A CORRECTION TO "ZEROS OF L-FUNCTIONS OUTSIDE THE CRITICAL STRIP"

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Abstract. We correct an error in Lemma 2.1. The main results remain valid with only cosmetic changes to the proof.

Lemma 2.1 in the paper is incorrect as claimed. Specifically, to obtain the $O(\sigma-1)$ error term, it is claimed in the final line of the proof that $\sum_{p>y} p^{-\sigma} \gg \frac{y^{1-\sigma}}{(\sigma-1)\log y}$ for $\sigma \in (1,2]$ and $y \geq \frac{3}{2}$, but this estimate holds with a uniform constant only if $\sigma - 1 \gg \frac{1}{\log y}$. In the lemma below we establish the weaker but uniform estimate $\sum_{p>y} p^{-\sigma} \gg \frac{y^{1-\sigma}}{\log y} \log \frac{2}{\sigma-1}$, so that Lemma 2.1 holds true with $O(\sigma-1)$ replaced by $O(\frac{1}{\log(2/(\sigma-1))})$. Lemma 2.1 is used in the proof of Proposition 3.3, but for that proof to go through it is only necessary that the error term tend to 0 as $\sigma \to 1^+$, so the corrected error term is sufficient.

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Lemma. *We have*

$$\sum_{p>y} p^{-\sigma} \gg \frac{y^{1-\sigma}}{\log y} \log \frac{2}{\sigma-1},$$

uniformly for $y \geq \frac{3}{2}$ and $\sigma \in (1,2]$.

Proof. By Mertens' theorem,

$$\sum_{p>y} p^{-\sigma} \geq \sum_{p} p^{-\sigma} - \sum_{p\leq y} p^{-1} \geq \log \frac{2}{\sigma-1} - \log\log y - C$$

for an absolute constant $C \geq 0$. If $\log \frac{2}{\sigma-1} \geq 2(\log\log y + C)$ then

$$\sum_{p>y} p^{-\sigma} \geq \frac{1}{2} \log \frac{2}{\sigma-1} \gg \frac{y^{1-\sigma}}{\log y} \log \frac{2}{\sigma-1}.$$

If $\log\log y + C + 1 \leq \log \frac{2}{\sigma-1} \leq 2(\log\log y + C)$ then

$$\sum_{p>y} p^{-\sigma} \geq 1 \geq \frac{\log \frac{2}{\sigma-1}}{2(\log\log y + C)} \gg \frac{y^{1-\sigma}}{\log y} \log \frac{2}{\sigma-1}.$$

If $\log \frac{2}{\sigma-1} \le \log\log y + C + 1$ then $1 - y^{1-\sigma} \ge 1 - \exp(-2e^{-C-1}) \gg 1$, so that

$$\begin{aligned}
\sum_{p>y} p^{-\sigma} &= \int_y^\infty t^{-\sigma}\, d\pi(t) = \sigma \int_y^\infty (\pi(t) - \pi(y)) t^{-1-\sigma}\, dt \\
&\ge \sigma \int_{2y}^{2y^2} (\pi(t) - \pi(y)) t^{-1-\sigma}\, dt \gg \frac{1}{\log y} \int_{2y}^{2y^2} t^{-\sigma}\, dt \\
&= \frac{(2y)^{1-\sigma}}{(\sigma-1)\log y} \big(1 - y^{1-\sigma}\big) \gg \frac{y^{1-\sigma}}{(\sigma-1)\log y} \gg \frac{y^{1-\sigma}}{\log y} \log \frac{2}{\sigma-1}.
\end{aligned}$$

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