O}_{\mathscr{B},z} \# D_{r_m}(U_z^{(e)}) \simeq \varinjlim_V \mathcal{A}_V \# D_{r_m}(U_z^{(e)}).$$

By Lemma 6.2.6 we have

$$\mathscr{I}_{\mathfrak{t},z}^{\mathrm{an}} = \varinjlim_V \mathcal{I}_{\mathfrak{t},m}^{\mathrm{an},z}(V)$$

for the ideal sheaf $\mathscr{I}_{\mathfrak{t}}^{\mathrm{an}} \subseteq \mathcal{O}_{\mathscr{B}} \# \underline{D}_{r_m}$. Consider the diagram of continuous K-linear maps

$$0 \longrightarrow \varinjlim_V \mathcal{I}_{\mathfrak{t},m}^{\mathrm{an},z}(V) \xrightarrow{\iota} \varinjlim_V \mathcal{A}_V \# D_{r_m}(U_z^{(e)}) \longrightarrow \varinjlim_V \mathcal{A}_V \# D_{r_m}(U_z^{(e)})/\mathcal{I}_{\mathfrak{t},m}^{\mathrm{an},z}(V) \longrightarrow 0,$$

which is short exact as a diagram of abstract K-vector spaces. The right-hand term is Hausdorff, as we have just seen. The injection ι has therefore closed image which is what we want. The case of the ideal $\mathscr{I}_{\chi,z}^{\mathrm{an}}$ follows similarly by using the sheaf $\mathcal{I}_{\chi}^{\mathrm{an},z}$ and the isomorphism (2). This finishes the proof of the corollary in the case $K = \mathbb{Q}_p$. A base change along the finite field extension $\mathbb{Q}_p \subseteq K$ yields the general case. □

We emphasize that the top vertical arrows in the commutative diagram (†) appearing in the preceding proof are injective and have dense image. Moreover, the top horizontal arrow is multiplicative. In particular, if the target of $\psi_{\mathcal{A}}$ were a ring, i.e., if the right ideal $\mathcal{I}_{\mathfrak{t}}^{\mathrm{an},z}(\mathscr{M}(\mathcal{A}))$ were two-sided, $\psi_{\mathcal{A}}$ would be a ring homomorphism.

Lemma 7.4.7. *Let $r = r_m$ for some $m \geq 0$ and keep the previous assumptions and notations. The right ideals $\mathscr{I}_{\mathfrak{t}}^{\mathrm{an}}$ and $\mathscr{I}_{\chi}^{\mathrm{an}}$ are two-sided ideals. Let $z \in \mathscr{B}$. The isomorphism* (1) *of the preceding corollary induces an isomorphism of K-algebras*

$$(\mathcal{O}_{\mathscr{B},z} \# D_{r_0}(U_z^{(e,-)})) \hat{\otimes}_L D_{r_0}(U_z^{(e,t)}) \xrightarrow{\simeq} (\mathcal{O}_{\mathscr{B},z} \# D_{r_0}(U_z^{(e)}))/\mathscr{I}_{\mathfrak{t},z}^{\mathrm{an}}.$$

Similarly, the isomorphism (2) *of the preceding corollary induces an isomorphism of K-algebras*

$$\mathcal{O}_{\mathscr{B},z} \# D_{r_0}(U_z^{(e,-)}) \xrightarrow{\simeq} (\mathcal{O}_{\mathscr{B},z} \# D_{r_0}(U_z^{(e)}))/\mathscr{I}_{\chi,z}^{\mathrm{an}}.$$

Proof. According to Section 8.3 the sheaves $\mathcal{O}_{\mathscr{B}} \# U(\mathfrak{g}_K)$ and $\mathcal{O}_{\mathscr{B}} \# \underline{D}_r$ have a natural G-equivariant structure such that the morphism $\mathcal{O}_{\mathscr{B}} \# U(\mathfrak{g}_K) \to \mathcal{O}_{\mathscr{B}} \# \underline{D}_r$ is equivariant. Moreover, the ideals $\mathcal{I}_{\mathscr{B},\mathfrak{t}}^{\mathrm{an}}$ and $\mathcal{I}_{\mathscr{B},\chi}^{\mathrm{an}}$ are G-stable. Hence, so are the right ideals $\mathscr{I}_{\mathrm{t}}^{\mathrm{an}}$ and $\mathscr{I}_{\chi}^{\mathrm{an}}$. That these ideals are two-sided can be checked stalkwise [Godement 1958, II.1.8]. We give the argument in the case $\mathscr{I}_{\mathrm{t}}^{\mathrm{an}}$. The other case follows in the same way. Recall that we have fixed a neighborhood basis of z consisting of strict affinoids V on which $U_z^{(e-1)}$ acts analytically. The corresponding inductive limit topology makes $\mathcal{O}_{\mathscr{B},z}\# D_r(U_z^{(e)})$ a separately continuous K-algebra and, hence, the multiplication map $D_r(U_z^{(e)}) \to \mathcal{O}_{\mathscr{B},z}\# D_r(U_z^{(e)}), \lambda \mapsto \lambda\cdot\partial$ is continuous for every $\partial \in \mathcal{O}_{\mathscr{B},z}\# D_r(U_z^{(e)})$. Fix $\partial \in \mathscr{I}_{\mathrm{t},z}^{\mathrm{an}}$. By [Schneider and Teitelbaum 2002, Lemma 3.1] and part (3) of the preceding corollary, we see that it suffices to prove that $\delta_g\cdot\partial \in \mathcal{O}_{\mathscr{B},z}\# D_r(U_z^{(e)})$ lies in the subspace $\mathscr{I}_{\mathrm{t},z}^{\mathrm{an}}$ for $g \in U_z^{(e)}$. Considering ∂ as an element of $\mathcal{O}_{\mathscr{B},z}\# D_r(U_z^{(e)})$ we may choose, by Proposition 2.4.1, an affinoid neighborhood V in our fixed neighborhood basis of z, such that $\partial \in \mathcal{A}_V\# D_r(U_z^{(e)})$. Using power series expansions for elements of completed distribution algebras (Section 2.2.3) we may write ∂ as an infinite sum $\partial = \sum_{\alpha\in\mathbb{N}_0^d} f_\alpha \hat{\otimes} \boldsymbol{b}^\alpha$, with $f_\alpha \in \mathcal{A}_V$ converging in the Banach algebra $\mathcal{A}_V\# D_r(U_z^{(e)})$. By definition of the skew multiplication (3.2.2) we have

$$\begin{aligned}\delta_g\cdot\partial &= \sum_{\alpha\in\mathbb{N}_0^d}(g.f_\alpha)\,\hat{\otimes}\,\delta_g\boldsymbol{b}^\alpha = \sum_{\alpha\in\mathbb{N}_0^d}(g.f_\alpha)\,\hat{\otimes}\,(\delta_g\boldsymbol{b}^\alpha\delta_g^{-1})\delta_g \\ &= \sum_{\alpha\in\mathbb{N}_0^d} g^*(f_\alpha\,\hat{\otimes}\,\boldsymbol{b}^\alpha)\delta_g = g^*(\partial)\delta_g,\end{aligned}$$

which is an element of $\mathscr{I}_{\mathrm{t}}^{\mathrm{an}}(V)$. Here, $g^* : \mathscr{I}_{\mathrm{t}}^{\mathrm{an}}(V) \xrightarrow{\sim} \mathscr{I}_{\mathrm{t}}^{\mathrm{an}}(V)$ is induced by the equivariant structure on the sheaf $\mathscr{I}_{\mathrm{t}}^{\mathrm{an}}$ (note that $U_z^{(e)}$ acts analytically on V). Passing to the stalk we obtain $\delta_g\cdot\partial \in \mathscr{I}_{\mathrm{t},z}^{\mathrm{an}}$. Thus, the right ideals $\mathscr{I}_{\mathrm{t}}^{\mathrm{an}}$ and $\mathscr{I}_{\chi}^{\mathrm{an}}$ are indeed two-sided ideals. Let $z \in \mathscr{B}$. Passing the isomorphisms (1) and (2) of the preceding corollary to the inductive limit over a neighborhood basis of z consisting of affinoids V on which $U_z^{(e-1)}$ acts analytically gives linear isomorphisms between K-algebras which are actually multiplicative; see the remark directly after the proof of the corollary. Hence, the lemma is proved. □

In the following we tacitly restrict to numbers r of the form $r = r_m$ for some $m \geq 0$. By the preceding lemma we may form the quotient sheaves

$$\mathscr{D}_{r,\mathrm{t}} := (\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)/\mathscr{I}_{\mathrm{t}}^{\mathrm{an}}, \quad \mathscr{D}_{r,\chi} := (\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)/\mathscr{I}_{\chi}^{\mathrm{an}}.$$

These are sheaves of (noncommutative) K-algebras on $\mathscr{B}$ and, at the same time, $\mathcal{O}_{\mathscr{B}}$-modules. We have a commutative diagram of morphisms

$$\begin{array}{ccc}\mathcal{D}^{\mathrm{an}}_{\mathscr{B},\mathrm{t}} & \longrightarrow & \mathcal{D}^{\mathrm{an}}_{\mathscr{B},\chi} \\ \downarrow & & \downarrow \\ \mathscr{D}_{r,\mathrm{t}} & \longrightarrow & \mathscr{D}_{r,\chi}\end{array} \qquad (7.4.8)$$

with surjective horizontal arrows. Moreover, it follows from (6.1.4) and the preceding lemma that the lower horizontal arrow induces an isomorphism

$$\mathscr{D}_{r,\mathfrak{t}}/(\ker \lambda)\mathscr{D}_{r,\mathfrak{t}} \xrightarrow{\sim} \mathscr{D}_{r,\chi}. \tag{7.4.9}$$

We have the following extension of the property 2 in [Beilinson and Bernstein 1981, Section 2, Lemme].

Proposition 7.4.10. *The morphism $\underline{D}_r \to \mathcal{O}_{\mathscr{B}}\#\underline{D}_r \to \mathscr{D}_{r,\chi}$ factors through $\underline{D}_r \to \underline{D}_{r,\theta}$.*

Proof. Letting $\mathscr{K}$ be the kernel of the morphism $\underline{D}_r \to \underline{D}_{r,\theta}$ the claim amounts to

$$\mathscr{K} \subseteq \ker(\underline{D}_r \to \mathscr{D}_{r,\chi}).$$

This can be checked stalkwise; i.e., we are reduced to showing that, for each $z \in \mathscr{B}$ the natural map $D_r(U_z^{(e)}) \to \mathcal{O}_{\mathscr{B},z}\#D_r(U_z^{(e)})/(\mathscr{I}_\chi)_z$ factors through $D_r(U_z^{(e)})_\theta$. The kernel of the map $D_r(U_z^{(e)}) \to D_r(U_z^{(e)})_\theta$ is generated by

$$I_\theta := \ker(U(\mathfrak{g}_K) \to U(\mathfrak{g}_K)_\theta)$$

and the ideal $(\mathscr{I}_\chi)_z$ is generated by the image of $\mathcal{I}^{\mathrm{an}}_{\mathscr{B},\chi,z}$. It therefore suffices to show that the natural map $U(\mathfrak{g}_K) \to \mathcal{O}_{\mathscr{B},z}\#U(\mathfrak{g}_K)$ maps I_θ into $\mathcal{I}^{\mathrm{an}}_{\mathscr{B},\chi,z}$. This follows from [loc. cit.]. $\square$

7.4.11. Let us finally make the structure of the stalks of the sheaves $\mathscr{D}_{r,\mathfrak{t}}$ and $\mathscr{D}_{r,\chi}$ at a point $z \in \mathscr{B}$ more explicit. According to Lemma 6.2.2 the local ring $\mathcal{O}_{\mathscr{B},z}$ is a field. For simplicity we put $\kappa(z) := \mathcal{O}_{\mathscr{B},z}$ and view this as a topological field of compact type. Note that the Berkovich point $z \in \mathscr{B} \subset X^{\mathrm{an}}$ canonically induces a norm topology on $\kappa(z)$ which is weaker than our topology. We shall not make use of this norm topology in the following.

By [loc. cit.] together with Section 5.1.1 we furthermore have $(\mathfrak{n}^\circ)_{\pi(z)} = \mathfrak{n}_{\pi(z)}$ and $(\mathfrak{b}^\circ)_{\pi(z)} = \mathfrak{b}_{\pi(z)}$ for the stalks of the sheaves $\mathfrak{n}^\circ$ and $\mathfrak{b}^\circ$ at $\pi(z) = \eta$ (the generic point of X). Since passage to the stalk is exact, this proves the following lemma. It gives back the isomorphisms of Lemma 7.4.7 in case $r = r_0$.

Lemma 7.4.12. *Assume $r = r_m$ for some $m \geq 0$. Let $z \in \mathscr{B}$. There is a canonical isomorphism*

$$\mathscr{D}_{r,\mathfrak{t},z} \xrightarrow{\sim} (\kappa(z)\hat{\otimes}_L D_r(U_z^{(e)}))/\mathfrak{n}_{\pi(z)}(\kappa(z)\hat{\otimes}_L D_r(U_z^{(e)})).$$

This isomorphism induces a canonical isomorphism between $\mathscr{D}_{r,\chi,z}$ and the λ-coinvariants of the $\mathfrak{t}_K$-module $(\kappa(z)\hat{\otimes}_L D_r(U_z^{(e)}))/\mathfrak{n}_{\pi(z)}(\kappa(z)\hat{\otimes}_L D_r(U_z^{(e)}))$. In particular,

$$\mathscr{D}_{r,\rho,z} \xrightarrow{\sim} (\kappa(z)\hat{\otimes}_L D_r(U_z^{(e)}))/\mathfrak{b}_{\pi(z)}(\kappa(z)\hat{\otimes}_L D_r(U_z^{(e)})).$$

8. From representations to sheaves

In this section, as well as in Sections 10 and 11, we assume that

$$L = \mathbb{Q}_p \quad \text{and} \quad e > e_{\mathrm{st}} \quad \text{and} \quad r = r_m = \sqrt[p^m]{1/p} \quad \text{for some } m \geq 0. \tag{8.0.1}$$

Our proposed "localization functor" from representations to sheaves associated to the pair $\sigma(\chi) = \theta$ will be a functor

$$\mathscr{L}_{r,\chi} : M \mapsto \mathscr{D}_{r,\chi} \otimes_{\underline{D}_{r,\theta}} \underline{M}_r$$

from (coadmissible) left $D(G)_\theta$-modules M to left $\mathscr{D}_{r,\chi}$-modules satisfying additional properties. Here $\underline{M}_r$ is a constructible sheaf replacing the constant sheaf $\underline{M}$ appearing in the Beilinson–Bernstein construction; see Theorem 5.2.2. It is a modest generalization of the sheaf $\underline{D}_r$ as follows.

8.1. ***A constructible sheaf of modules.*** Suppose we are given any (left) $D(G)$-module M. Let $F \subseteq \mathscr{B}$ be a facet. We may regard M as a $D(U_F^{(e)})$-module via the natural map $D(U_F^{(e)}) \to D(G)$. We put

$$M_r(U_F^{(e)}) := D_r(U_F^{(e)}) \otimes_{D(U_F^{(e)})} M,$$

a (left) $D_r(U_F^{(e)})$-module. If $F' \subseteq \mathscr{B}$ is another facet such that $F' \subset \overline{F}$ the map

$$\sigma_r^{F'F} \otimes \mathrm{id} : M_r(U_{F'}^{(e)}) \to M_r(U_F^{(e)}), \quad \delta \otimes m \mapsto \sigma_r^{F'F}(\delta) \otimes m$$

is a module homomorphism relative to $\sigma_r^{F'F}$ and inherits the homomorphic properties from $\sigma_r^{F'F}$ (Lemma 4.3.5). Again, we may define a sheaf of K-vector spaces $\underline{M}_r$ on $\mathscr{B}$ in a completely analogous way as the sheaf $\underline{D}_r$ by replacing each $D_r(U_F^{(e)})$ and each $D_r(U_z^{(e)})$ by $M_r(U_F^{(e)})$ and $M_r(U_z^{(e)})$ respectively. In particular, $\underline{M}_r$ restricted to a facet F is the constant sheaf with value $M_r(U_F^{(e)})$ and therefore $\underline{M}_r$ is a constructible sheaf. If $s \in D_r(U_z^{(e)})$, $m \in M_r(U_z^{(e)})$ the "pointwise multiplication" $(s \cdot m)(z) := s(z)m(z)$ makes $\underline{M}_r$ a $\underline{D}_r$-module.

Lemma 8.1.1. *If M is a $D(G)_\theta$-module then $\underline{M}_r$ is a $\underline{D}_{r,\theta}$-module via the morphism $\underline{D}_r \to \underline{D}_{r,\theta}$.*

Proof. This is easy to see. □

8.2. ***A localization functor.***

8.2.1. As usual, $\mathscr{D}_{r,\chi} \otimes_{\underline{D}_{r,\theta}} \underline{M}_r$ denotes the sheaf associated to the presheaf $V \mapsto \mathscr{D}_{r,\chi}(V) \otimes_{\underline{D}_{r,\theta}(V)} \underline{M}_r(V)$ on $\mathscr{B}$. The construction $M \mapsto \underline{M}_r$ is functorial in M and commutes with arbitrary direct sums. Thus the correspondence

$$\mathscr{L}_{r,\chi} : M \mapsto \mathscr{D}_{r,\chi} \otimes_{\underline{D}_{r,\theta}} \underline{M}_r$$

is a covariant functor from (left) $D(G)_\theta$-modules to (left) $\mathscr{D}_{r,\chi}$-modules. It commutes with arbitrary direct sums. We call it tentatively a *localization functor* associated to χ.

We emphasize that the functor $\mathscr{L}_{r,\chi}$ depends on the choice of the level e. As we did before we suppress this dependence in the notation. As a second remark, let $\mathcal{M}$ be an arbitrary $\mathscr{D}_{r,\chi}$-module and $f:\mathscr{L}_{r,\chi}(M)\to\mathcal{M}$ a morphism. The composite

$$M\to\Gamma(\mathscr{B},\underline{M}_r)\to\Gamma(\mathscr{B},\mathscr{L}_{r,\chi}(M))\xrightarrow{f}\Gamma(\mathscr{B},\mathcal{M})$$

is a K-linear map. We therefore have a natural transformation of functors

$$\mathrm{Hom}_{\mathscr{D}_{r,\chi}}(\mathscr{L}_{r,\chi}(\cdot),.)\to\mathrm{Hom}_K(\cdot,\Gamma(\mathscr{B},\cdot)).$$

Generally, it is far from being an equivalence.

We compute the stalks of the localization $\mathscr{L}_{r,\chi}(M)$ for a *coadmissible* module M. In this case $(\underline{M}_r)_z$ is finitely generated over the Banach algebra $D_r(U_z^{(e)})_\theta$ and therefore has a unique structure as a Banach module over $D_r(U_z^{(e)})_\theta$. Let $z\in\mathscr{B}\subset X^{\mathrm{an}}$ with residue field $\kappa(z)$. Recall that $\pi(z)$ equals the generic point of X.

Proposition 8.2.2. *Let M be a coadmissible left $D(G)_\theta$-module and let $z\in\mathscr{B}$. The morphism $\underline{M}_r\to\mathscr{L}_{r,\chi}(M)$ induces an isomorphism between the λ-coinvariants of the $\mathfrak{t}_K$-module*

$$(\kappa(z)\hat{\otimes}_L(\underline{M}_r)_z)/\mathfrak{n}_{\pi(z)}(\kappa(z)\hat{\otimes}_L(\underline{M}_r)_z)$$

and the stalk $\mathscr{L}_{r,\chi}(M)_z$. In particular, if $\theta=\theta_0$ we have

$$(\kappa(z)\hat{\otimes}_L(\underline{M}_r)_z)/\mathfrak{b}_{\pi(z)}(\kappa(z)\hat{\otimes}_L(\underline{M}_r)_z)\xrightarrow{\sim}\mathscr{L}_{r,\rho}(M)_z.$$

Proof. Let N be an arbitrary finitely generated $D_r(U_z^{(e)})$-module. According to Lemma 7.4.12 the space $\mathscr{D}_{r,\mathfrak{t},z}\otimes_{D_r(U_z^{(e)})}N$ may be written as

$$\Big[\big((\kappa(z)\hat{\otimes}_L D_r(U_z^{(e)}))/\mathfrak{n}_{\pi(z)}(\kappa(z))\hat{\otimes}_L D_r(U_z^{(e)})\big)\otimes_{\kappa(z)\hat{\otimes}_L D_r(U_z^{(e)})}\kappa(z)\hat{\otimes}_L D_r(U_z^{(e)})\Big]\otimes_{D_r(U_z^{(e)})}N.$$

Since N is a complete Banach module this may be identified with

$$(\kappa(z)\hat{\otimes}_L N)/\mathfrak{n}_{\pi(z)}(\kappa(z)\hat{\otimes}_L N)$$

by associativity of the completed tensor product. The resulting isomorphism

$$(\kappa(z)\hat{\otimes}_L N)/\mathfrak{n}_{\pi(z)}(\kappa(z)\hat{\otimes}_L N)\xrightarrow{\sim}\mathscr{D}_{r,\mathfrak{t},z}\otimes_{D_r(U_z^{(e)})}N$$

is functorial in N. According to the second part of loc.cit. we obtain a functorial homomorphism

$$(\lambda\text{-coinvariants of }(\kappa(z)\hat{\otimes}_L N)/\mathfrak{n}_{\pi(z)}(\kappa(z)\hat{\otimes}_L N))\to\mathscr{D}_{r,\chi,z}\otimes_{D_r(U_z^{(e)})}N,$$

which is an isomorphism in the case $N = D_r(U_z^{(e)})$. Note that the target is a right exact functor in N. Similarly, the source is also a right exact functor in N. To see this, it suffices to note that the functor which sends N to $\kappa(z)\hat{\otimes}_L N$ is exact. Indeed, any short exact sequence of finitely generated $D_r(U_z^{(e)})$-modules is a strict exact sequence relative to the unique Banach topology on such modules (cf. [Schneider and Teitelbaum 2003, Proposition 2.1.iii]) and so the claim follows from a well-known result of L. Gruson [1966, 3.2, Corollaire 1]. Since the source and the target are both right exact functors in N commuting with finite direct sums, we may use a finite free presentation of N to obtain that it is an isomorphism in general. The assertion of the proposition follows by taking $N = (\underline{M}_r)_z$. □

Corollary 8.2.3. *Let χ be dominant and regular. The functor $\mathscr{L}_{r,\chi}$, restricted to coadmissible modules, is exact.*

Proof. Exactness can be checked at a point $z \in \mathscr{B}$, where the functor in question equals the composite of three functors. The first functor equals

$$N \mapsto D_r(U_z^{(e)})_\theta \otimes_{D(U_z^{(e)})_\theta} N$$

on the category of coadmissible $D(U_z^{(e)})_\theta$-modules. It is exact by [Schneider and Teitelbaum 2003, Remark 3.2]. The second functor equals $N \mapsto \kappa(z)\hat{\otimes}_L N$ on the category of finitely generated $D_r(U_z^{(e)})_\theta$-modules. It is exact as we have explained in the proof of the preceding proposition. The natural inclusion $U(\mathfrak{g})_\theta \to D_r(U_z^{(e)})_\theta$ allows one to consider $\kappa(z)\hat{\otimes}_L N$ as a $U(\kappa(z) \otimes_L \mathfrak{g})_\theta$-module. The Beilinson–Bernstein stalk functor at $\pi(z)$ of the corresponding localization on the flag variety $X_{\kappa(z)}$ (note that the natural embedding $k(\pi(z)) \to \kappa(z)$ gives a canonical lift of $\pi(z)$ to a $\kappa(z)$-rational point of $X_{\kappa(z)}$) is given by the λ-coinvariants of the $\kappa(z) \otimes_L \mathfrak{h}$-module

$$(\kappa(z)\hat{\otimes}_L N)/\mathfrak{n}_{\pi(z)}(\kappa(z)\hat{\otimes}_L N)$$

according to Theorem 5.2.2(iii). By part (ii) of the same theorem, this functor is exact if χ is dominant and regular. □

Lemma 8.2.4. *Let $z \in \mathscr{B}$. If N is a finitely generated $D_r(U_z^{(e)})_\theta$-module which is finite dimensional over K, then the natural map*

$$\mathcal{D}^{\mathrm{an}}_{\mathscr{B},\chi,z} \otimes_{U(\mathfrak{g}_K)_\theta} N \xrightarrow{\sim} \mathscr{D}_{r,\chi,z} \otimes_{D_r(U_z^{(e)})_\theta} N$$

is an isomorphism which is functorial in modules of this kind.

Proof. We adopt the notation of Proposition 6.4.8 and write

$$\varinjlim_V (\mathcal{A}_V \# D_r(U_z^{(e)})) \xrightarrow{\sim} \mathcal{O}_{\mathscr{B},z} \# D_r(U_z^{(e)}),$$

an isomorphism of K-algebras according to Proposition 2.4.1. By Proposition 2.1

of [Schneider and Teitelbaum 2003] the finitely generated module

$$(\mathcal{A}_V \# D_r(U_z^{(e)})) \otimes_{D_r(U_z^{(e)})} N$$

has a unique Banach topology. We thus have canonical $\mathcal{A}_V$-linear isomorphisms

$$(\mathcal{A}_V \hat{\otimes}_L D_r(U_z^{(e)})) \otimes_{D_r(U_z^{(e)})} N \simeq \mathcal{A}_V \hat{\otimes}_L N = \mathcal{A}_V \otimes_L N.$$

Passage to the inductive limit yields, by Proposition 2.4.1, the $\mathcal{O}_{\mathscr{B},z}$-linear map

$$(\mathcal{O}_{\mathscr{B},z} \hat{\otimes} D_r(U_z^{(e)})) \otimes_{D_r(U_z^{(e)})} N \simeq \mathcal{O}_{\mathscr{B},z} \otimes_L N = (\mathcal{O}_{\mathscr{B},z} \# U(\mathfrak{g}_K)) \otimes_{U(\mathfrak{g}_K)} N.$$

The target maps canonically to $\mathcal{D}^{\mathrm{an}}_{\mathscr{B},\chi,z} \otimes_{U(\mathfrak{g}_K)_\theta} N$ and the composed map annihilates all elements of the form $\xi \hat{\otimes} n$ with $n \in N$ and $\xi \in \mathcal{I}^{\mathrm{an}}_{\mathscr{B},\chi,z}$. Since such ξ generate $\mathscr{I}^{\mathrm{an}}_{\chi,z}$ the composed map factors therefore into a map

$$\mathscr{D}_{r,\chi,z} \otimes_{D_r(U_z^{(e)})_\theta} N \to \mathcal{D}^{\mathrm{an}}_{\mathscr{B},\chi,z} \otimes_{U(\mathfrak{g}_K)_\theta} N.$$

This gives the required inverse map. □

Corollary 8.2.5. *Let M be a left $D(G)_\theta$-module such that* $\dim_K M_r(U_z^{(e)}) < \infty$ *for all $z \in \mathscr{B}$. The natural morphism of sheaves*

$$\mathcal{D}^{\mathrm{an}}_{\mathscr{B},\chi} \otimes_{U(\mathfrak{g}_K)_\theta} \underline{M}_r \xrightarrow{\sim} \mathscr{D}_{r,\chi} \otimes_{\underline{D}_{r,\theta}} \underline{M}_r = \mathscr{L}_{r,\chi}(M)$$

induced from (7.4.8) is an isomorphism.

Proof. Let $z \in \mathscr{B}$. Applying the preceding lemma to $N := M_r(U_z^{(e)})$ we see that the morphism is an isomorphism at the point z. This proves the claim. □

8.3. *Equivariance.*

8.3.1. Consider for a moment an arbitrary ringed space $(Y, \mathcal{A})$, where $\mathcal{A}$ is a sheaf of (not necessarily commutative) K-algebras on Y. Let Γ be an abstract group acting (from the right) on $(Y, \mathcal{A})$. In other words, for every $g, h \in \Gamma$ and every open subset $U \subseteq Y$ there is an isomorphism of K-algebras $g^* : \mathcal{A}(U) \xrightarrow{\sim} \mathcal{A}(g^{-1}U)$ compatible in an obvious sense with restriction maps and satisfying $(gh)^* = h^* g^*$.

A Γ-*equivariant* $\mathcal{A}$-module (see [Jantzen 2003, II.F.5]) is a (left) $\mathcal{A}$-module $\mathcal{M}$ equipped, for any open subset $U \subseteq Y$ and for $g \in G$, with K-linear isomorphisms $g^* : \mathcal{M}(U) \xrightarrow{\sim} \mathcal{M}(g^{-1}U)$ compatible with restriction maps and such that $g^*(am) = g^*(a) g^*(m)$ for $a \in \mathcal{A}(U)$, $m \in \mathcal{M}(U)$. If $g, h \in G$ we require $(gh)^* = h^* g^*$.

An obvious example is $\mathcal{M} = \mathcal{A}$. If $\mathcal{M}$ is equivariant we have a K-linear isomorphism $\mathcal{M}_z \xrightarrow{\sim} \mathcal{M}_{g^{-1}z}$ between the stalks of the sheaf $\mathcal{M}$ at z and $g^{-1}z$ for any $g \in G$. Finally, a morphism of equivariant modules is an $\mathcal{A}$-linear map compatible with the Γ-actions. The equivariant modules form an abelian category.

8.3.2. After these preliminaries we go back to the situation discussed in the previous section. We keep all the assumptions from this section. The group G naturally acts on the ringed space $(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$. Moreover, G acts on $\mathfrak{g}$ and $U(\mathfrak{g})$ via the adjoint action as usual. It follows from the classical argument [Miličić 1993a, Section 3] that the sheaves

$$\mathcal{O}_{X^{\mathrm{an}}}\#U(\mathfrak{g}), \mathcal{I}_{\chi}^{\mathrm{an}} \quad \text{and} \quad \mathcal{D}_{\chi}^{\mathrm{an}} := (\mathcal{O}_{X^{\mathrm{an}}}\#U(\mathfrak{g}))/\mathcal{I}_{\chi}^{\mathrm{an}}$$

(as defined in Section 6) are equivariant $\mathcal{O}_{X^{\mathrm{an}}}$-modules. Of course, here

$$g^* : \mathcal{D}_{\chi}^{\mathrm{an}}(U) \xrightarrow{\sim} \mathcal{D}_{\chi}^{\mathrm{an}}(g^{-1}U)$$

is even a K-algebra isomorphism for all $g \in G$ and open subsets $U \subseteq X^{\mathrm{an}}$.

On the other hand, the group G acts on the ringed space $(\mathscr{B}, \mathcal{O}_{\mathscr{B}})$ and the natural map $\vartheta_{\boldsymbol{B}} : \mathscr{B} \to X^{\mathrm{an}}$ is G-equivariant, see Theorem 6.2.1. Since our functor $\vartheta_{\boldsymbol{B}}^{-1}$ preserves G-equivariance the $\mathcal{O}_{\mathscr{B}}$-modules

$$\mathcal{O}_{\mathscr{B}}\#U(\mathfrak{g}_K), \mathcal{I}_{\mathscr{B},\chi}^{\mathrm{an}} \quad \text{and} \quad \mathcal{D}_{\mathscr{B},\chi}^{\mathrm{an}} = (\mathcal{O}_{\mathscr{B}}\#U(\mathfrak{g}_K))/\mathcal{I}_{\mathscr{B},\chi}^{\mathrm{an}}$$

are G-equivariant. Again, here $g^* : \mathcal{D}_{\mathscr{B},\chi}^{\mathrm{an}}(U) \xrightarrow{\sim} \mathcal{D}_{\mathscr{B},\chi}^{\mathrm{an}}(g^{-1}U)$ is a K-algebra isomorphism for all $g \in G$ and open subsets $U \subseteq \mathscr{B}$. Recall from Definition 7.2.1 the sheaf of K-algebras $\mathcal{O}_{\mathscr{B}}\#\underline{D}_r$.

Proposition 8.3.3. *The $\mathcal{O}_{\mathscr{B}}$-module $\mathcal{O}_{\mathscr{B}}\#\underline{D}_r$ is G-equivariant. For $g \in G$ the map g^* is a K-algebra isomorphism.*

Proof. Given $g \in G$ and $z \in \mathscr{B}$ we have the group isomorphism

$$g^{-1}(\,\cdot\,)g : U_z^{(e)} \xrightarrow{\sim} U_{g^{-1}z}^{(e)}$$

by (4.1.6). Since it is compatible with variation of the level e it is compatible with the p-valuations $\mathring{\omega}_z$ and $\mathring{\omega}_{g^{-1}z}$. It induces therefore an isometric isomorphism of Banach algebras

$$g^{-1}(\,\cdot\,)g : D_r(U_z^{(e)}) \xrightarrow{\sim} D_r(U_{g^{-1}z}^{(e)}). \tag{8.3.4}$$

The induced map

$$\mathcal{O}_{\mathscr{B},z}\hat{\otimes}_L D_r(U_z^{(e)}) \xrightarrow{\sim} \mathcal{O}_{\mathscr{B},z}\hat{\otimes}_L D_r(U_{g^{-1}z}^{(e)})$$

is multiplicative with respect to the skew multiplication and we obtain an isomorphism of K-algebras

$$g^* : (\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)_z \xrightarrow{\sim} (\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)_{g^{-1}z} \tag{8.3.5}$$

according to Lemma 7.2.3. Since we have the identity $g\mathfrak{x}g^{-1} = \mathrm{Ad}(g)(\mathfrak{x})$ in $D(G)$

this isomorphism fits into the commutative diagram

$$\begin{array}{ccc} (\mathcal{O}_{\mathscr{B}}\#U(\mathfrak{g}))_z & \xrightarrow{\simeq} & (\mathcal{O}_{\mathscr{B}}\#U(\mathfrak{g}))_{g^{-1}z} \\ \downarrow & & \downarrow \\ (\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)_z & \xrightarrow{\simeq} & (\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)_{g^{-1}z} \end{array}$$

where the vertical arrows are the inclusions from (6.4.12). Recall the sheaf $\mathcal{F}$ appearing in Lemma 7.2.2. Let $\Omega \subseteq \mathscr{B}$ be an open subset. The isomorphisms (8.3.5) for $z \in \Omega$ assemble to a K-algebra isomorphism

$$g^* : \mathcal{F}(\Omega) \xrightarrow{\sim} \mathcal{F}(g^{-1}\Omega),\ s \mapsto [z \mapsto (g^*)^{-1}(s(gz))]$$

compatible with restriction maps and satisfying $(gh)^* = h^* g^*$ for $g, h \in G$. It now suffices to see that g^* maps the subspace $(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)(\Omega)$ into $(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)(g^{-1}\Omega)$. Let $s \in (\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)(\Omega)$. If F is a facet in $\mathscr{B}$ we let $\Omega = \bigcup_{i\in I} \Omega_i$ be a datum for s with respect to F. If $F \cap \Omega_i \neq \varnothing$ we consider $g^{-1}V_i$ and $(g^*)^{-1}(s_i)$ and obtain a datum $g^{-1}\Omega = \bigcup_{i\in I} g^{-1}\Omega_i$ for the section $(g^*)^{-1}sg \in \mathcal{F}(g^{-1}\Omega)$ with respect to the facet $g^{-1}F$. Indeed, the axiom (a) of Definition 7.2.1 for the section $(g^*)^{-1}sg$ follows from the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathscr{B}}(U) & \xrightarrow{\iota_z} & \mathcal{O}_{\mathscr{B},z} \\ \downarrow{\scriptstyle (g^*)^{-1}} & & \downarrow{\scriptstyle (g^*)^{-1}} \\ \mathcal{O}_{\mathscr{B}}(gU) & \xrightarrow{\iota_{gz}} & \mathcal{O}_{\mathscr{B},gz} \end{array}$$

valid for any open subset $U \subseteq \mathscr{B}$ containing z. Moreover, we have a commutative diagram

$$\begin{array}{ccc} D_r(U_{F'}^{(e)}) & \xrightarrow{\sigma_r^{F'F}} & D_r(U_F^{(e)}) \\ \downarrow{\scriptstyle (g^*)^{-1}} & & \downarrow{\scriptstyle (g^*)^{-1}} \\ D_r(U_{gF'}^{(e)}) & \xrightarrow{\sigma_r^{gF'gF}} & D_r(U_{gF}^{(e)}) \end{array}$$

whenever F', F are two facets in $\mathscr{B}$ with $F' \subseteq \overline{F}$. From this the axiom (b) for the section $(g^*)^{-1}sg$ follows easily. □

It follows from the preceding proof that the morphism $\mathcal{O}_{\mathscr{B}}\#U(\mathfrak{g}) \to \mathcal{O}_{\mathscr{B}}\#\underline{D}_r$ from Proposition 7.2.6 is equivariant. The equivariant structure of $\mathcal{I}^{\mathrm{an}}_{\mathscr{B},\chi}$ therefore implies that the ideal sheaf $\mathscr{I}^{\mathrm{an}}_{\chi}$ of $\mathcal{O}_{\mathscr{B}}\#\underline{D}_r$ is naturally equivariant. This yields the following corollary.

Corollary 8.3.6. *The $\mathcal{O}_{\mathscr{B}}$-module $\mathscr{D}_{r,\chi}$ is equivariant. The map g^* is a K-algebra isomorphism for any $g \in G$. The morphism $\mathcal{D}^{\mathrm{an}}_{\mathscr{B},\chi} \to \mathscr{D}_{r,\chi}$ from (7.4.8) is equivariant.*

The above discussion shows that there is a natural right action of G on the ringed space $(\mathscr{B}, \mathscr{D}_{r,\chi})$. We let $\mathrm{Mod}_G(\mathscr{D}_{r,\chi})$ be the abelian category of G-equivariant (left) $\mathscr{D}_{r,\chi}$-modules.

8.3.7. Using very similar arguments we may use the isomorphisms (8.3.4) appearing in the above proof to define an equivariant structure on the sheaves $\underline{D}_r$ and $\underline{D}_{r,\theta}$. As before we suppose $\sigma(\chi) = \theta$. If M is a $D(G)$-module (resp. $D(G)_\theta$-module) with $m \in M$ and $g \in G$ we put $g.m := \delta_{g^{-1}} m$. This defines a K-linear isomorphism

$$g^* : M_r(U_z^{(e)}) \xrightarrow{\sim} M_r(U_{g^{-1}z}^{(e)})$$

via $g^*(\delta \otimes m) := g^*(\delta) \otimes gm$ for any $\delta \in D_r(U_z^{(e)})$. As in the case of $\underline{D}_r$ these isomorphisms lift to an equivariant structure on the sheaf $\underline{M}_r$. Since these isomorphisms are compatible with the isomorphisms (8.3.4) we obtain that $\underline{M}_r$ is an equivariant $\underline{D}_r$-module (resp. $\underline{D}_{r,\theta}$-module). We now define $g^*(\partial \otimes m) := g^*(\partial) \otimes g^*(m)$ for local sections ∂ and m of $\mathscr{D}_{r,\chi}$ and $\underline{M}_r$ respectively. Since the morphism $\underline{D}_{r,\theta} \to \mathscr{D}_{r,\chi}$ induced by Proposition 7.2.5 is equivariant this yields an equivariant structure on $\mathscr{L}_{r,\chi}(M)$. If $M \to N$ is a $D(G)_\theta$-linear map the resulting morphism $\mathscr{L}_{r,\chi}(M) \to \mathscr{L}_{r,\chi}(N)$ is easily seen to be equivariant. This shows

Corollary 8.3.8. *The functor $\mathscr{L}_{r,\chi}$ takes values in* $\mathrm{Mod}_G(\mathscr{D}_{r,\chi})$.

9. Comparison with the Schneider–Stuhler construction

In this section we assume $L = \mathbb{Q}_p$, $e > e_{\mathrm{st}}, e_{cl}$ and $r \in [r_0, 1)$. We will work in this section with the trivial infinitesimal character, i.e., $\lambda := \lambda_0$ and $\theta := \theta_0$.

9.1. *Preliminaries on smooth distributions.*

9.1.1. Let M be a co-admissible $D(G)$-module such that the associated locally analytic representation $V = M'_b$ is smooth. In the previous section, we have associated to M a sheaf $\underline{M}_r$ on the Bruhat–Tits building $\mathscr{B}$. On the other hand, we also have the sheaf $\underset{\approx}{V}$ on $\mathscr{B}$ constructed in [Schneider and Stuhler 1997, 4.6]. We now show that for $r < p^{-1/(p-1)}$, the two sheaves $\underset{\approx}{\check{V}}$ and $\underline{M}_r$ are canonically isomorphic. Here, $\check{V}$ denotes the smooth dual. We remark straightaway that V is admissible-smooth [Schneider and Teitelbaum 2003, Theorem 6.6] and hence, so is $\check{V}$ [Cartier 1979, 1.5(c)].

Suppose H is a uniform locally $\mathbb{Q}_p$-analytic group with $\mathbb{Q}_p$-Lie algebra $\mathfrak{h}$. Let $D^\infty(H)$ denote the quotient of $D(H)$ by the ideal generated by $\mathfrak{h}$. Let $\mathcal{C}_H^\infty$ denote the category of coadmissible $D^\infty(H)$-modules. If $U_r(\mathfrak{h})$ denotes the closure of $U(\mathfrak{h})$ inside $D_r(H)$ we put

$$H_{(r)} := H \cap U_r(\mathfrak{h}).$$

Lemma 9.1.2. *The set $H_{(r)}$ is an open normal subgroup of H constituting, for $r \uparrow 1$, a neighborhood basis of $1 \in H$.*

Proof. As the norm $\|\cdot\|_r$ on $D_r(H)$ does not depend on the choice of ordered basis the inversion map $h \mapsto h^{-1}$ induces an automorphism of $D_r(H)$. It induces an automorphism of $U_r(\mathfrak{h})$, which implies that $H_{(r)}$ is a subgroup of H. A similar argument with the conjugation automorphism $h \mapsto ghg^{-1}$ for a $g \in H$ implies that this subgroup is normal in H. For the remaining assertions we choose $m \geq 0$ such that $r_m = \sqrt[p^m]{r_0} \geq r$ and consider $D(P_{m+1}(H))$. The inclusion $D(P_{m+1}(H)) \subseteq D(H)$ gives rise to an isometric embedding

$$D_{r_0}(P_{m+1}(H)) \hookrightarrow D_{r_m}(H)$$

(final remark in Section 2.2.3). Since $U(\mathfrak{h})$ is norm-dense inside $D_{r_0}(P_{m+1}(H))$ it follows that $P_{m+1}(H) \subset U_{r_m}(\mathfrak{h}) \subseteq U_r(\mathfrak{h})$, which implies $P_{m+1}(H) \subseteq H_{(r)}$ and therefore $H_{(r)}$ is open. Finally, if $r \uparrow 1$ then $r_m \uparrow 1$ whence $m \uparrow \infty$. Since the lower p-series $\{P_m(H)\}_m$ constitutes a neighborhood basis of $1 \in H$ the last assertion of the lemma follows. □

By the proof of Theorem 6.6 in [Schneider and Teitelbaum 2003], the lemma implies a canonical K-algebra isomorphism $D^\infty(H) \simeq \varprojlim_r K[H/H_{(r)}]$ coming from restricting distributions to the subspace of K-valued locally constant functions on H.

Proposition 9.1.3. (i) *We have $D_r(H) \otimes_{D(H)} D^\infty(H) \simeq K[H/H_{(r)}]$ as right $D^\infty(H)$-modules.*

(ii) *If $M \in \mathcal{C}_H^\infty$ and $V = M_b'$ denotes the corresponding smooth representation then $D_r(H) \otimes_{D(H)} M \simeq (\check{V})_{H_{(r)}}$ as K-vector spaces. Here, $(\cdot)_{H_{(r)}}$ denotes $H_{(r)}$-coinvariants and $(\check{\cdot})$ denotes the smooth dual.*

Proof. The first statement follows from $D_r(H) = \bigoplus_{h \in H/H_{(r)}} \delta_h U_r(\mathfrak{h})$ as right $U_r(\mathfrak{h})$-modules by passing to quotients modulo the ideals generated by $\mathfrak{h}$. The second statement follows from (i) by observing the general identities $K[H/N] \otimes_{D^\infty(H)} M = \mathrm{Hom}_K(V^N, K) = (\check{V})_N$ valid for any normal open subgroup N of H. □

Corollary 9.1.4. *If $M \in \mathcal{C}_H^\infty$ and $r_0 \leq r < p^{-1/p-1}$ then $D_r(H) \otimes_{D(H)} M \simeq (\check{V})_H$.*

Proof. We have $U_r(\mathfrak{h}) = D_r(H)$ for such an r and therefore $H_{(r)} = H$. □

9.2. ***The comparison isomorphism.***

9.2.1. Let us return to our sheaf $M \mapsto \underline{M}_r$. We assume in the following that

$$r_0 \leq r < p^{-1/p-1}.$$

Let F be a facet in X. If we apply the above corollary to the uniform group $U_F^{(e)}$ we obtain a canonical linear isomorphism

$$f_r^F : M(U_F^{(e)}) = D_r(U_F^{(e)}) \otimes_{D(U_F^{(e)})} M \xrightarrow{\sim} (\check{V})_{U_F^{(e)}}.$$

If $F \subseteq \overline{F'}$ for two facets F, F' in X it follows that

$$f_r^{F'} \circ \sigma_r^{FF'} = \mathrm{pr}^{FF'} \circ f_r^F \tag{9.2.2}$$

where $\mathrm{pr}^{FF'} : (\check{V})_{U_F^{(e)}} \to (\check{V})_{U_{F'}^{(e)}}$ denotes the natural projection.

Proposition 9.2.3. *Given an open subset $\Omega \subseteq X$, the collection of maps f_r^z for $z \in \Omega$ induces a K-linear isomorphism $\underline{M}_r(\Omega) \simeq \underset{\approx}{\check{V}}(\Omega)$ compatible with restriction maps whence a canonical isomorphism of sheaves*

$$\underline{M}_r \xrightarrow{\sim} \underset{\approx}{\check{V}}$$

which is natural in admissible V.

Proof. Given $z \in \mathscr{B}$ we have the isomorphism

$$f_r^z : M_r(U_z^{(e)}) \xrightarrow{\sim} (\check{V})_{U_z^{(e)}},$$

as explained above. These maps assemble to a K-linear isomorphism, say f_r^Ω, between the space of maps

$$s : \Omega \to \dot{\bigcup_{z\in\Omega}} M_r(U_z^{(e)})$$

such that $s(z) \in M_r(U_z^{(e)})$ for all $z \in \mathscr{B}$ and the space of maps

$$s : \Omega \to \dot{\bigcup_{z\in\Omega}} (\check{V})_{U_z^{(e)}}$$

such that $s(z) \in (\check{V})_{U_z^{(e)}}$ for all $z \in \mathscr{B}$. It is clearly compatible with restriction. It therefore suffices to show that it descends to an isomorphism between the subspaces $\underline{M}_r(\Omega)$ and $\underset{\approx}{\check{V}}(\Omega)$ respectively. Since $\underline{M}_r$ and $\underset{\approx}{\check{V}}$ are sheaves it suffices to verify this over the open sets $\Omega \cap \mathrm{St}(F)$ for facets $F \subset \mathscr{B}$. We may therefore fix a facet $F \subset \mathscr{B}$ and assume that $\Omega \subseteq \mathrm{St}(F)$. Restricting to members Ω_i with $\Omega_i \cap F \neq \varnothing$ of a datum for s with respect to F and using the sheaf property a second time we may assume that the covering $\{\Omega\}$ of $\Omega = \Omega \cap \mathrm{St}(F)$ is a datum for s with respect to F satisfying $\Omega \cap F \neq \varnothing$. Let $s \in M_r(U_F^{(e)})$ be the corresponding element of the datum. We let $\check{v}$ be any preimage in $\check{V}$ of $f_r^F(s) \in (\check{V})_{U_F^{(e)}}$. The value of the function $f_r^\Omega(s)$ at $z \in \Omega$ is then given by

$$f_r^z(s(z)) = f_r^{F'}(\sigma_r^{FF'}(s)) \overset{(9.2.2)}{=} \mathrm{pr}^{FF'}(f_r^F(s)) = \text{class of } \check{v} \in (\check{V})_{U_{F'}^{(e)}},$$

where $F' \in \mathrm{St}(F)$ is the unique open facet containing z. This means $f_r^\Omega(s) \in \underset{\approx}{\check{V}}(\Omega)$.

Conversely, let $\check{s} \in \underset{\approx}{\check{V}}(\Omega)$ and consider $s := (f_r^{\Omega})^{-1}(\check{s})$. Let $F \subset \mathscr{B}$ be a facet. Any defining open covering $\Omega = \bigcup_{i \in I} \Omega_i$ with vectors $\check{v}_i \in \check{V}$ for the section $\check{s}$ induces an open covering $\Omega \cap \mathrm{St}(F) = \bigcup_{i \in I} \Omega_{i,F}$, where $\Omega_{i,F} := \Omega_i \cap \mathrm{St}(F)$. If $F \cap \Omega_{i,F} \neq \varnothing$ we let $s_i \in M_r(U_F^{(e)})$ be the inverse image of the class of $\check{v}_i$ under $(f_r^F)^{-1}$. We claim that this gives a datum for s with respect to F. Indeed, for any $z \in \Omega_{i,F} \cap F$ we compute

$$s(z) = (f_r^z)^{-1}(\check{s}(z)) = (f_r^z)^{-1}(\text{class of } \check{v}_i) = s_i,$$

which settles the axiom (a) for s. Similarly, for any $z' \in \Omega_{i,F}$ the value of $s(z')$ equals

$$\begin{aligned}(f_r^{z'})^{-1}(\check{s}(z')) &= (f_r^{F'})^{-1}(\text{class of } \check{v}_i)\\ &= (f_r^{F'})^{-1}(\mathrm{pr}^{FF'}(\check{v}_i)) \overset{(9.2.2)}{=} \sigma_r^{FF'}((f_r^F)^{-1}(\check{v}_i)) = \sigma_r^{FF'}(s_i)\end{aligned}$$

where F' denotes the unique open facet of $\mathrm{St}(F)$ containing z'. This proves (b) for s. All in all $s \in \underline{M}_r(\Omega)$. This proves the proposition. □

Lemma 9.2.4. *Let M be a coadmissible $D^\infty(G)$-module. Then M is a $D(G)_{\theta_0}$-module.*

Proof. We have to show that the canonical map $D(G) \to D^\infty(G)$ factors through $D(G)_{\theta_0}$. The kernel of $D(G) \to D^\infty(G)$ is the two sided ideal generated by $\mathfrak{g}$. The intersection of this latter ideal with $Z(\mathfrak{g}_K)$ equals $\ker \theta_0$ (see the example on page 1411). It follows that the map $Z(\mathfrak{g}_K) \to D^\infty(G)$ factors through θ_0. □

Theorem 9.2.5. *Let $r = r_0$. Suppose M is a coadmissible $D^\infty(G)$-module. Then there is a canonical isomorphism of $\mathcal{O}_{\mathscr{B}}$-modules*

$$C^{SS} : \mathcal{O}_{\mathscr{B}} \otimes_L \underset{\approx}{\check{V}} \xrightarrow{\sim} \mathscr{L}_{r_0,\rho}(M)$$

which is natural in such M. Here, as above, $V = M'_b$.

Proof. Since $\mathfrak{g}M = 0$ there is a canonical isomorphism

$$\mathcal{O}_{\mathscr{B}} \otimes_L \underline{M}_{r_0} \xrightarrow{\sim} \mathcal{D}^{\mathrm{an}}_{\mathscr{B}\chi} \otimes_{U(\mathfrak{g}_K)_\theta} \underline{M}_{r_0}.$$

Arguing stalkwise the assertion follows from Corollary 8.2.5 and Proposition 9.2.3. □

10. Compatibility with the Beilinson–Bernstein localization

Throughout this section we suppose that the conditions (8.0.1) are fulfilled.

Let V denote a finite dimensional algebraic representation of $\boldsymbol{G}$. Then V gives rise to a $U(\mathfrak{g})$-module. Let $M = V'$ denote the dual of V. It is a coadmissible $D(G)$-module. Suppose the $U(\mathfrak{g}_K)$-module underlying M is a $U(\mathfrak{g}_K)_\theta$-module.

Recall that to any $U(\mathfrak{g}_K)_\theta$-module M, Beilinson and Bernstein associate a $\mathcal{D}_\chi$-module which will be denoted $\Delta(M)$ (see Section 5). We can pull this back under the natural map $\pi : X^{\mathrm{an}} \to X$ to get a $\mathcal{D}_\chi^{\mathrm{an}}$-module $\Delta(M)^{\mathrm{an}}$. Finally, we may apply the functor $\vartheta_{\boldsymbol{B}}^{-1}$ to this module. Denote the latter $\mathcal{O}_{\mathscr{B}}$-module by $\Delta(M)^{\mathrm{an}}_{\mathscr{B}}$. One has the following description of $\Delta(M)^{\mathrm{an}}$ and $\Delta(M)^{\mathrm{an}}_{\mathscr{B}}$:

$$\Delta(M)^{\mathrm{an}} = \mathcal{D}_\chi^{\mathrm{an}} \otimes_{U(\mathfrak{g}_K)_\theta} M,$$
$$\Delta(M)^{\mathrm{an}}_{\mathscr{B}} = \mathcal{D}^{\mathrm{an}}_{\mathscr{B},\chi} \otimes_{U(\mathfrak{g}_K)_\theta} M.$$

The second identity follows from the compatibility between tensor products with restriction functors [Kashiwara and Schapira 1990, Proposition 2.3.5]. On the other hand, any finite dimensional algebraic representation V gives rise to a $D(G)$-module M, where $M = V'$. If V is a $U(\mathfrak{g}_K)_\theta$-module, then M is a $D(G)_\theta$-module. In particular, the results of Section 8 allow us to associate to M the $\mathscr{D}_{r,\chi}$-module $\mathscr{L}_{r,\chi}(M)$. Recall that this module is given by

$$\mathscr{L}_{r,\chi}(M) = \mathscr{D}_{r,\chi} \otimes_{\underline{D}_{r,\theta}} \underline{M}_r$$

Now the canonical morphism $\mathcal{D}^{\mathrm{an}}_{\mathscr{B},\chi} \to \mathscr{D}_{r,\chi}$ induces a morphism

$$C^{BB} : \mathcal{D}^{\mathrm{an}}_{\mathscr{B},\chi} \otimes_{U(\mathfrak{g}_K)_\theta} M \to \mathscr{D}_{r,\chi} \otimes_{\underline{D}_{r,\theta}} \underline{M}_r.$$

Recall that $r = r_m$ for some m.

Theorem 10.1.1. *There is $r(M) \in [r_0, 1)$ such that for $r \geq r(M)$ (i.e., $m \gg 0$ sufficiently large) the canonical morphism*

$$C^{BB} : \Delta(M)^{\mathrm{an}}_{\mathscr{B}} \xrightarrow{\sim} \mathscr{L}_{r,\chi}(M)$$

is an isomorphism of $\mathcal{D}^{\mathrm{an}}_{\mathscr{B},\chi}$-modules.

Proof. Let F be a facet in $\mathscr{B}$ such that $F \subseteq \overline{\mathscr{C}}$. By Proposition 4.2.10 of [Emerton 2011] the $D(U_F^{(e)})$-module M decomposes into a *finite* direct sum of irreducible $D(U_F^{(e)})$-modules M_i. Since all M_i are coadmissible $D(U_F^{(e)})$-modules there exists $r(F) \in [r_0, 1)$ such that

$$M_{i,r} := D_r(U_F^{(e)}) \otimes_{D(U_F^{(e)})} M_i \neq 0$$

for all $r \geq r(F)$ and all i. By Theorem A in [Schneider and Teitelbaum 2002, Section 3] the $D(U_F^{(e)})$-equivariant map $M_i \to M_{i,r}$, $m \mapsto 1 \otimes m$ has dense image and is therefore surjective. Since M_i is irreducible the map is therefore bijective whenever $r \geq r(F)$. It follows $M \xrightarrow{\sim} M_r(U_F^{(e)})$ for $r \geq r(F)$. Given $g \in G$ we can use the G-equivariance of the sheaf $\underline{M}_r$ to express the canonical map

$M \to M_r(U^{(e)}_{g^{-1}F})$ as the composite

$$M \xrightarrow{g\cdot} M \xrightarrow{\sim} M_r(U^{(e)}_F) \xrightarrow{g^*} M_r(U^{(e)}_{g^{-1}F}).$$

It is therefore bijective. Put $r(M) := \max_{F\subseteq\overline{\mathscr{C}}} r(F)$. Then $M \xrightarrow{\sim} M_r(U^{(e)}_F)$ for all $F \subset \mathscr{B}$ and all $r \geq r(M)$. Identifying M with its constant sheaf on $\mathscr{B}$ the natural morphism $M \xrightarrow{\sim} \underline{M}_r$ is therefore an isomorphism for all $r \geq r(M)$. On the other hand, arguing stalkwise gives, by Lemma 8.2.4, a canonical isomorphism

$$\mathcal{D}^{\mathrm{an}}_{\mathscr{B},\chi} \otimes_{U(\mathfrak{g}_K)_\theta} \underline{M}_r \xrightarrow{\sim} \mathscr{D}_{r,\chi} \otimes_{\underline{D}_{r,\theta}} \underline{M}_r. \qquad \square$$

11. A class of examples

Throughout this section we suppose that the conditions (8.0.1) are fulfilled.

11.1.1. Let $\mathcal{O}$ be the classical BGG-category for the reductive Lie algebra $\mathfrak{g}_K$ relative to the choice of Borel subalgebra $\mathfrak{b}_K$ [Bernstein et al. 1976]. Since this category was originally defined for complex semisimple Lie algebras only we briefly repeat what we mean by it here. The category $\mathcal{O}$ equals the full subcategory of all (left) $U(\mathfrak{g}_K)$-modules consisting of modules M such that

(i) M is finitely generated as $U(\mathfrak{g}_K)$-module;

(ii) the action of $\mathfrak{t}_K$ on M is semisimple and locally finite;

(iii) the action of $\mathfrak{n}_K$ on M is locally finite.

Recall here that $\mathfrak{t}_K$ acts locally finitely on some module M if $U(\mathfrak{t}_K).m$ is finite dimensional for all $m \in M$ (similar for $\mathfrak{n}_K$).

Let $\mathcal{O}_{\mathrm{alg}}$ be the full abelian subcategory of $\mathcal{O}$ consisting of those $U(\mathfrak{g}_K)$-modules whose $\mathfrak{t}_K$-weights are integral, i.e., are contained in the lattice $X^*(\boldsymbol{T}) \subset \mathfrak{t}^*_K$.

11.1.2. In [Orlik and Strauch 2010a] the authors study an exact functor

$$M \mapsto \mathcal{F}^G_B(M)$$

from $\mathcal{O}_{\mathrm{alg}}$ to admissible locally analytic G-representations. It maps irreducible modules to (topologically) irreducible representations. The image of $\mathcal{F}^G_B$ comprises a wide class of interesting representations containing all principal series representations and many representations arising from homogeneous vector bundles on p-adic symmetric spaces. In this final section we wish to study the localizations of representations in this class. We restrict our attention to modules $M \in \mathcal{O}_{\mathrm{alg},\theta}$ having fixed central character θ. Let $\chi \in \mathfrak{t}^*_K$ be such that $\sigma(\chi) = \theta$.

11.1.3. To start with let $U(\mathfrak{g}_K, B)$ be the smallest subring of $D(G)$ containing $U(\mathfrak{g}_K)$ and $D(B)$. The $\mathfrak{b}$-action on any $M \in \mathcal{O}_{\mathrm{alg}}$ integrates to an algebraic, and hence, locally analytic B-action on M and one has a canonical $D(G)$-module isomorphism

$$\mathcal{F}^G_B(M)'_b \xrightarrow{\sim} D(G) \otimes_{U(\mathfrak{g}_K,B)} M =: N$$

[Orlik and Strauch 2010a, Proposition 3.6]. Of course, N is a $D(G)_\theta$-module. We may therefore consider its localization $\mathscr{L}_{r,\chi}(N)$ on $\mathscr{B}$. We recall that the stalk $\mathscr{L}_{r,\chi}(N)_z$ at a point z is a quotient of $\kappa(z)\hat{\otimes}(\underline{N}_r)_z$ (see Proposition 8.2.2) and therefore has its quotient topology. We finally say a morphism of sheaves to $\mathscr{L}_{r,\chi}(N)$ has *dense image* if this holds stalkwise at all points.

On the other hand, we may form

$$GM := K[G] \otimes_{K[B]} M.$$

It may be viewed as a $U(\mathfrak{g}_K)$-module via $x.(g \otimes m) := g \otimes \mathrm{Ad}(g^{-1})(x).m$ for $g \in G, m \in M, x \in \mathfrak{g}_K$. Since $K[G]$ is a free right $K[B]$-module, GM equals the direct sum of $U(\mathfrak{g}_K)$-submodules $gM := g \otimes M$ indexed by a system of coset representatives g for G/B. Since the group $\boldsymbol{G}$ is connected, the adjoint action of $G = \boldsymbol{G}(L)$ fixes the center $Z(\mathfrak{g}_K) \subset U(\mathfrak{g}_K)$ [Demazure and Gabriel 1970, II, Section 6.1.5] and therefore GM still has central character θ. Let us consider its Beilinson–Bernstein module $\Delta(GM)$ over X. The linear map $gM \xrightarrow{\sim} M$ given by $g \otimes m \mapsto m$ is an isomorphism and equivariant with respect to the automorphism $\mathrm{Ad}(g^{-1})$ of $U(\mathfrak{g}_K)$. It follows that, given an open subset $V \subseteq X$, we have a linear isomorphism $\Delta(gM)(V) \xrightarrow{\sim} \Delta(M)(g^{-1}V)$ given by $\delta \otimes (g \otimes m) \mapsto g^*(\delta) \otimes m$ for a local section δ of $\mathcal{D}_\chi$ and $m \in M$. Here g^* refers to the G-equivariant structure on $\mathcal{D}_\chi$ 8.3.2. The same argument works for the analytifications $\Delta^{\mathrm{an}}(gM)$ and $\Delta^{\mathrm{an}}(M)$. In particular, the stalks $\Delta^{\mathrm{an}}(gM)_z \simeq \Delta^{\mathrm{an}}(M)_{g^{-1}z}$ are isomorphic vector spaces for any $z \in \mathscr{B}$ and any $g \in G$.

Lemma 11.1.4. $\Delta^{\mathrm{an}}(M)|_{\mathscr{B}} = 0 \iff \Delta^{\mathrm{an}}(GM)|_{\mathscr{B}} = 0.$

Proof. Suppose $\Delta^{\mathrm{an}}(M)|_{\mathscr{B}} = 0$. Let $g \in G$. For any $z \in \mathscr{B}$ we compute $\Delta^{\mathrm{an}}(gM)_z \simeq \Delta^{\mathrm{an}}(M)_{g^{-1}z} = 0$, whence $\Delta^{\mathrm{an}}(gM)|_{\mathscr{B}} = 0$. This yields $\Delta^{\mathrm{an}}(GM)|_{\mathscr{B}} = 0$, since $\Delta^{\mathrm{an}}(\,\cdot\,)_{\mathscr{B}}$ commutes with arbitrary direct sums. The converse is clear. □

Lemma 11.1.5. *There is a canonical morphism of $\mathscr{D}_{r,\chi}$-modules*

$$\mathscr{D}_{r,\chi} \otimes_{\mathcal{D}^{\mathrm{an}}_{\mathscr{B},\chi}} \Delta(GM)^{\mathrm{an}}_{\mathscr{B}} \to \mathscr{L}_{r,\chi}(\mathcal{F}^G_B(M)')$$

functorial in M and with dense image.

Proof. The morphism is induced from the functorial map

$$P : GM \to D(G) \otimes_{U(\mathfrak{g}_K,B)} M = N$$

via the inclusions $K[B] \subset D(B)$ and $K[G] \subset D(G)$. Let us show that the morphism has dense image. We claim first that the map P has dense image with respect to the canonical topology on the coadmissible module N. Let G_0 be the (hyper-) special maximal compact open subgroup of G equal to the stabilizer of the origin $x_0 \in A$. Let $B_0 := B \cap G_0$. The Iwasawa decomposition $G = G_0 \cdot B$ implies $K[G] = K[G_0] \otimes_{K[B_0]} K[B]$ and similarly for distributions $D(\cdot)$. Let $G_0 M := K[G_0] \otimes_{K[B_0]} M$ and $N_0 := D(G_0) \otimes_{U(\mathfrak{g}_K, B_0)} M$. Then $G_0 M \simeq GM$ as $K[G_0]$-modules and $N \simeq N_0$ as $D(G_0)$-modules via the obvious maps. Write $D(G_0) = \varprojlim_r D_r(G_0)$ with some Banach algebra completions $D_r(G_0)$. The map P induces maps $P_r : G_0 M \to D_r(G_0) \otimes_{U(\mathfrak{g}_K, B_0)} M$. Since $K[G_0] \subset D_r(G_0)$ is dense, the definition of the Banach topology on the target implies that P_r has dense image. Passing to the limit over r shows that P has dense image. Let $z \in \mathscr{B}$. Then the map P composed with the map $N \to \underline{N}_{r,z}$ has dense image [Schneider and Teitelbaum 2003, Section 3, Theorem A]. Now we are done: the map

$$\mathscr{D}_{r,\chi,z} \otimes_{\mathcal{D}^{\mathrm{an}}_{\mathscr{B},\chi,z}} \Delta(GM)^{\mathrm{an}}_{\mathscr{B},z} \to \mathscr{L}_{r,\chi}(N)_z,$$

pulled back to $\Delta(GM)^{\mathrm{an}}_{\mathscr{B},z}$, may be written as

$$\big((\kappa(z)\hat{\otimes}_L GM)/\mathfrak{n}_{\pi(z)}(\kappa(z)\hat{\otimes}_L GM)\big)_{\lambda\text{-coinv}} \to \big((\kappa(z)\hat{\otimes}_L \underline{N}_{r,z})/\mathfrak{n}_{\pi(z)}(\kappa(z)\hat{\otimes}_L \underline{N}_{r,z})\big)_{\lambda\text{-coinv}}$$

by Theorem 5.2.2 and Proposition 8.2.2. Consequently, it has dense image. □

11.1.6. We now look closer at the case $\theta = \theta_0$ and $\chi = \rho$. Let $V := \mathrm{ind}_B^G(1)$ be the smooth induction of the trivial character of B. Its smooth dual $\check{V}$ equals the smooth induction $\mathrm{ind}_B^G(\delta_B^{-1})$, where $\delta_B : B \to \mathbb{Q}^\times \subseteq K^\times$ is the modulus character of the locally compact group B. We choose e large enough so that the Schneider–Stuhler sheaf $\underset{\approx}{\check{V}}$ of $\check{V}$ is nonzero [Schneider and Stuhler 1997, Theorem IV.4.1].

The finitely many irreducible modules in $\mathcal{O}_{\mathrm{alg},\theta_0}$ are given by the irreducible quotients L_w of the Verma modules M_w of highest weight $-w(\rho) - \rho$ for $w \in W$. The cardinality of the latter set of weights is $|W|$. As usual, w_0 denotes the longest element in W. Let $w \in W$. Let $\mathcal{M}_w$ and $\mathcal{L}_w$ be the Beilinson–Bernstein localizations over X of M_w and L_w respectively. Let $\iota_w : X_w \hookrightarrow X$ be the inclusion of the Bruhat cell $\boldsymbol{B}w\boldsymbol{B}/\boldsymbol{B}$ into X and let $\mathcal{O}_{X_w}$ be its structure sheaf with its natural (left) D_{X_w}-module structure. Let $\mathcal{N}_w = \iota_{w*}\mathcal{O}_w$ be its D-module push-forward to X. Since $\mathcal{O}_{X_w}$ is a holonomic module and ι_w is an affine morphism, $\mathcal{N}_w$ may be viewed as an D_X-module (rather than just a complex of such) [Hotta et al. 2008, 3.4].

Proposition 11.1.7. *Let $w \in W$ and $\mathcal{L}_w^{\mathrm{an}}$ be the analytification of $\mathcal{L}_w$. Then $\mathcal{L}_{w_0}^{\mathrm{an}}|_{\mathscr{B}} = \mathcal{O}_{\mathscr{B}}$ and $\mathcal{L}_w^{\mathrm{an}}|_{\mathscr{B}} = 0$ for $w \neq w_0$.*

Proof. By [loc. cit., Lemma 12.3.1] the sheaf $\mathcal{N}_w$ has support contained in X_w. By [loc. cit., Proposition 12.3.2(i)] the module $\mathcal{L}_w$ injects into $\mathcal{N}_w$. Now let $w \neq w_0$. Let $\eta \in X$ be the generic point of X and X_η^{an} the fiber of $\pi : X^{\mathrm{an}} \to X$ over η. Since

$\eta \notin X_w$ one has $(\mathcal{N}_w)_\eta = 0$ and therefore $\mathcal{N}_w^{\mathrm{an}}|_{X_\eta^{\mathrm{an}}} = 0$. Lemma 6.2.2 states that $\mathscr{B} \subset X_\eta^{\mathrm{an}}$ whence $\mathcal{L}_w^{\mathrm{an}}|_{\mathscr{B}} = 0$. The converse is clear: the module L_{w_0} equals the trivial one-dimensional $U(\mathfrak{g})$-module having localization $\mathcal{L}_{w_0} = \mathcal{O}_X$ (e.g., by the Borel–Weil theorem). Hence, $\mathcal{L}_{w_0}^{\mathrm{an}}|_{\mathscr{B}} = \mathcal{O}_{\mathscr{B}}$. □

Corollary 11.1.8. *Let $w \in W$. Then $\mathscr{L}_{r,\rho}(\mathcal{F}_B^G(L_w)') \neq 0$ if and only if $w = w_0$.*

Proof. Let $w \neq w_0$. The preceding proposition together with the first lemma above yields $\Delta^{\mathrm{an}}(\mathrm{GL}_w)|_{\mathscr{B}} = 0$. The second lemma then yields $\mathscr{L}_{r,\rho}(\mathcal{F}_B^G(L_w)') = 0$. Conversely, let $w = w_0$. We have $\mathcal{F}_B^G(L_{w_0}) = \mathrm{ind}_B^G(1) = V$, the smooth induction of the trivial B-representation [Orlik and Strauch 2010a]. By the choice of e we have $\underset{\approx}{\check{V}} \neq 0$. Let $z \in \mathscr{B}$ be a point such that $\check{V}_{U_z^{(e)}} \neq 0$. With $N := V'$ and $(U_z^{(e)})_{(r)} := U_z^{(e)} \cap U_r(U_z^{(e)})$, Proposition 9.1.3 yields a surjection

$$(\underline{N}_r)_z = D_r(U_z^{(e)}) \otimes_{D(U_z^{(e)})} N = \check{V}_{(U_z^{(e)})_{(r)}} \to \check{V}_{U_z^{(e)}}$$

between the two spaces of coinvariants which implies $(\underline{N}_r)_z \neq 0$. It follows that $\mathscr{L}_{r,\rho}(N)_z = \kappa(z) \otimes_L (\underline{N}_r)_z \neq 0$ (Proposition 8.2.2), which means $\mathscr{L}_{r,\rho}(N)|_{\mathscr{B}} \neq 0$. □

Recall that any $U(\mathfrak{g}_K)$-module $M \in \mathcal{O}$ is of finite length.

Proposition 11.1.9. *Let $M \in \mathcal{O}_{\mathrm{alg},\theta_0}$. Let $n \geq 0$ be the Jordan–Hölder multiplicity of the trivial representation in the module M and let $V = \mathrm{ind}_B^G(1)$. There is a (noncanonical) isomorphism of $\mathcal{O}_{\mathscr{B}}$-modules*

$$\mathscr{L}_{\rho,r}(\mathcal{F}_B^G(M)') \xrightarrow{\sim} \mathscr{L}_{\rho,r}(V'^{\oplus n})$$

with both sides equal to zero in case $n = 0$.

Proof. Let $\underset{\approx,r}{\check{V}}$ be the constructible sheaf of K-vector spaces on $\mathscr{B}$ which is constructed in the same way as $\underset{\approx}{\check{V}}$ but using the groups $(U_F^{(e)})_{(r)}$ instead of $U_F^{(e)}$ for all facets F. The very same arguments as in the case $r = r_0$ (Theorem 9.2.5) show that the $\mathcal{O}_{\mathscr{B}}$-module $\mathscr{L}_{\rho,r}(V')$ is isomorphic to the module $\mathcal{O}_{\mathscr{B}} \otimes_L \underset{\approx,r}{\check{V}}$. In particular, it is a free $\mathcal{O}_{\mathscr{B}}$-module.

We now prove the claim of the proposition by induction on n. Let $n = 0$. By exactness of the functors $\mathcal{F}_B^G(\,\cdot\,)'$ and $\mathscr{L}_{\rho,r}$ a Jordan–Hölder filtration of M induce a filtration of $\mathscr{L}_{\rho,r}(\mathcal{F}_B^G(M)')$ whose graded pieces vanish by the preceding corollary. Thus $\mathscr{L}_{\rho,r}(\mathcal{F}_B^G(M)') = 0$. Let $n = 1$. Using a Jordan–Hölder filtration of M and the case $n = 0$ we may assume that the trivial representation sits in the top graded piece of M. Applying the case $n = 0$ a second time gives the claim. Assume now $n \geq 2$. Using again a Jordan–Hölder filtration of M we have an exact sequence

$$0 \to M_1 \to M \to M_2 \to 0$$

in $\mathcal{O}_{\mathrm{alg},\theta_0}$, where M_i has multiplicity $n_i \geq 1$. Applying the induction hypothesis to M_1 and M_2 yields an exact sequence of $\mathcal{O}_{\mathscr{B}}$-modules

$$0 \to \mathscr{L}_{\rho,r}(V'^{\oplus n_1}) \to \mathscr{L}_{\rho,r}(\mathcal{F}^G_B(M)') \to \mathscr{L}_{\rho,r}(V'^{\oplus n_2}) \to 0.$$

By our first remark this sequence is (noncanonically) split. Since $\mathscr{L}_{\rho,r}$ commutes with direct sums, this completes the induction. □

When $r = r_0$ the statement of the preceding Proposition 11.1.9 can be made more concrete, because in this case the sheaf $\mathscr{L}_{\rho,r_0}(V'^{\oplus n})$ equals the sum over n copies of $\mathcal{O}_{\mathscr{B}} \otimes_L \underset{\approx}{\check{V}}$ with $\check{V} = \mathrm{ind}^G_B(\delta_B^{-1})$; see Theorem 9.2.5.

Appendix: Analyticity of group actions near points on the building

In this appendix we give a proof of Lemma 6.2.6 about the analyticity of group actions near points on the building. Before doing so we would like to remark that we have not used anything special about these points, except that they correspond to supremum norms on affinoid subdomains. It is certainly possible to prove more general statements in similar settings.

Proof of Lemma 6.2.6. Step 1. Recall that G acts transitively on the set of apartments in $\mathscr{B}$, and that we denote by $A = (X_*(\boldsymbol{T})/X_*(\boldsymbol{C})) \otimes_{\mathbb{Z}} \mathbb{R}$ the apartment which corresponds to the torus $\boldsymbol{T}$; see 4.1.2. The affine Weyl group determined by $\boldsymbol{T}$ acts on A, and this action has a relatively compact fundamental domain, which we denote by D. Because of the identity $gU_z^{(e)}g^{-1} = U_{gz}^{(e)}$ in (4.1.6), it suffices to prove the assertion of Lemma 6.2.6 for those z which lie in the closure $\bar{D}$ of D. For any fixed e, the set of groups $\{U_z^{(e)} \mid z \in \bar{D}\}$ is finite, as $\bar{D}$ is compact. Recall that for fixed z the groups $U_z^{(e)}$ form a fundamental system of neighborhoods of 1 in G; see Proposition 4.1.7. Hence, given any affinoid subgroup $\mathbb{U} \subset \boldsymbol{G}^{\mathrm{an}}$, there is $e_{\mathrm{st}} \geq e_{\mathrm{uni}}$ such that $\mathbb{U}_z^{(e)} \subset \mathbb{U}(L)$ for all $z \in \bar{D}$ and all $e \geq e_{\mathrm{st}}$. In Step 3 below we exhibit a certain condition for an affinoid subgroup $\mathbb{U} \subset \boldsymbol{G}^{\mathrm{an}}$. This condition is fulfilled by any sufficiently small affinoid subgroup $\mathbb{U}$. We will then show that there is a fundamental system $\{V_n\}_{n\geq 0}$ as in Lemma 6.2.4 such that $\mathbb{U}$ acts analytically on every V_n, in the sense of Section 6.2.5.

Step 2. According to [Rémy et al. 2010, 2.17], the map $\vartheta_{\boldsymbol{B}} : \mathscr{B} \to X^{\mathrm{an}}$ maps the apartment A into the analytification of the open subscheme $U_1 = \boldsymbol{N}^-\boldsymbol{B}/\boldsymbol{B} \subset X$, which is isomorphic to $\boldsymbol{N}^-$ (notation as in 5.1.3, 5.1.5). Put $\Psi = -\Phi^+(\boldsymbol{G}, \boldsymbol{T})$. The choice of a Chevalley basis for $\mathfrak{g}$ gives coordinates $(X_\alpha)_{\alpha\in\Psi}$ on $\boldsymbol{N}^-$, hence on U_1. The points of the apartment A, considered as a subset of U_1^{an}, can then be described as norms on the algebra $L[(X_\alpha)_{\alpha\in\Psi}] = \mathcal{O}_X(U_1)$ as follows. To $z \in A$ there corresponds the norm

$$L[(X_\alpha)_{\alpha\in\Psi}] \ni \sum_{\nu\in\mathbb{N}^\Psi} a_\nu X^\nu \quad\mapsto\quad \sup_\nu |a_\nu| \prod_{\alpha\in\Psi} e^{\nu(\alpha)\langle z,\alpha\rangle};$$

see [Rémy et al. 2010, 2.17]. Here $\langle z,\alpha\rangle$ is the canonical pairing between cocharacters and characters. The norm just described is the supremum norm on the polydisc $\mathbb{D}(r)$ with polyradius $r=(e^{\langle z,\alpha\rangle})_{\alpha\in\Psi}$. $\mathbb{D}(r)$ is an affinoid domain in U_1^{an}, and it is strictly affinoid if (and only if) all $e^{\langle z,\alpha\rangle}$ are in the extended value group $\sqrt{|L^*|}$. As $\bar{D}$ is a compact subset of A, there are numbers $R_0>1>r_0$ in $|L^*|$ such that

$$r_0 < \inf_{\substack{z\in\bar{D}\\ \alpha\in\Psi}} e^{\langle z,\alpha\rangle} \quad\text{and}\quad \sup_{\substack{z\in\bar{D}\\ \alpha\in\Psi}} e^{\langle z,\alpha\rangle} < R_0.$$

In particular, $\bar{D}$ lies in the interior of the (strictly) affinoid polydisc $\mathbb{D}(R_0)$ with polyradius $(R_0,\ldots,R_0)$. By [Berkovich 1990, 3.4.6], U_1^{an} is an open subset of X^{an}. The polydisc $\mathbb{D}(R_0)$ is thus a neighborhood of $\bar{D}$. Because of this we will henceforth work on $\mathbb{D}(R_0)$.

Step 3. Let $\mathbb{C}_p$ be the completion of an algebraic closure of L. Let $\|\cdot\|$ be the maximum norm on $U_1^{\mathrm{an}}(\mathbb{C}_p)=\mathbb{C}_p^\Psi$, i.e., $\|(x_\alpha)_{\alpha\in\Psi}\|=\max_\alpha|x_\alpha|$. Fix $r_1\in(0,r_0)$. We claim that there is a connected strictly affinoid subgroup $\mathbb{U}\subset \boldsymbol{G}^{\mathrm{an}}$ which leaves $\mathbb{D}(R_0)$ stable, and such that

$$\text{for all } g\in\mathbb{U}(\mathbb{C}_p) \text{ and all } x\in\mathbb{D}(R_0)(\mathbb{C}_p) \text{ one has } \|g(x)-x\|\le r_1. \tag{11.1.10}$$

To see this, let $\Lambda^-\subset \mathrm{Lie}(\boldsymbol{N}^-)$ and $\Lambda^+\subset\mathfrak{b}$ be o_L-lattices, and put $\Lambda_m^-=p^m\Lambda^-$ and $\Lambda_m^+=p^m\Lambda^+$. For m large enough Λ_m^-, Λ_m^+, and $\Lambda_m:=\Lambda_m^-\oplus\Lambda_m^+$ will be o_L-Lie subalgebras. After possibly increasing m, these lattices can be exponentiated to give good analytic subgroups[8] $\mathbb{U}_m^-=\exp_G(\Lambda_m^-)^{\mathrm{an}}\subset\boldsymbol{N}^{-,\mathrm{an}}$, $\mathbb{U}_m^+=\exp_G(\Lambda_m^+)^{\mathrm{an}}\subset\boldsymbol{B}^{\mathrm{an}}$, and $\mathbb{U}_m=\exp_G(\Lambda_m)^{\mathrm{an}}\subset\boldsymbol{G}^{\mathrm{an}}$. Increasing m further (if necessary) ensures that $\mathbb{U}_m$ has an Iwahori decomposition, i.e., $\mathbb{U}_m=\mathbb{U}_m^-\times\mathbb{U}_m^+$ (this follows from the existence of "coordinates of the second kind"). Next consider $\mathbb{D}(R_0)$ as an affinoid subdomain in the group $\boldsymbol{N}^{-,\mathrm{an}}$, the analytification of $\boldsymbol{N}^-$. Then, for any given positive integer m_1 there will be $m_2\gg m_1$ such that $x^{-1}\mathbb{U}_{m_2}x\subset\mathbb{U}_{m_1}$ for all $x\in\mathbb{D}(R_0)$, because $\mathbb{D}(R_0)\subset\boldsymbol{N}^{-,\mathrm{an}}$ is bounded. Furthermore, because $\mathbb{D}(R_0)$ is defined in terms of the Chevalley basis, we can find m_1 such that $\mathbb{D}(R_0)$ (as a subset in $\boldsymbol{N}^{-,\mathrm{an}}$) is stable by right multiplication by $\mathbb{U}_{m_1}^-$. Enlarging m_1 if necessary we even have $\|xh-x\|\le r_1$ for all $x\in\mathbb{D}(R_0)$ and $h\in\mathbb{U}_{m_1}^-$ (each coordinate X_α of xh will be very close to that of x if h is close to the identity). Now fix $g\in\mathbb{U}_{m_2}$ and $x\in\mathbb{D}(R_0)$, considered as an element of $\boldsymbol{N}^{-,\mathrm{an}}$. Write $x^{-1}gx=u^-u^+$ with $u^-\in\mathbb{U}_{m_1}^-$ and $u^+\in\mathbb{U}_{m_1}^+\subset\boldsymbol{B}^{\mathrm{an}}$. Then, as elements of $U_1^{\mathrm{an}}\subset X^{\mathrm{an}}$ we have $g(x)=gx\boldsymbol{B}^{\mathrm{an}}=xu^-u^+\boldsymbol{B}^{\mathrm{an}}=xu^-\boldsymbol{B}^{\mathrm{an}}$. This shows that $\mathbb{D}(R_0)$ is stable under the left action of $\mathbb{U}_{m_2}$ and the inequality in

[8] In the sense of [Emerton 2011].

(11.1.10) is satisfied. With this choice of m_2 we let $\mathbb{U} = \mathbb{U}_{m_2}$. We put

$$\delta = \frac{r_1}{\inf_{z \in \bar{D}, \alpha \in \Psi} e^{\langle z, \alpha \rangle}},$$

which is less than 1.

Step 4. From now on we fix a point $z \in \bar{D}$, which we think of as a supremum norm on the polydisc $\mathbb{D}(r)$ with polyradius $r = (e^{\langle z, \alpha \rangle})_{\alpha \in \Psi}$. We remark that $|X_\alpha(z)| = e^{\langle z, \alpha \rangle}$. It follows from the very definition of the topology on the affinoid space $\mathbb{D}(R_0)$ that a fundamental system of neighborhoods of z is given by finite intersections of sets of the form

$$V_{f,c,C} = \{x \in \mathbb{D}(R_0) \mid c \le |f(x)| \le C\},$$

where $f \in \mathcal{O}(\mathbb{D}(R_0)) = L\langle R_0^{-1} X \rangle$ and $c < |f(z)| < C$ [Berkovich 1990, 2.2.3(iii)]. A particular example of such a neighborhood is the annulus

$$\mathbb{A}_{s,t} = \{x \in \mathbb{D}(R_0) \mid \forall \alpha \in \Psi : s_\alpha \le |X_\alpha(x)| \le t_\alpha\} = \bigcap_\alpha V_{X_\alpha, s_\alpha, t_\alpha},$$

where $s = (s_\alpha)_{\alpha \in \Psi}$ and $t = (t_\alpha)_{\alpha \in \Psi}$ are such that $s_\alpha < e^{\langle z, \alpha \rangle} < t_\alpha$ for all $\alpha \in \Psi$.

Let $\underline{r}_0 = (r_0, \ldots, r_0)$ be the tuple indexed by Ψ which has all components equal to r_0. Given a neighborhood $V_{f,c,C}$, we are now going to find real numbers $c' < C'$ in $\sqrt{|L^*|}$, and a tuple $r' = (r'_\alpha)_{\alpha \in \Psi} \in \sqrt{|L^*|}^{\Psi}$ such that

$$V' = V_{f,c',C'} \cap \mathbb{A}_{\underline{r}_0, r'}$$

(i) is a neighborhood of z,

(ii) is contained in $V_{f,c,C}$, and

(iii) is stable under the action of $\mathbb{U}$.

Step 5. It is straightforward to see that one can find real numbers $c' < C'$ in $\sqrt{|L^*|}$ with the properties

$$c < c' < |f(z)| < C' < C \quad \text{and} \quad C'\delta < c',$$

where δ is as in Step 3. Furthermore, as f has supremum norm less than C' on the disk $\mathbb{D}(r)$, we can find $r' = (r'_\alpha)_{\alpha \in \Psi} \in \sqrt{|L^*|}^{\Psi}$ such that

- for all $\alpha \in \Psi$: $r'_\alpha > e^{\langle z, \alpha \rangle}$,
- f has supremum norm less or equal to C' on the disc $\mathbb{D}(r')$.

We remark that the affinoid group $\mathbb{U}$ acts on the strictly affinoid annulus $\mathbb{A}_{\underline{r}_0, r'}$, because $r_1 < r_0$. Moreover, the strictly affinoid domain $V' = V_{f,c',C'} \cap \mathbb{A}_{\underline{r}_0, r'}$ is a neighborhood of z. Our aim is to show that $\mathbb{U}$ also acts on V'. To see this, it is enough to work with $\mathbb{C}_p$-valued points. Write f as a power series, $f(X) = \sum_{\nu \in \mathbb{N}^\Psi} a_\nu X^\nu$.

Then we have $|a_\nu|(r')^\nu \leq C'$. Consider $x \in V'(\mathbb{C}_p)$ and $g \in \mathbb{U}(\mathbb{C}_p)$. Expand f around x

$$f(x') = f(x) + \sum_{\nu \neq 0} b_\nu (x'-x)^\nu.$$

Then we also have $|b_\nu|(r')^\nu \leq C'$ for all ν. Put $x' = g(x)$ and get $f(g(x)) = f(x) + \sum_{\nu \neq 0} b_\nu (g(x)-x)^\nu$. Using the inequality $\|g(x)-x\| \leq r_1$ we find

$$|b_\nu(g(x)-x)^\nu| = |b_\nu|(r')^\nu \frac{|g(x)-x|^\nu}{(r')^\nu} \leq C' \cdot \frac{r_1^{|\nu|}}{(r')^\nu} < C'\delta < c'.$$

We conclude that $|f(g(x)) - f(x)| < c'$ and thus

$$|f(g(x))| = |f(x) + f(g(x)) - f(x)| = |f(x)|.$$

This shows that $\mathbb{U}$ acts on the (strictly) affinoid neighborhood V'.

Step 6. In the general case, consider a neighborhood of z of the form $V = V_1 \cap \cdots \cap V_m$ with $V_i = V_{f_i, c_i, C_i}$. Then we find for each V_i a neighborhood V_i' stable under $\mathbb{U}$, as in Step 5. The intersection $V' = V_1' \cap \cdots \cap V_m'$ will then be a neighborhood which is stable by the action of $\mathbb{U}$.

Step 7. Now let $W_1 \supset W_2 \supset \cdots$ be a sequence of neighborhoods of z as in Lemma 6.2.4. Use Step 6 to find an strictly affinoid neighborhood $W_1' \subset W_1$ of z on which $\mathbb{U}$ acts. W_1' is not necessarily irreducible. But irreducible and connected components coincide here [Berkovich 1990, 3.1.8] (use that X^{an} is a normal space, by [loc. cit., 3.4.3]), and $\mathbb{U}$, being connected, will stabilize the connected component of W_1' containing z. Call this connected component V_1. It is again a strictly affinoid neighborhood of z. Then choose n such that W_n is contained in the topological interior of V_1, and let $W_n' \subset W_n$ be a neighborhood of z on which $\mathbb{U}$ acts (by Step 6). Let V_2 be the connected component of W_n' containing z. Continuing this way we construct from $(W_n)_n$ a descending sequence of irreducible strictly affinoid neighborhoods $(V_n)_n$ with the same properties as that in Lemma 6.2.4, but with the additional property that $\mathbb{U}$ acts on each V_n. This finishes the proof of Lemma 6.2.6. □

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Complétés universels de représentations de $\mathrm{GL}_2(\mathbb{Q}_p)$

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Soit Π une représentation unitaire de $\mathrm{GL}_2(\mathbb{Q}_p)$, topologiquement de longueur finie. Nous décrivons la sous-représentation Π^{an} de ses vecteurs localement analytiques, et sa filtration par rayon d'analyticité, en termes du (φ, Γ)-module qui lui est associé via la correspondance de Langlands locale p-adique, et nous en déduisons que le complété universel de Π^{an} n'est autre que Π.

Let Π be a unitary representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, topologically of finite length. We describe the subrepresentation Π^{an} made of its locally analytic vectors, and its filtration by radius of analyticity, in terms of the (φ, Γ)-module attached to Π via the p-adic local Langlands correspondence, and we deduce that the universal completion of Π^{an} is Π itself.

Introduction

0.A. ***Notations.*** Soit p un nombre premier. On fixe une clôture algébrique $\overline{\mathbb{Q}}_p$ de $\mathbb{Q}_p$, et on note $\mathscr{G}_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ le groupe de Galois absolu de $\mathbb{Q}_p$. On note Γ le groupe de Galois $\mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$ de l'extension cyclotomique. Le caractère cyclotomique $\chi : \mathscr{G}_{\mathbb{Q}_p} \to \mathbb{Z}_p^*$ induit un isomorphisme de groupes topologiques

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$\Gamma \simeq \mathbb{Z}_p^*$, dont l'inverse $a \mapsto \sigma_a$ est caractérisé par $\sigma_a(\zeta) = \zeta^a$ pour $a \in \mathbb{Z}_p^*$ et $\zeta \in \mu_{p^\infty}$.

On fixe une extension finie L de $\mathbb{Q}_p$, et on note $\mathscr{O}_L$ l'anneau de ses entiers et k_L son corps résiduel. Soit $\widehat{\mathscr{T}}(L)$ l'ensemble des caractères continus $\delta : \mathbb{Q}_p^* \to L^*$, et, pour $\delta \in \widehat{\mathscr{T}}(L)$, notons $w(\delta)$ son *poids*, défini par $w(\delta) = \delta'(1)$, dérivée [1] de δ en 1. Si δ est *unitaire* (i.e., si δ est à valeurs dans $\mathscr{O}_L^*$), la théorie locale du corps de classes associe à δ un caractère continu de $\mathscr{G}_{\mathbb{Q}_p}$, que l'on note encore δ. Le poids de Hodge–Tate généralisé de ce caractère galoisien est alors $w(\delta)$. On note juste $x \in \widehat{\mathscr{T}}(L)$ le caractère induit par l'inclusion de $\mathbb{Q}_p$ dans L, et $|x|$ le caractère envoyant $x \in \mathbb{Q}_p^*$ sur $p^{-v_p(x)}$. Pour fixer les idées, le caractère $x|x|$ correspond au caractère cyclotomique χ et son poids est 1 ; on le note χ la plupart du temps.

Soit $G = \mathrm{GL}_2(\mathbb{Q}_p)$. Si $\delta \in \widehat{\mathscr{T}}(L)$, on note $\mathrm{Rep}_L(\delta)$ la catégorie dont les objets sont les L-espaces de Banach Π, munis d'une action continue de G telle que :

- Π a pour caractère central δ.
- L'action de G est *unitaire*, i.e., il existe une valuation v_Π sur Π, qui définit topologie de Π et telle que $v_\Pi(g \cdot v) = v_\Pi(v)$ pour tous $g \in G$ et $v \in \Pi$.
- Π est *résiduellement de longueur finie*, i.e., si v_Π est comme ci-dessus, la réduction mod p de la boule unité Π_0 de Π pour v_Π est un $\mathscr{O}_L[G]$-module de longueur finie.

Un morphisme entre deux objets Π_1 et Π_2 de $\mathrm{Rep}_L(\delta)$ est une application L-linéaire, continue et G-équivariante.

On note $\mathrm{Rep}_L(G)$ la catégorie des représentations de G unitaires, résiduellement de longueur finie, admettant un caractère central ; c'est la réunion des $\mathrm{Rep}_L(\delta)$ pour $\delta \in \widehat{\mathscr{T}}(L)$.

Remarque 0.1. (i) $\mathrm{Rep}_L(\delta)$ est vide quand δ n'est pas unitaire.

(ii) Il découle des travaux de Barthel et Livné [2] et de Breuil [5] que tout objet de $\mathrm{Rep}_L(\delta)$ est une représentation de Banach *admissible* (au sens de [27]) de G.

(iii) Tout objet Π de $\mathrm{Rep}_L(\delta)$ est topologiquement de longueur finie, car $\Pi_0/p\Pi_0$ l'est. En fait, on peut décrire $\mathrm{Rep}_L(\delta)$ de manière équivalente comme la catégorie des L-représentations de Banach unitaires et admissibles de G, à caractère central et topologiquement de longueur finie (cf. [25] pour $p \geq 5$ et [13] pour le cas général). Cette hypothèse de finitude pour $\Pi_0/p\Pi_0$ sert à assurer que le (φ, Γ)-module attaché à Π par la correspondance de Langlands locale p-adique est de dimension finie.

0.B. ***Complétions unitaires et vecteurs localement analytiques.*** Si $\delta \in \widehat{\mathscr{T}}(L)$ et si $\Pi \in \mathrm{Rep}_L(\delta)$, on note Π^{an} l'espace des vecteurs localement analytiques [28; 20]

1. δ est automatiquement localement analytique, donc la définition a un sens.

de Π. C'est l'espace des vecteurs $v \in \Pi$ dont l'application orbite

$$o_v : G \to \Pi, \quad g \mapsto g \cdot v$$

est localement analytique. C'est une sous-représentation de Π et, d'après un résultat général de Schneider et Teitelbaum [28, th. 7.1], le sous-espace Π^{an} est dense [2] dans Π.

L'espace Π^{an} a une topologie naturelle, induite par l'injection $\Pi^{\mathrm{an}} \to \mathscr{C}^{\mathrm{an}}(G, \Pi)$, envoyant $v \in \Pi^{\mathrm{an}}$ sur o_v. Cette topologie est nettement plus forte que celle induite par l'inclusion $\Pi^{\mathrm{an}} \subset \Pi$. Le résultat principal de cet article est alors le suivant (rappelons que $G = \mathrm{GL}_2(\mathbb{Q}_p)$) :

Théorème 0.2. *Si* $\Pi \in \mathrm{Rep}_L(G)$, *alors* Π *est le complété unitaire universel de* Π^{an}, *i.e., pour toute L-représentation W, de Banach unitaire, l'application naturelle* $\mathrm{Hom}^{\mathrm{cont}}_{L[G]}(\Pi, W) \to \mathrm{Hom}^{\mathrm{cont}}_{L[G]}(\Pi^{\mathrm{an}}, W)$, *induite par l'injection* $\Pi^{\mathrm{an}} \to \Pi$, *est un isomorphisme.*

Remarque 0.3. (i) La notion de complété universel a été dégagée par Emerton [19]. Le théorème ci-dessus répond, dans le cas de $\mathrm{GL}_2(\mathbb{Q}_p)$, à l'une de ses questions, à savoir si le même énoncé est valable pour $\mathrm{GL}_n(\mathbb{Q}_p)$ ou, plus généralement, pour un groupe réductif déployé sur $\mathbb{Q}_p$ (l'application $\mathrm{Hom}^{\mathrm{cont}}_{L[G]}(\Pi, W) \to \mathrm{Hom}^{\mathrm{cont}}_{L[G]}(\Pi^{\mathrm{an}}, W)$ est injective pour tout groupe de Lie p-adique G et toute représentation de Banach admissible Π de G, car Π^{an} est dense dans Π dans ces cas [28]).

(ii) Dans l'autre sens, la situation est nettement plus compliquée : si Π est une représentation localement analytique admissible de $\mathrm{GL}_2(\mathbb{Q}_p)$ admettant un complété unitaire universel $\widehat{\Pi}$, la sous-représentation $\widehat{\Pi}^{\mathrm{an}}$ de $\widehat{\Pi}$ n'est pas forcément égale à Π. Le cas des composantes de Jordan–Hölder de la série principale analytique est assez éclairant. Si $\delta_1, \delta_2 : \mathbb{Q}_p^* \to L^*$ sont des caractères continus (et donc localement analytiques), on note $\mathrm{Ind}^{\mathrm{an}}(\delta_1 \otimes \delta_2)$ l'espace des fonctions $\phi : G \to L$, localement analytiques, telles que $\phi\big(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) g\big) = \delta_1(a)\delta_2(d)\phi(g)$ pour tous $a, d \in \mathbb{Q}_p^*$, $b \in \mathbb{Q}_p$ et $g \in G$, que l'on munit de l'action de G définie par $(h \cdot \phi)(g) = \phi(gh)$; si $\delta_2 = x^k \delta_1$ avec $k \in \mathbb{N}$, alors $\mathrm{Ind}^{\mathrm{an}}(\delta_1 \otimes \delta_2)$ contient une sous-représentation $W(\delta_1, \delta_2)$ de dimension $k+1$ et le quotient $\mathrm{St}^{\mathrm{an}}(\delta_1, \delta_2)$ est une *steinberg analytique*. En utilisant les résultats de [6; 11; 12; 19; 22] on voit qu'il peut, en particulier, se passer les phénomènes suivants :

- $\widehat{\Pi} = 0$ et donc $\widehat{\Pi}^{\mathrm{an}} = 0$: c'est le cas si le caractère central n'est pas unitaire (ce qui équivaut à $v_p(\delta_1(p)) + v_p(\delta_2(p)) \neq 0$) ou s'il est unitaire mais $v_p(\delta_1(p)) > 0$.

2. Nous donnons une nouvelle preuve de cette densité pour les objets de $\mathrm{Rep}_L(\delta)$, en utilisant la théorie des (φ, Γ)-modules (cf. cor. 0.13).

- $\widehat{\Pi}$ n'est pas admissible (c'est le cas des steinberg analytiques avec $k \geq 1$).
- $\widehat{\Pi}$ est non nul et admissible, mais $\widehat{\Pi}^{\mathrm{an}}$ est strictement plus grand que Π : c'est le cas si $\Pi = \mathrm{Ind}^{\mathrm{an}}(\delta_1 \otimes \delta_2)$, si le caractère central est unitaire, et si $v_p(\delta_2(p)) > 0$ et $w(\delta_2) - w(\delta_1) \notin \mathbb{N}$.

Mentionnons un corollaire immédiat du th. 0.2, qui ne semble pas facile à démontrer directement :

Corollaire 0.4. *Le foncteur* $\Pi \mapsto \Pi^{\mathrm{an}}$ *de la catégorie* $\mathrm{Rep}_L(\delta)$ *dans la catégorie des* L*-représentations localement analytiques admissibles de* G *est pleinement fidèle.*

Remarque 0.5. (i) En utilisant ce corollaire et les résultats de [16], on déduit que Π^{an} admet un caractère infinitésimal pour tout objet absolument irréductible Π de $\mathrm{Rep}_L(G)$ (cela n'est pas une conséquence formelle du résultat principal de [loc. cit.], car Π^{an} peut fort bien ne pas être irréductible si Π est absolument irréductible).

(ii) Comme nous l'a fait remarquer Paškūnas, ce corollaire n'est pas vrai pour des représentations de Banach unitaires admissibles d'un groupe de Lie p-adique quelconque : si $G = \mathbb{Z}_p$ et $\Pi = \mathscr{C}(G, L)$, les endomorphismes de Π sont les mesures sur $\mathbb{Z}_p$, alors que ceux de Π^{an} sont les distributions. Il semble raisonnable de penser qu'il reste vrai si on se restreint aux représentations absolument irréductibles et donc que Π^{an} admet un caractère infinitésimal si Π est irréductible.

La suite de cette introduction explique les étapes de la preuve du th. 0.2, dont la correspondance de Langlands locale p-adique [10] pour G est l'ingrédient clé.

0.C. *Un raffinement du foncteur* $\Pi \mapsto \Pi^{\mathrm{an}}$. Dans ce paragraphe, on considère un groupe de Lie p-adique G arbitraire, un sous-groupe H de G qui est un pro-p-groupe uniforme, et une L-représentation de Banach admissible Π de G, pas forcément unitaire. On choisit un système minimal de générateurs topologiques $h_1, \ldots, h_d$ de H et on note

$$b^\alpha = (h_1 - 1)^{\alpha_1} \cdots (h_d - 1)^{\alpha_d} \in \mathbb{Z}_p[H]$$

pour $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$. Pour tout entier $h \geq 1$, on note

$$r_h = \frac{1}{p^{h-1}(p-1)}$$

et

$$\Pi_H^{(h)} = \left\{ v \in \Pi \mid \lim_{|\alpha| \to \infty} p^{-r_h |\alpha|} b^\alpha v = 0 \right\},$$

où $|\alpha| = \alpha_1 + \cdots + \alpha_d$.

Alors $\Pi_H^{(h)}$ est naturellement un banach qui ne dépend pas du choix des générateurs $h_1, \ldots, h_d$ et qui est stable par H. Par ailleurs, les $b^\alpha v$ sont les coefficients de Mahler de o_v :

$$o_v(h_1^{x_1}\cdots h_d^{x_d}) = \sum_{\alpha\in\mathbb{N}^d}\binom{x_1}{\alpha_1}\cdots\binom{x_d}{\alpha_d}\cdot b^\alpha v \quad \text{pour tout } (x_1,\dots,x_d)\in\mathbb{Z}_p^d.$$

Il résulte donc du théorème d'Amice [1] que Π^{an} est la limite inductive des $\Pi_H^{(h)}$, et cela pour tout sous-groupe H de G qui est un pro-p-groupe uniforme. Le résultat suivant (cor. IV.14 et prop. IV.11) peut, au langage près, se trouver dans [28].

Théorème 0.6. *Soient H un pro-p sous-groupe uniforme de G et $h\geq 1$.*

(i) *Le foncteur $\Pi\mapsto\Pi_H^{(h)}$ de la catégorie des L-représentations de Banach admissibles de G dans la catégorie des L-banach est exact.*

(ii) *On a $\Pi_H^{(h+1)}=\Pi_{H^p}^{(h)}$ pour toute représentation de Banach Π de G.*

On dit que Π est *cohérente* (ou H-cohérente si on veut préciser le sous-groupe H de référence) s'il existe h_0 tel que, pour tout $h\geq h_0$, on ait

$$\Pi_H^{(h+1)} = \sum_{H^p\subset gHg^{-1}} g\cdot\Pi_H^{(h)}.$$

Comme $g\cdot\Pi_H^{(h)}=\Pi_{gHg^{-1}}^{(h)}$ et $\Pi_{H_1}^{(h)}\subset\Pi_{H_2}^{(h)}$ si $H_2\subset H_1$, on déduit du théorème précédent que le terme de droite est toujours contenu dans celui de gauche. Le même théorème permet de montrer que la cohérence est une propriété stable par extensions, ce qui joue un rôle important dans la preuve du th. 0.2. Notre intérêt pour la notion de cohérence vient du résultat suivant :

Proposition 0.7. *Si Π est H-cohérente, alors Π^{an} admet un complété unitaire universel. Plus précisément, si $\Pi_0^{(h)}$ est la boule unité de $\Pi_H^{(h)}$ et $\mathscr{L}_h=\sum_{g\in G}g\cdot\Pi_0^{(h)}$, alors pour tout h assez grand on a un isomorphisme de $L[G]$-modules de Banach*

$$\widehat{\Pi^{\mathrm{an}}}\simeq L\otimes_{\mathscr{O}_L}(\varprojlim\mathscr{L}_h/p^n\mathscr{L}_h).$$

Au vu de la proposition précédente, le th. 0.2 est une conséquence du résultat suivant et de la densité de Π^{an} dans Π :

Théorème 0.8. *Si $G=\mathrm{GL}_2(\mathbb{Q}_p)$ et si $\Pi\in\mathrm{Rep}_L(G)$, alors :*

(i) *Π est cohérente.*

(ii) *Si Π_0 est un $\mathscr{O}_L$-réseau de Π, ouvert, borné et G-stable, alors $\mathscr{L}_h$ est commensurable avec $\Pi_0\cap\Pi^{\mathrm{an}}$ pour tout h assez grand.*

La preuve de ce théorème utilise de manière cruciale la théorie des (φ,Γ)-modules. Plus précisément, si D est le (φ,Γ)-module attaché à Π par la correspondance de Langlands locale p-adique, on décrit l'espace $\Pi^{(h)}$ (et même la boule unité $\Pi_0^{(h)}$) directement en termes de D. Cela demande d'étendre et de raffiner bon nombre de résultats des chap. II, IV et V de [10], et les paragraphes suivants expliquent de quelle manière plus en détail.

0.D. ***Description de*** $\mathbf{Rep}_L(\delta)$ ***en termes de*** (φ, Γ)***-modules.*** Soient $\mathscr{R}$ l'anneau de Robba,[3] $\mathscr{E}^\dagger$ le sous-anneau de $\mathscr{R}$ des éléments bornés (c'est un corps) et $\mathscr{E}$ le complété de $\mathscr{E}^\dagger$ pour la valuation p-adique. On munit ces anneaux d'actions continues de Γ et d'un frobenius φ, commutant entre elles, en posant $\varphi(T) = (1+T)^p - 1$ et $\sigma_a(T) = (1+T)^a - 1$ si $a \in \mathbb{Z}_p^*$.

Si $\Lambda \in \{\mathscr{E}, \mathscr{E}^\dagger, \mathscr{R}\}$, on note $\Phi\Gamma^{\rm et}(\Lambda)$ la catégorie des (φ, Γ)-modules étales sur Λ. Ce sont des Λ-modules libres de type fini D, munis d'actions de φ et Γ, continues, semi-linéaires, qui commutent et telles que φ soit de pente nulle. Ces catégories sont toutes équivalentes à la catégorie des L-représentations de $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ (cf. [21; 8; 23]) ; en particulier elles sont équivalentes entre elles, et on note $D^\dagger \in \Phi\Gamma^{\rm et}(\mathscr{E}^\dagger)$, $D_{\rm rig} \in \Phi\Gamma^{\rm et}(\mathscr{R})$ les (φ, Γ)-modules attachés à $D \in \Phi\Gamma^{\rm et}(\mathscr{E})$, de telle sorte que $D = \mathscr{E} \otimes_{\mathscr{E}^\dagger} D^\dagger$ et $D_{\rm rig} = \mathscr{R} \otimes_{\mathscr{E}^\dagger} D^\dagger$.

Si δ est un caractère unitaire et si $D \in \Phi\Gamma^{\rm et}(\mathscr{E})$, on peut construire [10, chap. II] un faisceau G-équivariant $U \to D \boxtimes_\delta U$ sur $\mathbb{P}^1(\mathbb{Q}_p)$, muni de l'action usuelle définie par

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \frac{ax+b}{cx+d},$$

dont les sections sur $\mathbb{Z}_p$ sont D (i.e., $D \boxtimes_\delta \mathbb{Z}_p = D$). Par ailleurs, si U est un ouvert compact de $\mathbb{P}^1$, l'extension par 0 induit une inclusion de $D \boxtimes_\delta U$ dans l'espace $D \boxtimes_\delta \mathbb{P}^1$ des sections globales. Les formules décrivant l'action de G sont très compliquées (et inutiles dans la plupart des situations) en général, mais on a par exemple, pour $z \in D = D \boxtimes_\delta \mathbb{Z}_p$, $a \in \mathbb{Z}_p^*$ et $b \in \mathbb{Z}_p$,

$$\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} z = \varphi(z), \quad \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} z = \sigma_a(z), \quad \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} z = (1+T)^b \cdot z.$$

On dispose aussi [10, chap. IV] d'un foncteur $\Pi \mapsto D(\Pi)$, contravariant, exact, de $\mathrm{Rep}_L(\delta)$ dans $\Phi\Gamma^{\rm et}(\mathscr{E})$. On note $\mathscr{C}_L(\delta)$ son image essentielle. Si $D \in \Phi\Gamma^{\rm et}(\mathscr{E})$, on note $\check{D}$ le dual de Cartier de D : si D est attaché à une représentation galoisienne V, alors $\check{D}$ est attaché à $V^* \otimes \chi$. Le résultat suivant fait le lien entre les constructions précédentes et décrit $\mathrm{Rep}_L(\delta)$ (à des morceaux de dimension finie près) en termes de (φ, Γ)-modules, ce qui est fondamental pour la preuve du th. 0.2.

Théorème 0.9. *Soit* $\delta : \mathbb{Q}_p^* \to \mathscr{O}_L^*$ *un caractère unitaire. Alors* :

(i) $\mathscr{C}_L(\delta)$ *est stable par sous-quotients.*

3. Il s'agit de l'anneau des séries de Laurent $\sum_{n\in\mathbb{Z}} a_n T^n$ à coefficients dans L, qui convergent sur une couronne du type $0 < v_p(T) \le r$, où r dépend de la série.

(ii) *Si* $D \in \mathscr{C}_L(\delta)$, *alors* $\check{D} \in \mathscr{C}_L(\delta^{-1})$.

(iii) *Il existe un foncteur covariant* $D \to \Pi_\delta(D)$ *de* $\mathscr{C}_L(\delta^{-1})$ *dans* $\mathrm{Rep}_L(\delta)$ *tel que, pour tout* $D \in \mathscr{C}_L(\delta)$, *on ait une suite exacte de G-modules topologiques*

$$0 \to \Pi_{\delta^{-1}}(\check{D})^* \to D \boxtimes_\delta \mathbb{P}^1 \to \Pi_\delta(D) \to 0.$$

(iv) *Les foncteurs* $\Pi \mapsto D(\Pi)$ *et* $D \to \Pi_\delta(\check{D})$ *induisent des anti-équivalences quasi-inverses exactes entre* $\mathrm{Rep}_L(\delta)/S$ *et* $\mathscr{C}_L(\delta)$, *où S est la sous-catégorie de* $\mathrm{Rep}_L(\delta)$ *formée des représentations de dimension finie.*

Ce théorème admet des versions entière et de torsion qui sont fort utiles ; il est essentiellement démontré dans [10], mais il n'est pas facile de l'en extraire sous cette forme. Nous reprenons et étendons les arguments de [loc. cit.] pour l'obtenir sous cette forme, mieux adaptée aux applications éventuelles (cf. [13] par exemple).

0.E. ***Description de*** $\mathbf{\Pi}^{(h)}$**.** Soit $\mathscr{E}^{(0,r_b]}$ le sous-anneau de $\mathscr{E}^\dagger$ des fonctions analytiques bornées sur la couronne $0 < v_p(T) \le r_b$. On note $D^{(0,r_b]}$ le plus grand sous-$\mathscr{E}^{(0,r_b]}$-module de type fini M de D tel que $\varphi(M) \subset \mathscr{E}^{(0,r_{b+1}]} \otimes_{\mathscr{E}^{(0,r_b]}} M$ (son existence est un résultat standard de la théorie des (φ, Γ)-modules). Le théorème de surconvergence [8; 4] montre que $D^{(0,r_b]}$ est libre de rang $\dim_{\mathscr{E}}(D)$ sur $\mathscr{E}^{(0,r_b]}$ et engendre D, si b est assez grand. Comme $D \boxtimes_\delta \mathbb{Z}_p = D$ et comme $\mathbb{P}^1 = \mathbb{Z}_p \cup \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\mathbb{Z}_p$, l'application

$$z \mapsto \left(\mathrm{Res}_{\mathbb{Z}_p} z, \mathrm{Res}_{\mathbb{Z}_p} \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) z\right)$$

est une injection de $D \boxtimes_\delta \mathbb{P}^1$ dans $D \times D$, ce qui permet de définir le module

$$D^{(0,r_b]} \boxtimes_\delta \mathbb{P}^1 = (D \boxtimes_\delta \mathbb{P}^1) \cap (D^{(0,r_b]} \times D^{(0,r_b]}).$$

Il est muni de la topologie induite par l'inclusion dans $D^{(0,r_b]} \times D^{(0,r_b]}$, le module $D^{(0,r_b]}$ étant muni de sa topologie naturelle.

Proposition 0.10. *Soit* $D \in \mathscr{C}_L(\delta^{-1})$. *Si b est assez grand, le sous-L-espace vectoriel* $D^{(0,r_b]} \boxtimes_\delta \mathbb{P}^1$ *de* $D \boxtimes_\delta \mathbb{P}^1$ *est stable sous l'action de* $\mathrm{GL}_2(\mathbb{Z}_p)$, *et* $\mathrm{GL}_2(\mathbb{Z}_p)$ *agit continûment pour la topologie naturelle de* $D^{(0,r_b]} \boxtimes_\delta \mathbb{P}^1$. *De plus,* $\Pi_{\delta^{-1}}(\check{D})^*$ *est un sous-module fermé de* $D^{(0,r_b]} \boxtimes_\delta \mathbb{P}^1$.

Soit $K_m = 1 + p^m \mathrm{M}_2(\mathbb{Z}_p)$ avec $m \ge 1$ (resp. $m \ge 2$ si $p = 2$). C'est un pro-p-groupe uniforme de dimension 4, auquel les constructions du § 0.C s'appliquent. Pour simplifier les notations, on note simplement

$$\Pi^{(b)} = \Pi_{K_m}^{(b-m)}$$

pour $b > m$ (le th. 0.6 montre que le terme de droite ne dépend pas du choix de $m < b$). Le résultat technique principal de l'article est alors la description de $\Pi^{(b)}$, pour tout b assez grand, si $\Pi \in \mathrm{Rep}_L(G)$. Le th. 0.9 implique en particulier que tout objet de $\mathrm{Rep}_L(G)$ est de la forme $\Pi_\delta(D)$ à des représentations de dimension finie près, et pour une représentation de la forme $\Pi_\delta(D)$, on a le résultat suivant.

Théorème 0.11. *Soit $D \in \mathscr{C}_L(\delta^{-1})$. L'inclusion de $D^{(0,r_b]} \boxtimes_\delta \mathbb{P}^1$ dans $D \boxtimes_\delta \mathbb{P}^1$ induit, si b est assez grand, une suite exacte de $\mathrm{GL}_2(\mathbb{Z}_p)$-modules topologiques*

$$0 \to \Pi_{\delta^{-1}}(\check{D})^* \to D^{(0,r_b]} \boxtimes_\delta \mathbb{P}^1 \to \Pi_\delta(D)^{(b)} \to 0.$$

Les méthodes utilisées pour l'étude de $\Pi^{(b)}$ sont sensiblement différentes de celles de [10, chap. V] : les arguments de bidualité de [loc. cit.] sont remplacés par une étude directe des rayons d'analyticité des vecteurs de $\Pi_\delta(D)$, à travers l'étude de la croissance des coefficients de Mahler de o_v. Cette étude est grandement facilitée par la prop. V.10, qui est aussi utilisée dans la preuve du cor. 0.16 ci-dessous.

Une conséquence immédiate du th. 0.11 est la généralisation suivante du résultat principal du chap. V de [10].

Corollaire 0.12. *Si $D \in \mathscr{C}_L(\delta^{-1})$, le sous-faisceau $U \mapsto D^\dagger \boxtimes_\delta U$ du faisceau $U \mapsto D \boxtimes_\delta U$ est stable par G, qui agit continûment pour la topologie naturelle de $D^\dagger$, et on a une suite exacte de G-modules topologiques*

$$0 \to \Pi_{\delta^{-1}}(\check{D})^* \to D^\dagger \boxtimes_\delta \mathbb{P}^1 \to \Pi_\delta(D)^{\mathrm{an}} \to 0.$$

Mentionnons que l'action de G sur le faisceau $U \mapsto D^\dagger \boxtimes_\delta U$ s'étend par continuité en une action sur un faisceau $U \mapsto D_{\mathrm{rig}} \boxtimes_\delta U$, et les sections globales $D_{\mathrm{rig}} \boxtimes_\delta \mathbb{P}^1$ de ce faisceau fournissent (chap. VI) une extension de $\Pi_\delta(D)^{\mathrm{an}}$ par $(\Pi_{\delta^{-1}}(\check{D})^{\mathrm{an}})^*$ qui est très utile [12; 14; 15] pour l'étude de $\Pi_\delta(D)^{\mathrm{an}}$.

Une autre conséquence est le résultat suivant qui renforce le théorème de Schneider et Teitelbaum sur la densité des vecteurs localement analytiques.

Corollaire 0.13. *Si $\Pi \in \mathrm{Rep}_L(G)$ il existe $m_0 \geq 2$ tel que $\Pi^{(b)}$ soit dense dans Π^{an} (et donc aussi dans Π) pour tout $b > m_0$.*

En utilisant le fait que les orbites des éléments de $\Pi^{(b)}$ sont somme de leur série de Taylor sur K_{b-1}, et le cor. 0.13, on en déduit le résultat suivant.

Corollaire 0.14. *Soient $\Pi_1, \Pi_2 \in \mathrm{Rep}_L(G)$ et soit $f : \Pi_1^{\mathrm{an}} \to \Pi_2^{\mathrm{an}}$ une application continue, linéaire et $\mathfrak{gl}_2$-équivariante. Alors il existe un sous-groupe ouvert compact H de G tel que f soit H-équivariante.*

Question 0.15. Les cor. 0.13 et 0.14 sont-ils valables pour les représentations de Banach admissibles, topologiquement de longueur finie, d'un groupe de Lie p-adique quelconque ?

Signalons aussi un sous-produit de la preuve, pour lequel nous ne connaissons pas de démonstration plus simple. Une telle démonstration simplifierait considérablement l'étude de $\Pi_\delta(D)^{\mathrm{an}}$.

Corollaire 0.16. *Soient* $\Pi \in \mathrm{Rep}_L(G)$ *et* $v \in \Pi$. *Si les applications* $x \mapsto \left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)v$ *et* $x \mapsto \left(\begin{smallmatrix}1 & 0\\ x & 1\end{smallmatrix}\right)v$ *sont localement analytiques* (*de* $\mathbb{Q}_p$ *dans* Π), *alors* $v \in \Pi^{\mathrm{an}}$.

I. Anneaux de fonctions analytiques

Ce chapitre peu éclairant introduit un certain nombre d'anneaux de séries de Laurent et établit certains résultats techniques dont on aura besoin dans le chap. V. Rappelons que L est une extension finie de $\mathbb{Q}_p$, dont on note $\mathscr{O}_L$ l'anneau des entiers. Pour $b \in \mathbb{N}^*$ on note

$$n_b = p^{b-1}(p-1) \quad \text{et} \quad r_b = 1/n_b.$$

I.A. ***Topologies sur les anneaux.*** On munit l'anneau

$$\mathscr{O}_{\mathscr{E}} = \left\{ \sum_{n\in\mathbb{Z}} a_n T^n \,\middle|\, a_n \in \mathscr{O}_L \text{ et } \lim_{n\to-\infty} v_p(a_n) = \infty \right\}$$

de la topologie faible, dont une base de voisinages de 0 est constituée des $p^n\mathscr{O}_{\mathscr{E}} + T^m\mathscr{O}_L[\![T]\!]$, avec $m, n \in \mathbb{N}$. On munit son corps des fractions

$$\mathscr{E} = \mathscr{O}_{\mathscr{E}}\left[\frac{1}{p}\right] = \bigcup_{n\geq 0} p^{-n}\mathscr{O}_{\mathscr{E}}$$

de la topologie limite inductive.

Si $a \geq b \geq 1$, on note $\mathscr{E}^{[r_a,r_b]}$ l'anneau des $f = \sum_{n\in\mathbb{Z}} a_n T^n$, analytiques sur la couronne $r_a \leq v_p(T) \leq r_b$, définies sur L, que l'on munit de la valuation

$$v^{[r_a,r_b]}(f) = \inf_{r_a \leq v_p(x) \leq r_b} v_p(f(x)) = \inf_{n\in\mathbb{Z}} (v_p(a_n) + \min(nr_a, nr_b)).$$

On note $\mathscr{O}_{\mathscr{E}}^{[r_a,r_b]}$ l'anneau de valuation de $\mathscr{E}^{[r_a,r_b]}$, et on pose

$$\mathscr{E}^{]0,r_b]} = \varprojlim_{a\geq b} \mathscr{E}^{[r_a,r_b]}$$

(c'est l'anneau des fonctions analytiques sur la couronne $0 < v_p(T) \leq r_b$, définies sur L). Soit $\mathscr{O}_{\mathscr{E}}^{\dagger,b}$ le complété de $\mathscr{O}_L[\![T]\!][p/T^{n_b}]$ pour la topologie p-adique. La preuve du résultat suivant est laissée au lecteur.

Lemme I.1. (i) $\mathscr{O}_{\mathscr{E}}^{\dagger,b}$ *est l'anneau des séries de Laurent* $\sum_{k\in\mathbb{Z}} a_k T^k \in \mathscr{O}_L[\![T, T^{-1}]\!]$ *telles que la suite*[4] $([v_p(a_k)] + kr_b)_{k\leq 0}$ *est positive et tend vers* $+\infty$ *quand* $k \to -\infty$.

(ii) $\mathscr{O}_{\mathscr{E}}^{[r_a,r_b]}$ *est l'anneau des séries de Laurent* $\sum_{k\in\mathbb{Z}} a_k T^k \in L[\![T, T^{-1}]\!]$ *telles que la suite* $(v_p(a_k) + \min(kr_b, kr_b))_{k\in\mathbb{Z}}$ *est positive et tend vers* $+\infty$ *quand* $k \to \pm\infty$.

4. On note [ ] la partie entière.

On note

$$\mathscr{E}^{(0,r_b]} = \mathscr{E}^{]0,r_b]} \cap \mathscr{E}$$

(c'est le sous-anneau de $\mathscr{E}^{]0,r_b]}$ formé des fonctions analytiques bornées) et $\mathscr{O}_{\mathscr{E}}^{(0,r_b]}$ le réseau de $\mathscr{E}^{(0,r_b]}$ formé des séries à coefficients dans $\mathscr{O}_L$. On déduit du lemme I.1 que $\mathscr{O}_{\mathscr{E}}^{(0,r_b]} = \mathscr{O}_{\mathscr{E}}^{\dagger,b}[1/T]$ et que $\mathscr{O}_{\mathscr{E}}^{\dagger,b}$ est séparé et complet pour la topologie T-adique. Cela munit $\mathscr{O}_{\mathscr{E}}^{(0,r_b]}$ d'une topologie naturelle et $\mathscr{E}^{(0,r_b]}$ de la topologie limite inductive, en écrivant

$$\mathscr{E}^{(0,r_b]} = \bigcup_{k\geq 0} p^{-k}\mathscr{O}_{\mathscr{E}}^{(0,r_b]}.$$

Enfin, l'anneau de Robba $\mathscr{R}$ est la réunion des $\mathscr{E}^{]0,r_b]}$, muni de la topologie limite inductive et le corps

$$\mathscr{E}^{\dagger} = \bigcup_{b\geq 1} \mathscr{E}^{(0,r_b]}$$

est le sous-anneau de $\mathscr{R}$ des éléments bornés. Il est dense dans $\mathscr{R}$ et $\mathscr{E}$ s'identifie au complété de $\mathscr{E}^{\dagger}$ pour la valuation p-adique. Si $\Lambda \in \{\mathscr{E}, \mathscr{R}\}$, on pose $\Lambda^+ = \Lambda \cap L[\![T]\!]$.

I.B. ***Quelques calculs.*** Les lemmes techniques suivants seront utilisés dans l'étude des vecteurs localement analytiques des représentations unitaires admissibles de $\mathrm{GL}_2(\mathbb{Q}_p)$.

Lemme I.2. *On a* $p\mathscr{O}_{\mathscr{E}} \cap \mathscr{E}^{\dagger} \subset \bigcup_{n\geq 1} \mathscr{O}_{\mathscr{E}}^{\dagger,n}$.

Démonstration. Soit $f = \sum_{n\in\mathbb{Z}} a_n T^n \in p\mathscr{O}_{\mathscr{E}} \cap \mathscr{E}^{\dagger}$. Il existe b tel que f converge sur $0 < v_p(T) \leq r_b$. On a donc $\lim_{k\to\infty} v_p(a_{-k}) - kr_b = \infty$. En particulier, il existe b_1 tel que si $k \geq n_{b_1}$, alors $v_p(a_{-k}) \geq 1 + kr_b$. Puisque $v_p(a_{-k}) \geq 1$ pour tout k, on en déduit que $[v_p(a_{-k})] \geq kr_{b+b_1}$ pour tout $k \geq 0$, donc $f \in \mathscr{O}_{\mathscr{E}}^{\dagger,b+b_1}$ (lemme I.1). □

Lemme I.3. (i) $\mathscr{O}_{\mathscr{E}}^{[r_a,r_b]} \cap \mathscr{O}_{\mathscr{E}}^{(0,r_b]} \subset \frac{1}{p}\mathscr{O}_{\mathscr{E}}^{\dagger,b}$.

(ii) *Si* $f \in \mathscr{O}_{\mathscr{E}}^{(0,r_b]}$ *satisfait* $v^{[r_a,r_b]}(f) \geq N$ *pour un* $N \in \mathbb{N}$, *alors* $f \in \frac{1}{p}T^{Nn_b}\mathscr{O}_{\mathscr{E}}^{\dagger,b}$.

Démonstration. (i) C'est une conséquence immédiate du lemme I.1.

(ii) Si $r_a \leq v_p(x) \leq r_b$, on a

$$v_p(f(x)) - Nn_b v_p(x) \geq v^{[r_a,r_b]}(f) - Nn_b r_b \geq 0,$$

donc $T^{-Nn_b} f \in \mathscr{O}_{\mathscr{E}}^{[r_a,r_b]} \cap \mathscr{O}_{\mathscr{E}}^{(0,r_b]}$ et on conclut en utilisant le (i). □

Lemme I.4. (i) *Si* $(f_k)_k$ *est une suite d'éléments de* $\mathscr{O}_{\mathscr{E}}^{\dagger,b}$ *qui converge vers* 0 *pour la topologie* p*-adique, alors la série* $\sum_{k\geq 0}(T^{n_a}/p)^k f_k$ *converge dans* $\mathscr{O}_{\mathscr{E}}^{[r_a,r_b]}$.

(ii) *Si* $f \in \mathscr{O}_{\mathscr{E}}^{[r_a,r_b]}$, *alors il existe une suite* $(f_k)_k$ *comme dans* (i) *et telle que*

$$pf = \sum_{k\geq 0}\Big(\frac{T^{n_a}}{p}\Big)^k f_k.$$

Démonstration. (i) Il suffit de constater que

$$v^{[r_a,r_b]}\Big(\Big(\frac{T^{n_a}}{p}\Big)^k f_k\Big) \geq v^{[r_a,r_b]}(f_k)$$

et que, par hypothèse, la dernière quantité est positive et tend vers ∞ pour $k \to \infty$.

(ii) Posons

$$pf = \sum_{k\in\mathbb{Z}} b_k T^k \qquad \text{et} \qquad g_k = \sum_{j=0}^{n_a-1} p^k b_{kn_a+j} T^j \quad \text{pour } k \geq 0.$$

Alors $g_k \in \mathscr{O}_{\mathscr{E}}^+$ tend vers 0 pour la topologie p-adique (car $v_p(b_k) + kr_a \geq 1$ pour tout k et $\lim_{k\to+\infty} v_p(b_k) + kr_a = +\infty$) et on a

$$\sum_{k\geq 0} b_k T^k = \sum_{k\geq 0} \Big(\frac{T^{n_a}}{p}\Big)^k g_k.$$

Pour conclure, il suffit de vérifier que $\sum_{k\leq 0} b_k T^k \in \mathscr{O}_{\mathscr{E}}^{\dagger,b}$. Cela découle du lemme I.1. □

I.C. *Actions de φ, ψ, Γ.* On munit les anneaux $\mathscr{E}^+$, $\mathscr{R}^+$, $\mathscr{O}_{\mathscr{E}}$, $\mathscr{E}$, $\mathscr{E}^\dagger$, $\mathscr{R}$ d'actions continues de $\Gamma = \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$ et d'un Frobenius φ, commutant entre elles, en posant $\varphi(T) = (1+T)^p - 1$ et $\sigma_a(T) = (1+T)^a - 1$ si $a \in \mathbb{Z}_p^*$. L'opérateur φ ne laisse pas stable les anneaux $\mathscr{O}_{\mathscr{E}}^{\dagger,b}$, $\mathscr{O}_{\mathscr{E}}^{(0,r_b]}$, $\mathscr{O}_{\mathscr{E}}^{[r_a,r_b]}$ et $\mathscr{E}^{]0,r_b]}$; il les envoie respectivement dans $\mathscr{O}_{\mathscr{E}}^{\dagger,b+1}$, $\mathscr{O}_{\mathscr{E}}^{(0,r_{b+1}]}$, $\mathscr{O}_{\mathscr{E}}^{[r_{a+1},r_{b+1}]}$ et $\mathscr{E}^{]0,r_{b+1}]}$. Ces anneaux sont, en revanche, stables sous l'action de Γ.

Le corps $\mathscr{E}$ est une extension de degré p de $\varphi(\mathscr{E})$, ce qui permet de définir un inverse à gauche ψ de φ par la formule

$$\psi(f) = p^{-1}\varphi^{-1}(\mathrm{Tr}_{\mathscr{E}/\varphi(\mathscr{E})}\, f).$$

Alors ψ laisse stable $\mathscr{O}_{\mathscr{E}}$ et $\mathscr{E}^\dagger$, s'étend par continuité à $\mathscr{R}$, et envoie les anneaux $\mathscr{O}_{\mathscr{E}}^{\dagger,b+1}$, $\mathscr{O}_{\mathscr{E}}^{(0,r_{b+1}]}$, $\mathscr{O}_{\mathscr{E}}^{[r_{a+1},r_{b+1}]}$ et $\mathscr{E}^{]0,r_{b+1}]}$ dans $\mathscr{O}_{\mathscr{E}}^{\dagger,b}$, $\mathscr{O}_{\mathscr{E}}^{(0,r_b]}$, $\mathscr{O}_{\mathscr{E}}^{[r_a,r_b]}$ et $\mathscr{E}^{]0,r_b]}$ respectivement. De plus, ψ commute à Γ et

$$\psi\Big(\sum_{i=0}^{p-1}(1+T)^i\varphi(f_i)\Big) = f_0$$

(tout élément de $\mathscr{E}$ ou $\mathscr{R}$ peut s'écrire sous cette forme, et une telle écriture est unique).

Le résultat suivant est parfaitement classique.

Lemme I.5. *$\varphi^n(T)/T^{p^n}$ est une unité de $\mathscr{O}_{\mathscr{E}}^{\dagger,b}$ si $b > n$.*

Démonstration. Voir le lemme II.5.2 de [8]. □

II. (φ, Γ)-modules

Ce chapitre est aussi préliminaire. On rappelle quelques résultats standard de la théorie des (φ, Γ)-modules et on établit deux résultats techniques qui seront utilisés dans l'étude des vecteurs localement analytiques de la représentation $\Pi_\delta(D)$.

II.A. ***(φ, Γ)-modules et faisceaux P^+-équivariants sur $\mathbb{Z}_p$.*** Soit A un anneau topologique, commutatif, muni d'un endomorphisme continu φ et d'une action continue de Γ, qui commutent. Un (φ, Γ)-module sur A est un A-module de type fini muni d'un endomorphisme semi-linéaire φ et d'une action semi-linéaire de Γ, commutant entre elles.

Un (φ, Γ)-module D sur $\mathscr{O}_{\mathscr{E}}$ est dit *étale* si $\varphi(D)$ engendre D sur $\mathscr{O}_{\mathscr{E}}$. Un (φ, Γ)-module D sur $\mathscr{E}$ est dit *étale* s'il admet un $\mathscr{O}_{\mathscr{E}}$-réseau stable par φ et Γ et qui est étale en tant que (φ, Γ)-module sur $\mathscr{O}_{\mathscr{E}}$.

On note $\Phi\Gamma^{\rm et}_{\rm tors}$ (resp. $\Phi\Gamma^{\rm et}(\mathscr{O}_{\mathscr{E}})$) la catégorie des (φ, Γ)-modules étales sur $\mathscr{O}_{\mathscr{E}}$, qui sont de torsion (resp. libres) comme $\mathscr{O}_{\mathscr{E}}$-module. Enfin, on note $\Phi\Gamma^{\rm et}(\mathscr{E})$ la catégorie des (φ, Γ)-modules étales sur $\mathscr{E}$.

Soit D un (φ, Γ)-module étale. Alors D est muni d'une action de P^+ donnée par

$$\begin{pmatrix} p^k a & b \\ 0 & 1 \end{pmatrix} \cdot z = (1+T)^b \varphi^k \circ \sigma_a(z)$$

si $k \in \mathbb{N}$, $a \in \mathbb{Z}_p^*$ et $b \in \mathbb{Z}_p$, et d'un inverse à gauche ψ de φ qui commute à l'action de Γ et qui est définie par $\psi\big(\sum_{i=0}^{p-1}(1+T)^i\varphi(x_i)\big) = x_0$. On utilise ces données pour associer à D un faisceau $U \mapsto D \boxtimes U$ sur $\mathbb{Z}_p$ (où U décrit les ouverts compacts de $\mathbb{Z}_p$), équivariant sous l'action de P^+, où P^+ agit sur $\mathbb{Z}_p$ par la formule $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \cdot x = ax + b$ habituelle. De manière précise :

- $D \boxtimes \mathbb{Z}_p = D$ et $D \boxtimes \varnothing = 0$,
- $D \boxtimes (i + p^k\mathbb{Z}_p) = \left(\begin{smallmatrix} p^k & i \\ 0 & 1 \end{smallmatrix}\right) D \subset D$
- La restriction ${\rm Res}_{i+p^k\mathbb{Z}_p} : D \boxtimes \mathbb{Z}_p \to D \boxtimes (i + p^k\mathbb{Z}_p)$ est définie par la formule

$${\rm Res}_{i+p^k\mathbb{Z}_p} = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \circ \varphi^k \circ \psi^k \circ \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix}.$$

Remarque II.1. Soit $\mathscr{C}$ le faisceau sur $\mathbb{Z}_p$ des fonctions continues à valeurs dans L et soit $\mathscr{D}_0$ le faisceau des mesures (i.e., le dual de $\mathscr{C}$). Le dictionnaire d'analyse fonctionnelle p-adique fournit une suite exacte $0 \to \mathscr{D}_0 \to D \to \mathscr{C} \otimes \chi^{-1} \to 0$ de faisceaux P^+-équivariants sur $\mathbb{Z}_p$ si $D = \mathscr{E}$ est le (φ, Γ)-module trivial (la torsion par χ^{-1} signifie que l'action de $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$ est multipliée par $\chi(a)^{-1}$).

II.B. ***Surconvergence.*** Soit $D \in \Phi\Gamma^{\rm et}(\mathscr{O}_{\mathscr{E}})$. Si $b \in \mathbb{N}^*$, on note $D^{\dagger,b}$ le plus grand sous-$\mathscr{O}_{\mathscr{E}}^{\dagger,b}$-module M de type fini de D tel que $\varphi(M) \subset \mathscr{O}_{\mathscr{E}}^{\dagger,b+1} \cdot M$ (le tout à l'intérieur de D). On renvoie à [4, prop. 4.2.6] pour une preuve du résultat suivant :

Proposition II.2. *Si $D \in \Phi\Gamma^{\mathrm{et}}(\mathscr{O}_{\mathscr{E}})$, il existe $m(D)$ tel que $D^{\dagger,m(D)}$ soit libre de rang $\mathrm{rg}_{\mathscr{O}_{\mathscr{E}}}(D)$ sur $\mathscr{O}_{\mathscr{E}}^{\dagger,m(D)}$, et $D^{\dagger,b} = \mathscr{O}_{\mathscr{E}}^{\dagger,b} \otimes_{\mathscr{O}_{\mathscr{E}}^{\dagger,m(D)}} D^{\dagger,m(D)}$ pour tout $b \geq m(D)$.*

La prop. II.2 permet de définir, pour $a \geq b \geq m(D)$, des modules $D^{(0,r_b]}$, $D^{[r_a,r_b]}$, $D^{]0,r_b]}$, $D^\dagger$ et D_{rig}, en tensorisant $D^{\dagger,m(D)}$ par $\mathscr{O}_{\mathscr{E}}^{(0,r_b]}$, $\mathscr{O}_{\mathscr{E}}^{[r_a,r_b]}$, $\mathscr{E}^{]0,r_b]}$, $\mathscr{E}^\dagger$, $\mathscr{R}$ respectivement. Ils ne dépendent pas du choix de $m(D)$ et ils sont libres de même rang que D sur les anneaux correspondants. Le choix d'une base permet de munir ces modules de topologies naturelles (induites par celles des anneaux de séries de Laurent, voir le § I.A), qui ne dépendent pas du choix de la base.

Tous les modules définis ci-dessus sont munis d'une action de Γ, les modules $D^\dagger$ et D_{rig} sont aussi munis d'actions de φ et ψ commutant à celle de Γ et vérifiant $\psi \circ \varphi = \mathrm{id}$. Le sous-faisceau $U \mapsto D^\dagger \boxtimes U$ de $U \mapsto D \boxtimes U$ est donc stable par P^+, et il s'étend en un faisceau $U \mapsto D_{\mathrm{rig}} \boxtimes U$. Par contre, φ ne préserve pas les autres modules : il envoie $D^{\dagger,b}$ dans $D^{\dagger,b+1}$, $D^{[r_a,r_b]}$ dans $D^{[r_{a+1},r_{b+1}]}$, et $D^{]0,r_b]}$ dans $D^{]0,r_{b+1}]}$. De manière analogue, ψ laisse stable $D^\dagger$ et D_{rig}, mais il envoie (pour $a \geq b \geq m(D)$) $D^{\dagger,b+1}$ dans $D^{\dagger,b}$, $D^{[r_{a+1},r_{b+1}]}$ dans $D^{[r_a,r_b]}$, et $D^{]0,r_{b+1}]}$ dans $D^{]0,r_b]}$.

Nous aurons besoin de l'estimée plus précise ci-dessous.

Lemme II.3. *Soit $D \in \Phi\Gamma^{\mathrm{et}}(\mathscr{O}_{\mathscr{E}})$. Il existe $l(D) \geq 1$ tel que, pour tous $a \in \mathbb{Z}$, $k \in \mathbb{N}^*$ et $b \geq m(D) + k$,*

$$\psi^k(T^a D^{\dagger,b}) \subset T^{[a/p^k]-l(D)} D^{\dagger,b-k}.$$

Démonstration. Si $a \in \mathbb{Z}$ et si $c = [a/p^k]$, le lemme I.5 montre que

$$\psi^k(T^a D^{\dagger,b}) \subset \psi^k(T^{p^k c} D^{\dagger,b}) = \psi^k(\varphi^k(T)^c D^{\dagger,b}) = T^c \psi^k(D^{\dagger,b}).$$

On peut donc se contenter de traiter le cas $a = 0$. Fixons une base $e_1, \ldots, e_d$ de $D^{\dagger,m(D)}$ sur $\mathscr{O}_{\mathscr{E}}^{\dagger,m(D)}$; c'est aussi une base de $D^{\dagger,b}$ sur $\mathscr{O}_{\mathscr{E}}^{\dagger,b}$ pour tout $b \geq m(D)$.

Soit $l \geq 1$ tel que p divise l et $\psi((1+T)^j e_i) \in T^{-l} D^{\dagger,m(D)}$ pour $(i, j) \in [1, d] \times [0, p-1]$. Alors $\psi(D^{\dagger,b}) \subset T^{-l} D^{\dagger,b-1}$ pour tout $b > m(D)$, car $\psi(\mathscr{O}_{\mathscr{E}}^{\dagger,b}) \subset \mathscr{O}_{\mathscr{E}}^{\dagger,b-1}$ et donc

$$D^{\dagger,b} = \sum_{i=1}^{d} \mathscr{O}_{\mathscr{E}}^{\dagger,b} \cdot e_i = \sum_{i=1}^{d} \sum_{j=0}^{p-1} (1+T)^j \varphi(\mathscr{O}_{\mathscr{E}}^{\dagger,b-1}) e_i.$$

Posons $l(D) = 2l$ et montrons par récurrence sur k que $\psi^k(D^{\dagger,b}) \subset T^{-l(D)} D^{\dagger,b-k}$ pour $b \geq m(D) + k$. Pour $k = 1$, on vient de le faire. Pour passer de k à $k+1$, on utilise l'hypothèse de récurrence et le lemme I.5, ce qui donne pour $b > m(D) + k$

$$\psi^{k+1}(D^{\dagger,b}) \subset \psi(\varphi(T)^{-l(D)/p} D^{\dagger,b-k}) = T^{-l(D)/p} \psi(D^{\dagger,b-k}).$$

On conclut en utilisant l'inclusion $\psi(D^{\dagger,b-k}) \subset T^{-l(D)/2} D^{\dagger,b-k-1}$ (cf. ci-dessus) et l'inégalité

$$l(D) \geq \frac{l(D)}{2} + \frac{l(D)}{p}. \qquad \square$$

II.C. *Dualité.* Le module $\Omega^1_{\mathscr{O}_{\mathscr{E}}}$ des $\mathscr{O}_L$-différentielles continues de $\mathscr{O}_{\mathscr{E}}$ est naturellement un (φ,Γ)-module étale libre de rang 1, une base étant $dT/(1+T)$ et les actions de φ et Γ étant [5]

$$\sigma_a\Big(\frac{dT}{1+T}\Big)=a\frac{dT}{1+T}\ \text{ si } a\in\mathbb{Z}_p^*\quad\text{et}\quad\varphi\Big(\frac{dT}{1+T}\Big)=\frac{dT}{1+T}.$$

Si D est un objet de $\Phi\Gamma^{\rm et}(\mathscr{O}_{\mathscr{E}})$, $\Phi\Gamma^{\rm et}(\mathscr{E})$ ou $\Phi\Gamma^{\rm et}_{\rm tors}$, on note $\check{D}$ le (φ,Γ)-module des morphismes $\mathscr{O}_{\mathscr{E}}$-linéaires de D dans

$$\mathscr{O}_{\mathscr{E}}\frac{dT}{1+T},\quad \mathscr{E}\frac{dT}{1+T}\quad\text{et}\quad(\mathscr{E}/\mathscr{O}_{\mathscr{E}})\frac{dT}{1+T},$$

respectivement ; les actions de φ et Γ étant définies par [6]

$$\langle\sigma_a(x),\sigma_a(y)\rangle=\sigma_a(\langle x,y\rangle)\ \text{ si } a\in\mathbb{Z}_p^*\quad\text{et}\quad\langle\varphi(x),\varphi(y)\rangle=\varphi(\langle x,y\rangle);$$

l'accouplement $\langle\ ,\ \rangle$ sur $\check{D}\times D$ étant l'accouplement naturel. Le foncteur $D\to\check{D}$ est involutif et exact. Par extension des scalaires et fonctorialité, l'accouplement $\langle\ ,\ \rangle$ induit des accouplements (pour $a\geq m(D)$)

$$\langle\ ,\ \rangle:\check{D}^{(0,r_a]}\times D^{(0,r_a]}\to\mathscr{E}^{(0,r_a]}\frac{dT}{1+T},\quad\langle\ ,\ \rangle:\check{D}^{\dagger,a}\times D^{\dagger,a}\to\mathscr{O}_{\mathscr{E}}^{\dagger,a}\frac{dT}{1+T},$$

et

$$\langle\ ,\ \rangle:\check{D}_{\rm rig}\times D_{\rm rig}\to\mathscr{R}\frac{dT}{1+T}.$$

L'application résidu

$$\text{rés}_0:\mathscr{O}_L[\![T,T^{-1}]\!]dT\to\mathscr{O}_L,\quad\text{rés}_0\Big(\Big(\sum\nolimits_{n\in\mathbb{Z}}a_nT^n\Big)dT\Big)=a_{-1}$$

induit une application $\text{rés}_0:\mathscr{O}_{\mathscr{E}}\frac{dT}{1+T}\to\mathscr{O}_L$ et donc des applications

$$\text{rés}_0:\mathscr{E}\frac{dT}{1+T}\to L\quad\text{et}\quad\text{rés}_0:\mathscr{E}/\mathscr{O}_{\mathscr{E}}\frac{dT}{1+T}\to L/\mathscr{O}_L.$$

Si $\check{z}\in\check{D}$ et $z\in D$, on pose

$$\{\check{z},z\}=\text{rés}_0(\langle\sigma_{-1}\cdot\check{z},z\rangle).$$

On obtient ainsi un accouplement à valeurs dans $L/\mathscr{O}_L$, $\mathscr{O}_L$ ou L si $D\in\Phi\Gamma^{\rm et}_{\rm tors}$, $D\in\Phi\Gamma^{\rm et}(\mathscr{O}_{\mathscr{E}})$ ou $D\in\Phi\Gamma^{\rm et}(\mathscr{E})$, respectivement. Cet accouplement est parfait, i.e., l'application ι qui envoie x sur $\iota(x)=(y\mapsto\{x,y\})$ identifie $\check{D}$ et D^* (le dual étant muni de la topologie de la convergence simple). On définit par la même formule un accouplement parfait $\{\ ,\ \}$ entre $\check{D}_{\rm rig}$ et $D_{\rm rig}$.

5. La formule $\varphi\big(\frac{dT}{1+T}\big)=p\frac{dT}{1+T}$, qui semblerait naturelle, ne fournit pas un (φ,Γ)-module étale.

6. La condition « D étale » est précisément ce qu'il faut pour garantir l'existence et l'unicité d'un tel φ sur $\check{D}$, si D est un (φ,Γ)-module sur $\mathscr{O}_{\mathscr{E}}$.

Le résultat suivant sera utilisé dans l'étude des vecteurs localement analytiques des objets de $\mathrm{Rep}_L(G)$.

Lemme II.4. *Si* $D \in \Phi\Gamma^{\mathrm{et}}(\mathscr{O}_{\mathscr{E}})$, $a_1, a_2 \in \mathbb{Z}$ *et* $b > \max(m(D), m(\check{D}))$, *alors*

$$\{T^{a_1}\check{D}^{\dagger,b}, T^{a_2}D^{\dagger,b}\} \subset \{x \in \mathscr{O}_L, v_p(x) \geq (a_1+a_2)r_b\}.$$

Démonstration. Soient $\check{z} \in \check{D}^{\dagger,b}$, $z \in D^{\dagger,b}$. Puisque $\sigma_{-1}(T)/T$ est inversible dans $\mathscr{O}_{\mathscr{E}}^{\dagger,b}$, il existe $f \in \mathscr{O}_{\mathscr{E}}^{\dagger,b}$ tel que

$$\left(\frac{\sigma_{-1}(T)}{T}\right)^{a_1}\langle\sigma_{-1}(\check{z}), z\rangle = f\,\frac{dT}{1+T}.$$

Puisque $\langle\ ,\ \rangle$ est $\mathscr{O}_{\mathscr{E}}^{\dagger,b}$-linéaire, on a

$$\{T^{a_1}\check{z}, T^{a_2}z\} = \mathrm{rés}_0\Big(T^{a_1+a_2}f\,\frac{dT}{1+T}\Big).$$

En écrivant $f = \sum_{n\in\mathbb{Z}} b_nT^n$, un petit calcul montre que

$$\mathrm{rés}_0\Big(T^{a_1+a_2}f\,\frac{dT}{1+T}\Big) = \sum_{j\geq 0}(-1)^j b_{-1-(a_1+a_2)-j},$$

la convergence de la série étant assurée par l'inégalité $v_p(b_n) \geq -nr_b$, si $n \leq 0$. Cette inégalité permet aussi de montrer que

$$v_p\Big(\mathrm{rés}_0\Big(T^{a_1+a_2}f\,\frac{dT}{1+T}\Big)\Big) \geq (a_1+a_2)r_b,$$

si $a_1+a_2 \geq 0$; le cas $a_1+a_2 < 0$ étant trivial, cela permet de conclure. □

II.D. *Les modules D^{nr}, $D^\sharp$ et $D^\natural$.* Les modules ci-dessous font l'objet d'une étude détaillée dans [9, chap. II].

Définition II.5. (i) Si $D \in \Phi\Gamma^{\mathrm{et}}(\mathscr{O}_{\mathscr{E}}) \cup \Phi\Gamma^{\mathrm{et}}_{\mathrm{tors}}$ on note $D^{\mathrm{nr}} = \bigcap_{n\geq 1}\varphi^n(D)$ et

$$D^{++} = \big\{x \in D \mid \lim_{n\to\infty}\varphi^n(x) = 0\big\}, \quad D^+ = D^{++} \oplus D^{\mathrm{nr}}.$$

(ii) Si $D \in \Phi\Gamma^{\mathrm{et}}_{\mathrm{tors}}$, on note $D^\natural$ et $D^\sharp$ les orthogonaux respectifs de $\check{D}^+$ et $\check{D}^{++}$, pour l'accouplement $\{\,,\}$. Si $D \in \Phi\Gamma^{\mathrm{et}}(\mathscr{O}_{\mathscr{E}})$, on pose $D^? = \varprojlim_k (D/p^kD)^?$, pour $? \in \{\natural, \sharp\}$.

On étend ces définitions aux (φ,Γ)-modules sur $\mathscr{E}$, en choisissant des réseaux stables par φ et Γ et en tensorisant par L (les objets obtenus ne dépendent pas des choix).

Proposition II.6. *Si* $D \in \Phi\Gamma^{\mathrm{et}}(\mathscr{O}_{\mathscr{E}}) \cup \Phi\Gamma^{\mathrm{et}}_{\mathrm{tors}}$, *alors :*

(i) *D^{nr} et $D^\sharp/D^\natural$ sont des $\mathscr{O}_L$-modules de type fini. Si D est de torsion, alors $\check{D}^{\mathrm{nr}}$ est le dual de $D^\sharp/D^\natural$.*

(ii) $D^\natural$ *et* $D^\sharp$ *sont des sous* $\mathscr{O}_L[\![T]\!]$*-modules compacts de* D, *qui engendrent* D *et sur lesquels* ψ *est surjectif.*

(iii) *Si* D *est de torsion ou si* D *est irréductible de rang* ≥ 2, *alors* $D^\sharp/D^\natural$ *est un* $\mathscr{O}_L$*-module de longueur finie.*

Démonstration. Toutes les références sont à [9]. Le (i) suit de la prop. II.2.2 et de la prop. II.5.19. Le (ii) découle de la prop. II.6.3. Enfin, (iii) est le cor. II.5.21. □

On déduit de la proposition ci-dessus que si $D \in \Phi\Gamma^{\rm et}(\mathscr{E})$, alors $D^{\rm nr}$ et $D^\sharp/D^\natural$ sont des L-espaces vectoriels de dimension finie et que $\check{D}^{\rm nr}$ est le L-dual de $D^\sharp/D^\natural$. De plus, si D est irréductible de dimension ≥ 2, alors $D^\natural = D^\sharp$ car $\check{D}^{\rm nr} = 0$. Cela est faux si D est de dimension 1, car dans ce cas $D^\sharp/D^\natural$ est un L-espace vectoriel de dimension 1 puisque $\check{D}^{\rm nr}$ est de dimension 1.

III. L'image du foncteur $\Pi \mapsto D(\Pi)$

Dans ce chapitre on démontre le th. 0.9 de l'introduction (ainsi que les versions entière et de torsion de ce théorème). Beaucoup des arguments qui suivent sont tirés de [9] et des chap. II et IV de [10] mais nous avons explicité certains résultats implicites dans [10] (comme ceux du § III.K qui ne sont rédigés que dans le cas de torsion dans [10] ou bien ceux du § III.M sur la compatibilité à la réduction modulo p), rajouté des sorites sur les invariants par $\mathrm{SL}_2(\mathbb{Q}_p)$, simplifié la démonstration de résultats clefs comme les th. III.21 et III.49, et introduit la notion de paire G-compatible qui rend la présentation des résultats plus agréable.

III.A. ***Représentations de G.*** Si A est un anneau commutatif et si H est un groupe topologique, une A-représentation de H est un $A[H]$-module à gauche. Une telle représentation Π est dite *lisse* si le stabilisateur de tout $v \in \Pi$ est ouvert dans H et *lisse admissible* si de plus Π^K est un A-module de type fini pour tout sous-groupe ouvert compact K de H.

Nous aurons besoin des catégories suivantes de représentations de $G = \mathrm{GL}_2(\mathbb{Q}_p)$:

- $\mathrm{Rep}_{\rm tors}(G)$ est la catégorie des $\mathscr{O}_L$-représentations lisses de G, de longueur finie et ayant un caractère central.[7] Tout $\Pi \in \mathrm{Rep}_{\rm tors}(G)$ est de torsion comme $\mathscr{O}_L$-module, et admissible d'après les travaux de Barthel et Livné [2] et de Breuil [5].
- $\mathrm{Rep}_{\mathscr{O}_L}(G)$ est la catégorie des $\mathscr{O}_L$-représentations Π de G, ayant un caractère central et telles que Π est un $\mathscr{O}_L$-module séparé et complet pour la topologie p-adique (i.e., $\Pi = \varprojlim \Pi/p^n\Pi$), sans p-torsion et tel que $\Pi/p^n\Pi \in \mathrm{Rep}_{\rm tors}(G)$ pour tout n.

7. Qui n'est pas forcément unique.

- $\mathrm{Rep}_L(G)$ est la catégorie des L-représentations de Banach de G qui admettent un $\mathcal{O}_L$-réseau ouvert, borné, stable par G et appartenant à $\mathrm{Rep}_{\mathcal{O}_L}(G)$, et donc $\mathrm{Rep}_L(G)$ est la catégorie des L-représentations de Banach de G, qui sont unitaires, admissibles au sens de [27], résiduellement de longueur finie [8] et à caractère central.

Si $\Pi \in \mathrm{Rep}_{\mathrm{tors}}(G)$ (resp. $\mathrm{Rep}_{\mathcal{O}_L}(G)$, $\mathrm{Rep}_L(G)$), on note Π^* le dual de Pontryagin (resp. le $\mathcal{O}_L$- ou L-dual continu) de Π, que l'on munit de la topologie faible (i.e., celle de la convergence simple) et de l'action évidente de G.

Si $\delta : \mathbb{Q}_p^* \to \mathcal{O}_L^*$ est un caractère unitaire, on note $\mathrm{Rep}_{\mathrm{tors}}(\delta)$ (resp. $\mathrm{Rep}_{\mathcal{O}_L}(\delta)$, $\mathrm{Rep}_L(\delta)$) la sous-catégorie de $\mathrm{Rep}_{\mathrm{tors}}(G)$ (resp. $\mathrm{Rep}_{\mathcal{O}_L}(G)$, $\mathrm{Rep}_L(G)$) des représentations sur lesquelles $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ agit par multiplication par $\delta(a)$. Par définition, $\mathrm{Rep}_{\mathrm{tors}}(G)$ (resp. $\mathrm{Rep}_{\mathcal{O}_L}(G)$, $\mathrm{Rep}_L(G)$) est la réunion des $\mathrm{Rep}_{\mathrm{tors}}(\delta)$ (resp. $\mathrm{Rep}_{\mathcal{O}_L}(\delta)$, $\mathrm{Rep}_L(\delta)$), pour δ unitaire.

Si η_1, η_2 sont des caractères continus de $\mathbb{Q}_p^*$, à valeurs dans k_L^* (resp. $\mathcal{O}_L^*$), soit

$$\mathrm{Ind}(\eta_1 \otimes \eta_2)$$

l'espace des fonctions $\phi : G \to k_L$ (resp. $\phi : G \to L$), continues, telles que

$$\phi\left(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) g\right) = \eta_1(a)\eta_2(d)\phi(g)$$

pour tous $a, d \in \mathbb{Q}_p^*$, $b \in \mathbb{Q}_p$ et $g \in G$, que l'on munit de l'action de G définie par $(h \cdot \phi)(g) = \phi(gh)$. Alors $\mathrm{Ind}(\eta_1 \otimes \eta_2)$ est un objet de $\mathrm{Rep}_{\mathrm{tors}}(\eta_1\eta_2)$ (resp. $\mathrm{Rep}_L(\eta_1\eta_2)$).

Le résultat suivant est parfaitement classique.

Proposition III.1. (i) *Si* $\eta_1 \neq \eta_2$, *la représentation* $\mathrm{Ind}(\eta_1 \otimes \eta_2)$ *est irréductible* (*resp. topologiquement irréductible*).

(ii) *Si* $\eta_1 = \eta_2$, *la fonction* $g \mapsto \eta_1 \circ \det g$ *engendre une sous-représentation de dimension* 1 *sur laquelle G agit à travers le caractère* $\eta_1 \circ \det g$, *et le quotient est une représentation irréductible* (*resp. topologiquement irréductible*) *de* G, *de la forme* $\mathrm{St} \otimes (\eta_1 \circ \det g)$, *où* St *est* la steinberg (*resp.* la steinberg continue).

Les composantes de Jordan–Hölder des $\mathrm{Ind}(\eta_1 \otimes \eta_2)$ sont dites *ordinaires* ; les objets absolument irréductibles de $\mathrm{Rep}_{\mathrm{tors}}(G)$ ou $\mathrm{Rep}_L(G)$ qui ne sont pas ordinaires sont dits *supersinguliers*. Il n'est pas très facile de construire des L-représentations supersingulières par de purs procédés de théorie des représentations, mais les th. III.4 et III.15 et la prop. III.33 en donnent une classification complète en termes de (φ, Γ)-modules.

8. La condition « résiduellement de longueur finie » implique « topologiquement de longueur finie » de manière évidente ; ces deux conditions sont en fait équivalentes (cf. [25] pour $p \geq 5$ et [13] pour le cas général).

III.B. ***Le foncteur*** $\Pi \mapsto D(\Pi)$**.** On note

$$P = \begin{pmatrix} \mathbb{Q}_p^* & \mathbb{Q}_p \\ 0 & 1 \end{pmatrix}, \quad P^+ = \begin{pmatrix} \mathbb{Z}_p - \{0\} & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix} \subset P$$

(P s'appelle le sous-groupe *mirabolique* de G, et P^+ en est un sous-semi-groupe).

Soit Π un objet de $\mathrm{Rep}_{\mathrm{tors}}(G)$. Si $W \subset \Pi$ est un sous-$\mathscr{O}_L$-module de type fini, stable sous l'action de $\mathrm{GL}_2(\mathbb{Z}_p)$ et qui engendre Π comme G-module (un tel W existe car Π est de longueur finie [10, lemme III.1.6]), on note :

- $D_W^\natural(\Pi)$ le dual de Pontryagin de $P^+ \cdot W$.
- $D_W^+(\Pi)$ l'ensemble des $\mu \in \Pi^*$ nuls sur $g \cdot W$ pour tout $g \in P - P^+$.

$D_W^+(\Pi)$ est stable par P^+ car $P - P^+$ est stable par multiplication par g^{-1} si $g \in P^+$; il admet donc une structure naturelle[9] de (φ, Γ)-module sur $\mathscr{O}_L[\![T]\!]$. On définit alors

$$D(\Pi) = \mathscr{O}_{\mathscr{E}} \otimes_{\mathscr{O}_L[\![T]\!]} D_W^+(\Pi)$$

et on montre [10, th. IV.2.13] que $D(\Pi) \in \Phi\Gamma^{\mathrm{et}}_{\mathrm{tors}}$ (la seule difficulté est de vérifier que $D(\Pi)$ est de longueur finie). De plus, $D(\Pi)$ ne dépend pas du choix de W et $\Pi \mapsto D(\Pi)$ est un foncteur exact contravariant de $\mathrm{Rep}_{\mathrm{tors}}(G)$ dans $\Phi\Gamma^{\mathrm{et}}_{\mathrm{tors}}$.

Si $\Pi \in \mathrm{Rep}_{\mathscr{O}_L}(G)$, on pose

$$D(\Pi) = \varprojlim_n D(\Pi / p^n \Pi).$$

Enfin, si $\Pi \in \mathrm{Rep}_L(G)$, on choisit un $\mathscr{O}_L$-réseau ouvert Π_0, borné et stable par G, et on pose $D(\Pi) = L \otimes_{\mathscr{O}_L} D(\Pi_0)$ (cela ne dépend pas du choix de Π_0). On obtient ainsi des foncteurs exacts contravariants

$$\mathrm{Rep}_{\mathscr{O}_L}(G) \mapsto \Phi\Gamma^{\mathrm{et}}(\mathscr{O}_{\mathscr{E}}) \quad \text{et} \quad \mathrm{Rep}_L(G) \mapsto \Phi\Gamma^{\mathrm{et}}(\mathscr{E}).$$

Remarque III.2. Il est clair que le foncteur $\Pi \mapsto D(\Pi)$ tue les objets de type fini sur $\mathscr{O}_L$ (ou L) mais, comme on le verra (th. III.4), c'est la seule information que l'on perd en utilisant ce foncteur, ce qui est assez remarquable car la construction de $D(\Pi)$ n'utilise que peu d'information sur Π.

III.C. ***Le résultat principal.*** Soit $\delta : \mathbb{Q}_p^* \to \mathscr{O}_L^*$ un caractère unitaire et soit D un (φ, Γ)-module étale.[10] Les constructions de [10, chap. II] fournissent un faisceau G-équivariant sur $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{Q}_p)$, dont l'espace des sections sur U est noté $D \boxtimes_\delta U$. Par construction, on a $D \boxtimes_\delta \mathbb{Z}_p = D$ (et la restriction du faisceau à $\mathbb{Z}_p$ muni de l'action de P^+ est le faisceau du § II.A) et le caractère central du G-module $D \boxtimes_\delta \mathbb{P}^1$ est δ.

9. Les actions de φ et Γ sont celles de $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ et $\left(\begin{smallmatrix} \mathbb{Z}_p^* & 0 \\ 0 & 1 \end{smallmatrix}\right)$; la structure de $\mathscr{O}_L[\![T]\!]$-module est induite par l'action de $\left(\begin{smallmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{smallmatrix}\right)$ et l'isomorphisme standard $\mathscr{O}_L[\![T]\!] \simeq \mathscr{O}_L\left[\!\left[\left(\begin{smallmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{smallmatrix}\right)\right]\!\right]$.

10. Cela signifie que D est un objet d'une des catégories $\Phi\Gamma^{\mathrm{et}}_{\mathrm{tors}}$, $\Phi\Gamma^{\mathrm{et}}(\mathscr{O}_{\mathscr{E}})$ ou $\Phi\Gamma^{\mathrm{et}}(\mathscr{E})$.

De plus, si U est un ouvert compact de $\mathbb{Q}_p$, l'extension par 0 permet de considérer $D \boxtimes_\delta U$ comme un sous-module de $D \boxtimes_\delta \mathbb{P}^1$. Le module $D \boxtimes_\delta U$ est alors stable sous l'action du stabilisateur de U ; en particulier, $D = D \boxtimes_\delta \mathbb{Z}_p$ est stable par $1 + p\mathrm{M}_2(\mathbb{Z}_p)$ puisque ce groupe stabilise $\mathbb{Z}_p \subset \mathbb{P}^1$.

Si $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, l'application $z \mapsto (\mathrm{Res}_{\mathbb{Z}_p}(z), \mathrm{Res}_{\mathbb{Z}_p}(wz))$ induit une injection de $D \boxtimes_\delta \mathbb{P}^1$ dans $D \times D$, ce qui permet de munir $D \boxtimes_\delta \mathbb{P}^1$ d'une structure de G-module topologique (D étant muni de la topologie faible). Plus précisément, si on note w_δ la restriction de l'action de w à $D^{\psi=0} = D \boxtimes_\delta \mathbb{Z}_p^*$, alors $D \boxtimes_\delta \mathbb{P}^1$ s'identifie au sous-ensemble de $D \times D$ des (z_1, z_2) vérifiant $\mathrm{Res}_{\mathbb{Z}_p^*} z_2 = w_\delta(\mathrm{Res}_{\mathbb{Z}_p^*} z_1)$.

Remarque III.3. Comme G est engendré par

$$\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \mathbb{Z}_p^* & 0 \\ 0 & 1 \end{pmatrix}, \quad w \quad \text{et} \quad \begin{pmatrix} 1 & 1+p\mathbb{Z}_p \\ 0 & 1 \end{pmatrix},$$

et comme $\mathbb{P}^1 = \mathbb{Z}_p \cup w \cdot p\mathbb{Z}_p$, l'action de G sur $D \boxtimes_\delta \mathbb{P}^1$ est complètement décrite par les formules suivantes.

- Si $z \in D = D \boxtimes_\delta \mathbb{Z}_p$, si $a \in \mathbb{Z}_p^*$ et si $b \in \mathbb{Z}_p$, on a

$$\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} z = \varphi(z), \quad \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} z = \sigma_a(z), \quad \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} z = (1+T)^b \cdot z.$$

- Si $z = (z_1, z_2) \in D \boxtimes_\delta \mathbb{P}^1$, on a $wz = (z_2, z_1)$, $\mathrm{Res}_{\mathbb{Z}_p}\big(w\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)z\big) = \delta(p)\psi(z_2)$ et, si $b \in p\mathbb{Z}_p$, $\mathrm{Res}_{p\mathbb{Z}_p}\big(w\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)z\big) = u_b(\mathrm{Res}_{p\mathbb{Z}_p}(z_2))$, où [11]

$$u_b = \delta(1+b) \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \circ w_\delta \circ \begin{pmatrix} (1+b)^{-2} & b(1+b)^{-1} \\ 0 & 1 \end{pmatrix} \circ w_\delta \circ \begin{pmatrix} 1 & (1+b)^{-1} \\ 0 & 1 \end{pmatrix}$$

  sur $D \boxtimes_\delta p\mathbb{Z}_p$.

Si $D \in \Phi\Gamma^{\mathrm{et}}(\mathscr{E})$ (resp. $\Phi\Gamma^{\mathrm{et}}(\mathscr{O}_{\mathscr{E}})$, $\Phi\Gamma^{\mathrm{et}}_{\mathrm{tors}}$), le module $D \boxtimes_\delta \mathbb{P}^1$ est l'extension d'un banach par le dual d'un banach (resp. de la boule unité d'un banach par le dual de la boule unité d'un banach, d'un module discret par un module compact). Il n'est, en général, pas possible de trouver une telle extension où les deux termes sont stables par G, et on dit que (D, δ) est *G-compatible* si c'est le cas. On dispose alors de deux éléments Π_1, Π_2 de $\mathrm{Rep}_L(G)$ (resp. $\mathrm{Rep}_{\mathscr{O}_L}(G)$, $\mathrm{Rep}_{\mathrm{tors}}(G)$) et d'une suite exacte $0 \to \Pi_1^* \to D \boxtimes_\delta \mathbb{P}^1 \to \Pi_2 \to 0$.

Soit $\mathscr{C}_{\mathrm{tors}}(\delta) \subset \Phi\Gamma^{\mathrm{et}}_{\mathrm{tors}}$ l'image de $\mathrm{Rep}_{\mathrm{tors}}(\delta)$ par le foncteur $\Pi \mapsto D(\Pi)$. On définit de manière analogue les catégories $\mathscr{C}_{\mathscr{O}_L}(\delta)$ et $\mathscr{C}_L(\delta)$. Il résulte du th. III.4 ci-dessous que les objets de ces catégories sont exactement les (φ, Γ)-modules D tels que (D, δ^{-1}) soit G-compatible.

11. La formule de [10, pag. 325] comporte quelques fautes de frappe.

Théorème III.4. *Si $\delta : \mathbb{Q}_p^* \to \mathscr{O}_L^*$ est un caractère unitaire, alors :*

(i) *$\mathscr{C}_{\rm tors}(\delta)$ est stable par sous-quotients.*

(ii) *Si $D \in \mathscr{C}_{\rm tors}(\delta)$, alors $\check{D} \in \mathscr{C}_{\rm tors}(\delta^{-1})$.*

(iii) *Il existe un foncteur covariant $D \mapsto \Pi_\delta(D)$ de $\mathscr{C}_{\rm tors}(\delta^{-1})$ dans* $\mathrm{Rep}_{\rm tors}(\delta)$ *tel que pour tout $D \in \mathscr{C}_{\rm tors}(\delta^{-1})$ on ait une suite exacte de G-modules topologiques*

$$0 \to \Pi_{\delta^{-1}}(\check{D})^* \to D \boxtimes_\delta \mathbb{P}^1 \to \Pi_\delta(D) \to 0.$$

(iv) *Les foncteurs $\Pi \mapsto D(\Pi)$ et $D \mapsto \Pi_\delta(\check{D})$ induisent des anti-équivalences quasi-inverses exactes entre* $\mathrm{Rep}_{\rm tors}(\delta)/S$ *et $\mathscr{C}_{\rm tors}(\delta)$, où S est la sous-catégorie de* $\mathrm{Rep}_{\rm tors}(\delta)$ *formée des représentations de type fini comme $\mathscr{O}_L$-module.*

(v) *Les résultats précédents restent valables si on remplace $\mathscr{C}_{\rm tors}$ par $\mathscr{C}_{\mathscr{O}_L}$ (resp. $\mathscr{C}_L$) et* $\mathrm{Rep}_{\rm tors}$ *par* $\mathrm{Rep}_{\mathscr{O}_L}$ *(resp.* Rep_L*) et $\mathscr{O}_L$ par $\mathscr{O}_L$ (resp. L) dans la définition de S.*

Remarque III.5. La suite exacte

$$0 \to \Pi_{\delta^{-1}}(\check{D})^* \to D \boxtimes_\delta \mathbb{P}^1 \to \Pi_\delta(D) \to 0$$

ne détermine pas uniquement $\Pi_\delta(D)$ et $\Pi_{\delta^{-1}}(\check{D})$ mais presque (en fait, si $D \in \mathscr{C}_{\rm tors}(\delta)$ (resp. $D \in \mathscr{C}_L(\delta)$) n'a pas de sous-quotient isomorphe à $k_{\mathscr{E}}(\eta)$ (resp. $\mathscr{E}(\eta)$), avec $\delta = \eta^2$ ou $\delta = \eta^2\chi^{-2}$, alors $\Pi_\delta(D)$ et $\Pi_{\delta^{-1}}(\check{D})$ sont uniquement déterminés par la suite exacte). Nous donnerons une construction explicite de ces représentations (déf. III.9) ce qui permet de restaurer l'unicité dans tous les cas.

La preuve de ce théorème occupe la quasi-totalité de ce chapitre. Les (i), (ii) et (iii) s'obtiennent en mélangeant le cor. III.22, les prop. III.29 et III.44, ainsi que les th. III.45 et III.49. Pour le (iv), il faut en plus utiliser la prop. III.31.

III.D. ***Paires G-compatibles.*** Soit D un (φ, Γ)-module étale. L'application

$$x \mapsto \left(\mathrm{Res}_{\mathbb{Z}_p}\begin{pmatrix} p^n & 0 \\ 0 & 1\end{pmatrix} x\right)_{n\geq 0}$$

induit un isomorphisme

$$D \boxtimes_\delta \mathbb{Q}_p \cong \{(x_n)_{n\in\mathbb{N}} \mid x_n \in D \text{ et } \psi(x_{n+1}) = x_n\},$$

et on munit $D \boxtimes_\delta \mathbb{Q}_p$ de la topologie induite par la topologie produit sur $D^{\mathbb{N}}$.

Remarque III.6. Comme on passe de $\mathbb{P}^1$ à $\mathbb{Q}_p$ en n'enlevant qu'un point, la restriction à $\mathbb{Q}_p$ est presque injective [10, prop. II.1.14] :

$$\mathrm{Ker}\left(\mathrm{Res}_{\mathbb{Q}_p} : D \boxtimes_\delta \mathbb{P}^1 \to D \boxtimes_\delta \mathbb{Q}_p\right) = \{(0, z_2) \mid z_2 \in D^{\rm nr}\}.$$

Si $? \in \{\natural, \sharp\}$, on pose

$$D^? \boxtimes_\delta \mathbb{Q}_p = (D \boxtimes_\delta \mathbb{Q}_p) \cap (D^?)^{\mathbb{N}}.$$

Si $D \in \Phi\Gamma^{\mathrm{et}}(\mathscr{O}_{\mathscr{E}}) \cup \Phi\Gamma^{\mathrm{et}}_{\mathrm{tors}}$, c'est un sous-module compact de $D \boxtimes_\delta \mathbb{Q}_p$.

Proposition III.7. *Soit* $D \in \Phi\Gamma^{\mathrm{et}}_{\mathrm{tors}} \cup \Phi\Gamma^{\mathrm{et}}(\mathscr{O}_{\mathscr{E}})$.

(i) *Si M est un sous $\mathscr{O}_L$-module fermé de $D \boxtimes_\delta \mathbb{Q}_p$, stable par P, il existe un sous-objet D_1 de D tel que*

$$D_1^\natural \boxtimes_\delta \mathbb{Q}_p \subset M \subset D_1^\sharp \boxtimes_\delta \mathbb{Q}_p.$$

En particulier, $M \subset D^\sharp \boxtimes_\delta \mathbb{Q}_p$ et $D^\natural \boxtimes_\delta \mathbb{Q}_p \subset M$, si $\mathrm{Res}_{\mathbb{Z}_p}(M)$ engendre D en tant que (φ, Γ)-module.

(ii) *$(D^\sharp \boxtimes_\delta \mathbb{Q}_p)/(D^\natural \boxtimes_\delta \mathbb{Q}_p)$ est isomorphe à $D^\sharp/D^\natural$ et est de type fini sur $\mathscr{O}_L$.*

(iii) *Le foncteur $D \mapsto D^\sharp \boxtimes_\delta \mathbb{Q}_p$ est exact.*

Démonstration. Le (i) correspond au th. III.3.8 de [9] (noter que le caractère δ ne joue aucun rôle quand on considère la restriction à P). Le (ii) correspond aux prop. III.3.1 et cor. III.3.2 de [9], et le (iii) au th. III.3.5 de [9]. □

On définit des sous-B-modules (fermés si $D \in \Phi\Gamma^{\mathrm{et}}_{\mathrm{tors}} \cup \Phi\Gamma^{\mathrm{et}}(\mathscr{O}_{\mathscr{E}})$) de $D \boxtimes_\delta \mathbb{P}^1$ par

$$D^\sharp \boxtimes_\delta \mathbb{P}^1 = \mathrm{Res}_{\mathbb{Q}_p}^{-1}(D^\sharp \boxtimes_\delta \mathbb{Q}_p), \quad (D^\natural \boxtimes_\delta \mathbb{P}^1)_{\mathrm{ns}} = \mathrm{Res}_{\mathbb{Q}_p}^{-1}(D^\natural \boxtimes_\delta \mathbb{Q}_p).$$

On pose $D^\natural \boxtimes_\delta \mathbb{P}^1 = (D^\natural \boxtimes_\delta \mathbb{P}^1)_{\mathrm{ns}}$ si $D \in \Phi\Gamma^{\mathrm{et}}_{\mathrm{tors}} \cup \Phi\Gamma^{\mathrm{et}}(\mathscr{E})$, et [12] on définit $D^\natural \boxtimes_\delta \mathbb{P}^1$ comme le saturé du $\mathscr{O}_L$-module $(D^\natural \boxtimes_\delta \mathbb{P}^1)_{\mathrm{ns}}$ si $D \in \Phi\Gamma^{\mathrm{et}}(\mathscr{O}_{\mathscr{E}})$.

Remarque III.8. $(D^\sharp \boxtimes_\delta \mathbb{P}^1)/(D^\natural \boxtimes_\delta \mathbb{P}^1)$ et $(D^\natural \boxtimes_\delta \mathbb{P}^1)/(D^\natural \boxtimes_\delta \mathbb{P}^1)_{\mathrm{ns}}$ sont de type fini (sur $\mathscr{O}_L$ ou L suivant les cas) et sont nuls si $D^\natural = D^\sharp$. En effet, $D^\sharp \boxtimes_\delta \mathbb{Q}_p$ est saturé, et donc $D^\sharp \boxtimes_\delta \mathbb{P}^1$ aussi, ce qui fait que

$$D^\natural \boxtimes_\delta \mathbb{P}^1 \subset D^\sharp \boxtimes_\delta \mathbb{P}^1$$

et que $(D^\sharp \boxtimes_\delta \mathbb{P}^1)/(D^\natural \boxtimes_\delta \mathbb{P}^1)$ est un quotient de $(D^\sharp \boxtimes_\delta \mathbb{P}^1)/(D^\natural \boxtimes_\delta \mathbb{P}^1)_{\mathrm{ns}}$. Or, par définition, $\mathrm{Res}_{\mathbb{Q}_p}$ induit une injection

$$(D^\sharp \boxtimes_\delta \mathbb{P}^1)/(D^\natural \boxtimes_\delta \mathbb{P}^1)_{\mathrm{ns}} \to (D^\sharp \boxtimes_\delta \mathbb{Q}_p)/(D^\natural \boxtimes_\delta \mathbb{Q}_p) \cong D^\sharp/D^\natural$$

(cf. (ii) de la prop. III.7).

Proposition-définition III.9. (D, δ) *est G-compatible si et seulement si le module $D^\natural \boxtimes_\delta \mathbb{P}^1$ est stable par G. Dans ce cas, on pose*

$$\Pi_\delta(D) = (D \boxtimes_\delta \mathbb{P}^1)/(D^\natural \boxtimes_\delta \mathbb{P}^1).$$

12. Le sous-module $(D^\natural \boxtimes_\delta \mathbb{P}^1)_{\mathrm{ns}}$ de $D \boxtimes_\delta \mathbb{P}^1$ n'est pas forcément saturé p-adiquement, voir la rem. VII.4.28 de [10].

Démonstration. Que (D, δ) soit G*-compatible* si et seulement si $D^\natural \boxtimes_\delta \mathbb{P}^1$ est stable par G résulte de [10, lemme II.2.5]. □

Remarque III.10. (i) Si $f : D_1 \to D_2$ est un morphisme dans une des catégories $\Phi\Gamma^{\rm et}_{\rm tors}$, $\Phi\Gamma^{\rm et}(\mathscr{O}_{\mathscr{E}})$, $\Phi\Gamma^{\rm et}(\mathscr{E})$, alors f induit un morphisme équivariant du faisceau attaché à (D_1, δ) dans le faisceau attaché à (D_2, δ) (cela résulte de la construction du faisceau $D \to D \boxtimes_\delta U$). En particulier, f induit des morphismes de G-modules (resp. B-modules) topologiques $f : D_1 \boxtimes_\delta \mathbb{P}^1 \to D_2 \boxtimes_\delta \mathbb{P}^1$ (resp. $f : D_1 \boxtimes_\delta \mathbb{Q}_p \to D_2 \boxtimes_\delta \mathbb{Q}_p$). Si $? \in \{\sharp, \natural\}$, alors f envoie $D_1^?$ dans $D_2^?$ et donc f envoie $D_1^? \boxtimes_\delta \mathbb{Q}_p$ dans $D_2^? \boxtimes_\delta \mathbb{Q}_p$ et $D_1^? \boxtimes_\delta \mathbb{P}^1$ dans $D_2^? \boxtimes_\delta \mathbb{P}^1$. Il en résulte que si (D_1, δ) et (D_2, δ) sont G-compatibles, alors f induit un morphisme G-équivariant de $\Pi_\delta(D_1)$ dans $\Pi_\delta(D_2)$.

(ii) Si (D, δ) est une paire G-compatible avec $D \in \Phi\Gamma^{\rm et}(\mathscr{O}_{\mathscr{E}})$ et si $(D^\natural \boxtimes_\delta \mathbb{P}^1)_{\rm ns}$ n'est pas saturé (ce qui se produit rarement ; mais voir [10, rem. VII.4.28] pour un exemple), il n'y a pas de morphisme naturel $\Pi_\delta(D) \to \Pi_\delta(D/\pi_L)$, mais seulement un morphisme

$$(D \boxtimes_\delta \mathbb{P}^1)/(D^\natural \boxtimes_\delta \mathbb{P}^1)_{\rm ns} \to \Pi_\delta(D/\pi_L).$$

(iii) Si (D, δ) est une paire G-compatible avec $D \in \Phi\Gamma^{\rm et}(\mathscr{E})$ et si D_0 est un réseau stable par φ et Γ dans D, alors (D_0, δ) est G-compatible et $\Pi_\delta(D_0)$ est un réseau ouvert, borné et G-stable dans $\Pi_\delta(D)$.

Proposition III.11. *Si (D, δ) est G-compatible, alors $\Pi_\delta(D)$ est un objet de* $\mathrm{Rep}_{\rm tors}(\mathrm{G})$, $\mathrm{Rep}_{\mathscr{O}_{\rm L}}(\mathrm{G})$ *ou* $\mathrm{Rep}_{\rm L}(\mathrm{G})$, *suivant que $D \in \Phi\Gamma^{\rm et}_{\rm tors}$, $D \in \Phi\Gamma^{\rm et}(\mathscr{O}_{\mathscr{E}})$, ou $D \in \Phi\Gamma^{\rm et}(\mathscr{E})$.*

Démonstration. Cf. [10, lemme II.2.10] : la seule difficulté est de prouver que les objets obtenus sont (résiduellement) de longueur finie (voir la prop. III.28 pour une justification de cette finitude). □

Remarque III.12. Soit (D, δ) une paire G-compatible.

(i) Si $D \in \Phi\Gamma^{\rm et}_{\rm tors} \cup \Phi\Gamma^{\rm et}(\mathscr{O}_{\mathscr{E}})$, alors $D^? \boxtimes_\delta \mathbb{P}^1$ est compact, si $? \in \{\natural, \sharp\}$. En effet, $z \mapsto (\mathrm{Res}_{\mathbb{Z}_p} z, \mathrm{Res}_{\mathbb{Z}_p} w \cdot z)$ permet d'identifier $D^\sharp \boxtimes_\delta \mathbb{P}^1$ à un sous-module fermé de $D^\sharp \times D^\sharp$, ce qui prouve qu'il est compact. Le même argument montre que $(D^\natural \boxtimes_\delta \mathbb{P}^1)_{\rm ns}$ est compact, et la rem. III.8 permet d'en déduire le résultat pour $D^\natural \boxtimes_\delta \mathbb{P}^1$.

(ii) Si $D \in \Phi\Gamma^{\rm et}_{\rm tors} \cup \Phi\Gamma^{\rm et}(\mathscr{O}_{\mathscr{E}})$ est non nul, alors $\Pi_\delta(D)$ n'est pas de type fini comme $\mathscr{O}_L$-module. En effet $D^\natural \boxtimes_\delta \mathbb{P}^1$ est compact et donc son intersection M avec $D = D \boxtimes_\delta \mathbb{Z}_p$ aussi, ainsi que l'image $\overline{M}$ de M dans $k_L \otimes D$. Il en résulte que $(k_L \otimes D)/\overline{M}$ est de dimension infinie sur k_L et donc que l'image de D dans $\Pi_\delta(D)$ n'est pas de type fini sur $\mathscr{O}_L$.

(iii) Si $D \in \Phi\Gamma^{\rm et}(\mathscr{E})$ est non nul, alors $\Pi_\delta(D)$ est de dimension infinie sur L (cela résulte du (ii) en tensorisant par L).

Proposition III.13. *Soit D un (φ, Γ)-module étale et soient $\delta, \eta : \mathbb{Q}_p^* \to \mathscr{O}_L^*$ des caractères unitaires. Si (D, δ) est une paire G-compatible, il en est de même de $(D(\eta), \delta\eta^2)$ et on a un isomorphisme de G-modules de Banach*

$$\Pi_{\delta\eta^2}(D(\eta)) \simeq \Pi_\delta(D) \otimes (\eta \circ \det).$$

Démonstration. C'est une conséquence de l'isomorphisme (cf. [10, prop. II.1.11]) $D(\eta) \boxtimes_{\delta\eta^2} \mathbb{P}^1 \cong (D \boxtimes_\delta \mathbb{P}^1) \otimes (\eta \circ \det)$. □

Proposition III.14. *Si $\Lambda \in \{k_{\mathscr{E}}, \mathscr{E}\}$ et si D est de rang 1 sur Λ, alors (D, δ) est G-compatible pour tout δ. Plus précisément, si δ_1, δ_2 sont deux caractères unitaires :*

(i) *On a un isomorphisme de G-modules topologiques*

$$\Lambda(\delta_1)^\natural \boxtimes_{\delta_1\delta_2\chi^{-1}} \mathbb{P}^1 \cong (\mathrm{Ind}(\delta_1 \otimes \delta_2\chi^{-1}))^* \otimes (\delta_1\delta_2\chi^{-1} \circ \det).$$

(ii) *L'application $z \mapsto \phi_z$, avec $\phi_z(g) = \mathrm{res}_0\left((\mathrm{Res}_{\mathbb{Z}_p}(wgz)\frac{dT}{1+T}\right)$, induit un isomorphisme de G-modules topologiques*

$$\Pi_{\delta_1\delta_2\chi^{-1}}(\Lambda(\delta_1)) \cong \mathrm{Ind}(\delta_2 \otimes \delta_1\chi^{-1}).$$

Démonstration. Il s'agit d'une traduction de l'analyse fonctionnelle sur $\mathbb{Z}_p$: voir la rem. II.1.1 de [10] ou la prop. 4.9 de [12]. □

Théorème III.15. (i) *Si D est de rang 2 et si δ_D est le caractère χ^{-1} det D, alors (D, δ_D) est G-compatible et si D est absolument irréductible, alors δ_D est l'unique caractère δ de $\mathbb{Q}_p^*$ tel que (D, δ) soit G-compatible.*

(ii) *Si D est absolument irréductible de rang ≥ 3, alors (D, δ) n'est G-compatible pour aucun choix de δ.*

Démonstration. La G-compatibilité de (D, δ_D) est le résultat principal du chap. IV de [10] (à part le cas où $p = 2$ et $\bar{D}^{\rm ss}$ est la somme de deux caractères égaux pour lequel on a besoin des résultats de [13]). Le reste de l'énoncé est une conséquence de [25] si $p \geq 5$, et de [13] en général. □

Remarque III.16. La G-compatibilité de (D, δ_D) est valable, plus généralement, pour une déformation d'un (φ, Γ)-module de rang 2, i.e., pour un (φ, Γ)-module de rang 2 sur $A \otimes_L \mathscr{E}$ où A est une L-algèbre artinienne. Ce genre de considération joue d'ailleurs un grand rôle dans la démonstration de la G-compatibilité de (D, δ_D) pour $D \in \Phi\Gamma^{\rm et}(\mathscr{E})$.

Proposition III.17. (i) *Soit $D \in \Phi\Gamma^{\rm et}(\mathscr{O}_{\mathscr{E}})$. Alors (D, δ) est G-compatible si et seulement si $(D/p^kD, \delta)$ est G-compatible pour tout $k \geq 0$.*

(ii) *Une paire (D, δ) est G-compatible si et seulement si $D^\sharp \boxtimes_\delta \mathbb{P}^1$ est stable par G.*

(iii) *Si* $D \in \Phi\Gamma^{\rm et}_{\rm tors} \cup \Phi\Gamma^{\rm et}(\mathscr{O}_{\mathscr{E}})$, *et si* (D, δ) *est* G*-compatible, le module* $D^\sharp \boxtimes_\delta \mathbb{P}^1$ *est le plus grand sous-module compact de* $D \boxtimes_\delta \mathbb{P}^1$ *stable par* G.

Démonstration. Ces énoncés sont contenus dans la prop. II.2.6 de [10] (et sa preuve). □

III.E. ***Le sous-module*** $\tilde{\boldsymbol{D}}^+$ ***de*** $\boldsymbol{D}^\natural \boxtimes_\delta \mathbb{P}^1$**.** On note $\tilde{\boldsymbol{E}}_{\mathbb{Q}_p}$ le complété de la clôture radicielle de $\mathbb{F}_p((T))$. Il est muni d'actions continues de φ et Γ (on a $\varphi(T) = T^p$ et $\sigma_a(T) = (1+T)^a - 1$). Soit $\tilde{\boldsymbol{A}}_{\mathbb{Q}_p} = W(\tilde{\boldsymbol{E}}_{\mathbb{Q}_p})$ l'anneau des vecteurs de Witt à coefficients dans $\tilde{\boldsymbol{E}}_{\mathbb{Q}_p}$. Si $x \in \tilde{\boldsymbol{E}}_{\mathbb{Q}_p}$, on note $[x]$ le représentant de Teichmüller de x dans $\tilde{\boldsymbol{A}}_{\mathbb{Q}_p}$. L'anneau $\tilde{\boldsymbol{A}}_{\mathbb{Q}_p}$ est naturellement muni d'actions de φ et Γ, que l'on étend par $\mathscr{O}_L$-linéarité à l'anneau $\tilde{\mathscr{O}}_{\mathscr{E}} := \mathscr{O}_L \otimes_{\mathbb{Z}_p} \tilde{\boldsymbol{A}}_{\mathbb{Q}_p}$. Si $b \in \mathbb{Q}_p$ et $n \geq 1$ est tel que $p^n b \in \mathbb{Z}_p$, on pose

$$[(1+T)^b] = \varphi^{-n}((1+T)^{p^n b}) = \varphi^{-n}\Big(\sum_{k=0}^{\infty}\binom{p^n b}{k}T^k\Big) \in \tilde{\mathscr{O}}_{\mathscr{E}}.$$

Cela ne dépend pas du choix de n et on a

$$[(1+T)^{b+c}] = [(1+T)^b]\cdot[(1+T)^c]$$

si $b, c \in \mathbb{Q}_p$.

Si D est un (φ, Γ)-module étale, on pose $\tilde{D} = \tilde{\mathscr{O}}_{\mathscr{E}} \otimes_{\mathscr{O}_{\mathscr{E}}} D$, que l'on munit d'une action du mirabolique P en posant, si $k \in \mathbb{Z}$, $a \in \mathbb{Z}_p^*$, $b \in \mathbb{Q}_p$,

$$\begin{pmatrix} p^k a & b \\ 0 & 1\end{pmatrix}\tilde{z} = [(1+T)^b]\varphi^k(\sigma_a(\tilde{z})).$$

Cette action laisse stable le sous-module $\tilde{D}^+$ de $\tilde{D}$, formé des $x \in \tilde{D}$ tels que la suite $(\varphi^n(x))_{n\in\mathbb{N}}$ soit bornée dans $\tilde{D}$.

Proposition III.18. *Si* $D \in \Phi\Gamma^{\rm et}_{\rm tors} \cup \Phi\Gamma^{\rm et}(\mathscr{E})$, *alors* $\tilde{D}/\tilde{D}^+$ *est un* $\mathscr{O}_L[B]$*-module* (*resp. un* $L[B]$*-module topologique*) *de longueur égale à celle de* D. *En particulier, si* D *est irréductible, il en est de même de* $\tilde{D}/\tilde{D}^+$ *comme* B*-module* (*topologique*).

Démonstration. C'est une reformulation de [9, prop. IV.5.6] et de sa preuve. □

Remarque III.19. La démonstration de la proposition citée repose sur le fait que $\tilde{D}/\tilde{D}^+$ est le dual de $\check{D}^\natural \boxtimes \mathbb{Q}_p$ (voir [9, prop. IV.5.4]), ce qui permet d'utiliser le (i) de la prop. III.7 (i.e., [9, th. III.3.8]) pour déterminer la longueur de $\tilde{D}/\tilde{D}^+$.

Soit $I_n = [0, 1[\cap (p^{-n}\mathbb{Z}_p \cap \mathbb{Q})$ et soit I la réunion croissante des I_n. C'est un système de représentants de $\mathbb{Q}_p/\mathbb{Z}_p$ et on montre [loc. cit., lemme IV.1.2] que tout élément z de $\tilde{D}$ s'écrit, de manière unique, sous la forme $z = \sum_{i\in I}[(1+T)^i]z_i$, avec $z_i \in D$ et $\lim_{i\in I} z_i = 0$. D'après [10, lemme II.1.16], la suite de terme général

$$\sum_{i\in I_n} \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} z_i \in D \boxtimes_\delta p^{-n}\mathbb{Z}_p \subset D \boxtimes_\delta \mathbb{P}^1$$

converge dans $D \boxtimes_\delta \mathbb{P}^1$ et on note $i(z)$ sa limite. On montre [9, lemme IV.2.2] que $i : \tilde{D} \to D \boxtimes_\delta \mathbb{P}^1$ est une injection P-équivariante, qui envoie $\tilde{D}^+$ dans $(D^\natural \boxtimes_\delta \mathbb{P}^1)_{\mathrm{ns}}$ car $\mathrm{Res}_{\mathbb{Z}_p} \tilde{D}^+ \subset D^\natural$. (Tout ceci ne suppose pas que (D, δ) soit G-compatible.)

III.F. *Dualité.* Si D est un (φ, Γ)-module étale, on étend [10, th. II.1.13] l'accouplement $\{\, ,\}$ sur $\check{D} \times D$ en un accouplement G-équivariant et parfait $\{\, ,\}_{\mathbb{P}^1}$ sur $(\check{D} \boxtimes_{\delta^{-1}} \mathbb{P}^1) \times (D \boxtimes_\delta \mathbb{P}^1)$, en posant

$$\{(\check{z}_1, \check{z}_2), (z_1, z_2)\}_{\mathbb{P}^1} = \{\check{z}_1, z_1\} + \{\psi(\check{z}_2), \psi(z_2)\}.$$

Dans cet accouplement, $\check{D} \boxtimes_{\delta^{-1}} U$ et $D \boxtimes_\delta V$ sont orthogonaux si U et V sont des ouverts compacts de $\mathbb{P}^1$ tels que $U \cap V = \varnothing$ (on se ramène à [9, prop. III.2.3] en utilisant la G-équivariance).

Lemme III.20. *Soit D un (φ, Γ)-module étale et $\delta : \mathbb{Q}_p^* \to \mathscr{O}_L^*$ un caractère unitaire. Alors l'orthogonal de $\tilde{D}^+$ dans $\check{D} \boxtimes_{\delta^{-1}} \mathbb{P}^1$ est inclus dans $\check{D}^\natural \boxtimes_{\delta^{-1}} \mathbb{P}^1$.*

Démonstration. Le cas $D \in \Phi\Gamma^{\mathrm{et}}(\mathscr{E})$ se déduit par tensorisation par L ; on suppose donc que $D \in \Phi\Gamma^{\mathrm{et}}_{\mathrm{tors}} \cup \Phi\Gamma^{\mathrm{et}}(\mathscr{O}_{\mathscr{E}})$. Soit N l'orthogonal de $\tilde{D}^+$ dans $\check{D} \boxtimes_{\delta^{-1}} \mathbb{P}^1$. Il est stable par P, car $\tilde{D}^+$ l'est.

Supposons que D est de torsion. Si $x = (x_1, x_2) \in N$, alors pour tout $y \in D^+ \subset \tilde{D}^+$ on a $\{x_1, y\} = \{x, y\}_{\mathbb{P}^1} = 0$, donc x_1 est orthogonal à D^+ et $x_1 \in \check{D}^\natural$. En appliquant ceci à $\left(\begin{smallmatrix} p^n & 0 \\ 0 & 1 \end{smallmatrix}\right) x$ pour tout $n \geq 0$, on obtient $x \in \check{D}^\natural \boxtimes_{\delta^{-1}} \mathbb{P}^1$ et donc $N \subset \check{D}^\natural \boxtimes_{\delta^{-1}} \mathbb{P}^1$.

Supposons maintenant que $D \in \Phi\Gamma^{\mathrm{et}}(\mathscr{O}_{\mathscr{E}})$ et soient $D_k = D/p^k D$ et c tel que p^c tue le conoyau de $\tilde{D}^+ \to \tilde{D}_k^+$ pour tout k (l'existence de c découle de [9, lemme IV.5.1]). Donc, si $x \in N$, alors $p^c x \pmod{p^k}$ est orthogonal à $\tilde{D}_k^+$ et, d'après le cas de torsion, on a $p^c x \pmod{p^k} \in \check{D}_k^\natural \boxtimes_{\delta^{-1}} \mathbb{P}^1$. En passant à la limite projective on obtient $p^c x \in (\check{D}^\natural \boxtimes_{\delta^{-1}} \mathbb{P}^1)_{\mathrm{ns}}$ et donc $x \in \check{D}^\natural \boxtimes_{\delta^{-1}} \mathbb{P}^1$, ce qui permet de conclure. □

Théorème III.21. *Si (D, δ) est une paire G-compatible, alors $\check{D}^\natural \boxtimes_{\delta^{-1}} \mathbb{P}^1$ est l'orthogonal de $\tilde{D}^+$ et de $D^\natural \boxtimes_\delta \mathbb{P}^1$ dans $\check{D} \boxtimes_{\delta^{-1}} \mathbb{P}^1$.*

Démonstration. En utilisant le lemme III.20 et l'inclusion $\tilde{D}^+ \subset D^\natural \boxtimes_\delta \mathbb{P}^1$, il suffit de montrer que $D^\natural \boxtimes_\delta \mathbb{P}^1$ est orthogonal à $\check{D}^\natural \boxtimes_{\delta^{-1}} \mathbb{P}^1$. Quitte à remplacer D par $D/p^k D$ et à passer à la limite, on peut supposer que D est de torsion.

Soit M l'orthogonal de $D^\natural \boxtimes_\delta \mathbb{P}^1$. Alors M est fermé (c'est un orthogonal) dans $\check{D}^\natural \boxtimes_{\delta^{-1}} \mathbb{P}^1$ (lemme III.20), et donc M est compact (rem. III.12). Il s'ensuit que $\mathrm{Res}_{\mathbb{Q}_p}(M)$ est un sous P-module compact de $\check{D}^\natural \boxtimes_\delta \mathbb{Q}_p$, et comme $\mathrm{Res}_{\mathbb{Z}_p}(M)$

contient $\check{D}^{++}$ (car M contient $\check{D}^{++} \subset \check{D}^{\natural} \boxtimes_{\delta^{-1}} \mathbb{P}^1$), qui engendre $\check{D}$, on en déduit (cf. le (i) de la prop. III.7) que $\mathrm{Res}_{\mathbb{Q}_p}(M) = \check{D}^{\natural} \boxtimes_{\delta} \mathbb{Q}_p$ et donc (rem. III.6) $\check{D}^{\natural} \boxtimes_{\delta^{-1}} \mathbb{P}^1 \subset M + (0, \check{D}^{\mathrm{nr}})$. Ainsi, $\check{D}^{\natural} \boxtimes_{\delta^{-1}} \mathbb{P}^1$ est lui-même compact [13] car $(0, \check{D}^{\mathrm{nr}})$ est de longueur finie sur $\mathscr{O}_L$ (prop. II.6). Le même argument montre alors que $\mathrm{Res}_{\mathbb{Q}_p}(\check{M}) = D^{\natural} \boxtimes_{\delta} \mathbb{Q}_p$, si $\check{M}$ est l'orthogonal de $\check{D}^{\natural} \boxtimes_{\delta^{-1}} \mathbb{P}^1$.

On a donc $\check{D}^{\natural} \boxtimes_{\delta} \mathbb{P}^1 \subset M + (0, \check{D}^{\mathrm{nr}})$ et $D^{\natural} \boxtimes_{\delta} \mathbb{P}^1 \subset \check{M} + (0, D^{\mathrm{nr}})$, et il reste à voir que $M + (0, \check{D}^{\mathrm{nr}})$ est orthogonal à $\check{M} + (0, D^{\mathrm{nr}})$. Or, on a vu que $\check{M} + (0, D^{\mathrm{nr}}) \subset D^{\natural} \boxtimes_{\delta} \mathbb{P}^1$ et, par définition, M est orthogonal à $D^{\natural} \boxtimes_{\delta} \mathbb{P}^1$, donc M est orthogonal à $\check{M} + (0, D^{\mathrm{nr}})$. En faisant la même chose avec $\check{M}$ et en utilisant le fait que D^{nr} est orthogonal à $\check{D}^{\mathrm{nr}}$ car $\check{D}^{\mathrm{nr}} \subset \check{D}^{+}$ et $D^{\mathrm{nr}} \subset D^{\natural}$, cela permet de conclure. □

Corollaire III.22. *Soit (D, δ) une paire G-compatible. Alors :*

(i) *$(\check{D}, \delta^{-1})$ est G-compatible.*

(ii) *On a un isomorphisme de G-modules topologiques $\Pi_{\delta}(D)^* \simeq \check{D}^{\natural} \boxtimes_{\delta^{-1}} \mathbb{P}^1$ et $\tilde{\check{D}}^{+}$ est dense [14] dans $\Pi_{\delta}(D)^*$.*

(iii) *On a une suite exacte de G-modules topologiques*

$$0 \to \Pi_{\delta^{-1}}(\check{D})^* \to D \boxtimes_{\delta} \mathbb{P}^1 \to \Pi_{\delta}(D) \to 0.$$

Démonstration. Cela découle du théorème précédent et du fait que $\{\ ,\ \}_{\mathbb{P}^1}$ est G-équivariant (pour le (i)) et parfait (pour le reste). □

III.G. *Un modèle de* $\Pi_{\delta}(D)$. L'application $\mathrm{Res}_{\mathbb{Q}_p} : D \boxtimes_{\delta} \mathbb{P}^1 \to D \boxtimes_{\delta} \mathbb{Q}_p$ est B-équivariante et son noyau est inclus dans $(D^{\natural} \boxtimes_{\delta} \mathbb{P}^1)_{\mathrm{ns}}$ par définition de ce module. Si $D \in \Phi\Gamma^{\mathrm{et}}(\mathscr{E})$ et si $? \in \{\natural, \sharp\}$, on pose $(D^{?} \boxtimes_{\delta} \mathbb{Q}_p)_{\mathrm{b}} = (D_0^{?} \boxtimes_{\delta} \mathbb{Q}_p) \otimes_{\mathscr{O}_L} L$, pour n'importe quel réseau D_0 de D, stable par φ et Γ.

Proposition III.23. *Soit (D, δ) une paire G-compatible. Posons $X = D^{\natural} \boxtimes_{\delta} \mathbb{Q}_p$ si $D \in \Phi\Gamma^{\mathrm{et}}_{\mathrm{tors}} \cup \Phi\Gamma^{\mathrm{et}}(\mathscr{O}_{\mathscr{E}})$ et $X = (D^{\natural} \boxtimes_{\delta} \mathbb{Q}_p)_{\mathrm{b}}$ si $D \in \Phi\Gamma^{\mathrm{et}}(\mathscr{E})$. Alors* $\mathrm{Res}_{\mathbb{Q}_p}$ *induit une suite exacte*

$$0 \to (0, D^{\mathrm{nr}}) \to (D^{\natural} \boxtimes_{\delta} \mathbb{P}^1)_{\mathrm{ns}} \to X \to 0.$$

Démonstration. L'exactitude à gauche est immédiate, celle au milieu résulte de la rem. III.6. Par définition, $\mathrm{Res}_{\mathbb{Q}_p}((D^{\natural} \boxtimes_{\delta} \mathbb{P}^1)_{\mathrm{ns}}) \subset X$. Pour démontrer la surjectivité, on peut supposer que $D \in \Phi\Gamma^{\mathrm{et}}_{\mathrm{tors}} \cup \Phi\Gamma^{\mathrm{et}}(\mathscr{O}_{\mathscr{E}})$. Alors $(D^{\natural} \boxtimes_{\delta} \mathbb{P}^1)_{\mathrm{ns}}$ est compact (rem. III.12) donc son image par $\mathrm{Res}_{\mathbb{Q}_p}$ est un sous-$P(\mathbb{Q}_p)$-module compact de $D^{\natural} \boxtimes_{\delta} \mathbb{Q}_p$, qui contient $\tilde{D}^{+}$; le (i) de la prop. III.7 permet donc de montrer que cette image contient $D^{\natural} \boxtimes_{\delta} \mathbb{Q}_p$. □

13. Cela n'a rien de trivial à cet instant, car nous ne savons pas encore que $(\check{D}, \delta^{-1})$ est G-compatible. C'est d'ailleurs ce que nous cherchons à montrer…

14. Rappelons que tous les duaux sont munis de la topologie faible.

Corollaire III.24. *L'application* $\mathrm{Res}_{\mathbb{Z}_p} : D^\natural \boxtimes_\delta \mathbb{P}^1 \to D^\natural$ *est surjective.*

Démonstration. D'après la prop. III.23 il suffit de prouver la surjectivité de l'application $\mathrm{Res}_{\mathbb{Z}_p} : X \to D^\natural$. Le cas $D \in \Phi\Gamma^{\mathrm{et}}(\mathscr{E})$ se déduit du cas $D \in \Phi\Gamma^{\mathrm{et}}(\mathscr{O}_{\mathscr{E}})$ en tensorisant par L. Le cas $D \in \Phi\Gamma^{\mathrm{et}}_{\mathrm{tors}} \cup \Phi\Gamma^{\mathrm{et}}(\mathscr{O}_{\mathscr{E}})$ découle de l'isomorphisme $X \simeq \varprojlim_\psi D^\natural$ et du fait que ψ est surjectif sur $D^\natural$. □

Corollaire III.25. *Soit* (D, δ) *une paire* G*-compatible telle que* $\check{D}^{\mathrm{nr}} = 0$.

(i) *Si* $D \in \Phi\Gamma^{\mathrm{et}}_{\mathrm{tors}}$, *on a un isomorphisme* $\Pi_\delta(D)^* \simeq \check{D}^\natural \boxtimes_{\delta^{-1}} \mathbb{Q}_p$ *de* B*-modules compacts.*

(ii) *Si* $D \in \Phi\Gamma^{\mathrm{et}}(\mathscr{E})$, *on a un isomorphisme* $\Pi_\delta(D)^* \simeq (\check{D}^\natural \boxtimes_{\delta^{-1}} \mathbb{Q}_p)_{\mathrm{b}}$ *de* B*-modules topologiques.*

Démonstration. Cela découle du cor. III.22 et de la prop. III.23. □

Corollaire III.26. *Si* (D, δ) *est* G*-compatible, avec* $D \in \Phi\Gamma^{\mathrm{et}}_{\mathrm{tors}} \cup \Phi\Gamma^{\mathrm{et}}(\mathscr{E})$, *alors l'inclusion de* $\tilde{D}$ *dans* $D \boxtimes_\delta \mathbb{P}^1$ *induit une suite exacte de* B*-modules topologiques*

$$0 \to \tilde{D}/\tilde{D}^+ \to \Pi_\delta(D) \to D^\sharp/D^\natural \to 0.$$

Démonstration. Commençons par le cas de torsion. Alors $D^\sharp/D^\natural$ est le dual (de Pontryagin) de $\check{D}^{\mathrm{nr}}$ et $\tilde{D}/\tilde{D}^+$ est le dual de $\check{D}^\natural \boxtimes_{\delta^{-1}} \mathbb{Q}_p$ (rem. III.19). En utilisant le th. III.21, on voit que la suite exacte demandée est obtenue en dualisant la suite exacte

$$0 \to (0, \check{D}^{\mathrm{nr}}) \to \check{D}^\natural \boxtimes_{\delta^{-1}} \mathbb{P}^1 \to \check{D}^\natural \boxtimes_\delta \mathbb{Q}_p \to 0$$

de la prop. III.23, ce qui permet de conclure. □

Supposons que $D \in \Phi\Gamma^{\mathrm{et}}(\mathscr{O}_{\mathscr{E}})$ et posons $D_k = D/p^k D$. Alors $D^\sharp/D^\natural$ est la limite projective des $D_k^\sharp/D_k^\natural$ et $\tilde{D}/\tilde{D}^+$ est la limite projective des $\tilde{D}_k/\tilde{D}_k^+$ ([9, lemme IV.5.3] pour ce dernier). L'application naturelle $\tilde{D}_{k+1} \to \tilde{D}_k$ est surjective, car D_{k+1} se surjecte sur D_k. Il en est donc de même de l'application $\tilde{D}_{k+1}/\tilde{D}_{k+1}^+ \to \tilde{D}_k/\tilde{D}_k^+$. Ainsi, en passant à la limite dans

$$0 \to \tilde{D}_k/\tilde{D}_k^+ \to \Pi_\delta(D_k) \to D_k^\sharp/D_k^\natural \to 0$$

on obtient bien une suite exacte

$$0 \to \tilde{D}/\tilde{D}^+ \to \varprojlim \Pi_\delta(D_k) \to D^\sharp/D^\natural \to 0.$$

On a $\Pi_\delta(L \otimes D) = L \otimes \Pi_\delta(D)$ puisque $\Pi_\delta(D)$ est le quotient de $\varprojlim \Pi_\delta(D_k)$ par son $\mathscr{O}_L$-module de torsion ; on en déduit le résultat pour un objet de $\Phi\Gamma^{\mathrm{et}}(\mathscr{E})$.

Remarque III.27. Dans le cas $D \in \Phi\Gamma^{\mathrm{et}}(\mathscr{O}_{\mathscr{E}})$, il résulte de la preuve ci-dessus qu'il faut modifier la suite exacte de la proposition en remplaçant $\Pi_\delta(D)$ par $\varprojlim \Pi_\delta(D_k)$.

Proposition III.28. *Si* (D, δ) *est* G*-compatible, avec* $D \in \Phi\Gamma^{\rm et}_{\rm tors} \cup \Phi\Gamma^{\rm et}(\mathscr{E})$*, alors* $\Pi_\delta(D)$ *est un* B*-module (topologiquement) de longueur finie, et donc aussi un* G*-module (topologiquement) de longueur finie.*

Démonstration. C'est une conséquence du cor. III.26, de la finitude de la longueur de $\tilde{D}/\tilde{D}^+$ (prop. III.18) et de celle de $D^\sharp/D^\natural$ (prop. II.6). □

III.H. *Presque exactitude de $D \to \Pi_\delta(D)$.* Si (D_1, δ) et (D_2, δ) sont des paires G-compatibles (noter que δ est le même dans les deux paires) et si $f : D_1 \to D_2$ est un morphisme de (φ, Γ)-modules, la rem. III.10 montre que f induit un morphisme G-équivariant continu $f : \Pi_\delta(D_1) \to \Pi_\delta(D_2)$.

Proposition III.29. *Soit* $0 \to D_1 \to D \to D_2 \to 0$ *une suite exacte dans une des catégories* $\Phi\Gamma^{\rm et}_{\rm tors}$, $\Phi\Gamma^{\rm et}(\mathscr{O}_{\mathscr{E}})$, $\Phi\Gamma^{\rm et}(\mathscr{E})$*. Si* (D, δ) *est une paire* G*-compatible, alors* (D_1, δ) *et* (D_2, δ) *sont des paires* G*-compatibles.*

Démonstration. Pour montrer que (D_1, δ) est G-compatible, suffit (prop. III.17) de montrer que [15] $D_1^\sharp \boxtimes_\delta \mathbb{P}^1$ est stable par G, ce qui résulte de ce que $D_1^\sharp \boxtimes_\delta \mathbb{P}^1 = (D_1 \boxtimes_\delta \mathbb{P}^1) \cap (D^\sharp \boxtimes_\delta \mathbb{P}^1)$ par exactitude du foncteur $D \mapsto D^\sharp \boxtimes_\delta \mathbb{Q}_p$ (prop. III.7).

Pour montrer que (D_2, δ) est G-compatible, on dualise la suite exacte $0 \to D_1 \to D \to D_2 \to 0$ et on obtient une suite exacte $0 \to \check{D}_2 \to \check{D} \to \check{D}_1 \to 0$. On conclut alors en utilisant ce que l'on vient de démontrer et le cor. III.22. □

Remarque III.30. La réciproque de la prop. III.29 est presque toujours fausse, la G-compatibilité étant une contrainte très forte.

Proposition III.31. *Soit* $0 \to D_1 \to D \to D_2 \to 0$ *une suite exacte dans* $\Phi\Gamma^{\rm et}_{\rm tors}$ *(resp.* $\Phi\Gamma^{\rm et}(\mathscr{O}_{\mathscr{E}})$, $\Phi\Gamma^{\rm et}(\mathscr{E})$*). Si* (D, δ) *est une paire* G*-compatible, les groupes de cohomologie du complexe*

$$0 \to \Pi_\delta(D_1) \to \Pi_\delta(D) \to \Pi_\delta(D_2) \to 0$$

sont des $\mathscr{O}_L$*-modules de longueur finie (resp. de type fini sur* $\mathscr{O}_L$*, de dimension finie sur* L*).*

Démonstration. Commençons par traiter le cas de (φ, Γ)-modules sur $\mathscr{O}_{\mathscr{E}}$. La suite $0 \to D_1 \boxtimes_\delta \mathbb{P}^1 \to D \boxtimes_\delta \mathbb{P}^1 \to D_2 \boxtimes_\delta \mathbb{P}^1 \to 0$ étant trivialement exacte, il suffit de prouver que la cohomologie du complexe $0 \to D_1^\natural \boxtimes_\delta \mathbb{P}^1 \to D^\natural \boxtimes_\delta \mathbb{P}^1 \to D_2^\natural \boxtimes_\delta \mathbb{P}^1 \to 0$ a les propriétés de finitude requises. L'exactitude du foncteur $D \mapsto D^\sharp \boxtimes_\delta \mathbb{Q}_p$ et le fait que $\Delta^\sharp \boxtimes_\delta \mathbb{Q}_p / \Delta^\natural \boxtimes_\delta \mathbb{Q}_p$ soit un sous-quotient de $\Delta^\sharp/\Delta^\natural$, et donc un $\mathscr{O}_L$-module de type fini si $\Delta \in \{D_1, D, D_2\}$, entraînent la finitude des groupes de cohomologie du complexe $0 \to D_1^\natural \boxtimes_\delta \mathbb{Q}_p \to D^\natural \boxtimes_\delta \mathbb{Q}_p \to D_2^\natural \boxtimes_\delta \mathbb{Q}_p \to 0$. Les suites exactes (pour $\Delta \in \{D_1, D, D_2\}$)

$$0 \to (0, \Delta^{\rm nr}) \to (\Delta^\natural \boxtimes_\delta \mathbb{P}^1)_{\rm ns} \to \Delta^\natural \boxtimes_\delta \mathbb{Q}_p \to 0$$

15. Si $D \in \Phi\Gamma^{\rm et}(\mathscr{E})$, remplacer $D^\sharp \boxtimes_\delta \mathbb{Q}_p$ par $(D^\sharp \boxtimes_\delta \mathbb{Q}_p)_{\rm b}$.

fournies par la prop. III.23 et la finitude des $\mathscr{O}_L$-modules Δ^{nr} montrent que les groupes de cohomologie du complexe $0 \to (D_1^\natural \boxtimes_\delta \mathbb{P}^1)_{\mathrm{ns}} \to (D^\natural \boxtimes_\delta \mathbb{P}^1)_{\mathrm{ns}} \to (D_2^\natural \boxtimes_\delta \mathbb{P}^1)_{\mathrm{ns}} \to 0$ sont de type fini sur $\mathscr{O}_L$.

- Si D_1, D, D_2 sont des objets de $\Phi\Gamma^{\mathrm{et}}_{\mathrm{tors}}$, cela permet de conclure.
- Si D_1, D, D_2 sont des objets de $\Phi\Gamma^{\mathrm{et}}(\mathscr{O}_{\mathscr{E}})$, on conclut en utilisant la finitude de $\Delta^\natural \boxtimes_\delta \mathbb{P}^1/(\Delta^\natural \boxtimes_\delta \mathbb{P}^1)_{\mathrm{ns}}$ (rem. III.8).
- Si D_1, D, D_2 sont des objets de $\Phi\Gamma^{\mathrm{et}}(\mathscr{E})$, il suffit de remplacer dans la démonstration $\Delta^? \boxtimes_\delta \mathbb{P}^1$ par $(\Delta^? \boxtimes_\delta \mathbb{P}^1)_{\mathrm{b}}$, si $\Delta = D_1, D, D_2$ et $? \in \{\natural, \sharp\}$. □

Remarque III.32. Si D_1, D, D_2 sont des objets de $\Phi\Gamma^{\mathrm{et}}_{\mathrm{tors}}$ ou de $\Phi\Gamma^{\mathrm{et}}(\mathscr{E})$, il résulte de la preuve que la suite $0 \to \Pi_\delta(D_1) \to \Pi_\delta(D) \to \Pi_\delta(D_2) \to 0$ est exacte si $D_j^{\mathrm{nr}} = 0$ et $\check{D}_j^{\mathrm{nr}} = 0$ pour $j = 1, 2$. En effet, dans ce cas Δ^{nr} et $\Delta^\natural/\Delta^\natural \cong (\check{\Delta}^{\mathrm{nr}})^*$ sont nuls si $\Delta \in \{D_1, D, D_2\}$, et donc les groupes de cohomologie du complexe sont nuls.

Proposition III.33. *Soit (D, δ) une paire G-compatible, avec $D \in \Phi\Gamma^{\mathrm{et}}_{\mathrm{tors}} \cup \Phi\Gamma^{\mathrm{et}}(\mathscr{E})$.*

(i) *Si $\Pi_\delta(D)$ est irréductible, alors D est irréductible.*

(ii) *Si D est de dimension ≥ 2, les assertions suivantes sont équivalentes :*

(a) *D est irréductible.*

(b) *$\Pi_\delta(D)$ est topologiquement irréductible comme G-module.*

(c) *$\Pi_\delta(D)$ est topologiquement irréductible comme B-module, B étant le Borel supérieur.*

Démonstration. (i) Si $0 \to D_1 \to D \to D_2 \to 0$ est une suite exacte dans $\Phi\Gamma^{\mathrm{et}}(\mathscr{E})$, la prop. III.31 et l'irréductibilité de $\Pi_\delta(D)$ montrent qu'une des représentations $\Pi_\delta(D_1)$ et $\Pi_\delta(D_2)$ est de dimension finie sur L. La rem. III.12 permet d'en déduire que $D_1 = 0$ ou $D_2 = 0$, et donc que D est irréductible.

(ii) Supposons que D est irréductible et montrons le (c). Comme $\dim_{\mathscr{E}} D \geq 2$, on a $D^\sharp = D^\natural$ et donc $\Pi_\delta(D) \simeq \tilde{D}/\tilde{D}^+$ en tant que B-modules topologique (cor. III.26). On conclut en utilisant la prop. III.18. L'implication (b) $\Longrightarrow$ (a) ayant été prouvée dans le (i), cela permet de conclure puisque l'implication (c) $\Longrightarrow$ (b) est triviale. □

Remarque III.34. Les conclusions de la proposition sont en défaut en rang 1.

(i) La représentation $\Pi_\delta(D)$ n'est jamais topologiquement irréductible comme B-module : il y a un quotient de dimension 1 puisque la représentation obtenue est une induite d'un caractère de B (prop. III.14).

(ii) Si $D = \mathscr{E}(\eta)$, et si $\delta = \eta^2$, alors $\Pi_\delta(D)$ n'est pas topologiquement irréductible comme G-module (il y a un sous-objet de dimension 1). Par contre, si $\delta \neq \eta^2$, alors $\Pi_\delta(D)$ est topologiquement irréductible.

Remarque III.35. Si (D,δ) est G-compatible avec $D\in\Phi\Gamma^{\rm et}_{\rm tors}\cup\Phi\Gamma^{\rm et}(\mathscr{E})$, et si $0\to D_1\to D\to D_2\to 0$ est exacte, les groupes de cohomologie de $0\to\Pi_\delta(D_1)\to\Pi_\delta(D)\to\Pi_\delta(D_2)\to 0$ sont des G-modules dont les composantes de Jordan–Hölder sont parmi celles de $D\boxtimes_\delta\mathbb{P}^1$. Comme, par ailleurs (prop. III.31), elles sont de type fini (sur $\mathscr{O}_L$ ou L), la suite $0\to\Pi_\delta(D_1)\to\Pi_\delta(D)\to\Pi_\delta(D_2)\to 0$ est exacte si $D\boxtimes_\delta\mathbb{P}^1$ n'a pas de composante de Jordan–Hölder de dimension finie (sur k_L ou sur L). En regroupant les résultats des prop. III.1, III.14, III.33 et du cor. III.22, on voit que c'est le cas si et seulement si D n'a pas de composante de Jordan–Hölder de la forme $k_{\mathscr{E}}(\eta)$ ou $L(\eta)$, avec $\delta=\eta^2$ ou $\delta=\eta^2\chi^{-2}$.

III.I. ***Invariants sous*** $\mathbf{SL_2(\mathbb{Q}_p)}$**.** Dans ce paragraphe on étudie les $\mathrm{SL}_2(\mathbb{Q}_p)$-invariants d'une représentation de $\mathrm{Rep}_{\rm tors}(G)$ ou $\mathrm{Rep}_L(G)$.

Lemme III.36. *Si $\Pi\in\mathrm{Rep}_{\rm tors}(G)$, alors $\Pi^{\mathrm{SL}_2(\mathbb{Q}_p)}$ est un $\mathscr{O}_L$-module de longueur finie.*

Démonstration. Soit $K_m=1+p^m\mathrm{M}_2(\mathbb{Z}_p)$. Comme Π est admissible, il suffit de montrer que $\Pi^{\mathrm{SL}_2(\mathbb{Q}_p)}\subset\Pi^{K_m}$ pour m assez grand. Soit δ un caractère central de Π et soit $n\geq 1$ tel que δ soit trivial sur $1+p^n\mathbb{Z}_p$. Si $x\in 1+p^{n+1}\mathbb{Z}_p$, il existe $y\in 1+p^n\mathbb{Z}_p$ tel que $x=y^2$. Si $v\in\Pi^{\mathrm{SL}_2(\mathbb{Q}_p)}$, alors

$$\begin{pmatrix}x&0\\0&1\end{pmatrix}v=\begin{pmatrix}y&0\\0&y\end{pmatrix}\begin{pmatrix}y&0\\0&y^{-1}\end{pmatrix}v=\delta(y)v=v.$$

Comme $K_{n+1}\subset\left(\begin{smallmatrix}1+p^{n+1}\mathbb{Z}_p&0\\0&1\end{smallmatrix}\right)\cdot\mathrm{SL}_2(\mathbb{Q}_p)$, on voit que l'on peut prendre $m=n+1$. □

Corollaire III.37. *Si $\Pi\in\mathrm{Rep}_{\mathscr{O}_L}(G)$ (resp. $\Pi\in\mathrm{Rep}_L(G)$), alors $\Pi^{\mathrm{SL}_2(\mathbb{Q}_p)}$ est un $\mathscr{O}_L$-module libre de type fini (resp. un L-espace vectoriel de dimension finie).*

Démonstration. On peut supposer que $\Pi\in\mathrm{Rep}_{\mathscr{O}_L}(G)$. Dans ce cas, le résultat est une conséquence du lemme précédent et du fait que $\Pi^{\mathrm{SL}_2(\mathbb{Q}_p)}$ est un $\mathscr{O}_L$-module saturé de Π. □

Remarque III.38. La démonstration ci-dessus utilise juste l'admissibilité. Or on a supposé que les représentations sont de longueur finie et toute composante de Jordan–Hölder de $\Pi^{\mathrm{SL}_2(\mathbb{Q}_p)}$ est de dimension au plus 2 sur k_L ou L car le sous-groupe engendré par $\mathrm{SL}_2(\mathbb{Q}_p)$ et le centre est d'indice 2 dans G. Cela prouve, non seulement que $\Pi^{\mathrm{SL}_2(\mathbb{Q}_p)}$ a les propriétés de finitude du lemme III.36 et du cor. III.37, mais aussi que $(\Pi^*)^{\mathrm{SL}_2(\mathbb{Q}_p)}$ a les mêmes propriétés.

Proposition III.39. *Soit $\Pi\in\mathrm{Rep}_{\mathscr{O}_L}(G)$. Si $\Pi^{\mathrm{SL}_2(\mathbb{Q}_p)}=0$, le $\mathscr{O}_L$-module*

$$((L/\mathscr{O}_L)\otimes\Pi)^{\mathrm{SL}_2(\mathbb{Q}_p)}$$

est de type fini, et donc inclus dans la p^n-torsion de $(L/\mathscr{O}_L)\otimes\Pi$ si n est assez grand.

Démonstration. Notons H le groupe $\mathrm{SL}_2(\mathbb{Q}_p)$ et Π_n la p^n-torsion de $(L/\mathscr{O}_L)\otimes\Pi$. Alors $\Pi_n\cong\Pi/p^n\Pi$ est un objet de $\mathrm{Rep}_{\mathrm{tors}}(G)$ et donc Π_n^H est de type fini sur $\mathscr{O}_L$ (lemme III.36). Il s'agit de prouver qu'il existe $n_0\in\mathbb{N}$ tel que $((L/\mathscr{O}_L)\otimes\Pi)^H=\Pi_{n_0}^H$. Dans le cas contraire, il existe une partie infinie I de $\mathbb{N}$ et, pour tout $n\in I$, un vecteur $v_n\in\Pi_n^H$ n'appartenant pas à Π_{n-1}. On peut donc trouver $x_n\in\Pi-p\Pi$ tel que $v_n=p^{-n}x_n \pmod{\Pi}$ et $gx_n-x_n\in p^n\Pi$ pour tout $g\in H$. Pour $n\in I\cap\,]j,\infty]$ on a $x_n \pmod{p^j}\in\Pi_j^H$, qui est un ensemble fini (lemme III.36). Par extraction diagonale, on obtient ainsi l'existence d'une sous-suite $(y_n)_n$ de $(x_n)_{n\in I}$ qui converge p-adiquement vers un $\alpha\in\Pi$. En passant à la limite dans la congruence $gx_n-x_n\in p^n\Pi$, on obtient $\alpha\in\Pi^H=0$. Mais cela contredit le fait que $y_n\notin p\Pi$ pour tout n, ce qui permet de conclure. □

Lemme III.40. *Soit M un $\mathscr{O}_L$-module tué par une puissance de p et muni d'une action $\mathscr{O}_L$-linéaire de $\mathrm{SL}_2(\mathbb{Q}_p)$. Alors $M/M^{\mathrm{SL}_2(\mathbb{Q}_p)}$ n'a pas de $\mathrm{SL}_2(\mathbb{Q}_p)$-invariants non triviaux.*

Démonstration. Il faut montrer que si $x\in M$ et $(g-1)(h-1)x=0$ pour tous $g,h\in\mathrm{SL}_2(\mathbb{Q}_p)$, alors $(g-1)x=0$ pour tout $g\in\mathrm{SL}_2(\mathbb{Q}_p)$. Pour $h=g^n$, on obtient $g^{n+1}(x)-g^n(x)=g(x)-x$, et donc $g^n(x)=n(g(x)-x)+x$ pour tous $g\in\mathrm{SL}_2(\mathbb{Q}_p)$ et $n\geq 0$. Par hypothèse il existe n qui tue M. On a alors $g^n(x)=x$ pour tout $g\in\mathrm{SL}_2(\mathbb{Q}_p)$. On conclut en utilisant le fait que $g\to g^n$ est bijective sur les sous-groupes unipotents de G, sous-groupes qui engendrent $\mathrm{SL}_2(\mathbb{Q}_p)$. □

Corollaire III.41. *Si $\Pi\in\mathrm{Rep}_L(G)$, alors $\Pi/\Pi^{\mathrm{SL}_2(\mathbb{Q}_p)}$ n'a pas de $\mathrm{SL}_2(\mathbb{Q}_p)$-invariants non triviaux.*

Démonstration. C'est une conséquence formelle du lemme III.40. □

Lemme III.42. (i) *Si $\Pi\in\mathrm{Rep}_{\mathrm{tors}}(G)\cup\mathrm{Rep}_{\mathscr{O}_L}(G)$ est un $\mathscr{O}_L$-module de type fini, alors $\Pi=\Pi^{\mathrm{SL}_2(\mathbb{Q}_p)}$.*

(ii) *Si $\Pi\in\mathrm{Rep}_L(G)$ est de dimension finie sur L, alors $\Pi=\Pi^{\mathrm{SL}_2(\mathbb{Q}_p)}$.*

Démonstration. Si $\Pi\in\mathrm{Rep}_{\mathrm{tors}}(G)$, cela découle de [10, lemme III.1.5]. Les autres cas s'en déduisent. □

Proposition III.43. *Si (D,δ) est G-compatible, avec $D\in\Phi\Gamma^{\mathrm{et}}_{\mathrm{tors}}\cup\Phi\Gamma^{\mathrm{et}}(\mathscr{O}_{\mathscr{E}})$ (resp. $D\in\Phi\Gamma^{\mathrm{et}}(\mathscr{E})$), alors*

$$\Pi_\delta(D)^{\mathrm{SL}_2(\mathbb{Q}_p)}=(D^\sharp\boxtimes_\delta\mathbb{P}^1)/(D^\natural\boxtimes_\delta\mathbb{P}^1)$$

et c'est la plus grande sous-représentation de type fini sur $\mathscr{O}_L$ (resp. de dimension finie sur L) de $\Pi_\delta(D)$.

Démonstration. On peut supposer que $D \in \Phi\Gamma^{\rm et}_{\rm tors} \cup \Phi\Gamma^{\rm et}(\mathscr{O}_{\mathscr{E}})$. D'après la prop. III.17, $X := (D^\sharp \boxtimes_\delta \mathbb{P}^1)/(D^\natural \boxtimes_\delta \mathbb{P}^1)$ est un sous-$\mathscr{O}_L[G]$-module de $\Pi_\delta(D)$, et il est de type fini sur $\mathscr{O}_L$ d'après la rem. III.8. On déduit du lemme III.42 que $X \subset \Pi_\delta(D)^{{\rm SL}_2(\mathbb{Q}_p)}$.

Pour montrer l'inclusion inverse, soit Y le sous-$\mathscr{O}_L$-module des $z \in D \boxtimes_\delta \mathbb{P}^1$ dont l'image dans $\Pi_\delta(D)$ est invariante par ${\rm SL}_2(\mathbb{Q}_p)$. Si $z \in Y$, alors $\big(\big(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\big)-1\big)x \in D^\natural \boxtimes_\delta \mathbb{P}^1$, pour $x \in \{z, wz\}$, et en appliquant ${\rm Res}_{\mathbb{Z}_p}$ on obtient ${\rm Res}_{\mathbb{Z}_p}(z), {\rm Res}_{\mathbb{Z}_p}(wz) \in (1/T)D^\natural$ car ${\rm Res}_{\mathbb{Z}_p}\big(\big(\big(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\big)-1\big)x\big) = T\ {\rm Res}_{\mathbb{Z}_p} x$. Donc Y est compact. Comme Y est stable par G, on obtient $Y \subset D^\sharp \boxtimes_\delta \mathbb{P}^1$ d'après le (ii) de la prop. III.17, ce qui permet de conclure. □

III.K. *Reconstruction de* Π. Le but de ce paragraphe est d'expliquer comment reconstruire Π à partir de $D(\Pi)$ (éventuellement à des morceaux près de type fini sur $\mathscr{O}_L$ ou L).

Soit $\Pi \in {\rm Rep}_{\rm tors}(\delta)$ et soit W comme dans le § III.B, dont on reprend les notations. La restriction à $P^+ \cdot W$ induit une injection de $D^+_W(\Pi)$ dans $D^\natural_W(\Pi)$, dont l'image est d'indice fini dans $D^\natural_W(\Pi)$ (cf. [10, lemme IV.1.4]). On a donc un isomorphisme de $\mathscr{O}_{\mathscr{E}}$-modules $D(\Pi) \simeq \mathscr{O}_{\mathscr{E}} \otimes_{\mathscr{O}_L[\![T]\!]} D^\natural_W(\Pi)$, ce qui permet de définir une application $\beta_{\mathbb{Z}_p} : \Pi^* \to D(\Pi)$, composée des

$$\Pi^* \to D^\natural_W(\Pi) \to \mathscr{O}_{\mathscr{E}} \otimes_{\mathscr{O}_L[\![T]\!]} D^\natural_W(\Pi) \simeq D(\Pi),$$

la première flèche étant la restriction à $P^+ \cdot W$, et la deuxième $x \mapsto 1 \otimes x$. On définit

$$\beta_{\mathbb{P}^1} : \Pi^* \to D(\Pi) \oplus D(\Pi), \quad \beta_{\mathbb{P}^1}(x) = (\beta_{\mathbb{Z}_p}(x), \beta_{\mathbb{Z}_p}(w \cdot x)).$$

Comme leurs noms l'indiquent, $\beta_{\mathbb{Z}_p}$ et $\beta_{\mathbb{P}^1}$ ne dépendent pas du choix de W et sont fonctorielles par fonctorialité de $\Pi \mapsto D(\Pi)$.

Soient maintenant $\Pi \in {\rm Rep}_{\mathscr{O}_L}(\delta)$ et $D = D(\Pi)$. On note Π_n le sous-module de p^n-torsion de $(L/\mathscr{O}_L) \otimes \Pi$ et on pose $D_n = D(\Pi_n) \simeq D/p^n$. Alors

$$\Pi^* = \varprojlim \Pi_n^* \quad \text{et} \quad D = \varprojlim D_n.$$

Les applications $\beta_{\mathbb{P}^1} : \Pi_n^* \to D_n \oplus D_n$ sont compatibles, d'où une application continue $\beta_{\mathbb{P}^1} : \Pi^* \to D \oplus D$. Le cas $\Pi \in {\rm Rep}_L(\delta)$ s'en déduit en prenant un réseau appartenant à ${\rm Rep}_{\mathscr{O}_L}(\delta)$.

Proposition III.44. *Soit* $\Pi \in {\rm Rep}_{\rm tors}(\delta) \cup {\rm Rep}_{\mathscr{O}_L}(\delta) \cup {\rm Rep}_L(\delta)$ *et soit* $D = D(\Pi)$.

(i) (D, δ^{-1}) *est une paire G-compatible.*

(ii) $\beta_{\mathbb{P}^1}$ *est un morphisme G-équivariant* $\Pi^* \to D^\sharp \boxtimes_{\delta^{-1}} \mathbb{P}^1$, *de noyau* $(\Pi^*)^{{\rm SL}_2(\mathbb{Q}_p)}$.

(iii) $\beta_{\mathbb{P}^1}$ *envoie l'orthogonal de* $\Pi^{{\rm SL}_2(\mathbb{Q}_p)}$ *dans* $D^\natural \boxtimes_{\delta^{-1}} \mathbb{P}^1$.

Démonstration. Le cas $\Pi \in \mathrm{Rep}_{\mathrm{tors}}(\delta)$ est le contenu du th. IV.4.7 de [10]. Le cas $\Pi \in \mathrm{Rep}_L(\delta)$ se déduit du cas $\Pi \in \mathrm{Rep}_{\mathscr{O}_L}(\delta)$. Supposons donc que $\Pi \in \mathrm{Rep}_{\mathscr{O}_L}(\delta)$ et considérons les objets D_n, Π_n introduits ci-dessus de sorte que

$$D \simeq \varprojlim D_n \quad \text{et} \quad \Pi^* \simeq \varprojlim \Pi_n^*.$$

Puisque chacune des paires (D_n, δ^{-1}) est G-compatible, le module $(D^\natural \boxtimes_{\delta^{-1}} \mathbb{P}^1)_{\mathrm{ns}} \simeq \varprojlim (D_n^\natural \boxtimes_{\delta^{-1}} \mathbb{P}^1)$ est stable par G, donc (D, δ^{-1}) est G-compatible. Un argument identique démontre le (ii) à partir du cas de torsion.

Passons à la preuve du (iii). Soit $H = \mathrm{SL}_2(\mathbb{Q}_p)$. Si $\tilde{\Pi} = \Pi/\Pi^H$, on a $D(\tilde{\Pi}) \simeq D(\Pi)$ (puisque Π^H est un $\mathscr{O}_L$-module de type fini d'après le lemme III.42, donc tué par le foncteur $\Pi \mapsto D(\Pi)$) et $\tilde{\Pi}^H = 0$ (cor. III.41). De plus, $\tilde{\Pi}^*$ est l'orthogonal de Π^H, et on est ramené à prouver que $\beta_{\mathbb{P}^1}(\tilde{\Pi}^*) \subset D(\tilde{\Pi})^\natural \boxtimes_{\delta^{-1}} \mathbb{P}^1$. Autrement dit, on peut supposer que $\Pi^H = 0$.

Notons Z_n l'orthogonal de Π_n^H dans Π_n^*, de telle sorte que la suite exacte

$$0 \to \Pi_n^H \to \Pi_n \to \Pi_n/\Pi_n^H \to 0$$

nous donne une exacte de $\mathscr{O}_L$-modules profinis $0 \to Z_n \to \Pi_n^* \to (\Pi_n^H)^* \to 0$. En passant à la limite projective, on obtient

$$0 \to \varprojlim Z_n \to \Pi^* \to (\varinjlim \Pi_n^H)^* \to 0.$$

La prop. III.39 montre qu'il existe N tel que p^N tue $(\varinjlim \Pi_n^H)^*$; on en déduit que donc $p^N \Pi^* \subset \varprojlim Z_n$. Comme $\beta_{\mathbb{P}^1}(Z_n) \subset D_n^\natural \boxtimes_{\delta^{-1}} \mathbb{P}^1$ d'après le cas de torsion, on obtient

$$\beta_{\mathbb{P}^1}(p^N \Pi^*) \subset \varprojlim (D_n^\natural \boxtimes_{\delta^{-1}} \mathbb{P}^1) = (D^\natural \boxtimes_{\delta^{-1}} \mathbb{P}^1)_{\mathrm{ns}}$$

et donc $\beta_{\mathbb{P}^1}(\Pi^*) \subset D^\natural \boxtimes_{\delta^{-1}} \mathbb{P}^1$, ce qui permet de conclure. □

Théorème III.45. *Si* $\Pi \in \mathrm{Rep}_{\mathrm{tors}}(\delta) \cup \mathrm{Rep}_{\mathscr{O}_L}(\delta)$ (*resp.* $\mathrm{Rep}_L(\delta)$), *la transposée* $\beta_{\mathbb{P}^1}^*$ *de* $\beta_{\mathbb{P}^1}$ *induit un morphisme* G*-équivariant*

$$\beta_{\mathbb{P}^1}^* : \Pi_\delta(\check{D}(\Pi)) \to \Pi/\Pi^{\mathrm{SL}_2(\mathbb{Q}_p)},$$

dont les noyau et conoyau sont de type fini sur $\mathscr{O}_L$ (*resp. de dimension finie sur* L). *Plus précisément*, $\mathrm{Coker}(\beta_{\mathbb{P}^1}^*)$ *est un quotient de* $((\Pi^*)^{\mathrm{SL}_2(\mathbb{Q}_p)})^*$.

Démonstration. On peut supposer que Π est une $\mathscr{O}_L$-représentation. Soit $\tilde{\Pi} = \Pi/\Pi^{\mathrm{SL}_2(\mathbb{Q}_p)}$. La prop. III.44 montre que

$$\beta_{\mathbb{P}^1}(\tilde{\Pi}^*) \subset D(\Pi)^\natural \boxtimes_{\delta^{-1}} \mathbb{P}^1 = \Pi_\delta(\check{D}(\Pi))^*$$

(la dernière égalité suit du cor. III.22). Puisque le noyau de $\beta_{\mathbb{P}^1}$ est un sous-$\mathscr{O}_L$-module de $(\Pi^*)^{\mathrm{SL}_2(\mathbb{Q}_p)}$, le conoyau de $\beta_{\mathbb{P}^1}^*$ est un quotient de $((\Pi^*)^{\mathrm{SL}_2(\mathbb{Q}_p)})^*$, qui est un $\mathscr{O}_L$-module de type fini (rem. III.38).

Pour conclure, il nous reste à prouver que $\mathrm{Coker}(\beta_{\mathbb{P}^1})$ est un $\mathscr{O}_L$-module de type fini. Comme $\tilde{\Pi}^*$ est compact, $M = \mathrm{Res}_{\mathbb{Q}_p}(\beta_{\mathbb{P}^1}(\tilde{\Pi}^*))$ est un sous-P-module compact de $D(\Pi)^\natural \boxtimes_{\delta^{-1}} \mathbb{Q}_p$. De plus, $\mathrm{Res}_{\mathbb{Z}_p}(M)$ engendre $D(\Pi)$ (car $\mathrm{Res}_{\mathbb{Z}_p}(\beta_{\mathbb{P}^1}(\Pi_n^*)) = \beta_{\mathbb{Z}_p}(\Pi_n^*)$ engendre $D(\Pi_n)$ par construction même) donc $M = D(\Pi)^\natural \boxtimes_{\delta^{-1}} \mathbb{Q}_p$ (cf. (i) de la prop. III.7). On en déduit (rem. III.6) que

$$D(\Pi)^\natural \boxtimes_{\delta^{-1}} \mathbb{P}^1 \subset \beta_{\mathbb{P}^1}(\tilde{\Pi}^*) + (0, D(\Pi)^{\mathrm{nr}}),$$

et donc $\mathrm{Coker}(\beta_{\mathbb{P}^1})$ est un quotient de $(0, D(\Pi)^{\mathrm{nr}})$ ce qui permet de conclure puisque $D(\Pi)^{\mathrm{nr}}$ est un $\mathscr{O}_L$-module de type fini. □

Corollaire III.46. *Tout objet de* $\mathrm{Rep}_{\mathrm{tors}}(G)$ *ou de* $\mathrm{Rep}_L(G)$ *est de longueur finie comme B-module* (*topologique*).

Démonstration. C'est une conséquence immédiate du th. III.45 et de la prop. III.28. □

Corollaire III.47. *Soit* $\Pi \in \mathrm{Rep}_L(G)$ *supersingulière, de caractère central* δ. *Alors* $D(\Pi)$ *est absolument irréductible de dimension* ≥ 2 *et on a des isomorphismes topologiques de G-modules*

$$\Pi^* \simeq D(\Pi)^\natural \boxtimes_{\delta^{-1}} \mathbb{P}^1, \quad \Pi \simeq \Pi_\delta(\check{D}(\Pi)).$$

Démonstration. Comme Π est irréductible de dimension infinie, on a $\Pi^{\mathrm{SL}_2(\mathbb{Q}_p)} = 0$ et $(\Pi^*)^{\mathrm{SL}_2(\mathbb{Q}_p)} = 0$. Le th. III.45 fournit une suite exacte $0 \to K \to \Pi_\delta(\check{D}(\Pi)) \to \Pi \to 0$, avec $\dim_L(K) < \infty$.

L'irréductibilité de $D(\Pi)$ est une conséquence du (i) de la prop. III.33.

Si $\dim_{\mathscr{E}}(D(\Pi)) = 1$, il découle du th. III.45 et de la prop. III.14 que Π est ordinaire, ce qui est contraire à l'hypothèse. Donc $\dim_{\mathscr{E}} D(\Pi) \geq 2$ et, puisque $D(\Pi)$ est irréductible, on a $\check{D}(\Pi)^\sharp = \check{D}(\Pi)^\natural$. La prop. III.43 permet de conclure que $K = 0$, et donc $\Pi \simeq \Pi_\delta(\check{D}(\Pi))$. On conclut en utilisant le cor. III.22. □

Remarque III.48. On déduit du cor. III.47 que si Π_1, Π_2 sont supersingulières, *de même caractère central* et si $D(\Pi_1) \simeq D(\Pi_2)$, alors $\Pi_1 \simeq \Pi_2$. Paškūnas [25, preuve du th. 10.4] a démontrée cette propriété d'injectivité du foncteur $\Pi \mapsto D(\Pi)$ par voie très détournée, mais son approche fournit plus d'informations. Il prouve que si $p \geq 5$ et si $\Pi \in \mathrm{Rep}_L(\delta)$ est supersingulière, alors $D(\Pi)$ est de dimension 2 sur $\mathscr{E}$ et $\delta = \chi^{-1} \det \check{D}(\Pi)$. En particulier, l'image par le foncteur $\Pi \mapsto D(\Pi)$ suffit à retrouver le caractère central, ce qui est assez surprenant car le (φ, Γ)-module attaché à un $\Pi \in \mathrm{Rep}_L(G)$ n'utilise que la restriction au mirabolique. Ce résultat est étendu à p quelconque dans [13].

III.L. ***Reconstruction de D.*** Le but de ce paragraphe est de démontrer que l'on peut récupérer D à partir de $\Pi_\delta(D)$ quand (D, δ) est une paire G-compatible. On note $K = \mathrm{GL}_2(\mathbb{Z}_p)$ le sous-groupe compact maximal de G et Z son centre.

Théorème III.49. *Soit (D,δ) une paire G-compatible, avec D dans une des catégories $\Phi\Gamma^{\mathrm{et}}_{\mathrm{tors}}$, $\Phi\Gamma^{\mathrm{et}}(\mathscr{O}_{\mathscr{E}})$ ou $\Phi\Gamma^{\mathrm{et}}(\mathscr{E})$. Alors on a un isomorphisme canonique de (φ,Γ)-modules $D(\Pi_\delta(D))\simeq \check{D}$.*

Démonstration. Le cas $D\in\Phi\Gamma^{\mathrm{et}}(\mathscr{E})$ se déduit du cas $D\in\Phi\Gamma^{\mathrm{et}}(\mathscr{O}_{\mathscr{E}})$ en tensorisant par L. Supposons que $D\in\Phi\Gamma^{\mathrm{et}}(\mathscr{O}_{\mathscr{E}})$ et posons $D_n=D/p^n\in\Phi\Gamma^{\mathrm{et}}_{\mathrm{tors}}$. On déduit de la prop. III.31 que l'application naturelle $D\to D_n$ induit un isomorphisme $D(\Pi_\delta(D)/p^n\Pi_\delta(D))=D(\Pi_\delta(D_n))$ pour tout n, ce qui montre qu'il suffit de traiter le cas $D\in\Phi\Gamma^{\mathrm{et}}_{\mathrm{tors}}$, ce que l'on supposera dans la suite.

Soit W l'image de $\widetilde{W}=\sum_{g\in K} g\cdot D^\sharp\subset D\boxtimes_\delta\mathbb{P}^1$ dans $\Pi_\delta(D)$.

Lemme III.50. *W est un sous-KZ-module de $\Pi_\delta(D)$, de longueur finie comme $\mathscr{O}_L$-module et W engendre $\Pi_\delta(D)$ comme $\mathscr{O}_L[G]$-module.*

Démonstration. Il est clair que W est stable par KZ. Soient $z_1,\dots,z_d\in D^\sharp$ tels que $D^\sharp=\bigcup_{i=1}^d(D^++z_i)$. Puisque $D^+\subset D^\natural\boxtimes_\delta\mathbb{P}^1$ et $D^\natural\boxtimes_\delta\mathbb{P}^1$ est stable par G, on a $gD^+\subset D^\natural\boxtimes_\delta\mathbb{P}^1$ pour tout $g\in G$. Si v_i est l'image de z_i dans $\Pi_\delta(D)$, on conclut que W est le sous-$\mathscr{O}_L[K]$-module de $\Pi_\delta(D)$ engendré par $v_1,\dots,v_d$. Comme $\Pi_\delta(D)$ est lisse, le $\mathscr{O}_L$-module W est de longueur finie.

Montrons enfin que W engendre $\Pi_\delta(D)$ comme $\mathscr{O}_L[G]$-module. Il suffit de vérifier que $D\boxtimes_\delta\mathbb{P}^1=\sum_{g\in G} g\cdot D^\sharp$ et comme $D\boxtimes_\delta\mathbb{P}^1=D+w\cdot D$, il suffit de prouver l'inclusion $D\subset\sum_{g\in G} g\cdot D^\sharp$. Or, si $z\in D$, alors $z_{n,i}:=\psi^n((1+T)^{-i}z)\in D^\sharp$ pour tout n assez grand, uniformément en $i\in\mathbb{Z}_p$, et

$$z=\sum_{i=0}^{p^n-1}(1+T)^i\varphi^n(z_{n,i})=\sum_{i=0}^{p^n-1}\begin{pmatrix}1&i\\0&1\end{pmatrix}\cdot\begin{pmatrix}p^n&0\\0&1\end{pmatrix}z_{n,i},$$

ce qui permet de conclure. □

Rappelons que D_W^+ est l'orthogonal de $\sum_{g\in P-P^+} g\cdot W$ dans $\Pi_\delta(D)^*=\check{D}^\natural\boxtimes_{\delta^{-1}}\mathbb{P}^1$. L'isomorphisme $\mathscr{O}_{\mathscr{E}}\otimes_{\mathscr{O}_L[\![T]\!]}D_W^+\cong\check{D}$ que l'on cherche à établir est une conséquence de la platitude de $\mathscr{O}_{\mathscr{E}}$ sur $\mathscr{O}_L[\![T]\!]$ et des deux lemmes suivants.

Lemme III.51. *On a $D_W^+\subset\check{D}^{++}$.*

Démonstration. Si $\check{z}\in D_W^+$, alors $\{\check{z},gkD^\sharp\}_{\mathbb{P}^1}=0$ pour tout $g\in P-P^+$ et $k\in K$, donc $\mathrm{Res}_{\mathbb{Z}_p}(k^{-1}g^{-1}\check{z})$ est orthogonal à $D^\sharp$. On en déduit que $\mathrm{Res}_{\mathbb{Z}_p}(gz)\in\check{D}^{++}$ pour tout $g\in M:=\{zkh^{-1}\mid z\in Z,\ k\in K,\ h\in P-P^+\}$. Si $n\geq 1$ et $0\leq i<p^n$ est un multiple de p, alors[16] $\begin{pmatrix}p^{-n}&0\\0&1\end{pmatrix}\cdot\begin{pmatrix}1&-i\\0&1\end{pmatrix}\cdot w\in M$. On en déduit que, pour tout

16. Pour $i=0$ cela découle de l'identité $\begin{pmatrix}p^{-n}&0\\0&1\end{pmatrix}\cdot w=\begin{pmatrix}p^{-n}&0\\0&p^{-n}\end{pmatrix}w\begin{pmatrix}p^{-n}&0\\0&1\end{pmatrix}^{-1}$, et si $i\neq 0$, de l'identité (avec $N=n-2v_p(i)$)

$$\begin{pmatrix}p^{-n}&0\\0&1\end{pmatrix}\cdot\begin{pmatrix}1&-i\\0&1\end{pmatrix}\cdot w=\begin{pmatrix}1/i&0\\0&1/i\end{pmatrix}\cdot\begin{pmatrix}-i^2p^{N-n}&0\\p^Ni&1\end{pmatrix}\begin{pmatrix}p^N&1/i\\0&1\end{pmatrix}^{-1}.$$

$0 \leq i < p^n$ multiple de p, l'on a

$$\psi^n((1+T)^{-i}\operatorname{Res}_{\mathbb{Z}_p}(w\check{z})) \in \check{D}^{++}.$$

En faisant $n \to \infty$ dans l'égalité

$$\operatorname{Res}_{p\mathbb{Z}_p}(w\check{z}) = \sum_{i<p^n,\ p|i} (1+T)^i \varphi^n(\psi^n((1+T)^{-i}\operatorname{Res}_{\mathbb{Z}_p}(w\check{z})),$$

on obtient $\operatorname{Res}_{p\mathbb{Z}_p}(w\check{z}) = 0$, i.e., $\check{z} \in \check{D}$. De plus, $\varphi(\check{z}) = \operatorname{Res}_{\mathbb{Z}_p}\big(\big(\begin{smallmatrix} p^{-1} & 0 \\ 0 & 1\end{smallmatrix}\big)^{-1}\check{z}\big) \in \check{D}^{++}$, donc $\check{z} \in \check{D}^{++}$, ce qui permet de conclure. □

Lemme III.52. *Si n est assez grand, $\varphi^n(T)\check{D}^{++} \subset D_W^+$.*

Démonstration. Il s'agit de prouver que $\varphi^n(T)\check{D}^{++}$ est orthogonal à $g \cdot \widetilde{W}$, si $g = \big(\begin{smallmatrix} a & b \\ 0 & 1\end{smallmatrix}\big)$ avec $v_p(a) < 0$ ou $v_p(b) < 0$. De manière équivalente, il s'agit de vérifier que $\check{D}^{++}$ est orthogonal à $\big(\big(\begin{smallmatrix} 1 & -p^n \\ 0 & 1\end{smallmatrix}\big) - 1\big) \cdot g \cdot \widetilde{W}$. L'argument est différent suivant que $v_p(a) \geq 0$ ou $v_p(a) < 0$.

• Si $v_p(a) \geq 0$, on a $v_p(b) < 0$. Or il existe un treillis [17] M de D tel que $\big(\begin{smallmatrix} a & 0 \\ 0 & 1\end{smallmatrix}\big) \cdot \widetilde{W}$ soit inclus dans $D + wM + (D^\natural \boxtimes_\delta \mathbb{P}^1)$ pour tout $a \in \mathbb{Z}_p^*$. Choisissons n assez grand pour que $\big(\big(\begin{smallmatrix} 1 & -p^n \\ 0 & 1\end{smallmatrix}\big) - 1\big) \cdot wM \subset D^\natural \boxtimes_\delta \mathbb{P}^1$. En écrivant $\big(\big(\begin{smallmatrix} 1 & -p^n \\ 0 & 1\end{smallmatrix}\big) - 1\big) \cdot g$ sous la forme

$$\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix} \cdot \left(\begin{pmatrix} 1 & -p^n \\ 0 & 1\end{pmatrix} - 1\right) \cdot \begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix},$$

on voit que

$$\left(\begin{pmatrix} 1 & -p^n \\ 0 & 1\end{pmatrix} - 1\right) \cdot g \cdot \widetilde{W} \subset (D \boxtimes_\delta (b+\mathbb{Z}_p)) + (D^\natural \boxtimes_\delta \mathbb{P}^1),$$

qui est orthogonal à $\check{D}^{++}$ car $D \boxtimes_\delta (b+\mathbb{Z}_p)$ l'est puisque $\mathbb{Z}_p \cap (b+\mathbb{Z}_p) = \varnothing$ et $D^\natural \boxtimes_\delta \mathbb{P}^1$ l'est puisque $\check{D}^{++}$ est inclus dans $\check{D}^\natural \boxtimes_{\delta^{-1}} \mathbb{P}^1$.

• Si $v_p(a) < 0$, on écrit $\big(\big(\begin{smallmatrix} 1 & -p^n \\ 0 & 1\end{smallmatrix}\big) - 1\big) \cdot g$ sous la forme $g \cdot \big(\big(\begin{smallmatrix} 1 & -a^{-1}p^n \\ 0 & 1\end{smallmatrix}\big) - 1\big)$, et on choisit n assez grand pour que $\big(\big(\begin{smallmatrix} 1 & c \\ 0 & 1\end{smallmatrix}\big) - 1\big) \cdot \widetilde{W} \subset (D^\natural \boxtimes_\delta \mathbb{P}^1)$ pour tout $c \in p^n\mathbb{Z}_p$. Les arguments ci-dessus montrent qu'alors $\check{D}^{++}$ est orthogonal à $\big(\big(\begin{smallmatrix} 1 & -p^n \\ 0 & 1\end{smallmatrix}\big) - 1\big) \cdot g \cdot \widetilde{W}$.

Ceci permet de conclure la démonstration du lemme et celle du th. III.49. □

Remarque III.53. Le module W utilisé dans la preuve du th. III.49 est raisonnablement naturel, et garde un sens si D n'est pas de torsion. Cela semble un bon candidat si on veut écrire $\Pi_\delta(D)$ comme quotient d'une induite de KZ à G.

III.M. ***Compatibilité avec la réduction modulo p.*** Dans ce paragraphe nous étendons un résultat de Berger [3] sur la compatibilité entre la correspondance de Langlands locale p-adique et celle modulo p.

17. Un treillis est un sous-$\mathscr{O}_L[[T]]$-module compact de D dont l'image modulo p^n est ouverte dans D/p^nD pour tout $n \in \mathbb{N}$ (il suffit qu'elle le soit dans $k_L \otimes D$).

Si $D \in \Phi\Gamma^{\mathrm{et}}(\mathscr{E})$ et si D_0 est un $\mathscr{O}_{\mathscr{E}}$-réseau de D stable par φ et Γ, on note $\bar{D}_0 = D_0 \otimes_{\mathscr{O}_L} k_L$ et $\bar{D}^{\mathrm{ss}}$ la semi-simplifiée de $\bar{D}_0$ (qui ne dépend pas du choix de D_0). De même, si $\Pi \in \mathrm{Rep}_L(G)$ et si Π_0 est un réseau ouvert, borné et G-invariant, on note $\overline{\Pi}_0 = \Pi_0 \otimes_{\mathscr{O}_L} k_L$ et $\overline{\Pi}^{\mathrm{ss}}$ la semi-simplifiée de $\overline{\Pi}_0$ (qui, à nouveau, ne dépend pas du choix de Π_0). Rappelons que, par définition de $\mathrm{Rep}_L(G)$, la représentation $\overline{\Pi}^{\mathrm{ss}}$ est de longueur finie. Remarquons aussi que, si (D, δ) est G-compatible, il en est de même de $(\bar{D}^{\mathrm{ss}}, \delta)$ (cela résulte du (i) de la prop. III.17 et de la prop. III.29).

Proposition III.54. *Soit (D, δ) une paire G-compatible, avec $D \in \Phi\Gamma^{\mathrm{et}}(\mathscr{E})$, et soit D_0 un $\mathscr{O}_{\mathscr{E}}$-réseau de D stable par φ et Γ. Si $\bar{D}_0$ n'a pas de composante de Jordan–Hölder isomorphe à $k_{\mathscr{E}}(\eta)$, avec $\eta^2 = \delta$ ou $\eta^2\chi^{-2} = \delta$ mod $\mathfrak{m}_L$, alors*

$$\overline{\Pi_\delta(D_0)} = \Pi_\delta(\bar{D}_0) \quad et \quad \overline{\Pi_\delta(D)}^{\mathrm{ss}} = \Pi_\delta(\bar{D}^{\mathrm{ss}}).$$

Démonstration. $(D_0^\natural \boxtimes_\delta \mathbb{P}^1)/(D_0^\natural \boxtimes_\delta \mathbb{P}^1)_{\mathrm{ns}}$ est un G-module de longueur finie sur $\mathscr{O}_L$ dont les composantes de Jordan–Hölder sont parmi celles de $\bar{D}_0 \boxtimes_\delta \mathbb{P}^1 = \overline{D_0 \boxtimes_\delta \mathbb{P}^1}$ (le foncteur $D \to D \boxtimes_\delta \mathbb{P}^1$ est trivialement exact). L'hypothèse sur les composantes de Jordan–Hölder de $\bar{D}_0$ implique que $\bar{D}_0 \boxtimes_\delta \mathbb{P}^1$ n'a pas de composante de dimension finie sur k_L (cf. la rem. III.35), et donc que $(D_0^\natural \boxtimes_\delta \mathbb{P}^1)_{\mathrm{ns}} = D_0^\natural \boxtimes_\delta \mathbb{P}^1$. On en déduit que $\overline{\Pi_\delta(D_0)} = \Pi_\delta(\bar{D}_0)$. L'isomorphisme $\overline{\Pi_\delta(D)}^{\mathrm{ss}} = \Pi_\delta(\bar{D}^{\mathrm{ss}})$ est, lui-aussi, une conséquence de la rem. III.35. □

Sans l'hypothèse sur les composantes de Jordan–Hölder de la prop. III.54, la situation est plus problématique mais on a quand même le résultat suivant.

Proposition III.55. *Si (D, δ) est une paire G-compatible, avec $D \in \Phi\Gamma^{\mathrm{et}}(\mathscr{E})$, et si $\check{D} \cong D \otimes \delta^{-1}$, alors $\overline{\Pi_\delta(D)}^{\mathrm{ss}} = \Pi_\delta(\bar{D}^{\mathrm{ss}})$.*

Démonstration. Nous allons constamment utiliser le cor. III.22 dans la suite. Soit D_0 un réseau stable dans D et soit $\Pi_0 = \Pi_\delta(D_0)$. Alors Π_0 est un réseau ouvert, G-invariant, et dans $\Pi = \Pi_\delta(D)$. L'hypothèse d'autodualité de D fournit un isomorphisme $\check{D}_0 \simeq D_0 \otimes \delta^{-1}$, donc

$$\Pi_{\delta^{-1}}(\check{D}_0) \simeq \Pi_{\delta^{-1}}(D_0 \otimes \delta^{-1}) \simeq \Pi_0 \otimes \delta^{-1},$$

le dernier isomorphisme étant une conséquence de la prop. III.13. On obtient donc une suite exacte de G-modules topologiques

$$0 \to \Pi_0^* \otimes \delta \to D_0 \boxtimes_\delta \mathbb{P}^1 \to \Pi_0 \to 0.$$

En réduisant modulo $\mathfrak{m}_L$, on obtient une suite exacte

$$0 \to \overline{\Pi}_0^\vee \otimes \delta \to \bar{D}_0 \boxtimes_\delta \mathbb{P}^1 \to \overline{\Pi}_0 \to 0.$$

Soit $\pi = \Pi_\delta(\bar{D}_0)$. Les mêmes arguments que ci-dessus fournissent une suite exacte de G-modules topologiques

$$0 \to \pi^\vee \otimes \delta \to \bar{D}_0 \boxtimes_\delta \mathbb{P}^1 \to \pi \to 0.$$

En utilisant le fait que les composantes de Jordan–Hölder de $\bar{D}_0 \boxtimes_\delta \mathbb{P}^1$, de dimension finie sur k_L, sont invariantes par $W \mapsto W^\vee \otimes \delta$, on en déduit qu'il y en a le même nombre de chaque dans π et dans $\overline{\Pi}_0$, à savoir la moitié du nombre de ces composantes dans $\bar{D}_0 \boxtimes_\delta \mathbb{P}^1$ car π et $\pi^\vee \otimes \delta$ en contiennent le même nombre, ainsi que $\overline{\Pi}_0$ et $\overline{\Pi}_0{}^\vee \otimes \delta$. Comme les composantes de Jordan–Hölder de dimension infinie d'un dual d'un objet de $\mathrm{Rep}_{\mathrm{tors}}(G)$ sont compactes alors que celles d'un objet de $\mathrm{Rep}_{\mathrm{tors}}(G)$ sont discrètes, on en déduit les égalités

$$\overline{\Pi}^{\mathrm{ss}} = \overline{\Pi}_0{}^{\mathrm{ss}} = \pi^{\mathrm{ss}}.$$

Il nous reste donc à prouver que $\pi^{\mathrm{ss}} = \Pi_\delta(\bar{D}^{\mathrm{ss}})$. Comme ci-dessus, cela revient à comprendre comment les composantes de dimension finie de $\bar{D}_0 \boxtimes_\delta \mathbb{P}^1$ se répartissent (notons que $(\bar{D}_0 \boxtimes_\delta \mathbb{P}^1)^{\mathrm{ss}} = \bar{D}^{\mathrm{ss}} \boxtimes_\delta \mathbb{P}^1$ par exactitude de $D \mapsto D \boxtimes_\delta \mathbb{P}^1$). Or l'hypothèse d'auto-dualité implique, comme ci-dessus, que π^{ss} et $\Pi_\delta(\bar{D}^{\mathrm{ss}})$ en contiennent le même nombre de chaque, ce qui permet de conclure. □

Remarque III.56. Les hypothèses de la prop. III.55 sont satisfaites si D est de rang 2 et si $\delta = \chi^{-1} \det D$ et, quitte à étendre les scalaires de L à son extension quadratique non ramifiée, il y a deux possibilités :

- $\bar{D}^{\mathrm{ss}}$ est irréductible et $\overline{\Pi_\delta(D)}^{\mathrm{ss}}$ est supersingulière ;
- $\bar{D}^{\mathrm{ss}} = k_{\mathscr{E}}(\delta_1) \oplus k_{\mathscr{E}}(\delta_2)$ et

$$\overline{\Pi_\delta(D)}^{\mathrm{ss}} = (\mathrm{Ind}_B^G(\delta_1 \otimes \chi^{-1}\delta_2))^{\mathrm{ss}} \oplus (\mathrm{Ind}_B^G(\delta_2 \otimes \chi^{-1}\delta_1))^{\mathrm{ss}}.$$

Dans le premier cas $\overline{\Pi_\delta(D)}^{\mathrm{ss}}$ est irréductible, dans le second elle est de longueur 2, sauf si $\delta_1\delta_2^{-1} = \chi^{\pm 1}$ où elle est de longueur 3 si $p \geq 5$, et de longueur 4 si $p = 2, 3$ (car alors $\chi = \chi^{-1}$ modulo p).

IV. Représentations localement analytiques

Dans ce chapitre, on revisite les travaux de Schneider et Teitelbaum [28] sur les représentations localement analytiques en étudiant de plus près la filtration naturelle par rayon d'analyticité (on fera attention que cette notion est légèrement différente de celle à laquelle on penserait naturellement (cf. la rem. IV.15)). On donne aussi (th. IV.6) une description, par dualité, des vecteurs localement analytiques d'une représentation de Banach.

IV.A. ***Groupes uniformes.*** On renvoie à [17] pour les preuves des résultats énoncés ci-dessous. Posons $\kappa = 1$ si $p > 2$ et $\kappa = 2$ sinon. Dans ce chapitre, H est un pro-p-groupe uniforme, i.e., un pro-p-groupe topologiquement de type fini, sans p-torsion et tel que $[H, H] \subset H^{p\kappa}$.

Si $i \geq 0$, soit $H_i = \{g^{p^i} \mid g \in H\} = H^{p^i}$. Alors H_i est un sous-groupe ouvert distingué de H, et $(H_i)_{i \geq 0}$ est un système fondamental de voisinages ouverts de 1. En posant $\omega(1) = \infty$ et $\omega(g) = i$ si $g \in H_{i-\kappa} \backslash H_{i-\kappa+1}$, on obtient une p-valuation (au sens de Lazard) satisfaisant l'hypothèse HYP de [28], ce qui nous permet d'utiliser directement les résultats de [loc. cit.]. Si $h_1, h_2, \ldots, h_d$ est un système minimal de générateurs topologiques de H, alors $\omega(h_i) = \kappa$ pour tout i, et l'application $\mathbb{Z}_p^d \to H$ définie par $(x_1, x_2, \ldots, x_d) \mapsto h_1^{x_1} h_2^{x_2} \cdots h_d^{x_d}$ est un homéomorphisme. De plus, on a

$$\omega(h_1^{x_1} h_2^{x_2} \cdots h_d^{x_d}) = \kappa + \min_{1 \leq i \leq d} v_p(x_i)$$

pour tous $x_1, \ldots, x_d \in \mathbb{Z}_p$.

On utilise les notations standard pour les d-uplets :

$$|\alpha| = \sum_{i=1}^{d} \alpha_i, \quad \binom{\alpha}{\beta} = \prod_i \binom{\alpha_i}{\beta_i}, \quad h^\alpha = \prod_i h_i^{\alpha_i}, \quad \text{etc.}$$

On écrit $\alpha \leq \beta$ si $\alpha_i \leq \beta_i$ pour tout i. Si $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, on note

$$b^\alpha = (h_1 - 1)^{\alpha_1} (h_2 - 1)^{\alpha_2} \cdots (h_d - 1)^{\alpha_d} \in \mathbb{Z}_p[H].$$

IV.B. ***Coefficients de Mahler.*** L'espace $\mathscr{C}(H)$ des fonctions continues sur H, à valeurs dans L, est une L-représentation de Banach de H, si on le munit de la norme sup et de l'action de H définie par $(g \cdot \phi)(x) = \phi(xg)$. Si $\phi \in \mathscr{C}(H)$, on note

$$a_\alpha(\phi) = (b^\alpha \phi)(1) = \sum_{\beta \leq \alpha} (-1)^{\alpha - \beta} \binom{\alpha}{\beta} \phi(h^\beta)$$

ses coefficients de Mahler, relativement au choix des coordonnées $h_1, \ldots, h_d$ sur H. Soit $\phi_\alpha \in \mathscr{C}(H)$ l'application définie par $\phi_\alpha(h^x) = \binom{x}{\alpha}$ pour $x \in \mathbb{Z}_p^d$. Un théorème classique de Mahler montre que $(\phi_\alpha)_{\alpha \in \mathbb{N}^d}$ est une base orthonormale de $\mathscr{C}(H)$, et pour tout $\phi \in \mathscr{C}(H)$,

$$\phi = \sum_{\alpha \in \mathbb{N}^d} a_\alpha(\phi) \cdot \phi_\alpha.$$

Définition IV.1. Pour tout $h \in \mathbb{N}^*$ on note

$$\mathrm{LA}^{(h)}(H) = \big\{\phi \in \mathscr{C}(H) \mid \lim_{|\alpha| \to \infty} (v_p(a_\alpha(\phi)) - r_h|\alpha|) = \infty\big\}.$$

C'est un banach pour la valuation $v^{(h)}$ définie par

$$v^{(h)}(\phi) = \inf_\alpha (v_p(a_\alpha(\phi)) - r_h|\alpha|).$$

D'après le théorème d'Amice [1], l'espace $\mathscr{C}^{\mathrm{an}}(H)$ des fonctions localement analytiques sur H est la limite inductive des espaces $\mathrm{LA}^{(h)}(H)$.

IV.C. ***Complétions de l'algèbre des mesures de H.*** Le but de ce paragraphe est de rappeler un certain nombre de constructions et résultats de [28], et d'établir quelques estimées techniques dont on aura besoin plus loin.

Soit $\Lambda(H)$ le dual faible de $\mathscr{C}(H)$; on a aussi $\Lambda(H) = L\otimes_{\mathscr{O}_L}(\varprojlim \mathscr{O}_L[H/H^{p^n}])$. C'est une algèbre (pour le produit de convolution) topologique localement compacte, et le § IV.B montre que tout élément de $\Lambda(H)$ s'écrit de manière unique

$$\lambda = \sum_{\alpha\in\mathbb{N}^d} c_\alpha b^\alpha,$$

avec $(c_\alpha)_\alpha$ une suite bornée dans L. La valeur de λ en $\phi\in\mathscr{C}(H)$ est donnée par

$$\langle\lambda,\phi\rangle = \sum_{\alpha\in\mathbb{N}^d} c_\alpha a_\alpha(\phi).$$

On note $\mathscr{D}_{1/p}(H)$ le complété de $\Lambda(H)$ pour la valuation d'algèbre (c'est-à-dire $v_{1/p}(\lambda\mu)\geq v_{1/p}(\lambda)+v_{1/p}(\mu)$) :

$$v_{1/p}\Big(\sum_\alpha c_\alpha b^\alpha\Big) = \inf_\alpha (v_p(c_\alpha)+\kappa|\alpha|)$$

et, si $h\in\mathbb{N}^*$, on définit [18]

$$\mathscr{D}_h(H) = \Big\{\sum_\alpha c_\alpha b^\alpha \in \mathscr{D}_{1/p}(H)\ \Big|\ \inf_\alpha (v_p(c_\alpha)+r_h|\alpha|) > -\infty\Big\},$$

que l'on munit d'une valuation d'algèbre $v^{(h)}$ en posant

$$v^{(h)}\Big(\sum_\alpha c_\alpha b^\alpha\Big) = \inf_\alpha (v_p(c_\alpha)+r_h|\alpha|).$$

Enfin, on note $\mathscr{D}(H)$ le dual topologique (fort) de $\mathscr{C}^{\mathrm{an}}(H)$. C'est aussi la limite projective des $\mathscr{D}_h(H)$. L'accouplement

$$\langle\lambda,\phi\rangle = \sum_{\alpha\in\mathbb{N}^d} c_\alpha a_\alpha(\phi)$$

identifie $\mathscr{D}_h(H)$ au dual topologique de $\mathrm{LA}^{(h)}(H)$, ce qui permet de définir une topologie faible sur $\mathscr{D}_h(H)$. Dans la suite on munit [19] $\mathscr{D}_h(H)$ de cette topologie faible. La différence avec la topologie forte (qui, elle, est induite par la valuation $v^{(h)}$)

18. Rappelons que $r_h = 1/(p^{h-1}(p-1))$.

19. L'algèbre $\mathscr{D}_h(H)$ peut aussi être munie d'une topologie d'algèbre de Banach en utilisant la valuation $v^{(h)}$. C'est d'ailleurs ce qui est fait dans [28]. Cette topologie est trop forte pour les applications que nous avons en vue.

est que $p^{-r_h|\alpha|}b^\alpha$ tend vers 0 pour la topologie faible dans $\mathscr{D}_h(H)$, mais pas pour la topologie forte.

Proposition IV.2. *La multiplication dans $\Lambda(H)$ s'étend par continuité à $\mathscr{D}_h(H)$ et l'inclusion naturelle $\Lambda(H) \to \mathscr{D}_h(H)$ est plate. De plus, l'inclusion $\Lambda(H) \to \mathscr{D}(H)$ est fidèlement plate.*

Démonstration. Tout ceci est démontré dans le chap. 4 de [28]. Pour faciliter la comparaison, notons que l'on utilise comme p-valuation celle introduite dans le § IV.A, et que si l'on pose $s_h = p^{-r_h\kappa^{-1}} \in]1/p, 1[\cap p^{\mathbb{Q}}$, alors $p^{-v^{(h)}}$ correspond à $\|\cdot\|_{s_h}$ de [loc. cit.], donc $\mathscr{D}_h(H)$ correspond à $D_{<s_h}(H, L)$ de [loc. cit.]. □

Lemme IV.3. (i) *Soit $h'_1, \dots, h'_d$ un autre système minimal de générateurs topologiques de H et soit $b'^\alpha = (h'_1-1)^{\alpha_1}\cdots(h'_d-1)^{\alpha_d}$. Il existe des $c_{\alpha,\beta} \in \mathscr{O}_L$ tels que $b'^\alpha = \sum_\beta c_{\alpha,\beta}b^\beta$ et $v_p(c_{\alpha,\beta}) \geq \max(0, \kappa(|\alpha|-|\beta|))$.*

(ii) *Pour tout $x \in H$ il existe des $c_{\alpha,\beta,x} \in \mathscr{O}_L$ tels que $xb^\alpha = \sum_\beta c_{\alpha,\beta,x}b^\beta$ et $v_p(c_{\alpha,\beta,x}) \geq \max(0, \kappa(|\alpha|-|\beta|))$.*

(iii) *Il existe des $c_{\alpha,\beta} \in \mathscr{O}_L$ tels que*

$$(h_1^p-1)^{\alpha_1}\cdots(h_d^p-1)^{\alpha_d} = \sum_\beta c_{\alpha,\beta}b^\beta$$

et $v_p(c_{\alpha,\beta}) \geq \max(0, r_1(p|\alpha|-|\beta|))$.

Démonstration. (i) L'existence des $c_{\alpha,\beta}$ et le fait qu'ils appartiennent à $\mathscr{O}_L$ suivent du fait que $b'^\alpha \in \mathscr{O}_L[H] \subset \mathscr{O}_L[\![H]\!]$. L'inégalité $v_p(c_{\alpha,\beta}) \geq \max(0, \kappa(|\alpha|-|\beta|))$ découle du fait que $v_{1/p}(g-1) \geq \kappa$ pour tout $g \in H$. La preuve du (ii) est identique et laissée au lecteur.

(iii) Puisque $v^{(1)}$ est une valuation d'algèbre sur $\Lambda(H)$, on a

$$\inf_\beta(v_p(c_{\alpha,\beta}) + r_1|\beta|) \geq \sum_{i=1}^d \alpha_i v^{(1)}(h_i^p-1).$$

Or

$$v^{(1)}(h_i^p-1) = v^{(1)}\left(\sum_{k=1}^p \binom{p}{k}(h_i-1)^k\right) = \inf_{k\leq p} v_p\left(\binom{p}{k}\right) + kr_1 = pr_1,$$

ce qui permet de conclure. □

IV.D. ***Le foncteur $\Pi \mapsto \Pi^{(h)}$.*** Soit Π une L-représentation de Banach de H et soit v_Π une valuation sur Π qui définit sa topologie. Si $h \geq 1$ on définit

$$\Pi^{(h)} = \left\{v \in \Pi \mid \lim_{|\alpha|\to\infty} v_\Pi(b^\alpha v) - r_h|\alpha| = \infty\right\},$$

et l'on munit de la valuation $v^{(h)}$ définie par

$$v^{(h)}(v) = \inf_{\alpha\in\mathbb{N}^d}(v_\Pi(b^\alpha v) - r_h|\alpha|).$$

Alors $\Pi^{(h)}$ est un L-banach, qui ne dépend pas du choix de v_Π (la valuation $v^{(h)}$ en dépend de manière évidente, mais changer v_Π remplace $v^{(h)}$ par une valuation équivalente).

Les $b^\alpha v$ sont les coefficients de Mahler de la fonction $o_v : H \to \Pi$ définie par $o_v(g) = g \cdot v$, et le théorème d'Amice montre que le sous-espace $\Pi^{\rm an}$ des vecteurs localement analytiques est la limite inductive des $\Pi^{(h)}$.

Proposition IV.4. *L'espace $\Pi^{(h)}$ et la valuation $v^{(h)}$ ne dépendent pas du choix du système minimal de générateurs topologiques $h_1, \dots, h_d$ de H, que l'on utilise pour définir b^α.*

Démonstration. Soit $h'_1, \dots, h'_d$ un autre système minimal de générateurs topologiques et soit $c_{\alpha,\beta}$ comme dans le lemme IV.3. Soient $v \in \Pi^{(h)}$, $M \in \mathbb{R}$ et N tels que $v_\Pi(b^\alpha v) - r_h|\alpha| \geq M$ pour tout $|\beta| \geq N$. Si $|\alpha| \geq N$, alors (en utilisant le lemme IV.3)

$$\inf_{|\beta|<N} (v_\Pi(c_{\alpha,\beta} b^\beta v) - r_h|\alpha|) \geq \inf_{|\beta|<N} v_\Pi(b^\beta v) + \kappa(|\alpha| - N) - r_h|\alpha|,$$

quantité qui dépasse M si $|\alpha|$ est assez grand, et

$$\inf_{|\beta|\geq N} (v_\Pi(c_{\alpha,\beta} b^\beta v) - r_h|\alpha|) \geq M + r_h(|\beta| - |\alpha|) + \max(0, \kappa(|\alpha| - |\beta|)) \geq M.$$

Comme $b'^\alpha v = \sum_\beta c_{\alpha,\beta} b^\beta v$, on déduit des inégalités précédentes que l'on a $\lim_{|\alpha|\to\infty} v_\Pi(b'^\alpha v) - r_h|\alpha| = \infty$ et (en prenant $N = 0$ et $M = v^{(h)}(v)$)

$$\inf_\alpha (v_\Pi(b^\alpha v) - r_h|\alpha|) \leq \inf_\alpha (v_\Pi(b'^\alpha v) - r_h|\alpha|).$$

Le résultat s'en déduit par symétrie. □

Proposition IV.5. *On a $\mathscr{C}(H)^{(h)} = \mathrm{LA}^{(h)}(H)$.*

Démonstration. Il est immédiat de vérifier que $\mathscr{C}(H)^{(h)} \subset \mathrm{LA}^{(h)}(H)$. Réciproquement, supposons que $\phi \in \mathrm{LA}^{(h)}(H)$. On veut montrer que

$$\lim_{|\alpha|\to\infty} \inf_{x\in H} v_p((b^\alpha \phi)(x)) - r_h|\alpha| = \infty.$$

Si $c_{\alpha,\beta,x}$ est comme dans le lemme IV.3, alors

$$(b^\alpha \phi)(x) = (x b^\alpha \phi)(1) = \sum_\beta c_{\alpha,\beta,x} a_\beta(\phi).$$

On conclut en utilisant le lemme IV.3, comme dans la preuve de la prop. IV.4. □

Théorème IV.6. (i) *Si $v \in \Pi$, alors $v \in \Pi^{(h)}$ si et seulement si la fonction $g \mapsto l(g \cdot v)$ appartient à $\mathrm{LA}^{(h)}(H)$ pour tout $l \in \Pi^*$.*

(ii) *Si $l_1, \dots, l_r$ engendrent Π^* comme $\Lambda(H)$-module, alors $v \mapsto \iota(v)$, où $\iota(v) \in \mathrm{LA}^{(h)}(H)^r$ est la fonction $g \mapsto (l_1(g \cdot v), \dots, l_r(g \cdot v))$, est un plongement fermé de $\Pi^{(h)}$ dans $\mathrm{LA}^{(h)}(H)^r$.*

Démonstration. Par définition, $v \in \Pi^{(h)}$ si et seulement si la suite $x_\alpha = p^{-r_h|\alpha|}b^\alpha v$ tend vers 0 dans $\Pi \otimes_L L(p^{r_h})$. Le (surprenant) lemme IV.7 ci-dessous montre que cela arrive si et seulement si $p^{-r_h|\alpha|}l(b^\alpha v)$ tend vers 0 pour tout $l \in \Pi^*$. On conclut la preuve du (i) en remarquant que $l(b^\alpha v) = a_\alpha(o_{l,v})$, où $o_{l,v} : H \to L$ est la fonction $g \mapsto l(gv)$.

Pour prouver le (ii), notons que Π est admissible (car Π^* est de type fini comme $\Lambda(H)$-module). Puisque $l_1, \dots, l_r$ engendrent Π^* comme $\Lambda(H)$-module, l'application transposée

$$\iota : \Pi \to \mathscr{C}(H)^r, \quad v \mapsto (g \mapsto (l_1(g \cdot v), \dots, l_r(g \cdot v)))$$

est un plongement fermé [27]. Il découle de la prop. IV.5 que ι envoie $\Pi^{(h)}$ dans $\mathrm{LA}^{(h)}(H)^r$ et il nous reste à montrer que l'application ι ainsi obtenue est un plongement fermé. Supposons que les $v_n \in \Pi^{(h)}$ sont tels que $\iota(v_n) \to f$ dans $\mathrm{LA}^{(h)}(H)^r$. Alors $\iota(v_n) \to f$ dans $\mathscr{C}(H)^r$, et donc il existe $v \in \Pi$ tel que $f = \iota(v)$. Ainsi $\iota(v) = f \in \mathrm{LA}^{(h)}(H)^r$. Autrement dit, les $g \mapsto l_i(gv)$ appartiennent à $\mathrm{LA}^{(h)}(H)$ pour tout i. Comme $l_1, \dots, l_r$ engendrent Π^* en tant que $\Lambda(H)$-module, et comme $\mathrm{LA}^{(h)}(H)$ est un $\Lambda(H)$-module, il s'ensuit que $g \mapsto l(gv)$ appartient à $\mathrm{LA}^{(h)}(H)$ pour tout $l \in \Pi^*$. Le (i) permet de conclure que $v \in \Pi^{(h)}$. □

Lemme IV.7. *Dans un espace localement convexe sur un corps sphériquement complet une suite converge vers* 0 *si et seulement si elle converge faiblement vers* 0.

Démonstration. Voir par exemple [26, th. 5.5.2]. □

Corollaire IV.8. $\Pi^{(h)}$ *est stable sous l'action de* H.

IV.E. ***De H à*** H^p**.** Le but de ce paragraphe est d'étudier la variation de la filtration par rayon d'analyticité quand on remplace H par un sous-groupe (prop. IV.11).

Proposition IV.9. *Soit* $\phi \in \mathrm{LA}^{(h)}(H^p)$ *et soit* $\tilde{\phi} \in \mathscr{C}(H)$ *l'extension par* 0 *de* ϕ. *Alors* $\tilde{\phi} \in \mathrm{LA}^{(h+1)}(H)$.

Démonstration. Il s'agit de montrer que $\lim_{|\alpha|\to\infty} v_p(a_\alpha(\tilde{\phi})) - r_{h+1}|\alpha| = \infty$. Si $\beta \in \mathbb{N}^d$, on a $h^\beta \in H^p$ si et seulement si p divise β (i.e., p divise chaque β_i). On en déduit que [20]

$$\begin{aligned} a_\alpha(\tilde{\phi}) &= \sum_{p\beta \le \alpha} (-1)^{\alpha - p\beta}\binom{\alpha}{p\beta}\phi(h^{p\beta}) \\ &= \sum_{p\beta \le \alpha} (-1)^{\alpha - p\beta}\binom{\alpha}{p\beta} \cdot \sum_\gamma a_\gamma(\phi)\binom{\beta}{\gamma} = \sum_\gamma a_\gamma(\phi) \cdot c_{\alpha,\gamma}, \end{aligned}$$

avec

20. Attention au fait que les coefficients de Mahler de ϕ sont calculés par rapport à $(h_1^p, \dots, h_d^p)$. On a donc $\phi(h^{p\beta}) = \sum_\gamma a_\gamma(\phi)\binom{\beta}{\gamma}$.

$$c_{\alpha,\gamma} = \sum_{p\beta\leq\alpha} (-1)^{\alpha-p\beta}\binom{\alpha}{p\beta}\binom{\beta}{\gamma}.$$

On conclut comme dans la preuve de la prop. IV.4, en utilisant le lemme ci-dessous. □

Lemme IV.10. *Si* $c_{\alpha,\gamma} = \sum_{p\beta\leq\alpha} (-1)^{\alpha-p\beta}\binom{\alpha}{p\beta}\binom{\beta}{\gamma}$, *alors*

$$v_p(c_{\alpha,\gamma}) > \frac{|\alpha|}{p} - d - |\gamma|.$$

Démonstration. Soit $f_k = \psi(T^k)$ pour $k \in \mathbb{N}$. Alors $f_k \in \mathbb{Z}_p[T]$ et

$$f_k(T^p - 1) = \frac{1}{p}\sum_{\zeta^p=1} (\zeta T - 1)^k = \sum_{pj\leq k} (-1)^{k-pj}\binom{k}{pj} T^{pj}.$$

On en déduit que pour tout $\alpha \in \mathbb{N}^d$ on a

$$\sum_{p\beta\leq\alpha} (-1)^{\alpha-p\beta}\binom{\alpha}{p\beta} T^\beta = \prod_{i=1}^d f_{\alpha_i}(T_i - 1).$$

En dérivant cette relation γ-fois et en évaluant en 1, on obtient

$$c_{\alpha,\gamma} = \frac{1}{\gamma!}\frac{d^\gamma}{dT^\gamma}(\psi(T_1^{\alpha_1})\cdots\psi(T_d^{\alpha_d}))(0) = b_{\alpha_1,\gamma_1}\cdots b_{\alpha_d,\gamma_d},$$

où

$$\psi(T^k) = \sum_{i\leq k/p} b_{k,i} T^i.$$

Le résultat découle de l'inégalité $v_p(b_{k,i}) > k/p - 1 - i$ (cf. [11, lemme I.8]). □

Proposition IV.11. *Si* Π_1 *est la restriction de* Π *à* H^p, *alors* $\Pi^{(h+1)} = \Pi_1^{(h)}$.

Démonstration. On note $h'_i = h_i^p$ et $b'^\alpha = (h'_1 - 1)^{\alpha_1}\cdots(h'_d - 1)^{\alpha_d}$. Alors $h'_1, \dots, h'_d$ forment un système minimal de générateurs topologiques de H^p. Commençons par montrer l'inclusion $\Pi^{(h+1)} \subset \Pi_1^{(h)}$. La prop. IV.4 montre qu'il suffit de prouver que $\lim_{|\alpha|\to\infty} v_\Pi(b'^\alpha v) - r_h|\alpha| = \infty$ pour tout $v \in \Pi^{(h+1)}$. La preuve est identique à celle de la prop. IV.4, en utilisant le lemme IV.3.

Soit maintenant $v \in \Pi_1^{(h)}$ et soient $(g_i)_{i\in I}$ tels que $H = \coprod_{i\in I} H^p g_i$. Soit enfin $l \in \Pi^*$ et notons $\phi : H \to L$ l'application $g \to l(gv)$. Puisque H^p est distingué dans H et $v \in \Pi_1^{(h)}$, on montre comme dans la preuve du cor. IV.8 que $(g_i\phi)|_{H^p} \in \mathrm{LA}^{(h)}(H^p)$. Si ϕ_i est l'extension par zéro de $(g_i\phi)|_{H^p}$, la prop. IV.9 montre que $\phi_i \in \mathrm{LA}^{(h+1)}(H)$. On en déduit que $\phi = \sum_{i\in I} g_i^{-1}\cdot\phi_i \in \mathrm{LA}^{(h+1)}(H)$, puisque $\mathrm{LA}^{(h+1)}(H)$ est stable par H. Le th. IV.6 permet de conclure que $v \in \Pi^{(h+1)}$. □

Remarque IV.12. On déduit de la preuve de la prop. IV.11 que, si on décompose H sous la forme $H = \coprod_{i\in I} H^p g_i$, l'application $\phi \mapsto ((g_i \cdot \phi)|_{H^p})_{i\in I}$ induit un isomorphisme de L-banach

$$\mathrm{LA}^{(h+1)}(H) \simeq \bigoplus_{i\in I} \mathrm{LA}^{(h)}(H^p, L).$$

IV.F. ***Exactitude du foncteur*** $\boldsymbol{\Pi \mapsto \Pi^{(h)}}$**.** A partir de maintenant, on suppose que Π une représentation de Banach *admissible* de H. Cela signifie que Π^* est un $\Lambda(H)$-module de type fini (et donc de présentation finie, puisque $\Lambda(H)$ est noethérien). Le choix d'une surjection $\Lambda(H)^r \to \Pi^*$ induit un plongement fermé $\iota : \Pi \to \mathscr{C}(H, L)^r$, ainsi qu'une surjection $\sigma : \mathscr{D}_h(H)^r \to \mathscr{D}_h(H) \otimes_{\Lambda(H)} \Pi^*$. On munit $\mathscr{D}_h(H) \otimes_{\Lambda(H)} \Pi^*$ de la topologie quotient, induite par σ (l'espace $\mathscr{D}_h(H)$ ayant la topologie faible définie dans le § IV.C). La topologie faible de Π^* est alors la topologie quotient induite par la surjection $\Lambda(H)^r \to \Pi^*$.

Si $v \in \Pi^{(h)}$ et si $\lambda = \sum_\alpha c_\alpha b^\alpha \in \mathscr{D}_h(H)$, la série $\sum_\alpha c_\alpha b^\alpha$ converge dans $\Pi^{(h)}$. Cela permet de munir $\Pi^{(h)}$ (et donc $(\Pi^{(h)})^*$) d'une structure de $\mathscr{D}_h(H)$-module. L'application naturelle $\Pi^* \to (\Pi^{(h)})^*$ induit une application continue[21] $\mathscr{D}_h(H)$-linéaire $\mathscr{D}_h(H) \otimes_{\Lambda(H)} \Pi^* \to (\Pi^{(h)})^*$. Par passage à la limite on obtient aussi une application $\mathscr{D}(H)$-linéaire $\mathscr{D}(H) \otimes_{\Lambda(H)} \Pi^* \to (\Pi^{\mathrm{an}})^*$.

Proposition IV.13. (i) *L'application $\mathscr{D}_h(H) \otimes_{\Lambda(H)} \Pi^* \to (\Pi^{(h)})^*$ est un isomorphisme de L-espaces vectoriels topologiques.*

(ii) *L'application $\mathscr{D}(H) \otimes_{\Lambda(H)} \Pi^* \to (\Pi^{\mathrm{an}})^*$ est un isomorphisme de L-espaces vectoriels topologiques.*

Démonstration. La preuve est fortement inspirée de la preuve du th. 7.1 de [28] (qui est précisément la partie (ii) de la proposition). Le (ii) se déduit du (i) par passage à la limite, et pour le (i) il suffit de montrer la bijectivité de l'application en question.

Commençons par la surjectivité. Le plongement fermé $\Pi^{(h)} \to (\mathrm{LA}^{(h)}(H))^r$ du th. IV.6 se dualise en une surjection $\mathscr{D}_h(H)^r \to (\Pi^{(h)})^*$. Par construction, cette surjection se factorise par la surjection $\mathscr{D}_h(H)^r \to \mathscr{D}_h(H) \otimes_{\Lambda(H)} \Pi^*$, ce qui permet de conclure.

Pour démontrer l'injectivité, compte tenu du théorème de Hahn–Banach et de la dualité de Schikhof [27], il suffit de prouver la surjectivité de l'application transposée $\Pi^{(h)} \to (\mathscr{D}_h(H) \otimes_{\Lambda(H)} \Pi^*)^*$. L'application naturelle $\Pi^* \to \mathscr{D}_h(H) \otimes_{\Lambda_L} \Pi^*$ est continue (Π^* est muni de la topologie faible), d'image dense, car $\Lambda(H) \to \mathscr{D}_h(H)$ a ces propriétés. Cela montre que si $F \in (\mathscr{D}_h(H) \otimes_{\Lambda(H)} \Pi^*)^*$, alors il existe un unique $v \in \Pi$ tel que $F(1 \otimes l) = l(v)$ pour tout $l \in \Pi^*$. La continuité de F combinée au fait que $p^{-|\alpha| r_h} b^\alpha$ tend vers 0 dans $\mathscr{D}_h(H)$ pour la topologie faible montrent que

21. On munit dans la suite Π^* et $(\Pi^{(h)})^*$ de la topologie faible de dual de Banach.

$p^{-r_h|\alpha|}l(b^\alpha v)$ tend vers 0 pour tout $l \in \Pi^*$. On déduit du th. IV.6 que $v \in \Pi^{(h)}$ et on conclut en utilisant la densité de l'image de $\Pi^* \to \mathscr{D}_h(H) \otimes_{\Lambda(H)} \Pi^*$.

Pour démontrer le (ii), posons $M = \mathscr{D}(H) \otimes_{\Lambda(H)} \Pi^*$. Puisque Π^* est un $\Lambda(H)$-module de présentation finie, M est un $\mathscr{D}(H)$-module coadmissible [28], et donc M est isomorphe à la limite inverse des $\mathscr{D}_h(H) \otimes_{\Lambda(H)} \Pi^*$. Le (ii) se déduit donc de (i) et de l'isomorphisme $\Pi^{\mathrm{an}} \simeq \varinjlim_h \Pi^{(h)}$ fourni par le théorème d'Amice. □

Corollaire IV.14. *Le foncteur $\Pi \mapsto \Pi^{(h)}$ est exact de la catégorie des L-représentations de Banach admissibles de H dans la catégorie des L-banach.*

Démonstration. C'est une conséquence directe du théorème de l'image ouverte, de la proposition précédente et de la platitude de $\mathscr{D}_h(H)$ sur $\Lambda(H)$ (prop. IV.2). □

Remarque IV.15. Il serait plus naturel d'étudier le foncteur $\Pi \mapsto \Pi_h$, où Π_h est l'espace des vecteurs $v \in \Pi$ tels que la fonction $x \mapsto h^x \cdot v$ soit analytique sur $x_0 + p^h\mathbb{Z}_p^d$ pour tout $x_0 \in \mathbb{Z}_p^d$. La raison pour ne pas prendre ce point de vue est que ce foncteur n'est pas exact, ce qui est désagréable pour les applications. En effet, si $\mathrm{LA}_h(H) = \mathscr{C}(H)_h$, le théorème d'Amice [1] montre que $\phi \in \mathrm{LA}_h(H)$ si et seulement si

$$\lim_{|\alpha|\to\infty} v_p(a_\alpha(\phi)) - \sum_{i=1}^{d} v_p\Big(\Big[\frac{\alpha_i}{p^h}\Big]!\Big) = \infty,$$

donc le dual topologique de $\mathrm{LA}_h(H)$ est l'algèbre à puissances divisées partielles. [22] Or cette algèbre n'est pas plate sur $\Lambda(H)$ si $d \geq 2$. On remarquera toutefois que $\mathrm{LA}_h(H)^*$ est en fait très proche de $\mathscr{D}_{<h+1}(H)$ puisque $v_p([n/p^h]!) \sim nr_{h+1}$, et donc $\Pi^{(h+1)}$ est très semblable à Π_h.

IV.G. ***Cohérence.*** On suppose à partir de maintenant que $G = \mathrm{GL}_n(\mathbb{Q}_p)$ et on note $K = \mathrm{GL}_n(\mathbb{Z}_p)$ et Z le centre de G. Soit $H = 1 + p^\kappa \mathrm{M}_n(\mathbb{Z}_p)$; c'est un pro-p-groupe uniforme de dimension n^2 (rappelons que $\kappa = 1$ si $p > 2$ et $\kappa = 2$ si $p = 2$). Si $g \in G$, on note $d(g, 1)$ le plus petit entier m tel que $H^{p^m} \subset gHg^{-1}$. Puisque KZ normalise H et H^{p^m}, $d(g, 1)$ ne dépend que de $KZgKZ$.

Proposition IV.16. *Soit Π une L-représentation de Banach de G. Alors $\Pi^{(h)}$ est stable par K et $g \cdot \Pi^{(h)} \subset \Pi^{(h+n)}$ si $d(g, 1) \leq n$.*

Démonstration. La première assertion se démontre comme le cor. IV.8, en utilisant le fait que H est distingué dans K. La seconde découle de la prop. IV.11. □

D'après la décomposition de Cartan, on a $G = \bigsqcup_{t\in T^+} KtK$, où T^+ est l'ensemble des matrices $\mathrm{diag}(p^{a_1}, p^{a_2}, \dots, p^{a_n})$, avec $a_1 \geq a_2 \geq \cdots \geq a_n \in \mathbb{Z}$. Pour tout entier positif m, on note T_m^+ le sous-ensemble de T^+ des matrices $\mathrm{diag}(p^{a_1}, p^{a_2}, \dots, p^{a_n})$ avec $a_1 - a_n \leq m$. Le résultat suivant permet de montrer que $d(g, 1)$ est la distance entre les sommets représentés par g et 1 dans l'immeuble de Bruhat–Tits de PGL_n.

22. Ses éléments sont de la forme $\sum_{\alpha\in\mathbb{N}^d} c_\alpha \frac{1}{[\alpha/p^h]!} b^\alpha$ où $(c_\alpha)_\alpha$ est une suite bornée de L.

Lemme IV.17. *On a* $d(g, 1) \leq m$ *si et seulement si* $g \in KT_m^+K$. *En particulier,* $d(g, 1) = 0$ *si et seulement si* $g \in KZ$.

Démonstration. Soit $g = k_1 t k_2$, avec $k_1, k_2 \in K$ et $t = \mathrm{diag}(p^{a_1}, \ldots, p^{a_n}) \in T^+$. Puisque H et H^{p^m} sont distingués dans K, on a $g^{-1}H^{p^m}g \subset H$ si et seulement si $t^{-1}H^{p^m}t \subset H$. On conclut en utilisant les égalités $t(x_{ij})_{i,j}t^{-1} = (p^{a_i - a_j}x_{ij})_{i,j}$ et $H^{p^m} = 1 + p^{m+\kappa}\mathrm{M}_n(\mathbb{Z}_p)$. □

Lemme IV.18. *On a* $d(g, 1) \leq l + m$ *si et seulement si l'on peut écrire* $g = g_1 g_2$, *avec* $d(g_1, 1) \leq l$ *et* $d(g_2, 1) \leq m$.

Démonstration. Si $H^{p^m} \subset g_1 H g_1^{-1}$ et $H^{p^n} \subset g_2 H g_2^{-1}$, alors $H^{p^{m+n}} \subset g_1 g_2 H (g_1 g_2)^{-1}$, donc $d(g_1 g_2, 1) \leq d(g_1, 1) + d(g_2, 1)$ pour tous $g_1, g_2 \in G$. Dans l'autre sens, il suffit d'utiliser le lemme IV.17 et l'égalité $T_l^+ \cdot T_m^+ = T_{l+m}^+$. □

Si W est un sous-$L[KZ]$-module d'un $L[G]$-module Π et si $h \in \mathbb{N}$, on pose

$$W^{[h]} = \sum_{d(g,1) \leq h} g \cdot W;$$

c'est un sous-$L[KZ]$-module de Π. Notons que $g \cdot W$ et $d(g, 1)$ ne dépendent que de l'image de g dans $S := G/KZ$.

Lemme IV.19. *L'ensemble* $\{s \in S \mid d(s, 1) \leq h\}$ *est fini.*

Démonstration. Soit $T_{h,0}^+$ l'ensemble des $t = \mathrm{diag}(p^{a_1}, \ldots, p^{a_n}) \in T_h^+$ tels que $a_n = 0$. Alors $T_{h,0}^+$ est un ensemble fini et $T_{h,0}^+ Z \supset T_h^+$. Si I_h est un système de représentants de K/K_h (avec $K_h = 1 + p^h \mathrm{M}_n(\mathbb{Z}_p)$), alors

$$KT_h^+K \subset \bigcup_{\substack{t \in T_{h,0}^+ \\ k \in I_h}} kK_h tKZ \subset \bigcup_{\substack{t \in T_{h,0}^+ \\ k \in I_h}} ktKZ,$$

la dernière inclusion étant une conséquence du fait que $t^{-1}K_h t \subset K$ pour tout $t \in T_{h,0}^+$. On conclut en utilisant le lemme IV.17. □

On suppose maintenant que Π est une L-représentation de Banach admissible de G, ayant un caractère central. Le sous-espace $\Pi^{(h)}$ de Π (défini en considérant Π comme une représentation du pro-p-groupe uniforme H) est stable par KZ, puisque KZ normalise H.

Définition IV.20. On dit que la représentation Π est *cohérente* s'il existe $m(\Pi)$ tel que $\Pi^{(h+k)} = (\Pi^{(h)})^{[k]}$, pour tous $h \geq m(\Pi)$ et $k \in \mathbb{N}$.

Remarque IV.21. (i) L'inclusion $(\Pi^{(h)})^{[k]} \subset \Pi^{(h+k)}$ est vraie pour n'importe quelle représentation de Banach Π (cela découle de la prop. IV.11).

(ii) Le lemme IV.18 montre que $W^{[k+1]} = (W^{[k]})^{[1]}$. Pour montrer que Π est cohérente, il suffit donc de vérifier que $\Pi^{(h+1)} = (\Pi^{(h)})^{[1]}$ pour tout h assez grand.

(iii) Si Π est cohérente, alors $\Pi^{(h)}$ engendre $\Pi^{\rm an}$, en tant que représentation de G, pour tout h assez grand (cela découle de ce que $\Pi^{\rm an}$ est la réunion des $\Pi^{(h)}$). On peut se demander si une propriété de ce genre est automatiquement vérifiée pour une représentation de Banach admissible de G (au moins dans le cas de longueur finie). C'est le cas pour $G = \mathrm{GL}_2(\mathbb{Q}_p)$ (cf. le th. VII.11).

Proposition IV.22. *Soit* $0 \to \Pi_1 \to \Pi \to \Pi_2 \to 0$ *une suite exacte de représentations de Banach admissibles de* G. *Si* Π *est cohérente, alors* Π_2 *l'est aussi ; réciproquement, si* Π_1 *est cohérente, alors* Π *est cohérente si et seulement si* Π_2 *l'est.*

Démonstration. C'est une conséquence de l'exactitude du foncteur $\Pi \mapsto \Pi^{(h)}$. □

V. Vecteurs analytiques des représentations unitaires de $\mathrm{GL}_2(\mathbb{Q}_p)$

Ce chapitre étend (et raffine) à toutes les paires G-compatibles les résultats de [10, chap. V], concernant l'espace $\Pi_\delta(D)^{\rm an}$ des vecteurs localement analytiques de la représentation $\Pi_\delta(D)$. L'approche est assez différente de celle de [10] même si le noyau technique (à savoir le § V.E et, en particulier, la lemme V.18) est le même.

V.A. ***Préliminaires.*** On fixe dans la suite une paire G-compatible (D, δ), avec $D \in \Phi\Gamma^{\rm et}(\mathscr{E})$, et un réseau D_0 de D, stable par φ et Γ. On note $\Pi = \Pi_\delta(D)$ et $\Pi_0 = \Pi_\delta(D_0)$. Alors Π_0 est un réseau de Π, ouvert, borné et stable par G. On munit Π de la valuation v_Π, à valeurs dans $v_p(L)$, faisant de Π_0 la boule unité de Π.

On renvoie au chap. I pour les anneaux de fonctions analytiques utilisés dans la suite. Rappelons que $\Lambda(\Gamma)$ est l'anneau des mesures à valeurs dans $\mathscr{O}_L$ sur Γ. Si R est un anneau de séries de Laurent (comme $\mathscr{O}_{\mathscr{E}}$, $\mathscr{R}$, $\mathscr{O}_{\mathscr{E}}^{\dagger,b}$, etc.), on peut remplacer la variable T par $\gamma - 1$ où γ est un générateur topologique de Γ (ou plutôt de l'image inverse de $1 + p^\kappa \mathbb{Z}_p$ dans Γ, où $\kappa = 1$ si $p > 2$ et 2 si $p = 2$) pour construire un anneau $R(\Gamma)$ (on renvoie le lecteur au n° 3 du §V.1 de [10] pour les détails).

Rappelons que $m(D_0)$ est un entier assez grand, qui ne dépend que de D_0. Il est en particulier choisi tel que la prop. II.2 s'applique à D_0, et donc

$$D_0^\natural \boxtimes_\delta \mathbb{P}^1 \subset D_0^{(0, r_{m(D_0)}]} \boxtimes_\delta \mathbb{P}^1$$

(cf. [9, cor. II.7.2]). Comme $D_0^\natural$ est compact, il existe $l_1 = l_1(D_0)$ tel que

$$D_0^\natural \subset T^{-l_1} D_0^{\dagger, m(D_0)},$$

donc $D_0^\natural \boxtimes_\delta \mathbb{P}^1 \subset (T^{-l_1} D_0^{\dagger, m(D_0)}) \boxtimes_\delta \mathbb{P}^1$, où l'on note

$$X \boxtimes_\delta \mathbb{P}^1 = (D \boxtimes_\delta \mathbb{P}^1) \cap (X \times X)$$

pour $X \in \{D^{(0,r_b]}, D^\dagger\}$ (pareil avec D_0 si $X \in \{T^a D_0^{\dagger,b}, D_0^{(0,r_b]}\}$). Quitte à augmenter encore l_1 et $m(D_0)$, on peut supposer qu'ils sont aussi associés à $(\check{D}_0, \delta^{-1})$, et que l'involution i_δ de $\mathscr{O}_L[\Gamma]$ qui envoie σ_a sur $\delta(a)\sigma_{1/a}$ s'étend en une involution continue de $\mathscr{O}_{\mathscr{E}}^{\dagger,b}(\Gamma)$, $\mathscr{O}_{\mathscr{E}}^{(0,r_b]}(\Gamma)$, $\mathscr{R}(\Gamma)$ pour $b \geq m(D_0)$ (voir [10, lemme V.2.3]).

V.B. ***Vecteurs analytiques et surconvergence.*** Si $m \geq 2$, on note K_m le sous-groupe $1 + p^m \mathrm{M}_2(\mathbb{Z}_p)$ de G,

$$a_m^+ = \begin{pmatrix} 1+p^m & 0 \\ 0 & 1 \end{pmatrix}, \quad u_m^+ = \begin{pmatrix} 1 & p^m \\ 0 & 1 \end{pmatrix},$$
$$a_m^- = \begin{pmatrix} 1 & 0 \\ 0 & 1+p^m \end{pmatrix}, \quad u_m^- = \begin{pmatrix} 1 & 0 \\ p^m & 1 \end{pmatrix}$$

et, pour $\alpha \in \mathbb{N}^4$, on note

$$b_m^\alpha = (a_m^+ - 1)^{\alpha_1} \cdot (a_m^- - 1)^{\alpha_2} \cdot (u_m^+ - 1)^{\alpha_3} \cdot (u_m^- - 1)^{\alpha_4} \in \mathbb{Z}_p[K_m].$$

Alors $(a_m^+, a_m^-, u_m^+, u_m^-)$ est un système minimal de générateurs topologiques du pro-p groupe uniforme K_m.

Définition V.1. Si $b \geq m + 1$, soit

$$\Pi^{(b)} = \big\{ v \in \Pi \mid \lim_{|\alpha| \to \infty} v_\Pi(b_m^\alpha \cdot v) - p^m r_b \cdot |\alpha| = \infty \big\},$$

que l'on munit de la valuation

$$v^{(b)}(v) = \inf_{\alpha \in \mathbb{N}^4} (v_\Pi(b_m^\alpha \cdot v) - p^m r_b \cdot |\alpha|),$$

qui en fait un banach.

Remarque V.2. (i) Le théorème d'Amice [1] montre que, pour tout m, on a un isomorphisme d'espaces vectoriels topologiques

$$\varinjlim_b \Pi^{(b)} \simeq \Pi^{\mathrm{an}}.$$

(ii) Puisque $p^m r_b = r_{b-m}$, l'espace $\Pi^{(b)}$ n'est autre que l'espace $\Pi_{K_m}^{(b-m)}$ de l'introduction (ou, avec les notations du chapitre précédent, l'espace $\Pi^{(b-m)}$ correspondant au sous-groupe $H = K_m$ de G). Il résulte de la prop. IV.11 que l'espace $\Pi^{(b)}$ ne dépend pas du choix de $m \leq b-1$ (la valuation $v^{(b)}$ en dépend, mais les valuations obtenues en faisant varier m sont toutes équivalentes).

Le th. V.3 ci-dessous décrit le L-banach $\Pi^{(b)}$ en fonction de D. Nous commençons par préciser un peu les topologies sur les espaces divers et variés apparaissant dans ce théorème. On dispose d'une pléiade d'anneaux de séries de Laurent (voir le chap. I), chacun ayant une topologie naturelle. Cela induit une topologie naturelle sur les modules libres de type fini sur ces anneaux (et qui ne dépend pas des choix de bases). En appliquant cette discussion aux modules $D_0^{\dagger,b}$, $D_0^{(0,r_b]}$, $D^{(0,r_b]}$, $D^\dagger$, $D_{\rm rig}$, etc, on obtient des topologies sur ces modules.

Si $X \in \{D_0^{(0,r_b]}, D^{(0,r_b]}, D^\dagger\}$, on munit $X \boxtimes_\delta \mathbb{P}^1$ de la topologie induite par l'inclusion $X \boxtimes_\delta \mathbb{P}^1 \subset X \times X$. Cette topologie est plus forte que celle induite par l'inclusion $X \boxtimes_\delta \mathbb{P}^1 \subset D_0 \boxtimes_\delta \mathbb{P}^1$ (ou $X \subset D \boxtimes_\delta \mathbb{P}^1$). Comme $D_0^\natural \boxtimes_\delta \mathbb{P}^1$ est fermé dans $D_0 \boxtimes_\delta \mathbb{P}^1$, il est aussi fermé dans $D_0^{(0,r_b]} \boxtimes_\delta \mathbb{P}^1$ pour $b > m(D_0)$, donc $D^\natural \boxtimes_\delta \mathbb{P}^1$ est fermé dans $D^{(0,r_b]} \boxtimes_\delta \mathbb{P}^1$ et dans $D^\dagger \boxtimes_\delta \mathbb{P}^1$. On munit alors $(D_0^{(0,r_b]} \boxtimes_\delta \mathbb{P}^1)/(D_0^\natural \boxtimes_\delta \mathbb{P}^1)$ et $(X \boxtimes_\delta \mathbb{P}^1)/(D^\natural \boxtimes_\delta \mathbb{P}^1)$ de la topologie quotient, pour $X \in \{D^{(0,r_b]}, D^\dagger\}$.

On fixe une paire G-compatible (D, δ), avec $D \in \Phi\Gamma^{\rm et}(\mathscr{E})$ et on note $\Pi = \Pi_\delta(D)$ et $\check{\Pi} = \Pi_{\delta^{-1}}(\check{D})$.

Théorème V.3. *Il existe $c = c(D, \delta)$ tel que pour tout $b \geq c$:*

(i) *Le sous-module $D^{(0,r_b]} \boxtimes_\delta \mathbb{P}^1$ de $D \boxtimes_\delta \mathbb{P}^1$ est stable par* ${\rm GL}_2(\mathbb{Z}_p)$, *qui agit continûment.*

(ii) *On a un isomorphisme canonique de* ${\rm GL}_2(\mathbb{Z}_p)$*-modules de Banach*

$$(D^{(0,r_b]} \boxtimes_\delta \mathbb{P}^1)/(D^\natural \boxtimes_\delta \mathbb{P}^1) \simeq \Pi^{(b)},$$

et donc une suite exacte de ${\rm GL}_2(\mathbb{Z}_p)$*-modules topologiques*

$$0 \to \check{\Pi}^* \to D^{(0,r_b]} \boxtimes_\delta \mathbb{P}^1 \to \Pi^{(b)} \to 0.$$

Avant de passer à la preuve du th. V.3, qui occupera le reste de ce chapitre, donnons-en quelques conséquences. Le résultat suivant découle formellement du th. V.3 et de la rem. V.2.

Corollaire V.4. (i) *Le sous-module $D^\dagger \boxtimes_\delta \mathbb{P}^1$ de $D \boxtimes_\delta \mathbb{P}^1$ est stable par G, qui agit continûment.*

(ii) *On a un isomorphisme canonique de G-modules topologiques*

$$(D^\dagger \boxtimes_\delta \mathbb{P}^1)/(D^\natural \boxtimes_\delta \mathbb{P}^1) \simeq \Pi^{\rm an},$$

et donc une suite exacte de G-modules topologiques

$$0 \to \check{\Pi}^* \to D^\dagger \boxtimes_\delta \mathbb{P}^1 \to \Pi^{\rm an} \to 0.$$

Le th. V.3 fournit un raffinement du théorème de Schneider et Teitelbaum pour les objets de ${\rm Rep}(\delta)$.

Corollaire V.5. *Si* $\Pi \in \mathrm{Rep}_L(G)$ *il existe* $m_0 \geq 2$ *tel que* $\Pi^{(b)}$ *soit dense dans* Π^{an} (*et donc aussi dans* Π) *pour tout* $b \geq m_0 + 1$.

Démonstration. Disons que Π est *bonne* si elle satisfait le corollaire. Trivialement, toute représentation de dimension finie est bonne. Si $\Pi = \Pi_\delta(D)$ pour une paire G-compatible (D, δ), alors Π est bonne : cela découle du th. V.3, du cor. V.4 et de la densité de $D^{(0,r_b]} \boxtimes_\delta \mathbb{P}^1$ dans $D^\dagger \boxtimes_\delta \mathbb{P}^1$ (qui découle de celle de $D^{(0,r_b]}$ dans $D^\dagger$, elle-même conséquence de la densité de $\mathscr{E}^{(0,r_b]}$ dans $\mathscr{E}^\dagger$).

En utilisant le th. III.45, le cor. III.37 et ce que l'on vient de démontrer, nous pouvons conclure dans le cas général grâce au lemme suivant qui est une conséquence de l'exactitude des foncteurs $\Pi \mapsto \Pi^{(b)}$ et $\Pi \mapsto \Pi^{\mathrm{an}}$ (cor. IV.14). □

Lemme V.6. *Soit*

$$0 \to \Pi_1 \to \Pi \to \Pi_2 \to 0$$

une suite exacte dans $\mathrm{Rep}_L(G)$, *avec* Π_1 *bonne. Si une des* Π *et* Π_2 *est bonne, alors l'autre l'est aussi.*

Corollaire V.7. *Soient* $\Pi_1, \Pi_2 \in \mathrm{Rep}_L(G)$ *et soit* $f : \Pi_1^{\mathrm{an}} \to \Pi_2^{\mathrm{an}}$ *une application continue, linéaire et* $\mathfrak{gl}_2$*-équivariante. Alors il existe un sous-groupe ouvert compact* H *de* G *tel que* f *soit* H*-équivariante.*

Démonstration. D'après le cor. V.5, il existe $b_1 \geq 3$ tel que $\Pi_1^{(b_1)}$ soit dense dans Π_1^{an}. La boule unité X_{b_1} de $\Pi_1^{(b_1)}$ est bornée dans Π_1^{an} et f étant continue, on en déduit que $f(X_{b_1})$ est une partie bornée de Π_2^{an}. Comme Π_2^{an} est la limite inductive des $\Pi_2^{(b)}$, les applications de transition étant injectives et compactes, on en déduit qu'il existe $b \geq b_1$ tel que $f(X_{b_1}) \subset \Pi_2^{(b)}$. Ainsi, $f(\Pi_1^{(b_1)}) \subset \Pi_2^{(b)}$.

Soient $H = K_b$ et $v \in \Pi_1^{(b_1)}$, et soit $h_1, \ldots, h_4$ un système minimal de générateurs topologiques de H. Comme $v \in \Pi_1^{(b)}$, la rem. IV.15 montre que l'application $o_v : x \mapsto h^x \cdot v$ est analytique sur $\mathbb{Z}_p^4$. On peut donc écrire

$$h^x \cdot v = \sum_{\alpha \in \mathbb{N}^4} \frac{\partial^\alpha o_v}{\partial x^\alpha}(0) x^\alpha$$

pour $x \in \mathbb{Z}_p^4$. Chaque terme $(\partial^\alpha o_v / \partial x^\alpha)(0)$ s'écrit $P_\alpha(a^+, a^-, u^+, u^-)v$, où a^+, a^-, u^+, u^- désignent l'action infinitésimale de

$$\begin{pmatrix} \mathbb{Z}_p^* & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & \mathbb{Z}_p^* \end{pmatrix}, \quad \begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix} \quad \text{et} \quad \begin{pmatrix} 1 & 0 \\ \mathbb{Z}_p & 1 \end{pmatrix},$$

et $P_\alpha \in L[X, Y, Z, T]$. En appliquant f et en utilisant l'hypothèse on obtient

$$\begin{aligned} f(h^x \cdot v) &= \sum_{\alpha \in \mathbb{N}^4} f(P_\alpha(a^+, a^-, u^+, u^-)v) x^\alpha \\ &= \sum_{\alpha \in \mathbb{N}^4} P_\alpha(a^+, a^-, u^+, u^-) f(v) x^\alpha = h^x \cdot f(v). \end{aligned}$$

On en déduit que $f(hv) = hf(v)$ pour $h \in H$ et $v \in \Pi_1^{(b_1)}$. Comme f est continue et $\Pi_1^{(b_1)}$ est dense dans Π_1^{an}, on en déduit que f est H-équivariante, ce qui permet de conclure. □

Le corollaire suivant est un sous-produit de la preuve du th. V.3.

Corollaire V.8. *Soit* $\Pi \in \mathrm{Rep}_L(G)$. *Si* $x \mapsto \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) v$ *et* $x \mapsto \left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right) v$ *sont localement analytiques sur* $\mathbb{Q}_p$, *alors* $v \in \Pi^{\mathrm{an}}$.

Démonstration. Commençons par le cas où $\Pi = \Pi_\delta(D)$ pour une paire G-compatible (D, δ). La preuve de la prop. V.10 ci-dessous n'utilise que la croissance des coefficients de Mahler des applications $x \mapsto \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) v$ et $x \mapsto \left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right) v$. Elle montre que v admet un relèvement à $D^\dagger \boxtimes_\delta \mathbb{P}^1$. Le cor. V.4 permet alors de conclure. □

Passons au cas général. Disons que $v \in \Pi$ est *presqu'analytique* si $x \mapsto \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) v$ et $x \mapsto \left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right) v$ sont localement analytiques, et que Π est *bonne* si tout vecteur presqu'analytique est localement analytique (notons que tout vecteur localement analytique est trivialement presqu'analytique). On déduit du th. III.45, du cor. III.37 et du premier paragraphe l'existence d'un morphisme $\beta_\Pi : \Pi_1 \to \Pi / \Pi^{\mathrm{SL}_2(\mathbb{Q}_p)}$, dont le noyau et le conoyau sont de dimension finie, et tel que Π_1 soit bonne. On conclut en utilisant le lemme suivant :

Lemme V.9. *Soit* $0 \to \Pi_1 \to \Pi \to \Pi_2 \to 0$ *une suite exacte dans* $\mathrm{Rep}_L(G)$.

(i) *Si* Π_1 *est de dimension finie et si* Π *ou* Π_2 *est bonne, alors l'autre l'est aussi.*

(ii) *Si* Π_2 *est de dimension finie et si* Π_1 *est bonne, alors* Π *est bonne.*

Démonstration. (i) Supposons d'abord que Π est bonne et soit $v \in \Pi_2$ presqu'analytique. Soit $\hat{v} \in \Pi$ un relèvement de v. Notons $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) - 1 \in \mathscr{O}_L[G]$. Comme v est presqu'analytique, il existe $r > 0$ tel que $p^{-rn} T^n(v)$ tende vers 0 dans Π_2, ce qui veut dire qu'il existe $x_n \in \Pi_1$ tels que $p^{-rn} T^n(\hat{v}) - x_n \to 0$ dans Π. Puisque Π_1 est de dimension finie, elle est tuée par T (lemme III.42) et donc $p^{-rn} T^{n+1}(\hat{v}) \to 0$ dans Π. Le théorème d'Amice entraîne que $x \mapsto \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \hat{v}$ est localement analytique. On obtient de même que $x \mapsto \left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right) \hat{v}$ est localement analytique, et donc $\hat{v}$ est localement analytique (car Π est bonne) et son image v dans Π_2 l'est aussi. Cela montre que Π_2 est bonne.

Le reste de l'énoncé est une conséquence de l'exactitude du foncteur $\Pi \mapsto \Pi^{\mathrm{an}}$. □

V.C. ***Relèvement à*** $\boldsymbol{D^{(0,r_b]} \boxtimes_\delta \mathbb{P}^1}$. La proposition ci-dessous montre que tout $v \in \Pi^{(b)}$ se relève à $D^{(0,r_b]} \boxtimes_\delta \mathbb{P}^1$. Rappelons que l_1 est choisi tel que $D_0^\natural \subset T^{-l_1} D_0^{\dagger,b}$ (cf. le § V.A).

Proposition V.10. *Soient* $b > m > m(D_0)$ *et* $v \in \Pi^{(b)}$. *Alors* v *admet un relèvement à* $(p^s T^{s'} D_0^{\dagger,b}) \boxtimes_\delta \mathbb{P}^1 \subset D^{(0,r_b]} \boxtimes_\delta \mathbb{P}^1$, *avec* $s = v^{(b)}(v)$ *et* $s' = -(p^m n_b + l_1)$.

Démonstration. On peut supposer que $v^{(b)}(v) \geq 0$, quitte à multiplier v par une puissance de p. Notons $\xi = (u_m^+)^{n_b}$, $\mu = (u_m^-)^{n_b}$ et

$$a_k = \min(v_\Pi(\xi^k v), v_\Pi(\mu^k v)),$$

de telle sorte que $\lim_{k\to\infty} a_k - p^m k = \infty$ et $a_k \geq p^m k + v^{(b)}(v)$ pour tout k (par définition de $\Pi^{(b)}$ et $v^{(b)}$).

Soient $X = D_0 \boxtimes_\delta \mathbb{P}^1$ et $Y = D_0^\natural \boxtimes_\delta \mathbb{P}^1$, de telle sorte que $\Pi_0 = X/Y$ est la boule unité de Π pour la valuation v_Π. Alors $p^{a_0}\Pi_0 \subset \Pi_0$, car $a_0 \geq 0$, et $v \in p^{a_0}\Pi_0$, puisque $v_\Pi(v) \geq a_0$. Ainsi, v possède un relèvement $z = (z_1, z_2) \in p^{a_0}X$. Nous aurons besoin du lemme suivant.

Lemme V.11. *Il existe une suite $(y_n)_{n\geq 0}$ d'éléments de Y telle que pour tout $n \geq 1$*

$$\xi^n z - \sum_{k=0}^{n-1} p^{a_k}\xi^{n-k-1} y_k \in p^{a_n} X.$$

Démonstration. On construit la suite en question par récurrence. Noter que X et Y sont stables par ξ et η (car ils sont stables par G), et que $p^n X \cap Y = p^n Y$ (pour $n \geq 0$), car Y est un sous-$\mathscr{O}_L$-module saturé de X par définition.

Supposons d'abord que $n = 1$. Si $a_1 < a_0$, on prend $y_0 = 0$, supposons donc que $a_1 \geq a_0$. Comme $v_\Pi(\xi v) \geq a_1 \geq 0$, on a $\xi z \in (p^{a_1}X + Y) \cap p^{a_0}X \subset p^{a_1}X + p^{a_0}Y$, ce qui montre l'existence de y_0. Supposons avoir trouvé $y_0, \ldots, y_{n-1}$ et écrivons

$$\xi^n z = \sum_{k=0}^{n-1} p^{a_k}\xi^{n-k-1} y_k + p^{a_n} u$$

pour un $u \in X$. Si $a_n > a_{n+1}$, on prend $y_n = 0$. Sinon, en appliquant ξ à l'égalité précédente on obtient $p^{a_n}\xi u \in \xi^{n+1}z + Y \subset p^{a_{n+1}}X + Y$ (la deuxième inclusion suit de $v_\Pi(\xi^{n+1}v) \geq a_{n+1}$). On en déduit que $p^{a_n}\xi u \in p^{a_{n+1}}X + p^{a_n}Y$ et on choisit $y_n \in Y$ tel que $p^{a_n}\xi u - p^{a_n}y_n \in p^{a_{n+1}}X$. Cela permet de conclure. □

Revenons à la preuve de la proposition. En appliquant $\mathrm{Res}_{\mathbb{Z}_p}$ à la relation du lemme V.11, et en utilisant le fait que ξ agit par multiplication par $\varphi^m(T)^{n_b}$, on obtient $z_1 - \sum_{k=0}^{n-1} A_k \in p^{a_n} D_0$, avec

$$A_k = \frac{p^{a_k}}{\varphi^m(T)^{(k+1)n_b}} \mathrm{Res}_{\mathbb{Z}_p}(y_k) = p^{a_k - p^m k} \frac{1}{\varphi^m(T)^{n_b}} \left(\frac{p^{p^m}}{\varphi^m(T)^{n_b}}\right)^k \mathrm{Res}_{\mathbb{Z}_p}(y_k)$$

Le lemme I.5 montre que $1/\varphi^m(T)^{n_b} \in T^{-p^m n_b}\mathscr{O}_{\mathscr{E}}^{\dagger,b}$. Puisque $p \in T^{n_b}\mathscr{O}_{\mathscr{E}}^{\dagger,b}$, on en déduit que $p^{p^m}/\varphi^m(T)^{n_b} \in \mathscr{O}_{\mathscr{E}}^{\dagger,b}$. En utilisant aussi le fait que $\mathrm{Res}_{\mathbb{Z}_p}(y_k) \in D_0^\natural \subset T^{-l_1}D_0^{\dagger,b}$, on obtient enfin

$$A_k \in p^{a_k - p^m k} T^{-l_1 - p^m n_b} D_0^{\dagger,b}.$$

Puisque $a_k - p^m k$ tend vers ∞ et est minoré par $v^{(b)}(v)$, et puisque $D_0^{\dagger,b}$ est complet pour la topologie p-adique, le paragraphe précédent montre que $\sum_{k\geq 0} A_k$ converge dans $p^{v^{(b)}(v)}T^{-l_1-p^m n_b}D_0^{\dagger,b}$. La relation $z_1 - \sum_{k=0}^{n-1} A_k \in p^{a_n}D_0$ montre que $\sum_{k\geq 0} A_k$ tend vers z_1 dans $\mathscr{O}_{\mathscr{E}}$. On en déduit que $\sum_{k\geq 0} A_k$ tend vers z_1 dans $p^{v^{(b)}(v)}T^{-l_1-p^m n_b}D_0^{\dagger,b}$, et donc que

$$z_1 \in p^{v^{(b)}(v)}T^{-l_1-p^m n_b}D_0^{\dagger,b}.$$

Les mêmes arguments (remplacer dans ce qui précède ξ par μ et $\mathrm{Res}_{\mathbb{Z}_p}$ par $\mathrm{Res}_{\mathbb{Z}_p}(w\cdot))$ donnent la même estimée pour z_2, ce qui permet de conclure. □

Remarque V.12. Supposons que $D^\natural = D^\sharp$ (c'est par exemple le cas si D est irréductible de dimension ≥ 2), de telle sorte que l'inclusion de $\tilde{D}$ dans $D\boxtimes_\delta \mathbb{P}^1$ induise un isomorphisme de B-modules de Banach $\Pi \simeq \tilde{D}/\tilde{D}^+$ (cor. III.26). En posant $X = \tilde{D}_0$ et $Y = \tilde{D}_0^+$, on vérifie sans mal que le lemme V.11 s'applique encore (le point est que X et Y sont stables par ξ et μ, et $Y \cap p^n X = p^n Y$ pour $n \geq 0$). Le reste de la preuve s'applique et montre que tout $v \in \Pi^{(b)}$ admet un relèvement à $\tilde{D}^{(0,r_b]}$ (cf. [4] pour la définition de ce module). Par contre, l'image dans Π d'un élément de $\tilde{D}^{(0,r_b]}$ n'est pas toujours localement analytique.

Il nous reste à montrer que l'image de $D^{(0,r_b]}\boxtimes_\delta \mathbb{P}^1$ dans Π est contenue dans $\Pi^{(b)}$, si m et b sont assez grands. Cela va demander un certain nombre d'estimées techniques, auxquelles sont dédiées les paragraphes V.D et V.E ci-dessous.

V.D. ***Vecteurs propres de ψ.*** Si $\alpha \in \mathscr{O}_L^*$, on pose

$$\mathscr{C}_0^\alpha = (1-\alpha\cdot\varphi)D_0^{\psi=\alpha} \subset D_0 \boxtimes_\delta \mathbb{Z}_p^*.$$

La proposition suivante est une version de la prop. V.2.1 de [10].[23] Voir le § V.A pour les objets i_δ, $\Lambda(\Gamma)$, etc.

Proposition V.13. *Soit $P \in \mathscr{O}_L[X]$ tel que $P(\psi) = 0$ sur $D_0^\sharp/D_0^\natural$. Soit $\alpha \in \mathscr{O}_L^*$ tel que α et $\beta := (\delta(p)\alpha)^{-1}$ ne soient pas des racines de P et α^{-1} et β^{-1} ne soient pas des valeurs propres de φ sur D^{nr}. Alors $w_\delta(\mathscr{C}_0^\alpha)\cap\mathscr{C}_0^\beta$ est d'indice fini dans $\mathscr{C}_0^\beta$.*

Démonstration. On commence par montrer que $w_\delta(\mathscr{C}_0^\alpha)\otimes_{\mathscr{O}_L} L = \mathscr{C}_0^\beta \otimes_{\mathscr{O}_L} L$. Soit $z \in D_0^{\psi=\alpha}$ et soit $z' = P(\alpha)z = P(\psi)z$. Comme

$$D_0^{\psi=\alpha} \subset D_0^\sharp \quad \text{et} \quad P(\psi)D_0^\sharp \subset D_0^\natural$$

23. Cette proposition n'est vraie qu'après tensorisation par L, le problème étant que $D_0^{\psi=1}$ n'est pas toujours contenu dans $D_0^\natural$, même sous les hypothèses de [loc. cit.]. Comme le montre la suite, cela ne change rien aux arguments.

par hypothèse, on a $z' \in D_0^\natural$, donc $(\alpha^{-n}z')_{n\geq 0} \in D_0^\natural \boxtimes_\delta \mathbb{Q}_p$. Comme $D_0^\natural \boxtimes_\delta \mathbb{P}^1$ se surjecte sur $D_0^\natural \boxtimes_\delta \mathbb{Q}_p$ (rem. III.6), il existe $x = (x_1, x_2) \in D_0^\natural \boxtimes_\delta \mathbb{P}^1$ tel que

$$\mathrm{Res}_{\mathbb{Z}_p}\left(\begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix} x\right) = \alpha^{-n} z'$$

pour tout $n \geq 0$. Alors (ibid.) $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)x - \alpha^{-1}x \in \mathrm{Ker}(\mathrm{Res}_{\mathbb{Q}_p}) = (0, D_0^{\mathrm{nr}})$ et un petit calcul montre que ceci entraîne $\psi(x_2) - \beta x_2 \in D_0^{\mathrm{nr}}$. Comme β^{-1} n'est pas valeur propre de $\varphi \in \mathrm{End}_L(D^{\mathrm{nr}})$, il existe $u \in D^{\mathrm{nr}}$ tel que $\beta\varphi(u) - u = \psi(x_2) - \beta x_2$. Alors $x_2 + \varphi(u) \in D_0^{\psi=\beta} \otimes_{\mathscr{O}_L} L$ et donc $\mathrm{Res}_{\mathbb{Z}_p^*}(x_2) = \mathrm{Res}_{\mathbb{Z}_p^*}(x_2 + \varphi(u)) \in \mathscr{C}_0^\beta \otimes_{\mathscr{O}_L} L$. Comme

$$\mathrm{Res}_{\mathbb{Z}_p^*}(x_2) = w_\delta(\mathrm{Res}_{\mathbb{Z}_p^*}(x_1)) = P(\alpha) w_\delta((1-\alpha\varphi)(z))$$

et $P(\alpha) \neq 0$, on conclut que $w_\delta(\mathscr{C}_0^\alpha \otimes_{\mathscr{O}_L} L) \subset \mathscr{C}_0^\beta \otimes_{\mathscr{O}_L} L$. Par symétrie et puisque w_δ est une involution, cette inclusion est une égalité.

Comme $\mathscr{C}_0^\alpha$ et $\mathscr{C}_0^\beta$ sont des $\Lambda(\Gamma)$-modules de type fini [9, corollaire VI.1.3] et comme w_δ est i_δ-semi-linéaire, le paragraphe précédent montre l'existence d'une constante $c = c(P, \alpha, D_0)$ telle que $w_\delta(\mathscr{C}_0^\beta) \subset p^{-c}\mathscr{C}_0^\alpha$. Soit alors $x \in \mathscr{C}_0^\beta$. On vient de voir qu'il existe $y \in D_0^{\psi=\alpha}$ tel que $w_\delta(x) = p^{-c}(1-\alpha\varphi)y$. Si y' est un autre choix, alors $y - y' \in D_0^{\varphi=1/\alpha}$. Comme $w_\delta(x) \in D_0$, on a $y \pmod{p^c} \in (D_0/p^c D_0)^{\varphi=1/\alpha}$. De plus, si $y \pmod{p^c} = 0$, alors $x \in w_\delta(\mathscr{C}_0^\alpha) \cap \mathscr{C}_0^\beta$. Ainsi, l'application $x \mapsto y \pmod{p^c}$ induit une injection de $\mathscr{C}_0^\beta/(\mathscr{C}_0^\beta \cap w_\delta(\mathscr{C}_0^\alpha))$ dans le quotient de $(D_0/p^c D_0)^{\varphi=1/\alpha}$ par l'image de $D_0^{\varphi=1/\alpha}$. Comme ce quotient est fini (car $(D_0/p^c D_0)^{\varphi=1/\alpha}$ est contenu dans $(D_0/p^c D_0)^{\mathrm{nr}}$, qui est fini), cela permet de conclure. □

Remarque V.14. (i) Comme $D_0^\sharp/D_0^\natural$ est un $\mathscr{O}_L$-module de type fini, on peut toujours trouver un polynôme non nul P comme dans la proposition précédente. Si D n'a pas de composante de Jordan–Hölder de rang 1, on peut même prendre P de la forme p^N car alors $D_0^\sharp/D_0^\natural$ est de torsion.

(ii) Soient $D \in \Phi\Gamma^{\mathrm{et}}(\mathscr{E})$ et δ formant une paire G-compatible, et soit $\check{\Pi} = \Pi_{\delta^{-1}}(\check{D})$. Si $\alpha \in \mathscr{O}_L^*$, notons $\check{\Pi}^*(\alpha)$ l'ensemble des $\mu \in \check{\Pi}^*$ vérifiant $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\mu = \alpha^{-1}\mu$. La preuve de la prop. V.13 montre que, pour tout $\alpha \in \mathscr{O}_L^*$, l'application $\mathrm{Res}_{\mathbb{Z}_p^*}$ envoie $\check{\Pi}^*(\alpha)$ sur un sous-L-espace vectoriel de codimension finie de $\mathscr{C}^\alpha = (1-\alpha\varphi)D^{\psi=\alpha}$; en particulier $\check{\Pi}^*(\alpha)$ est un $\Lambda(\Gamma)[1/p]$-module de type fini dont le quotient par son sous-module de torsion est libre de rang égal à celui de D. Plus précisément, si α n'est pas valeur propre de φ sur D^{nr} et si $(\delta(p)\alpha)^{-1}$ n'est pas valeur propre de ψ sur $D^\sharp/D^\natural$, ce qui est le cas pour tout α si D n'a pas de composante de Jordan–Hölder de rang 1, alors $\mathrm{Res}_{\mathbb{Z}_p}$ (resp. $\mathrm{Res}_{\mathbb{Z}_p^*}$) induit un isomorphisme de $\check{\Pi}^*(\alpha)$ sur $D^{\psi=\alpha}$ (resp. $\mathscr{C}^\alpha$) et donc $\check{\Pi}^*(\alpha)$ est un $\Lambda(\Gamma)[1/p]$-module libre de rang égal à celui de D.

V.E. ***L'action de G sur $T^a D_0^{\dagger,b} \boxtimes_\delta \mathbb{P}^1$.*** Le but de ce paragraphe est de contrôler l'action de G sur les modules $(T^a D_0^{\dagger,b}) \boxtimes_\delta \mathbb{P}^1$, plus précisément de démontrer la prop. V.19 ci-dessous. La plupart des arguments sont adaptés de [10, chap. V]. On fixe une base $e_1, e_2, \dots, e_d$ de $D_0^{\dagger,m(D_0)}$ sur $\mathscr{O}_{\mathscr{E}}^{\dagger,m(D_0)}$ (c'est aussi une base de $D_0^{\dagger,b}$ sur $\mathscr{O}_{\mathscr{E}}^{\dagger,b}$ pour tout $b \geq m(D_0)$). Les constantes $c, c_1, c_2, \dots, m_1, m_2, \dots$ ci-dessous ne dépendent que de D_0, δ et du choix de la base $e_1, e_2, \dots, e_d$.

Proposition V.15. *Il existe $m_1 \geq m(D_0)$ tel que w_δ laisse stable $D_0^{(0,r_b]} \boxtimes_\delta \mathbb{Z}_p^*$ pour tout $b \geq m_1$.*

Démonstration. On choisit P, α et β comme dans la prop. V.13 et on note M le $\Lambda(\Gamma)$-module $w_\delta(\mathscr{C}_0^\alpha) \cap \mathscr{C}_0^\beta$. On choisit ensuite $m_1 \geq m(D_0)$ tel que pour tout $b \geq m_1$:

- $\mathscr{C}_0^?$ est inclus dans $D_0^{(0,r_b]} \boxtimes_\delta \mathbb{Z}_p^*$ et $\mathscr{O}_{\mathscr{E}}^{(0,r_b]}(\Gamma) \otimes_{\Lambda(\Gamma)} \mathscr{C}_0^? = D_0^{(0,r_b]} \boxtimes_\delta \mathbb{Z}_p^*$, si $? \in \{\alpha, \beta\}$ (un tel m_1 existe ; cf. [10, corollaire V.1.13]).
- L'inclusion de M dans $\mathscr{C}_0^\beta$ induit un isomorphisme

$$\mathscr{O}_{\mathscr{E}}^{(0,r_b]}(\Gamma) \otimes_{\Lambda(\Gamma)} M = \mathscr{O}_{\mathscr{E}}^{(0,r_b]}(\Gamma) \otimes_{\Lambda(\Gamma)} \mathscr{C}_0^\beta$$

  (cette condition est automatique car $\mathscr{C}_0^\beta / M$ est tué par une puissance de $\sigma_{1+p} - 1$ puisqu'il de longueur finie sur $\mathscr{O}_L$ d'après la prop. V.13, et $\sigma_{1+p} - 1$ est inversible dans $\mathscr{O}_{\mathscr{E}}^{(0,r_b]}(\Gamma)$).

Alors $\mathscr{O}_{\mathscr{E}}^{(0,r_b]}(\Gamma) \otimes_{\Lambda(\Gamma)} M = D_0^{(0,r_b]} \boxtimes_\delta \mathbb{Z}_p^*$ et $w_\delta(M) \subset \mathscr{C}_0^\alpha \subset D_0^{(0,r_b]} \boxtimes_\delta \mathbb{Z}_p^*$ (par définition de M), ce qui permet de conclure, en utilisant la i_δ-semi-linéarité de w_δ. □

Corollaire V.16. *Si $b > m_1$, alors $D_0^{(0,r_b]} \boxtimes_\delta \mathbb{P}^1$ est stable par $\mathrm{GL}_2(\mathbb{Z}_p)$ et $D^\dagger \boxtimes_\delta \mathbb{P}^1$ est stable par G.*

Démonstration. Cf. [10, lemme II.1.10] : c'est une conséquence formelle des formules donnant l'action de G sur $D_0 \boxtimes_\delta \mathbb{P}^1$ (cf. la rem. III.3), de la prop. V.15 et des inclusions

$$\psi(D_0^{(0,r_b]}) \subset D_0^{(0,r_{b-1}]}, \quad \varphi(D_0^{(0,r_{b-1}]}) \subset D_0^{(0,r_b]} \quad \text{et} \quad \sigma_a(D_0^{(0,r_b]}) \subset D_0^{(0,r_b]},$$

qui impliquent que $D_0^{(0,r_b]}$ est stable par $\mathrm{Res}_{p\mathbb{Z}_p} = \varphi \circ \psi$. □

Si $b \geq m > m_1$, on note $\tau_m = \sigma_{1+p^m} - 1$ et

$$M_m^{\dagger,b} = (1+T)\varphi^m(D_0^{\dagger,b}) \quad \text{et} \quad M_m^{(0,r_b]} = (1+T)\varphi^m(D_0^{(0,r_b]}).$$

Remarquons que $\left(\begin{smallmatrix} 1+p^m & 0 \\ 0 & 1 \end{smallmatrix}\right) - 1$ agit comme τ_m sur D_0 et $\left(\begin{smallmatrix} 1 & p^m \\ 0 & 1 \end{smallmatrix}\right) - 1$ agit par multiplication par $\varphi^m(T)$. Le résultat suivant ([10, prop. V.1.14] et sa preuve) compare les deux actions, ce qui est fondamental pour la suite. Rappelons que l'on a fixé une base $(e_i)_i$ du $\mathscr{O}_{\mathscr{E}}^{\dagger,b}$-module $D_0^{\dagger,b}$, et que $\Gamma_m = \chi^{-1}(1+p^m\mathbb{Z}_p)$. Si $m \geq 2$, on définit des anneaux $\mathscr{O}_{\mathscr{E}}^{\dagger,b}(\Gamma_m)$, etc., en remplaçant simplement la variable T par τ_m.

Proposition V.17. *Il existe $m_2 > m_1$ tel que pour tous $b \geq m \geq m_2$ on ait :*

(i) *τ_m est bijectif sur $M_m^{(0,r_b]}$ et τ_m^a induit une bijection de $M_m^{\dagger,b}$ sur $\varphi^m(T)^a \cdot M_m^{\dagger,b}$, pour tout $a \in \mathbb{Z}$.*

(ii) *$M_m^{\dagger,b}$ (resp. $M_m^{(0,r_b]}$) est un $\mathscr{O}_{\mathscr{E}}^{\dagger,b}(\Gamma_m)$-module (resp. $\mathscr{O}_{\mathscr{E}}^{(0,r_b]}(\Gamma_m)$-module) libre de base $((1+T)\varphi^m(e_i))_i$.*

On fixe un tel m_2 et on le note simplement m. Voir le § V.B pour les notations K_m, a_m^+, a_m^-, b_m^α, etc.

Lemme V.18. *Il existe une constante $c \geq 1$ telle que :*

(i) *$w_\delta(\tau_m^a M_m^{\dagger,b}) \subset \tau_m^{a-c} M_m^{\dagger,b}$ pour tous $b \geq m$ et $a \in \mathbb{Z}$.*

(ii) *$(g-1)^n(\tau_m^a M_m^{\dagger,b}) \subset \tau_m^{n+a-c} M_m^{\dagger,b}$ pour tous $b \geq m$, $a \in \mathbb{Z}$, $n \geq 0$ et $g \in K_m$.*

Démonstration. (i) Notons $f_i := (1+T)\varphi^m(e_i) \in D_0^{(0,r_{2m}]}$. Alors (prop. V.15) $w_\delta(f_i) \in D_0^{(0,r_{2m}]} = D_0^{\dagger,2m}[1/T]$. On fixe c' tel que

$$w_\delta(f_i) \in \varphi^m(T)^{-c'+l(D_0)} D_0^{\dagger,2m}$$

pour $1 \leq i \leq d$ (voir le lemme II.3 pour $l(D_0)$). Comme w_δ commute à $\mathrm{Res}_{1+p^m\mathbb{Z}_p}$ (car $1+p^m\mathbb{Z}_p$ est stable par w), il existe $g_i \in D_0$ tels que $w_\delta(f_i) = (1+T)\varphi^m(g_i)$. Alors $\varphi^m(g_i) \in \varphi^m(T)^{-c'+l(D_0)} D_0^{\dagger,2m}$, donc $g_i \in T^{-c'} D_0^{\dagger,m}$ (utiliser le lemme II.3 et l'identité $g_i = \psi^m(\varphi^m(g_i))$) et donc finalement $w_\delta(f_i) \in \varphi^m(T)^{-c'} M_m^{\dagger,m} = \tau_m^{-c'} M_m^{\dagger,m}$ (prop. V.17).

Soient enfin $b \geq m$, $a \in \mathbb{Z}$ et notons $X = M_m^{\dagger,b}$. Comme les e_i forment une base de $D_0^{\dagger,b}$ sur $\mathscr{O}_{\mathscr{E}}^{\dagger,b}$, les f_i forment une base de X sur $\mathscr{O}_{\mathscr{E}}^{\dagger,b}(\Gamma_m)$ (prop. V.17). En utilisant la i_δ-semi-linéarité de w_δ et le fait que $w_\delta(f_i) \in \tau_m^{-c'} X$, on obtient $w_\delta(\tau_m^a X) \subset \tau_m^{a-c'} X$, ce qui permet de conclure.

(ii) On va montrer que $c = 8c'$ marche (avec c' comme dans la preuve de (i), dont on garde les notations). Le (i) de prop. V.17 montre que $(g-1)^n(\tau_m^a X) = \tau_m^{a+n} X$ si $a \in \mathbb{Z}$ et $g \in \{a_m^+, u_m^+\}$. En combinant cela avec le (i), on obtient pour $g \in \{u_m^+, a_m^+\}$

$$(wgw-1)^n(\tau_m^a X) = w(g-1)^n w(\tau_m^a X) \subset w(\tau_m^{a+n-c'} X) \subset \tau_m^{a+n-2c'} X$$

et donc $b_m^\alpha(\tau_m^a X) \subset \tau_m^{|\alpha|+a-c} X$.

Soit maintenant $g \in K_m$ quelconque et écrivons $(g-1)^n = \sum_{\alpha \in \mathbb{N}^4} c_\alpha b^\alpha$ dans $\Lambda(K_m)$. Alors $c_\alpha \in \mathbb{Z}_p$ et $v_p(c_\alpha) \geq n - |\alpha|$ quand $|\alpha| < n$. Comme p est multiple de $\tau_m^{n_b}$ (et donc de τ_m) dans $\mathscr{O}_{\mathscr{E}}^{\dagger,b}(\Gamma_m)$, on obtient $c_\alpha b^\alpha(\tau_m^a X) \subset \tau_m^{\max(n,|\alpha|)+a-c} X$ pour tout $\alpha \in \mathbb{N}^4$. Comme X est complet pour la topologie τ_m-adique (car $\mathscr{O}_{\mathscr{E}}^{\dagger,b}(\Gamma_m)$ l'est), cela permet de conclure que $(g-1)^n(\tau_m^a X) \subset \tau_m^{n+a-c} X$, ce qui finit la preuve. □

Proposition V.19. *Il existe $c_1 > c$ tel que :*

(i) *Pour tout $a \in \mathbb{Z}$ on a $w_\delta((T^a D_0^{\dagger,b})^{\psi=0}) \subset (T^{a-c_1} D_0^{\dagger,b})^{\psi=0}$.*

(ii) *Pour tous $b > 2m$, $a \in \mathbb{Z}$, $n \geq 1$ et $g \in K_m$,*

$$(g-1)^n (T^a D_0^{\dagger,b}) \subset T^{a+p^m n - c_1} D_0^{\dagger,b}.$$

Démonstration. On va montrer que l'on peut prendre $c_1 = p^m(1+c+l(D_0))$. Fixons $b > 2m$, $a \in \mathbb{Z}$, $n \geq 1$ et notons, pour simplifier, $q = [a/p^m]$ et $Y = \tau_m^{q-l(D_0)} M_m^{\dagger,b-m}$.

Lemme V.20. *Soit A (resp. B) l'ensemble des $i \in \{0, 1, \ldots, p^m - 1\}$ tels que p ne divise pas i (resp. p divise i). Alors*

$$T^a D_0^{\dagger,b} \subset \sum_{i\in A} \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} Y + \sum_{i \in B} \begin{pmatrix} 1 & i-1 \\ 0 & 1 \end{pmatrix} Y \quad \textit{et} \quad (T^a D_0^{\dagger,b})^{\psi=0} \subset \sum_{i\in A} \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} Y.$$

Démonstration. Soit $z \in T^a D_0^{\dagger,b}$ et posons $z_i = \psi^m((1+T)^{-i} z)$, de telle sorte que $z = \sum_{i=0}^{p^m-1} (1+T)^i \cdot \varphi^m(z_i)$ et $z_i \in T^{q-l(D_0)} D_0^{\dagger,b-m}$ (lemme II.3). On en déduit (prop. V.17) que $x_i = (1+T)\varphi^m(\sigma_{1/i}(z_i))$ (pour $i \in A$) et $y_i = (1+T)\varphi^m(z_i)$ (pour $i \in B$) sont des éléments de Y et on conclut en remarquant que

$$z = \sum_{i \in A} \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} x_i + \sum_{i\in B} \begin{pmatrix} 1 & i-1 \\ 0 & 1 \end{pmatrix} y_i.$$

La deuxième assertion s'en déduit, car si $\psi(z) = 0$, alors $z_i = 0$ pour tout $i \in B$. □

Revenons à la preuve de la prop. V.19. En appliquant le (ii) du lemme V.18, la prop. V.17(i) et le lemme I.5 (dans cet ordre) on obtient, pour $g \in K_m$,

$$\begin{aligned}(g-1)^n(Y) \subset \tau_m^{q-l(D_0)-c+n} M_m^{\dagger,b-m} &= \varphi^m(T)^{q-c-l(D_0)+n} M_m^{\dagger,b-m} \\ &\subset T^{p^m(q-c-l(D_0)+n)} D_0^{\dagger,b} \subset T^{a+p^m n - c_1} D_0^{\dagger,b}.\end{aligned}$$

On conclut pour le (ii) en utilisant le lemme V.20 et le fait que K_m est distingué dans $\mathrm{GL}_2(\mathbb{Z}_p)$. Le (i) se démontre de la même façon, en utilisant le (i) du lemme V.18. □

V.F. ***Fin de la preuve du th. V.3.*** On note $p_\Pi : D \boxtimes_\delta \mathbb{P}^1 \to \Pi$ la projection naturelle. Elle envoie $D_0 \boxtimes_\delta \mathbb{P}^1$ dans Π_0. Couplée avec la prop. V.10, la proposition ci-dessous permet de conclure quant à la preuve du th. V.3.

Proposition V.21. *Il existe une constante c_2 telle que si $a \in \mathbb{Z}$, $b > 2m+1$ et $z \in (T^a D_0^{\dagger,b}) \boxtimes_\delta \mathbb{P}^1$, alors $v := p_\Pi(z) \in \Pi^{(b)}$ et $v^{(b)}(v) \geq a r_b - c_2$.*

Démonstration. Si $z = (z_1, z_2) \in (T^a D_0^{\dagger,b}) \boxtimes_\delta \mathbb{P}^1$, alors $z = z_1 + w \cdot \varphi(\psi(z_2))$ et $\varphi(\psi(z_2)) \in T^{a-p(1+l(D))} D_0^{\dagger,b}$ (lemme II.3). Comme de plus $C r_b \leq C$, il suffit de démontrer la proposition pour $z \in T^a D_0^{\dagger,b}$. Nous aurons besoin du lemme suivant.

Lemme V.22. *Si $z \in D_0 \boxtimes_\delta \mathbb{P}^1$, alors $v_\Pi(p_\Pi(z)) \geq k$ si et seulement si $\{\check{z}, z\}_{\mathbb{P}^1} \in p^k \mathscr{O}_L$ pour tout $\check{z} \in \check{D}_0^\natural \boxtimes_{\delta^{-1}} \mathbb{P}^1$.*

Démonstration. Par dualité de Schikhof [27], le vecteur $v = p_\Pi(z)$ de Π_0 est dans $p^k \Pi_0$ si et seulement si $l(v) \in p^k \mathscr{O}_L$ pour tout $l \in \Pi_0^*$. Le résultat suit du fait que $\{\,,\}_{\mathbb{P}^1}$ induit un isomorphisme $\Pi_0^* = \check{D}_0^\natural \boxtimes_{\delta^{-1}} \mathbb{P}^1$. □

Revenons à la démonstration de la prop. V.21. Les lemmes V.22 et IV.7, la K_m-équivariance de $\{\,,\}$ et l'inclusion $D_0^\natural \subset T^{-l_1} D_0^{\dagger, b-1}$ ramènent la preuve de la prop. V.21 à celle de l'assertion suivante : il existe une constante C telle que

$$\lim_{|\alpha| \to \infty} v_p(\{b_m^\alpha \check{z}, z\}) - p^m r_b |\alpha| = \infty, \quad \inf_\alpha \big(v_p(\{b_m^\alpha \check{z}, z\}) - p^m r_b |\alpha|\big) \geq a r_b - C$$

pour tous $a \in \mathbb{Z}$, $b > 2m+1$, $z \in T^a D_0^{\dagger, b}$ et $\check{z} \in T^{-l_1} \check{D}_0^{\dagger, b-1}$.

Nous allons montrer que $C = l_1 + 4c_1$ convient. Comme $\check{z} \in T^{-l_1} \check{D}_0^{\dagger, b-1}$ et $T^{-l_1} \check{D}_0^{\dagger, b-1} \subset T^{-l_1} \check{D}_0^{\dagger, b}$, on déduit de la prop. V.19 que

$$b_m^\alpha \check{z} \in T^{p^m|\alpha| - C} \check{D}_0^{\dagger, b-1} \subset T^{p^m|\alpha| - C} \check{D}_0^{\dagger, b}.$$

L'inégalité $\inf_\alpha \big(v_p(\{b_m^\alpha \check{z}, z\}) - p^m r_b |\alpha|\big) \geq a r_b - C$ découle alors du lemme II.4.

Il nous reste à montrer que $\lim_{|\alpha| \to \infty} v_p(\{b_m^\alpha \check{z}, z\}) - p^m r_b |\alpha| = \infty$. Tout élément f de $\mathscr{O}_{\mathscr{E}}^{\dagger, b}$ peut s'écrire sous la forme

$$f = \sum_{k \geq 0} f_k \Big(\frac{p}{T^{n_b}}\Big)^k,$$

avec $f_k \in \mathscr{O}_{\mathscr{E}}^{\dagger, b-1}$ tendant vers 0 pour la topologie p-adique, donc on peut écrire

$$z = \sum_{k \geq 0} p^{u_k} y_k \Big(\frac{p}{T^{n_b}}\Big)^k,$$

avec $y_k \in T^a D_0^{\dagger, b-1}$ et $u_k \in \mathbb{N}$ tendant vers ∞. Notons

$$x_{k,\alpha} = \Big\{b_m^\alpha \check{z},\, p^{u_k} y_k \Big(\frac{p}{T^{n_b}}\Big)^k\Big\}.$$

On a

$$p^{u_k} y_k \Big(\frac{p}{T^{n_b}}\Big)^k \in p^{k+u_k} T^{a - k n_b} D_0^{\dagger, b-1} \subset p^{k+u_k} T^{a - k n_b} D_0^{\dagger, b}$$

et, comme on l'a déjà vu,

$$b_m^\alpha \check{z} \in T^{p^m|\alpha| - C} \check{D}_0^{\dagger, b-1} \subset T^{p^m|\alpha| - C} \check{D}_0^{\dagger, b}.$$

Combinées avec le lemme II.4, l'égalité $n_b r_{b-1} = p$ et les inégalités $a r_b, a r_{b-1} \geq -|a|$ et $C r_{b-1}, C r_b \leq C$, les relations précédentes donnent

$$v_p(x_{k,\alpha}) - p^m r_b |\alpha| \geq u_k - |a| - C + \max(0, (p-1)(p^m |\alpha| r_b - k)).$$

Un petit exercice d'analyse réelle montre alors que $\inf_k(v_p(x_{k,\alpha}) - p^m r_b|\alpha|)$ tend vers $+\infty$ quand $|\alpha| \to \infty$, ce qui permet de conclure. □

VI. Le module $D_{\mathrm{rig}} \boxtimes_\delta \mathbb{P}^1$ et l'espace $\Pi_\delta(D)^{\mathrm{an}}$

On fixe dans ce chapitre une paire G-compatible (D, δ), avec $D \in \Phi\Gamma^{\mathrm{et}}(\mathscr{E})$, et on note $\Pi = \Pi_\delta(D)$ et $\check{\Pi} = \Pi_{\delta^{-1}}(\check{D})$. On construit une extension non triviale $D_{\mathrm{rig}} \boxtimes_\delta \mathbb{P}^1$ de Π^{an} par $(\check{\Pi}^{\mathrm{an}})^*$. Cette construction n'est pas utilisée dans le chapitre suivant, consacré à la preuve du th. 0.2, mais est très utile pour une étude fine de Π^{an} (cf. [12; 14; 15], par exemple).

VI.A. ***Continuité de l'action de w_δ.*** Soit D_0 un $\mathscr{O}_{\mathscr{E}}$-réseau de D stable par φ et Γ. Soit m comme après la prop. V.17 et soient $a \geq b > 2m$. Fixons une base $e_1, e_2, \dots, e_d$ de $D_0^{\dagger,b}$ sur $\mathscr{O}_{\mathscr{E}}^{\dagger,b}$. C'est aussi une base de $D^{]0,r_b]}$ sur $\mathscr{E}^{]0,r_b]}$, ce qui nous permet de poser

$$v^{[r_a,r_b]}(z) = \min_{1\leq i\leq d} v^{[r_a,r_b]}(f_i) \quad \text{si} \quad z = \sum_{i=1}^{d} f_i e_i \in D^{]0,r_b]}.$$

Rappelons que $D \boxtimes \mathbb{Z}_p^* = D^{\psi=0}$ (et de même si on remplace D par $D^{(0,r_b]}$ pour b assez grand).

Proposition VI.1. *Il existe une constante c telle que pour tous $a \geq b > 2m$ et tout $z \in (D^{(0,r_b]})^{\psi=0}$ on ait*

$$v^{[r_a,r_b]}(w_\delta(z)) \geq v^{[r_a,r_b]}(z) - c.$$

Démonstration. On peut multiplier z par une puissance de p sans changer l'inégalité, donc on peut supposer que $z \in D_0^{(0,r_b]}$ et $[v^{[r_a,r_b]}(z)] = N \geq 1$. Nous aurons besoin du lemme suivant :

Lemme VI.2. *Soient $a, b, N \in \mathbb{N}^*$ tels que $a \geq b$ et soit $f \in \mathscr{O}_{\mathscr{E}}^{(0,r_b]}$. Si $v^{[r_a,r_b]}(f) \geq N$, alors*

$$f \in \sum_{i=0}^{N-1} p^{N-1-i} T^{in_a} \mathscr{O}_{\mathscr{E}}^{\dagger,b}.$$

Démonstration. Écrivons

$$f = \sum_{n<0} a_n T^n + \sum_{n=0}^{n_a-1} a_n T^n + \sum_{n=n_a}^{2n_a-1} a_n T^n + \cdots + \sum_{n\geq(N-1)n_a} a_n T^n.$$

Par hypothèse $v_p(a_n) + nr_a \geq N$ et $v_p(a_n) + nr_b \geq N$ pour tout n. En particulier $v_p(a_n) > N$ si $n < 0$, donc $\sum_{n<0} a_n T^n \in p^{N-1}\mathscr{O}_{\mathscr{E}}^{\dagger,b}$ (lemme I.1). Ensuite, si $0 \leq n < n_a$, on a $v_p(a_n) > N-1$, donc $\sum_{n=0}^{n_a-1} a_n T^n \in p^{N-1}\mathscr{O}_{\mathscr{E}}^+ \subset p^{N-1}\mathscr{O}_{\mathscr{E}}^{\dagger,b}$. Le même argument montre que

$$\sum_{n=n_a}^{2n_a-1} a_n T^n \in p^{N-2} T^{n_a}\mathscr{O}_{\mathscr{E}}^{\dagger,b}, \quad \dots, \sum_{n\geq (N-1)n_a} a_n T^n \in T^{(N-1)n_a}\mathscr{O}_{\mathscr{E}}^{\dagger,b}. \qquad \square$$

Revenons à la preuve de la prop. VI.1. D'après le lemme VI.2 ci-dessus, on a $z \in \sum_{i=0}^{N-1} p^{N-1-i} T^{in_a} D_0^{\dagger,b}$. Puisque $\psi(z) = 0$, on a

$$z = \mathrm{Res}_{\mathbb{Z}_p^*}(z) \in \sum_{i=0}^{N-1} p^{N-1-i}\,\mathrm{Res}_{\mathbb{Z}_p^*}(T^{in_a} D_0^{\dagger,b}).$$

Le lemme II.3 fournit une constante c_2 telle que

$$\mathrm{Res}_{\mathbb{Z}_p^*}(T^{in_a} D_0^{\dagger,b}) \subset (T^{in_a-c_2} D_0^{\dagger,b})^{\psi=0}$$

pour tous $a \geq b > 2m$ et tout i. Le (i) de la prop. V.19 fournit une constante c_1 telle que $w_\delta((T^d D_0^{\dagger,b})^{\psi=0}) \subset T^{d-c_1} D_0^{\dagger,b}$ pour tous $b > 2m$ et $d \in \mathbb{Z}$. On a donc, avec $c = c_1 + c_2$,

$$w_\delta(z) \in \sum_{i=0}^{N-1} p^{N-1-i} T^{in_a-c} D_0^{\dagger,b},$$

et donc

$$v^{[r_a,r_b]}(w_\delta(z)) \geq \inf_{0\leq i<N}(N-1-i+(in_a-c)r_a) \geq N-1-c > v^{[r_a,r_b]}(z)-c-2,$$

d'où le résultat. $\square$

Corollaire VI.3. *L'involution w_δ de $(D^{(0,r_b]})^{\psi=0}$ s'étend de manière unique en une involution continue de $(D^{]0,r_b]})^{\psi=0}$ pour tout $b > 2m$.*

Démonstration. Le module

$$(D^{(0,r_b]})^{\psi=0} = \bigoplus_{i=1}^{p-1}(1+T)^i\varphi(D^{(0,r_{b-1}]})$$

est dense dans

$$(D^{]0,r_b]})^{\psi=0} = \bigoplus_{i=1}^{p-1}(1+T)^i\varphi(D^{]0,r_{b-1}]}),$$

puisque $D^{(0,r_{b-1}]}$ l'est dans $D^{]0,r_{b-1}]}$. Cela démontre l'unicité de l'extension éventuelle de w_δ. L'existence est une conséquence de la proposition précédente, de la densité de $(D^{(0,r_b]})^{\psi=0}$ dans $(D^{]0,r_b]})^{\psi=0}$ et de la complétude de $(D^{]0,r_b]})^{\psi=0}$. $\square$

Le cor. VI.3 fournit une involution continue w_δ sur $D_{\mathrm{rig}}^{\psi=0} = \bigcup_{b>2m}(D^{]0,r_b]})^{\psi=0}$, qui étend l'involution w_δ sur $(D^\dagger)^{\psi=0}$. On définit alors, de la manière usuelle

$$D_{\mathrm{rig}} \boxtimes_\delta \mathbb{P}^1 = \{(z_1,z_2) \in D_{\mathrm{rig}} \times D_{\mathrm{rig}} \mid \mathrm{Res}_{\mathbb{Z}_p^*}(z_2) = w_\delta(\mathrm{Res}_{\mathbb{Z}_p^*}(z_1))\},$$

que l'on munit de la topologie induite par l'inclusion $D_{\mathrm{rig}} \boxtimes_\delta \mathbb{P}^1 \subset D_{\mathrm{rig}} \times D_{\mathrm{rig}}$. Notons que l'application $z \mapsto (\mathrm{Res}_{\mathbb{Z}_p}(z), \psi(\mathrm{Res}_{\mathbb{Z}_p}(wz)))$ induit un isomorphisme d'espaces vectoriels topologiques $D_{\mathrm{rig}} \boxtimes_\delta \mathbb{P}^1 \simeq D_{\mathrm{rig}} \times D_{\mathrm{rig}}$, l'application inverse étant donnée par

$$(z_1, z_2) \mapsto (z_1, \varphi(z_2) + w_\delta(\mathrm{Res}_{\mathbb{Z}_p^*}(z_1))).$$

La densité de $D^\dagger$ dans D_{rig} entraîne donc celle de $D^\dagger \boxtimes_\delta \mathbb{P}^1$ dans $D_{\mathrm{rig}} \boxtimes_\delta \mathbb{P}^1$.

VI.B. *L'action de G sur $D_{\mathrm{rig}} \boxtimes_\delta \mathbb{P}^1$.*

Proposition VI.4. *L'action de G sur $D^\dagger \boxtimes_\delta \mathbb{P}^1$ s'étend par continuité en une action continue de G sur $D_{\mathrm{rig}} \boxtimes_\delta \mathbb{P}^1$.*

Démonstration. Les formules du squelette d'action (voir la rem. III.3) permettent de définir une action de G sur $D_{\mathrm{rig}} \boxtimes_\delta \mathbb{P}^1$ (le fait qu'il s'agit bien d'une action découle de la densité de $D^\dagger \boxtimes_\delta \mathbb{P}^1$ dans $D_{\mathrm{rig}} \boxtimes_\delta \mathbb{P}^1$ et du fait que ces formules définissent une action de G sur $D^\dagger \boxtimes_\delta \mathbb{P}^1$). La continuité de l'action se démontre de la même manière que la prop. VI.1, en utilisant le (ii) de la prop. V.19. □

On renvoie au § IV.C pour les algèbres $\mathscr{D}(K_m)$ et $\mathscr{D}_h(K_m)$, et au § V.B pour les b_m^α.

Proposition VI.5. *Il existe une constante c telle que, pour tous $a \geq b > 2m$, $z \in D^{]0,r_b]}$ et $\alpha \in \mathbb{N}^4$, on ait*

$$v^{[r_a,r_b]}(b_m^\alpha z) \geq v^{[r_a,r_b]}(z) + p^m|\alpha|r_a - c.$$

Démonstration. La preuve est entièrement analogue à celle de la prop. VI.1, en utilisant le lemme VI.2 et le (ii) de la prop. V.19. □

Corollaire VI.6. *Pour tous $a \geq b > 2m$, $z \in D^{]0,r_b]}$ et $\lambda = \sum_{\alpha\in\mathbb{N}^4} c_\alpha b_m^\alpha \in \mathscr{D}_{a-m}(K_m)$, la série $\sum_\alpha c_\alpha b_m^\alpha z$ converge dans $D^{]0,r_b]}$ et*

$$v^{[r_a,r_b]}\Big(\sum_\alpha c_\alpha b_m^\alpha z\Big) \geq v^{[r_a,r_b]}(z) + v^{(a-m)}(\lambda) - c.$$

Démonstration. Une suite de $D^{]0,r_b]}$ converge dans $D^{]0,r_b]}$ si et seulement si elle converge pour la valuation $v^{[r_a,r_b]}$ pour tous $a \geq b$. Le résultat suit donc de la proposition précédente et de la définition de $v^{(a-m)}$. □

Proposition VI.7. *Soit H un sous-groupe ouvert compact de G.*

(i) *Si b est assez grand, l'action de H sur $D^{]0,r_b]} \boxtimes_\delta \mathbb{P}^1$ s'étend en une structure de $\mathscr{D}(H)$-module topologique.*

(ii) *$D_{\mathrm{rig}} \boxtimes_\delta \mathbb{P}^1$ est un $\mathscr{D}(H)$-module topologique.*

Démonstration. Comme H est commensurable à $K_m = 1 + p^m \mathrm{M}_2(\mathbb{Z}_p)$ (avec m comme ci-dessus), on peut supposer que $H = K_m$. Si $z = (z_1, z_2) \in D^{]0,r_b]} \boxtimes_\delta \mathbb{P}^1$, on peut écrire z sous la forme $z = z_1 + w \cdot \mathrm{Res}_{p\mathbb{Z}_p}(z_2)$, avec $z_1, \mathrm{Res}_{p\mathbb{Z}_p}(z_2) \in D^{]0,r_b]}$. En passant à la limite projective (sur a) dans le cor. VI.6, on obtient une application continue $\mathscr{D}(K_m) \times (D^{]0,r_b]} \boxtimes_\delta \mathbb{P}^1) \to D^{]0,r_b]} \boxtimes_\delta \mathbb{P}^1$, définie par

$$(\lambda, z) \mapsto \sum_{\alpha \in \mathbb{N}^4} c_\alpha b_m^\alpha z \quad \text{si } \lambda = \sum_{\alpha \in \mathbb{N}^4} c_\alpha b_m^\alpha.$$

Cette application étend la structure de $L[K_m]$-module de $D^{]0,r_b]} \boxtimes_\delta \mathbb{P}^1$, et comme $L[K_m]$ est dense dans $\mathscr{D}(K_m)$, cela prouve que $D^{]0,r_b]} \boxtimes_\delta \mathbb{P}^1$ est un $\mathscr{D}(K_m)$-module (topologique d'après ce qui précède). Ceci démontre le (i) et, le (ii) étant une conséquence immédiate du (i), cela permet de conclure. □

Proposition VI.8. *Soit H un sous-groupe ouvert compact de G, qui stabilise l'ouvert compact $U \subset \mathbb{P}^1(\mathbb{Q}_p)$. Alors $D_{\mathrm{rig}} \boxtimes_\delta U$ est un sous-$\mathscr{D}(H)$-module de $D_{\mathrm{rig}} \boxtimes_\delta \mathbb{P}^1$ et $\mathrm{Res}_U(\lambda \cdot z) = \lambda \cdot \mathrm{Res}_U(z)$ pour tous $z \in D_{\mathrm{rig}} \boxtimes_\delta \mathbb{P}^1$ et $\lambda \in \mathscr{D}(H)$.*

Démonstration. Cela découle de la continuité de l'action de $\mathscr{D}(H)$, de la densité de $L[H]$ dans $\mathscr{D}(H)$ et de la H-équivariance de l'application Res_U. □

VI.C. *Description de Π^{an} via $D_{\mathrm{rig}} \boxtimes_\delta \mathbb{P}^1$.* L'accouplement $\{\ ,\ \}_{\mathbb{P}^1}$ sur le produit $(\check{D}^\dagger \boxtimes_{\delta^{-1}} \mathbb{P}^1) \times (D^\dagger \boxtimes_\delta \mathbb{P}^1)$ s'étend en un accouplement G-équivariant parfait (voir la discussion qui précède [10, prop. V.2.10])

$$\{\ ,\ \}_{\mathbb{P}^1} : (\check{D}_{\mathrm{rig}} \boxtimes_{\delta^{-1}} \mathbb{P}^1) \times (D_{\mathrm{rig}} \boxtimes_\delta \mathbb{P}^1) \to L.$$

Théorème VI.9. $(\Pi^{\mathrm{an}})^*$ *est isomorphe comme G-module topologique à l'orthogonal de $D^\natural \boxtimes_\delta \mathbb{P}^1$ dans $\check{D}_{\mathrm{rig}} \boxtimes_{\delta^{-1}} \mathbb{P}^1$.*

Démonstration. Soit M l'orthogonal de $D^\natural \boxtimes_\delta \mathbb{P}^1$ dans $\check{D}_{\mathrm{rig}} \boxtimes_{\delta^{-1}} \mathbb{P}^1$. Notons que M est un sous-$\mathscr{D}(H)$-module de $\check{D}_{\mathrm{rig}} \boxtimes_{\delta^{-1}} \mathbb{P}^1$ pour tout sous-groupe ouvert compact H de G (cela suit de la stabilité de $D^\natural \boxtimes_\delta \mathbb{P}^1$ par H, de la H-équivariance et continuité de $\{\ ,\ \}_{\mathbb{P}^1}$, de la densité de $L[H]$ dans $\mathscr{D}(H)$ et de la continuité de $\{\ ,\ \}_{\mathbb{P}^1}$). Nous aurons besoin du lemme suivant :

Lemme VI.10. *Pour tout $\check{z} \in \check{D}_{\mathrm{rig}} \boxtimes_{\delta^{-1}} \mathbb{P}^1$ l'application $D^\dagger \boxtimes_\delta \mathbb{P}^1 \to L$, donnée par $z \mapsto \{\check{z}, z\}_{\mathbb{P}^1}$, est continue. De plus, l'application $\check{D}_{\mathrm{rig}} \boxtimes_{\delta^{-1}} \mathbb{P}^1 \to (D^\dagger \boxtimes_\delta \mathbb{P}^1)^*$ ainsi obtenue est continue.*

Démonstration. Il suffit de vérifier que pour tout $\check{z} \in \check{D}_{\mathrm{rig}}$ l'application $z \mapsto \{\check{z}, z\}$ est une forme linéaire continue sur $D^\dagger$ et que l'application $\check{D}_{\mathrm{rig}} \to (D^\dagger)^*$ ainsi obtenue est continue. En revenant aux définitions des topologies de $D^\dagger$ et $\check{D}_{\mathrm{rig}}$, la continuité de $\check{D}_{\mathrm{rig}} \to (D^\dagger)^*$ découle de l'inégalité

$$v_p(\{\check{z}, z\}) \geq k r_b + v^{[r_a, r_b]}(\check{z}) - 2$$

pour $a \geq b > 2m$, $k \geq 1$, $z \in T^k D_0^{\dagger,b}$ et $\check{z} \in D^{]0,r_b]}$. Pour démontrer cette inégalité on se ramène par densité et L-linéarité (et en utilisant le (ii) du lemme I.3) à $\check{z} \in (1/p)T^{Nn_b} D_0^{\dagger,b}$, avec N la partie entière de $v^{[r_a,r_b]}(\check{z}) \geq 0$. L'inégalité suit alors du lemme II.4. □

Revenons à la preuve du th. VI.9. On déduit du lemme VI.10, de l'isomorphisme $\Pi^{\rm an} \simeq (D^\dagger \boxtimes_\delta \mathbb{P}^1)/(D^\natural \boxtimes_\delta \mathbb{P}^1)$ (cor. V.4) et de la définition de M, une application linéaire continue $\phi : M \to (\Pi^{\rm an})^*$, induite par $\{\ ,\ \}_{\mathbb{P}^1}$. Explicitement, on a $\langle \phi(\check{z}), v\rangle = \{\check{z}, z_v\}_{\mathbb{P}^1}$ pour tout relèvement $z_v \in D^\dagger \boxtimes_\delta \mathbb{P}^1$ de $v \in \Pi^{\rm an}$ et tout $\check{z} \in M$. L'application ϕ est G-équivariante, puisque $\{\ ,\ \}_{\mathbb{P}^1}$ l'est. Nous allons montrer que ϕ est un homéomorphisme, en construisant son inverse.

Commençons par constater que ϕ est injective car un élément de ${\rm Ker}(\phi)$ est orthogonal à $D^\dagger \boxtimes_\delta \mathbb{P}^1$ et donc à $D_{\rm rig} \boxtimes_\delta \mathbb{P}^1$ par densité de $D^\dagger \boxtimes_\delta \mathbb{P}^1$, et donc est nul puisque $\{\ ,\ \}_{\mathbb{P}^1}$ est un accouplement parfait.

Le (ii) de la prop. VI.7 montre que l'inclusion $\Pi^* \simeq \check{D}^\natural \boxtimes_{\delta^{-1}} \mathbb{P}^1 \subset \check{D}_{\rm rig} \boxtimes_{\delta^{-1}} \mathbb{P}^1$ induit une application $\mathscr{D}(H)$-linéaire continue $\xi : \mathscr{D}(H) \otimes_{\Lambda(H)} \Pi^* \to \check{D}_{\rm rig} \boxtimes_{\delta^{-1}} \mathbb{P}^1$. Puisque $\check{\Pi}^*$ et Π^* sont orthogonaux et M est un sous-$\mathscr{D}(H)$-module de $\check{D}_{\rm rig} \boxtimes_{\delta^{-1}} \mathbb{P}^1$, l'image de ξ est contenue dans M.

La prop. IV.13 fournit un isomorphisme d'espaces vectoriels topologiques ι : $(\Pi^{\rm an})^* \simeq \mathscr{D}(H) \otimes_{\Lambda(H)} \Pi^*$ (H étant par exemple ${\rm GL}_2(\mathbb{Z}_p)$), et la composée $\iota \circ \phi \circ \xi$ est l'identité car c'est l'identité sur le sous-espace dense Π^* de $\mathscr{D}(H) \otimes_{\Lambda(H)} \Pi^*$.

Comme ϕ est injective, cela implique que son inverse est $\xi \circ \iota$, ce qui permet de conclure. □

Corollaire VI.11. $(\Pi^{\rm an})^*$ *et* $(\check{\Pi}^{\rm an})^*$ *sont exactement orthogonaux pour l'accouplement* $\{\ ,\ \}_{\mathbb{P}^1}$.

Démonstration. Le th. VI.9 et le cor. III.22 montrent que $(\Pi^{\rm an})^*$ est l'orthogonal de $\check{\Pi}^*$ dans $\check{D}_{\rm rig} \boxtimes_{\delta^{-1}} \mathbb{P}^1$. Or $\check{\Pi}^*$ est un sous-espace dense de $(\check{\Pi}^{\rm an})^*$ (par densité de $\check{\Pi}^{\rm an}$ dans $\check{\Pi}$ combinée à la réflexivité de $\check{\Pi}^{\rm an}$ et au théorème de Hahn–Banach), donc $(\Pi^{\rm an})^*$ est en fait l'orthogonal de $(\check{\Pi}^{\rm an})^*$, ce qui permet de conclure. □

Corollaire VI.12. *L'injection* $(\check{\Pi}^{\rm an})^* \to D_{\rm rig} \boxtimes_\delta \mathbb{P}^1$ *fournie par le th. VI.9 induit une suite exacte de G-modules topologiques*

$$0 \to (\check{\Pi}^{\rm an})^* \to D_{\rm rig} \boxtimes_\delta \mathbb{P}^1 \to \Pi^{\rm an} \to 0.$$

Démonstration. D'après le th. VI.9, $(\check{\Pi}^{\rm an})^*$ est un sous-espace fermé de $D_{\rm rig} \boxtimes_\delta \mathbb{P}^1$. Soit Y le quotient. Puisque $\{\ ,\ \}_{\mathbb{P}^1}$ induit une dualité parfaite entre $\check{D}_{\rm rig} \boxtimes_{\delta^{-1}} \mathbb{P}^1$ et $D_{\rm rig} \boxtimes_\delta \mathbb{P}^1$, on obtient un isomorphisme topologique de Y^* sur l'orthogonal de $(\check{\Pi}^{\rm an})^*$ dans $\check{D}_{\rm rig} \boxtimes_{\delta^{-1}} \mathbb{P}^1$, donc sur $(\Pi^{\rm an})^*$ (corollaire précédent). On a donc un isomorphisme de G-modules topologiques $Y^* \simeq (\Pi^{\rm an})^*$ et on conclut en observant que Π^{an} et Y sont réflexifs (pour le dernier, cela découle de ce que $\check{D}_{\mathrm{rig}} \boxtimes_{\delta^{-1}} \mathbb{P}^1$ satisfait Hahn–Banach). □

Corollaire VI.13. *Il existe $m = m(D)$ tel que* $(\Pi^{\mathrm{an}})^* \subset \check{D}^{]0,r_m]} \boxtimes_{\delta^{-1}} \mathbb{P}^1$.

Démonstration. $(\Pi^{\mathrm{an}})^*$ est un espace de Fréchet et le corollaire précédent fournit une injection continue dans $\check{D}_{\mathrm{rig}} \boxtimes_{\delta^{-1}} \mathbb{P}^1$, qui est la réunion croissante des espaces de Fréchet $\check{D}^{]0,r_m]} \boxtimes_{\delta^{-1}} \mathbb{P}^1$. Le résultat s'ensuit. □

Corollaire VI.14. *Soit $z \in D_{\mathrm{rig}}$ tel que $P(\varphi)z = 0$ pour un polynôme non nul $P \in L[X]$. Alors $fz \in (\check{\Pi}^{\mathrm{an}})^*$ pour tout $f \in \mathscr{R}^+$.*

Démonstration. Comme fz est à support $\mathbb{Z}_p$, on a $\{fz, y\}_{\mathbb{P}^1} = \{fz, \mathrm{Res}_{\mathbb{Z}_p} y\}$ pour tout $y \in \check{D}^\natural \boxtimes_{\delta^{-1}} \mathbb{P}^1$, et il suffit donc de vérifier que fz est orthogonal à $\check{D}^\natural$. Par linéarité et densité de $L[T]$ dans $\mathscr{R}^+$, on peut supposer que $f = (1+T)^k$, avec $k \in \mathbb{N}$. Comme $\check{D}^\natural$ est un $\mathscr{E}^+$-module et $\{(1+T)^k \check{z}, (1+T)^k z\} = \{\check{z}, z\}$, on peut supposer que $f = 1$. Le résultat suit alors de ce que $P(\psi)D^\natural = D^\natural$ [9, prop. II.5.15] et de ce que φ et ψ sont adjoints pour $\{\ ,\ \}$. □

VII. Complétés unitaires universels

VII.A. ***Réseaux invariants minimaux.*** Soit G un groupe de Lie p-adique et soit Π une représentation continue de G sur un L-espace vectoriel localement convexe. Rappelons qu'une L-représentation de Banach B de G est dite unitaire si G préserve une valuation définissant la topologie de B. Le *complété unitaire universel* $\widehat{\Pi}$ de Π est (s'il existe) une L-représentation de Banach unitaire de G, munie d'une application L-linéaire continue, G-équivariante $\iota : \Pi \to \widehat{\Pi}$, qui est universelle au sens suivant : pour toute L-représentation de Banach unitaire B de G, l'application

$$\mathrm{Hom}^{\mathrm{cont}}_{L[G]}(\widehat{\Pi}, B) \to \mathrm{Hom}^{\mathrm{cont}}_{L[G]}(\Pi, B), \quad f \mapsto f \circ \iota,$$

est une bijection. Autrement dit, tout morphisme continu $\Pi \to B$ se factorise de manière unique à travers $\iota : \Pi \to \widehat{\Pi}$.

Remarque VII.1. (i) Il découle facilement de la définition que si $\widehat{\Pi}$ existe, alors l'image de ι est dense dans $\widehat{\Pi}$, et que $\widehat{\Pi}$ est unique à isomorphisme unique près.

(ii) Même si $\widehat{\Pi}$ existe, il n'y a aucune raison a priori pour que $\widehat{\Pi} \neq 0$, et classifier les représentations de G ayant un complété universel non nul est un problème difficile et fondamental [7].

(iii) Si Π est topologiquement irréductible et si $\widehat{\Pi}$ existe et est non nul, alors Π admet une valuation invariante par G. En effet, dans ce cas l'application naturelle $\Pi \to \widehat{\Pi}$ est injective, ce qui permet de considérer la restriction de la valuation sur $\widehat{\Pi}$ à Π.

Si Π est un L-espace vectoriel, un *réseau* de Π est un sous-$\mathscr{O}_L$-module de Π qui engendre le L-espace vectoriel Π (on ne demande pas à un réseau d'être séparé pour la topologie p-adique ; un réseau peut donc contenir des sous-L-espaces vectoriels). La remarque suivante d'Emerton [19, lemma 1.3] sera utile pour la suite :

Lemme VII.2. Π *admet un complété universel* $\widehat{\Pi}$ *si et seulement si* Π *contient un* $\mathscr{O}_L$*-réseau* M *avec les propriétés suivantes* :

- M *est ouvert dans* Π *et stable sous l'action de* G.
- M *est minimal pour ces propriétés, i.e.,* M *est contenu dans un homothétique de tout sous-*$\mathscr{O}_L$*-réseau ouvert de* Π *stable par* G.

De plus, dans ce cas

$$\widehat{\Pi} = L \otimes_{\mathscr{O}_L} \varprojlim (M/p^n M).$$

VII.B. *Le complété universel de* $\mathbf{LA}(\mathbb{Z}_p)$. Soit $U(\mathbb{Z}_p)$ le groupe $\left(\begin{smallmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{smallmatrix}\right)$. Considérons la $U(\mathbb{Z}_p)$-représentation de Banach unitaire admissible $\mathscr{C}(\mathbb{Z}_p, L)$, l'action de $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ étant donnée par $\big(\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \cdot \phi\big)(x) = \phi(x-b)$. L'espace des vecteurs localement analytiques de $\mathscr{C}(\mathbb{Z}_p, L)$ est l'espace $\mathrm{LA}(\mathbb{Z}_p, L)$ des fonctions localement analytiques sur $\mathbb{Z}_p$, à valeurs dans L. Le résultat suivant montre que $\mathscr{C}(\mathbb{Z}_p, L)$ n'est pas le complété unitaire universel de $\mathrm{LA}(\mathbb{Z}_p, L)$, et donc qu'on n'a pas forcément $\widehat{\Pi^{\mathrm{an}}} = \Pi$ pour une représentation de Banach unitaire admissible Π.

Proposition VII.3. $\mathrm{LA}(\mathbb{Z}_p, L)$ *n'a pas de complété unitaire universel.*

Démonstration. D'après le lemme VII.2, il suffit de montrer que tout réseau ouvert et $\mathbb{Z}_p$-stable $\mathscr{L}$ dans $\mathrm{LA}(\mathbb{Z}_p, L)$ contient un réseau ouvert, $\mathbb{Z}_p$-stable et non commensurable avec $\mathscr{L}$. Soit $\mathscr{L}$ un tel réseau et, pour $n \geq 1$, notons X_n la boule unité de $\mathrm{LA}^{(n)}(\mathbb{Z}_p)$ (pour lequel on renvoie au § IV.B). Nous aurons besoin du lemme suivant.

Lemme VII.4. *Soit* $(b_n)_{n\geq 1}$ *une suite croissante dans* $\mathbb{N}$ *et soient* $k, a \in \mathbb{N}^*$ *tels que* $p^a X_k \subset \sum_{n\geq 1} p^{b_n} X_n$. *Alors* $b_k \leq 3a$.

Démonstration. Comme $X_1 \subset X_2 \subset \cdots \subset X_{k-1}$ et $X_n \subset \mathscr{C}(\mathbb{Z}_p, \mathscr{O}_L)$ pour $n \geq 1$, on a

$$p^a X_k \subset X_{k-1} + p^{b_k} \mathscr{C}(\mathbb{Z}_p, \mathscr{O}_L).$$

Soient $N = 2a \cdot p^{k-1}(p-1)$ et $\phi = p^{2a}\binom{x}{N}$. Alors $\phi \in X_k$, et donc $p^a\phi = \phi_1 + p^{b_k}\phi_2$, avec $\phi_1 \in X_{k-1}$ et $\phi_2 \in \mathscr{C}(\mathbb{Z}_p, \mathscr{O}_L)$. En regardant les N-ièmes coefficients de Mahler on obtient la relation $a_N(\phi_1) + p^{b_k} a_N(\phi_2) = p^{3a}$, et comme $v_p(a_N(\phi_1)) \geq N r_{k-1} = 2pa > 3a$, on en déduit que $b_k = 3a - v_p(a_N(\phi_2)) \leq 3a$, ce qui permet de conclure. □

Revenons à la preuve de la prop. VII.3. Puisque $\mathscr{L} \cap \mathrm{LA}^{(n)}(\mathbb{Z}_p)$ est un voisinage de 0 dans $\mathrm{LA}^{(n)}(\mathbb{Z}_p)$, il contient $p^{a_n} X_n$ pour un certain $a_n \in \mathbb{N}$. Posons $b_n = 4\max(a_1, \ldots, a_n) + n$ pour $n \geq 1$ et $\mathscr{L}' = \sum_{n\geq 1} p^{b_n} X_n$. Alors $\mathscr{L}'$ est un réseau

$\mathbb{Z}_p$-invariant dans $\mathrm{LA}(\mathbb{Z}_p, L)$, contenu dans $\mathscr{L}$. Il découle du lemme VII.4 que $\mathscr{L}'$ n'est pas commensurable avec $\mathscr{L}$, ce qui permet de conclure. □

On peut aussi voir $\mathscr{C}(\mathbb{Z}_p, L)$ et $\mathrm{LA}(\mathbb{Z}_p, L)$ comme des représentations du semi-groupe $P^+ = \left(\begin{smallmatrix} \mathbb{Z}_p-\{0\} & \mathbb{Z}_p \\ 0 & 1 \end{smallmatrix}\right)$, via

$$\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \cdot \phi\right)(x) = \mathbf{1}_{b+a\mathbb{Z}_p}(x)\, \phi\left(\frac{x-b}{a}\right).$$

La preuve du résultat suivant est une bonne préparation pour celle du th. VII.11.

Proposition VII.5. *$\mathscr{C}(\mathbb{Z}_p, L)$ est le complété unitaire universel de $\mathrm{LA}(\mathbb{Z}_p, L)$ en tant que P^+-représentation.*

Démonstration. Soit $\mathscr{L} = \mathrm{LA}(\mathbb{Z}_p, L) \cap \mathscr{C}(\mathbb{Z}_p, \mathscr{O}_L)$. Nous allons montrer que $\mathscr{L}$ est un réseau ouvert, P^+-stable et minimal. En utilisant le lemme VII.2 et la densité de $\mathrm{LA}(\mathbb{Z}_p, L)$ dans $\mathscr{C}(\mathbb{Z}_p, L)$, cela permet de conclure.

Soit $\mathscr{L}'$ un réseau ouvert et P^+-invariant dans $\mathrm{LA}(\mathbb{Z}_p, L)$. Soit X_n comme dans la preuve de la prop. VII.3. Quitte à remplacer $\mathscr{L}'$ par $p^N\mathscr{L}'$ pour un $N \in \mathbb{Z}$, on peut supposer que $X_1 \subset \mathscr{L}'$. Si $n \geq 2$ et $\phi \in X_n$, alors

$$\phi = \sum_{i=0}^{p-1} \begin{pmatrix} p & i \\ 0 & 1 \end{pmatrix} \phi_i, \quad \text{avec } \phi_i(x) = \phi(px+i),$$

et $\phi_i \in X_{n-1}$ pour $i = 0, \ldots, p-1$. Donc $X_n \subset \sum_{i=0}^{p-1} \left(\begin{smallmatrix} p & i \\ 0 & 1 \end{smallmatrix}\right) X_{n-1}$, ce qui permet de conclure que $\sum_{n\geq 1} X_n \subset \mathscr{L}'$. Enfin, on vérifie sans mal, en utilisant les coefficients de Mahler, que $\mathscr{L} \subset p^{-1}\sum_{n\geq 1} X_n$, ce qui permet de conclure. □

VII.C. ***Cohérence et complétion universelle.*** On suppose, dans ce paragraphe, que $G = \mathrm{GL}_n(\mathbb{Q}_p)$. Rappelons que $S = G/KZ$. On renvoie au § IV.G pour la définition de $d(s, 1)$ quand $s \in S$.

Proposition VII.6. *Si Π est cohérente, alors $\Pi^{\mathrm{an}} = \bigcup_h \Pi^{(h)}$ admet un complété universel $\widehat{\Pi^{\mathrm{an}}}$. Plus précisément, si $h \geq m(\Pi)$ et si $\Pi_0^{(h)}$ est la boule unité de $\Pi^{(h)}$ pour la valuation $v^{(h)}$, alors*

$$M := \sum_{g\in G} g \cdot \Pi_0^{(h)}$$

est un réseau ouvert de Π^{an} et $\widehat{\Pi^{\mathrm{an}}}$ est le complété de Π relativement à ce réseau.

Démonstration. Soient $k \geq h \geq m(\Pi)$. Pour chaque $s \in S$ tel que $d(s, 1) \leq k-h$ on choisit un relèvement $\hat{s}$ à G. L'application continue

$$\bigoplus_{d(s,1)\leq k-h} \Pi^{(h)} \to \Pi^{(k)}, \quad (x_s)_{d(s,1)\leq k-h} \mapsto \sum_{d(s,1)\leq k-h} \hat{s} \cdot x_s$$

est surjective par hypothèse. On en déduit que $\sum_{d(s,1)\leq k-h} s\cdot\Pi_0^{(h)}$ est un réseau ouvert de $\Pi^{(k)}$. En passant à la limite inductive, il s'ensuit que $M=\sum_{s\in S} s\cdot\Pi_0^{(h)}$ est un réseau ouvert de Π, invariant par G par construction. Par ailleurs, si $\mathscr{L}$ est un réseau ouvert de $\Pi^{\rm an}$, alors $\mathscr{L}\cap\Pi^{(h)}$ est un réseau ouvert de $\Pi^{(h)}$, donc il existe k tel que $\mathscr{L}\supset p^k\Pi_0^{(h)}$. Si de plus $\mathscr{L}$ est invariant par G, alors $\mathscr{L}$ contient p^kM. Ainsi, M est un réseau ouvert G-invariant et minimal à homothétie près. Le lemme VII.2 permet de conclure. □

Remarque VII.7. Il n'est pas difficile de voir, en reprenant les arguments ci-dessus, que la valuation $v^{(h)}$ sur $\Pi^{(h)}$ est génératrice au sens d'Emerton [19, définition 1.13]. Autrement dit, la cohérence implique l'existence d'une valuation génératrice (il suffirait que Π soit engendré algébriquement par $\Pi^{(h)}$ pour h assez grand) et donc la prop. VII.6 peut aussi se déduire de la prop. 1.14 de [19].

VII.D. *Fonctorialité.* Si $u\in{\rm Hom}_{L[G]}^{\rm cont}(\Pi_1,\Pi_2)$, et si Π_1 et Π_2 admettent des complétés universels, alors il existe un unique morphisme $\hat u\in{\rm Hom}_{L[G]}^{\rm cont}(\widehat\Pi_1,\widehat\Pi_2)$, tel que $\hat u\circ\iota_1=\iota_2\circ u$, où $\iota_i:\Pi_i\to\widehat\Pi_i$ est l'application canonique.

Proposition VII.8. *Soit* $0\to\Pi_1\to\Pi\to\Pi_2\to 0$ *une suite exacte stricte de représentations de* G *sur des* L*-espaces vectoriels localement convexes. Si* $\widehat\Pi$ *existe, alors :*

(i) $\widehat\Pi_2$ *existe aussi, et le morphisme* $\widehat\Pi\to\widehat\Pi_2$ *induit par* $\Pi\to\Pi_2$ *est surjectif.*

(ii) *Si de plus* $\widehat\Pi_1$ *existe, alors* $\Im(\widehat\Pi_1\to\widehat\Pi)$ *est dense dans* ${\rm Ker}(\widehat\Pi\to\widehat\Pi_2)$.

Démonstration. (i) Soit M un réseau ouvert, G-stable et minimal (à homothétie près) de Π, comme dans le lemme VII.2. Comme $\Pi\to\Pi_2$ est stricte et surjective, l'image M_2 de M dans Π_2 est un réseau ouvert de Π_2, stable par G. Si M_2' est un autre réseau ouvert de Π_2, stable par G, et si M' est l'image inverse de M_2' dans Π, alors $M'\cap M$ est un réseau ouvert de Π qui est stable par G ; comme M est minimal, il existe $k\in\mathbb{N}$ tel que $M'\cap M$ contienne p^kM, et donc M' contient p^kM_2. Il s'ensuit que M_2 est minimal (à homothétie près) et le résultat suit du lemme VII.2, qui montre aussi que $\varprojlim M_2/p^nM_2$ est un réseau ouvert borné de $\widehat\Pi_2$.

(ii) $M_1=\Pi_1\cap M$, est un réseau ouvert de Π_1, stable par G, et on a une suite exacte $0\to M_1\to M\to M_2\to 0$. En passant aux séparés complétés pour la topologie p-adique, puis en inversant p, on en déduit la suite exacte

$$0\to L\otimes_{\mathscr{O}_L}(\varprojlim M_1/p^nM_1)\to\widehat\Pi\to\widehat\Pi_2\to 0.$$

L'application naturelle $M_1\to\varprojlim M_1/p^nM_1$ induit un morphisme continu

$$f:\Pi_1\to L\otimes_{\mathscr{O}_L}(\varprojlim M_1/p^nM_1),$$

d'image dense. Par définition de $\hat{\Pi}_1$, f est induit par un unique morphisme φ : $\widehat{\Pi}_1 \to L \otimes_{\mathscr{O}_L} (\varprojlim M_1/p^n M_1)$. Puisque f est à image dense, il en est de même de φ, ce qui permet de conclure. □

Corollaire VII.9. *On garde les notations et hypothèses de la prop. VII.8, et on suppose de plus que $\widehat{\Pi}_1$ est admissible. Alors on a une suite exacte de L-espaces vectoriels $\widehat{\Pi}_1 \to \widehat{\Pi} \to \widehat{\Pi}_2 \to 0$.*

Démonstration. Soient $X = \mathrm{Ker}(\widehat{\Pi} \to \widehat{\Pi}_2)$ et H un sous-groupe ouvert compact de G. La prop. VII.8 montre l'existence d'un morphisme d'image dense $f : \widehat{\Pi}_1 \to X$. Ainsi, X^* est un sous-$\Lambda(H)$-module de $(\widehat{\Pi}_1)^*$, qui est de type fini par admissibilité de $\widehat{\Pi}_1$. Comme $\Lambda(H)$ est noethérien, X est admissible, et l'image de f est fermée [27]. Donc f est surjectif, ce qui permet de conclure. □

Remarque VII.10. (i) $\widehat{\Pi}_1 \to \widehat{\Pi}$ n'est pas toujours injective. Considérons par exemple une représentation π de $\mathrm{GL}_2(\mathbb{Q}_p)$, lisse et supercuspidale. Alors le complété universel $\hat{\pi}$ de π existe et est une représentation de Banach non admissible [18, 5.1.18]. Mais π admet [24] une famille de complétions unitaires topologiquement irréductibles. Si Π est un tel complété, alors π s'injecte dans Π^{an}, mais $\hat{\pi}$ ne s'injecte pas dans $\widehat{\Pi^{\mathrm{an}}} = \Pi$ (cette égalité étant une conséquence du th. 0.2).

(ii) On aurait pu aussi considérer une extension $E_{\mathscr{L}}$ de la représentation $W(\delta_1, \delta_2)$ par la steinberg analytique $\mathrm{St}^{\mathrm{an}}(\delta_1, \delta_2)$ de la rem. 0.3 de l'introduction (ces extensions sont paramétrées [12] par $\mathscr{L} \in \mathbb{P}^1(L)$). Ici encore, le complété universel de $E_{\mathscr{L}}$ est un quotient de celui de $\mathrm{St}^{\mathrm{an}}(\delta_1, \delta_2)$ par un sous-espace non trivial, et donc le complété universel de $\mathrm{St}^{\mathrm{an}}(\delta_1, \delta_2)$ ne s'injecte pas dans celui de $E_{\mathscr{L}}$.

(iii) Dans les deux exemples précédents, $\widehat{\Pi_1}$ n'est pas admissible, et nous ne connaissons pas d'exemple avec $\widehat{\Pi_1}$ admissible.

VII.E. *Le complété universel de* $\mathbf{\Pi^{an}}$. Supposons dorénavant que $G = \mathrm{GL}_2(\mathbb{Q}_p)$.

Théorème VII.11. *Si $\Pi \in \mathrm{Rep}_L(G)$, alors Π^{an} est cohérente et son complété universel est Π.*

Démonstration. La preuve va demander quelques préliminaires. On commence par supposer que $\Pi = \Pi_\delta(D)$ pour un $D \in \Phi\Gamma^{\mathrm{et}}(\mathscr{E})$. Soit D_0 un réseau de D et soit $\Pi_0 = \Pi_\delta(D_0)$, un réseau de Π, ouvert, borné et G-stable. Pour tout $b > m(D)$ on note X_b le sous-$\mathscr{O}_L$-module $(D_0^{\dagger,b} \boxtimes_\delta \mathbb{P}^1)/(D_0^\natural \boxtimes_\delta \mathbb{P}^1)$ de Π_0. Soit $\Pi_0^{(b)}$ la boule unité de $\Pi^{(b)}$ pour la valuation $v^{(b)}$. Les prop. V.10 et V.21, et le fait que $p \in T^{n_b} \mathscr{O}_{\mathscr{E}}^{\dagger,b}$, montrent qu'il existe $b_0 > m(D)$ et une constante c tels que $p^c \Pi_0^{(b)} \subset X_b \subset p^{-c} \Pi_0^{(b)}$ pour tout $b \geq b_0$.

24. Cela découle de la compatibilité entre la correspondance de Langlands classique et celle p-adique, voir [10, th. 0.21].

Lemme VII.12. *Il existe une constante c_1 telle que pour tout $b \geq b_0$*

$$p^{c_1} X_b \subset \sum_{d(g,1) \leq b-b_0} g \cdot X_{b_0}.$$

Démonstration. Il existe c_1 tel que $w_\delta(\mathrm{Res}_{\mathbb{Z}_p^*}(D_0^{\dagger,b})) \subset p^{-c_1} D_0^{\dagger,b}$ pour tout $b \geq b_0$ (utiliser la prop. V.19 et le fait que $p \in T^{n_b} \mathscr{O}_{\mathscr{E}}$). On a donc $p^{c_1} D_0^{\dagger,b} \subset D_0^{\dagger,b} \boxtimes_\delta \mathbb{P}^1$. Ensuite, tout $z \in D_0^{\dagger,b}$ s'écrit sous la forme

$$z = \sum_{i=0}^{p^{b-b_0}-1} \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^{b-b_0} & 0 \\ 0 & 1 \end{pmatrix} u_i,$$

avec $u_i = \psi^{b-b_0}((1+T)^{-i} z) \in D_0^{\dagger,b_0}$. Puisque tout $x \in D_0^{\dagger,b} \boxtimes_\delta \mathbb{P}^1$ s'écrit $x = \mathrm{Res}_{\mathbb{Z}_p}(x) + w \cdot \mathrm{Res}_{p\mathbb{Z}_p}(w \cdot x)$, on en déduit que

$$p^{c_1}(D_0^{\dagger,b} \boxtimes_\delta \mathbb{P}^1) \subset \sum_{d(g,1) \leq b-b_0} g \cdot (D_0^{\dagger,b_0} \boxtimes_\delta \mathbb{P}^1),$$

ce qui permet de conclure. □

Lemme VII.13. *On a $\Pi^{\mathrm{an}} \cap \Pi_0 \subset p^{-(c+c_1+1)} \sum_{g \in G} g \cdot \Pi_0^{(b_0)}$.*

Démonstration. Soit $v \in p \cdot (\Pi_0 \cap \Pi^{\mathrm{an}})$. Alors v a un relèvement z à $pD_0 \boxtimes_\delta \mathbb{P}^1$ et, puisque $v \in \Pi^{\mathrm{an}}$, le cor. V.4 montre que $z \in D^\dagger \boxtimes_\delta \mathbb{P}^1$. Puisque $pD_0 \cap D^\dagger \subset \bigcup_{b \geq b_0} D_0^{\dagger,b}$ (cela se déduit du lemme I.2), on conclut que $v \in X_b$ pour un certain $b \geq b_0$. Or, le lemme précédent et l'inclusion $X_{b_0} \subset p^{-c} \Pi_0^{(b_0)}$ entraînent

$$X_b \subset p^{-c-c_1} \sum_{d(g,1) \leq b-b_0} g \cdot \Pi_0^{(b_0)} \subset p^{-c-c_1} \sum_{g \in G} g \cdot \Pi_0^{(b_0)},$$

ce qui permet de conclure. □

Revenons à la preuve du th. VII.11. En tensorisant par L l'inclusion du lemme VII.12 on obtient

$$\Pi^{(b)} \subset \sum_{d(g,1) \leq b-b_0} g \cdot \Pi^{(b_0)}$$

pour tout $b \geq b_0$, ce qui montre que Π^{an} est cohérente. La prop. VII.6 montre que $\widehat{\Pi^{\mathrm{an}}}$ existe, et c'est le complété de Π^{an} par rapport au réseau minimal (à homothétie près) $\sum_{g \in G} g \cdot \Pi_0^{(b_0)}$. Le lemme VII.13 montre que $M = \Pi^{\mathrm{an}} \cap \Pi_0$ est commensurable à $\sum_{g \in G} g \cdot \Pi_0^{(b_0)}$, donc $\widehat{\Pi^{\mathrm{an}}}$ est le complété de Π^{an} par rapport au réseau M. Puisque Π^{an} est dense dans Π, l'injection naturelle $M/p^n M \to \Pi_0/p^n \Pi_0$ est un isomorphisme. En passant à la limite et en inversant p, on obtient $\widehat{\Pi^{\mathrm{an}}} = \Pi$.

Jusque là nous avons supposé que $\Pi = \Pi_\delta(D)$ pour une paire G-compatible (D, δ). Supposons maintenant que $\Pi \in \mathrm{Rep}_L(G)$ est quelconque. Disons que Π est *bonne* si elle est cohérente et égale au complété unitaire universel de ses vecteurs localement analytiques. Notons que si Π est une représentation cohérente, alors

$\widehat{\Pi^{an}}$ existe (prop. VII.6) et l'injection $\Pi^{an} \to \Pi$ induit un morphisme $\widehat{\Pi^{an}} \to \Pi$. Notons aussi qu'une représentation de dimension finie est bonne.

Le th. III.45 nous fournit une paire G-compatible (D, δ) et une application $\beta : \Pi_\delta(D) \to \Pi/\Pi^{SL_2(\mathbb{Q}_p)}$, dont le noyau et le conoyau sont de dimension finie sur L. Comme $\Pi^{SL_2(\mathbb{Q}_p)}$ est de dimension finie (cor. III.37), et comme $\Pi_\delta(D)$ est bonne d'après ce qui précède, le résultat s'obtient en appliquant plusieurs fois le lemme suivant.

Lemme VII.14. *Soit* $0 \to \Pi_1 \to \Pi \to \Pi_2 \to 0$ *une suite exacte dans* $\mathrm{Rep}_L(G)$. *Si* Π_1 *est bonne et si* Π *ou* Π_2 *est bonne, l'autre l'est aussi.*

Démonstration. L'exactitude du foncteur $\Pi \mapsto \Pi^{an}$ nous fournit une suite exacte $0 \to \Pi_1^{an} \to \Pi^{an} \to \Pi_2^{an} \to 0$. Puisque la cohérence est stable par quotient et extension (prop. IV.22), les hypothèses faites entraînent la cohérence de Π_1, Π et Π_2 et donc (prop. VII.6) l'existence de $\widehat{\Pi_1^{an}}$, $\widehat{\Pi^{an}}$ et $\widehat{\Pi_2^{an}}$. De plus, comme Π_1 est bonne, $\widehat{\Pi_1^{an}} \simeq \Pi_1$ est admissible. Le cor. VII.9 fournit donc une suite exacte $\widehat{\Pi_1^{an}} \to \widehat{\Pi^{an}} \to \widehat{\Pi_2^{an}} \to 0$, s'insérant dans un diagramme commutatif

$$\begin{array}{ccccccccc} & & \widehat{\Pi_1^{an}} & \longrightarrow & \widehat{\Pi^{an}} & \longrightarrow & \widehat{\Pi_2^{an}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Pi_1 & \longrightarrow & \Pi & \longrightarrow & \Pi_2 & \longrightarrow & 0 \end{array}$$

Par hypothèse la flèche verticale de gauche est un isomorphisme. Une chasse au diagramme montre que si une des flèches verticales restantes est un isomorphisme, l'autre l'est aussi, ce qui démontre le lemme et conclut la preuve du th. VII.11. □

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On moduli spaces for quasitilted algebras

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We prove that if a quasitilted algebra is tame, then the associated moduli spaces are products of projective spaces. Together with an earlier result of Chindris this gives a geometric characterization of the tame quasitilted algebras. In our proof we use knowledge of the representation theory of the tame quasitilted algebras and a construction of semi-invariants as determinants.

Throughout the article $\Bbbk$ is an algebraically closed field of characteristic 0. By $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_+$ we denote the sets of integers, nonnegative integers and positive integers, respectively. Finally, if $i, j \in \mathbb{Z}$, then $[i, j] := \{k \in \mathbb{Z} \mid i \leq k \leq j\}$ (in particular, $[i, j] = \varnothing$ if $i > j$).

There is a well-known dichotomy for finite-dimensional algebras due to Drozd [1980]: every algebra is either tame or wild, but not both. Here a finite-dimensional algebra is called tame if, for each dimension d, the indecomposable d-dimensional modules form finitely many one-parameter families. On the other hand, an algebra Λ is wild if the classification of Λ-modules is as difficult as the classification of the pairs of two (noncommuting) endomorphisms of a finite-dimensional vector space (the latter problem is considered to be hopeless).

The above definitions of tame and wild algebras are of a geometric nature. This encourages people to look for characterizations of representation type, which use properties of geometric objects associated with them, for example, module varieties (some results of this type can be found in [Bobiński and Skowroński 1999; Skowroński and Zwara 1998]). In particular, Skowroński and Weyman [2000] have proved that a hereditary algebra Λ is tame if and only if all the corresponding rings of semi-invariants are complete intersections. Inspired by this result Chindris [2009; 2011; 2013] has initiated a programme, whose aim is to characterize representation type in terms of (rational) invariant theory (see also [Domokos 2011] for an earlier result in this direction). As a result of his studies he and Carroll published the following conjecture, which they attribute to Weyman:

Conjecture [Carroll and Chindris 2012]. *Let Λ be an algebra. Then the following conditions are equivalent*:

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(1) Λ *is of tame representation type.*

(2) *For any dimension vector* $\boldsymbol{d}$, *for any irreducible component* $\mathscr{C}$ *of the variety* $\operatorname{mod}_\Lambda(\boldsymbol{d})$ *of* Λ*-modules of dimension vector* $\boldsymbol{d}$, *and for any weight* θ *such that* $\mathscr{C}_\theta^{\mathrm{ss}} \neq \varnothing$, $\mathscr{M}(\mathscr{C})_\theta^{\mathrm{ss}}$ *is a product of projective spaces.*

Here $\mathscr{C}_\theta^{\mathrm{ss}}$ denotes the open subset of θ-semistable modules in $\mathscr{C}$ and $\mathscr{M}(\mathscr{C})_\theta^{\mathrm{ss}}$ denotes the associated moduli space (see Section 5).

This conjecture has no chance of holding in such generality. Obvious counterexamples are local wild algebras. There is also a counterexample due to Ringel of a triangular (no cycles in the Gabriel quiver) wild algebra, such that all the associated moduli spaces are points.

The aim of this paper is to verify this conjecture for the quasitilted algebras. The quasitilted algebras form an important class of finite-dimensional algebras. Using covering techniques, the study of some classes of algebras can be reduced to the study of quasitilted algebras. In particular, every self-injective algebra of polynomial growth is a socle deformation of an orbit algebra of the repetitive algebra of a tame quasitilted algebra with positive semidefinite Euler form (for more in this direction see the survey article [Skowroński 2006]).

The main result of the paper is the following:

Theorem 1. *Let* Λ *be a quasitilted algebra. Then the following conditions are equivalent*:

(1) Λ *is of tame representation type.*

(2) *For any dimension vector* $\boldsymbol{d}$, *for any irreducible component* $\mathscr{C}$ *of* $\operatorname{mod}_\Lambda(\boldsymbol{d})$, *and for any weight* θ *such that* $\mathscr{C}_\theta^{\mathrm{ss}} \neq \varnothing$, $\mathscr{M}(\mathscr{C})_\theta^{\mathrm{ss}}$ *is a product of projective spaces.*

The implication (2) $\Longrightarrow$ (1) has been proved for the tilted algebras by Chindris [2013, Proposition 4.1]. In fact, as has been explained to me by Chindris, his proof of the implication (2) $\Longrightarrow$ (1) generalizes to the quasitilted algebras. More precisely, Chindris [2013] used a result of Kerner [1989] stating that every wild tilted algebra has a convex subalgebra which is wild concealed. In the case of the quasitilted algebras one has to use results of Lenzing and Skowroński [1996] (every wild quasitilted algebra has a convex subalgebra which is wild almost concealed-canonical) and Meltzer [1996] (every wild almost concealed-canonical algebra has a convex subcategory which is wild concealed). Thus in the paper we concentrate on the proof of the following theorem:

Theorem 2. *Let* Λ *be a tame quasitilted algebra. Then for any dimension vector* $\boldsymbol{d}$, *for any irreducible component* $\mathscr{C}$ *of* $\operatorname{mod}_\Lambda(\boldsymbol{d})$, *and for any weight* θ *such that* $\mathscr{C}_\theta^{\mathrm{ss}} \neq \varnothing$, $\mathscr{M}(\mathscr{C})_\theta^{\mathrm{ss}}$ *is a product of projective spaces.*

The paper is organized as follows. In Section 1 we recall basic facts about quivers and their representations. Section 2 is devoted to a short introduction of quasitilted algebras. Next, in Sections 3 and 4 we introduce module varieties and semi-invariants, respectively. Moreover, in Section 4 some reduction results for semi-invariants are obtained. Finally, in Section 5 we recall King's construction of moduli spaces and in Section 6 we prove that the moduli spaces for the tame quasitilted algebras are products of projective spaces.

1. Quivers and their representations

In this section we present facts about quivers and their representations, which we use in the paper. As a general background we suggest [Assem et al. 2006; Auslander et al. 1997; Ringel 1984].

By a quiver Q we mean a finite set Q_0 (called the set of vertices of Q) together with a finite set Q_1 (called the set of arrows of Q) and two maps $s, t : Q_1 \to Q_0$, which assign to each arrow α its starting vertex $s\alpha$ and its terminating vertex $t\alpha$, respectively. By a path of length $n \in \mathbb{N}_+$ in a quiver Q we mean a sequence $\sigma = (\alpha_1, \dots, \alpha_n)$ of arrows such that $s\alpha_i = t\alpha_{i+1}$ for each $i \in [1, n-1]$. In the above situation we put $\ell\sigma := n$, $s\sigma := s\alpha_n$ and $t\sigma := t\alpha_1$. We treat every arrow in Q as a path of length 1. Moreover, for each vertex x we have a trivial path $\mathbf{1}_x$ at x such that $\ell\mathbf{1}_x := 0$ and $s\mathbf{1}_x := x =: t\mathbf{1}_x$. For the rest of the paper we assume that the considered quivers do not have oriented cycles, where by an oriented cycle we mean a path σ of positive length such that $s\sigma = t\sigma$.

Let Q be a quiver. By the path algebra $\Bbbk Q$ of Q, we mean the vector space with a basis formed by the paths in Q and multiplication induced by the concatenation of paths. If x and y are vertices of Q, we put $\Bbbk Q(x, y) := \mathbf{1}_y \Bbbk Q \mathbf{1}_x$; i.e., $\Bbbk Q(x, y)$ is the space spanned by the paths with the starting vertex x and the terminating vertex y. The (finite-dimensional) $\Bbbk Q$-modules may be identified with the $\Bbbk$-representations of Q, where by a $\Bbbk$-representation of Q we mean V consisting of finite-dimensional $\Bbbk$-vector spaces $V(x)$, $x \in Q_0$, and $\Bbbk$-linear maps $V(\alpha) : V(s\alpha) \to V(t\alpha)$, $\alpha \in Q_1$. In particular, if M is a $\Bbbk Q$-module and V is the corresponding representation, then $V(x) := \mathbf{1}_x M$ for each $x \in Q_0$. We will usually identify $\Bbbk Q$-modules with the corresponding representations of $\Bbbk$. If V and W are representations of a quiver Q, then a morphism $\varphi : V \to W$ is given by linear maps $\varphi(x) : V(x) \to W(x)$, $x \in Q_0$, such that $W(\alpha)\varphi(s\alpha) = \varphi(t\alpha)V(\alpha)$ for each $\alpha \in Q_1$. We denote the category of $\Bbbk$-representations of Q by $\operatorname{rep} Q$. If V is a representation, $x, y \in Q_0$, and $\omega \in \Bbbk Q(x, y)$, then one defines $V(\omega) : V(x) \to V(y)$ in an obvious way. Given a representation V of Q we denote by $\mathbf{dim}\, V$ its dimension vector, defined by the formula $(\mathbf{dim}\, V)(x) := \dim_{\Bbbk} V(x)$, for $x \in Q_0$. Observe that $\mathbf{dim}\, V \in \mathbb{N}^{Q_0}$ for each representation V of Q. We call the elements of $\mathbb{N}^{Q_0}$ dimension vectors. If $\boldsymbol{d}$

is a dimension vector, then we denote by $\operatorname{supp} \boldsymbol{d}$ the subquiver of Q induced by the vertices x such that $\boldsymbol{d}(x) \neq 0$. A dimension vector $\boldsymbol{d}$ is called connected if the quiver $\operatorname{supp} \boldsymbol{d}$ is connected. A dimension vector $\boldsymbol{d}$ is called sincere if $\operatorname{supp} \boldsymbol{d} = Q$.

By a bound quiver (Q, I) we mean a quiver Q together with an ideal I of $\Bbbk Q$ such that $I \subseteq \langle Q_1 \rangle^2$, where by $\langle Q_1 \rangle$ we denote the ideal of $\Bbbk Q$ generated by the arrows. Given a bound quiver (Q, I), we call the algebra $\Bbbk Q/I$ the path algebra of (Q, I). Note that if (Q, I) is a bound quiver, then the $\Bbbk Q/I$-modules may be identified with the representations V of Q such that $V(\omega) = 0$ for each $\omega \in I \cap \bigl(\bigcup_{x,y \in Q_0} \Bbbk Q(x, y)\bigr)$. If Λ is the path algebra of a bound quiver (Q, I), then we call Q the Gabriel quiver of Λ. Gabriel proved that (up to isomorphism) Q is uniquely determined by Λ. Moreover, Gabriel's theorem implies that each quasitilted algebra is Morita equivalent to the path algebra of a bound quiver (since we only consider quivers without oriented cycles, we also need [Happel et al. 1996, Proposition III.1.1(b)] for this result). Thus from now on all algebras considered are the path algebras of bound quivers. Observe that if J is an ideal in an algebra Λ, then the Gabriel quiver of Λ/J is a subquiver of the Gabriel quiver of Λ (here this is important that there are no oriented cycles in the considered quivers). If (Q, I) is a bound quiver, then an algebra Λ' is called a convex subalgebra of $\Bbbk Q/I$ if there exists a convex subquiver Q' of Q such that $\Lambda' = \Bbbk Q'/(I \cap \Bbbk Q')$. Recall that a subquiver Q' of Q is called convex if for every path $(\alpha_1, \dots, \alpha_n)$ in Q with $s\alpha_n, t\alpha_1 \in Q'_0$ we have $\alpha_i \in Q'_1$ for each $i \in [1, n]$ (and, consequently, $s\alpha_i \in Q'_0$ for each $i \in [1, n-1]$).

Let Λ be an algebra with Gabriel quiver Q. For a vertex x of Q we put $P_\Lambda(x) := \Lambda \mathbf{1}_x$. Then $P_\Lambda(x)$ is an indecomposable projective Λ-module and every indecomposable projective Λ-module is (up to isomorphism) of this form. If V is a Λ-module, then $\operatorname{Hom}_\Lambda(P_\Lambda(x), V) = V(x)$ for each $x \in Q_0$. In particular, $\operatorname{Hom}_\Lambda(P_\Lambda(x), P_\Lambda(y)) = \Bbbk Q(y, x)$ for any $x, y \in Q_0$. Moreover, if $x, y \in Q_0$, $\omega \in \Bbbk(y, x)$ and V is a Λ-module, then

$$\operatorname{Hom}_\Lambda(\omega, V) : \operatorname{Hom}_\Lambda(P_\Lambda(y), V) \to \operatorname{Hom}_\Lambda(P_\Lambda(x), V)$$

is just $V(\omega) : V(y) \to V(x)$.

For an algebra Λ we denote by $\operatorname{mod} \Lambda$ the category of Λ-modules. Next, if Λ is an algebra and Λ^{op} is the opposite algebra of Λ, then we denote by D_Λ the duality $\operatorname{mod} \Lambda \to \operatorname{mod} \Lambda^{\mathrm{op}}$ given by

$$\mathrm{D}_\Lambda(M) := \operatorname{Hom}_k(M, k) \quad (M \in \operatorname{mod} \Lambda).$$

Finally, for an algebra Λ we denote by τ_Λ the corresponding Auslander–Reiten translation, which assigns to each Λ-module M another Λ-module $\tau_\Lambda M$ (see [Assem et al. 2006, Section IV.2] for a definition). We will need the following consequence of the Auslander–Reiten formula [Assem et al. 2006, Theorem IV.2.13]: if M and N

are Λ-modules and $\operatorname{pdim}_\Lambda M \le 1$, then

$$\dim_{\Bbbk} \operatorname{Ext}^1_\Lambda(M, N) = \dim_{\Bbbk} \operatorname{Hom}_\Lambda(N, \tau_\Lambda M). \tag{1}$$

Let Λ be an algebra with Gabriel quiver Q. Since there are no cycles in Q, $\operatorname{gldim} \Lambda < \infty$. Consequently, we may define the bilinear form $\langle -, - \rangle_\Lambda : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \to \mathbb{Z}$ by the condition

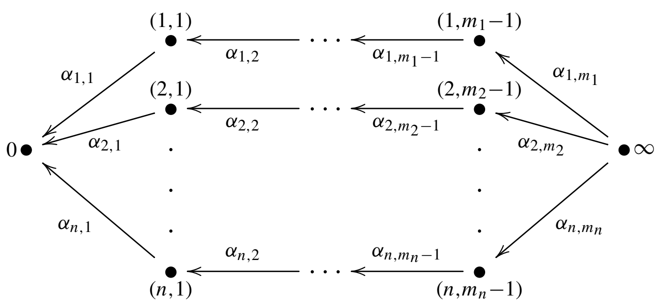
$$\langle \mathbf{dim}\, M, \mathbf{dim}\, N \rangle_\Lambda = \sum_{i \in \mathbb{N}} \dim_k \operatorname{Ext}^i_\Lambda(M, N)$$

for all Λ-modules M and N. We denote the corresponding quadratic form, called the Euler form, by χ_Λ.

2. Quasitilted algebras

A module T over an algebra Λ is called tilting if $\operatorname{pdim}_\Lambda T \le 1$, $\operatorname{Ext}^1_\Lambda(T, T) = 0$, and T is a direct sum of n pairwise nonisomorphic indecomposable Λ-modules, where n is the number of vertices of the Gabriel quiver of Λ. By a tilted algebra we mean the opposite algebra of the endomorphism algebra of a tilting module over the path algebra of a quiver.

An algebra Λ is called quasitilted if Λ is the opposite algebra of the endomorphism algebra of a tilting object in a connected hereditary abelian $\Bbbk$-category with finite-dimensional homomorphism and extension spaces. Equivalently, $\operatorname{gldim} \Lambda \le 2$ and either $\operatorname{pdim}_\Lambda X \le 1$ or $\operatorname{idim}_\Lambda X \le 1$ for each indecomposable Λ-module X (see [Happel et al. 1996, Theorem 2.3]). Two prominent examples of quasitilted algebras are the tilted algebras introduced above and the Ringel canonical algebras $\Lambda(\boldsymbol{m}, \boldsymbol{\lambda})$, where $\boldsymbol{m} = (m_1, \dots, m_n)$, $n \ge 3$, is a sequence of integers greater than 1 and $\boldsymbol{\lambda} = (\lambda_3, \dots, \lambda_n)$. In the above situation $\Lambda(\boldsymbol{m}, \boldsymbol{\lambda})$ is the path algebra of the quiver



modulo the ideal generated by the relations

$$\alpha_{1,1} \cdots \alpha_{1,m_1} + \lambda_i \alpha_{2,1} \cdots \alpha_{2,m_2} - \alpha_{i,1} \cdots \alpha_{i,m_i}, \quad i \in [3, n].$$

Due to [Happel 2001, Theorem 3.1] every quasitilted algebra is either a tilted algebra or is of canonical type (i.e., is derived equivalent to a canonical algebra).

The structure of the module categories over tilted algebras has been investigated in [Kerner 1989], while the structure of the module categories over quasitilted algebras of canonical type has been studied in [Lenzing and Skowroński 1996]. We also refer to [Skowroński 1998] for a characterization of the tame quasitilted algebras and to [Ringel 1984] for a description of the module categories over so-called tubular algebras, which form an important subclass of the tame quasitilted algebras. We list some consequences of these investigations.

Let Λ be a tame quasitilted algebra with Gabriel quiver Q. If $\boldsymbol{d}$ is a dimension vector, then there exists an indecomposable Λ-module with dimension vector $\boldsymbol{d}$ if and only if $\boldsymbol{d}$ is a root of χ_Λ, i.e., $\boldsymbol{d}$ is a connected nonzero dimension vector such that $\chi_\Lambda(\boldsymbol{d}) \in \{0, 1\}$. We call a root $\boldsymbol{d}$ isotropic if $\chi_\Lambda(\boldsymbol{d}) = 0$. We call a root $\boldsymbol{d}$ a Schur root if there exists a Λ-module X (necessarily indecomposable) with dimension vector $\boldsymbol{d}$ and trivial endomorphism algebra. Let $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ be two isotropic roots with $\operatorname{supp} \boldsymbol{d}_1 \cap \operatorname{supp} \boldsymbol{d}_2 \neq \varnothing$. Then $\operatorname{Hom}_\Lambda(X_1, X_2) \neq 0$ for all indecomposable Λ-modules X_1 and X_2 with $\mathbf{dim}\, X_1 = \boldsymbol{d}_1$ and $\mathbf{dim}\, X_2 = \boldsymbol{d}_2$, or $\operatorname{Hom}_\Lambda(X_2, X_1) \neq \varnothing$ for all indecomposable Λ-modules X_1 and X_2 with $\mathbf{dim}\, X_1 = \boldsymbol{d}_1$ and $\mathbf{dim}\, X_2 = \boldsymbol{d}_2$, or $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ are multiplicities of the same isotropic Schur root.

Now assume that Λ is a canonical algebra. The indecomposable Λ-modules can be divided into three classes, $\mathscr{L}_\Lambda$, $\mathscr{T}_\Lambda$ and $\mathscr{R}_\Lambda$: the class $\mathscr{L}_\Lambda$ is formed by the indecomposable Λ-modules X such that $\dim X(0) > \dim X(\infty)$, the class $\mathscr{T}_\Lambda$ is formed by the indecomposable Λ-modules X such that $\dim X(0) = \dim X(\infty)$, and the class $\mathscr{R}_\Lambda$ is formed by the indecomposable Λ-modules X such that $\dim X(0) < \dim X(\infty)$. An algebra Λ is called (almost) concealed-canonical if Λ is the opposite algebra of the endomorphism algebra of a tilting module T, which is a direct sum of indecomposable modules from $\mathscr{L}_\Lambda$ ($\mathscr{L}_\Lambda \cup \mathscr{T}_\Lambda$, respectively).

3. Module varieties

Let Λ be the path algebra of a bound quiver (Q, I). For a dimension vector $\boldsymbol{d}$ we denote by $\operatorname{mod}_\Lambda(\boldsymbol{d})$ the set of representations M of (Q, I) (recall that we identify the Λ-modules with the representations of (Q, I)) such that $M(x) = \Bbbk^{\boldsymbol{d}(x)}$ for each $x \in Q_0$. This set can be naturally identified with a closed subset of the affine space $\operatorname{rep}_Q(\boldsymbol{d}) := \prod_{\alpha \in Q_1} \mathbb{M}(\boldsymbol{d}(t\alpha), \boldsymbol{d}(s\alpha))$; thus it has the structure of an affine variety (note that under this identification $\operatorname{rep}_Q(\boldsymbol{d}) = \operatorname{mod}_{\Bbbk Q}(\boldsymbol{d})$). The reductive group $\operatorname{GL}(\boldsymbol{d}) := \prod_{x \in Q_0} \operatorname{GL}(\boldsymbol{d}(x))$ acts on $\operatorname{mod}_\Lambda(\boldsymbol{d})$ via

$$(g * M)(\alpha) := g(t\alpha) \cdot M(\alpha) \cdot g(s\alpha)^{-1} \quad (g \in \operatorname{GL}(\boldsymbol{d}),\ \alpha \in Q_1).$$

If $M \in \operatorname{mod}_\Lambda(\boldsymbol{d})$, then we denote its orbit with respect to this action by $\mathbb{O}(M)$. One has $\mathbb{O}(M) = \mathbb{O}(N)$ if and only if $M \simeq N$.

Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be closed irreducible subsets of varieties $\mathrm{mod}_\Lambda(\boldsymbol{d}_1)$ and $\mathrm{mod}_\Lambda(\boldsymbol{d}_2)$, respectively. By $\mathscr{C}_1 \oplus \mathscr{C}_2$ we denote the closure of the set consisting of all $M \in \mathrm{mod}_\Lambda(\boldsymbol{d}_1 + \boldsymbol{d}_2)$ such that $M \simeq M_1 \oplus M_2$ for some $M_1 \in \mathscr{C}_1$ and $M_2 \in \mathscr{C}_2$. In the above situation we call $\mathscr{C}_1$ and $\mathscr{C}_2$ summands of $\mathscr{C}$.

An irreducible component $\mathscr{C}$ of $\mathrm{mod}_\Lambda(\boldsymbol{d})$ is called indecomposable if the indecomposable modules in $\mathscr{C}$ form a dense subset of $\mathscr{C}$. If $\mathscr{C}$ is an irreducible component of $\mathrm{mod}_\Lambda(\boldsymbol{d})$, then there exist uniquely determined (up to ordering) indecomposable irreducible components $\mathscr{C}_1, \dots, \mathscr{C}_n$ of $\mathscr{C}$ such that

$$\mathscr{C} = \mathscr{C}_1 \oplus \cdots \oplus \mathscr{C}_n$$

[Crawley-Boevey and Schröer 2002, Theorem 1.1] (see also [de la Peña 1991]). We call the above presentation the generic decomposition of $\mathscr{C}$. Moreover, if, for $i \in [1, n]$, $\mathscr{C}_i \subseteq \mathrm{mod}_\Lambda(\boldsymbol{d}_i)$, then we call $\boldsymbol{d}_1, \dots, \boldsymbol{d}_n$ the generic summands of $\boldsymbol{d}$ at $\mathscr{C}$.

Now we present a description of the indecomposable irreducible components in the case of the tame quasitilted algebras, which follows from [Bobiński and Skowroński 1999]. First, if $\boldsymbol{d}$ is a dimension vector, then there is at most one indecomposable irreducible component of $\mathrm{mod}_\Lambda(\boldsymbol{d})$. Thus if it exists we denote it by $\mathscr{C}(\boldsymbol{d})$. Moreover, there exists an indecomposable irreducible component of $\mathrm{mod}_\Lambda(\boldsymbol{d})$ if and only if $\boldsymbol{d}$ is a Schur root. Moreover, if $\boldsymbol{d}$ is not isotropic, then $\mathscr{C}(\boldsymbol{d})$ is an orbit closure, i.e., there exists a Λ-module M such that $\mathscr{C}(\boldsymbol{d}) = \overline{\mathbb{O}(M)}$.

4. Semi-invariants

Let Q be a quiver, $\boldsymbol{d}$ a dimension vector, and $\mathscr{C}$ a $\mathrm{GL}(\boldsymbol{d})$-invariant closed subset of $\mathrm{rep}_Q(\boldsymbol{d})$. The action of $\mathrm{GL}(\boldsymbol{d})$ on $\mathscr{C}$ induces an action on the coordinate ring $\Bbbk[\mathscr{C}]$ via

$$(g * f)(M) := f(g^{-1} * M) \quad (g \in \mathrm{GL}(\boldsymbol{d}),\ f \in \Bbbk[\mathscr{C}],\ M \in \mathscr{C}).$$

If $\mathscr{C}$ is irreducible, then there is a unique closed orbit in $\mathscr{C}$, that of the semisimple module with dimension vector $\boldsymbol{d}$, hence there are only trivial $\mathrm{GL}(\boldsymbol{d})$-invariant regular functions on $\mathscr{C}$, i.e., $\Bbbk[\mathscr{C}]^{\mathrm{GL}(\boldsymbol{d})} = \Bbbk$. However, one may still have nontrivial semi-invariants. A regular function $f \in \Bbbk[\mathscr{C}]$ is called a semi-invariant of weight $\theta \in \mathbb{Z}^{Q_0}$ if

$$g * f = \chi^\theta(g) \cdot f$$

for each $g \in \mathrm{GL}(\boldsymbol{d})$. Here $\chi^\theta : \mathrm{GL}(\boldsymbol{d}) \to \Bbbk^\times$ is given by

$$\chi^\theta(g) := \prod_{x \in Q_0} \det\nolimits^{\theta(\boldsymbol{e}_x)}(g(x)) \quad (g \in \mathrm{GL}(\boldsymbol{d})),$$

where $\boldsymbol{e}_x$, $x \in Q_0$, are the standard basis vectors of $\mathbb{Z}^{Q_0}$. We denote the space of semi-invariants of weight θ by $\mathrm{SI}[\mathscr{C}]_\theta$. One easily observes that $\theta(\boldsymbol{d}) = 0$ provided $\mathrm{SI}[\mathscr{C}]_\theta \neq 0$.

We present a method of constructing semi-invariants, which in the case of quivers is due to Schofield [1991], and has been generalized to the case of bound quivers independently by Derksen and Weyman [2002] and Domokos [2002] (we also refer to the latter two articles for proofs). Let Q be a quiver and $\boldsymbol{d}$ a dimension vector. Fix sequences $\boldsymbol{x} = (x_1, \dots, x_n)$ and $\boldsymbol{y} = (y_1, \dots, y_m)$ of vertices of Q. Put

$$\Bbbk Q(\boldsymbol{x}, \boldsymbol{y}) := \prod_{\substack{i \in [1,n] \\ j \in [1,m]}} \Bbbk Q(x_i, y_j).$$

If $\phi = (\phi_{i,j})_{i \in [1,n], j \in [1,m]} \in \Bbbk Q(\boldsymbol{x}, \boldsymbol{y})$ and $M \in \mathrm{rep}_Q(\boldsymbol{d})$ for a dimension vector $\boldsymbol{d}$, then we obtain a map

$$M(\phi) := [M(\phi_{i,j})]_{\substack{i \in [1,n] \\ j \in [1,m]}} : M(\boldsymbol{x}) := \bigoplus_{i \in [1,n]} M(x_i) \to M(\boldsymbol{y}) := \bigoplus_{j \in [1,m]} M(y_j).$$

If, in addition, $\sum_{i \in [1,n]} \boldsymbol{d}(x_i) = \sum_{j \in [1,m]} \boldsymbol{d}(y_j)$, then we may define a regular function $c_{\boldsymbol{d}}^{\phi} : \mathrm{rep}_Q(\boldsymbol{d}) \to \Bbbk$ by

$$c_{\boldsymbol{d}}^{\phi}(M) := \det M(\phi) \quad (M \in \mathrm{rep}_Q(\boldsymbol{d})).$$

Then $c_{\boldsymbol{d}}^{\phi}$ is a semi-invariant of weight θ^{ϕ}, where

$$\theta^{\phi}(\boldsymbol{c}) := \sum_{i \in [1,n]} \boldsymbol{c}(x_i) - \sum_{j \in [1,m]} \boldsymbol{c}(y_j) \quad (\boldsymbol{c} \in \mathbb{Z}^{Q_0})$$

(note that, in particular, $\theta^{\phi}(\boldsymbol{d}) = 0$). If $\mathscr{C}$ is a $\mathrm{GL}(\boldsymbol{d})$-invariant closed subset of $\mathrm{rep}_Q(\boldsymbol{d})$, then we denote the restriction $c_{\boldsymbol{d}}^{\phi}|_{\mathscr{C}}$ of $c_{\boldsymbol{d}}^{\phi}$ to $\mathscr{C}$ by $c_{\mathscr{C}}^{\phi}$.

We list some obvious consequences (cf. [Derksen and Weyman 2000, Lemma 1]).

Lemma 4.1. *Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be sequences of vertices of a quiver Q, $\phi \in \Bbbk Q(\boldsymbol{x}, \boldsymbol{y})$, $\boldsymbol{d}$ a dimension vector such that $\theta^{\phi}(\boldsymbol{d}) = 0$, and $M = M_1 \oplus M_2 \in \mathrm{mod}_\Lambda(\boldsymbol{d})$.*

(1) *If $\theta^{\phi}(\mathbf{dim}\, M_1) \neq 0$ (hence, equivalently, $\theta^{\phi}(\mathbf{dim}\, M_2) \neq 0$), then $c_{\boldsymbol{d}}^{\phi}(M) = 0$.*

(2) *If $\theta^{\phi}(\mathbf{dim}\, M_1) = 0$ (hence, equivalently, $\theta^{\phi}(\mathbf{dim}\, M_2) = 0$), then $c_{\boldsymbol{d}}^{\phi}(M) = c_{\mathbf{dim}\, M_1}^{\phi}(M_1) \cdot c_{\mathbf{dim}\, M_2}^{\phi}(M_2)$.* □

If we have sequences $\boldsymbol{x}$, $\boldsymbol{x}'$, $\boldsymbol{y}$ and $\boldsymbol{y}'$ of vertices of a quiver Q, $\phi \in \Bbbk(\boldsymbol{x}, \boldsymbol{y})$ and $\phi' \in \Bbbk(\boldsymbol{x}', \boldsymbol{y}')$, then we may define an element $\phi \oplus \phi' \in \Bbbk Q(\boldsymbol{x} \cdot \boldsymbol{x}', \boldsymbol{y} \cdot \boldsymbol{y}')$ in the obvious way, where $\boldsymbol{x} \cdot \boldsymbol{x}'$ and $\boldsymbol{y} \cdot \boldsymbol{y}'$ are the concatenations of the respective sequences. Observe that

$$M(\phi \oplus \phi') = \begin{bmatrix} M(\phi) & 0 \\ 0 & M(\phi') \end{bmatrix} : M(\boldsymbol{x}) \oplus M(\boldsymbol{x}') \to M(\boldsymbol{y}) \oplus M(\boldsymbol{y}')$$

for each $M \in \operatorname{rep}_Q(\boldsymbol{d})$. Consequently, we get the following.

Lemma 4.2. *Let $\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y}$ and $\boldsymbol{y}'$ be sequences of vertices of a quiver Q, $\phi \in \Bbbk(\boldsymbol{x}, \boldsymbol{y})$ and $\phi' \in \Bbbk(\boldsymbol{x}', \boldsymbol{y}')$. If $\boldsymbol{d}$ is a dimension vector and $\theta^{\phi}(\boldsymbol{d}) = 0 = \theta^{\phi'}(\boldsymbol{d})$, then*

$$c_{\boldsymbol{d}}^{\phi \oplus \phi'} = c_{\boldsymbol{d}}^{\phi} \cdot c_{\boldsymbol{d}}^{\phi'}.$$

In particular, $c_{\boldsymbol{d}}^{\phi \oplus \phi'}$ is a semi-invariant of weight $\theta^{\phi} + \theta^{\phi'}$. □

We can interpret the above construction using projective presentations. Let Λ be a factor algebra of $\Bbbk Q$ for a quiver Q. As above, let $\boldsymbol{x} = (x_1, \dots, x_n)$ and $\boldsymbol{y} = (y_1, \dots, y_m)$ be sequences of vertices of a quiver Q, and $\phi \in \Bbbk Q(\boldsymbol{x}, \boldsymbol{y})$. If we put

$$P_{\Lambda}(\boldsymbol{x}) := \bigoplus_{i \in [1,n]} P_{\Lambda}(x_i) \quad \text{and} \quad P_{\Lambda}(\boldsymbol{y}) := \bigoplus_{j \in [1,m]} P_{\Lambda}(y_j),$$

then we may view ϕ as a map $P_{\Lambda}(\boldsymbol{y}) \to P_{\Lambda}(\boldsymbol{x})$ (note that every map between projective Λ-modules is of this form, for some $\boldsymbol{x}$, $\boldsymbol{y}$ and ϕ). Observe that

$$\theta^{\phi}(\mathbf{dim}\, M) = \dim_{\Bbbk} \operatorname{Hom}_{\Lambda}(P_{\Lambda}(\boldsymbol{x}), M) - \dim_{\Bbbk} \operatorname{Hom}_{\Lambda}(P_{\Lambda}(\boldsymbol{y}), M)$$

for each Λ-module M. If $M \in \operatorname{mod}_{\Lambda}(\boldsymbol{d})$ for a dimension vector $\boldsymbol{d}$, then $M(\phi)$ may be identified with the induced map

$$\operatorname{Hom}_{\Lambda}(\phi, M) : \operatorname{Hom}_{\Lambda}(P_{\Lambda}(\boldsymbol{x}), M) \to \operatorname{Hom}_{\Lambda}(P_{\Lambda}(\boldsymbol{y}), M).$$

This implies in particular that if $\theta^{\phi}(\boldsymbol{d}) = 0$, then $c_{\boldsymbol{d}}^{\phi}(M) \neq 0$ if and only if $\operatorname{Hom}_{\Lambda}(\operatorname{Coker} \phi, M) = 0$. We associate a semi-invariant with $\operatorname{Coker} \phi$ (independently of ϕ). In order to simplify the presentation, we make some additional assumptions.

Let Λ be an algebra with Gabriel quiver Q. Moreover, let V be a Λ-module with projective dimension at most 1. If $\phi : Q \to P$ is a projective presentation of V such that ϕ is a monomorphism, then $\theta^{\phi} = \langle \mathbf{dim}\, V, - \rangle_{\Lambda}$, hence is independent of ϕ. We denote this weight by θ^V. Moreover, if $\theta^V(\boldsymbol{d}) = 0$, and ϕ and ϕ' are projective presentations of V such that ϕ and ϕ' are monomorphisms, then $c_{\operatorname{mod}_{\Lambda}(\boldsymbol{d})}^{\phi}$ and $c_{\operatorname{mod}_{\Lambda}(\boldsymbol{d})}^{\phi'}$ coincide up to a nonzero scalar. Thus we may define $c_{\Lambda,\boldsymbol{d}}^V \in \Bbbk[\operatorname{mod}_{\Lambda}(\boldsymbol{d})]$ by $c_{\Lambda,\boldsymbol{d}}^V := c_{\operatorname{mod}_{\Lambda}(\boldsymbol{d})}^{\phi}$, where ϕ is a chosen projective presentation of V. Then $c_{\Lambda,\boldsymbol{d}}^V$ is a semi-invariant of weight θ^V and $c_{\Lambda,\boldsymbol{d}}^V(M) \neq 0$ if and only if $\operatorname{Hom}_{\Lambda}(V, M) = 0$ or, equivalently, $\operatorname{Hom}_{\Lambda}(M, \tau_{\Lambda} V) = 0$ (for the latter statement we need (1)). If $\mathscr{C}$ is a closed $\operatorname{GL}(\boldsymbol{d})$-invariant subset of $\operatorname{mod}_{\Lambda}(\boldsymbol{d})$, then we denote by $c_{\Lambda,\mathscr{C}}^V$ the restriction of $c_{\Lambda,\boldsymbol{d}}^V$ to $\mathscr{C}$.

It will be often useful to associate semi-invariants to modules of projective dimension at most 1 in a "regular" way. Thus assume in addition that $\boldsymbol{c}$ is a dimension vector such that $\mathscr{P}_{\Lambda}(\boldsymbol{c}) \neq \varnothing$, where $\mathscr{P}_{\Lambda}(\boldsymbol{c})$ is the subset of $\operatorname{mod}_{\Lambda}(\boldsymbol{c})$ consisting of the modules of projective dimension at most 1. Let $\boldsymbol{x}$ be a sequence of vertices of Q such that $P_\Lambda(\boldsymbol{x}) = \bigoplus_{x\in Q_0} P_\Lambda(x)^{\boldsymbol{c}(x)}$. Since there exists an epimorphism $P_\Lambda(\boldsymbol{x}) \twoheadrightarrow V$ for each Λ-module V with dimension vector $\boldsymbol{c}$ and $\mathcal{P}_\Lambda(\boldsymbol{c}) \neq \varnothing$, there exists a sequence $\boldsymbol{y}$ of vertices of Q such that $\mathbf{dim}\, P_\Lambda(\boldsymbol{x}) - \mathbf{dim}\, P_\Lambda(\boldsymbol{y}) = \boldsymbol{c}$. Let $\mathcal{X}_\Lambda(\boldsymbol{c}) \subseteq \Bbbk Q(\boldsymbol{x}, \boldsymbol{y})$ be the set of monomorphisms $P_\Lambda(\boldsymbol{y}) \hookrightarrow P_\Lambda(\boldsymbol{x})$. Obviously, if $\phi \in \mathcal{X}_\Lambda(\boldsymbol{c})$, then $\operatorname{Coker}\phi \in \mathcal{P}_\Lambda(\boldsymbol{c})$. On the other hand, if $V \in \mathcal{P}_\Lambda(\boldsymbol{c})$, then there exists $\phi \in \mathcal{X}_\Lambda(\boldsymbol{d})$ such that $\operatorname{Coker}\phi \simeq V$. In fact we have even more:

Lemma 4.3. *Let Λ be an algebra and $\boldsymbol{c}$ a dimension vector. If $\mathcal{V}$ is a nonempty open subset of $\mathcal{P}_\Lambda(\boldsymbol{c})$ and*

$$\mathcal{U} := \{\phi \in \mathcal{X}_\Lambda(\boldsymbol{c}) \mid \operatorname{Coker}\phi \in \mathcal{V}\},$$

then $\mathcal{U}$ is a nonempty open subset of $\mathcal{X}_\Lambda(\boldsymbol{c})$.

Proof. Let $P := \bigoplus_{x\in Q_0} P_\Lambda(x)^{\boldsymbol{c}(x)}$. Let $\mathcal{Y}_\Lambda(\boldsymbol{c})$ be the set of $\psi = (\psi(x))_{x\in Q_0}$ such that, for each $x \in Q_0$, $\psi(x) : P(x) \to \Bbbk^{\boldsymbol{c}(x)}$ is a linear map. We denote by $\mathcal{Z}_\Lambda(\boldsymbol{c})$ the set of pairs (ϕ, ψ) such that $\phi \in \mathcal{X}_\Lambda(\boldsymbol{c})$, $\psi \in \mathcal{Y}_\Lambda(\boldsymbol{c})$ and $\psi \circ \phi = 0$. If $\pi : \mathcal{Z}_\Lambda(\boldsymbol{c}) \to \mathcal{X}_\Lambda(\boldsymbol{c})$ is the canonical projection, then π is a vector bundle. Consequently, if $\mathcal{Z}'_\Lambda(\boldsymbol{c})$ is the set of pairs $(\phi, \psi) \in \mathcal{Z}_\Lambda(\boldsymbol{c})$ such that ψ is a surjection and π' is the restriction of π to $\mathcal{Z}'_\Lambda(\boldsymbol{c})$, then π' is locally trivial (with fiber isomorphic to $\mathrm{GL}(\boldsymbol{c})$). In particular, if $\mathcal{W}$ is an open subset of $\mathcal{Z}'_\Lambda(\boldsymbol{c})$, then $\pi'(\mathcal{W})$ is an open subset of $\mathcal{X}_\Lambda(\boldsymbol{c})$.

There exists a regular map $\Theta : \mathcal{Z}'_\Lambda(\boldsymbol{c}) \to \operatorname{mod}_\Lambda(\boldsymbol{c})$ such that $\Theta(\phi, \psi) \simeq \operatorname{Coker}\phi$ for all $(\phi, \psi) \in \mathcal{Z}'_\Lambda(\boldsymbol{c})$ (the proof is analogous to the proof of [Richmond 2001, Lemma 9], hence we omit it). Since $\mathcal{U} = \pi(\Theta^{-1}(\mathcal{V}))$, the claim follows. □

We will also need the following:

Lemma 4.4. *Let Λ be an algebra, $\boldsymbol{d}$ a dimension vector, $\mathcal{C}$ a $\mathrm{GL}(\boldsymbol{d})$-invariant irreducible closed subset of $\operatorname{mod}_\Lambda(\boldsymbol{d})$, and $\boldsymbol{c}$ a dimension vector such that $\mathcal{P}_\Lambda(\boldsymbol{c}) \neq \varnothing$.*

(1) *If $\mathcal{U}$ is a nonempty open subset of $\mathcal{X}_\Lambda(\boldsymbol{c})$, then*

$$\operatorname{span}\{c_{\mathcal{C}}^{\phi} \mid \phi \in \mathcal{U}\} = \operatorname{span}\{c_{\mathcal{C}}^{\phi} \mid \phi \in \mathcal{X}_\Lambda(\boldsymbol{c})\}.$$

(2) *If $\mathcal{V}$ is a nonempty open subset of $\mathcal{P}_\Lambda(\boldsymbol{c})$, then*

$$\operatorname{span}\{c_{\Lambda,\mathcal{C}}^{V} \mid V \in \mathcal{V}\} = \operatorname{span}\{c_{\Lambda,\mathcal{C}}^{V} \mid V \in \mathcal{P}_\Lambda(\boldsymbol{c})\}.$$

Proof. Using Lemma 4.3, it is sufficient to prove the first assertion. Let $\phi_1, \dots, \phi_m \in \mathcal{X}_\Lambda(\boldsymbol{c})$ be such that $c_{\mathcal{C}}^{\phi_1}, \dots, c_{\mathcal{C}}^{\phi_m}$ form a basis of $\operatorname{span}\{c_{\mathcal{C}}^{\phi} \mid \phi \in \mathcal{X}_\Lambda(\boldsymbol{c})\}$. There exist $M_1, \dots, M_m \in \mathcal{C}$ such that

$$\det[c_{\mathcal{C}}^{\phi_i}(M_j)]_{1\le i,j\le m} \neq 0.$$

It suffices to show there exist $\psi_1, \dots, \psi_m \in \mathcal{U}$ such that

$$\det[c_{\mathcal{C}}^{\psi_i}(M_j)]_{1\le i,j\le m} \neq 0.$$

However, the regular function

$$\phi : \mathscr{X}_\Lambda(\boldsymbol{c})^m \to \Bbbk, \quad (\psi_1, \dots, \psi_m) \mapsto \det[c_{\mathscr{C}}^{\psi_i}(M_j)]_{1 \le i, j \le m},$$

is not a zero function, hence the claim follows. □

Now we use the above construction to describe generating sets of semi-invariants. We present two such sets. Depending on the situation, it will be more convenient to use one of them.

Let Λ be an algebra with Gabriel quiver Q, $\boldsymbol{d}$ a dimension vector, $\mathscr{C}$ an irreducible component of $\operatorname{mod}_\Lambda(\boldsymbol{d})$, and θ a weight such that $\mathrm{SI}[\mathscr{C}]_\theta \neq 0$. There exists unique $\boldsymbol{c}_\theta \in \mathbb{Z}^{Q_0}$ such that $\theta = \langle \boldsymbol{c}_\theta, - \rangle_{\Bbbk Q}$. Since $\mathrm{SI}[\operatorname{rep}_Q(\boldsymbol{d})]_\theta \neq 0$, we may assume that $\boldsymbol{c}_\theta$ is a dimension vector. We explain this more precisely.

If θ' and θ'' are weights, then $\mathrm{SI}[\operatorname{rep}_Q(\boldsymbol{d})]_{\theta'}$ and $\mathrm{SI}[\operatorname{rep}_Q(\boldsymbol{d})]_{\theta''}$ are equal and both nonzero if and only if θ' and θ'' are $\boldsymbol{d}$-equivalent, i.e., $\theta'(x) = \theta''(x)$ for all $x \in (\operatorname{supp} \boldsymbol{d})_0$. Now [Derksen and Weyman 2000, Theorem 1] (see also [Schofield and van den Bergh 2001, Theorem 2.3]) implies that there exists a dimension vector $\boldsymbol{c}$ such that the weights θ and $\langle \boldsymbol{c}, - \rangle_{\Bbbk Q}$ are $\boldsymbol{d}$-equivalent. Consequently, we may assume that we only consider weights of this form.

It is clear that $\mathscr{P}_{\Bbbk Q}(\boldsymbol{c}_\theta) \neq \varnothing$ (the category $\operatorname{rep} Q$ is hereditary); hence also $\mathscr{X}_{\Bbbk Q}(\boldsymbol{c}_\theta) \neq \varnothing$. It follows from [Chindris 2009, Corollary 2.5] that the semi-invariants $c_{\boldsymbol{d}}^\phi$, for $\phi \in \mathscr{X}_{\Bbbk Q}(\boldsymbol{c}_\theta)$, span $\mathrm{SI}[\operatorname{rep}_Q(\boldsymbol{d})]_\theta$. Since $\mathscr{C}$ is a closed $\mathrm{GL}(\boldsymbol{d})$-invariant subset of $\operatorname{rep}_Q(\boldsymbol{d})$ and $\operatorname{char} \Bbbk = 0$, it follows that the semi-invariants $c_{\mathscr{C}}^\phi$, $\phi \in \mathscr{X}_{\Bbbk Q}(\boldsymbol{c}_\theta)$, span $\mathrm{SI}[\mathscr{C}]_\theta$.

We list some consequences:

Lemma 4.5. *Let Λ be an algebra, $\boldsymbol{d}$ a dimension vector, $\mathscr{C}$ an irreducible component of* $\operatorname{mod}_\Lambda(\boldsymbol{d})$, *and θ a weight such that* $\mathrm{SI}[\mathscr{C}]_\theta \neq 0$. *If $\mathscr{C}'$ is a summand of $\mathscr{C}$, then* $\mathrm{SI}[\mathscr{C}']_\theta \neq 0$.

Proof. Write $\mathscr{C} = \mathscr{C}' \oplus \mathscr{C}''$. By assumption there exist $\phi \in \mathscr{X}_{\Bbbk Q}(\boldsymbol{c}_\theta)$ and $M \in \mathscr{C}$ such that $c_{\mathscr{C}}^\phi(M) \neq 0$. Without loss of generality we may assume that $M = M' \oplus M''$ for $M' \in \mathscr{C}'$ and $M'' \in \mathscr{C}''$. Lemma 4.1(1) implies that $\theta(\mathbf{dim}\, M_1) = 0 = \theta(\mathbf{dim}\, M_2)$. Consequently, Lemma 4.1(2) implies that $c_{\mathscr{C}}^\phi(M) = c_{\mathscr{C}'}^\phi(M') \cdot c_{\mathscr{C}''}^\phi(M'')$. In particular, $c_{\mathscr{C}'}^\phi(M') \neq 0$. □

Lemma 4.6. *Let Λ be an algebra, $\boldsymbol{d}$ a dimension vector, $\mathscr{C}$ an irreducible component of* $\operatorname{mod}_\Lambda(\boldsymbol{d})$, *and θ a weight such that* $\mathrm{SI}[\mathscr{C}]_\theta \neq 0$. *Then there exists an open subset $\mathscr{U}$ of $\mathscr{X}_{\Bbbk Q}(\boldsymbol{c}_\theta)$ such that $c_{\mathscr{C}}^\phi \neq 0$ for all $\phi \in \mathscr{U}$.*

Proof. There exist $\phi_0 \in \mathscr{X}_{\Bbbk Q}(\boldsymbol{c}_\theta)$ and $M \in \mathscr{C}$ such that $c_{\mathscr{C}}^{\phi_0}(M) \neq 0$. We define a function $\Phi : \mathscr{X}_{\Bbbk Q}(\boldsymbol{c}_\theta) \to \Bbbk$ by

$$\Phi(\phi) := c_{\mathscr{C}}^\phi(M) \quad (\phi \in \mathscr{X}_{\Bbbk Q}(\boldsymbol{c}_\theta)).$$

This is a regular function and we take $\mathcal{U} := \Phi^{-1}(\Bbbk^\times)$. □

Now we present the second construction. As above let Λ be an algebra with Gabriel quiver Q, $\boldsymbol{d}$ a dimension vector, $\mathcal{C}$ an irreducible component of $\operatorname{mod}_\Lambda(\boldsymbol{d})$, and θ a weight such that $\mathrm{SI}[\mathcal{C}]_\theta \neq 0$. Next, let $\Lambda_{\mathcal{C}} := \Lambda / \operatorname{Ann} \mathcal{C}$, where $\operatorname{Ann} \mathcal{C} := \bigcap_{M \in \mathcal{C}} \operatorname{Ann} M$. Consequently, $\mathcal{C}$ is a faithful irreducible component of $\operatorname{mod}_{\Lambda_{\mathcal{C}}}(\boldsymbol{d})$. Again there exists $\boldsymbol{c}_{\theta,\mathcal{C}} \in \mathbb{Z}^{Q_0}$ such that $\theta = \langle \boldsymbol{c}_{\theta,\mathcal{C}}, - \rangle_{\Lambda_{\mathcal{C}}}$. Since $\mathrm{SI}[\mathcal{C}]_\theta \neq 0$, $\boldsymbol{c}_{\theta,\mathcal{C}}$ is a dimension vector and $\mathcal{P}_{\mathcal{C}}(\theta) := \mathcal{P}_{\Lambda_{\mathcal{C}}}(\boldsymbol{c}_{\theta,\mathcal{C}}) \neq \varnothing$ (see [Derksen and Weyman 2002, Theorem 1]). Moreover, [Derksen and Weyman 2002, Theorem 1] also says that $\mathrm{SI}[\mathcal{C}]_\theta$ is spanned by the semi-invariants $c^V_{\Lambda_{\mathcal{C}},\mathcal{C}}$, $V \in \mathcal{P}_{\mathcal{C}}(\theta)$.

It is known that $\overline{\mathcal{P}}_{\mathcal{C}}(\theta)$ is an irreducible component of $\operatorname{mod}_{\Lambda_{\mathcal{C}}}(\boldsymbol{c}_{\theta,\mathcal{C}})$ [Barot and Schröer 2001, Proposition 3.1]. It is quite easy to observe that the generic decomposition of $\overline{\mathcal{P}}_{\mathcal{C}}(\theta)$ is of the form

$$\overline{\mathcal{P}}_{\mathcal{C}}(\theta) = \overline{\mathcal{P}}_{\Lambda_{\mathcal{C}}}(\boldsymbol{c}_1) \oplus \cdots \oplus \overline{\mathcal{P}}_{\Lambda_{\mathcal{C}}}(\boldsymbol{c}_n)$$

for some dimension vectors $\boldsymbol{c}_1, \dots, \boldsymbol{c}_n$ such that $\boldsymbol{c}_{\theta,\mathcal{C}} = \boldsymbol{c}_1 + \cdots + \boldsymbol{c}_n$. Obviously $\boldsymbol{c}_1, \dots, \boldsymbol{c}_n$ are the generic summands of $\boldsymbol{c}_{\theta,\mathcal{C}}$ (at $\overline{\mathcal{P}}_{\mathcal{C}}(\theta)$). If we put $\theta_i := \langle \boldsymbol{c}_i, - \rangle_{\Lambda_{\mathcal{C}}}$, then we call the presentation

$$\theta = \theta_1 + \cdots + \theta_n$$

the generic decomposition of θ at $\mathcal{C}$.

As a first consequence we get the following:

Lemma 4.7. *Let Λ be an algebra, $\boldsymbol{d}$ a dimension vector, $\mathcal{C}$ an irreducible component of* $\operatorname{mod}_\Lambda(\boldsymbol{d})$, *and θ a weight such that* $\mathrm{SI}[\mathcal{C}]_\theta \neq 0$. *If $\theta = \theta_1 + \cdots + \theta_n$ is the generic decomposition of θ at $\mathcal{C}$, then the image of the map*

$$\mathrm{SI}[\mathcal{C}]_{\theta_1} \times \cdots \times \mathrm{SI}[\mathcal{C}]_{\theta_n} \to \mathrm{SI}[\mathcal{C}]_\theta, \quad (f_1, \dots, f_n) \mapsto f_1 \cdots f_n,$$

spans $\mathrm{SI}[\mathcal{C}]_\theta$. *In particular*, $\mathrm{SI}[\mathcal{C}]_{\theta_i} \neq 0$ *for each $i \in [1, n]$.*

Proof. Let $\boldsymbol{c}_1, \dots, \boldsymbol{c}_n$ be the dimension vectors corresponding to the weights $\theta_1, \dots, \theta_n$, respectively, in the sense explained above. The set $\mathcal{V}$ of $V \in \mathcal{P}_{\mathcal{C}}(\theta)$ such that $V \simeq V_1 \oplus \cdots \oplus V_n$ for $V_1 \in \mathcal{P}_{\Lambda_{\mathcal{C}}}(\boldsymbol{c}_1), \dots, V_n \in \mathcal{P}_{\Lambda_{\mathcal{C}}}(\boldsymbol{c}_n)$ contains an open subset of $\mathcal{P}_{\mathcal{C}}(\theta)$. Lemma 4.4(2) implies that $\mathrm{SI}[\mathcal{C}]_\theta$ is spanned by the semi-invariants $c^{V_1 \oplus \cdots \oplus V_n}_{\Lambda_{\mathcal{C}},\mathcal{C}}$, $V_1 \in \mathcal{P}_{\Lambda_{\mathcal{C}}}(\boldsymbol{c}_1), \dots, V_n \in \mathcal{P}_{\Lambda_{\mathcal{C}}}(\boldsymbol{c}_n)$. Moreover, Lemma 4.2 implies that

$$c^{V_1 \oplus \cdots \oplus V_n}_{\Lambda_{\mathcal{C}},\mathcal{C}} = c^{V_1}_{\Lambda_{\mathcal{C}},\mathcal{C}} \cdots c^{V_n}_{\Lambda_{\mathcal{C}},\mathcal{C}}$$

for all $V_1 \in \mathcal{P}_{\Lambda_{\mathcal{C}}}(\boldsymbol{c}_1), \dots, V_n \in \mathcal{P}_{\Lambda_{\mathcal{C}}}(\boldsymbol{c}_n)$, hence the claim follows. □

As a next consequence we obtain the following useful fact:

Lemma 4.8. *Let Λ be an algebra, $\boldsymbol{d}$ a dimension vector, $\mathscr{C}$ an irreducible component of* $\operatorname{mod}_\Lambda(\boldsymbol{d})$, *and θ a weight such that* $\mathrm{SI}[\mathscr{C}]_\theta \neq 0$. *If $\mathscr{P}_\mathscr{C}(\theta)$ contains a dense orbit, then* $\dim_\Bbbk \mathrm{SI}[\mathscr{C}]_\theta = 1$.

Proof. If $\mathcal{O}(V)$ is a dense orbit in $\mathscr{P}_\mathscr{C}(\theta)$, then Lemma 4.4(2) implies that $\mathrm{SI}[\mathscr{C}]_\theta$ is spanned by the semi-invariant $c^V_{\Lambda_\mathscr{C},\mathscr{C}}$, hence the claim follows. □

Consequently, we get the following:

Corollary 4.9. *Let Λ be an algebra, $\boldsymbol{d}$ a dimension vector, $\mathscr{C}$ an irreducible component of* $\operatorname{mod}_\Lambda(\boldsymbol{d})$, *and θ a weight such that* $\mathrm{SI}[\mathscr{C}]_\theta \neq 0$. *If $\boldsymbol{c}'$ is a generic summand of $\boldsymbol{c}_{\theta,\mathscr{C}}$ at $\overline{\mathscr{P}}_\mathscr{C}(\theta)$ such that $\mathscr{P}_{\Lambda_\mathscr{C}}(\boldsymbol{c}')$ contains a dense orbit, then*

$$\mathrm{SI}[\mathscr{C}]_\theta \simeq \mathrm{SI}[\mathscr{C}]_{\theta-\theta'},$$

where $\theta' := \langle \boldsymbol{c}', - \rangle_{\Lambda_\mathscr{C}}$.

Proof. Lemma 4.7 implies that $\mathrm{SI}[\mathscr{C}]_{\theta'} \neq 0$. Together with Lemma 4.8 this implies that $\dim_\Bbbk \mathrm{SI}[\mathscr{C}]_{\theta'} = 1$. Fix a nonzero semi-invariant $f \in \mathrm{SI}[\mathscr{C}]_{\theta'}$. Lemma 4.7 implies that the map

$$\mathrm{SI}[\mathscr{C}]_{\theta-\theta'} \to \mathrm{SI}[\mathscr{C}]_\theta, \quad c \mapsto f \cdot c,$$

is surjective. Since $\mathscr{C}$ is irreducible, this is also injective, and the claim follows. □

5. Moduli spaces

Let Λ be an algebra, $\boldsymbol{d}$ a dimension vector, and $\mathscr{C}$ an irreducible component of $\operatorname{mod}_\Lambda(\boldsymbol{d})$. If θ is a weight, then a Λ-module $M \in \mathscr{C}$ is called θ-semistable if there exists $f \in \mathrm{SI}[\mathscr{C}]_{p\theta}$, for some $p \in \mathbb{N}_+$, such that $f(M) \neq 0$. King [1994] has proved that M is θ-semistable if and only if $\theta(\mathbf{dim}\, M) = 0$ and $\theta(\mathbf{dim}\, N) \leq 0$ for each submodule N of M. We denote by $\mathscr{C}^{\mathrm{ss}}_\theta$ the set of θ-semistable Λ-modules in $\mathscr{C}$. King has also constructed a coarse moduli $\mathcal{M}(\mathscr{C})^{\mathrm{ss}}_\theta$ for the θ-semistable modules in $\mathscr{C}$ (up to an equivalence, which identifies modules which have the same simple composition factors within the category of θ-semistable modules). By definition,

$$\mathcal{M}(\mathscr{C})^{\mathrm{ss}}_\theta = \operatorname{Proj}\biggl(\bigoplus_{p\in\mathbb{N}} \mathrm{SI}[\mathscr{C}]_{p\theta}\biggr).$$

Lemma 5.1. *Let Λ be an algebra, $\boldsymbol{d}$ a dimension vector, $\mathscr{C}$ an irreducible component of* $\operatorname{mod}_\Lambda(\boldsymbol{d})$, *and θ a weight such that $\mathscr{C}^{\mathrm{ss}}_\theta \neq \varnothing$. If $\mathscr{C} = \mathscr{C}_1 \oplus \mathscr{C}_2$ for irreducible components $\mathscr{C}_1$ and $\mathscr{C}_2$ of* $\operatorname{mod}_\Lambda(\boldsymbol{d}_1)$ *and* $\operatorname{mod}_\Lambda(\boldsymbol{d}_2)$, *respectively, and $\mathscr{C}_2$ is an orbit closure, then*

$$\mathcal{M}(\mathscr{C})^{\mathrm{ss}}_\theta \simeq \mathcal{M}(\mathscr{C}_1)^{\mathrm{ss}}_\theta.$$

Proof. Without loss of generality we may assume that $\mathrm{SI}[\mathscr{C}]_\theta \neq 0$. Then we will show that

$$\mathrm{SI}[\mathscr{C}]_{p\theta} \simeq \mathrm{SI}[\mathscr{C}_1]_{p\theta}$$

for each $p \in \mathbb{N}$.

Let $\mathscr{C}_2 = \overline{\mathcal{O}(M)}$. Consider the map $\Phi : \mathscr{C}_1 \to \mathscr{C}$ given by

$$\Phi(N) := N \oplus M \quad (N \in \mathscr{C}_1).$$

We will show that $\Phi^* : \Bbbk[\mathscr{C}] \to \Bbbk[\mathscr{C}_1]$ induces an isomorphism $\Phi_p^* : \mathrm{SI}[\mathscr{C}]_{p\theta} \simeq \mathrm{SI}[\mathscr{C}_1]_{p\theta}$ for each $p \in \mathbb{N}$.

Fix $p \in \mathbb{N}$. Since $\mathrm{GL}(\boldsymbol{d}) \times (\mathscr{C}_1 \oplus \{M\})$ is a dense subset of $\mathscr{C}$, it is clear that Φ_p^* is a monomorphism. Thus it remains to show that Φ_p^* is an epimorphism. Let $\mathscr{X} := \mathscr{X}_{\Bbbk Q}(p\theta)$, where Q is the Gabriel quiver of Λ. Using Lemma 4.4(1), it suffices to show that there exists an open subset $\mathscr{U}$ of $\mathscr{X}$ such that $c_{\mathscr{C}_1}^\phi$ is in the image of Φ_p^* for each $\phi \in \mathscr{U}$.

It follows from Lemma 4.5 that $\mathrm{SI}[\mathscr{C}_2]_{p\theta} \neq 0$. Using Lemma 4.6, we obtain that there exists an open subset $\mathscr{U}$ of $\mathscr{X}$ such that $c_{\mathscr{C}_2}^\phi \neq 0$ for each $\phi \in \mathscr{U}$. In particular, $c_{\mathscr{C}_2}^\phi(M) \neq 0$ for each $\phi \in \mathscr{U}$. Now it follows from Lemma 4.1 that

$$c_{\mathscr{C}_1}^\phi = \Phi_p^*\left(\frac{1}{c_{\mathscr{C}_2}^\phi(M)} c_{\mathscr{C}}^\phi\right)$$

for each $\phi \in \mathscr{U}$. □

6. Moduli spaces for the tame quasitilted algebras

The aim of this section is to prove Theorem 2. Let Λ be a tame quasitilted algebra, $\boldsymbol{d}$ a dimension vector, $\mathscr{C}$ an irreducible component of $\mathrm{mod}_\Lambda(\boldsymbol{d})$, and θ a weight such that $\mathscr{C}_\theta^{\mathrm{ss}} \neq \varnothing$. We show that $\mathscr{M}(\mathscr{C})_\theta^{\mathrm{ss}}$ is a product of projective spaces. Let $\Lambda' := \Lambda / \operatorname{Ann} \mathscr{C}$.

We know that there exist Schur roots $\boldsymbol{d}_1, \dots, \boldsymbol{d}_n$ such that

$$\mathscr{C} = \mathscr{C}_1 \oplus \cdots \oplus \mathscr{C}_n,$$

where $\mathscr{C}_i := \mathscr{C}(\boldsymbol{d}_i)$, $i \in [1, n]$. Using Lemma 5.1, we may assume that $\boldsymbol{d}_1, \dots, \boldsymbol{d}_n$ are isotropic.

Now let $\boldsymbol{c}_1, \dots, \boldsymbol{c}_m$ be the generic summands of $\boldsymbol{c}_{\theta,\mathscr{C}}$ at $\overline{\mathscr{P}}_{\Lambda'}(\boldsymbol{c}_{\theta,\mathscr{C}})$. Using Corollary 4.9, we may assume that for each $j \in [1, m]$, $\mathscr{P}_{\Lambda'}(\boldsymbol{c}_j)$ does not contain a dense orbit. Since $\overline{\mathscr{P}}_{\Lambda'}(\boldsymbol{c}_j)$ is an indecomposable irreducible component, this implies that there exist infinitely many indecomposable Λ'-modules of dimension vector $\boldsymbol{c}_j$, for each $j \in [1, m]$. This also means that, for each $j \in [1, m]$, there exist infinitely many indecomposable Λ-modules of dimension vector $\boldsymbol{c}_j$, hence

$\boldsymbol{c}_1, \dots, \boldsymbol{c}_m$ are isotropic roots of χ_Λ (using more detailed knowledge of mod Λ one could also show that they are Schur roots, but we will not use this).

Before we formulate the next lemma let us recall that if $i \in [1, n]$, $j \in [1, m]$, $M \in \mathscr{C}_i$ and $V \in \mathscr{P}_{\Lambda'}(\boldsymbol{c}_j)$, then $c^V_{\mathscr{C}_i}(M) = 0$ if and only if $\operatorname{Hom}_\Lambda(V, M) \neq 0$ or $\operatorname{Hom}_\Lambda(M, \tau_{\Lambda'} V) \neq 0$.

Lemma 6.1. *In the above notation, either* $\boldsymbol{c}_j$ *is a multiplicity of* $\boldsymbol{d}_i$ *or the intersection* $\operatorname{supp} \boldsymbol{d}_i \cap \operatorname{supp} \boldsymbol{c}_j$ *is empty for all* $i \in [1, n]$ *and* $j \in [1, m]$.

Proof. Fix $i \in [1, n]$ and $j \in [1, m]$. Note that Lemmas 4.5 and 4.7 imply that $\operatorname{SI}[\mathscr{C}_i]_{\theta'} \neq 0$, where $\theta' := \langle \boldsymbol{c}_j, - \rangle_{\Lambda'}$. Assume that neither $\boldsymbol{c}_j$ is a multiplicity of $\boldsymbol{d}_i$ nor $\operatorname{supp} \boldsymbol{d}_i \cap \operatorname{supp} \boldsymbol{c}_j = \varnothing$. Then Section 2 implies that one of the following holds:

(1) $\operatorname{Hom}_\Lambda(V, M) \neq 0$ for each indecomposable Λ-module M with dimension vector $\boldsymbol{d}_i$ and each indecomposable Λ-module V with dimension vector $\boldsymbol{c}_j$.

(2) $\operatorname{Hom}_\Lambda(M, V) \neq 0$ for each indecomposable Λ-module M with dimension vector $\boldsymbol{d}_i$ and each indecomposable Λ-module V with dimension vector $\boldsymbol{c}_j$.

In the first case we immediately obtain that $\operatorname{SI}[\mathscr{C}_i]_{\theta'} = 0$, a contradiction. We show that we get the same conclusion in the second case.

Since Λ, hence also Λ', are tame and there are infinitely many indecomposable Λ'-modules in $\mathscr{P}_{\Lambda'}(\boldsymbol{c}_j)$, [Crawley-Boevey 1988, Theorem D] implies that there is a nonempty open subset $\mathscr{V}$ of $\mathscr{P}_{\Lambda'}(\boldsymbol{c}_j)$ such that $\tau_{\Lambda'} V \simeq V$ for each $V \in \mathscr{V}$. In particular, $\operatorname{\mathbf{dim}} \tau_{\Lambda'} V = \boldsymbol{c}_j$ for each $V \in \mathscr{V}$. Thus (2) together with Lemma 4.4(2) implies that $\operatorname{SI}[\mathscr{C}_i]_{\theta'} = 0$, and this finishes the proof. □

Let I' be the set of $i \in [1, n]$ such that there exists $j \in [1, m]$ with $\boldsymbol{c}_j$ a multiplicity of $\boldsymbol{d}_i$. Let $I'' := [1, n] \setminus I$. Lemma 6.1 implies that $\operatorname{supp} \boldsymbol{d}_p \cap \operatorname{supp} \boldsymbol{d}_q = \varnothing$ if $p \in I'$ and $q \in I''$ (since $\operatorname{supp} \boldsymbol{d}_p = \operatorname{supp} \boldsymbol{c}_j$ for some $j \in J$). Thus $\mathscr{C} = \mathscr{C}_1 \times \mathscr{C}_2$, where

$$\mathscr{C}_1 := \bigoplus_{i \in I'} \mathscr{C}(\boldsymbol{d}_i) \quad \text{and} \quad \mathscr{C}_2 := \bigoplus_{i \in I''} \mathscr{C}(\boldsymbol{d}_i).$$

Consequently,

$$\mathscr{M}(\mathscr{C})^{\mathrm{ss}}_\theta = \mathscr{M}(\mathscr{C}_1)^{\mathrm{ss}}_\theta \times \mathscr{M}(\mathscr{C}_2)^{\mathrm{ss}}_\theta.$$

If $I' \neq [1, n] \neq I''$, then we get Theorem 2 by induction. Hence we have only two cases to consider: either $I' = [1, n]$ or $I'' = [1, n]$.

First assume that $I'' = [1, n]$. Since $\mathscr{C}$ is a faithful component over Λ', $\boldsymbol{d}$ is a sincere dimension vector over Λ'. On the other hand, $\operatorname{supp} \boldsymbol{c}_j \cap \operatorname{supp} \boldsymbol{d} = \varnothing$, hence $\boldsymbol{c}_j = 0$. Consequently, $\theta = 0$. Thus

$$\operatorname{SI}[\mathscr{C}]_{p\theta} = \Bbbk[\mathscr{C}]^{\operatorname{GL}(\boldsymbol{d})} = \Bbbk$$

for each $p \in \mathbb{N}$, and

$$\mathscr{M}(\mathscr{C})^{\mathrm{ss}}_\theta = \operatorname{Proj}(\Bbbk[T]) = \{*\}.$$

Now assume that $I' = [1, n]$. We can make another reduction in this case. Let $\boldsymbol{d}'_1, \dots, \boldsymbol{d}'_l$ be the pairwise different vectors among $\boldsymbol{d}_1, \dots, \boldsymbol{d}_n$. For each $p \in [1, l]$, let I_p be the set of $i \in [1, n]$ such that $\boldsymbol{d}_i = \boldsymbol{d}'_p$. Let $\mathscr{C}'_p := \bigoplus_{i \in I_p} \mathscr{C}(\boldsymbol{d}_i)$ for $p \in [1, l]$. Lemma 6.1 again implies that $\operatorname{supp} \boldsymbol{d}'_p \cap \operatorname{supp} \boldsymbol{d}'_q = \varnothing$ if $p, q \in [1, l]$ and $p \neq q$. Consequently,

$$\mathscr{C} = \mathscr{C}'_1 \times \cdots \times \mathscr{C}'_l$$

and

$$\mathscr{M}(\mathscr{C})^{\mathrm{ss}}_{\theta} = \mathscr{M}(\mathscr{C}'_1)^{\mathrm{ss}}_{\theta} \times \cdots \times \mathscr{M}(\mathscr{C}'_l)^{\mathrm{ss}}_{\theta}.$$

If $l > 1$, then Theorem 2 follows by induction again, thus we may assume $l = 1$. In this case [Bobiński and Skowroński 1999, Theorem 2] implies that $\operatorname{mod}_{\Lambda}(\boldsymbol{d})$ is irreducible, hence $\mathscr{C} = \operatorname{mod}_{\Lambda}(\boldsymbol{d})$. Thus Theorem 2 is a result of the following:

Proposition 6.2. *Let Λ be a tame quasitilted algebra and let $\boldsymbol{h}$ be an isotropic Schur root of χ_{Λ}. If $n, p \in \mathbb{N}_+$, then*

$$\mathscr{M}(\operatorname{mod}_{\Lambda}(n\boldsymbol{h}))^{\mathrm{ss}}_{p\langle \boldsymbol{h}, - \rangle_{\Lambda}} \simeq \mathbb{P}^n_{\Bbbk}.$$

Proof. This is a part of [Domokos and Lenzing 2002, Theorem 7.1]. One may also give a more direct proof, using a description of the semi-invariants for concealed-canonical algebras (the support of $\boldsymbol{h}$ is a concealed-canonical algebra) which implies that $\bigoplus_{q \in \mathbb{N}} \mathrm{SI}[\operatorname{mod}_{\Lambda}(n\boldsymbol{h})]_{q\langle \boldsymbol{h}, - \rangle_{\Lambda}}$ is the polynomial ring in $n + 1$ variables (see [Bobiński 2015, Proposition 6.2]). □

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