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Let $H$ be a Hopf algebra and let $B$ be a Koszul $H$-module algebra. We provide necessary and sufficient conditions for a filtered algebra to be a Poincaré–Birkhoff–Witt (PBW) deformation of the smash product algebra $B \# H$. Many examples of these deformations are given.

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Introduction

Given a Hopf algebra ($H$-)action on a Koszul algebra $B$, the aim of this work is to provide necessary and sufficient conditions for a certain filtered algebra, namely $\mathcal{D}_{B,\kappa}$ in Notation 0.3 below, to be a Poincaré–Birkhoff–Witt (PBW) deformation of the smash product algebra $B \# H$, i.e., $\text{gr} \mathcal{D}_{B,\kappa} \cong B \# H$ (Definition 0.1). One well-studied case is that of group actions on polynomial rings, where many algebras of interest arise as such deformations; see for example [Crawley-Boevey and Holland 1998; Drinfeld 1986; Etingof and Ginzburg 2002; Lusztig 1989; Ram and Shepler 2003; Shepler and Witherspoon 2012a]. For group actions on other Koszul algebras, see [Levandovskyy and Shepler 2014; Naidu and Witherspoon 2014; Shepler and Witherspoon 2012b; Shroff 2014]. There are some results involving Hopf algebra actions, such as those of Khare [2007], when $H$ is cocommutative and $B$ is a polynomial algebra. More specifically, the case when $H = U(\mathfrak{g})$, with $\mathfrak{g}$ the Lie algebra of a (not necessarily connected) reductive algebraic group, was studied by Etingof, Gan, and Ginzburg [Etingof et al. 2005], and by Khare and Tikaradze [2010] where $\mathfrak{g} = \mathfrak{sl}_2$. Results for an action of the quantized enveloping algebra $H = U_q(\mathfrak{sl}_2)$ on the quantum plane are provided by Gan and Khare [2007].

The goal of this paper is to provide a general theorem encompassing all of the above known classes of examples from the literature. Specifically, Theorem 3.1 gives PBW deformation conditions for $B \# H$, and it only requires the following

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of $H$ and $B$: (1) the antipode of the Hopf algebra $H$ is bijective, (2) the Koszul $H$-module algebra $B$ is connected ($B_0 = k$), and (3) the $H$-action on $B$ preserves the grading of $B$. We then apply our theorem to several different choices of Hopf algebras acting on Koszul algebras to obtain nontrivial PBW deformations, both known and new. Our work indicates that such examples abound.

Many ring-theoretic properties are preserved under PBW deformation. To discuss this, let us consider the following definition:

**Definition 0.1.** Let $D = \bigcup_{i \geq 0} F_i$ be a filtered algebra with $\{0\} \subseteq F_0 \subseteq F_1 \subseteq \cdots \subseteq D$. We say that $D$ is a Poincaré–Birkhoff–Witt (PBW) deformation of an $\mathbb{N}$-graded algebra $A$ if $A$ is isomorphic to the associated graded algebra $\text{gr}_{F} D = \bigoplus_{i \geq 0} F_i / F_{i-1}$, as $\mathbb{N}$-graded algebras.

Now if $\text{gr}_{F} D$ is an integral domain, prime, or (right) noetherian, then so is $D$. Moreover, if $D$ is affine with the standard filtration $F'$, then the Gelfand–Kirillov (GK) dimensions of $D$ and of $\text{gr}_{F'} D$ are equal; $\text{GKdim}(\text{gr}_{F} D) \leq \text{GKdim}(D)$ for a general filtration $F$ of $D$. The Krull dimension and global dimension of $\text{gr}_{F} D$ serve as upper bounds for the corresponding dimensions of $D$. These ring-theoretic results can all be found in [McConnell and Robson 2001]. Homological properties preserved under PBW deformation have also been investigated; see [Berger and Taillefer 2007] and [Wu and Zhu 2013] regarding the Calabi–Yau property, for instance. The representation theory of some classes of PBW deformations of smash product algebras has been thoroughly studied in the literature and still remains an active area of research. Some examples of PBW deformations whose representation theory is of interest include rational Cherednik algebras, symplectic reflection algebras, and various types of Hecke algebras (see, for example, [Drinfeld 1986; Etingof et al. 2005; Etingof and Ginzburg 2002; Lusztig 1989; Ram and Shepler 2003], and for more recent work, see [Ding and Tsymbaliuk 2013; Losev and Tsymbaliuk 2014; Tikaradze 2010; Tsymbaliuk 2014]).

In order to state the main result, we need the following notation and terminology.

Let $k$ be a field of arbitrary characteristic and let an unadorned $\otimes$ mean $\otimes_k$. Let $\mathbb{N}$ denote the natural numbers, including 0. Recall that an $\mathbb{N}$-graded algebra is Koszul if its trivial module $k$ admits a linear minimal graded free resolution; see [Polishchuk and Positselski 2005, Chapter 2] for more details.

**Notation 0.2** $(H, B, I, \kappa, \kappa^C, \kappa^L)$. Let $V$ be a finite-dimensional vector space over $k$.

(i) Let $H$ be a Hopf algebra with the standard structure notation $(H, m, \Delta, u, \epsilon, S)$. Here, we assume that the antipode $S$ of $H$ is bijective.

(ii) Let $B = T_k(V)/(I)$ be an $\mathbb{N}$-graded, Koszul, left $H$-module algebra $B = \bigoplus_{j \geq 0} B_j$ with $B_0 = k$ and $I \subseteq V \otimes V$. We assume that the action of $H$ preserves the grading and the subspace $I$ of $V \otimes V$. So in this case, $V$ is an $H$-module.
(iii) Take $\kappa : I \rightarrow H \oplus (V \otimes H)$ to be a $k$-bilinear map, where $\kappa$ is the sum of its constant and linear parts $\kappa^C : I \rightarrow H$ and $\kappa^L : I \rightarrow V \otimes H$, respectively.

**Notation 0.3** ($\mathcal{D}_{B, \kappa}$). Let $\mathcal{D}_{B, \kappa}$ be the filtered $k$-algebra given by

$$\mathcal{D}_{B, \kappa} = \frac{T_k(V) \# H}{(r - \kappa(r))_{r \in I}}.$$ 

Here, we assign the elements of $H$ degree 0.

Our main result is given as follows:

**Theorem 3.1.** The algebra $\mathcal{D}_{B, \kappa}$ is a PBW deformation of $B \# H$ if and only if the following conditions hold:

(a) $\kappa$ is $H$-invariant (Definition 1.4); and

If $\dim_k V \geq 3$, then the following equations hold for the maps $\kappa^C \otimes \text{id} - \text{id} \otimes \kappa^C$ and $\kappa^L \otimes \text{id} - \text{id} \otimes \kappa^L$, which are defined on the intersection $(I \otimes V) \cap (V \otimes I)$:

(b) $\text{Im}(\kappa^L \otimes \text{id} - \text{id} \otimes \kappa^L) \subseteq I$;

(c) $\kappa^L \circ (\kappa^L \otimes \text{id} - \text{id} \otimes \kappa^L) = - (\kappa^C \otimes \text{id} - \text{id} \otimes \kappa^C)$;

(d) $\kappa^C \circ (\text{id} \otimes \kappa^L - \kappa^L \otimes \text{id}) \equiv 0$.

In the case that $H$ is cocommutative and $B$ is the symmetric algebra $S(V)$, this result was proven by Khare [2007, Theorem 2.1], via the diamond lemma. Our proof is a generalization of that of [Braverman and Gaitsgory 1996, Lemma 0.4, Theorem 0.5] (where $H = k$) and of [Shepler and Witherspoon 2012b, Theorem 5.4] (where $H$ is a group algebra).

Background information on Hopf algebra (co)actions, Hochschild cohomology, and deformations of algebras are provided in Section 1. In Section 2, we produce a free resolution of the smash product algebra $B \# H$; see Construction 2.5 and Theorem 2.10. This resolution is adapted from Guccione and Guccione [2002]; Negron [2014] independently constructed a similar resolution. Our resolution is used in the proof of Theorem 3.1, which is given in Section 3. Many examples of PBW deformations of $B \# H$ are provided in Section 4, including/involving:

- (Example 4.1) the Crawley-Boevey–Holland algebras;
- (Examples 4.2 and 4.4) some actions of semisimple, noncommutative, nonco-commutative Hopf algebras on skew polynomial rings;
- (Examples 4.13 and 4.16) actions of the Sweedler and the Taft algebras on the polynomial ring $k[u, v]$;
- (Example 4.18) the quantized symplectic oscillator algebras of rank 1.

All of the examples of $B \# H$ above have nontrivial PBW deformations.
1. Background material

We begin by discussing Hopf (co)actions on algebras and (co)modules and end with a discussion on deformations of algebras. For further background on these topics, we refer the reader to [Montgomery 1993] and [Braverman and Gaitsgory 1996; Gerstenhaber 1964], respectively.

1A. Hopf algebra (co)actions.

Definition 1.1. (i) For a left $H$-module $M$, we denote the $H$-action by $\cdot : H \otimes M \to M$, that is, by $h \cdot m \in M$ for all $h \in H$, $m \in M$. Similarly for all $h \in H$ and $m \in M$, we denote the right $h$-action on $m$ by $m \cdot h$.

(ii) Given a Hopf algebra $H$ and an algebra $A$, we say that $H$ acts on $A$ (from the left, as a Hopf algebra) if $A$ is a left $H$-module and

$$h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b) \quad \text{and} \quad h \cdot 1_A = \epsilon(h)1_A$$

for all $h \in H, a, b \in A$, where the comultiplication is given by $\Delta(h) = \sum h_1 \otimes h_2$ (Sweedler’s notation). In this case, we say that $A$ is a left $H$-module algebra.

(iii) For any left $H$-comodule $M$, we denote the left $H$-coaction by $\rho(M) \subseteq H \otimes M$, where $\rho(m) = \sum m_{-1} \otimes m_0$ for $m_{-1} \in H$ and $m, m_0 \in M$. Likewise, the right $H$-coaction on a right $H$-comodule $M$ is given by $\rho(m) = \sum m_0 \otimes m_1$ for $m, m_0 \in M$ and $m_1 \in H$.

Note that $H$ is naturally an $H$-bimodule via left and right multiplication. This yields a left $H$-adjoint action on $H$ given by

$$h \cdot \ell := \sum h_1 \ell S(h_2)$$ (1-2)

for $h, \ell \in H$. Moreover, if $V$ is a left $H$-module, we give $V \otimes H$ an $H$-bimodule structure as follows: $h(v \otimes \ell) = \sum (h_1 \cdot v) \otimes h_2 \ell$ and $(v \otimes \ell)h = v \otimes \ell h$ for all $h, \ell \in H$ and $v \in V$. A left $H$-adjoint action on $V \otimes H$ arises by combining these:

$$h \cdot (v \otimes \ell) := \sum (h_1 \cdot v) \otimes h_2 \ell S(h_3).$$ (1-3)

The left $H$-adjoint actions in (1-2) and (1-3) extend to the standard left $H$-adjoint action on $A = B # H$ (where $B = T_k(V)/(I)$ as in Notation 0.2(ii)), via Definition 1.1, since the action of $H$ preserves $I$.

Now we discuss the $H$-invariance of the map $\kappa$ (Notation 0.2(iii)), which is one of the necessary conditions for the filtered algebra $\mathcal{D}_{B,\kappa}$ (Notation 0.3) to be a PBW deformation of $B \# H$.

Definition 1.4. Recall Notation 0.2. We say that the map $\kappa$ is $H$-invariant if $h \cdot (\kappa(r)) = \kappa(h \cdot r)$ in $H \oplus (V \otimes H)$ for any relation $r \in I$ and $h \in H$. 
**1B. Deformations of algebras and Hochschild cohomology.** In this part, we remind the reader of the notion of a deformation of a $k$-algebra $A$ and how Hochschild cohomology plays a role in its construction. This is seminal work of Gerstenhaber [1964], adapted to our graded setting as in [Braverman and Gaitsgory 1996].

**Definition 1.5** ($A_j$, $A^{(j)}$). Let $A$ be an associative algebra and let $t$ be an indeterminate. A deformation of $A$ over $k[t]$ is an associative $k[t]$-algebra $A_t$ over $k[t]$ which is isomorphic to $A[t]$ as $k$-vector spaces, with multiplication given by

$$a_1 \ast a_2 = \mu_0(a_1 \otimes a_2) + \mu_1(a_1 \otimes a_2)t + \mu_2(a_1 \otimes a_2)t^2 + \cdots$$

for all $a_1, a_2 \in A$. Here, $\mu_i : A \otimes A \rightarrow A$ is a $k$-linear map, referred to as the $i$-th multiplication map. Moreover, $\mu_0(a_1 \otimes a_2) = a_1a_2$ is the usual product in $A$.

Now assume that $A$ is graded by $\mathbb{N}$. A graded deformation of $A$ over $k[t]$ is an algebra $A_t$ as above, which is itself graded by $\mathbb{N}$, setting $\deg(t) = 1$. The map $\mu_i$ is homogeneous of degree $-i$ in this case. A $j$-th-level graded deformation of $A$ is a graded associative algebra $A^{(j)}$ over $k[t]/(t^{j+1})$ that is isomorphic to $A[t]/(t^{j+1})$ as $k$-vector spaces, with multiplication given by

$$a_1 \ast a_2 = \mu_0(a_1 \otimes a_2) + \mu_1(a_1 \otimes a_2)t + \cdots + \mu_j(a_1 \otimes a_2)t^j.$$ 

The maps $\mu_i : A \otimes A \rightarrow A$ are extended to be linear over $k[t]/(t^{j+1})$.

The associativity of $\ast$ for the deformation $A_t$ imposes conditions on the maps $\mu_i$. Specifically, for each degree $i$, the following equation must hold for all $a_1, a_2, a_3 \in A$:

$$\sum_{j=0}^{i} \mu_j(\mu_{i-j}(a_1 \otimes a_2) \otimes a_3) = \sum_{j=0}^{i} \mu_j(a_1 \otimes \mu_{i-j}(a_2 \otimes a_3)).$$

We use Hochschild cohomology to study these equations.

**Definition 1.7** ($B_*(A)$). Let $A$ be a $k$-algebra and let $M$ be an $A$-bimodule, or equivalently, an $A^e$-module. Here, $A^e := A \otimes A^{\text{op}}$. The Hochschild cohomology of $M$ is $\text{HH}^n(A, M) := \text{Ext}^n_{A^e}(A, M)$. Moreover, this cohomology may be derived from the bar resolution $B_*(A)$ of the $A^e$-module $A$:

$$B_*(A) : \cdots \xrightarrow{\delta_3} A^\otimes 4 \xrightarrow{\delta_2} A^\otimes 3 \xrightarrow{\delta_1} A \otimes A \xrightarrow{\delta_0} A \rightarrow 0,$$

where

$$\delta_n(a_0 \otimes \cdots \otimes a_{n+1}) := \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_ia_{i+1} \otimes \cdots \otimes a_{n+1}$$

for all $n \geq 0$ and $a_0, \ldots, a_{n+1} \in A$. When $M = A$, write $\text{HH}^n(A)$ for $\text{HH}^n(A, A)$. Moreover, if $A$ is ($\mathbb{N}$-)graded, then $\text{HH}^n(A)$ inherits the grading of $A$: If $A = \bigoplus_i A_i$, then $\text{HH}^n(A) = \bigoplus_i \text{HH}^{n,i}(A).$
Note that $\text{Hom}_k(A \otimes^n, A) \cong \text{Hom}_{A^e}(A \otimes^{n+2}, A)$, since the $A^e$-module $A \otimes^{n+2}$ is induced from the $k$-module $A \otimes^n$. We will identify these two Hom spaces often without further comment. Now we make some remarks about the multiplication maps $\mu_i$.

**Remark 1.8.** Using (1-6) for $i = 1$, we see that

\begin{equation}
\mu_1(a_1 \otimes a_2)a_3 + \mu_1(a_1a_2 \otimes a_3) = a_1\mu_1(a_2 \otimes a_3) + \mu_1(a_1 \otimes a_2a_3)
\end{equation}

for all $a_1, a_2, a_3 \in A$. In other words, $\mu_1$ is a Hochschild 2-cocycle on the bar resolution of $A$; that is, $\delta^3(\mu_1) := \mu_1 \circ \delta_3$ vanishes. (Here we have identified the input $a_1 \otimes a_2 \otimes a_3$ with $1 \otimes a_1 \otimes a_2 \otimes a_3 \otimes 1$ to apply $\delta_3$, under the identification of Hom spaces described above.)

Next, using (1-6) for $i = 2$, we see that

\begin{equation}
\mu_2(a_1 \otimes a_2)a_3 + \mu_1(\mu_1(a_1 \otimes a_2) \otimes a_3) + \mu_2(a_1a_2 \otimes a_3) = a_1\mu_2(a_2 \otimes a_3) + \mu_1(a_1 \otimes \mu_1(a_2 \otimes a_3)) + \mu_2(a_1 \otimes a_2a_3).
\end{equation}

Therefore

\begin{equation}
\delta^3_3(\mu_2)(a_1 \otimes a_2 \otimes a_3) = \mu_1(\mu_1(a_1 \otimes a_2) \otimes a_3) - \mu_1(a_1 \otimes \mu_1(a_2 \otimes a_3))
\end{equation}

for all $a_1, a_2, a_3 \in A$. In other words, $\mu_2$ is a cochain on the bar resolution of $A$ whose coboundary is given by the right-hand side of (1-10).

For all $i \geq 1$, (1-6) is equivalent to

\begin{equation}
\delta^3_3(\mu_i)(a_1 \otimes a_2 \otimes a_3) = \sum_{j=1}^{i-1} \mu_j(\mu_{i-j}(a_1 \otimes a_2) \otimes a_3) - \mu_j(a_1 \otimes \mu_{i-j}(a_2 \otimes a_3)).
\end{equation}

That is, $\mu_i$ is a cochain on the bar resolution of $A$ whose coboundary is given by the right-hand side of (1-11).

**Definition 1.12.** The right-hand side of (1-11) is the $(i - 1)$-th obstruction of the deformation $A_t$ of $A$ from **Definition 1.5**. An $(i - 1)$-th-level graded deformation (defined by maps $\mu_1, \ldots, \mu_{i-1}$) lifts to an $i$-th-level graded deformation if there exists a map $\mu_i$ for which $\mu_1, \ldots, \mu_{i-1}, \mu_i$ define an $i$-th-level graded deformation.

The next proposition makes clear the choice of terminology in the above definition. Ultimately, one is interested in a deformation of $A$ over $k[t]$ and its specializations at particular values of $t$. The $i$-th-level graded deformations are steps in this direction.

**Proposition 1.13 [Braverman and Gaitsgory 1996, Proposition 1.5].** All obstructions to lifting an $(i - 1)$-th-level graded deformation to the next level lie in $\text{HH}^{3-i}(A)$. An $(i - 1)$-th-level deformation lifts to the $i$-th-level if and only if its $(i - 1)$-th obstruction cocycle is zero in cohomology, i.e., there is a map $\mu_i$ such that (1-11) holds for all $a_1, a_2, a_3$ in $A$. 
The connection between graded deformations and PBW deformations is well known; the following statement is a consequence of the canonical embedding of \( A \) as a \( k \)-linear direct summand of \( A[t] \), with splitting map given by specialization at \( t = 0 \).

**Proposition 1.14** [Braverman and Gaitsgory 1996, Remark 1.4]. Given a graded algebra \( A \) and a graded deformation \( A_t \) of \( A \), then \( A_t \) specialized at \( t = 1 \) is a PBW deformation of \( A \).

Now we explain that the two notions of deformation of \( B \# H \) coincide; recall Notations 0.2 and 0.3. The following result is well known in related contexts, but we include some details for the reader’s convenience.

**Proposition 1.15.** The following statements are equivalent.

1. The algebra \( D_{B,\kappa} := \left( T_k(V) \# H \right)/(r - \kappa(r))_{r \in I} \) is a PBW deformation of \( B \# H \).
2. The algebra \( D_{B,\kappa,i} := \left( T_k(V) \# H \right)[t]/(r - \kappa^0(r)t - \kappa^C(r)t^2)_{r \in I} \) is a graded deformation of \( B \# H \) over \( k[t] \).

**Proof.** Assume that \( D_{B,\kappa} \) is a PBW deformation of \( B \# H \). By its definition, \( D_{B,\kappa,i} \) is an associative algebra, and so we need only see that it is isomorphic to \( B \# H[t] \) as a vector space. To this end, use the PBW property to define a \( k \)-linear map \( \pi : B \# H \to T_k(V) \# H \) whose composition with the quotient map onto \( D_{B,\kappa} \) is an isomorphism of filtered vector spaces. Extend \( \pi \) to a \( k[t] \)-linear map from \( B \# H[t] \) to \( T_k(V) \# H[t] \). Its composition with the quotient map to \( D_{B,\kappa,i} \) is an isomorphism of \( k \)-vector spaces; one sees this by a degree argument.

Conversely, assume that \( D_{B,\kappa,i} \) is a graded deformation of \( B \# H \) over \( k[t] \). We may specialize to \( t = 1 \) to obtain \( D_{B,\kappa} \). Now apply Proposition 1.14 to conclude that \( D_{B,\kappa} \) is a PBW deformation of \( B \# H \). \( \square \)

### 2. Resolutions for smash product algebras

In this section, let \( A \) denote the smash product \( B \# H \), which is an \( \mathbb{N} \)-graded algebra: 
\[ A = \bigoplus_{j \geq 0} (B_j \otimes H) \]. Thus \( A_0 \cong H \). The aim is to construct a free \( A^e \)-resolution \( X_* \) of the \( A^e \)-module \( A \) from resolutions of \( H \) and of \( B \) (denoted by \( C_* \) and \( D_* \), respectively). This construction simultaneously generalizes results of Guccione and Guccione [2002] and of Shepler and Witherspoon [2012b, Section 4]. A similar resolution was constructed independently by Negron [2014].

**Definition 2.1** \((C_*, C_i, C'_i)\). For \( i \geq 0 \), let \( C_i \) denote the \( H^e \)-module \( H^\otimes(i+2) \). The left \( H \)-comodule structure \( \rho : C_i \to H \otimes C_i \) is given by 
\[ \rho(h^0 \otimes h^1 \otimes \cdots \otimes h^{i+1}) := \sum h^0_1 \cdots h^{i+1}_1 \otimes h^0_2 \cdots \otimes h^{i+1}_2 \in H \otimes C_i \]
for all \( h^0, \ldots, h^{i+1} \in H \). For \( h \in H \), the left and right \( h \)-actions on an element \( x \in C_i \) are given respectively by left and right multiplication by \( h \) in the leftmost and rightmost factors of \( x \). Now, let
be the bar resolution $B_\ast(H)$ of $H$ (Definition 1.7), which is an $H^e$-free resolution of $H$.

There is an isomorphism of free $H^e$-modules $C_i \cong H \otimes C_i' \otimes H$, where $C_i' = H^\otimes i$ if $i \geq 1$ and $C_0' = k$. We give each $C_i'$ the $H$-comodule structure inducing that on $C_i$ under the usual tensor product of comodules.

**Remark 2.2.** The resolution $C_\ast$ satisfies the following conditions:

(i) The right $H$-action and left $H$-coaction on $C_i$ commute in the sense that for all $x \in C_i$ and $h \in H$

$$
\sum (x \cdot h)_{-1} \otimes (x \cdot h)_0 = \sum x_{-1}h_1 \otimes (x_0 \cdot h_2).
$$

That is, each $C_i$ is a Hopf module (for which the action is a left action and the coaction is a right coaction).

(ii) The differentials are left $H$-comodule homomorphisms.

**Definition 2.3** ($D_\ast$, $D_i$, $D'_i$). Recall that $B$ is a Koszul algebra. Let

$$
\cdots \longrightarrow D_1 \longrightarrow D_0 \longrightarrow B \longrightarrow 0
$$

be the *Koszul resolution* of $B$ as a $B^e$-module: $D_0 = B \otimes B$, $D_1 = B \otimes V \otimes B$, $D_2 = B \otimes I \otimes B$, and for each $n \geq 3$, $D_i = B \otimes D'_i \otimes B$, where

$$
D'_i = \bigcap_{j=0}^{i-2} (V^\otimes j \otimes I \otimes V^\otimes (i-2-j)).
$$

Each $D_i$ is a subspace of $B^\otimes (i+2)$, and the differential on the Koszul resolution is the one induced by the canonical embedding of the Koszul resolution into the bar resolution of $B$.

**Remark 2.4.** The resolution $D_\ast$ satisfies the following conditions:

(i) Each $B^e$-module $D_i$ is a left $H$-module and the differentials are $H$-module homomorphisms.

(ii) The left actions of $B$ and $H$ on $D_i$ are compatible in the sense that they induce a left action of $A = B \# H$ on $D_i$.

(iii) In addition, the right $B$-action on $D_i$ is compatible with the left $H$-action on $D_i$ in the sense that for all $h \in H$, $y = b^0 \otimes y' \otimes b^1 \in D_i$ for $y' \in D'_i$ and $b^0, b^1, b \in B$,

$$
h \cdot (y \cdot b) = \sum (h_1 \cdot y) \cdot (h_2 \cdot b) = \left[\sum (h_1 \cdot b^0) \otimes (h_2 \cdot y') \otimes (h_3 \cdot b^1)\right] \cdot (h_4 \cdot b) = \sum (h_1 \cdot b^0) \otimes (h_2 \cdot y') \otimes (h_3 \cdot b^1)b = (h \cdot y) \cdot b.
$$
We will next extend the actions on the modules in each of these two sequences so that it be \(H\)-coinvariant; that is, the comodule structure is given by maps \(\rho_i : D_i \rightarrow H \otimes D_i\), where \(\rho_i(y) = 1 \otimes y\) for all \(y \in D_i\). The maps \(\rho_i\) are maps of left \(H\)-modules, if we give \(H \otimes D_i\) the tensor product \(H\)-module structure, where the factor \(H\) has the adjoint \(H\)-module structure. See Section 1A.

**Construction 2.5** \((X_\ast)\). We wish to combine the two resolutions, \(C_\ast\) and \(D_\ast\), from Definitions 2.1 and 2.3, to form a resolution \(X_\ast\) of \(\Lambda = B \# H\) by \(A\)-bimodules, via a tensor product. To that end, we first apply \((A \otimes_H -)\) to \(C_\ast\). Note that \(A\) is free as a right \(H\)-module (under multiplication) and that \(A \otimes_H H \cong A\). The following sequence of \(A \otimes H^{\text{op}}\)-modules is therefore exact:

\[
\cdots \rightarrow A \otimes_H C_1 \rightarrow A \otimes_H C_0 \rightarrow A \rightarrow 0.
\]

Similarly, we apply \((- \otimes_B A\) to \(D_\ast\). Note that \(A\) is free as a left \(B\)-module, and that \(B \otimes_B A \cong A\). The following sequence of \(B \otimes A^{\text{op}}\)-modules is therefore exact:

\[
\cdots \rightarrow D_1 \otimes_B A \rightarrow D_0 \otimes_B A \rightarrow A \rightarrow 0.
\]

We will next extend the actions on the modules in each of these two sequences so that they become \(A^e\)-modules. Then, we will take their tensor product over \(A\).

We extend the right \(H\)-module structure on \(A \otimes_H C_\ast\) to a right \(A\)-module structure by defining a right action of \(B\) on \(A \otimes_H C_\ast\): for all \(a \in A\), \(x \in C_i\), \(b \in B\), we set

\[
(a \otimes_H x) \cdot b := \sum a(x_{-1} \cdot b) \otimes_H x_0.
\]

This does indeed make \(A \otimes_H C_i\) into a right \(B\)-module, and, by combining with the right action of \(H\), gives a right action of \(A\) on \(A \otimes_H C_i\). Note that for \(x = x^0 \otimes \cdots \otimes x^{i+1} \in C_i\) (with \(x^0, \ldots, x^{i+1} \in H\),

\[
\rho(hx) = \sum (hx)_{-1} \otimes (hx)_0 = \sum h_1 x^0_1 \cdot \cdots \cdot x^{i+1}_1 \otimes h_2 x^0_2 \otimes \cdots \otimes x^{i+1}_2
\]

\[
= \sum h_1 x_1 \otimes h_2 x_0.
\]

The action is well-defined: If \(h \in H\), then

\[
(ah \otimes_H x) \cdot b \overset{(2-6)}{=} \sum ah(x_{-1} \cdot b) \otimes_H x_0 = \sum a(h_1 x_{-1} \cdot b) \otimes_H h_2 x_0 \overset{(2-6)}{=} (a \otimes_H hx) \cdot b.
\]

Since the differentials on \(C_\ast\) are \(H\)-comodule homomorphisms (Remark 2.2(ii)), this action commutes with the differentials.

We extend the left \(B\)-module structure on \(D_i \otimes_B A\) to a left \(A\)-module structure by defining a left action of \(H\) by

\[
h \cdot (y \otimes_B a) := \sum (h_1 \cdot y) \otimes_B h_2 a
\]
for all \( h \in H, y \in D_i, a \in A \). It is well-defined, since for all \( h \in H, b \in B \), we have by the definitions in Section 1A that

\[
\begin{align*}
    h \cdot (yb \otimes_B a) \equiv (2-7) & \sum (h_1 \cdot (yb)) \otimes_B (h_2 aS(h_3)) = \sum (h_1 \cdot y)(h_2 \cdot b) \otimes_B h_3 aS(h_4) \\
    & = \sum (h_1 \cdot y) \otimes_B (h_2 \cdot b)h_3 aS(h_4) = \sum (h_1 \cdot y) \otimes_B (h_2 \cdot (ba)) \\
    & \equiv (2-7) h \cdot (y \otimes_B ba).
\end{align*}
\]

The left \( H \)-action on \( D_i \) is compatible with the right \( B \)-action on \( D_i \) by Remark 2.4(iii). Again, this action commutes with the differentials, since the differentials on \( D_* \) are \( H \)-module homomorphisms (Remark 2.4(i)).

We may now consider \( A \otimes_H C_* \) and \( D_* \otimes_B A \) to be complexes of \( A^e \)-modules via the \( A \)-bimodule structure defined above. We take their tensor product over \( A \), letting \( X_* := (A \otimes_H C_*) \otimes_A (D_* \otimes_B A) \); that is, for all \( i, j \geq 0 \),

\[
\begin{align*}
    (2-8) \quad X_{i,j} :&= (A \otimes_H C_i) \otimes_A (D_j \otimes_B A),
\end{align*}
\]

with horizontal and vertical differentials

\[
\begin{align*}
    d^h_{i,j} : X_{i,j} & \rightarrow X_{i-1,j} \quad \text{and} \quad d^v_{i,j} : X_{i,j} \rightarrow X_{i,j-1}
\end{align*}
\]

given by \( d^h_{i,j} := d^C_i \otimes \text{id} \) and \( d^v_{i,j} := (-1)^i \text{id} \otimes d^D_j \).

Finally, let \( X_* \) be the total complex of \( X_* :\)

\[
\begin{align*}
    \cdots & \rightarrow X_2 \rightarrow X_1 \rightarrow X_0 \rightarrow A \rightarrow 0,
\end{align*}
\]

with \( X_n = \bigoplus_{i+j=n} X_{i,j} \).

**Theorem 2.10.** We have the following statements:

(a) For each \( i, j \), the \( A^e \)-module \( X_{i,j} \) is isomorphic to \( A \otimes C'_i \otimes D'_j \otimes A \).

(b) The complex \( X_* \) given in (2-9) is a free resolution of the \( A^e \)-module \( A \).

**Proof.** (a) Write \( C_i \cong H \otimes C'_i \otimes H \) and \( D_j \cong B \otimes D'_j \otimes B \) for vector spaces \( C'_i \) and \( D'_j \), as in Definitions 2.1 and 2.3. Then

\[
\begin{align*}
    X_{i,j} & \cong (A \otimes_H H \otimes C'_i \otimes H) \otimes_A (B \otimes D'_j \otimes B \otimes_B A) \\
    & \cong (A \otimes C'_i \otimes H) \otimes_A (B \otimes D'_j \otimes A).
\end{align*}
\]

We will show that this is isomorphic to \( A \otimes C'_i \otimes D'_j \otimes A \) as an \( A^e \)-module. First, define a map as follows:

\[
\begin{align*}
    (2-11) \quad (A \otimes C'_i \otimes H) \times (B \otimes D'_j \otimes A) & \rightarrow A \otimes C'_i \otimes D'_j \otimes A, \\
    (a \otimes x \otimes h, b \otimes y \otimes a') & \mapsto \sum a(x_1h \cdot b) \otimes x_0 \otimes (h_2 \cdot y) \otimes h_3 a'.
\end{align*}
\]

for all \( h \in H, y \in D_i, a \in A \). It is well-defined, since for all \( h \in H, b \in B \), we have by the definitions in Section 1A that
for all \( a, a' \in A \), \( x \in C'_i \), \( y \in D'_j \), \( h \in H \), \( b \in B \). This map is \( k \)-bilinear by definition, and we will check that it is \( A \)-balanced. First, let \( b' \in B \). We rewrite \((a \otimes x \otimes h) \cdot b'\) as follows. First, using \( A \otimes C'_i \otimes H \cong A \otimes H C_i \), identify this element with \( a \otimes_H (1 \otimes x \otimes h) \in A \otimes_H C_i \). By (2-6),

\[
(a \otimes_H (1 \otimes x \otimes h)) \cdot b' = \sum a((1 \otimes x \otimes h)_{-1} \cdot b') \otimes_H (1 \otimes x \otimes h)_0.
\]

By Definition 2.1, and by identifying \( x \in C'_i \) with \( x^1 \otimes x^2 \otimes \cdots \otimes x^i \), we have that

\[
\rho(1 \otimes x \otimes h) = \sum (1 \otimes x \otimes h)_{-1} \otimes (1 \otimes x \otimes h)_0 = \sum (x^1 h_1 x^2 \cdots x^i h_1) \otimes (1 \otimes x^1_2 x^2 \cdots x^i_2 \otimes h_2).
\]

So, \((1 \otimes x \otimes h)_{-1} = x_{-1} h_1 \) and \((1 \otimes x \otimes h)_0 = x_0 \otimes h_2 \). Now \( C_i \cong H \otimes C'_i \otimes H \) as an \( H \)-comodule, so

\[
((a \otimes x \otimes h) \cdot b', b \otimes y \otimes a') = \sum (a(x_{-1} h_1 \cdot b') \otimes x_0 \otimes h_2, b \otimes y \otimes a')
\]

\[
\mapsto \sum a(x_{-2} h_1 \cdot b')(x_{-1} h_2 \cdot b) \otimes x_0 \otimes (h_3 \cdot y) \otimes h_4 a'.
\]

On the other hand,

\[
((a \otimes x \otimes h, b' \cdot (b \otimes y \otimes a')) = (a \otimes x \otimes h, b'b \otimes y \otimes a')
\]

\[
\mapsto \sum a(x_{-1} h_1 \cdot (b'b)) \otimes x_0 \otimes (h_2 \cdot y) \otimes h_3 a',
\]

which is the same as the previous image since \( B \) is an \( H \)-module algebra. Now let \( \ell \in H \). Then

\[
((a \otimes x \otimes h) \cdot \ell, b \otimes y \otimes a') = (a \otimes x \otimes h \ell, b \otimes y \otimes a')
\]

\[
\mapsto \sum a(x_{-1} h_1 \ell_1 \cdot b) \otimes x_0 \otimes (h_2 \ell_2 \cdot y) \otimes h_3 \ell_3 a'.
\]

On the other hand,

\[
(a \otimes x \otimes h, \ell \cdot (b \otimes y \otimes a')) = \sum (a \otimes x \otimes h, (\ell_1 \cdot b) \otimes (\ell_2 \cdot y) \otimes \ell_3 a')
\]

\[
\mapsto \sum a(x_{-1} h_1 \ell_1 \cdot b) \otimes x_0 \otimes (h_2 \ell_2 \cdot y) \otimes h_3 \ell_3 a',
\]

which is the same as the previous image. Therefore, there is an induced map

\[
(A \otimes C'_i \otimes H) \otimes_A (B \otimes D'_j \otimes A) \rightarrow A \otimes C'_i \otimes D'_j \otimes A.
\]

Now, we verify that the map below is an inverse map of (2-11):

(2-12) \( a \otimes x \otimes y \otimes a' \mapsto (a \otimes x \otimes 1) \otimes_A (1 \otimes y \otimes a') \).

It is clear that first applying (2-12) and then (2-11) yields the identity map on \( A \otimes C'_i \otimes D'_j \otimes A \). On the other hand, the image of first applying (2-11) then (2-12) to \((a \otimes x \otimes h, b \otimes y \otimes a')\) is
\[ \sum (a(x_{-1}h_1 \cdot b) \otimes x_0 \otimes 1) \otimes_A (1 \otimes (h_2 \cdot y) \otimes h_3a') \]

\[ = \sum (a(x_{-1}h_1 \cdot b) \otimes x_0 \otimes 1) \otimes_A (\epsilon(h_2) \otimes (h_3 \cdot y) \otimes h_4a') \]

\[ = \sum (a(x_{-1}h_1 \cdot b) \otimes x_0 \otimes 1) \otimes_A (h_2 \cdot (1 \otimes y \otimes a')) \]

\[ = \sum (a(x_{-1}h_1 \cdot b) \otimes x_0 \otimes h_2) \otimes_A (1 \otimes y \otimes a') \]

\[ \overset{(2-6)}{=} (a \otimes x \otimes h) \otimes_A (1 \otimes y \otimes a') \]

\[ = (a \otimes x \otimes h) \otimes_A (b \otimes y \otimes a'). \]

Therefore the two \( A^e \)-modules \( X_{ij} \) and \( A \otimes C'_i \otimes D'_i \otimes A \) are isomorphic, as claimed.

(b) We wish to apply the Künneth theorem to show that the complex \( X_* \) is a free resolution of the \( A^e \)-module \( A \). To that end, we check that each term in the complex \( D_* \otimes_B A \) is a free left \( A \)-module and that the image of each differential in the complex is also projective as a left \( A \)-module. First, write each \( D_i \otimes_B A \cong (B \otimes D'_i \otimes B) \otimes_B A \cong B \otimes D'_i \otimes A \). Define a \( k \)-linear map \( f : A \otimes D'_i \otimes B \rightarrow B \otimes D'_i \otimes A \) by

\[ f(rh \otimes y \otimes b) = \sum r \otimes (h_1 \cdot y) \otimes h_2b \]

for \( h \in H, y \in D'_i \), and \( r, b \in B \). Give \( A \otimes D'_i \otimes B \) the structure of a left \( A \)-module by requiring \( A \) to act by left multiplication on the leftmost factor. Clearly this is a free left \( A \)-module. The map \( f \) is an \( A \)-module homomorphism by the definition of the left \( A \)-action on \( B \otimes D'_i \otimes A \); see (2-7). We claim that the following map is an inverse map, so that \( f \) is an isomorphism of \( A \)-modules: let \( S^{-1} \) denote the (composition) inverse of the antipode \( S \) of \( H \). Let \( g : B \otimes D'_i \otimes A \rightarrow A \otimes D'_i \otimes B \) be the \( k \)-linear map defined by

\[ g(r \otimes y \otimes hb) = \sum rh_2 \otimes (S^{-1}(h_1) \cdot y) \otimes b. \]

Since for each \( h \in H \) we have \( \sum h_2S^{-1}(h_1) = \epsilon(h) = \sum S^{-1}(h_2)h_1 \) (see, e.g., [Radford 2012, Proposition 7.1.10]), the function \( g \) is indeed the inverse of \( f \). Thus, each term in the complex \( D_* \otimes_B A \) is a free left \( A \)-module.

That the image of each differential is projective as a left \( A \)-module may be proved inductively, starting on one end of the complex

\[ \cdots \rightarrow D_1 \otimes_B A \xrightarrow{d_1 \otimes \text{id}} D_0 \otimes_B A \xrightarrow{d_0 \otimes \text{id}} A \rightarrow 0, \]

as follows. Since \( A \) is a projective left \( A \)-module and \( d_0 \otimes \text{id} \) is surjective, the map splits, implying that \( \text{Ker}(d_0 \otimes \text{id}) = \text{Im}(d_1 \otimes \text{id}) \) is a direct summand of the free left \( A \)-module \( D_0 \otimes_B A \). Therefore it is projective. Again, since \( \text{Im}(d_1 \otimes \text{id}) \) is projective, the map \( d_1 \otimes \text{id} \) from \( D_1 \otimes_B A \) to its image splits so that \( \text{Ker}(d_1 \otimes \text{id}) = \text{Im}(d_2 \otimes \text{id}) \).
is a direct summand of the free left $A$-module $D_1 \otimes_B A$. Continuing in this way, we see that $\text{Im}(d_i \otimes \text{id})$ is a free left $A$-module for each $i$.

The Künneth theorem [Weibel 1994, Theorem 3.6.3] then gives for each $n$ an exact sequence

$$0 \longrightarrow \bigoplus_{i+j=n} H_i(A \otimes_H C_\ast) \otimes_A H_j(D_\ast \otimes_B A) \longrightarrow H_n((A \otimes_H C_\ast) \otimes_A (D_\ast \otimes_B A)) \longrightarrow \bigoplus_{i+j=n-1} \text{Tor}^A_i(H_i(A \otimes_H C_\ast), H_j(D_\ast \otimes_B A)) \rightarrow 0.$$ 

Now $A \otimes_H C_\ast$ and $D_\ast \otimes_B A$ are exact except in degree 0, where their homologies are each $A$; that is, $H_0(A \otimes_H C_\ast) = A$ and $H_0(D_\ast \otimes_B A) = A$. Therefore the only potentially nonzero Tor term is when $i = 0 = j$, or $\text{Tor}^A_1(A, A)$, yet this equals 0 since $A$ is flat over $A$. So for each $n$, we have

$$H_n((A \otimes_H C_\ast) \otimes_A (D_\ast \otimes_B A)) \cong \bigoplus_{i+j=n} H_j(A \otimes_H C_\ast) \otimes_A H_i(D_\ast \otimes_B A).$$

Again the right side is only nonzero when $i = 0 = j$, so we have

$$H_0((A \otimes_H C_\ast) \otimes_A (D_\ast \otimes_B A)) \cong H_0(A \otimes_H C_\ast) \otimes_A H_0(D_\ast \otimes_B A) \cong A \otimes_A A \cong A$$

and $H_n((A \otimes_H C_\ast) \otimes_A (D_\ast \otimes_B A)) = 0$ for all $n > 0$. Thus we have proven that $X_\ast$ is an $A^e$-free resolution of $A$. \hfill \Box

We next relate the resolution $X_\ast$ of $A$ (from Construction 2.5) to the bar resolution $B_\ast(A)$ of $A$:

**Lemma 2.13.** There exist degree-preserving chain maps between $X_\ast$ and the bar resolution $B_\ast(A)$ of $A$,

$$\phi_n : X_\ast \longrightarrow B_\ast(A) \quad \text{and} \quad \psi_n : B_\ast(A) \longrightarrow X_\ast,$$

such that $\psi_n \phi_n$ is the identity map on the $A^e$-submodule $X_{0,n}$ of $X_n$ for each $n \geq 0$.

**Proof.** Recall by Notation 0.2 that $B$ is generated by the vector space $V$, with quadratic relations $I \subseteq V \otimes V$. First we prove by induction on $n$ that there are degree-preserving maps $\phi_n : X_n \rightarrow A^\otimes(n+2)$ and $\psi_n : A^\otimes(n+2) \rightarrow X_n$ commuting with the differentials. For clarity, we denote the differential on the bar resolution of $A$ by $\delta$. We have the diagram

$$X_\ast : \cdots \longrightarrow X_3 \xrightarrow{d_3} X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} A \longrightarrow 0$$

$$\begin{array}{c}
\psi_3 \phi_3 \\
\downarrow \\
\delta_3 \\
\downarrow \\
\psi_2 \phi_2 \\
\downarrow \\
\delta_2 \\
\downarrow \\
\psi_1 \phi_1 \\
\downarrow \\
\delta_1 \\
\downarrow \\
\psi_0 \phi_0 \\
\downarrow \\
\delta_0 \\
\downarrow \\
X_\ast: \cdots \longrightarrow A^\otimes5 \xrightarrow{\delta_5} A^\otimes4 \xrightarrow{\delta_4} A^\otimes3 \xrightarrow{\delta_3} A \otimes A \xrightarrow{\delta_2} A \longrightarrow 0
\end{array}$$

where $B_n(A) = A^\otimes(n+2)$ and $X_n = \bigoplus_{i+j=n} X_{i,j}$, with $X_{i,j}$ defined in (2-8); see also Theorem 2.10(a).
Define $\phi_0 = \text{id} \otimes \text{id} = \psi_0$, the identity map from $A \otimes A$ to itself. We wish to define $\phi_n$ so that when restricted to $X_0$, it corresponds to the standard embedding of the Koszul complex into the bar complex: for $n = 1$, this is the embedding of $A \otimes V \otimes A$ into $A \otimes A \otimes A$ via the containment of $V$ in $A$. We may define $\phi_1$ on $X_1 = X_{0,1} \oplus X_{1,0} \cong (A \otimes V \otimes A) \oplus (A \otimes H \otimes A)$ by $\phi_1(1 \otimes v \otimes 1) = 1 \otimes v \otimes 1$ and $\phi_1(1 \otimes h \otimes 1) = 1 \otimes h \otimes 1$ for all $v \in V$, $h \in H$. Note that for $n \geq 2$,

$$
(2-14) \quad X_{0,n} \cong A \otimes \left( \bigcap_{i=0}^{n-2} V^i \otimes I \otimes V^{n-i-2} \right) \otimes A,
$$

which is a free $A^e$-submodule of $A^{\otimes (n+2)}$. For each $i, j$ with $i + j = n$, choose a basis of the vector space $C_i^j \otimes D_i^j$ (whose elements are homogeneous of degree $j$, as $H$ is declared to have degree 0). By hypothesis, $\phi_{n-1}$ is degree-preserving, and $d_n$ is degree-preserving by construction. So, applying $\phi_{n-1}d_n$ to these basis elements of $C_i^j \otimes D_i^j$ produces elements of degree $j$ in the kernel of $\delta_{n-1}$, that is, the image of $\delta_n$. We define $\phi_n$ by choosing (arbitrary) corresponding elements in the inverse image of $\text{Im}(\delta_n)$. If we start with an element in $X_{0,n}$, we may choose its image in $A^{\otimes (n+2)}$ under the canonical embedding of $X_{0,n}$ into $A^{\otimes (n+2)}$ (see (2-14)). Given $X_{i,j}$ and $X_{i',j'}$ with $i + j = i' + j' = n$ and $i \neq 0$, $i' \neq 0$, elements of $X_{i,j}$ have degree $j$ and elements of $X_{i',j'}$ have degree $j'$. So their images under $\phi_n$ may be chosen independently, and in particular, independently of those of $X_{0,n}$. Thus, we have the maps $\phi_n$, as desired.

Now we show that $\psi_n$ may be chosen so that $\psi_n\phi_n$ is the identity map on $X_{0,n}$. In degree 1, we have summands $X_{0,1} \cong A \otimes V \otimes A$ and $X_{1,0} \cong A \otimes H \otimes A$. Note that $V \oplus H$ is a direct summand of $A$ as a vector space. We may therefore define $\psi_1(1 \otimes v \otimes 1) = 1 \otimes v \otimes 1$ in $X_{0,1}$ for all $v \in V$, and $\psi_1(1 \otimes h \otimes 1) = 1 \otimes h \otimes 1$ in $X_{1,0}$ for all $h \in H$. We also have that $\psi_1$ is the identity map on elements of the form $1 \otimes z \otimes 1$, for $z$ ranging over a basis of a chosen complement of $V \oplus H$ as a vector subspace of $A$. This complement may be chosen arbitrarily, subject to the condition that $d_1\psi_1(1 \otimes z \otimes 1) = \psi_0\delta_1(1 \otimes z \otimes 1)$. Since $\psi_0$, $d_1$, $\delta_1$ all have degree 0 as maps, one may also choose $\psi_1$ to have degree 0. In particular, note that $\psi_1\phi_1$ is the identity map on $X_{0,1}$. Now let $n \geq 2$ and assume that $\psi_{n-2}$ and $\psi_{n-1}$ have been defined to be degree-0 maps for which $d_{n-1}\psi_{n-1} = \psi_{n-2}\delta_{n-1}$ and $\psi_{n-1}\phi_{n-1}$ is the identity map on $X_{0,n-1}$. To define $\psi_n$, first note that $A^{\otimes (n+2)}$ contains the space $X_{0,n}$ as an $A^e$-submodule (see (2-14)) and the image of each $X_{i,j}$ under $\phi_n$ ($n = i + j$, $i \geq 1$). By construction, their images intersect in 0, the image of $X_{0,n}$ under $\phi_n$ is free and, moreover, $\phi_n$ is injective on restriction to $X_{0,n}$. Choose a set of free generators of $\phi_n(X_{0,n})$, and choose a set of free generators of its complement in $A^{\otimes (n+2)}$. For each chosen generator $x$ of $X_{0,n}$, we define $\psi_n(\phi_n(x))$ to be $x$. On the complement of $\phi_n(X_{0,n})$, define $\psi_n$ arbitrarily, subject to being a chain map of degree 0. Thus,
ψ_nφ_n is the identity map on X_{0,n}. Now for all x ∈ X_{0,n}, since d_n(x) ∈ X_{0,n-1}, we have that ψ_{n-1}φ_{n-1}d_n(x) = d_n(x), by induction. As δ_nφ_n(x) = φ_{n-1}d_n(x), it follows that d_nψ_nφ_n(x) = ψ_{n-1}δ_nφ_n(x). So ψ_n also extends the chain map from degree n − 1 to degree n, as desired.

3. Poincaré–Birkhoff–Witt theorem for Hopf algebra actions

Consider the algebra $\mathcal{D}_{B,\kappa}$ from Notation 0.3. The goal of this section is to prove our main result, Theorem 3.1. We provide necessary and sufficient conditions for $\mathcal{D}_{B,\kappa}$ to be a PBW deformation of $B \# H$ (Definition 0.1) as follows:

**Theorem 3.1.** The algebra $\mathcal{D}_{B,\kappa}$ is a PBW deformation of $B \# H$ if and only if the following conditions hold:

1. $\kappa$ is $H$-invariant (Definition 1.4);
2. If $\dim_k V \geq 3$, then the following equations hold for the maps $\kappa^C \otimes \text{id} - \text{id} \otimes \kappa^C$ and $\kappa^L \otimes \text{id} - \text{id} \otimes \kappa^L$, which are defined on the intersection $(I \otimes V) \cap (V \otimes I)$:
   - $\text{Im}(\kappa^L \otimes \text{id} - \text{id} \otimes \kappa^L) \subseteq I$;
   - $(\kappa^L \circ (\kappa^L \otimes \text{id} - \text{id} \otimes \kappa^L)) = -(\kappa^C \otimes \text{id} - \text{id} \otimes \kappa^C)$;
   - $(\kappa^C \circ (\text{id} \otimes \kappa^L - \kappa^L \otimes \text{id})) \equiv 0$.

Recall Notation 0.2: $B$ is generated by the $k$-vector space $V$ with quadratic relations $I \subset V \otimes V$, so $B = T_k(V)/(I)$. Moreover, consider:

**Notation 3.2** $(U, T_H(U), R, P)$. Let $U := V \otimes H$, which is an $H$-bimodule under the actions defined in Section 1A. Set $R = I \otimes H$, similarly an $H$-bimodule, and an $H$-subbimodule of $U \otimes H U$. Let $P = \{r \otimes 1 - \kappa(r) | r \in I\}$ be the relation space of $\mathcal{D}_{B,\kappa}$, generating an $H$-submodule of $H \oplus U \oplus (U \otimes H U)$ in the tensor algebra

$$T_H(U) = H \oplus U \oplus (U \otimes H U) \oplus (U \otimes H U \otimes H U) \oplus \cdots.$$ 

Note that $U^{\otimes n}_H \cong V^{\otimes n} \otimes H$ as $k$-vector spaces. We see that $\pi(P) = R$, where the map $\pi$ is the projection onto the homogeneous quadratic part of $P$.

Consider the following preliminary results:

**Lemma 3.3.** Since $T_H(U)$ is canonically isomorphic to $T_k(V) \# H$, we have that

$$T_H(U)/(P) \cong \mathcal{D}_{B,\kappa}$$ and $$T_H(U)/(R) \cong (T_k(V) \# H)/(I) \cong B \# H,$$

where $(I)$ is identified with the ideal of $T_k(V) \# H$ generated by $I$.

Hence, $\mathcal{D}_{B,\kappa}$ is a PBW deformation of $B \# H$ if and only if $T_H(U)/(P)$ is a PBW deformation of $T_H(U)/(R)$. 


Lemma 3.4 [Shepler and Witherspoon 2012b, Lemma 5.2]. If $T_H(U)/(P)$ is a PBW deformation of $T_H(U)/(R)$, then the following conditions hold for maps $\alpha : R \to U$ and $\beta : R \to H$ for which $P = \{ x - \alpha(x) - \beta(x) \mid x \in R \}$:

(i) $\text{Im}(\alpha \otimes_H \text{id} - \text{id} \otimes_H \alpha) \subseteq R$.
(ii) $\alpha \circ (\alpha \otimes_H \text{id} - \text{id} \otimes_H \alpha) = -(\beta \otimes_H \text{id} - \text{id} \otimes_H \beta)$.
(iii) $\beta \circ (\text{id} \otimes_H \alpha - \alpha \otimes_H \text{id}) \equiv 0$.

Here, the maps $\alpha \otimes_H \text{id} - \text{id} \otimes_H \alpha$ and $\beta \otimes_H \text{id} - \text{id} \otimes_H \beta$ are defined on the subspace $(R \otimes_H U) \cap (U \otimes_H R)$ of $T_H(U)$.

Remark 3.5. Given the maps $\kappa^L : I \to V \otimes H$ and $\kappa^C : I \to H$ as in Notation 0.2, we see that $\alpha = \kappa^L \otimes \text{id}_H$ and $\beta = \kappa^C \otimes \text{id}_H$.

Lemma 3.6. Consider the algebra

$$(T_H(U)/(P))_t := \frac{T_H(U)[t]}{(x - \alpha(x)t - \beta(x)t^2)_{x \in R}}.$$

We have that $(T_H(U)/(P))_t$ is a PBW deformation of $T_H(U)/(R)$ over $k[t]$ if and only if $\mathcal{D}_{B,k,t}$ (of Proposition 1.15) is a PBW deformation of $B \# H$ over $k[t]$.

Proof. This follows from Lemma 3.3 and Remark 3.5.

Now we provide the proof of Theorem 3.1. A somewhat shorter proof would suffice in case $H$ is semisimple: The first proof of [Shepler and Witherspoon 2012a, Theorem 3.1] may be generalized from semisimple group algebras to semisimple Hopf algebras. In that context, one has on hand a much smaller resolution than that which we will use below.

Proof of Theorem 3.1. Note that we will employ the identifications given in the lemmas and remark above, sometimes without comment. Namely, results from Section 2 will be used here where, for instance, $I$ is identified so that $R = I \otimes H$ and $B \# H$ is identified with $T_H(U)/(R)$.

Necessity of conditions (a)–(d). Let us first show that conditions (a)–(d) are necessary. Assume that $\mathcal{D}_{B,k}$ is a PBW deformation of $B \# H$, and take $Q$ to be the relation space of $\mathcal{D}_{B,k}$. Then, for all $h \in H$ and $r \in I$, we have that $h \cdot r - h \cdot (\kappa(r)) \in Q$. (Refer to Section 1A for the definition of these actions.) We also have that $h \cdot h - \kappa(h \cdot r) \in Q$, so $h \cdot (\kappa(r)) - \kappa(h \cdot r) \in Q$. This implies that $h \cdot (\kappa(r)) = \kappa(h \cdot r)$ in $\mathcal{D}_{B,k}$, since $Q$ cannot contain nonzero elements in degree less than two. Thus, condition (a) holds. Moreover, by Lemma 3.3, $T_H(U)/(P)$ satisfies the PBW property.

Now by applying Lemma 3.4, we see that conditions (i), (ii), (iii) hold for $T_H(U)/(P)$. These conditions are equivalent to conditions (b), (c), (d) in Theorem 3.1 for the algebra $\mathcal{D}_{B,k}$ by Notation 3.2 and Remark 3.5. Thus, if $\mathcal{D}_{B,k}$ is a PBW deformation of $B \# H$, then conditions (a)–(d) of Theorem 3.1 must hold.
Sufficiency of conditions (a)–(d). Conversely, let us assume that conditions (a)–(d) of Theorem 3.1 hold for the algebra \( D_{B,k} \). Equivalently, by Notation 3.2, Lemma 3.3, and Remark 3.5, we assume the following for the algebra \( T_H(U)/(P) \):

- The maps \( \alpha \) and \( \beta \) are \( H \)-invariant.
- Conditions (i), (ii), (iii) of Lemma 3.4 hold.

The goal is to show that \( D_{B,k} \) is a PBW deformation of \( B \# H \), which, by Proposition 1.15, is equivalent to showing that \( D_{B,k,t} \) (of Proposition 1.15) is a graded deformation of \( B \# H \) over \( k[t] \). Hence, by Lemma 3.6, the goal is then equivalent to verifying that the algebra \( (T_H(U)/(P))_t \) is a graded deformation of \( T_H(U)/(R) \) over \( k[t] \). We thus have the following strategy:

- Let \( A \) denote \( T_H(U)/(R) \).
- Construct multiplication maps, \( \mu_i : A \otimes A \to A \), as in Definition 1.5, subject to the restraints listed in Remark 1.8.
- Form the graded deformation \( A_t \) of \( A \) as in Definition 1.5.
- Conclude that \( A' := T_H(U)/(P) \cong (A_t)|_{t=1} \) is a PBW deformation of \( A \) by Proposition 1.15.

We generalize the proof in [Shepler and Witherspoon 2012b] from group actions to Hopf algebra actions. Namely, we use the free resolution \( X_* \) of the \( A^e \)-module \( A \) in Construction 2.5 to define the maps \( \mu_i \). Recall that \( X_* \) is constructed from \( C_* = B_*(A) \), the bar resolution of \( H \), and from \( D_* \), the Koszul resolution of \( B \).

Extending \( \alpha \) and \( \beta \) to be maps on \( X_* \). We first extend \( \alpha \) and \( \beta \) to be maps on \( X_* \) as follows. In degree 2, \( X_2 \) contains as a direct summand \( X_{0,2} \cong A \otimes I \otimes A \); see (2-14). As \( \alpha, \beta \) are \( H \)-bilinear by \( H \)-invariance, we may extend them to \( A^e \)-module maps from \( A \otimes R \otimes_H A \cong A \otimes I \otimes A \) to \( A \) by composing with the multiplication map. By abuse of notation, denote these extended maps by \( \alpha, \beta \) as well. Extend \( \alpha \) and \( \beta \) yet further by setting them equal to 0 on the summands \( X_{2,0} \) and \( X_{1,1} \) of \( X_2 \) so that they become maps \( \alpha, \beta : X_2 \to A \). More precisely, \( \alpha, \beta \in \text{Hom}_{A^e}(X_2, A) \cong \text{Hom}_k(X'_2, A) \) for \( X_2 \cong A \otimes X'_2 \otimes A \).

Construction of the multiplication map \( \mu_1 \). To build \( \mu_1 \in \text{Hom}_k(A \otimes A, A) \cong \text{Hom}_{A^e}(A^{\otimes 3}, A) \), recall that it must be a Hochschild 2-cocycle as in (1-9). We will show that \( \alpha : X_2 \to A \) is a Hochschild 2-cocycle on \( X_* \), that is, \( d_3^*(\alpha) = 0 \). Recall the chain maps of Lemma 2.13. We set \( \mu_1 = \psi_2^*(\alpha) \), which will be a Hochschild 2-cocycle on \( B_*(A) \), that is, \( \delta_3^*(\mu_1) = 0 \).

To show that \( d_3^*(\alpha) : X_3 \to A \) is the zero map, first note that \( X_3 = X_{0,3} \oplus X_{1,2} \oplus X_{2,1} \oplus X_{3,0} \) from (2-9) and that the images of \( X_{2,1} \) and \( X_{3,0} \) under \( d_3 \) lie in \( X_{1,1} \oplus X_{2,0} \). Since \( \alpha|_{X_{1,1} \oplus X_{2,0}} \equiv 0 \) by the extension above, it suffices to show that \( d_3^*(\alpha)|_{X_{0,3}} \) and \( d_3^*(\alpha)|_{X_{1,2}} \) are zero maps.
Rewriting condition (i) of Lemma 3.4, we see that it implies that $\alpha$ is 0 on the image of the differential on $X_{0,3}$ as follows: let $\sum_i 1 \otimes x_i \otimes y_i \otimes z_i \otimes 1 \in A \otimes ((I \otimes V) \cap (V \otimes I)) \otimes A = X_{0,3}$; see (2-14). Then

$$\alpha\left(d_3\left(\sum_i 1 \otimes x_i \otimes y_i \otimes z_i \otimes 1\right)\right) = \alpha\left(\sum_i x_i \otimes y_i \otimes z_i \otimes 1 - \sum_i 1 \otimes x_i y_i \otimes z_i \otimes 1 + \sum_i 1 \otimes x_i \otimes y_i z_i \otimes 1 - \sum_i 1 \otimes x_i \otimes y_i \otimes z_i\right) = \sum_i (x_i \alpha(y_i \otimes z_i) - \alpha(x_i \otimes y_i)z_i).$$

(To see this, note that applying the multiplication map of $A$ to elements in $I$ yields 0.) Thus $d_3^+(\alpha) = \text{id} \otimes \alpha - \alpha \otimes \text{id}$ on $X_{0,3}$; here, we identify $\text{id} \otimes \alpha - \alpha \otimes \text{id}$ with $m \circ (\text{id} \otimes \alpha - \alpha \otimes \text{id})$, where $m$ is the multiplication map on $A$. We see that condition (i) indeed implies (in fact, is equivalent to) $d_3^+(\alpha)|_{X_{0,3}} = 0$.

Next, we claim that $\alpha$ being $H$-invariant implies that $\alpha$ is also 0 on the image of the differential on $X_{1,2}$. Let $a, b \in A$, $h \in H$, and $r \in I$, and consider $a \otimes h \otimes r \otimes b$ as an element of $X_{1,2} \cong A \otimes H \otimes I \otimes A$ by Theorem 2.10(a). By the definition of the differential on $X_{1,2}$,

$$d(a \otimes h \otimes r \otimes b) = d((a \otimes h \otimes 1) \otimes (1 \otimes r \otimes b)) = d(a \otimes h \otimes 1) \otimes (1 \otimes r \otimes b) - (a \otimes h \otimes 1) \otimes d(1 \otimes r \otimes b).$$

The second term lies in $X_{1,1}$, but $\alpha$ is 0 on $X_{1,1}$ by definition. Therefore,

$$\alpha(d(a \otimes h \otimes r \otimes b)) = \alpha((ah \otimes 1 - a \otimes h) \otimes (1 \otimes r \otimes b)) = \alpha\left(ah \otimes r \otimes b - \sum a \otimes (h_1 \cdot r) \otimes h_2 b\right) = ahb(r) - \sum a\alpha(h_1 \cdot r)h_2 b.$$

Since $\alpha$ is $H$-invariant, we have that

$$h\alpha(r) = \sum h_1 \epsilon(h_2)\alpha(r) = \sum h_1 \alpha(r)\epsilon(h_2) = \sum h_1 \alpha(r)S(h_2)h_3 = \sum \alpha(h_1 \cdot r)h_2,$$

where the last equality used the fact that $\alpha$ is $H$-invariant. Thus, $\alpha$ is zero on the image of $d = d_3$ on $X_{1,2}$ by (3-7). It follows that $\alpha$ is a Hochschild 2-cocycle on $X_*$. Now, let $\mu_1 = \psi_2^\star(\alpha)$, where $\psi_\star$ is a chain map satisfying the conditions of Lemma 2.13. We conclude that

$$\delta_3^\star(\mu_1) = \delta_3^\star(\psi_2^\star(\alpha)) = \psi_3^\star(d_3^\star(\alpha)) \equiv 0,$$
as desired. So, we have a first-level graded deformation $A_{(1)}$ of $A$ with first multiplication map $\mu_1 : A \otimes A \to A$.

As an aside, we also get that $\phi_2^*(\mu_1) = \alpha$ as cochains. To see this, first note that since $\alpha$ is homogeneous of degree $-1$ by its definition, so is $\mu_1$. Let $x \in X_{0,2}$. By Lemma 2.13, $\psi_2 \phi_2(x) = x$, and thus

$$\mu_1 \phi_2(x) = \alpha \psi_2 \phi_2(x) = \alpha(x).$$

Now let $y$ be a free generator of $X_{1,1}$ or of $X_{2,0}$, which may always be chosen to have degree 1 or 0, respectively. Then $\psi_2 \phi_2(y)$ has degree 1 or 0 respectively, implying that its component in $X_{0,2}$ is 0. It follows that $\mu_1 \phi_2(y) = \alpha \psi_2 \phi_2(y) = 0 = \alpha(y)$; the last equation follows from the extension of $\alpha$ to $X_*$. Therefore $\phi_2^*(\mu_1) = \alpha$.

**Construction of the multiplication map $\mu_2$.** Given $\mu_1$ as above, note that the map $\mu_2$ must satisfy (1-10); that is, $\delta_3^*(\mu_2) = \mu_1 \circ (\mu_1 \otimes \text{id} - \text{id} \otimes \mu_1)$ as cochains on the bar resolution $B_*(A)$ of $A$. We will show that a modification of $\psi_2^*(\beta)$ is such a map, as follows:

First, note that $\beta = \phi_2^*(\psi_2^*(\beta))$ as cochains, by an argument similar to that above for $\alpha$. Moreover, condition (ii) implies that $d_3^*(\beta) = \alpha \circ (\alpha \otimes \text{id} \otimes \text{id} \otimes \mu_1)$ as cochains on $X_{0,3}$. Let

$$\gamma = \delta_3^* \psi_2^*(\beta) - \mu_1 \circ (\mu_1 \otimes \text{id} - \text{id} \otimes \mu_1).$$

Then $\phi_3^*(\gamma)$ is zero on $X_{0,3}$: $\phi_3^* \delta_3^* \psi_2^*(\beta) = d_3^*(\beta)$ and $\phi_3^*(\mu_1 \circ (\mu_1 \otimes \text{id} - \text{id} \otimes \mu_1)) = \alpha \circ (\alpha \otimes \text{id} - \text{id} \otimes \alpha)$ by Lemma 3.4(ii). To see the last statement, note that the image of $\phi_3$ on $X_{0,3}$ is contained in $A \otimes (I \otimes V) \cap (V \otimes I) \otimes A$ with $\phi^*(\mu_1) = \alpha$. We also see that $\phi_3^*(\gamma)$ is 0 on $X_{2,1}$ and $X_{3,0}$ since it is a map of degree $-2$. We claim it is also 0 on $X_{1,2}$ as follows. As an $A^e$-module, the image of $X_{1,2}$ under $\phi_3$ is generated by elements of degree 2. Since $\mu_1 = \psi_2^*(\alpha)$, it is zero on elements of degree less than two, and so the map $\mu_1 \circ (\mu_1 \otimes \text{id} - \text{id} \otimes \mu_1)$ must be 0 on the image of $X_{1,2}$ under $\phi_3$. Since $\beta$ is $H$-invariant, and thus $\beta$ is a cocycle (see the argument following (3-7)) and $\phi_2^* \psi_2^*(\beta) = \beta$, we have that $\phi_3^* \delta_3^* \psi_2^*(\beta) = \phi_3^*(d_3^* \phi_2^* \psi_2^*(\beta)) = \phi_3^*(\beta) = 0$ on $X_{1,2}$. Therefore $\phi_3^*(\gamma)$ is 0 on $X_{1,2}$.

We have shown that $\phi_3^*(\gamma)$ is 0 on all of $X_3$, and so $\gamma$ must be a coboundary on the bar resolution $B_*(A)$ of $A$. Thus there is a 2-cochain $\mu$ of degree $-2$ on the bar resolution with

$$\delta_3^*(\mu) = \gamma.$$  

Consider $\psi_2^*(\beta) - \mu$, yet note that $\phi_2^*(\psi_2^*(\beta) - \mu)$ may not agree with $\beta$ on $X_2$. We need a statement for the next step of constructing $\mu_3$. Now,

$$d_3^* \phi_2^*(\mu) = \phi_3^* \delta_3^*(\mu) = \phi_3^*(\gamma) = 0,$$
so the 2-cochain $\phi_2^*(\mu)$ is a cocycle on the complex $X_\ast$. Thus, $\phi_2^*(\mu)$ lifts to a cocycle $\mu'$ of degree $-2$ on the bar complex $B_\ast(A)$. In other words, $\phi_2^*(\psi_2^*(\beta) - \mu + \mu')$ agrees with $\beta$ on $X_2$.

Moreover, $\delta_3^*(\mu') = 0$, and by (3-8) and (3-9), we have that $\delta_3^*(\psi_2^*(\beta) - \mu) = \mu_1 \circ (\mu_1 \otimes \text{id} - \text{id} \otimes \mu_1)$. So,

\[(3-10) \quad \delta_3^*(\psi_2^*(\beta) - \mu + \mu') - \mu_1 \circ (\mu_1 \otimes \text{id} - \text{id} \otimes \mu_1) = 0\]
on the bar resolution $B_\ast(A)$ of $A$.

Thus, setting $\mu_2$ equal to $\psi_2^*(\beta) - \mu + \mu'$, we have maps $\mu_1, \mu_2$ to obtain a second-level graded deformation $A_{(2)}$ of $A$ extending $A_{(1)}$.

**Construction of the multiplication map $\mu_3$.** Recall the restraint on $\mu_3$ given in (1-11): $\mu_3$ is a cochain on $B_\ast(A)$ whose coboundary is given by $\mu_1 \circ (\mu_2 \otimes \text{id} - \text{id} \otimes \mu_2) + \mu_2 \circ (\mu_1 \otimes \text{id} - \text{id} \otimes \mu_1)$. We construct $\mu_3$ as follows.

By (3-10) and condition (iii) of Lemma 3.4, we have that $\mu_2 \circ (\mu_1 \otimes \text{id} - \text{id} \otimes \mu_1)$ is 0 on the image of $\phi$. By degree considerations, $\mu_1 \circ (\mu_2 \otimes \text{id} - \text{id} \otimes \mu_2)$ is always 0 on the image of $\phi$. Therefore, the obstruction

$$\mu_2 \circ (\mu_1 \otimes \text{id} - \text{id} \otimes \mu_1) + \mu_1 \circ (\mu_2 \otimes \text{id} - \text{id} \otimes \mu_2)$$

is a coboundary. Thus there exists a 2-cochain $\mu_3$, necessarily having degree $-3$, satisfying the restraint given above, and the deformation lifts to the third level.

**Construction of the multiplication maps $\mu_i$ for $i \geq 4$.** The obstruction for a third-level graded deformation $A_{(3)}$ of $A$ to lift to the fourth level lies in $\text{HH}^{3,-4}(A)$ by Proposition 1.13. We apply $\phi_3^*$ to this obstruction to obtain a cochain on $X_3$. Since there are no cochains of degree $-4$ on $X_3$ by definition (as it is generated by elements of degree 3 or less), $\phi_3^*$ applied to the obstruction is automatically zero. Therefore, the deformation may be continued to the fourth level. Similar arguments show that it can be continued to the fifth level, and so on.

**Construction of $A_t$.** Let $A_t$ be the graded deformation of $A$ that we obtain in this manner (Definition 1.5). Then, $A_t$ is the $k$-vector space $A[t]$ with multiplication defined for all $a_1, a_2 \in A$ by

$$a_1 * a_2 = a_1 a_2 + \mu_1 (a_1 \otimes a_2) t + \mu_2 (a_1 \otimes a_2) t^2 + \mu_3 (a_1 \otimes a_2) t^3 + \cdots,$$

where $a_1 a_2$ is the product in $A$ and each $\mu_i : A \otimes A \to A$ is a $k$-linear map of homogeneous degree $-i$. Now for any $r$ in $R$, $\mu_1(r) = (\psi_2^* \alpha)(r)$ and $\mu_2(r) = (\psi_2^*(\beta) - \mu + \mu')(r)$ by construction, and $\mu_i(r) = 0$ for $i \geq 3$ since deg$(r) = 2$.

**Conclusion that $A' := T_H(U)/(P)$ is a PBW deformation of $A = T_H(U)/(R)$.** Now we show that $A' := T_H(U)/(P)$ is isomorphic, as a filtered algebra, to the fiber of the deformation $A_t$ at $t = 1$ as follows. Let $A'' = (A_t)|_{t=1}$. Then $A''$ is generated by $V$ and $H$ and one thus obtains a surjective algebra homomorphism
$T_H(U) \cong T_k(V) \# H \rightarrow A''$. The elements of $P$ lie in the kernel (by definition of $A''$), and thus this map induces a surjective algebra homomorphism $A' = T_H(U)/(P) \rightarrow A''$. This map is in fact an isomorphism of filtered algebras by a dimension argument in each degree. Therefore $A'$ is a PBW deformation of $A$, since $A''$ is one (Proposition 1.14).

4. Examples

For our examples, we restrict $k$ to be an algebraically closed field of characteristic zero. There are many interesting examples, both known and new, in this setting. Less is known about Hopf actions on Koszul algebras and corresponding deformations in positive characteristic.

As an application of Theorem 3.1, we provide various examples of PBW deformations $\mathcal{D}_{B,\kappa}$ of smash products $B \# H$; recall Notations 0.2 and 0.3. We do this by describing deformation parameter(s) $\kappa = \kappa^C + \kappa^L$ below. In particular, Examples 4.1, 4.2, and 4.4 involve semisimple Hopf actions, and Examples 4.13, 4.16, and 4.18 involve nonsemisimple Hopf actions on (skew) polynomial rings. Recall that skew polynomial rings are Koszul by [Polishchuk and Positselski 2005, Example 4.2.1 and Theorem 4.3.1].

4A. Semisimple Hopf actions. We begin by revisiting the well-known PBW deformations of Crawley-Boevey and Holland [1998].

Example 4.1. Take $H = k\Gamma$, for $\Gamma$ a finite subgroup of $SL_2(k)$, and $B = k[u, v]$. For $g = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \Gamma$, let the action of $g$ on $B$ be given by $g \cdot u = au + cv$ and $g \cdot v = bu + dv$.

By [Crawley-Boevey and Holland 1998], the deformation parameter $\kappa$ of the PBW deformation $\mathcal{D}_{B,\kappa}$ of $B \# H$ must be in the center of $k\Gamma$, which we verify again with Theorem 3.1. We assume here that $\kappa^L \equiv 0$, as in that work.

Since $\dim_k V = 2$, only condition (a) of Theorem 3.1 applies. So we have for all $g \in \Gamma$ that $g \cdot (\kappa(uv - vu)) = \kappa(g \cdot (uv - vu))$. Now since the determinant of $g$ is 1, $g \cdot (\kappa(uv - vu)) = \kappa(uv - vu)$, and the image of $\kappa$ lies in the center of $k\Gamma$. That is,

$$\mathcal{D}_{B,\kappa} = \frac{k\langle u, v \rangle \# k\Gamma}{(uv - vu - \lambda)}$$

is a PBW deformation of $k[u, v] \# k\Gamma$ if and only if $\lambda \in Z(k\Gamma)$.

It is worth pointing out that there are analogues of Crawley-Boevey–Holland algebras when working in positive characteristic; see the work of Emily Norton [2013] for some examples that are quite different from those in characteristic zero.

The following two Hopf actions were produced by Walton in joint work with Kenneth Chan, Ellen Kirkman, and James Zhang [Chan et al. 2012]. We thank them for permitting us to use these results.
Example 4.2. Let $H := H_8$ be the unique noncommutative noncocommutative semisimple 8-dimensional Hopf algebra [Kac and Paljutkin 1966; Masuoka 1995], and let $B = k\langle u, v \rangle / (u^2 + v^2)$ (which is isomorphic to the skew polynomial ring $k\langle u, v \rangle / (uv + vu)$). The Hopf algebra $H_8$ is generated by $x, y, z$, and the relations are 

$$x^2 = y^2 = 1, \quad xy = yx, \quad zx = yz, \quad zy = xz, \quad z^2 = \frac{1}{2}(1 + x + y - xy).$$

The rest of the structure of $H_8$ and the left $H_8$-action on $B$ are given by

$$\Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes y, \quad \Delta(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z), \quad \epsilon(x) = \epsilon(y) = \epsilon(z) = 1, \quad S(x) = x, \quad S(y) = y, \quad S(z) = z,$$

$$x \cdot u = -u, \quad x \cdot v = v, \quad y \cdot u = u, \quad y \cdot v = -v, \quad z \cdot u = v, \quad z \cdot v = u.$$

Let $r := u^2 + v^2$ and note that

$$x \cdot r = r, \quad y \cdot r = r, \quad z \cdot r = r,$$

so the ideal of relations of $B$, $I = \langle r \rangle$, is $H$-stable.

Since $\dim_k V = 2$, only condition (a) of Theorem 3.1 applies. We begin by computing $\kappa^C$. Let

$$\kappa^C(r) = \gamma_0 + \gamma_1 x + \gamma_2 y + \gamma_3 xy + \gamma_4 z + \gamma_5 xz + \gamma_6 yz + \gamma_7 xyz$$

for some scalars $\gamma_i \in k$. Since $h \cdot (\kappa^C(r)) = \sum h_1(\kappa^C(r)) S(h_2)$ (see Section 1A), both $x \cdot (\kappa^C(r)) = \kappa^C(r)$ and $y \cdot (\kappa^C(r)) = \kappa^C(r)$ imply that $\gamma_7 = \gamma_4$ and $\gamma_6 = \gamma_5$. Moreover,

$$z \cdot (\kappa^C(r)) = \gamma_0 + \gamma_2 x + \gamma_1 y + \gamma_3 xy + \gamma_4 z + \gamma_5 xz + \gamma_5 yz + \gamma_4 xyz = \kappa^C(r),$$

which implies that $\gamma_2 = \gamma_1$. Thus,

$$(4-3) \quad \kappa^C(r) =: g(\gamma_0, \gamma_1, \gamma_3, \gamma_4, \gamma_5) = \gamma_0 + \gamma_1 (x + y) + \gamma_3 xy + \gamma_4 (z + xyz) + \gamma_5 (xz + yz).$$

On the other hand, let $\kappa^L(r) = u \otimes f + v \otimes f' \in V \otimes H$ with

$$f = \delta_0 + \delta_1 x + \delta_2 y + \delta_3 xy + \delta_4 z + \delta_5 xz + \delta_6 yz + \delta_7 xyz,$$

$$f' = \delta'_0 + \delta'_1 x + \delta'_2 y + \delta'_3 xy + \delta'_4 z + \delta'_5 xz + \delta'_6 yz + \delta'_7 xyz,$$

for some scalars $\delta_i, \delta'_i \in k$. Note that $h \cdot (\kappa^L(r)) = \sum h_1 \cdot u \otimes h_2 f S(h_3) + \sum h_1 \cdot v \otimes h_2 f' S(h_3)$ (see Section 1A). Since $x \cdot (\kappa^L(r)) = \kappa^L(r)$, it follows that:

$$\delta_0 = \delta_1 = \delta_2 = \delta_3 = 0, \quad \delta_4 = -\delta_7, \quad \delta_5 = -\delta_6 \quad \delta'_4 = \delta'_7, \quad \text{and} \quad \delta'_5 = \delta'_6.$$

By considering the coefficient of $u$ in the equation $y \cdot (\kappa^L(r)) = \kappa^L(r)$, we now find that $f = 0$. Similarly, by considering the coefficient of $v$ in the equation $y \cdot (\kappa^L(r)) = \kappa^L(r)$, we find that $f' = 0$. Hence, $\kappa^L(r) = 0$. 
Thus the deformation parameter \( \kappa \) of \( D_{B, \kappa} \) equals its constant part \( \kappa^C \), which depends on five scalar parameters as described above. In short,

\[
D_{B, \kappa} = \frac{k \langle u, v \rangle \# H_8}{(u^2 + v^2 - \kappa(u^2 + v^2))}
\]

is a PBW deformation of \((k \langle u, v \rangle)/(u^2 + v^2)) \# H_8\) if and only if \( \kappa(u^2 + v^2) = g(\gamma_0, \gamma_1, \gamma_3, \gamma_4, \gamma_5) \) as given in (4-3). This yields a five-parameter family of PBW deformations of \( B \# H_8 \).

**Example 4.4.** Let \( H \) be \( H_{d; 1} \), one of the 16-dimensional semisimple Hopf algebras classified in [Kashina 2000], and let \( B \) be the skew polynomial ring

\[
B = \frac{k \langle t, u, v, w \rangle}{(r_{tu} := tu - ut, \quad r_{tv} := tv - vt, \quad r_{tw} := tw + wt, \quad r_{uv} := uv - vu, \quad r_{uw} := uw + wu, \quad r_{vw} := vw - vw)}.
\]

The Hopf algebra \( H_{d; 1} \) is generated by \( x, y, z \), subject to the relations

\[
x^4 = y^2 = z^2 = 1, \quad yx = xy, \quad zx = xyz, \quad zy = yz.
\]

The rest of the structure of \( H_{d; 1} \) and the left \( H_{d; 1} \)-action on \( B \) are given by

\[
\Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes y, \quad \Delta(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x^2 + y \otimes 1 - y \otimes x^2)(z \otimes z),
\]

\[
\epsilon(x) = \epsilon(y) = \epsilon(z) = 1, \quad S(x) = x^3, \quad S(y) = y, \quad S(z) = \frac{1}{2}(1 + x^2 + y - x^2 y)z,
\]

\[
x \cdot t = it, \quad y \cdot t = -t, \quad z \cdot t = u, \quad x \cdot u = -iu, \quad y \cdot u = -u, \quad z \cdot u = t,
\]

\[
x \cdot v = v, \quad y \cdot v = -v, \quad z \cdot v = w, \quad x \cdot w = -w, \quad y \cdot w = -w, \quad z \cdot w = v,
\]

where \( i \) is a primitive fourth root of unity in \( k \). Note that

\[
x \cdot r_{tu} = r_{tu}, \quad x \cdot r_{tv} = ir_{tv}, \quad x \cdot r_{tw} = -i r_{tw},
\]

\[
x \cdot r_{uv} = -ir_{uv}, \quad x \cdot r_{uw} = ir_{uw}, \quad x \cdot r_{vw} = -r_{vw},
\]

\[
y \cdot r_{tu} = r_{tu}, \quad y \cdot r_{tv} = r_{tv}, \quad y \cdot r_{tw} = r_{tw},
\]

\[
y \cdot r_{uv} = r_{uv}, \quad y \cdot r_{uw} = r_{uw}, \quad y \cdot r_{vw} = r_{vw},
\]

\[
z \cdot r_{tu} = r_{tu}, \quad z \cdot r_{tv} = r_{tv}, \quad z \cdot r_{tw} = r_{tw},
\]

\[
z \cdot r_{uv} = r_{uv}, \quad z \cdot r_{uw} = r_{uw}, \quad z \cdot r_{vw} = -r_{vw}.
\]

So, the ideal of relations \( I = \langle r_{tu}, r_{tv}, r_{tw}, r_{uv}, r_{uw}, r_{vw} \rangle \) of \( B \) is \( H \)-stable.

Now we compute the possible values \( \kappa^C(r) \in H \) for all \( r \in I \), under condition (a) of Theorem 3.1. Take \( \kappa^C(r) = g(\gamma) \in H \) given by

\[
g(\gamma) = \gamma_0 + \gamma_1 x + \gamma_2 x^2 + \gamma_3 x^3 + \gamma_4 y + \gamma_5 x y + \gamma_6 x^2 y + \gamma_7 x^3 y
\]

\[
+ \gamma_8 z + \gamma_9 x z + \gamma_{10} x^2 z + \gamma_{11} x^3 z + \gamma_{12} y z + \gamma_{13} x y z + \gamma_{14} x^2 y z + \gamma_{15} x^3 y z,
\]
where \( \gamma_i \in k \). Note that \( h \cdot g(\gamma) = \sum h_1 g(\gamma) S(h_2) \). With the assistance of Affine, a subpackage of Maxima, we have the following computations:

\[
(4-5) \quad x \cdot g(\gamma) = xg(\gamma)x^3 = \gamma_0 + \gamma_1 x + \gamma_2 x^2 + \gamma_3 x^3 + \gamma_4 y + \gamma_5 xy
\]
\[
+ \gamma_6 x^2 y + \gamma_7 x^3 y + \gamma_8 yz + \gamma_9 xyz + \gamma_{10} x^2 yz
\]
\[
+ \gamma_{11} x^3 yz + \gamma_{12} z + \gamma_{13} xz + \gamma_{14} x^2 z + \gamma_{15} x^3 z;
\]

\[
(4-6) \quad y \cdot g(\gamma) = yg(\gamma)y = g(\gamma);
\]

\[
(4-7) \quad z \cdot g(\gamma) = \frac{1}{2} (zg(\gamma)S(z) + zg(\gamma)S(x^2 z) + yzg(\gamma)S(z) - yzg(\gamma)S(x^2 z))
\]
\[
= \gamma_0 + \gamma_1 xy + \gamma_2 x^2 + \gamma_3 x^3 y + \gamma_4 y + \gamma_5 x
\]
\[
+ \gamma_6 x^2 y + \gamma_7 x^3 y + \gamma_8 z + \gamma_9 xyz + \gamma_{10} x^2 z
\]
\[
+ \gamma_{11} x^3 yz + \gamma_{12} yz + \gamma_{13} xz + \gamma_{14} x^2 yz + \gamma_{15} x^3 z.
\]

For \( r_{tu} \), let \( \kappa^C(r_{tu}) = g(\gamma) \). We have that \( x \cdot \kappa^C(r_{tu}) = \kappa^C(r_{tu}) \) and \( y \cdot \kappa^C(r_{tu}) = \kappa^C(r_{tu}) \) imply that \( \gamma_8 = \gamma_{12}, \gamma_9 = \gamma_{13}, \gamma_{10} = \gamma_{14}, \gamma_{11} = \gamma_{15} \). Moreover, \( z \cdot \kappa^C(r_{tu}) = \kappa^C(r_{tu}) \) implies that \( \gamma_1 = \gamma_{15}, \gamma_3 = \gamma_7, \gamma_9 = \gamma_{13}, \gamma_{11} = \gamma_{15} \). Therefore,

\[
(4-8) \quad \kappa^C(r_{tu}) = g(\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_6, \gamma_8, \gamma_9, \gamma_{10}, \gamma_{11})
\]
\[
= \gamma_0 + \gamma_1 (x + xy) + \gamma_2 x^2 + \gamma_3 (x^3 + x^3 y) + \gamma_4 y + \gamma_5 x
\]
\[
+ \gamma_6 x^2 y + \gamma_7 x^3 y + \gamma_8 z + \gamma_9 xyz + \gamma_{10} x^2 z
\]
\[
+ \gamma_{11} x^3 yz + \gamma_{12} yz + \gamma_{13} xz + \gamma_{14} x^2 yz + \gamma_{15} x^3 z.
\]

For \( r_{vw} \), let \( \kappa^C(r_{vw}) = g(\gamma') \). We have that \( x \cdot \kappa^C(r_{vw}) = -\kappa^C(r_{vw}) \) and \( y \cdot \kappa^C(r_{vw}) = \kappa^C(r_{vw}) \) implies that \( \gamma_6' = \cdots = \gamma_7' = 0, \gamma_8' = -\gamma_1', \gamma_9' = -\gamma_1', \gamma_{10}' = -\gamma_1', \gamma_{11}' = -\gamma_1' \). Moreover, we have that \( z \cdot \kappa^C(r_{vw}) = -\kappa^C(r_{vw}) \). So the conditions on \( \gamma_i' \) in (4-7) then imply that \( \gamma_i' = 0 \) for \( i = 0, \ldots, 7, 8, 10, 12, 14 \) with \( \gamma_6' = -\gamma_1', \gamma_{11}' = -\gamma_1' \). Thus,

\[
(4-9) \quad \kappa^C(r_{vw}) = g(\gamma_0', \gamma_1') = \gamma_0' (xz - xzy) + \gamma_1' (x^3 z - x^3 yz).
\]

For \( r \neq r_{tu}, r_{vw} \), we have that \( x \cdot \kappa^C(r) = \pm i \kappa^C(r) \) implies that \( \kappa^C(r) = 0 \).

We compute \( \kappa^L(r) \) under condition (a) of Theorem 3.1. Fix \( r \in I \) and let

\[
\kappa^L(r) = t \otimes f_I + u \otimes f_U + v \otimes f_V + w \otimes f_W \in V \otimes H
\]

for some \( f_I, f_U, f_V, f_W \in H \). Since \( y \) is central in \( H \) and \( y \cdot r = r \) for each relation \( r \), we have that

\[
\kappa^L(r) = y \cdot \kappa^L(r)
\]
\[
= y \cdot t \otimes yf_I S(y) + y \cdot u \otimes yf_U S(y) + y \cdot v \otimes yf_V S(y) + y \cdot w \otimes yf_W S(y)
\]
\[
= -t \otimes f_I - u \otimes f_U - v \otimes f_V - w \otimes f_W = -\kappa^L(r).
\]

Thus, \( \kappa^L(r) = 0 \).
To finish, we apply to \( \kappa \) conditions (b)–(d) of Theorem 3.1. Since \( \kappa^L(r) = 0 \) for all \( r \in I \), only condition (c) is pertinent. Namely, we only need to impose \( \kappa^C \otimes \text{id} = \text{id} \otimes \kappa^C \) as maps on \( (V \otimes V) \cap (V \otimes I) \). This intersection is a 4-dimensional \( k \)-vector space with basis
\[
\begin{array}{c}
s_{tuw} := tuv - tvu - utv + uvtt + vtu - vtu, \\
s_{tuw} := tuw + twu - utw - uwt + wtu - wut,
\end{array}
\]
\[
\begin{array}{c}
s_{tvw} := tvw - twv - wtv + wtv - wtv - wtv,
\end{array}
\]
\[
\begin{array}{c}
s_{uvw} := uvw - uvw - uwv - vuv + wvu.
\end{array}
\]
Since \( \kappa^C(r_{tv}) = \kappa^C(r_{uw}) = 0 \), we get that
\[
(4-10) \quad (\kappa^C \otimes \text{id} - \text{id} \otimes \kappa^C)(s_{tvw}) = \kappa^C(r_{uv})t - \kappa^C(r_{tv})u + \kappa^C(r_{tw})u - t\kappa^C(r_{uv}) + u\kappa^C(r_{tv}) - v\kappa^C(r_{tw})
\]
\[
= \kappa^C(r_{tu})u - v\kappa^C(r_{tw}) = 0
\]
Identify \( b \in V \) with \( b \# 1 \in A \) and \( h \in H \) with \( 1 \# h \in A \). Recall that in \( A \) we have
\[
(1 \# h)(b \# 1) = \sum (h_1 \cdot b) \# h_2.
\]
Now by using (4-8) and by setting (4-10) equal to 0, we get that
\[
(4-11) \quad \kappa^C(r_{tu}) = g(\gamma_0, \gamma_2) = \gamma_0 + \gamma_2 x^2.
\]
Moreover,
\[
(\kappa^C \otimes \text{id} - \text{id} \otimes \kappa^C)(s_{tvw}) = -\kappa^C(r_{uv})t + \kappa^C(r_{tv})u - \kappa^C(r_{tw})u - t\kappa^C(r_{uv}) + u\kappa^C(r_{tv}) - w\kappa^C(r_{tw})
\]
\[
= \kappa^C(r_{tu})u - w\kappa^C(r_{tw}) = 0
\]
imposes no new restrictions on \( \kappa^C(r_{tu}) \), nor do \( (\kappa^C \otimes \text{id} - \text{id} \otimes \kappa^C)(s_{tvw}) = 0 \), \( (\kappa^C \otimes \text{id} - \text{id} \otimes \kappa^C)(s_{uvw}) = 0 \). Therefore, \( \kappa^C(r_{tu}) \) is given by (4-11).

To compute \( \kappa^C(r_{vw}) \), consider the calculation
\[
(4-12) \quad (\kappa^C \otimes \text{id} - \text{id} \otimes \kappa^C)(s_{tvw}) = \kappa^C(r_{uv})t - \kappa^C(r_{tv})u + \kappa^C(r_{tw})u - t\kappa^C(r_{uv}) + v\kappa^C(r_{tv}) - w\kappa^C(r_{tw})
\]
\[
= \kappa^C(r_{tv})u - t\kappa^C(r_{tv}) = \kappa^C(r_{tv}) - \kappa^C(r_{tv}).
\]
Now by using (4-9) and by setting (4-12) equal to 0, we get that \( \kappa^C(r_{uv}) = 0 \).

Therefore, the filtered algebra \( \mathcal{D}_{B,\kappa} \) is a PBW deformation of \( B \# H_{a;1} \) if and only if the deformation parameter \( \kappa = \kappa^C \) of \( \mathcal{D}_{B,\kappa} \) is defined by (4-11) for the relation \( r_{tu} \), and \( \kappa^C(r) = 0 \) for \( r \neq r_{tu} \). Hence, we have a two-parameter family of PBW deformations of \( B \# H_{a;1} \).
4B. Nonsemisimple Hopf actions. Here, we consider nonsemisimple Hopf actions to illustrate Theorem 3.1. We begin with an example of a Taft algebra action.

Example 4.13. Let $H = T(n)$, the $n^2$-dimensional nonsemisimple Taft algebra. We take $n \geq 3$ here and we consider the (slightly different) case $n = 2$ (the Sweedler algebra) in Example 4.16 below. Let $B = k[u, v]$. The Hopf algebra $T(n)$ is generated by $g, x$ and the relations are $g^n = 1, x^n = 0$, and $xg = \zeta gx$, for $\zeta \in k^\times$ a primitive $n$-th root of unity. The rest of the structure of $T(n)$ and the left $T(n)$-action on $B$ are given by

$$\Delta(g) = g \otimes g, \quad \Delta(x) = g \otimes x + x \otimes 1, \quad \epsilon(g) = 1, \quad \epsilon(x) = 0,$$

$$S(g) = g^{-1} = g^{n-1}, \quad S(x) = -g^{-1}x,$$

$$g \cdot u = u, \quad g \cdot v = \zeta^{-1}v, \quad x \cdot u = 0, \quad x \cdot v = u.$$

Let $r := uv - vu$ and note that $g \cdot r = \zeta^{-1}r$ and $x \cdot r = 0$. Hence, the ideal of relations $I = \langle r \rangle$ of $B$ is $H$-stable.

Since $\dim_k V = 2$, only condition (a) of Theorem 3.1 applies. Now, we compute $\kappa^C$. Let $\kappa^C(r) = \sum_{i,j=0}^{n-1} \gamma_{ij} g^i x^j$. Since $h \cdot (\kappa^C(r)) = \sum h_1 (\kappa^C(r)) S(h_2)$ for $h \in H$, we have that the equality $g \cdot (\kappa^C(r)) = \zeta^{-1}\kappa^C(r)$ implies that all terms equal zero except when $j = 1$; hence

$$\kappa^C(r) = \gamma_0 x + \gamma_1 gx + \cdots + \gamma_{n-1} g^{n-1}x$$

for $\gamma_i \in k$. Also, the equality $x \cdot (\kappa^C(r)) = 0$ implies that all terms equal zero except for $i = n - 1$, so

$$(4-14) \quad \kappa^C(r) = \gamma g^{n-1}x$$

for $\gamma \in k$.

On the other hand, let $\kappa^L(r) = u \otimes f + v \otimes f' \in V \otimes H$ for $f = \sum_{i,j=0}^{n-1} \lambda_{ij} g^i x^j$ and $f' = \sum_{i,j=0}^{n-1} \lambda'_{ij} g^i x^j$. Notice that $h \cdot (\kappa^L(r)) = \sum h_1 \cdot u \otimes h_2 f S(h_3) + \sum h_1 \cdot v \otimes h_2 f' S(h_3)$ (see Section 1A). Since $g \cdot (\kappa^L(r)) = \zeta^{-1}\kappa^L(r)$, all terms equal zero except possibly those in the first sum for which $j = 1$ and those in the second sum for which $j = 0$. Therefore $\kappa^L(r) = u \otimes f + v \otimes f'$ for

$$f = \lambda_0 x + \lambda_1 gx + \cdots + \lambda_{n-1} g^{n-1}x \quad \text{and} \quad f' = \lambda'_0 + \lambda'_1 g + \cdots + \lambda'_{n-1} g^{n-1}$$

with $\lambda_i, \lambda'_j \in k$. Applying $x$, we obtain

$$0 = x \cdot \kappa^L(r) = (g \cdot u) \otimes (gf S(x) + xf S(1)) + (x \cdot u) \otimes f$$

$$+ (g \cdot v) \otimes (gf' S(x) + xf' S(1)) + (x \cdot v) \otimes f'$$

$$= u \otimes (-gfg^{n-1}x + xf + f') + \zeta^{-1} v \otimes (-gf'g^{n-1}x + xf').$$
It follows that
\[-gfg^{n-1}x + xf + f' = 0 \quad \text{and} \quad -gf'g^{n-1}x + xf' = 0.\]
Since \(f'\) is in the group algebra \(kG(T(n)) \cong k\mathbb{Z}_n\) and \(g^n = 1\), the second equation implies that \(xf' = f'x\), and so \(f' = \lambda_0\) is constant. The first equation further implies that \(f' = 0\) and that all terms of \(f\) are equal to zero except for \(i = n - 1\). Thus,
\[(4-15) \quad \kappa^L(r) = u \otimes \lambda g^{n-1}x\]
for \(\lambda \in k\).

In summary,
\[\mathcal{D}_{B,\kappa} = \frac{k\langle u, v \rangle \# T(n)}{(uv - vu - \kappa(uv - vu))}\]
is a PBW deformation of \(k[u, v] \# T(n)\) if and only if the deformation map \(\kappa\) equals \(\kappa^C + \kappa^L\) as given in (4-14) and (4-15). So, we have a two-parameter family of PBW deformations of \(k[u, v] \# T(n)\).

**Example 4.16.** Let \(H\) be \(H_{Sw} = T(2)\), the 4-dimensional nonsemisimple Sweedler algebra, which is a Taft algebra with \(n = 2\). Let \(B = k[u, v]\). Retaining the notation from Example 4.13, the Hopf algebra \(H_{Sw}\) is generated by \(g, x\) and acts on \(B\) by \(g \cdot u = u, g \cdot v = -v, x \cdot u = 0, x \cdot v = u\). Similar to Example 4.13, let \(r := uv - vu\) and note that \(g \cdot r = -r\) and \(x \cdot r = 0\). So, \(I = \langle r \rangle\) is \(H\)-stable.

Let \(\kappa^C(r) = \gamma_0 + \gamma_1 g + \gamma_2 x + \gamma_3 gx\). We have that \(g \cdot (\kappa^C(r)) = -\kappa^C(r)\) implies that \(\gamma_0 = \gamma_1 = 0\). Moreover, \(x \cdot (\kappa^C(r)) = 0\) does not yield new restrictions on \(\kappa^C(r)\).

Thus, for \(\gamma, \gamma' \in k\), we get that \(\kappa^C(r) = \gamma x + \gamma' gx\). In the same fashion as Example 4.13, we also get that \(\kappa^L(r) = u \otimes (\lambda x + \lambda' gx)\) for \(\lambda, \lambda' \in k\).

In summary,
\[\mathcal{D}_{B,\kappa} = \frac{k\langle u, v \rangle \# H_{Sw}}{(uv - vu - \kappa(uv - vu))}\]
is a PBW deformation of \(k[u, v] \# H_{Sw}\) if and only if the deformation map \(\kappa\) equals \(\kappa^C + \kappa^L\), where
\[\kappa^C(uv - vu) = \gamma x + \gamma' gx \quad \text{and} \quad \kappa^L(uv - vu) = u \otimes (\lambda x + \lambda' gx)\]
for \(\gamma, \gamma', \lambda, \lambda' \in k\). Thus, we have a four-parameter family of PBW deformations of \(k[u, v] \# H_{Sw}\).

**Remark 4.17.** The invariant ring resulting from the action of \(H_{Sw}\) on \(k[u, v]\) is isomorphic to the polynomial ring \(k[u, v^2]\), that is to say, \(k[u, v]_{H_{Sw}}\) is regular. Recall that the Shephard–Todd–Chevalley theorem states that when given a finite group \((G-)\) action on a commutative polynomial ring \(R\) that is linear and faithful, \(R^G\) is regular if and only if \(G\) is a reflection group. Our results would then suggest that \(H_{Sw}\) is a “reflection Hopf algebra”. Ram and Shepler [2003] showed that there are
no nontrivial PBW deformations of $k[v_1, \ldots, v_n]#kG$ for many complex reflection groups $G$; such deformations are referred to as graded Hecke algebras. Now by broadening their setting to Hopf actions on (possibly noncommutative) regular algebras, we consider new objects in representation theory: Hopf analogues of graded Hecke algebras. Nontrivial examples of these objects exist as we showed in the example above. Further examples and a general explanation of this phenomenon are worthy of further investigation.

Now we consider the well-known Hopf action of $U_q(\mathfrak{sl}_2)$ on $k\langle u, v \rangle/(uv - qvu)$, where $q \in k^\times$ with $q^2 \neq 1$. A PBW deformation of $(k\langle u, v \rangle/(uv - qvu))#U_q(\mathfrak{sl}_2)$ was studied by Gan and Khare [2007]; we recover their result below. Such algebras are known as quantized symplectic oscillator algebras of rank 1.

**Example 4.18.** Fix $q \in k^\times$, with $q^2 \neq 1$. Let $H$ be the Hopf algebra $U_q(\mathfrak{sl}_2)$, and $B = k\langle u, v \rangle/(uv - qvu)$. As in [Brown and Goodearl 2002, I.6.2], we take $U_q(\mathfrak{sl}_2)$ to be generated by $E, F, K, K^{-1}$ with relations:

$$EF - FE = (q - q^{-1})^{-1}(K - K^{-1}), \quad KEK^{-1} = q^2E,$$
$$KFK^{-1} = q^{-2}F, \quad KKK^{-1} = K^{-1}K = 1.$$ 

So, $U_q(\mathfrak{sl}_2)$ has a $k$-vector space basis $\{E^iF^jK^m\}_{i, j \in \mathbb{N}; m \in \mathbb{Z}}$. The rest of the structure of $U_q(\mathfrak{sl}_2)$ and the left $U_q(\mathfrak{sl}_2)$-action on $B$ is given by:

$$\Delta(E) = E \otimes 1 + K \otimes E, \quad \Delta(F) = F \otimes K^{-1} + 1 \otimes F,$$
$$\Delta(K) = K \otimes K, \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1},$$
$$\epsilon(E) = 0, \quad \epsilon(F) = 0, \quad \epsilon(K) = 1, \quad \epsilon(K^{-1}) = 1,$$
$$S(E) = -K^{-1}E, \quad S(F) = -FK, \quad S(K) = K^{-1}, \quad S(K^{-1}) = K,$$
$$E \cdot u = 0, \quad F \cdot u = v, \quad K \cdot u = qu, \quad K^{-1} \cdot u = q^{-1}u,$$
$$E \cdot v = u, \quad F \cdot v = 0, \quad K \cdot v = q^{-1}v, \quad K^{-1} \cdot v = qv.$$

Let $r := uv - qvu$ and note that $E \cdot r = F \cdot r = 0$ and $K \cdot r = K^{-1} \cdot r = r$. Hence, the ideal of relations $I = \langle r \rangle$ of $B$ is $H$-stable.

Since $\dim_k V = 2$, only condition (a) of Theorem 3.1 applies. Let us compute $\kappa^C(r)$. Since $K \cdot \kappa^C(r) = \kappa^C(K \cdot r) = \kappa^C(r)$, we have that $K \kappa^C(r)S(K) = \kappa^C(r)$ (see Section 1A). Hence, $K \kappa^C(r) = \kappa^C(r)K$. Moreover,

$$0 = \kappa^C(E \cdot r) = E \cdot \kappa^C(r) = E\kappa^C(r)S(1) + K\kappa^C(r)S(E),$$

so $E\kappa^C(r) = \kappa^C(r)E$. Likewise, $F \cdot \kappa^C(r) = 0$ implies that $F\kappa^C(r) = \kappa^C(r)F$. So, $\kappa^C(r)$ is in the center of $U_q(\mathfrak{sl}_2)$. For $q$ not a root of unity, the center of $U_q(\mathfrak{sl}_2)$ is
generated by the *quantum Casimir element* [Kassel 1995, Theorem VI.4.8],

\[ C_q = EF + \frac{q^{-1}K + qK^{-1}}{(q - q^{-1})^2} = FE + \frac{qK + q^{-1}K^{-1}}{(q - q^{-1})^2}, \]

whereas for \( q \) a root of unity, the elements \( E^e, F^e, K^e \) also belong to the center of \( U_q(\mathfrak{sl}_2) \), where \( e = \text{ord}(q^2) \).

To compute \( \kappa^L(r) \), let \( \kappa^L(r) = u \otimes \sum \gamma_{ijm} E^i F^j K^m + v \otimes \sum \gamma'_{ijm} E^i F^j K^m \) for \( \gamma_{ijm}, \gamma'_{ijm} \in k \). Then,

\[
\kappa^L(r) = \kappa^L(K \cdot r) \\
= \sum K \cdot u \otimes \gamma_{ijm} K(E^i F^j K^m)K^{-1} + \sum K \cdot v \otimes \gamma'_{ijm} K(E^i F^j K^m)K^{-1} \\
= \sum qu \otimes \gamma_{ijm}q^{2(i-j)} E^i F^j K^m + \sum q^{-1}v \otimes \gamma'_{ijm}q^{2(i-j)} E^i F^j K^m \\
= \sum q^{2(i-j)+1}u \otimes \gamma_{ijm} E^i F^j K^m + \sum q^{2(i-j)-1}v \otimes \gamma'_{ijm} E^i F^j K^m.
\]

Thus, given \( m \in \mathbb{Z}/n\mathbb{Z} \), define the subspace \( V_m \subset U_q(\mathfrak{sl}_2) \) to be the \( k \)-span of all monomials \( E^i F^j K^\ell \) such that \( j - i \equiv m \mod n \). Then \( \kappa^L(uv - qvu) \in u \otimes V_{2-1} + v \otimes V_{-2-1} \) if \( q \) is a primitive root of unity of odd order, and \( \kappa^L(uv - qvu) = 0 \) otherwise.

Therefore,

\[
\mathcal{D}_{B,\kappa} = \frac{k \langle u, v \rangle \# U_q(\mathfrak{sl}_2)}{(uv - qvu - \kappa(uv - qvu))}
\]

is a PBW deformation of \( (k \langle u, v \rangle/(uv - qvu)) \# U_q(\mathfrak{sl}_2) \) if and only if \( \kappa = \kappa^C + \kappa^L \), where \( \kappa^C(uv - qvu) \) is in the center of \( U_q(\mathfrak{sl}_2) \) and \( \kappa^L(uv - qvu) \) is given as above.

More generally, there is a standard \( U_q(\mathfrak{sl}_n) \)-action on a \( q \)-polynomial ring \( B \) in \( n \) variables.

**Question 4.19.** Are there nontrivial PBW deformations of the resulting smash product algebra \( B \# U_q(\mathfrak{sl}_n) \)?

These would be *quantized symplectic oscillator algebras of rank* \( n - 1 \), and merit further investigation.

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References


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<td>Bounded gaps between primes with a given primitive root</td>
<td>1769</td>
</tr>
<tr>
<td>PAUL POLLACK</td>
<td></td>
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