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Preston Wake

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Eisenstein Hecke algebras and conjectures in Iwasawa theory

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We formulate a weak Gorenstein property for the Eisenstein component of the p-adic Hecke algebra associated to modular forms. We show that this weak Gorenstein property holds if and only if a weak form of Sharifi's conjecture and a weak form of Greenberg's conjecture hold.

1. Introduction

In this paper, we study the relationship between the Iwasawa theory of cyclotomic fields and certain ring-theoretic properties of the Hecke algebra acting on modular forms. This continues work started in our previous paper [Wake 2013].

The philosophy of our work is that simplicity of the Iwasawa theory should correspond to simplicity of Hecke algebras. This philosophy comes from remarkable conjectures formulated by Sharifi [2011].

In [Wake 2013], we showed, under some assumptions, that if the Hecke algebra for modular forms is Gorenstein, then the plus part of the corresponding ideal class group is zero. In particular, we gave an example to show that this Hecke algebra is not always Gorenstein.

Since the Hecke algebra is not always Gorenstein, it is natural to ask if there is a weaker ring-theoretic property that we can expect the Hecke algebra to have. In the present work, we formulate such a weaker property based on whether certain localizations of the Hecke algebra are Gorenstein. In a vague sense, we think of this condition as something like "the obstructions to Gorenstein-ness are finite".

We show that this weak Gorenstein property holds if and only if a weak form of Sharifi's conjecture and a weak form of Greenberg's conjecture both hold. In particular, the weak Gorenstein property holds in every known example.

We make a few remarks before stating our results more precisely.

Notation. In order to state our results more precisely, we introduce some notation, coinciding with that of [Wake 2013].

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Let $p \geq 5$ be a prime, and let N be an integer such that $p \nmid \varphi(N)$ and $p \nmid N$. Let $\theta : (\mathbb{Z}/Np\mathbb{Z})^\times \to \overline{\mathbb{Q}}_p^\times$ be an even character and let $\chi = \omega^{-1}\theta$, where $\omega : (\mathbb{Z}/Np\mathbb{Z})^\times \to (\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{Z}_p^\times$ denotes the Teichmüller character. We assume that θ satisfies the same conditions as in [Fukaya and Kato 2012] — namely that (1) θ is primitive, (2) if $\chi|_{(\mathbb{Z}/p\mathbb{Z})^\times} = 1$, then $\chi|_{(\mathbb{Z}/N\mathbb{Z})^\times}(p) \neq 1$, and (3) if $N = 1$, then $\theta \neq \omega^2$.

A subscript θ or χ will denote the eigenspace for that character for the $(\mathbb{Z}/Np\mathbb{Z})^\times$-action (see Section 1C).

Let $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_{p,N}^\times]]_\theta$ be the Iwasawa algebra, where $\mathbb{Z}_{p,N}^\times = \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$. Let $\mathfrak{m}_\Lambda$ be the maximal ideal of Λ.

Let $\mathfrak{H}$ (resp. $\mathfrak{h}$) be the θ-Eisenstein component of the Hecke algebra for Λ-adic modular forms (resp. cusp forms). Let $\mathcal{I}$ (resp. I) be the Eisenstein ideal of $\mathfrak{H}$ (resp. $\mathfrak{h}$), and let $I_{\mathfrak{H}} \supset \mathcal{I}$ be the preimage of I in $\mathfrak{H}$. Let H be the cohomology group on which $\mathfrak{h}$ acts (see Section 3A).

Let $\mathbb{Q}_\infty = \mathbb{Q}(\zeta_{Np^\infty})$; let M be the maximal abelian p-extension of $\mathbb{Q}_\infty$ unramified outside Np, and let L be the maximal abelian p-extension of $\mathbb{Q}_\infty$ unramified everywhere. Let $\mathfrak{X} = \mathrm{Gal}(M/\mathbb{Q}_\infty)$ and $X = \mathrm{Gal}(L/\mathbb{Q}_\infty)$.

1A. *Statement of results.*

1A1. *Weakly Gorenstein Hecke algebras.* We define what it means for the Hecke algebras $\mathfrak{h}$ and $\mathfrak{H}$ to be weakly Gorenstein. In the case of $\mathfrak{h}$, the definition comes from a condition that appears in work of Fukaya and Kato [2012, Section 7.2.10] on Sharifi's conjecture, and is related to a condition that appears in [Sharifi 2007].

Definition 1.1. We say that $\mathfrak{h}$ is *weakly Gorenstein* if $\mathfrak{h}_\mathfrak{p}$ is Gorenstein for every prime ideal $\mathfrak{p} \in \mathrm{Spec}(\mathfrak{h})$ of height 1 such that $I \subset \mathfrak{p}$.

We say that $\mathfrak{H}$ is *weakly Gorenstein* if $\mathfrak{H}_\mathfrak{p}$ is Gorenstein for every prime ideal $\mathfrak{p} \in \mathrm{Spec}(\mathfrak{H})$ of height 1 such that $I_{\mathfrak{H}} \subset \mathfrak{p}$.

In general, neither the algebra $\mathfrak{h}$ nor the algebra $\mathfrak{H}$ is Gorenstein. However, we conjecture that they are both weakly Gorenstein:

Conjecture 1.2. The Hecke algebras $\mathfrak{h}$ and $\mathfrak{H}$ are weakly Gorenstein.

1A2. *Relation to ideal class groups.* These ring-theoretic properties of Hecke algebras are related to ideal class groups via Sharifi's conjecture [2011]. Sharifi has constructed a map

$$\Upsilon : X_\chi(1) \to H^-/IH^-$$

which he conjectures to be an isomorphism.

A weaker conjecture is that Υ is a pseudoisomorphism — recall that a morphism of Λ-modules is called a pseudoisomorphism if its kernel and cokernel are both finite. If Υ is a pseudoisomorphism, then $\mathfrak{h}$ is weakly Gorenstein if and only X_χ

is pseudocyclic (see Section 5A below). We have the following analogous result for $\mathfrak{H}$. In the statement of the theorem, ξ_χ is a characteristic power series for $X_\chi(1)$ as a Λ-module.

Theorem 1.3. *Consider the following conditions*:

(1) $\mathfrak{H}$ *is weakly Gorenstein.*

(2) $\operatorname{coker}(\Upsilon)$ *is finite.*

(3) $X_\theta/\xi_\chi X_\theta$ *is finite.*

Condition (1) *holds if and only if conditions* (2) *and* (3) *both hold.*

Remark 1.4. Note that if $X_\chi = 0$, then all three conditions hold trivially. Indeed, if $X_\chi = 0$, then $\mathfrak{H} = \Lambda$, the domain and codomain of Υ are 0, and ξ_χ is a unit (see [Wake 2013, Remark 1.3]).

Remark 1.5. The conditions (2) and (3) are conjectured to hold in general (see Section 1B). In particular, they hold in all known examples.

Remark 1.6. Condition (2) is equivalent to the condition that Υ is an injective pseudoisomorphism (see Proposition 7.4).

Remark 1.7. Condition (3) is strange: ξ_χ is the opposite of the usual p-adic zeta function that is related to X_θ. That is, X_θ is annihilated by $\xi_{\chi^{-1}}$, and not (at least not for any obvious reason) by ξ_χ.

The proof of Theorem 1.3 will be given in Section 7.

1A3. *Strong and weak versions of Sharifi's conjecture.* One consequence of Sharifi's conjecture is that $X_\chi(1) \cong H^-/IH^-$ as Λ-modules. Since X_χ has no p-torsion, this would imply that H^-/IH^- has no p-torsion, which Sharifi [2011, Remark, p. 51] explicitly conjectured.

A theorem of Ohta implies that if $\mathfrak{H}$ is Gorenstein, then $X_\chi(1) \cong H^-/IH^-$ (see Theorem 5.11 below). Moreover, Ohta also proves that $\mathfrak{H}$ is Gorenstein under a certain hypothesis ([Ohta 2007, Theorem I], for example). Sharifi [2011, Proposition 4.10] used this as evidence for his conjecture.

Since it is now known that $\mathfrak{H}$ is not always Gorenstein [Wake 2013, Corollary 1.4], one may wonder if Sharifi's conjecture should be weakened to the statement "Υ is a pseudoisomorphism" (see Conjecture 4.2 below). Fukaya and Kato [2012] have partial results on this version of the conjecture. When neither $\mathfrak{h}$ nor $\mathfrak{H}$ is Gorenstein, we know of no evidence for Sharifi's conjecture that Υ is an isomorphism (and not just a pseudoisomorphism); we hope that our next result can be used to provide evidence. This result concerns a module $H^-/I\widetilde{H}^-_{\mathrm{DM}}$. As explained in Section 5, $H^-/I\widetilde{H}^-_{\mathrm{DM}}$ measures how much the ring $\mathfrak{H}$ is "not Gorenstein".

For a finitely generated Λ-module M, let

$$d_{\mathfrak{m}_\Lambda}(M) = \dim_{\Lambda/\mathfrak{m}_\Lambda}(M/\mathfrak{m}_\Lambda M).$$

Note that $d_{\mathfrak{m}_\Lambda}(M)$ is the minimal number of generators of M as a Λ-module.

Theorem 1.8. *Assume that $X_\theta \neq 0$ and that $\mathfrak{h}$ is weakly Gorenstein. Then we have*

$$d_{\mathfrak{m}_\Lambda}(H^-/I\widetilde{H}^-_{\mathrm{DM}}) \geq d_{\mathfrak{m}_\Lambda}(X_\chi),$$

with equality if and only if Υ is an isomorphism.

If, in addition, $\#(X_\theta) = \#(\Lambda/\mathfrak{m}_\Lambda)$, then Υ is an isomorphism if and only if

$$\#(H^-/I\widetilde{H}^-_{\mathrm{DM}}) = \#(\Lambda/\mathfrak{m}_\Lambda)^{d_{\mathfrak{m}_\Lambda}(X_\chi)}.$$

This theorem may be used to provide evidence for Sharifi's conjecture that Υ is an isomorphism in two ways. The first way is philosophical: although the ring $\mathfrak{H}$ is not always Gorenstein, we may like to believe that $\mathfrak{H}$ is "as close to being Gorenstein as possible". This translates to the belief that $H^-/I\widetilde{H}^-_{\mathrm{DM}}$ is as small as possible; the theorem says that $H^-/I\widetilde{H}^-_{\mathrm{DM}}$ is smallest when Υ is an isomorphism.

The second way is a method for providing computational evidence: Theorem 1.8 may allow one to compute examples where Υ is an isomorphism but where $\mathfrak{H}$ is not Gorenstein. We now outline a scheme for doing this. First, one finds an imaginary quadratic field with noncyclic p-class group; this provides a character χ of order 2 such that $X_\theta \neq 0$ and such that $\mathfrak{H}$ is not Gorenstein (see [Wake 2013, Corollary 1.4]). However, $\mathfrak{h}$ will be weakly Gorenstein by Lemma 5.8 below (or else we have found a counterexample to a famous conjecture!). The assumptions for Theorem 1.8 are then satisfied. Then, if one can compute $H^-/I\widetilde{H}^-_{\mathrm{DM}}$ and X_χ sufficiently well, one can verify that $d_{\mathfrak{m}_\Lambda}(H^-/I\widetilde{H}^-_{\mathrm{DM}}) = d_{\mathfrak{m}_\Lambda}(X_\chi)$.

The proof of Theorem 1.8 will be given in Section 8.

1B. ***Relation to known results and conjectures.*** Our results are related to previous results and conjectures of various authors, including Fukaya and Kato, Greenberg, Ohta, Sharifi, Skinner and Wiles and the present author. In the main text, we try to survey these results and conjectures. However, since this is an area with many conjectures, and many of the results are about the interrelation of the conjectures or proofs of special cases of the conjectures, the reader may find it difficult to see what is known, what is unknown, and what exactly is conjectured.

In this section, we try to write down the conjectures and results in a compact but clear fashion. This involves creating an unorthodox naming convention, which we hope will aid in understanding the connections between the statements. The reader may wish to skip this section, and use it as a reference when reading the main text.

1B1. *Naming convention.* We use C(Y) to denote a conjecture about Y, Q(Y) to denote a question about Y (a statement that is not conjectured to be true or false), and A(Y) to denote an assumption about Y (a statement that is *known* to be false in general).

Numbered statement are listed in increasing order of logical strength. For example, Q(Y II) is a questionable statement about Y that implies C(Y I), a conjectural statement about Y.

1B2. *Finiteness conditions.* We consider the following statements about finiteness and cyclicity of class groups:

C(Fin I): $X_\theta/\xi_\chi X_\theta$ is finite.
C(Fin II): X_θ is finite.
A(Fin III): $X_\theta = 0$.
A(Fin IV): $\mathfrak{X}_\theta = 0$.
C(Cyc I): $\mathfrak{X}_\theta \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is cyclic as a $\Lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$-module.
A(Cyc II): $\mathfrak{X}_\theta$ is cyclic as a Λ-module.
C(Cyc' I): $X_\chi \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is cyclic as a $\mathbb{Z}_p[[\mathbb{Z}_{p,N}^\times]]_\chi \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$-module.
A(Cyc' II): X_χ is cyclic as a $\mathbb{Z}_p[[\mathbb{Z}_{p,N}^\times]]_\chi$-module.

There are implications A(Fin III) $\Longrightarrow$ A(Cyc II) and C(Fin II) $\Longrightarrow$ C(Cyc I). In the case $N = 1$, A(Fin III) is actually a conjecture, known as the Kummer–Vandiver conjecture. The conjectures C(Fin II), C(Cyc I) and C(Cyc' I) are due to Greenberg [2001, Conjecture 3.5]. Note that there is no relation between the conjectures C(Cyc' I) and C(Cyc I) for our fixed choices of χ and θ (the modules are *not* adjoint — see Proposition 2.2).

As far as we know, the conjecture C(Fin I) has never been considered before.

1B3. *Gorenstein conditions.* We consider the following statements about Hecke algebras:

C($\mathfrak{h}$ I): $\mathfrak{h}$ is weakly Gorenstein.
A($\mathfrak{h}$ II): $\mathfrak{h}$ is Gorenstein.
C($\mathfrak{H}$ I): $\mathfrak{H}$ is weakly Gorenstein.
A($\mathfrak{H}$ II): $\mathfrak{H}$ is Gorenstein.

The fact that $\mathfrak{h}$ is not always Gorenstein is can be deduced from results of Ohta (following ideas of Kurihara [1993] and Harder and Pink [1992], who considered the case $N = 1$). Ohta ([2007, Corollary 4.2.13], for example) proved the implication A($\mathfrak{h}$ II) $\Longrightarrow$ A(Cyc' II).

The fact that $\mathfrak{H}$ is not always Gorenstein is [Wake 2013, Corollary 1.4]. The weakly Gorenstein conjectures are ours (although this paper shows that they are the consequence of conjectures by other authors).

1B4. *Conjectures of Sharifi type.* We consider the following versions of Sharifi's conjecture. They concern maps ϖ and Υ that were defined by Sharifi.

C(Υ I): coker(Υ) is finite.
C(Υ II): Υ is an isomorphism.
C(S. I): The maps Υ and ϖ are pseudoisomorphisms.
C(S. II): The maps Υ and ϖ are inverse isomorphisms modulo torsion.
C(S. III): The maps Υ and ϖ are inverse isomorphisms.

Note that C(S. I) implies C(Υ I), and C(S. III) is equivalent to C(S. II) + C(Υ II).

See [Sharifi 2011, Conjectures 4.12, 5.2 and 5.4] for the original statements of the conjecture and [Fukaya and Kato 2012, Section 7.1] for some modified statements.

1B5. *A question about zeta functions.* We consider the following statement about p-adic zeta functions, which appears in [Fukaya and Kato 2012]:

Q(ξ): The factorization of ξ_χ in Λ has no prime element occurring with multiplicity > 1.

This statement holds in every known example (see [Greenberg 2001, p. 12]). It is the author's impression that this statement is believed to hold in general, but that there is not enough evidence to call it a conjecture.

1B6. *Relations between the conditions.* Fukaya and Kato have recently made progress towards Sharifi's conjecture. They showed the implications C($\mathfrak{h}$ I) $\Longrightarrow$ C(S. I) and Q(ξ) $\Longrightarrow$ C(S. II) [2012, Theorem 7.2.6]. They also show that if Q(ξ) and at least one of A($\mathfrak{h}$ II) or A($\mathfrak{H}$ II) hold, then C(S. III) holds [2012, Corollary 7.2.7]. Moreover, it can be shown that, if C(Υ I), then C(Cyc' I) is equivalent to C($\mathfrak{h}$ I) (cf. Section 5.1 below). Therefore, their results imply that C($\mathfrak{h}$ I) is equivalent to C(S. I) + C(Cyc' I).

Sharifi [2011, Proposition 4.10], using [Ohta 2003], has shown that A($\mathfrak{H}$ II) $\Longrightarrow$ C(Υ II). As far as we know, there are no results on C(S. III) when neither A($\mathfrak{h}$ II) nor A($\mathfrak{H}$ II) hold.

Ohta [2003] has also shown that A(Fin IV) $\Longrightarrow$ A($\mathfrak{H}$ II). Similar results were obtained by Skinner and Wiles [1997] by a different method.

In our previous work [Wake 2013], we showed that C(Υ II) and A(Fin III) together imply A($\mathfrak{H}$ II), and moreover that if $X_\chi \neq 0$, then A($\mathfrak{H}$ II) implies A(Fin III).

The main result of this paper is that C($\mathfrak{H}$ I) is equivalent to C(Υ I)+C(Fin I).

1C. ***Conventions.*** If $\phi : G \to \overline{\mathbb{Q}}_p^\times$ is a character of a group G, we let $\mathbb{Z}_p[\phi]$ denote the $\mathbb{Z}_p$-algebra generated by the values of ϕ, on which G acts through ϕ. If M is a $\mathbb{Z}_p[G]$-module, denote by M_ϕ the ϕ-eigenspace:

$$M_\phi = M \otimes_{\mathbb{Z}_p[G]} \mathbb{Z}_p[\phi].$$

For a field K, let $G_K = \mathrm{Gal}(\overline{K}/K)$ be the absolute Galois group. For a $G_{\mathbb{Q}}$-module M, let M^+ and M^- denote the eigenspaces of complex conjugation.

We fix a system of primitive Np^r-th roots of unity (ζ_{Np^r}) with the property that $\zeta_{Np^{r+1}}^p = \zeta_{Np^r}$.

2. Conjectures in Iwasawa theory

2A. ***Iwasawa theory of cyclotomic fields.*** We review some important results from the classical Iwasawa theory of cyclotomic fields. Nice references for this material include [Greenberg 2001], [Greither 1992], and [Washington 1997].

2A1. *Class groups and Galois groups.* The main object of study is the inverse limit of the p-power torsion part of the ideal class group $\mathrm{Cl}(\mathbb{Q}(\zeta_{Np^r}))$. By class field theory, there is an isomorphism

$$X \cong \varprojlim \mathrm{Cl}(\mathbb{Q}(\zeta_{Np^r}))\{p\},$$

where, as in the introduction, $X = \mathrm{Gal}(L/\mathbb{Q}_\infty)$ with L the maximal abelian pro-p-extension of $\mathbb{Q}_\infty$ unramified everywhere, and where $(-)\{p\}$ denotes the p-Sylow subgroup.

A closely related object is $\mathfrak{X} = \mathrm{Gal}(M/\mathbb{Q}_\infty)$, where M is the maximal abelian pro-p-extension of $\mathbb{Q}_\infty$ unramified outside Np. We will explain the relation between X and $\mathfrak{X}$ below.

2A2. *Iwasawa algebra.* The natural action of $\mathrm{Gal}(\mathbb{Q}(\zeta_{Np^r})/\mathbb{Q})$ on $\mathrm{Cl}(\mathbb{Q}(\zeta_{Np^r}))\{p\}$ makes X a module over the group ring $\varprojlim \mathbb{Z}_p[\mathrm{Gal}(\mathbb{Q}(\zeta_{Np^r})/\mathbb{Q})]$.

We fix a choice of isomorphism $\mathrm{Gal}(\mathbb{Q}(\zeta_{Np^r})/\mathbb{Q}) \cong (\mathbb{Z}/Np^r\mathbb{Z})^\times$, and this induces an isomorphism $\varprojlim \mathbb{Z}_p[\mathrm{Gal}(\mathbb{Q}(\zeta_{Np^r})/\mathbb{Q})] \cong \mathbb{Z}_p[\![\mathbb{Z}_{p,N}^\times]\!]$, where we define $\mathbb{Z}_{p,N}^\times = \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$. Note that the surjection $\mathbb{Z}_{p,N}^\times \to (\mathbb{Z}/Np\mathbb{Z})^\times$ splits canonically. We use this to identify $\mathbb{Z}_{p,N}^\times$ with $\Gamma \times (\mathbb{Z}/Np\mathbb{Z})^\times$, where Γ is the torsion-free part of $\mathbb{Z}_{p,N}^\times$ (note that $\Gamma \cong \mathbb{Z}_p$).

The ring $\mathbb{Z}_p[\![\mathbb{Z}_{p,N}^\times]\!]$ is, in general, a product of rings. To simplify things, we consider only a particular eigenspace for the action of the torsion subgroup $(\mathbb{Z}/Np\mathbb{Z})^\times$ of $\mathbb{Z}_{p,N}^\times$. We define $\Lambda = \mathbb{Z}_p[\![\mathbb{Z}_{p,N}^\times]\!]_\theta$. There are isomorphisms $\Lambda \cong \mathcal{O}[\![\Gamma]\!] \cong \mathcal{O}[\![T]\!]$, where $\mathcal{O}$ is the $\mathbb{Z}_p$-algebra generated by the values of θ. Note that $\mathcal{O}$ is the valuation ring of a finite extension of $\mathbb{Q}_p$, and so Λ is a noetherian regular local ring of dimension 2 with finite residue field.

2A3. *The operators τ and ι.* We introduce two operations ι and τ on the rings $\mathbb{Z}_p[\![\mathbb{Z}_{p,N}^\times]\!]$ and Λ, and related functors $M \mapsto M^\#$ and $M \mapsto M(r)$. This is a technical part, and the reader may wish to ignore any instance of these on a first reading.

Let $\iota\colon \mathbb{Z}_p[\![\mathbb{Z}_{p,N}^\times]\!] \to \mathbb{Z}_p[\![\mathbb{Z}_{p,N}^\times]\!]$ by the involution given by $c \mapsto c^{-1}$ on $\mathbb{Z}_{p,N}^\times$. Let $\tau : \mathbb{Z}_p[\![\mathbb{Z}_{p,N}^\times]\!] \to \mathbb{Z}_p[\![\mathbb{Z}_{p,N}^\times]\!]$ be the morphism induced by $[c] \mapsto \bar{c}[c]$ for $c \in \mathbb{Z}_{p,N}^\times$, where $[c] \in \mathbb{Z}_p[\![\mathbb{Z}_{p,N}^\times]\!]$ is the group element and where $\bar{c} \in \mathbb{Z}_p^\times$ is the projection of c.

Note that ι and τ do not commute, but $\iota\tau = \tau^{-1}\iota$. In particular, $\tau^r\iota$ is an involution for any $r \in \mathbb{Z}$.

For a $\mathbb{Z}_p[\![\mathbb{Z}_{p,N}^\times]\!]$-module M, we let $M^\#$ (resp. $M(r)$) be the same abelian group, with $\mathbb{Z}_p[\![\mathbb{Z}_{p,N}^\times]\!]$-action changed by ι (resp. τ^r). Note that the functors $M \mapsto M^\#$ and $M \mapsto M(r)$ are exact.

2A4. *p-adic zeta functions and characteristic ideals.* We define $\xi_{\chi^{-1}}, \xi_\chi \in \Lambda$ to be generators of the principal ideals $\mathrm{Char}_\Lambda(X^\#_{\chi^{-1}}(1))$ and $\mathrm{Char}_\Lambda(X_\chi(1))$ respectively (see the Appendix for a review of characteristic ideals).

The Iwasawa main conjecture (now a theorem of Mazur and Wiles [1984]) states that (a certain choice of) $\xi_{\chi^{-1}}$ and ξ_χ can be constructed by p-adically interpolating values of Dirichlet L-functions.

Remark 2.1. In [Wake 2013], we viewed X_χ and $X_{\chi^{-1}}$ as Λ-modules via the isomorphisms

$$\tau : \mathbb{Z}_p[\![\mathbb{Z}_{p,N}^\times]\!]_\chi \xrightarrow{\sim} \Lambda, \quad \iota\tau : \mathbb{Z}_p[\![\mathbb{Z}_{p,N}^\times]\!]_{\chi^{-1}} \xrightarrow{\sim} \Lambda.$$

We learned from the referee that this is an unusual choice of notation, and so we have adopted the above convention, which we learned is more standard.

The element ξ of [Wake 2013] is the ξ_χ of this paper. However, the element denoted $\xi_{\chi^{-1}}$ in that paper would be denoted $\iota\tau\xi_{\chi^{-1}}$ in this paper. We hope this doesn't cause confusion.

2A5. *Adjoints.* For a finitely generated Λ-module M, let $\mathrm{E}^i(M) = \mathrm{Ext}^i_\Lambda(M, \Lambda)$. These are called the (*generalized*) *Iwasawa adjoints* of M.

This theory is important to us because of the following fact, which is well-known to experts.

Proposition 2.2. *The $\mathbb{Z}_p[\![\mathbb{Z}_{p,N}^\times]\!]$-modules $X_{\chi^{-1}}$ and $\mathfrak{X}_\theta$ are both torsion and have no nonzero finite submodule, and we have*

$$\mathfrak{X}_\theta \cong \mathrm{E}^1(X_{\chi^{-1}}(-1)).$$

In particular, we have $\mathrm{Char}_\Lambda(\mathfrak{X}_\theta) = (\xi_{\chi^{-1}})$ *as ideals in* Λ.

Proof. The first sentence is explained in [Wake 2013, Corollary 4.4]. The second sentence follows from the fact that for any finitely generated, torsion Λ-module M, there is a pseudoisomorphism $E^1(M) \to M^\#$ [Neukirch et al. 2008, Proposition 5.5.13, p. 319]. Since $\mathrm{Char}_\Lambda(-)$ is a pseudoisomorphism invariant, we have

$$\mathrm{Char}_\Lambda\big(\mathrm{E}^1(X_{\chi^{-1}}(-1))\big) = \mathrm{Char}_\Lambda\big((X_{\chi^{-1}}(-1))^\#\big) = \mathrm{Char}_\Lambda\big(X^\#_{\chi^{-1}}(1)\big) = (\xi_{\chi^{-1}}),$$

and so the second sentence follows from the first. $\square$

2A6. *Exact sequence.* There is an exact sequence

$$\Lambda/\xi_{\chi^{-1}} \longrightarrow \mathfrak{X}_\theta \longrightarrow X_\theta \longrightarrow 0, \tag{2-3}$$

coming from class field theory and Coleman power series (see, e.g., [Wake 2013, Sections 3 and 5]). From Proposition 2.2 and the fact that $\Lambda/\xi_{\chi^{-1}}$ has no finite submodule, we can see that X_θ is finite (resp. zero) if and only if the leftmost arrow in (2-3) is injective (resp. an isomorphism).

2B. ***Finiteness and cyclicity of class groups.*** We discuss some statements of finiteness and cyclicity of ideal class groups.

2B1. *Kummer–Vandiver conjecture.* We first consider the case $N = 1$.

Conjecture 2.4 (Kummer–Vandiver). Assume $N = 1$. Then $X^+ = 0$.

Lemma 2.5. *Assume $N = 1$. If* Conjecture 2.4 *is true*, *then $\mathfrak{X}_\theta$ and $X_{\chi^{-1}}$ are cyclic.*

Proof. If $X_\theta = 0$, we see from (2-3) that $\mathfrak{X}_\theta$ is cyclic. We wish to show that $X_{\chi^{-1}}$ is cyclic. It is enough to show that $X_{\chi^{-1}}(-1)$ is cyclic, and we claim that this follows from the fact that $\mathfrak{X}_\theta$ is cyclic by Proposition 2.2 and standard arguments from [Neukirch et al. 2008]. Indeed, we have isomorphisms

$$X_{\chi^{-1}}(-1) \cong \mathrm{E}^1\big(\mathrm{E}^1(X_{\chi^{-1}}(-1))\big) \cong \mathrm{E}^1(\mathfrak{X}_\theta)$$

coming from [Neukirch et al. 2008, Proposition 5.5.8(iv), p. 316] and Proposition 2.2, respectively. So it is enough to show that $\mathrm{E}^1(\mathfrak{X}_\theta)$ is cyclic whenever $\mathfrak{X}_\theta$ is. But this is clear from [Neukirch et al. 2008, Proposition 5.5.3(iv), p. 313], which says that the projective dimension of $\mathfrak{X}_\theta$ is 1; if $\mathfrak{X}_\theta$ is generated by one element, then there is exactly one relation, and the dual of the resulting presentation gives a cyclic presentation of $\mathrm{E}^1(\mathfrak{X}_\theta)$. □

2B2. *Greenberg's conjecture.* For general $N > 1$, there are examples where X_χ is not cyclic, and so X^+ is not always zero. However, it may still be true that X^+ is finite:

Conjecture 2.6 [Greenberg 2001, Conjecture 3.4]. The module X^+ is finite.

The following lemma may be proved in the same manner as Lemma 2.5:

Lemma 2.7. *The following are equivalent*:

(1) *X_θ is finite.*

(2) *The map $\Lambda/\xi_{\chi^{-1}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathfrak{X}_\theta \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ induced by the map in* (2-3) *is an isomorphism.*

3. Hecke algebras and modular forms

In this section, we introduce Hecke algebras for modular forms to the story.

3A. ***Hecke algebras and Eisenstein ideals.*** We introduce the objects from the theory of modular forms that we will need. See [Wake 2014] for a more detailed treatment of this theory.

3A1. *Modular curves and Hecke operators.* Let $Y_1(Np^r)/\mathbb{Q}$ be the moduli space for pairs (E, P), where $E/\mathbb{Q}$ is an elliptic curve and $P \in E$ is a point of order Np^r. Let $X_1(Np^r)/\mathbb{Q}$ be the compactification of $Y_1(Np^r)$ obtained by adding cusps.

There is an action of $(\mathbb{Z}/Np^r\mathbb{Z})^\times$ on $Y_1(Np^r)$, where $a \in (\mathbb{Z}/Np^r\mathbb{Z})^\times$ acts by $a(E, P) = (E, aP)$. This is called the action of diamond operators $\langle a \rangle$. There are also Hecke correspondences $T^*(n)$ on $Y_1(Np^r)$ and $X_1(Np^r)$. We consider these as endomorphisms of the cohomology.

We define the Hecke algebra of $Y_1(Np^r)$ to be the algebra generated by the $T^*(n)$ for all integers n and the $\langle a \rangle$ for all $a \in (\mathbb{Z}/Np^r\mathbb{Z})^\times$. We define the Eisenstein ideal to be the ideal of the Hecke algebra generated by $1 - T^*(l)$ for all primes $l \mid Np$ and by $1 - T^*(l) + l\langle l \rangle^{-1}$ for all primes $l \nmid Np$.

3A2. *Ordinary cohomology.* Let

$$H' = \varprojlim H^1(\bar{X}_1(Np^r), \mathbb{Z}_p)^{\text{ord}}_\theta$$

and

$$\widetilde{H}' = \varprojlim H^1(\bar{Y}_1(Np^r), \mathbb{Z}_p)^{\text{ord}}_\theta,$$

where the superscript "ord" denotes the ordinary part for the dual Hecke operator $T^*(p)$, and the subscript refers to the eigenspace for the diamond operators.

3A3. *Eisenstein parts.* Let $\mathfrak{h}'$ and $\mathfrak{H}'$ be the algebras of dual Hecke operators acting on H' and $\widetilde{H}'$, respectively. Let I and $\mathcal{I}$ be the Eisenstein ideals of $\mathfrak{h}'$ and $\mathfrak{H}'$. Let $\mathfrak{H}$ denote the *Eisenstein component* $\mathfrak{H} = \mathfrak{H}'_\mathfrak{m}$, the localization at the unique maximal ideal $\mathfrak{m}$ containing $\mathcal{I}$. We can define the Eisenstein component $\mathfrak{h}$ of $\mathfrak{h}'$ analogously. Let $\widetilde{H} = \widetilde{H}' \otimes_{\mathfrak{H}'} \mathfrak{H}$ and $H = H' \otimes_{\mathfrak{h}'} \mathfrak{h}$ be the Eisenstein components.

There is a natural surjection $\mathfrak{H} \twoheadrightarrow \mathfrak{h}$ by restriction. Let $I_\mathfrak{H} \subset \mathfrak{H}$ be the kernel of the composite map $\mathfrak{H} \twoheadrightarrow \mathfrak{h} \twoheadrightarrow \mathfrak{h}/I$. Note that $\mathcal{I} \subsetneq I_\mathfrak{H}$.

3B. ***Properties of the Hecke modules.*** We first recall some properties of the Hecke modules $\widetilde{H}$ and H and Hecke algebras $\mathfrak{H}$ and $\mathfrak{h}$. See [Fukaya and Kato 2012, Section 6] for a simple and self-contained exposition of this.

3B1. *Control theorem.* There are natural maps $\Lambda \to \mathfrak{h}$ and $\Lambda \to \mathfrak{H}$ given by diamond operators. It is a theorem of Hida that these maps are finite and flat. In particular, $\mathfrak{h}$ and $\mathfrak{H}$ are noetherian local rings of dimension 2 with (the same) finite residue field.

Let $\mathfrak{h}^\vee$ (resp. $\mathfrak{H}^\vee$) denote the $\mathfrak{h}$-module (resp. $\mathfrak{H}$-module) $\mathrm{Hom}_\Lambda(\mathfrak{h}, \Lambda)$ (resp. $\mathrm{Hom}_\Lambda(\mathfrak{H}, \Lambda)$). We will call these the *dualizing modules* for the respective algebras.

3B2. *Eichler–Shimura isomorphisms.* Ohta ([2007, Section 4.2], for example) has proven theorems on the Hecke module structure of $\widetilde{H}$ and H. See [Wake 2014] for a different approach. The main result we need is the following, which appears in this form in [Fukaya and Kato 2012, Section 6.3]:

Theorem 3.1. *There are isomorphisms of Hecke-modules* $H^+ \cong \mathfrak{h}$, $H^- \cong \mathfrak{h}^\vee$ *and* $\widetilde{H}^- \cong \mathfrak{H}^\vee$.

3B3. *Boundary at the cusps.* The cokernel of the natural map $H \to \widetilde{H}$ is described as the boundary at cusps. Ohta [2003, Theorem 1.5.5] has shown that the module of cusps is free of rank one as a Λ-module. That is, there is an exact sequence of $\mathfrak{H}$-modules

$$0 \longrightarrow H \longrightarrow \widetilde{H} \longrightarrow \Lambda \longrightarrow 0.$$

Moreover, there is a canonical element $\{0, \infty\} \in \widetilde{H}$ that gives a generator of $\widetilde{H}/H$. This is proven in [Sharifi 2011, Lemma 4.8], following [Ohta 2003, Theorem 2.3.6] (cf. [Fukaya and Kato 2012, Section 6.2.5]).

3B4. *Relation between Hecke and Iwasawa algebras.* The following is a consequence of the Iwasawa main conjecture. See [Fukaya et al. 2014, Section 2.5.3] for a nice explanation.

Proposition 3.2. *The natural inclusions* $\Lambda \to \mathfrak{H}$ *and* $\Lambda \to \mathfrak{h}$ *induce isomorphisms*

$$\Lambda \xrightarrow{\sim} \mathfrak{H}/\mathcal{I} \quad \textit{and} \quad \Lambda/\xi_\chi \xrightarrow{\sim} \mathfrak{h}/I.$$

3B5. *Drinfeld–Manin modification.* Let $\widetilde{H}_{\mathrm{DM}} = \widetilde{H} \otimes_{\mathfrak{H}} \mathfrak{h}$. By the previous two paragraphs, there is an exact sequence of $\mathfrak{h}$-modules

$$0 \longrightarrow H \longrightarrow \widetilde{H}_{\mathrm{DM}} \longrightarrow \Lambda/\xi_\chi \longrightarrow 0.$$

By abuse of notation, we let $\{0, \infty\} \in \widetilde{H}_{\mathrm{DM}}$ be the image of $\{0, \infty\} \in \widetilde{H}$.

4. Sharifi's conjecture

In this section, we will discuss some remarkable conjectures that were formulated by Sharifi [2011]. Sharifi gave a conjectural construction of a map

$$\varpi : H^-/IH^- \to X_\chi(1),$$

and constructed a map

$$\Upsilon : X_\chi(1) \to H^-/IH^-.$$

Conjecture 4.1 (Sharifi). The maps Υ and ϖ are inverse isomorphisms.

We refer to [Sharifi 2011] and [Fukaya and Kato 2012] for the original constructions, and [Fukaya et al. 2014] for a nice survey of the known results. There is also the following weaker version, which appears as [Fukaya and Kato 2012, Conjecture 7.1.2]. It allows for the possibility that the p-torsion part (tor) of H^-/IH^- is nonzero (note that Conjecture 4.1 implies that (tor) $= 0$, and that Sharifi [2011, Remark, p. 51] specifically notes this).

Conjecture 4.2. The maps Υ and ϖ are inverses up to torsion. That is, $\Upsilon \circ \varpi$ is the identity map on $(H^-/IH^-)/(\text{tor})$ and $\varpi \circ \Upsilon$ is the identity map on $X_\chi(1)$.

The following is [Fukaya and Kato 2012, Theorem 7.2.6(2)]. This result is not needed in the remainder of the paper, except to say that Conjecture 4.2 holds in every known example.

Theorem 4.3. *If ξ_χ has no multiple roots, then* Conjecture 4.2 *is true. In particular, if ξ_χ has no multiple roots and H^-/IH^- has no nonzero finite submodule, then* Conjecture 4.1 *is true.*

The paper [Fukaya and Kato 2012] also has results on Sharifi's conjecture when $H^-/IH^- \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is generated by one element. This is related to Gorenstein conditions on Hecke algebras, the subject of the next section.

5. Gorenstein Hecke algebras

In this section we discuss to what extent the Hecke algebras $\mathfrak{h}$ and $\mathfrak{H}$ are Gorenstein. The relevant characterization of being Gorenstein is the following:

Definition 5.1. Let k be a regular local ring, and let $k \to R$ be a finite, flat ring homomorphism. Then R is *Gorenstein* if $\mathrm{Hom}_k(R, k)$ is a free R-module of rank 1.

This definition is seen to be equivalent to the usual one from homological algebra, but is more useful for our purposes. In our applications we will take $k = \Lambda$ and $R = \mathfrak{h}$, $\mathfrak{H}$ or their localizations. Asking whether $\mathfrak{h}$ or $\mathfrak{H}$ is Gorenstein is the same as asking whether $\mathfrak{h}^\vee$ or $\mathfrak{H}^\vee$ is free of rank 1. This is relevant in light of Theorem 3.1.

5A. *Conditions on* $\mathfrak{h}$.

We consider under what conditions $\mathfrak{h}$ is Gorenstein or weakly Gorenstein.

5A1. *Gorenstein.* The following lemma is explained in [Fukaya and Kato 2012, Section 7.2.12]:

Lemma 5.2. *The following are equivalent*:

(1) $\mathfrak{h}$ *is Gorenstein.*

(2) H^-/IH^- *is cyclic as an* $\mathfrak{h}$*-module.*

(3) H^-/IH^- *is a free* $\mathfrak{h}/I$*-module of rank* 1.

Proof. This follows from the fact that $H^- \cong \mathfrak{h}^\vee$, that $\mathfrak{h}^\vee$ is a faithful $\mathfrak{h}$-module, and Nakayama's lemma. □

One may ask whether $\mathfrak{h}$ is always Gorenstein. The following result is based on ideas of Kurihara [1993] and Harder and Pink [1992], who proved it in the case $N = 1$. The result in this form was proven by Ohta.

Theorem 5.3. *Suppose that $\mathfrak{h}$ is Gorenstein. Then $X_\chi(1)$ is cyclic as a Λ-module.*

Proof. This is proven, for example, in [Ohta 2007, Corollary 4.2.13], where it is the implication "(ii) $\Longrightarrow$ (i)". Note that the proof of "(ii) $\Longrightarrow$ (i)" given there does not require the assumption that $\mathfrak{H}$ is Gorenstein. □

As remarked in Section 2B2, there are examples where X_χ is not cyclic and therefore where $\mathfrak{h}$ is not Gorenstein. One could also ask if the converse holds.

Lemma 5.4. *Suppose that $H^-/IH^- \cong X_\chi(1)$. Then $\mathfrak{h}$ is Gorenstein if and only if $X_\chi(1)$ is cyclic as a Λ-module.*

Proof. This follows from Lemma 5.2. □

In particular, we have the following:

Corollary 5.5. *Assume $N = 1$, and assume Sharifi's conjecture* (Conjecture 4.1) *and the Kummer–Vandiver theorem* (Conjecture 2.4). *Then $\mathfrak{h}$ is Gorenstein.*

5A2. *Weakly Gorenstein.* We recall that $\mathfrak{h}$ is said to be *weakly Gorenstein* if $\mathfrak{h}_\mathfrak{p}$ is Gorenstein for every prime ideal $\mathfrak{p} \in \operatorname{Spec}(\mathfrak{h})$ of height 1 such that $I \subset \mathfrak{p}$. This definition is relevant in light of the following lemma:

Lemma 5.6 [Fukaya and Kato 2012, Section 7.2.10]. *The following are equivalent*:

(1) *$\mathfrak{h}$ is weakly Gorenstein.*

(2) *$(H^-/IH^-) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is cyclic as an $\mathfrak{h}/I \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$-module.*

(3) *$(H^-/IH^-) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a free $\mathfrak{h}/I \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$-module of rank* 1.

The following result on Sharifi's conjecture assumes that $\mathfrak{h}$ is weakly Gorenstein:

Theorem 5.7 [Fukaya and Kato 2012, Theorem 7.2.8(1)]. *Assume that $\mathfrak{h}$ is weakly Gorenstein. Then $\Upsilon\colon X_\chi(1) \to (H^-/IH^-)/(\mathrm{tor})$ and $\varpi\colon (H^-/IH^-)/(\mathrm{tor}) \to X_\chi(1)$ are isomorphisms.*

Their work also implies the following result on the converse:

Lemma 5.8. *Assume that* $\operatorname{coker}(\Upsilon)$ *is finite and that $X_\chi(1) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is cyclic. Then $\mathfrak{h}$ is weakly Gorenstein.*

Proof. If $\operatorname{coker}(\Upsilon)$ is finite, then $\Upsilon : X_\chi(1) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to (H^-/IH^-) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is surjective, so this follows from Lemma 5.6. □

Since this lemma applies whenever Conjecture 4.2 and Conjecture 2.6 hold, we should conjecture that $\mathfrak{h}$ is always weakly Gorenstein:

Conjecture 5.9. The ring $\mathfrak{h}$ is weakly Gorenstein.

5B. ***Conditions on $\mathfrak{H}$.*** We consider under what conditions $\mathfrak{H}$ is Gorenstein or weakly Gorenstein.

5B1. *Gorenstein.* Recall that $\widetilde{H}^-_{\mathrm{DM}}/H^- \cong \mathfrak{h}/I$. In particular, the natural inclusion $I\widetilde{H}^-_{\mathrm{DM}} \subset \widetilde{H}^-_{\mathrm{DM}}$ lands in H^-. The following proposition is proven in [Wake 2013]. It can also be proven along the same lines as the proof of Proposition 5.13.

Proposition 5.10. *The following are equivalent*:

(1) *$\mathfrak{H}$ is Gorenstein.*

(2) *$\widetilde{H}^-_{\mathrm{DM}}$ is a free $\mathfrak{h}$-module of rank* 1.

(3) $I\widetilde{H}^-_{\mathrm{DM}} = H^-$.

It was proven by Ohta [2007, Theorem I] that $\mathfrak{H}$ is Gorenstein if $\mathfrak{X}_\theta = 0$. Similar results were obtained earlier by Skinner and Wiles [1997].

The following theorem illustrates the importance of the condition that $\mathfrak{H}$ is Gorenstein. It was first proven by Sharifi, following [Ohta 2003].

Theorem 5.11. *Suppose that $\mathfrak{H}$ is Gorenstein. Then Υ is an isomorphism.*

Proof. This is proven in [Sharifi 2011, Proposition 4.10], where the assumption "$p \nmid B_{1,\theta^{-1}}$" is not needed in the proof — all that is needed is the weaker assumption that $\mathfrak{H}$ is Gorenstein. □

Sharifi used this as evidence for his conjecture. However, it is not true that $\mathfrak{H}$ is always Gorenstein; the following is the main result of [Wake 2013]:

Theorem 5.12. *If $\mathfrak{H}$ is Gorenstein, then either $X_\theta = 0$ or $X_\chi = 0$. Moreover, there are examples where $X_\theta \neq 0$ and $X_\chi \neq 0$, and so $\mathfrak{H}$ is not always Gorenstein.*

5B2. *Weakly Gorenstein.* Recall that $\mathfrak{H}$ is said to be *weakly Gorenstein* if $\mathfrak{H}_\mathfrak{p}$ is Gorenstein for every prime ideal $\mathfrak{p} \in \mathrm{Spec}(\mathfrak{H})$ such that $I_\mathfrak{H} \subset \mathfrak{p}$.

Proposition 5.13. *Let $\mathcal{P} \subset \mathrm{Spec}(\mathfrak{H})$ be the set of height* 1 *prime ideals $\mathfrak{p}$ such that $I_\mathfrak{H} \subset \mathfrak{p}$. The following are equivalent*:

(1) *$H^-/I\widetilde{H}^-_{\mathrm{DM}}$ is finite.*

(2) *As a module over $\mathfrak{h} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, $\widetilde{H}^-_{\mathrm{DM}}/I\widetilde{H}^-_{\mathrm{DM}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is generated by* $\{0, \infty\}$.

(3) *For any $\mathfrak{p} \in \mathcal{P}$, $(\widetilde{H}^-)_\mathfrak{p}$ is generated by* $\{0, \infty\}$.

(4) *For any $\mathfrak{p} \in \mathcal{P}$, $(\widetilde{H}^-)_\mathfrak{p}$ is generated by* 1 *element.*

(5) *For any $\mathfrak{p} \in \mathcal{P}$, $(\widetilde{H}^-)_\mathfrak{p}$ is free of rank* 1 *as an $\mathfrak{H}_\mathfrak{p}$-module.*

(6) *$\mathfrak{H}$ is weakly Gorenstein.*

Proof. (1) $\Longrightarrow$ (2): Follows from taking $\otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ in the exact sequence

$$0 \to H^-/I\widetilde{H}^-_{\mathrm{DM}} \to \widetilde{H}^-_{\mathrm{DM}}/I\widetilde{H}^-_{\mathrm{DM}} \to \widetilde{H}^-_{\mathrm{DM}}/H^- \to 0.$$

(2) $\Longrightarrow$ (3): Since $\mathfrak{H}/I_{\mathfrak{H}} \xrightarrow{\sim} \mathfrak{h}/I$, we have that

$$\widetilde{H}^-_{\mathrm{DM}}/I\widetilde{H}^-_{\mathrm{DM}} \cong \widetilde{H}^-_{\mathrm{DM}} \otimes_{\mathfrak{h}} \mathfrak{h}/I \cong \widetilde{H}^- \otimes_{\mathfrak{H}} \mathfrak{H}/I_{\mathfrak{H}}.$$

For any $\mathfrak{p} \in \mathcal{P}$, we have that p is invertible in $\mathfrak{H}_{\mathfrak{p}}$, so $(\widetilde{H}^-_{\mathrm{DM}}/I\widetilde{H}^-_{\mathrm{DM}})_{\mathfrak{p}} = (\widetilde{H}^-/I_{\mathfrak{H}}\widetilde{H}^-)_{\mathfrak{p}}$ is generated by $\{0, \infty\}$. By Nakayama's lemma, we have (3).

(3) $\Longrightarrow$ (4): Clear.

(4) $\Longrightarrow$ (5): By Theorem 3.1, $(\widetilde{H}^-)_{\mathfrak{p}}$ is a dualizing module for $\mathfrak{H}_{\mathfrak{p}}$, and so it is faithful. Then, if it is generated by 1 element, it is free.

(5) $\Longleftrightarrow$ (6): Since $(\widetilde{H}^-)_{\mathfrak{p}}$ is a dualizing module for $\mathfrak{H}_{\mathfrak{p}}$, this is clear.

(5) $\Longrightarrow$ (1): Note that $H^-/I\widetilde{H}^-_{\mathrm{DM}}$ is a $\mathfrak{H}/I_{\mathfrak{H}}$-module. To show (1), it suffices (by Lemma A.1) to show that $H^-/I\widetilde{H}^-_{\mathrm{DM}}$ is not supported on any nonmaximal prime ideals of $\mathfrak{H}/I_{\mathfrak{H}}$. Since the nonmaximal prime ideals of $\mathfrak{H}/I_{\mathfrak{H}}$ are exactly the images under $\mathfrak{H} \twoheadrightarrow \mathfrak{H}/I_{\mathfrak{H}}$ of elements of $\mathcal{P}$, it is enough to show that $\mathrm{Supp}_{\mathfrak{H}}(H^-/I\widetilde{H}^-_{\mathrm{DM}}) \cap \mathcal{P}$ is empty.

Let $\mathfrak{p} \in \mathcal{P}$. By (5) we see that $(\widetilde{H}^-/I_{\mathfrak{H}}\widetilde{H}^-)_{\mathfrak{p}}$ is free of rank 1 as an $(\mathfrak{H}/I_{\mathfrak{H}})_{\mathfrak{p}}$-module. But, since $\mathfrak{H}/I_{\mathfrak{H}} \xrightarrow{\sim} \mathfrak{h}/I$, $(\widetilde{H}^-_{\mathrm{DM}}/H^-)_{\mathfrak{p}}$ is also free of rank 1 as an $(\mathfrak{H}/I_{\mathfrak{H}})_{\mathfrak{p}}$-module. Then the natural surjective map

$$(\widetilde{H}^-_{\mathrm{DM}}/I\widetilde{H}^-_{\mathrm{DM}})_{\mathfrak{p}} = (\widetilde{H}^-/I_{\mathfrak{H}}\widetilde{H}^-)_{\mathfrak{p}} \twoheadrightarrow (\widetilde{H}^-_{\mathrm{DM}}/H^-)_{\mathfrak{p}}$$

must be an isomorphism. This implies that the kernel $(H^-/I\widetilde{H}^-_{\mathrm{DM}})_{\mathfrak{p}}$ is zero. □

6. Pairing with cyclotomic units

In this section, we recall some results from [Wake 2013] that will be used in the proof of Theorem 1.3.

6A. ***The Kummer pairing.*** As in [Wake 2013, Section 3.2], we will make use of a pairing between $\mathfrak{X}$ and global units. Let E denote the pro-p part of the closure of the global units in $\varprojlim(\mathbb{Z}[\zeta_{Np^r}] \otimes \mathbb{Z}_p)^\times$.

There is a pairing of $\mathbb{Z}_p[[\mathbb{Z}_{p,N}^\times]]$-modules

$$[\ ,\ ]_{\mathrm{Kum}} : E \times \mathfrak{X}^\#(1) \to \mathbb{Z}_p[[\mathbb{Z}_{p,N}^\times]].$$

It is essentially defined as the "Λ-adic version" of the pairing

$$\mathbb{Z}_p[\zeta_{Np^r}]^\times \times \mathfrak{X} \to \mu_{p^r}, \quad (u, \sigma) \mapsto \frac{\sigma(u^{1/p^r})}{u^{1/p^r}}.$$

We refer to [Wake 2013, Section 3.2] for the detailed definition.

6A1. *The map ν.* The Kummer pairing gives a homomorphism of Λ-modules $E_\theta \to \mathrm{Hom}(\mathfrak{X}_{\chi^{-1}}, \Lambda^\#(1))$. There is a special element $\mathbf{1-\zeta} \in E_\theta$, namely, the image of $(1-\zeta_{Np^r})_r \in \varprojlim(\mathbb{Z}[\zeta_{Np^r}]^\times \otimes \mathbb{Z}_p)$.

We define ν to be the image of $\mathbf{1-\zeta}$ under the Kummer pairing. So

$$\mathfrak{X}_{\chi^{-1}} \xrightarrow{\nu} \Lambda^\#(1)$$

is a morphism of $\mathbb{Z}_p[\![\mathbb{Z}_{p,N}^\times]\!]_{\chi^{-1}}$-modules.

The importance of ν comes from the following lemma, which relates ν to X^+:

Lemma 6.1. *There exists a natural commutative diagram of $\mathbb{Z}_p[\![\mathbb{Z}_{p,N}^\times]\!]$-modules with exact rows*:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U_{\chi^{-1}}(-1) & \longrightarrow & \mathfrak{X}_{\chi^{-1}}(-1) & \longrightarrow & X_{\chi^{-1}}(-1) & \longrightarrow & 0 \\ & & \downarrow \wr & & \downarrow \nu(-1) & & & & \\ 0 & \longrightarrow & \Lambda^\# & \longrightarrow & \Lambda^\# & \longrightarrow & \Lambda^\#/\iota\xi_{\chi^{-1}} & \longrightarrow & 0 \end{array}$$

Let $\bar{\nu}: X_{\chi^{-1}}(-1) \to \Lambda^\#/\iota\xi_{\chi^{-1}}$ be the induced map, and let C denote $\mathrm{coker}(\bar{\nu})$. *Then we have an equality of characteristic ideals*

$$\mathrm{Char}_\Lambda(X_\theta) = \mathrm{Char}_\Lambda(C^\#).$$

Proof. The existence of the commutative diagram is from [Wake 2013, Lemma 4.5] (note that the element denoted by $\xi_{\chi^{-1}}$ in that work would be denoted $\iota\tau\xi_{\chi^{-1}}$ here).

Let $C = \mathrm{coker}(\bar{\nu})$. By [Wake 2013, Proposition 4.8(1) and Lemma 4.6], there is an exact sequence

$$0 \longrightarrow \mathrm{E}^1(C) \longrightarrow \Lambda/\xi_{\chi^{-1}} \longrightarrow \mathfrak{X}_\theta \longrightarrow X_\theta \longrightarrow 0.$$

Since $\mathrm{Char}_\Lambda(\mathfrak{X}_\theta) = (\xi_{\chi^{-1}})$, we have $\mathrm{Char}_\Lambda(E^1(C)) = \mathrm{Char}_\Lambda(X_\theta)$. We claim that $\mathrm{Char}_\Lambda(E^1(C)) = \mathrm{Char}_\Lambda(C^\#)$. This follows from [Neukirch et al. 2008, Proposition 5.5.13, p. 319], as in the proof of Proposition 2.2 above. □

6A2. *The map ν'.* Let $\nu' : \mathfrak{X}_{\chi^{-1}} \to (\Lambda/\xi_\chi)^\#(1)$ be the composite

$$\mathfrak{X}_{\chi^{-1}} \xrightarrow{\nu} \Lambda^\#(1) \to \Lambda^\#(1)/\iota\tau\xi_\chi\Lambda^\#(1) = (\Lambda/\xi_\chi)^\#(1).$$

Lemma 6.2. *We have that* $\mathrm{coker}(\nu')$ *is finite if and only if $X_\theta/\xi_\chi X_\theta$ is finite. Moreover*, $\mathrm{coker}(\nu') = 0$ *if and only if $X_\chi = 0$ or $X_\theta = 0$.*

Proof. Let $C = \mathrm{coker}(\bar{\nu})$. From Lemma 6.1, we see that $C \cong \mathrm{coker}(\nu(-1))$. We see from the definition of ν' that

$$\mathrm{coker}(\nu') \cong C(1)/\tau^{-1}\iota\xi_\chi C(1) = (C/\iota\xi_\chi C)(1).$$

Note that since $(C/\iota\xi_\chi C)^\# \cong C^\#/\xi_\chi C^\#$, we have that $\mathrm{coker}(\nu')$ is finite if and only if $C^\#/\xi_\chi C^\#$ is finite.

Recall from Lemma 6.1 that $\mathrm{Char}_\Lambda(C^\#) = \mathrm{Char}_\Lambda(X_\theta)$. We apply Lemma A.4 to the case $M = \Lambda/\xi_\chi$, $N = C^\#$, and $N' = X_\theta$ to get that $C^\#/\xi_\chi C^\#$ is finite if and only if $X_\theta/\xi_\chi X_\theta$ is finite. This completes the proof of the first statement.

For the second statement, notice that if $X_\chi = 0$, then ξ_χ is a unit, so $\mathrm{coker}(\nu') = 0$. If $X_\chi \neq 0$, then ξ_χ is not a unit and, by Nakayama's lemma, $\mathrm{coker}(\nu') = 0$ if and only if $C = 0$. It remains to prove that $C = 0$ if and only if $X_\theta = 0$, and this follows from [Wake 2013]. Indeed, if $X_\theta = 0$, then [Wake 2013, Proposition 4.8(2)] implies that $C = 0$. Conversely, if $C = 0$, then [Wake 2013, Proposition 4.8(1), Corollary 4.7] together imply that X_θ is finite, and then we can apply [Wake 2013, Proposition 4.8(2)] to conclude that $X_\theta = 0$. □

6B. ***The map ν as an extension class.*** Fukaya and Kato [2012, Section 9.6] gave an interpretation of ν as an extension class. We review this here, and refer to [Wake 2013, Section 2.3] for more details.

There is an exact sequence

$$0 \longrightarrow H^+/IH^+ \longrightarrow \widetilde{H}_{\mathrm{DM}}/K \longrightarrow \widetilde{H}_{\mathrm{DM}}/H \longrightarrow 0, \tag{6-3}$$

where K is the kernel of the natural map $H \to H^+/IH^+$ (and so $K \cong H^- \oplus IH^+$ as $\mathfrak{h}$-modules). It can be shown that $H \to H^+/IH^+$ respects the $G_{\mathbb{Q}}$-action ([Sharifi 2011]; cf. [Fukaya and Kato 2012, Proposition 6.3.2]), and so (6-3) is an extension of $\widetilde{H}_{\mathrm{DM}}/H$ by H^+/IH^+ as $\mathfrak{h}[G_{\mathbb{Q}}]$-modules. By considering this extension as a Galois cocycle, we obtain a homomorphism of $\mathbb{Z}_p[\![\mathbb{Z}_{p,N}^\times]\!]_{\chi^{-1}}$-modules

$$\Theta : \mathfrak{X}_{\chi^{-1}} \to (\Lambda/\xi_\chi)^\#(1).$$

By [Fukaya and Kato 2012, Theorem 9.6.3], we have:

Theorem 6.4 [Wake 2013, Proposition 3.4]. *We have* $\nu' = \Theta$.

7. Relationship between the Hecke and Iwasawa sides

The goal of this section is to complete the proof of Theorem 1.3.

7A. ***The key diagram.*** First we consider a commutative diagram coming from the maps of Section 6. Let $\nu'' = (\nu')^\#(1) : \mathfrak{X}^\#_{\chi^{-1}}(1) \to \Lambda/\xi_\chi$, so that ν'' is a map of Λ-modules. Then we have the diagram of Λ-modules

$$\begin{array}{ccc} \mathfrak{X}^\#_{\chi^{-1}}(1) \otimes_\Lambda X_\chi(1) & \xrightarrow{\nu''\otimes 1} & \Lambda/\xi_\chi \otimes_\Lambda X_\chi(1) \\ \downarrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle\Upsilon} \\ H^-/IH^- & = & H^-/IH^- \end{array}$$

where $\Phi = \Theta^{\#}(1) \otimes \Upsilon$. It is commutative by Theorem 6.4. This is a slight reformulation of the diagram (∗) in [Wake 2013, Section 1.3]. We record the result of applying the Snake Lemma to this diagram as a lemma:

Lemma 7.1. *There is an exact sequence*

$$\ker(\Phi) \longrightarrow \ker(\Upsilon) \longrightarrow \operatorname{coker}(\nu'') \otimes_\Lambda X_\chi(1) \longrightarrow \operatorname{coker}(\Phi) \longrightarrow \operatorname{coker}(\Upsilon) \longrightarrow 0.$$

7A1. *Some lemmas.*

Lemma 7.2. *We have* $\operatorname{coker}(\Phi) = H^-/I\widetilde{H}^-_{\mathrm{DM}}$.

Proof. This is a slight reformulation of [Wake 2013, Proposition 2.2], which states that the image of Φ is $I\{0, \infty\}$. Since $\{0, \infty\}$ generates $\widetilde{H}^-_{\mathrm{DM}}/H^-$, we see that the images of $I\{0, \infty\}$ and $I\widetilde{H}^-_{\mathrm{DM}}$ in H^-/IH^- are the same. □

Lemma 7.3. *We have that* $\operatorname{coker}(\nu'') \otimes_\Lambda X_\chi(1)$ *is finite if and only if* $X_\theta/\xi_\chi X_\theta$ *is finite.*

Proof. Let $C'' = \operatorname{coker}(\nu'')$. Since $\nu'' = (\nu')^{\#}(1)$, it is clear that C'' is finite if and only if $\operatorname{coker}(\nu')$ is finite. By Lemma 6.2 it suffices to show that $C'' \otimes_\Lambda X_\chi(1)$ is finite if and only if C'' is finite.

Now apply Lemma A.3 to $M = C''$ and $N = X_\chi(1)$ to get that $C'' \otimes_\Lambda X_\chi(1)$ is finite if and only if $C''/\operatorname{Char}_\Lambda(X_\chi(1))C''$ is finite. However, $\operatorname{Char}_\Lambda(X_\chi(1)) = (\xi_\chi)$, which annihilates C''. So $C'' = C''/\operatorname{Char}_\Lambda(X_\chi(1))C''$ and the lemma follows. □

Proposition 7.4. *Suppose that* $\operatorname{coker}(\Upsilon)$ *is finite. Then* Υ *is injective.*

Proof. It is well-known (see [Fukaya and Kato 2012, Section 7.1.3]) that

$$\operatorname{Fitt}_\Lambda(H^-/IH^-) \subset (\xi_\chi).$$

We apply Lemma A.7 to the case $M = X_\chi(1)$, $N = H^-/IH^-$ and $f = \Upsilon$. It says that if $\operatorname{coker}(\Upsilon)$ is finite, then $\ker(\Upsilon)$ is finite. But $X_\chi(1)$ has no finite submodule, so the result follows. □

7A2. *The proof of Theorem 1.3.* We can now prove Theorem 1.3, which we restate here for convenience.

Theorem 1.3. *Both* $\operatorname{coker}(\Upsilon)$ *and* $X_\theta/\xi_\chi X_\theta$ *are finite if and only if* $\mathfrak{H}$ *is weakly Gorenstein.*

Proof. First assume that $\operatorname{coker}(\Upsilon)$ and $X_\theta/\xi_\chi X_\theta$ are finite. By Lemma 7.3 we have that $\operatorname{coker}(\nu'') \otimes_\Lambda X_\chi(1)$ is finite. By Lemma 7.1, we see that $\operatorname{coker}(\Phi)$ is finite. By Lemma 7.2, $H^-/I\widetilde{H}^-_{\mathrm{DM}}$ is finite. By Proposition 5.13, we have that $\mathfrak{H}$ is weakly Gorenstein.

Now assume that $\mathfrak{H}$ is weakly Gorenstein. Then, as above, $\operatorname{coker}(\Phi)$ is finite. By Lemma 7.1, we see that $\operatorname{coker}(\Upsilon)$ is finite. By Proposition 7.4, we see that $\ker(\Upsilon) = 0$. Again using Lemma 7.1 we see that $\operatorname{coker}(\nu'') \otimes_\Lambda X_\chi(1) \subset \operatorname{coker}(\Phi)$, and so $\operatorname{coker}(\nu'') \otimes_\Lambda X_\chi(1)$ is finite. By Lemma 7.3, we have that $X_\theta/\xi_\chi X_\theta$ is finite. □

8. Application to Sharifi's conjecture

For a finitely generated Λ-module M, let

$$d_{\mathfrak{m}_\Lambda}(M) = \dim_{\Lambda/\mathfrak{m}_\Lambda}(M/\mathfrak{m}_\Lambda M).$$

Note that, by Nakayama's lemma, $d_{\mathfrak{m}_\Lambda}(M)$ is the minimal number of generators of M. In particular, $d_{\mathfrak{m}_\Lambda}(M) = 0$ if and only if $M = 0$.

We can now prove Theorem 1.8, which we restate here for convenience:

Theorem 1.8. *Assume that $X_\theta \neq 0$ and that $\mathfrak{h}$ is weakly Gorenstein.*

Then we have

$$d_{\mathfrak{m}_\Lambda}(H^-/I\widetilde{H}^-_{\mathrm{DM}}) \geq d_{\mathfrak{m}_\Lambda}(X_\chi(1)),$$

with equality if and only if Υ is an isomorphism.

If, in addition, $\#(X_\theta) = \#(\Lambda/\mathfrak{m}_\Lambda)$, then Υ is an isomorphism if and only if

$$\#(H^-/I\widetilde{H}^-_{\mathrm{DM}}) = \#(\Lambda/\mathfrak{m}_\Lambda)^{d_{\mathfrak{m}_\Lambda}(X_\chi(1))}.$$

Proof. By Theorem 5.7, we have that $\mathrm{coker}(\Upsilon)$ is finite. By Proposition 7.4, we have $\ker(\Upsilon) = 0$. By Lemma 7.1, we have an exact sequence

$$0 \longrightarrow \mathrm{coker}(\nu'') \otimes_\Lambda X_\chi(1) \longrightarrow \mathrm{coker}(\Phi) \longrightarrow \mathrm{coker}(\Upsilon) \longrightarrow 0.$$

Theorem 5.7 implies that $\mathrm{coker}(\Upsilon) \xrightarrow{\sim} (\mathrm{tor}) \to \mathrm{coker}(\Phi)$ gives a spitting of this sequence. This gives us an isomorphism

$$H^-/I\widetilde{H}^-_{\mathrm{DM}} = \mathrm{coker}(\Phi) \cong (\mathrm{coker}(\nu'') \otimes_\Lambda X_\chi(1)) \oplus \mathrm{coker}(\Upsilon),$$

and so

$$\begin{aligned} d_{\mathfrak{m}_\Lambda}(H^-/I\widetilde{H}^-_{\mathrm{DM}}) &= d_{\mathfrak{m}_\Lambda}(\mathrm{coker}(\nu'') \otimes_\Lambda X_\chi(1)) + d_{\mathfrak{m}_\Lambda}(\mathrm{coker}(\Upsilon)) \\ &= d_{\mathfrak{m}_\Lambda}(\mathrm{coker}(\nu''))d_{\mathfrak{m}_\Lambda}(X_\chi(1)) + d_{\mathfrak{m}_\Lambda}(\mathrm{coker}(\Upsilon)). \end{aligned}$$

We claim that in fact

$$d_{\mathfrak{m}_\Lambda}(H^-/I\widetilde{H}^-_{\mathrm{DM}}) = d_{\mathfrak{m}_\Lambda}(X_\chi(1)) + d_{\mathfrak{m}_\Lambda}(\mathrm{coker}(\Upsilon)),$$

from which the first statement of the theorem follows.

To prove the claim, note that it is clear if $d_{\mathfrak{m}_\Lambda}(X_\chi(1)) = 0$. Now assume $d_{\mathfrak{m}_\Lambda}(X_\chi(1)) \neq 0$. Then we claim that $d_{\mathfrak{m}_\Lambda}(\mathrm{coker}(\nu'')) = 1$. Indeed, since $\mathrm{coker}(\nu'')$ is cyclic it suffices to show $\mathrm{coker}(\nu'') \neq 0$. But since $X_\chi(1) \neq 0$, Lemma 6.2 implies that $\mathrm{coker}(\nu'') \neq 0$ if and only if $X_\theta \neq 0$, which we are assuming. This completes the proof of the claim and of the first statement.

For the second statement, notice that the assumption can only occur if $X_\theta \cong \Lambda/\mathfrak{m}_\Lambda$. By [Wake 2013, Proposition 4.8], this implies that $\operatorname{coker}(\overline{\nu})^\# \cong \Lambda/\mathfrak{m}_\Lambda$. As in Lemma 6.2, where we computed $\operatorname{coker}(\nu')$ in terms of $\operatorname{coker}(\overline{\nu})$, we compute

$$\operatorname{coker}(\nu'') \cong \operatorname{coker}(\overline{\nu})^\# / \xi_\chi \operatorname{coker}(\overline{\nu})^\#,$$

so

$$\operatorname{coker}(\nu'') \cong \begin{cases} \Lambda/\mathfrak{m}_\Lambda & \text{if } X_\chi(1) \neq 0, \\ 0 & \text{if } X_\chi(1) = 0. \end{cases}$$

In either case,

$$\operatorname{coker}(\nu'') \otimes_\Lambda X_\chi(1) \cong (\Lambda/\mathfrak{m}_\Lambda)^{d_{\mathfrak{m}_\Lambda}(X_\chi(1))},$$

and the statement follows from the established isomorphism

$$H^-/I\widetilde{H}^-_{\mathrm{DM}} \cong (\operatorname{coker}(\nu'') \otimes_\Lambda X_\chi(1)) \oplus \operatorname{coker}(\Upsilon). \qquad \square$$

Appendix: some commutative algebra

We review some lemmas from commutative algebra that are used in the body of the paper. The results of this appendix are well-known; we include them for completeness.

Finite modules. We begin with a review of some generalities about finite modules (meaning modules of finite cardinality). Let $(A, \mathfrak{m})$ be a noetherian local ring, and assume that the residue field $A/\mathfrak{m}$ is finite. For an A-module M, we use the notation $\operatorname{Supp}_A(M)$ for the set $\{\mathfrak{p} \in \operatorname{Spec}(A) \mid M_\mathfrak{p} \neq 0\}$.

Lemma A.1. *Let M be a finitely generated A-module. The following are equivalent*:

(1) $\mathfrak{m}^n M = 0$ *for some n.*

(2) *M is finite.*

(3) *M is an Artinian A-module.*

(4) $\operatorname{Supp}_A(M) \subset \{\mathfrak{m}\}$.

Proof. For (4) $\Longrightarrow$ (1), since M is finitely generated, it is enough to prove the case where $M \cong A/I$ for an ideal I. By (4) we have that $\operatorname{Spec}(A/I) \subset \{\mathfrak{m}\}$. This implies that A/I is Artinian, which implies that $\mathfrak{m}^n(A/I) = 0$ for some n. The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (1) $\Longrightarrow$ (4) are clear. $\square$

Corollary A.2. *Suppose M and N are finitely generated A-modules. Then $M \otimes_A N$ is finite if and only if* $\operatorname{Supp}_A(N) \cap \operatorname{Supp}_A(M) \subset \{\mathfrak{m}\}$.

Proof. This is clear from Lemma A.1, since

$$\operatorname{Supp}_A(M \otimes_A N) = \operatorname{Supp}_A(N) \cap \operatorname{Supp}_A(M). \qquad \square$$

Λ-modules. Let Λ be a noetherian regular local ring of dimension 2 with finite residue field. For example, let $\Lambda = \mathbb{O}[\![T]\!]$, where $\mathbb{O}$ is the valuation ring of a finite extension of $\mathbb{Q}_p$.

Characteristic ideals. For a finitely generated torsion Λ-module M, define the characteristic ideal of M to be

$$\mathrm{Char}_\Lambda(M) = \prod_{\mathfrak{p}} \mathfrak{p}^{l_\mathfrak{p}(M)},$$

where $\mathfrak{p}$ ranges over all height-1 primes of Λ and $l_\mathfrak{p}(M)$ is the length of $M_\mathfrak{p}$ as a $\Lambda_\mathfrak{p}$-module. Note that $l_\mathfrak{p}(M) > 0$ if and only if $\mathfrak{p} \in \mathrm{Supp}_\Lambda(M)$.

It follows from the definition that Char_Λ is multiplicative on exact sequences and that $\mathrm{Char}_\Lambda(M)$ is a principal ideal. By Lemma A.1, $\mathrm{Char}_\Lambda(M) = \Lambda$ if and only if $\Lambda/\mathrm{Char}_\Lambda(M)$ is finite if and only if M is finite. We have the following consequence of Corollary A.2:

Lemma A.3. *Let N and M be finitely generated Λ-modules and suppose that N is torsion. Then $M \otimes_\Lambda N$ is finite if and only if $M/\mathrm{Char}_\Lambda(N)M$ is finite.*

Proof. This is clear from Corollary A.2, as $\mathrm{Supp}_\Lambda(N) = \mathrm{Supp}_\Lambda(\Lambda/\mathrm{Char}_\Lambda(N))$. □

In the body of the paper, we often use Lemma A.3 in the following form:

Lemma A.4. *Let N, N', and M be finitely generated Λ-modules, and suppose that N and N' are torsion and that $\mathrm{Char}_\Lambda(N) = \mathrm{Char}_\Lambda(N')$. Then $M \otimes_\Lambda N$ is finite if and only if $M \otimes_\Lambda N'$ is finite.*

Fitting ideals. Let R be a commutative, noetherian ring. For a finitely generated R-module M, we define $\mathrm{Fitt}_R(M) \subset R$, the Fitting ideal of M, as follows. Let

$$R^m \xrightarrow{A} R^n \longrightarrow M \longrightarrow 0$$

be a presentation of M. Then $\mathrm{Fitt}_R(M)$ is defined to be the R-module generated by all the (n,n)-minors of the matrix A. This does not depend on the choice of resolution (see [Mazur and Wiles 1984, Appendix]).

The following lemma is a result of the independence of resolution:

Lemma A.5. *If $\phi : R \to R'$ is a ring homomorphism and M is an R-module, then*

$$\mathrm{Fitt}_{R'}(M \otimes_R R') \subset R'$$

is the ideal generated by $\phi(\mathrm{Fitt}_R(M))$.

We consider the case $R = \Lambda$. The following relation to Char_Λ is a well-known:

Lemma A.6. *If M is finitely generated and torsion, then $\mathrm{Char}_\Lambda(M)$ is the unique principal ideal such that $\mathrm{Fitt}_\Lambda(M) \subset \mathrm{Char}_\Lambda(M)$ and $\mathrm{Char}_\Lambda(M)/\mathrm{Fitt}_\Lambda(M)$ is finite.*

Proof. Using Lemma A.5, we see that for any prime $\mathfrak{p}$ of Λ,

$$\mathrm{Fitt}_{\Lambda_\mathfrak{p}}(M_\mathfrak{p}) = \mathrm{Fitt}_\Lambda(M)_\mathfrak{p}.$$

Using that $\Lambda_\mathfrak{p}$ is a DVR for a height-1 prime $\mathfrak{p}$, we have $\mathrm{Fitt}_{\Lambda_\mathfrak{p}}(M_\mathfrak{p}) = \mathfrak{p}^{l_\mathfrak{p}(M)}$ by the structure theorem for modules over a PID. We have then $\mathrm{Fitt}_\Lambda(M)_\mathfrak{p} = \mathrm{Char}_\Lambda(M)_\mathfrak{p}$ for all $\mathfrak{p}$ of height 1. Lemma A.1 then implies that $\mathrm{Char}_\Lambda(M)/\,\mathrm{Fitt}_\Lambda(M)$ is finite.

For uniqueness, suppose $\mathrm{Fitt}_\Lambda(M) \subset (f)$ has finite quotient. Then $\mathrm{Fitt}_\Lambda(M)_\mathfrak{p} = (f)_\mathfrak{p}$ for each height-1 prime $\mathfrak{p}$. This determines the prime factorization of f. □

Using this relation, we can deduce the following:

Lemma A.7. *Let M and N be two finitely generated torsion Λ-modules. Assume that* $\mathrm{Fitt}_\Lambda(N) \subset \mathrm{Char}_\Lambda(M)$. *If a morphism $f : M \to N$ has finite cokernel, then it has finite kernel.*

Proof. Indeed, if f has finite cokernel, then

$$\mathrm{Char}_\Lambda(M) = \mathrm{Char}_\Lambda(\ker(f))\,\mathrm{Char}_\Lambda(N), \quad \text{and so} \quad \mathrm{Char}_\Lambda(M) \subset \mathrm{Char}_\Lambda(N).$$

Since $\mathrm{Char}_\Lambda(N)/\,\mathrm{Fitt}_\Lambda(N)$ is finite, this implies that $\mathrm{Char}_\Lambda(M)/\,\mathrm{Fitt}_\Lambda(N)$ is finite. By Lemma A.6, this implies that $\mathrm{Char}_\Lambda(N) = \mathrm{Char}_\Lambda(M)$. The result follows by multiplicativity of characteristic ideals. □

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pwake@math.uchicago.edu *Department of Mathematics, University of Chicago, Eckhart Hall, Chicago, IL 60637, United States*

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