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On direct images of pluricanonical bundles

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We show that techniques inspired by Kollár and Viehweg's study of weak positivity, combined with vanishing for log-canonical pairs, lead to new generation and vanishing results for direct images of pluricanonical bundles. We formulate the strongest such results as Fujita conjecture-type statements, which are then shown to govern a range of fundamental properties of direct images of pluricanonical and pluriadjoint line bundles, like effective vanishing theorems, weak positivity, or generic vanishing.

1. Introduction

The purpose of this paper is twofold: on the one hand we show that techniques inspired by Kollár and Viehweg's study of weak positivity, combined with vanishing theorems for log-canonical pairs, lead to new consequences regarding generation and vanishing properties for direct images of pluricanonical bundles. On the other hand, we formulate the strongest such results as Fujita conjecture-type statements, which are then shown to govern a range of fundamental properties of direct images of pluricanonical and pluriadjoint line bundles, like effective vanishing theorems, weak positivity, or generic vanishing.

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Vanishing, regularity, and Fujita-type statements. All varieties we consider in this paper are defined over an algebraically closed field of characteristic zero. Recall to begin with the following celebrated conjecture.

Conjecture 1.1 (Fujita). If X is a smooth projective variety of dimension n, and L is an ample line bundle on X, then $\omega_X \otimes L^{\otimes l}$ is globally generated for $l \geq n+1$.

It is well known that Fujita's conjecture holds in the case when L is ample and globally generated, based on Kodaira vanishing and the theory of Castelnuovo–Mumford regularity, and that this can be extended to the relative setting as follows:

Proposition 1.2 (Kollár). *Let $f : X \to Y$ be a morphism of projective varieties, with X smooth and Y of dimension n. If L is an ample and globally generated line bundle on Y, then*

$$R^i f_*\omega_X \otimes L^{\otimes n+1}$$

is 0*-regular, and therefore globally generated for all i.*

Recall that a sheaf $\mathscr{F}$ on Y is 0*-regular* with respect to an ample and globally generated line bundle L if

$$H^i(Y, \mathscr{F} \otimes L^{\otimes -i}) = 0 \quad \text{for all } i > 0.$$

The Castelnuovo–Mumford Lemma says that every 0-regular sheaf is globally generated (see, e.g., [Lazarsfeld 2004a, Theorem 1.8.3]); the proposition is then a consequence of Kollár's vanishing theorem, recalled as Theorem 2.2 below.

An extension of Fujita's general conjecture to the relative case was formulated by Kawamata [1982, Conjecture 1.3], and proved in dimension up to four; the statement is that Proposition 1.2 should remain true for any L ample, at least as long as the branch locus of the morphism f is a divisor with simple normal crossings support (when the sheaves $R^i f_*\omega_X$ are locally free [Kollár 1986]). However, at least for $i = 0$, we propose the following unconditional extension of Conjecture 1.1:

Conjecture 1.3. Let $f : X \to Y$ be a morphism of smooth projective varieties, with Y of dimension n, and let L be an ample line bundle on Y. Then, for every $k \geq 1$, the sheaf

$$f_*\omega_X^{\otimes k} \otimes L^{\otimes l}$$

is globally generated for $l \geq k(n+1)$.

Our main result in this direction is a proof of a stronger version of Conjecture 1.3 in the case of ample and globally generated line bundles, generalizing Proposition 1.2 for $i = 0$ to arbitrary powers.

Theorem 1.4. *Let $f : X \to Y$ be a morphism of projective varieties, with X smooth and Y of dimension n. If L is an ample and globally generated line bundle on Y, and $k \geq 1$ an integer, then*

$$f_*\omega_X^{\otimes k} \otimes L^{\otimes l}$$

is 0-regular, and therefore globally generated, for $l \geq k(n+1)$.

This follows in fact from a more general effective vanishing theorem for direct images of powers of canonical bundles, which is Kollár vanishing when $k = 1$; see Corollary 2.9. We also observe in Proposition 2.13 that just knowing the (klt version of the) Fujita-type Conjecture 1.3 for $k = 1$ would imply a similar vanishing theorem when L is only ample. Using related methods, we find analogous statements in the contexts of pluriadjoint bundles and of log-canonical pairs as well. We will call a *fibration* a surjective morphism whose general fiber is irreducible.

Variant 1.5. *Let $f : X \to Y$ be a fibration between projective varieties, with X smooth and Y of dimension n. Let M be a nef and f-big line bundle on X. If L is an ample and globally generated line bundle on Y, and $k \geq 1$ an integer, then*

$$f_*(\omega_X \otimes M)^{\otimes k} \otimes L^{\otimes l}$$

is 0-regular, and therefore globally generated, for $l \geq k(n+1)$.

Variant 1.6. *Let $f : X \to Y$ be a morphism of projective varieties, with X normal and Y of dimension n, and consider a log-canonical $\mathbb{R}$-pair (X, Δ) on X. Consider a line bundle B on X such that $B \sim_{\mathbb{R}} k(K_X + \Delta + f^*H)$ for some $k \geq 1$, where H is an ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on Y. If L is an ample and globally generated line bundle on Y, and $k \geq 1$ an integer, then*

$$f_*B \otimes L^{\otimes l} \quad \textit{with } l \geq k(n+1-t)$$

is 0-regular, and so globally generated, where $t := \sup\{s \in \mathbb{R} \mid H - sL \text{ is ample}\}$.[1]

All of these results are consequences of our main technical result, stated next. It can be seen both as an effective vanishing theorem for direct images of powers, and as a partial extension of Ambro–Fujino vanishing (recalled as Theorem 2.3 below) to arbitrary log-canonical pairs.

Theorem 1.7. *Let $f : X \to Y$ be a morphism of projective varieties, with X normal and Y of dimension n, and consider a log-canonical $\mathbb{R}$-pair (X, Δ) on X. Consider a line bundle B on X such that $B \sim_{\mathbb{R}} k(K_X + \Delta + f^*H)$ for some $k \geq 1$, where H*

[1] This is of course a generalization of Theorem 1.4. We chose to state it separately in order to preserve the simplicity of the main point, as will be done a few times throughout the paper.

is an ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on Y. If L is an ample and globally generated line bundle on Y, then

$$H^i(Y, f_*B \otimes L^{\otimes l}) = 0 \quad \textit{for all } i > 0 \textit{ and } l \geq (k-1)(n+1-t) - t + 1,$$

where $t := \sup\{s \in \mathbb{R} \mid H - sL \textit{ is ample}\}$.

The proof of this result relies on a variation of a method used by Viehweg in the study of weak positivity, and on the use of the Ambro–Fujino vanishing theorem. Shifting emphasis from weak positivity to vanishing turns out to lead to stronger statements, as was already pointed out by Kollár [1986, §3] in the case $k = 1$; his point of view, essentially based on regularity, is indeed a crucial ingredient in the applications.

One final note in this regard is that all the vanishing theorems used in the paper hold for higher direct images as well. At the moment we do not know, however, how to obtain statements similar to those above for higher direct images, for instance for $R^i f_*\omega_X^{\otimes k}$ with $i > 0$.

Applications. The Fujita-type statements in Theorem 1.4 and its variants turn out to govern a number of fundamental properties of direct images of pluricanonical and pluriadjoint bundles. Besides the vanishing statements discussed above, we sample a few here, and refer to the main body of the paper for full statements. To begin with, we deduce in Section 4 an effective version of Viehweg's weak positivity theorem for sheaves of the form $f_*\omega_{X/Y}^{\otimes k}$ for arbitrary $k \geq 1$, just as Kollár did in the case $k = 1$; we leave the rather technical statement, Theorem 4.2, for the main text. The same method applies to pluriadjoint bundles (see Theorem 4.4) and in this case even the noneffective weak positivity consequence stated below is new. The case $k = 1$ is again due to Kollár and Viehweg; see also [Höring 2010, Theorem 3.30] for a nice exposition.

Theorem 1.8. *If $f : X \to Y$ is a fibration between smooth projective varieties, and M is a nef and f-big line bundle on X, then $f_*(\omega_{X/Y} \otimes M)^{\otimes k}$ is weakly positive for every $k \geq 1$.*

With this result at hand, Viehweg's machinery for studying Iitaka's conjecture can be applied to deduce the adjoint bundle analogue of his result on the additivity of the Kodaira dimension over a base of general type.

Theorem 1.9. *Let $f : X \to Y$ be a fibration between smooth projective varieties, and let M be a nef and f-big line bundle on X. We denote by F the general fiber of f, and by M_F the restriction of M to F. Then:*

(i) *If L is an ample line bundle on Y, and $k > 0$, then*

$$\kappa((\omega_X \otimes M)^{\otimes k} \otimes f^*L) = \kappa(\omega_F \otimes M_F) + \dim Y.$$

(ii) *If Y is of general type*, *then*

$$\kappa(\omega_X \otimes M) = \kappa(\omega_F \otimes M_F) + \dim Y.$$

In a different direction, the method involved in the proof of Theorem 1.4 (more precisely Corollary 2.9) leads to a generic vanishing statement for pluricanonical bundles. Let $f : X \to A$ be a morphism from a smooth projective variety to an abelian variety. Hacon [2004] showed that the higher direct images $R^i f_* \omega_X$ satisfy generic vanishing, i.e., are GV-sheaves on A; see Definition 5.1. This refines the well-known generic vanishing theorem of Green and Lazarsfeld [1987], and is crucial in studying the birational geometry of irregular varieties. In Section 5 we deduce the following statement, which is somewhat surprising given our previous knowledge about the behavior of powers of ω_X.

Theorem 1.10. *If $f : X \to A$ is a morphism from a smooth projective variety to an abelian variety*, *then $f_* \omega_X^{\otimes k}$ is a GV-sheaf for every $k \geq 1$.*

We also present a self-contained proof of this theorem based on an effective result, Proposition 5.2, which is weaker than Corollary 2.9, but has a more elementary proof of independent interest. Theorem 1.10 leads in turn to vanishing and generation consequences that are stronger than those for morphisms to arbitrary varieties; see Corollary 5.4. Similar statements are given for log-canonical pairs and for adjoint bundles in Variants 5.5 and 5.6.

2. Vanishing and freeness for direct images of pluri-log-canonical bundles

In this section we address results related to Conjecture 1.3, via vanishing theorems for direct images of pluricanonical bundles. The most general result we prove is for log-canonical pairs; this is of interest from a different perspective as well, as it partially extends a vanishing theorem of Ambro and Fujino.

Motivation and background. To motivate the main technical result, recall that, given an ample line bundle L on a smooth projective variety of dimension n, Conjecture 1.1 implies that $\omega_X \otimes L^{\otimes n+1}$ is a nef line bundle; this is in fact follows unconditionally from the fundamental theorems of the minimal model program. As a consequence, Kodaira vanishing implies that for every $k \geq 1$ one has

$$H^i(X, \omega_X^{\otimes k} \otimes L^{\otimes k(n+1)-n}) = 0 \quad \text{for all } i > 0, \tag{2.1}$$

an effective vanishing theorem for powers of ω_X.

We will look for similar results for direct images. Recall first that for $k = 1$ there is a well-known analogue of Kodaira vanishing, for all higher direct images.

Theorem 2.2 (Kollár vanishing [Kollár 1986, Theorem 2.1]). *Let $f : X \to Y$ be a morphism of projective varieties*, *with X smooth. If L is an ample line bundle on Y*, *then*

$$H^j(Y, R^i f_*\omega_X \otimes L) = 0 \quad \textit{for all } i \textit{ and all } j > 0.$$

Moreover, of great use for the minimal model program are extensions of vanishing and positivity theorems to the log situation, in particular to log-canonical pairs; see, e.g., [Fujino 2011; 2014b]. For instance, Kollár's theorem above has an extension to this situation due to Ambro and Fujino; see, e.g., [Ambro 2003, Theorem 3.2] and [Fujino 2011, Theorem 6.3] (where a relative version can be found as well).

Theorem 2.3 (Ambro and Fujino). *Let $f : X \to Y$ be a morphism between projective varieties, with X smooth and Y of dimension n. Let (X, Δ) be a log-canonical log-smooth $\mathbb{R}$-pair,*[2] *and consider a line bundle B on X such that $B \sim_{\mathbb{R}} K_X + \Delta + f^*H$, where H is an ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on Y. Then*

$$H^j(Y, R^i f_*B) = 0 \quad \textit{for all } i \textit{ and all } j > 0.$$

Just as Proposition 1.2 follows from Theorem 2.2 via the Castelnuovo–Mumford Lemma, so Theorem 2.3 has the following consequence:

Lemma 2.4. *Under the hypotheses of Theorem 2.3, consider in addition an ample and globally generated line bundle L on Y. Then*

$$R^i f_*B \otimes L^{\otimes n}$$

is 0*-regular, and therefore globally generated, for all i.*

Main technical result. We will now prove our main vanishing theorem, which can be seen as an extension of both vanishing of type (2.1) for powers of canonical bundles, for L ample and globally generated, and of Ambro–Fujino vanishing (Theorem 2.3) to log-canonical pairs with arbitrary Cartier index.

Proof of Theorem 1.7. Step 1. We will first show that we can reduce to the case when X is smooth, Δ has simple normal crossings support, and the image of the adjunction morphism

$$f^*f_*B \longrightarrow B$$

is a line bundle. *A priori* the image is $\mathfrak{b} \otimes B$, where $\mathfrak{b}$ is the relative base ideal of B. We consider a birational modification

$$\mu : \widetilde{X} \longrightarrow X$$

which is a common log-resolution of $\mathfrak{b}$ and (X, Δ). On $\widetilde{X}$ we can write

$$K_{\widetilde{X}} - \mu^*(K_X + \Delta) = P - N,$$

[2] This means that Δ is an effective $\mathbb{R}$-divisor with simple normal crossings support, and with the coefficient of each component at most equal to 1.

where P and N are effective $\mathbb{R}$-divisors with simple normal crossings support, without common components, and such that P is exceptional and all coefficients in N are at most 1. We consider the line bundle

$$\widetilde{B} := \mu^* B \otimes \mathcal{O}_{\widetilde{X}}(k\lceil P \rceil).$$

Note that by definition we have

$$\widetilde{B} \sim_{\mathbb{R}} k(K_{\widetilde{X}} + N + \lceil P \rceil - P + \mu^* f^* H).$$

Since $\lceil P \rceil$ is μ-exceptional, we have $\mu_* \widetilde{B} \simeq B$ for all k. Moreover,

$$\Delta_{\widetilde{X}} := N + \lceil P \rceil - P$$

is log-canonical with simple normal crossings support on $\widetilde{X}$.

Going back to the original notation, we can thus assume that X is smooth and Δ has simple normal crossings support, and the image sheaf of the adjunction morphism is of the form $B \otimes \mathcal{O}_X(-E)$ for a divisor E such that $E + \Delta$ has simple normal crossings support.

Step 2. Now since L is ample, there is a smallest integer $m \geq 0$ such that $f_* B \otimes L^{\otimes m}$ is globally generated, and so using the adjunction morphism we have that $B \otimes \mathcal{O}_X(-E) \otimes f^* L^{\otimes m}$ is globally generated as well. We can then write

$$B \otimes f^* L^{\otimes m} \simeq \mathcal{O}_X(D + E),$$

where D is an irreducible smooth divisor, not contained in the support of $E + \Delta$, and such that $D + E + \Delta$ has simple normal crossings support. Rewriting this in divisor notation, we have

$$k(K_X + \Delta + f^* H) + m f^* L \sim_{\mathbb{R}} D + E,$$

and hence

$$(k-1)(K_X + \Delta + f^* H) \sim_{\mathbb{R}} \frac{k-1}{k} D + \frac{k-1}{k} E - \frac{k-1}{k} \cdot m f^* L. \tag{2.5}$$

Note that Δ and E may have common components in their support, which may cause trouble later on; therefore their coefficients need to be adjusted conveniently. Let's start by writing

$$\Delta = \sum_{i=1}^{l} a_i D_i, \quad a_i \in \mathbb{R} \text{ with } 0 < a_i \leq 1,$$

and

$$E = \sum_{i=1}^{l} s_i D_i + E_1, \quad s_i \in \mathbb{N},$$

where the support of E_1 and that of Δ have no common components.

Observe now that for every effective Cartier divisor $E' \preceq E$ we have

$$f_*(B \otimes \mathcal{O}_X(-E')) \simeq f_*B. \tag{2.6}$$

Indeed, it is enough to have this for E itself; but this is the base locus of B relative to f, so by construction we have

$$f^*f_*B \to B \otimes \mathcal{O}_X(-E) \hookrightarrow B,$$

and so the isomorphism follows by pushing forward to get the commutative diagram

$$\begin{tikzcd} f_*B \arrow[r] \arrow[rr, bend left, "\mathrm{id}"] & f_*(B \otimes \mathcal{O}_X(-E)) \arrow[r, hook] & f_*B. \end{tikzcd}$$

Define now

$$\gamma_i := a_i + \frac{k-1}{k} \cdot s_i \quad \text{for } i = 1, \ldots, l.$$

We claim that we can find for each i an integer b_i such that

$$0 \le \gamma_i - b_i \le 1 \quad \text{and} \quad 0 \le b_i \le s_i.$$

This is the same as $\gamma_i - 1 \le b_i \le \gamma_i$, while on the other hand, $\gamma_i < 1 + s_i$, so it is clear that such integers exist. We define

$$E' := \sum_{i=1}^{l} b_i D_i + \left\lfloor \frac{k-1}{k} E_1 \right\rfloor \preceq E,$$

and for this divisor (2.6) applies.

Step 3. Using (2.5), for any integer l we can now write

$$B - E' + lf^*L \sim_{\mathbb{R}} K_X + \Delta + \frac{k-1}{k}E - E' + \frac{k-1}{k}D + f^*\Big(H + \Big(l - \frac{k-1}{k} \cdot m\Big)L\Big).$$

We first note that the $\mathbb{R}$-divisor

$$H' := H + \Big(l - \frac{k-1}{k} \cdot m\Big)L$$

on Y is ample provided $l + t - ((k-1)/k)m > 0$. On the other hand, the effective $\mathbb{R}$-divisor with simple normal crossings support

$$\Delta' := \Delta + \frac{k-1}{k}E - E' + \frac{k-1}{k}D$$

on X is log-canonical. Indeed, the only coefficients that could cause trouble are those of the D_i. Note however that these are equal to $\gamma_i - b_i$, which are between 0 and 1 by our choice of b_i. Putting everything together, it means that on X (which is now smooth) we have written

$$B - E' + lf^*L \sim_{\mathbb{R}} K_X + \Delta' + f^*H',$$

where Δ' is log-canonical with simple normal crossings support, and H' is ample on Y. The pushforward of the left-hand side is $f_*B \otimes L^{\otimes l}$, while for the right-hand side we can now apply Theorem 2.3 to conclude that

$$H^i(Y, f_*B \otimes L^{\otimes l}) = 0 \quad \text{for all } i > 0 \text{ and } l > \frac{k-1}{k} \cdot m - t. \tag{2.7}$$

We therefore have that for every $l > ((k-1)/k)m - t + n$ the sheaf $f_*B \otimes L^{\otimes l}$ is 0-regular, hence globally generated. Given our minimal choice of m, we conclude that for the smallest integer l_0 which is greater than $((k-1)/k)m - t$ we have $m \leq l_0 + n$. This implies

$$m \leq l_0 + n \leq \frac{k-1}{k} \cdot m + n + 1 - t,$$

which is equivalent to $m \leq k(n+1-t)$, and in particular the vanishing in (2.7) holds for

$$l \geq (k-1)(n+1-t) - t + 1. \qquad \square$$

Note that the inequality $m \leq k(n+1-t)$ obtained above implies the statement of Variant 1.6. Just as with the statement of Theorem 1.7 compared to that of the Ambro–Fujino Theorem 2.3, one notes that, even for $k = 1$, Variant 1.6 is slightly more general than the case $i = 0$ in Lemma 2.4. This is not surprising, but particular to zeroth direct images, as after passing to a log-resolution of the log-canonical pair there is no need to appeal to local vanishing for higher direct images; the Ambro–Fujino theorem and the lemma cannot be stated in this form in the case $i > 0$.

Special cases. We spell out the most important special cases of Theorem 1.7. They are obtained by taking $H = L$ in the statement of Theorem 1.7, so that $t = 1$.

Corollary 2.8. *Let $f : X \to Y$ be a morphism of projective varieties, with X normal and Y of dimension n. Consider a log-canonical pair (X, Δ) and an integer $k > 0$ such that $k(K_X + \Delta)$ is Cartier. If L is an ample and globally generated line bundle on Y, then*

$$H^i(Y, f_*\mathcal{O}_X(k(K_X+\Delta)) \otimes L^{\otimes l}) = 0 \quad \textit{for all } i > 0 \textit{ and } l \geq k(n+1) - n.$$

In particular, we have an extension of (2.1) to direct images, and of Proposition 1.2 for $i = 0$ to arbitrary k:

Corollary 2.9. *Let $f : X \to Y$ be a morphism of projective varieties, with X smooth and Y of dimension n. If L is an ample and globally generated line bundle on Y, and $k > 0$ is an integer, then*

$$H^i(Y, f_*\omega_X^{\otimes k} \otimes L^{\otimes l}) = 0 \quad \textit{for all } i > 0 \textit{ and } l \geq k(n+1) - n.$$

Remark. Note that if we perform the proof of Theorem 1.7 only in the "classical" case considered in Corollary 2.9, the second step is unnecessary since $\Delta = 0$, while Δ' in the third step is klt. This means that one does not need to appeal to the Ambro–Fujino vanishing theorem, but rather to the klt version of Theorem 2.2, still due to Kollár; see for instance [Kollár 1995, Theorem 10.19].

The regularity statement in the introduction is an immediate consequence.

Proof of Theorem 1.4. We note that $k(n+1) = k(n+1) - n + n$, and apply the vanishing statement in Corollary 2.9 by successively subtracting n powers of L. $\square$

A rephrasing of Theorem 1.4 is a useful uniform global generation statement involving powers of relative canonical bundles.

Corollary 2.10. *Let $f : X \to Y$ be a morphism of smooth projective varieties, with Y of dimension n. If L is an ample and globally generated line bundle on Y, $k \geq 1$ an integer, and $A := \omega_Y \otimes L^{\otimes n+1}$, then*

$$f_*\omega_{X/Y}^{\otimes k} \otimes A^{\otimes k}$$

is globally generated.

Question. The arguments leading to Corollary 2.9, and more generally Theorem 1.7 and its applications, do not extend to higher direct images. It is natural to ask, however, whether the statements do hold for all $R^i f_*\omega_X^{\otimes k}$ and analogues, just as Theorem 2.2 and Theorem 2.3 do.

Example: the main conjecture over curves. We record one case when the main Fujita-type Conjecture 1.3 can be shown to hold, namely when the base of the morphism has dimension one. This is not hard to check, but it uses important special facts about vector bundles on curves.

Proposition 2.11. *Let $f : X \to C$ be a morphism of smooth projective varieties, with C a curve, and let L be an ample line bundle on C. Then, for every $k \geq 1$, the vector bundle*

$$f_*\omega_X^{\otimes k} \otimes L^{\otimes m}$$

is globally generated for $m \geq 2k$.

Proof. First, note that the sheaf in question is locally free (since C is a curve). We can rewrite it as

$$f_*\omega_X^{\otimes k} \otimes L^{\otimes m} \simeq f_*\omega_{X/C}^{\otimes k} \otimes \omega_C^{\otimes k} \otimes L^{\otimes m}.$$

Now Theorem 1 of [Kawamata 2002] says that $f_*\omega_{X/C}^{\otimes k}$ is a semipositive vector bundle on C, while

$$\deg \omega_C^{\otimes k} \otimes L^{\otimes m} \geq k(2g-2) + m \deg L \geq 2g,$$

with g the genus of C, as $\deg L > 0$. The statement then follows from the following general result. □

Lemma 2.12. *Let E be a semipositive vector bundle and L a line bundle of degree at least $2g$ on a smooth projective curve C of genus g. Then $E \otimes L$ is globally generated.*

Proof. It is enough to show that, for every $p \in C$, one has

$$H^1(C, E \otimes L \otimes \mathscr{O}_C(-p)) = 0,$$

or equivalently, by Serre duality, that there are no nontrivial homomorphisms

$$E \longrightarrow \omega_C \otimes \mathscr{O}_C(p) \otimes L^{-1}.$$

But the semipositivity of E means precisely that it cannot have any quotient line bundle of negative degree. □

Remark. For curves of genus at least 1, the argument in Proposition 2.11 shows, in fact, that $f_*\omega_X^{\otimes k} \otimes L^{\otimes 2}$ is always globally generated.

Relative Fujita conjecture and vanishing for ample line bundles. It is worth observing that it suffices to know Conjecture 1.3 and its variants for $k = 1$ in order to obtain vanishing theorems for twists by line bundles that are assumed to be just ample, and not necessarily globally generated. For simplicity we spell out only the case of pluricanonical bundles, i.e., the analogue of Corollary 2.9.

Proposition 2.13. *Assume that Conjecture 1.3 holds for $k = 1$.*[3] *Then for any morphism $f : X \to Y$ of smooth projective varieties with Y of dimension n, any ample line bundle L on Y, and any integer $k \geq 2$, one has*

$$H^i(Y, f_*\omega_X^{\otimes k} \otimes L^{\otimes l}) = 0 \quad \textit{for all } i > 0 \textit{ and } l \geq k(n+1) - n.$$

[3] Or, more precisely, its klt version: if Y is smooth of dimension n, L is ample on Y, and (X, Δ) is a klt pair such that $B = K_X + \Delta + \alpha f^*L$ is Cartier for some $\alpha \in \mathbb{R}$, then $f_*B \otimes L^{\otimes l}$ is globally generated for any $l + \alpha \geq n + 1$.

Proof. This is a corollary of the proof of Corollary 2.9. Indeed, the only time we used that L is globally generated and not just ample was to deduce the global generation of a sheaf of the form $f_*B \otimes L^{\otimes l}$ from its 0-regularity with respect to L, where B is $\mathbb{Q}$-linearly equivalent to something of the form $K_X + \Delta + \alpha f^*L$ with (X, Δ) klt and $\alpha \in \mathbb{Q}$; see also the remark on page 2282. The klt version of Conjecture 1.3 for $k = 1$ would then serve as a replacement. □

A natural version of Conjecture 1.3 can be stated in the log-canonical case,[4] with the same effect regarding the result of Theorem 1.7, but this would take us far beyond what is currently known.

3. Vanishing and freeness for direct images of pluriadjoint bundles

We now switch our attention to direct images of powers of line bundles of the form $\omega_X \otimes M$, where M is a nef and relatively big line bundle. Recall first that Proposition 1.2 has the following analogue:

Proposition 3.1. *Let $f : X \to Y$ be a fibration between projective varieties, with X smooth and Y of dimension n. Consider a nef and f-big line bundle M on X, and (X, Δ) a klt pair with Δ an $\mathbb{R}$-divisor with simple normal crossings support. If B is a line bundle on X such that $B \sim_{\mathbb{R}} K_X + M + \Delta + f^*H$ for some ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor H on Y, then*

$$H^i(Y, f_*B) = 0 \quad \text{for all } i > 0.$$

In particular, if L is an ample and globally generated line bundle on Y, then

$$f_*B \otimes L^{\otimes n}$$

is 0-regular, and therefore globally generated.

Proof. We include the well-known proof for completeness, as it is usually given in the case $\Delta = 0$ (see, e.g., [Höring 2010, Lemma 3.28]). Note first that $M + f^*H$ continues to be a nef and f-big $\mathbb{R}$-divisor on X. The local version of the Kawamata–Viehweg vanishing theorem (see [Lazarsfeld 2004b, Remarks 9.1.22 and 9.1.23]) applies then to give

$$R^i f_*B = 0 \quad \text{for all } i > 0.$$

We conclude that it is enough to show

$$H^i(X, B) = 0 \quad \text{for all } i > 0.$$

This will follow from the global $\mathbb{R}$-version of Kawamata–Viehweg vanishing as soon as we show that $M + f^*H$ is in fact a big divisor. Since it is nef, it suffices

[4] In the absolute case, an Angehrn–Siu type statement has been obtained by Kollár [1997, Theorem 5.8] in the klt case, and further extended by Fujino [2010, Theorem 1.1] to the log-canonical setting.

to check that $(M+f^*H)^m>0$, where $m=\dim X$. Now $(M+f^*H)^m$ is a linear combination with positive coefficients of terms of the form

$$M^s\cdot f^*H^{m-s},$$

which are all nonnegative. Moreover, since M is f-big, the term $M^{m-n}\cdot f^*H^n$ is strictly positive, which gives the conclusion. □

We now prove an analogue of Corollary 2.9 in this context. Just as with Theorem 1.4, Variant 1.5 is its immediate consequence.

Theorem 3.2. *Let $f: X\to Y$ be a fibration between projective varieties, with X smooth and Y of dimension n. Let M be a nef and f-big line bundle on X. If L is an ample and globally generated line bundle on Y, and $k\geq 1$ an integer, then*

$$H^i(Y,\, f_*(\omega_X\otimes M)^{\otimes k}\otimes L^{\otimes l})=0\quad \textit{for all } i>0 \textit{ and } l\geq k(n+1)-n.$$

Proof. The strategy is similar to that of the proof of Theorem 1.7, so we will be brief in some of the steps. We consider the minimal $m\geq 0$ such that $f_*(\omega_X\otimes M)^{\otimes k}\otimes L^{\otimes m}$ is globally generated. Using the adjunction morphism

$$f^*f_*(\omega_X\otimes M)^{\otimes k}\to(\omega_X\otimes M)^{\otimes k},$$

after possibly blowing up, we can write

$$(\omega_X\otimes M)^{\otimes k}\otimes f^*L^{\otimes m}\simeq\mathscr{O}_X(D+E),$$

with D smooth and $D+E$ a divisor with simple normal crossings support. In divisor notation, we obtain

$$K_X+M\sim_{\mathbb{Q}}\frac{1}{k}D+\frac{1}{k}E-\frac{m}{k}f^*L. \tag{3.3}$$

For any integer $l\geq 0$, using (3.3) we can then write the equivalence

$$\begin{aligned} &k(K_X+M)-\Big\lfloor\frac{k-1}{k}E\Big\rfloor+lf^*L\\ &\quad=K_X+M+(k-1)(K_X+M)-\Big\lfloor\frac{k-1}{k}E\Big\rfloor+lf^*L\\ &\quad\sim_{\mathbb{Q}}K_X+M+\Delta+\Big(l-\frac{k-1}{k}\cdot m\Big)f^*L,\end{aligned}$$

where

$$\Delta=\frac{k-1}{k}D+\frac{k-1}{k}E-\Big\lfloor\frac{k-1}{k}E\Big\rfloor$$

is a boundary divisor with simple normal crossings support. Since E is the base divisor of $(\omega_X\otimes M)^{\otimes k}$ relative to f, just as in the proof of Theorem 1.7 it follows that

$$f_*\mathscr{O}_X\Big(k(K_X+M)-\Big\lfloor\frac{k-1}{k}E\Big\rfloor+lf^*L\Big)\simeq f_*(\omega_X\otimes M)^{\otimes k}\otimes L^{\otimes l}.$$

On the other hand, on the right-hand side we can apply Proposition 3.1, to deduce that

$$H^i(Y, f_*(\omega_X \otimes M)^{\otimes k} \otimes L^{\otimes l}) = 0 \quad \text{for all } i > 0 \text{ and } l > \frac{k-1}{k} \cdot m.$$

We conclude that $f_*(\omega_X \otimes M)^{\otimes k} \otimes L^{\otimes l}$ is globally generated for $l > ((k-1)/k)m + n$. Since m was chosen minimal, we conclude as in Theorem 1.7 that $m \leq k(n+1)$, and that vanishing holds for all $l \geq k(n+1) - n$. □

Remark. Fujita's conjecture and all similar statements have more refined numerical versions, replacing $L^{\otimes n+1}$ by any ample line bundle A such that $A^{\dim V} \cdot V > (\dim V)^{\dim V}$ for any subvariety $V \subseteq X$. Similarly, the analogues of Conjecture 1.3 and Proposition 2.13 make sense replacing ω_X by $\omega_X \otimes M$ as well.

4. Effective weak positivity, and additivity of adjoint Iitaka dimension

Recall the following fundamental definition (see, e.g., [Viehweg 1983, §1]):

Definition 4.1. A torsion-free coherent sheaf $\mathscr{F}$ on a projective variety X is *weakly positive* on a nonempty open set $U \subseteq X$ if for every ample line bundle A on X and every $a \in \mathbb{N}$, the sheaf $S^{[ab]}\mathscr{F} \otimes A^{\otimes b}$ is generated by global sections at each point of U for b sufficiently large. (Here $S^{[p]}\mathscr{F}$ denotes the reflexive hull of the symmetric power $S^p\mathscr{F}$.) As noted in [Viehweg 1983, Remark 1.3], it is not hard to see that it is enough to check this definition for a fixed line bundle A.

Kollár [1986, §3] introduced an approach to proving the weak positivity of sheaves of the form $f_*\omega_{X/Y}$ based on his vanishing theorem for $f_*\omega_X$, which in particular gives effective statements. Here we first provide a complement to Kollár's result, using Theorem 1.4, in order to make this approach work for all $f_*\omega_{X/Y}^{\otimes k}$ with $k \geq 1$. Concretely, below is the analogue of [Kollár 1986, Theorem 3.5(i)]; the proof is very similar, and we only sketch it for convenience.

Theorem 4.2. *Let $f : X \to Y$ be a surjective morphism of smooth projective varieties, with generically reduced fibers in codimension one.*[5] *Let L be an ample and globally generated line bundle on Y, and $A = \omega_Y \otimes L^{\otimes n+1}$, where $n = \dim Y$. Then for every $s \geq 1$, the sheaf*

$$f_*(\omega_{X/Y}^{\otimes k})^{[\otimes s]} \otimes A^{\otimes k}$$

is globally generated over a fixed open set U containing the smooth locus of f; here $f_*(\omega_{X/Y}^{\otimes k})^{[\otimes s]}$ denotes the reflexive hull of $f_*(\omega_{X/Y}^{\otimes k})^{\otimes s}$.

[5] This means that there exists a closed subset $Z \subset Y$ of codimension at least two such that over $Y - Z$ the fibers of f are generically reduced. This condition is realized for instance if there is such a Z such that over $Y - Z$ the branch locus of f is smooth, and its preimage is a simple normal crossings divisor; see [Kollár 1986, Lemma 3.4].

Proof. As in [Viehweg 1983, §3] and in the proof of [Kollár 1986, Theorem 3.5], based on Viehweg's fiber product construction one can show that there is an open set $U \subset Y$, whose complement $Y - U$ has codimension at least two, over which there exists a morphism

$$\varphi : f_*^{(s)}(\omega_{X^{(s)}/Y}^{\otimes k}) \longrightarrow f_*(\omega_{X/Y}^{\otimes k})^{[\otimes s]}$$

which is an isomorphism over the smooth locus of f. Here $\mu : X^{(s)} \to X^s$ is a desingularization of the unique irreducible component X^s of the s-fold fiber product of X over Y which dominates Y; we have natural morphisms $f^s : X^s \to Y$ and $f^{(s)} = f^s \circ \mu : X^{(s)} \to Y$. The reason one can do this for any $k \geq 1$ is this: the hypothesis on the morphism implies that X^s is normal and Gorenstein over such a U (contained in the flat locus of f) with complement of small codimension; see also [Höring 2010, Lemma 3.12]. In particular, for every $k \geq 1$ there is a morphism

$$t : \mu_*\omega_{X^{(s)}/Y}^{\otimes k} \longrightarrow \omega_{X^s/Y}^{\otimes k}$$

which induces φ.

Now, without changing the notation, we can pass to a compactification of $X^{(s)}$, and the morphism φ extends to a morphism of sheaves on Y, since it is defined in codimension one and the sheaf on the right is reflexive. Corollary 2.10 says that

$$f_*^{(s)}(\omega_{X^{(s)}/Y}^{\otimes k}) \otimes A^{\otimes k}$$

is globally generated for all s and k, which implies that

$$f_*(\omega_{X/Y}^{\otimes k})^{[\otimes s]} \otimes A^{\otimes k}$$

is generated by global sections over the locus where φ is an isomorphism. □

Corollary 4.3 [Viehweg 1983, Theorem III]. *If $f : X \to Y$ is a surjective morphism of smooth projective varieties, then $f_*\omega_{X/Y}^{\otimes k}$ is weakly positive for every $k \geq 1$.*

This follows in standard fashion from Theorem 4.2, by passing to semistable reduction along the lines of [Viehweg 1983, Lemma 3.2 and Proposition 6.1]. This was already noted by Kollár [1986, Corollary 3.7 and the preceding comments] in the case $k = 1$. As mentioned above, the theorem has the advantage of producing an effective bound, at least for sufficiently nice morphisms. We note also that Fujino [2014a] has used the argument above in order to deduce results on the semipositivity of direct images of pluricanonical bundles.

We now switch our attention to the context of direct images of adjoint line bundles of the form $\omega_X \otimes M$, where M is a nef and f-big line bundle for a fibration $f : X \to Y$. Given Theorem 3.2, we are now able to use the cohomological approach to weak positivity for higher powers of adjoint bundles as well. Concretely, Theorem 1.8

again follows via Viehweg's semistable reduction methods from the following analogue of the effective Theorem 4.2.

Theorem 4.4. *Let $f : X \to Y$ be a fibration between smooth projective varieties, with generically reduced fibers in codimension one. Let M be a nef and f-big line bundle on X, L an ample and globally generated line bundle on Y, and $A = \omega_Y \otimes L^{\otimes n+1}$ with $n = \dim Y$. Then*

$$f_*((\omega_{X/Y} \otimes M)^{\otimes k})^{[\otimes s]} \otimes A^{\otimes k}$$

is globally generated over a fixed nonempty open set U for any $s \geq 1$.

Proof. Using the notation in the proof of Theorem 4.2, over the same open subset $U \subset Y$ with complement of codimension at least two, one has a morphism which is generically an isomorphism:

$$\varphi : f_*^{(s)}((\omega_{X^{(s)}/Y} \otimes M^{(s)})^{\otimes k}) \longrightarrow f_*((\omega_{X/Y} \otimes M)^{\otimes k})^{[\otimes s]}. \tag{4.5}$$

Here $M^{(s)}$ is the line bundle on the desingularization $X^{(s)}$ defined inductively as

$$M^{(s)} := p_1^* M \otimes p_2^* M^{(s-1)},$$

with p_1 and p_2 the projections of $X^{(s)}$ to X and $X^{(s-1)}$, respectively. The morphism in (4.5) is obtained as a consequence of flatness and the projection formula; an excellent detailed discussion of the case $k = 1$, as well as of this whole circle of ideas, can be found in [Höring 2010, §3.D], in particular Lemma 3.15 and Lemma 3.24. The case $k > 1$ follows completely analogously, given the morphism t in the proof of Theorem 4.2.

Finally, Variant 1.5 immediately gives the analogue of Corollary 2.10 for twists by nef and relatively big line bundles, implying that $f_*^{(s)}((\omega_{X^{(s)}/Y} \otimes M^{(s)})^{\otimes k}) \otimes A^{\otimes k}$ is globally generated for all s and k. Combined with the reflexivity of the right-hand side, this leads to the desired conclusion. □

We conclude by noting that Corollary 4.3 has a natural extension to the setting of log-canonical pairs; see [Campana 2004, §4], and also [Fujino 2014b, §6]. It is an interesting and delicate problem to obtain an analogue of Theorem 4.2 in this setting as well.

Subadditivity of Iitaka dimension for adjoint bundles. Theorem 1.8 allows us to make use of an argument developed by Viehweg in order to provide the analogue in the adjoint setting of [Viehweg 1983, Corollary IV] on the subadditivity of Kodaira dimension for fibrations with base of general type.

Proof of Theorem 1.9. Note that the $\leq$ inequalities are consequences of the easy addition formula; see [Mori 1987, Corollary 1.7]. The proof of the reverse inequalities closely follows the ideas of Viehweg [1983] based on the use of weak positivity, as streamlined by Mori with the use of a result of Fujita; we include it below for completeness. Namely, we will apply the following lemma (but not directly for the line bundles on the left-hand side in (i) and (ii)).

Lemma 4.6 [Fujita 1977, Proposition 1; Mori 1987, Lemma 1.14]. *Let $f : X \to Y$ be a fibration with general fiber F, and N a line bundle on X. Then there exists a big line bundle L on Y and an integer $m > 0$ with $f^*L \hookrightarrow N^{\otimes m}$ if and only if*

$$\kappa(N) = \kappa(N_F) + \dim Y.$$

To make use of this, note first that, according to [Viehweg 1983, Lemma 7.3], there exists a smooth birational modification $\tau : Y' \to Y$ and a resolution X' of $X \times_Y Y'$ giving a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "\tau'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "\tau"] & Y \end{tikzcd}$$

with the property that every effective divisor B on X' that is exceptional for f' lies in the exceptional locus of τ'. Note that in this case $\tau'_*\omega_{X'}^{\otimes k}(kB) \simeq \omega_X^{\otimes k}$ for every $k \geq 0$. Also, τ'^*M is still nef and f'-big.

Fix now an ample line bundle L on Y, and consider the big line bundle $L' = \tau^*L$ on Y'. By Theorem 1.8 we have that for any $k > 0$ (which we can assume to be such that $f'_*(\omega_{X'/Y'} \otimes \tau'^*M)^{\otimes k} \neq 0$) there exists $b > 0$ such that

$$S^{[2b]} f'_*(\omega_{X'/Y'} \otimes \tau'^*M)^{\otimes k} \otimes L'^{\otimes b}$$

is generically globally generated. On the other hand, there exists an effective divisor B on X', exceptional for f', such that the reflexive hull of

$$f'_*(\omega_{X'/Y'} \otimes \tau'^*M)^{\otimes p}$$

is equal to

$$f'_*(\omega_{X'/Y'}(B) \otimes \tau'^*M)^{\otimes p}$$

for every $p \leq kb$. Using the nontrivial map induced by multiplication of sections on the fibers, we obtain that

$$f'_*(\omega_{X'/Y'}(B) \otimes \tau'^*M)^{\otimes 2kb} \otimes L'^{\otimes b}$$

has a nonzero section, and hence we obtain an inclusion

$$f'^*L'^{\otimes b} \hookrightarrow (\omega_{X'/Y'}(B) \otimes \tau'^*M)^{\otimes 2kb} \otimes f'^*L'^{\otimes 2b}.$$

According to Lemma 4.6, we obtain that

$$\kappa\big((\omega_{X'/Y'}(B)\otimes\tau'^{*}M)^{\otimes k}\otimes f'^{*}L'\big)=\kappa(\omega_{F'}\otimes(\tau'^{*}M)_{F'})+\dim Y'$$
$$=\kappa(\omega_F\otimes M_F)+\dim Y,$$

where F' is the general fiber of f'.

To deduce (i), note that, as we have observed that $\tau'_{*}\omega_{X'}^{\otimes k}(kB)\simeq\omega_X^{\otimes k}$, we have

$$\tau'_{*}\big((\omega_{X'/Y'}(B)\otimes\tau'^{*}M)^{\otimes k}\otimes f'^{*}L'\big)\simeq(\omega_{X/Y}\otimes M)^{\otimes k}\otimes f^{*}L.$$

To deduce (ii), since Y' is of general type, recall that by Kodaira's lemma there exists an inclusion $L'\hookrightarrow\omega_{Y'}^{\otimes r}$ for some $r>0$. This implies that

$$\kappa(\omega_X\otimes M)=\kappa(\omega_{X'}(B)\otimes\tau'^{*}M)\geq\kappa\big((\omega_{X'/Y'}(B)\otimes\tau'^{*}M)^{\otimes r}\otimes f'^{*}L'\big),$$

which is equal to $\kappa(\omega_F\otimes M_F)+\dim Y$ by the above. □

5. Generic vanishing for direct images of pluricanonical bundles

We concentrate now on the case of morphisms $f: X\to A$, where X is a smooth projective variety and A is an abelian variety. We denote by P the normalized Poincaré bundle on the product $A\times\mathrm{Pic}^0(A)$, and by P_α its restriction to the slice $A\times\{\alpha\}$; this is of course just a different name for the point $\alpha\in\mathrm{Pic}^0(A)$.

Definition 5.1 [Pareschi and Popa 2011a, Definition 3.1]. A coherent sheaf $\mathscr{F}$ on X is called a *GV-sheaf* (with respect to the given morphism f) if it satisfies

$$\operatorname{codim}\{\alpha\in\mathrm{Pic}^0(A)\mid H^k(X,\mathscr{F}\otimes f^{*}P_\alpha)\neq 0\}\geq k$$

for every $k\geq 0$.

If f is generically finite, then by a special case of the generic vanishing theorem of Green and Lazarsfeld [1987], ω_X is a GV-sheaf. This was generalized by Hacon [2004] to the effect that for an arbitrary f the higher direct images $R^i f_*\omega_X$ are GV-sheaves on A for all i. On the other hand, there exist simple examples showing that even when f is generically finite, the powers $\omega_X^{\otimes k}$ with $k\geq 2$ are not necessarily GV-sheaves; see [Pareschi and Popa 2011a, Example 5.6]. Therefore Theorem 1.10 in the introduction is a quite surprising application of the methods in this paper.

Proof of Theorem 1.10. Let M be a very high power of an ample line bundle on $\widehat{A}$, and let $\varphi_M:\widehat{A}\to A$ be the isogeny induced by M. According to a criterion of Hacon [2004, Corollary 3.1], the assertion will be proved if we manage to show that

$$H^i(\widehat{A},\varphi_M^{*}f_*\omega_X^{\otimes k}\otimes M)=0\quad\text{for all } i>0.$$

Equivalently, we need to show that

$$H^i(\widehat{A}, g_*\omega_{X_1}^{\otimes k} \otimes M) = 0 \quad \text{for all } i > 0,$$

where $g : X_1 \to \widehat{A}$ is the base change of $f : X \to A$ via φ_M. We can, however, perform another base change $\mu : \widehat{A} \to \widehat{A}$ by a multiplication map of large degree, such that $\mu^* M \simeq L^{\otimes d}$, where L is an ample line bundle, which we can also assume to be globally generated, and d is arbitrarily large. The situation is summarized in the diagram

$$\begin{tikzcd} X_2 \arrow[r] \arrow[d, "h"] & X_1 \arrow[r] \arrow[d, "g"] & X \arrow[d, "f"] \\ \widehat{A} \arrow[r, "\mu"] & \widehat{A} \arrow[r, "\varphi_M"] & A \end{tikzcd}$$

It is then enough to show that

$$H^i(\widehat{A}, h_*\omega_{X_2}^{\otimes k} \otimes L^{\otimes d}) = 0 \quad \text{for all } i > 0.$$

Note that we cannot apply Serre vanishing here, as all of our constructions depend on the original choice of M. However, we can conclude if we know that there exists a bound $d = d(n, k)$, i.e., depending only on $n = \dim A$ and k, such that the vanishing in question holds for any morphism h.

At this stage we can of course apply Corollary 2.9, which allows us to take $d \geq k(n+1) - n$. We stress, however, that as long as we know that such a uniform bound for d exists, for this argument its precise shape does not matter. We therefore choose to present below a weaker but more elementary result that does not need vanishing theorems for $\mathbb{Q}$-divisors, making the argument self-contained.

Indeed, Proposition 5.2 below shows that there exists a morphism $\varphi : Z \to \widehat{A}$ with Z smooth projective, and $m \leq n + k$, such that $h_*\omega_{X_2}^{\otimes k} \otimes L^{\otimes m(k-1)}$ is a direct summand in $\varphi_*\omega_Z$. Applying Kollár vanishing (Theorem 2.2), we deduce that

$$H^i(\widehat{A}, h_*\omega_{X_2}^{\otimes k} \otimes L^{\otimes d}) = 0 \quad \text{for all } \ i > 0 \text{ and all } d \geq (n+k)(k-1) + 1,$$

which suffices to conclude the proof. □

Proposition 5.2. *Let $f : X \to Y$ be a morphism of projective varieties, with X smooth and Y of dimension n. Let L be an ample and globally generated line bundle on Y, and $k \geq 1$ an integer. Then there exists a smooth projective variety Z with a morphism $\varphi : Z \to Y$, and an integer $0 \leq m \leq n + k$, such that $f_*\omega_X^{\otimes k} \otimes L^{\otimes m(k-1)}$ is a direct summand in $\varphi_*\omega_Z$.*

Proof. This is closer in spirit to the arguments towards weak positivity used in [Viehweg 1983, §5]. Note first that $f_*\omega_X^{\otimes k} \otimes L^{\otimes pk}$ is globally generated for some sufficiently large p. Denote by m the minimal $p \geq 0$ for which this is satisfied.

We are going to use a branched covering construction to show that $m \leq n+k$. First, consider the adjunction morphism

$$f^* f_* \omega_X^{\otimes k} \longrightarrow \omega_X^{\otimes k}.$$

After blowing up on X, if necessary, we can assume that the image sheaf is of the form $\omega_X^{\otimes k} \otimes \mathscr{O}_X(-E)$ for a divisor E with normal crossing support. As $f_* \omega_X^{\otimes k} \otimes L^{\otimes mk}$ is globally generated, we have that the line bundle

$$\omega_X^{\otimes k} \otimes f^* L^{\otimes mk} \otimes \mathscr{O}_X(-E)$$

is globally generated as well. It is therefore isomorphic to $\mathscr{O}_X(D)$, where D is an irreducible smooth divisor, not contained in the support of E, such that $D+E$ still has normal crossings. We have arranged that

$$(\omega_X \otimes f^* L^{\otimes m})^{\otimes k} \simeq \mathscr{O}_X(D+E),$$

and so we can take the associated covering of X branched along $D+E$ and resolve its singularities. This gives us a generically finite morphism $g : Z \to X$ of degree k, and we denote $\varphi = f \circ g : Z \to Y$.

Now by a well-known calculation of Esnault and Viehweg [Viehweg 1983, Lemma 2.3], the direct image $g_* \omega_Z$ contains the sheaf

$$\omega_X \otimes (\omega_X \otimes f^* L^{\otimes m})^{\otimes k-1} \otimes \mathscr{O}_X\Big(-\Big\lfloor \frac{k-1}{k}(D+E)\Big\rfloor\Big) \simeq \omega_X^{\otimes k} \otimes f^* L^{\otimes m(k-1)} \otimes \mathscr{O}_X\Big(-\Big\lfloor \frac{k-1}{k} E\Big\rfloor\Big)$$

as a direct summand. If we now apply f_*, we find that

$$f_*\Big(\omega_X^{\otimes k} \otimes \mathscr{O}_X\Big(-\Big\lfloor \frac{k-1}{k} E\Big\rfloor\Big)\Big) \otimes L^{\otimes m(k-1)} \tag{5.3}$$

is a direct summand of $\varphi_* \omega_Z$. At this point we observe, as in the proof of Theorem 1.7, that, since E is the relative base locus of $\omega_X^{\otimes k}$, we have

$$f_*\Big(\omega_X^{\otimes k} \otimes \mathscr{O}_X\Big(-\Big\lfloor \frac{k-1}{k} E\Big\rfloor\Big)\Big) \simeq f_* \omega_X^{\otimes k}.$$

In other words, $f_* \omega_X^{\otimes k} \otimes L^{\otimes m(k-1)}$ is a direct summand in $\varphi_* \omega_Z$. Applying Proposition 1.2, we deduce in turn that $f_* \omega_X^{\otimes k} \otimes L^{\otimes m(k-1)+n+1}$ is globally generated. By our minimal choice of m, this is only possible if

$$m(k-1) + n + 1 \geq (m-1)k + 1,$$

which is equivalent to $m \leq n+k$. □

Remark. With slightly more clever choices, the integer m in Proposition 5.2 can be chosen to satisfy $m \leq n+2$, but the effective vanishing consequence is still weaker than that obtained in Corollary 2.9. Note also that one can show analogous results in the case of log-canonical pairs and of adjoint bundles, with only small additional technicalities.

Going back to the case when the base is an abelian variety, once we know generic vanishing the situation is in fact much better than what we obtained for morphisms to arbitrary varieties.

Corollary 5.4. *If $f : X \to A$ is a morphism from a smooth projective variety to an abelian variety, for every ample line bundle L on A and every $k \geq 1$ one has*:

(i) $f_*\omega_X^{\otimes k}$ *is a nef sheaf on A.*

(ii) $H^i(A, f_*\omega_X^{\otimes k} \otimes L) = 0$ *for all $i > 0$.*

(iii) $f_*\omega_X^{\otimes k} \otimes L^{\otimes 2}$ *is globally generated.*

Proof. For (i), note that every GV-sheaf is nef by [Pareschi and Popa 2011b, Theorem 4.1]. Part (ii) follows from the more general fact that the tensor product of a GV-sheaf with an IT_0 locally free sheaf is IT_0; see [ibid., Proposition 3.1]. Finally, (iii) follows from [Pareschi and Popa 2003, Theorem 2.4], as by (ii) $f_*\omega_X^{\otimes k} \otimes L$ is an M-regular sheaf on A. □

Question. It is again natural to ask whether, given a morphism $f : X \to A$, the higher direct images $R^i f_*\omega_X^{\otimes k}$ are GV-sheaves for all i.

The exact same method, with appropriate technical modifications, gives the following analogues for log-canonical pairs and pluriadjoint bundles, either based on Corollary 2.8 and Theorem 3.2, or on the analogues of Proposition 5.2; we will not repeat the argument.

Variant 5.5. *Let $f : X \to A$ be a morphism from a normal projective variety to an abelian variety. If (X, Δ) is a log-canonical pair and $k \geq 1$ is any integer such that $k(K_X + \Delta)$ is Cartier, then $f_*\mathcal{O}_X(k(K_X + \Delta))$ is a GV-sheaf for every $k \geq 1$.*

Variant 5.6. *Let $f : X \to A$ be a fibration between a smooth projective variety and an abelian variety, and M a nef and f-big line bundle on X. Then $f_*(\omega_X \otimes M)^{\otimes k}$ is a GV-sheaf for every $k \geq 1$.*

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