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On the normalized arithmetic Hilbert function

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Let $X \subset \mathbb{P}^N_{\overline{\mathbb{Q}}}$ be a subvariety of dimension n, and let $\mathcal{H}_{\text{norm}}(X;\,\cdot)$ be the normalized arithmetic Hilbert function of X introduced by Philippon and Sombra. We show that this function admits the asymptotic expansion

$$\mathcal{H}_{\text{norm}}(X;D) = \frac{\hat{h}(X)}{(n+1)!}D^{n+1} + o(D^{n+1}), \quad \forall D \gg 1,$$

where $\hat{h}(X)$ is the normalized height of X. This gives a positive answer to a question raised by Philippon and Sombra.

1. Introduction

In [Philippon and Sombra 2008], the authors introduce an arithmetic Hilbert function defined for any subvariety in $\mathbb{P}^N$, the projective space of dimension N over $\overline{\mathbb{Q}}$. This function measures the binary complexity of the subvariety. In the case of toric subvarieties, a result of Philippon and Sombra shows that the asymptotic behavior of the associated normalized arithmetic Hilbert function is related to the normalized height of the subvariety considered; see [Philippon and Sombra 2008, Proposition 0.4]. This result is an important step toward the proof of the main theorem of the same paper, which is an explicit formula for the normalized height of projective translated toric varieties; see [Philippon and Sombra 2008, Théorème 0.1].

In [Philippon and Sombra 2008, Question 2.2], the authors ask if the normalized arithmetic Hilbert function admits an asymptotic expansion similar to the toric case. More precisely, given X a subvariety of dimension n in $\mathbb{P}^N$, the projective space of dimension N over $\overline{\mathbb{Q}}$, can we find a real $c(X) \geq 0$ such that

$$\mathcal{H}_{\text{norm}}(X;D) = \frac{c(X)}{(n+1)!}D^{n+1} + o(D^{n+1})?$$

If so, do we have $c(X) = \hat{h}(X)$, where $\hat{h}(X)$ is the normalized height of X?

In this article, we give an affirmative answer to this question.

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Theorem 1.1. *Let $X \subset \mathbb{P}^N$ be a subvariety of dimension n in $\mathbb{P}^N$. Then the normalized arithmetic Hilbert function associated to X admits the asymptotic expansion*

$$\mathcal{H}_{\mathrm{norm}}(X; D) = \frac{\hat{h}(X)}{(n+1)!} D^{n+1} + o(D^{n+1}), \quad \forall D \gg 1.$$

The notion of normalized height plays an important role in the diophantine approximation on tori, particularly in Bogomolov's and generalized Lehmer's problems; see [David and Philippon 1999; Amoroso and David 2003]. A result of Zhang [1992] shows that a subvariety X with a vanishing normalized height is necessarily a union of toric subvarieties.

Gillet and Soulé [1992] proved an arithmetic Hilbert–Samuel formula as a consequence of the arithmetic Riemann–Roch theorem. Roughly speaking, this formula describes the asymptotic behavior of the arithmetic degree of a hermitian module defined by the global sections of the tensorial power of a positive hermitian line bundle on an arithmetic variety. Moreover, the leading term is given by the arithmetic degree of the hermitian line bundle. Later Abbès and Bouche [1995] gave a new proof for this result without using the arithmetic Riemann–Roch theorem. Randriambololona [2006] extended the result Gillet and Soulé to the case of coherent sheaf provided as a subquotient of a metrized vector bundle on an arithmetic variety.

Notation. Let $\mathbb{Q}$ be the field of rational numbers, $\mathbb{Z}$ the ring of integers, K a number field and $\mathcal{O}_K$ its ring of integers. For N and D two integers in $\mathbb{N}$ we define $\mathbb{N}_D^{N+1} := \{a \in \mathbb{N}^{N+1} : a_0 + \cdots + a_N = D\}$, and we let $\mathbb{C}[x_0, \dots, x_N]_D$ (resp. $K[x_0, \dots, x_N]_D$) denote the complex vector space (resp. K-vector space) of homogeneous polynomials of degree D in $\mathbb{C}[x_0, \dots, x_N]$ (resp. in $K[x_0, \dots, x_N]$).

For any prime number p we denote by $|\cdot|_p$ the p-adic absolute value on $\mathbb{Q}$ such that $|p|_p = p^{-1}$ and by $|\cdot|_\infty$, or simply $|\cdot|$, the standard absolute value. Let $M_\mathbb{Q}$ be the set of these absolute values. We denote by M_K the set of absolute values of K extending the absolute values of $M_\mathbb{Q}$, and by M_K^∞ the subset in M_K of archimedean absolute values.

We denote by $\mathbb{P}^N$ the projective space over $\overline{\mathbb{Q}}$ of dimension N. A variety is assumed reduced and irreducible.

2. The proof of Theorem 1.1

We keep the same notation as in [Philippon and Sombra 2008]. Let ω be the Fubini–Study form on $\mathbb{P}^N(\mathbb{C})$. For any $k \in \mathbb{N}_{\geq 1} \cup \{\infty\}$, we denote by h_k the hermitian metric on $\mathcal{O}(1)$ given by

$$h_k(\cdot\,,\cdot) = \frac{|\cdot|^2}{(|x_0|^{2k} + \cdots + |x_N|^{2k})^{1/2k}} \quad \text{and} \quad h_\infty(\cdot\,,\cdot) = \frac{|\cdot|^2}{\max(|x_0|, \dots, |x_N|)^2},$$

and we let $\overline{\mathcal{O}(1)}_k := (\mathcal{O}(1), h_k)$ and $\omega_k := c_1(\mathcal{O}(1), h_k)$ for any $k \in \mathbb{N} \cup \{\infty\}$. Note that $\omega_k = (1/k)[k]^*\omega$, where $[k] : \mathbb{P}^N(\mathbb{C}) \to \mathbb{P}^N(\mathbb{C}),\ [x_0 : \cdots : x_N] \mapsto [x_0^k : \cdots : x_N^k]$. Observe that the sequence $(\omega_k)_{k\in\mathbb{N}_{\geq 1}}$ converges weakly to the current ω_∞. We consider the normalized volume form

$$\Omega_k := \omega_k^{\wedge N}, \quad \forall k \in \mathbb{N}_{\geq 1} \cup \{\infty\}.$$

For any $k \in \mathbb{N}_{\geq 1} \cup \{\infty\}$, the metrics of $\overline{\mathcal{O}(1)}_k$ and Ω_k define a scalar product $\mathbb{C}[x_0, \ldots, x_N]_D$ denoted by $\langle \cdot, \cdot \rangle_k$ and given by

$$\langle f, g \rangle_k := \int_{\mathbb{P}^N(\mathbb{C})} h_k^{\otimes D}(f, g)\Omega_k, \tag{1}$$

for any $f = \sum_a f_a x^a,\ g = \sum_a g_a x^a$ in $\mathbb{C}[x_0, \ldots, x_N]_D$ with $f_a, g_a \in \mathbb{C}$. We denote by $\|\cdot\|_k$ the associated norm for any $k \in \mathbb{N}_{\geq 1} \cup \{\infty\}$. Note that $\langle f, g \rangle_\infty = \sum_a f_a \bar{g}_a$ and $\|x^a\|_\infty = 1$ for any $a \in \mathbb{N}_D^{N+1}$ and $D \in \mathbb{N}$.

Let $X \subset \mathbb{P}^N$ be a subvariety defined over a number field K. Define an embedding $\sigma_v : K \to \mathbb{C}$, where $v \in M_K^\infty$. For any $p_1, \ldots, p_l \in K[x_0, \ldots, x_N]_D$, we set

$$\|p_1 \wedge \cdots \wedge p_l\|_{k,v} := \|\sigma_v(p_1) \wedge \cdots \wedge \sigma_v(p_l)\|_k, \quad \forall k \in \mathbb{N} \cup \{\infty\}.$$

Define $\mathcal{O}(D) := \mathcal{O}(1)^{\otimes D}$. We let $M := \Gamma(\Sigma, \mathcal{O}(D)_{|\Sigma})$ be the $\mathcal{O}_K$-module of global sections of $\mathcal{O}(D)_{|\Sigma}$, where Σ is the Zariski closure of X in $\mathbb{P}^N_{\mathcal{O}_K}$. For any $v \in M_K^\infty$, we set $\Gamma(\Sigma, \mathcal{O}(D)_{|\Sigma})_{\sigma_v} := \Gamma(\Sigma, \mathcal{O}(D)_{|\Sigma}) \otimes_{\sigma_v} \mathbb{C}$. We consider the following restriction map:

$$\pi : \Gamma(\mathbb{P}^N_{\mathcal{O}_K}, \mathcal{O}(D))_{\sigma_v} \to \Gamma(\Sigma, \mathcal{O}(D)_{|\Sigma})_{\sigma_v} \to 0.$$

The space $\Gamma(\mathbb{P}^N_{\mathcal{O}_K}, \mathcal{O}(D))_{\sigma_v}$ is identified canonically to $K_\sigma[x_0, \ldots, x_N]_D$. For any $k \in \mathbb{N}_{\geq 1} \cup \{\infty\}$, this space can be endowed by the scalar product induced by Ω_k and h_k and denoted by $\langle \cdot, \cdot \rangle_{k,v}$, where

$$\langle f, g \rangle_{k,v} = \langle \sigma_v(f), \sigma_v(g) \rangle_k$$

for any $f, g \in \Gamma(\mathbb{P}^N_{\mathcal{O}_K}, \mathcal{O}(D))_{\sigma_v}$. Since $\mathcal{O}(1)$ is ample, there exists a $D_0 \in \mathbb{N}$ such that, for any $D \geq D_0$, the restriction map is surjective. Let $D \geq D_0$. For any $k \in \mathbb{N} \cup \{\infty\}$, we denote by $\|\cdot\|_{k,v,\mathrm{quot}}$ the quotient norm induced by π and $\|\cdot\|_{k,v}$. Following [Philippon and Sombra 2008, p. 348], we endow $\Gamma(\Sigma, \mathcal{O}(D)_{|\Sigma})_{\sigma_v}$ with $\|\cdot\|_{k,v,\mathrm{quot}}$, for any $k \in \mathbb{N}_{\geq 1} \cup \{\infty\}$. By this construction, M can be equipped with a structure of a hermitian $\mathcal{O}_K$-module, denoted by $\bar{M}_k$. If $f_1, \ldots, f_s \in M$, is a K-basis for $M \otimes_{\mathcal{O}_K} K$, then

$$\begin{aligned}\widehat{\deg}(\bar{M}_k) &= \widehat{\deg}(\overline{\Gamma(\Sigma, \mathcal{O}(D)_{|\Sigma})}_k) \\ &:= \frac{1}{[K:\mathbb{Q}]}\Big(\log \mathrm{Card}\big(\textstyle\bigwedge^s M/(f_1 \wedge \cdots \wedge f_s)\big) - \displaystyle\sum_{v:K\to\mathbb{C}} \log \|f_1 \wedge \cdots f_s\|_{k,v}\Big).\end{aligned}$$

The normalized arithmetic Hilbert function. Let $X \subset \mathbb{P}^N$ be a subvariety defined over a number field K and let $I := I(X) \subset K[x_0, \dots, x_N]$ be its ideal of definition. We set

$$\mathcal{H}_{\mathrm{geom}}(X; D) := \dim_K\big(K[x_0, \dots, x_N]/I\big)_D = \binom{D+N}{N} - \dim_K(I_D).$$

The function $\mathcal{H}_{\mathrm{geom}}(X; \cdot)$ is known as *the classical geometric Hilbert function*. Philippon and Sombra [2008] introduced an arithmetic analogue of this function. Define $m := \mathcal{H}_{\mathrm{geom}}(X; D)$, $l := \dim_K(I_D)$ and let

$$\textstyle\bigwedge^l K[x_0, \dots, x_N]_D$$

be the l-th exterior power product of $K[x_0, \dots, x_N]_D$. For $f \in \bigwedge^l K[x_0, \dots, x_N]_D$ and $v \in M_K$ we denote by $|f|_v$ the sup-norm of the coefficients of f at the place v, with respect to the standard basis of $\bigwedge^l K[x_0, \dots, x_N]_D$.

Definition 2.1 [Philippon and Sombra 2008, Définition 2.1]. Let $p_1, \dots, p_l$ be a K-basis of I_D. We set

$$\mathcal{H}_{\mathrm{norm}}(X; D) = \sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log |p_1 \wedge \cdots \wedge p_m|_v.$$

By the product formula, this definition does not depend on the choice of the basis; also it is invariant under finite extensions of K. We call $\mathcal{H}_{\mathrm{norm}}(X; \cdot)$ the normalized arithmetic Hilbert function of X.

Following Philippon and Sombra, this arithmetic Hilbert function measures, for any $D \in \mathbb{N}$, the binary complexity of the K-vector space of forms of degree D in $K[x_0, \dots, x_N]$ modulo I. As pointed out by Philippon and Sombra [2008, Proposition 0.4], when X is a toric variety, the asymptotic behavior of its associated normalized arithmetic Hilbert function is related to $\hat{h}(X)$, the normalized height of X. The authors ask the following question:

Given X a subvariety in $\mathbb{P}^N$ of dimension n, can we find a real $c(X) \geq 0$ such that

$$\mathcal{H}_{\mathrm{norm}}(X; D) = \frac{c(X)}{(n+1)!} D^{n+1} + o(D^{n+1})?$$

If so, do we have $c(X) = \hat{h}(X)$?

We recall the following proposition, which gives a dual formulation for $\mathcal{H}_{\mathrm{norm}}$.

Proposition 2.2. *Let $q_1, \dots, q_m \in K[x_0, \dots, x_N]_D^\vee$ be a K-basis of* $\mathrm{Ann}(I_D)$. *Then*

$$\mathcal{H}_{\mathrm{norm}}(X; D) = \sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log |q_1 \wedge \cdots \wedge q_m|_v.$$

Proof. See [Philippon and Sombra 2008, Proposition 2.3]. □

For any $k \in \mathbb{N}_{\geq 1} \cup \{\infty\}$, we consider the arithmetic function

$$\begin{aligned}\mathcal{H}_{\text{arith}}(X; D, k) \\ := \sum_{v \in M_K^\infty} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log \|p_1 \wedge \cdots \wedge p_l\|_{k,v} \\ + \sum_{v \in M_K \setminus M_K^\infty} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log |p_1 \wedge \cdots \wedge p_l|_v + \tfrac{1}{2} \log(\gamma(N, D, k)),\end{aligned} \quad (2)$$

where $p_1, \ldots, p_l$ is a K-basis of I_D and

$$\gamma(N; D, k) := \prod_{a \in \mathbb{N}_D^{N+1}} \langle a, a \rangle_k^{-1}. \quad (3)$$

For $k = 1$, notice that $\mathcal{H}_{\text{arith}}(X; \cdot, 1)$ corresponds, up to a constant, to the arithmetic function $\mathcal{H}_{\text{arith}}(X; \cdot)$ considered in [Philippon and Sombra 2008, p. 346].

Similarly to $\mathcal{H}_{\text{norm}}$, the function $\mathcal{H}_{\text{arith}}$ admits a dual formulation. The scalar product $\langle \cdot, \cdot \rangle_k$ induces the following linear isomorphism:

$$\eta_k : \mathbb{C}[x_0, \ldots, x_N] \to \mathbb{C}[x_0, \ldots, x_N]^\vee, \quad f \mapsto \langle \cdot, f \rangle_k.$$

Thus $\mathbb{C}[x_0, \ldots, x_N]^\vee$ can be endowed with the dual scalar product, given by

$$\langle \eta_k(f), \eta_k(g) \rangle_k := \langle f, g \rangle_k, \quad \forall f, g \in \mathbb{C}[x_0, \ldots, x_N]_D.$$

We can check easily that, for any $k \in \mathbb{N} \cup \{\infty\}$, we have

$$\|\theta\|'_k := \sup_{g \in \mathbb{C}[x_0, \ldots, x_N] \setminus \{0\}} \frac{|\theta(g)|}{\|g\|_k} = \|f\|_k,$$

where $f \in \mathbb{C}[x_0, \ldots, x_N]$ is such that $\theta = \eta_k(f)$. Then $\|\theta\|'^2_k = \langle \theta, \theta \rangle_k$ for any $\theta \in \mathbb{C}[x_0, \ldots, x_N]^\vee$. It follows that

$$\langle \theta, \zeta \rangle_k = \sum_b \langle x^b, x^b \rangle_k^{-1} \theta_b \bar{\zeta}_b. \quad (4)$$

This product extends to $\wedge^m(\mathbb{C}[x_0, \ldots, x_N]_D^\vee)$ as follows:

$$\langle \theta_1 \wedge \cdots \wedge \theta_m, \zeta_1 \wedge \cdots \wedge \zeta_m \rangle_k := \det(\langle \theta_i, \zeta_j \rangle_k)_{1 \leq i, j \leq m}.$$

Proposition 2.3. *Let $q_1, \ldots, q_m \in K[x_0, \ldots, x_N]_D^\vee$ be a K-basis of* $\operatorname{Ann}(I_D)$. *Then*

$$\begin{aligned}\mathcal{H}_{\text{arith}}(X; D, k) \\ = \sum_{v \in M_K^\infty} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log \|q_1 \wedge \cdots \wedge q_m\|_{k,v}^\vee + \sum_{v \in M_K \setminus M_K^\infty} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log |q_1 \wedge \cdots \wedge q_m|_v.\end{aligned}$$

Proof. The proof is similar to [Philippon and Sombra 2008, Proposition 2.5]. □

Lemma 2.4. *There exists a D_1 such that, for any $D \geq D_1$ and any $k \in \mathbb{N}$, we have*

$$\mathcal{H}_{\mathrm{arith}}(X; D, k) = \widehat{\deg}(\overline{\Gamma(\Sigma, \mathcal{O}(D)_{|\Sigma})}_k) - \tfrac{1}{2}\mathcal{H}_{\mathrm{geom}}(X; D)\log\binom{D+N}{N}.$$

Proof. The proof is similar to [Philippon and Sombra 2008, Lemme 2.6]. Let $\mathcal{I}$ be the ideal sheaf of Σ and let $\Gamma(\mathbb{P}^N_{\mathcal{O}_K}, \mathcal{I}\mathcal{O}(D))$ be the $\mathcal{O}_K$-module of global sections of $\mathcal{I}\mathcal{O}(D)$, endowed with the scalar products induced by the scalar product $\langle \cdot, \cdot \rangle_k$. We claim that there exists an integer D_1, which does not depend on k, such that, for any $D \geq D_1$, we have

$$\widehat{\deg}(\overline{\Gamma(\Sigma, \mathcal{O}(D)_{|\Sigma})}_k) = \widehat{\deg}(\overline{\Gamma(\mathbb{P}^N_{\mathcal{O}_K}, \mathcal{O}(D))}_k) - \widehat{\deg}(\overline{\Gamma(\mathbb{P}^N_{\mathcal{O}_K}, \mathcal{I}\mathcal{O}(D))}_k).$$

Indeed, we can find a $D_1 \in \mathbb{N}$ such that, for all $D \geq D_1$, the following sequence is exact:

$$0 \to \Gamma(\mathbb{P}^N_{\mathcal{O}_K}, \mathcal{I}\mathcal{O}(D)_{|\Sigma}) \to \Gamma(\mathbb{P}^N_{\mathcal{O}_K}, \mathcal{O}(D)) \to \Gamma(\Sigma, \mathcal{O}(D)_{|\Sigma}) \to 0.$$

Then by [Randriambololona 2001, Lemme 2.3.6], the sequence of hermitian $\mathcal{O}_K$-modules

$$0 \to \overline{\Gamma(\mathbb{P}^N_{\mathcal{O}_K}, \mathcal{I}\mathcal{O}(D)_{|\Sigma})}_k \to \overline{\Gamma(\mathbb{P}^N_{\mathcal{O}_K}, \mathcal{O}(D))}_k \to \overline{\Gamma(\Sigma, \mathcal{O}(D)_{|\Sigma})}_k \to 0$$

is exact, where the metrics of $\overline{\Gamma(\mathbb{P}^N_{\mathcal{O}_K}, \mathcal{I}\mathcal{O}(D)_{|\Sigma})}_k$ and $\overline{\Gamma(\Sigma, \mathcal{O}(D)_{|\Sigma})}_k$ are induced by the metric of $\overline{\Gamma(\mathbb{P}^N_{\mathcal{O}_K}, \mathcal{O}(D))}_k$.

We have

$$\widehat{\deg}(\overline{\Gamma(\mathbb{P}^N_{\mathcal{O}_K}, \mathcal{O}(D))}_k) = \tfrac{1}{2}\log(\gamma(N; D, k)) + \tfrac{1}{2}\binom{D+N}{N}\log\binom{N+D}{N}. \tag{5}$$

As in the proof of [Philippon and Sombra 2008, Lemme 2.6], and keeping the same notation, we have

$$\begin{aligned}\widehat{\deg}&(\overline{\Gamma(\Sigma, \mathcal{O}(D)_{|\Sigma})}_k)\\ &= \tfrac{1}{2}\log(\gamma(N; D, k)) + \tfrac{1}{2}\mathcal{H}_{\mathrm{geom}}(X; D)\log\binom{N+D}{N}\\ &\quad + \sum_{v \in M_K^\infty}\frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]}\log\|p_1 \wedge \cdots \wedge p_l\|^\vee_{k,v}\\ &\quad - \frac{1}{[K : \mathbb{Q}]}\log \mathrm{Card}\big(\textstyle\bigwedge^l(I_{\mathcal{O}_K})/(p_1 \wedge \cdots \wedge p_l)\big).\end{aligned} \tag{6}$$

The last term in (6) does not depend on the metric. It is computed in [Philippon and Sombra 2008, p. 349]; we have

$$\frac{1}{[K:\mathbb{Q}]}\log\mathrm{Card}\big(\textstyle\bigwedge^l(I_{\mathcal{O}_K})/(p_1\wedge\cdots\wedge p_l)\big) = -\sum_{v\in M_K\setminus M_K^\infty}\frac{[K_v:\mathbb{Q}_v]}{[K:\mathbb{Q}]}\log|p_1\wedge\cdots\wedge p_l|_v. \quad\square$$

By [Randriambololona 2006, Théorème A], we have

$$\widehat{\deg}(\overline{\Gamma(\Sigma,\mathcal{O}(D)_{|\Sigma})_k}) = \frac{h_{\overline{\mathcal{O}(1)}_k}(X)}{(n+1)!}D^{n+1}+o(D^{n+1}),\quad \forall D\gg 1, \tag{7}$$

where $h_{\overline{\mathcal{O}(1)}_k}(X)$ denotes the height of the Zariski closure of X in $\mathbb{P}^N_{\mathcal{O}_K}$ with respect to $\overline{\mathcal{O}(1)}_k$. Since $\frac{1}{2}\mathcal{H}_{\mathrm{geom}}(X;D)\log\binom{D+N}{N}=o(D^{n+1})$ for $D\gg 1$, by Lemma 2.4, we get

$$\mathcal{H}_{\mathrm{arith}}(X;D,k) = \frac{h_{\overline{\mathcal{O}(1)}_k}(X)}{(n+1)!}D^{n+1}+o(D^{n+1}),\quad \forall D\gg 1. \tag{8}$$

Let $q_1,\dots,q_m\in K[x_0,\dots,x_N]^\vee$ be a K-basis of $\mathrm{Ann}(I_D)$. For any finite subset M in $\mathbb{N}^{N+1}_D$ of cardinal m, we set $q_M := (q_{jb})_{1\le j\le m, b\in M}\in K^{m\times m}$ where the q_{jb} are such that $q_j=\sum_{b\in\mathbb{N}^{N+1}_D}q_{jb}(x^b)^\vee$. For any $v\in M_K^\infty$, we have

$$\begin{aligned}|q_1\wedge\cdots\wedge q_m|_v &= \max\{|\det(q_M)|_v : M\subset\mathbb{N}^{N+1}_D,\ \mathrm{Card}(M)=m\}\\ &\le\Bigg(\sum_{M;\mathrm{Card}(M)=m}\Bigg(\prod_{b\in M}\langle b,b\rangle^{-1}_{v,k}\Bigg)|\det(q_M)|_v^2\Bigg)^{1/2}.\end{aligned} \tag{9}$$

(We use the inequality $\langle x^a,x^a\rangle_k=\int_{\mathbb{P}^N(\mathbb{C})}h_{\overline{\mathcal{O}(D)}_k}(x^a,x^a)\Omega_k\le 1$ for any $a\in\mathbb{N}^{N+1}_D$, which follows from $h_{\overline{\mathcal{O}(D)}_k}(x^a,x^a)\le h_{\overline{\mathcal{O}(D)}_\infty}(x^a,x^a)\le 1$ on $\mathbb{P}^N(\mathbb{C})$ and the facts that $\Omega_k>0$ on $\mathbb{P}^N(\mathbb{C})$ and $\int_{\mathbb{P}^N(\mathbb{C})}\Omega_k=1$.)

Then

$$|q_1\wedge\cdots\wedge q_m|_v\le\|q_1\wedge\cdots\wedge q_m\|^\vee_{k,v},\quad \forall k\in\mathbb{N}. \tag{10}$$

By Propositions 2.2 and 2.3, we get

$$\mathcal{H}_{\mathrm{norm}}(X;D)\le\mathcal{H}_{\mathrm{arith}}(X;D,k),\quad\forall k\in\mathbb{N}. \tag{11}$$

By (8), the previous inequality gives

$$\limsup_{D\to\infty}\frac{(n+1)!}{D^{n+1}}\mathcal{H}_{\mathrm{norm}}(X;D)\le h_{\overline{\mathcal{O}(1)}_k}(X),\quad\forall k\in\mathbb{N}. \tag{12}$$

We know that $(h_k)_{k\in\mathbb{N}}$ converges uniformly to h_∞ on $\mathbb{P}^N(\mathbb{C})$. Fix $0 < \varepsilon < 1$. Then there exists a $k_0 \in \mathbb{N}$ such that, for any $k \geq k_0$, we have

$$(1-\varepsilon)^{2D} \leq \frac{(\max(|x_0|_v, \dots, |x_N|_v))^{2D}}{(|x_0|_v^{2k} + \dots + |x_N|_v^{2k})^{D/k}} \leq (1+\varepsilon)^{2D}, \quad \forall x \in \mathbb{P}^N(\mathbb{C}),\ \forall D \in \mathbb{N}.$$

Thus, for any $k \geq k_0,\ D \in \mathbb{N}_{\geq 1}$ and $a \in \mathbb{N}_D^{N+1}$, we get

$$\langle x^a, x^a\rangle_k \geq (1-\varepsilon)^{2D} \int_{\mathbb{P}^N(\mathbb{C})} h_\infty^{\otimes D}(x^a, x^a)\omega_k^N. \tag{13}$$

We have

$$\begin{aligned}
&\int_{\mathbb{P}^N(\mathbb{C})} h_\infty^{\otimes D}(x^a, x^a)\omega_k^N \\
&\qquad = \int_{\mathbb{C}^N} \frac{|z^{2a}|}{\max(1, |z_1|, \dots, |z_N|)^{2D}} \frac{k^N \prod_{i=1}^N |z_i|^{2(k-1)} \prod_{i=1}^N dz_i \wedge d\bar{z}_i}{\left(1+\sum_{i=1}^N |z_i^{2k}|\right)^{N+1}} \\
&\qquad = 2^N \int_{(\mathbb{R}^+)^N} \frac{k^N r^{a+k-1}}{\max(1, r_1, \dots, r_N)^{2D}} \frac{\prod_{i=1}^N dr_i}{\left(1+\sum_{i=1}^N r_i^k\right)^{N+1}} \\
&\qquad = 2^N \int_{(\mathbb{R}^+)^N} \frac{r^{a/k}}{\max_i(1, r_1, \dots, r_N)^{D/k}} \frac{\prod_{i=1}^N dr_i}{\left(1+\sum_{i=1}^N r_i^k\right)^{N+1}} \\
&\qquad = 2^N \sum_{j=0}^N \int_{E_j} \frac{r^{a/k}}{\max_i(1, r_1, \dots, r_N)^{D/k}} \frac{\prod_{i=1}^N dr_i}{\left(1+\sum_{i=1}^N r_i\right)^{N+1}},
\end{aligned}$$

where $E_j := \{x \in (\mathbb{R}^+)^N : x_j \geq 1,\ x_l \leq x_j \text{ for } l = 1, \dots, N\}$ for $j = 1, \dots, N$ and $E := \{x \in (\mathbb{R}^+)^N : x_l \leq 1, \text{ for } l = 1, \dots, N\}$. Using the function

$$(\mathbb{R}^{*+})^N \to (\mathbb{R}^{*+})^N, \quad x = (x_1, \dots, x_N) \mapsto \left(\frac{x_1}{x_j}, \dots, \frac{x_{j-1}}{x_j}, \frac{1}{x_j}, \dots, \frac{x_n}{x_j}\right),$$

for $j = 1, \dots, N$, we can show that there exists a $b^{(j)} = (b_1^{(j)}, \dots, b_N^{(j)}) \in \mathbb{N}^N$ such that

$$\int_{E_j} \frac{r^{a/k}}{\max_i(1, r_1, \dots, r_N)^{D/k}} \frac{\prod_{i=1}^N dr_i}{\left(1+\sum_{i=1}^N r_i\right)^{N+1}} = \int_E r^{b^{(j)}/k} \frac{\prod_{i=1}^N dr_i}{\left(1+\sum_{i=1}^N r_i\right)^{N+1}}. \tag{14}$$

We set $b^{(0)} := a$. Then

$$\int_{\mathbb{P}^N(\mathbb{C})} h_\infty^D(x^a, x^a)\omega_k^N = 2^N \sum_{j=0}^N \int_E r^{b^{(j)}/k} \frac{\prod_{i=1}^N dr_i}{\left(1+\sum_{i=1}^N r_i\right)^{N+1}}. \tag{15}$$

Let $0 < \delta < 1$, and set $E_\delta := \{x \in E : x_l \geq \delta \text{ for } l = 1, \ldots, N\}$. From (13) and (15), we obtain

$$\langle x^a, x^a\rangle_k \geq (1-\varepsilon)^{2D} 2^N \sum_{j=0}^{N} \int_{E_\delta} r^{b^{(j)}/k} \frac{\prod_{i=1}^N dr_i}{\bigl(1+\sum_{i=1}^N r_i\bigr)^{N+1}} \geq (1-\varepsilon)^{2D} 2^N (N+1)\delta^{D/k}\mu_\delta,$$

where $\mu_\delta := \int_{E_\delta} \prod_{i=1}^N dr_i / \bigl(1+\sum_{i=1}^N r_i\bigr)^{N+1}$.

Thus,

$$\langle x^a, x^a\rangle_k^{-1} \leq (1-\varepsilon)^{-2D}\delta^{-D/k}{\mu_\delta}^{-1}, \quad \forall k \geq k_0,\ \forall D \in \mathbb{N}_{\geq 1},\ \forall a \in \mathbb{N}_D^{N+1}. \tag{16}$$

Then, for any $k \geq k_0$ and $D \geq D_1$,

$$\begin{aligned}\|q_1 \wedge \cdots \wedge q_m\|_{k,v}^{\vee} &\leq \Biggl(\sum_{M;\,\mathrm{Card}(M)=m} \Bigl(\prod_{b\in M} \langle b, b\rangle_{v,k}^{-1}\Bigr)\Biggr)^{1/2} |q_1 \wedge \cdots \wedge q_m|_v \\ &\leq \mathrm{Card}\{M \subset \mathbb{N}_D^{N+1} : \mathrm{Card}(M) = m\}^{1/2} \\ &\qquad \times (1-\varepsilon)^{-mD}\delta^{-mD/k}\mu_\delta^{-m}|q_1 \wedge \cdots \wedge q_m|_v \\ &\leq \mathrm{Card}(\mathbb{N}_D^{N+1})(1-\varepsilon)^{-mD}\delta^{-mD/k}\mu_\delta^{-m}|q_1 \wedge \cdots \wedge q_m|_v \\ &= \binom{N+D}{N}^{1/2}(1-\varepsilon)^{-mD}\delta^{-mD/k}\mu_\delta^{-m}|q_1 \wedge \cdots \wedge q_m|_v,\end{aligned} \tag{17}$$

where the second line follows by (16).

Therefore,

$$\begin{aligned}&\mathcal{H}_{\mathrm{arith}}(X; D, k) \\ &\quad \leq \mathcal{H}_{\mathrm{norm}}(X; D) + \tfrac{1}{2}\log\binom{N+D}{N} - D\mathcal{H}_{\mathrm{geom}}(X; D)\log(1-\varepsilon) \\ &\qquad - \frac{D\mathcal{H}_{\mathrm{geom}}(X; D)}{k}\log\delta - \mathcal{H}_{\mathrm{geom}}(X; D)\log\mu_\delta.\end{aligned} \tag{18}$$

By (8), we obtain that

$$h_{\overline{\mathcal{O}(1)}_k}(X) \leq \liminf_{D\to\infty} \frac{(n+1)!}{D^{n+1}}\mathcal{H}_{\mathrm{norm}}(X; D) + O(\varepsilon) + \frac{\log\delta}{k}O(1), \quad \forall k \geq k_0. \tag{19}$$

Gathering (12) and (19), we conclude that, for any $0 < \varepsilon < 1$, there exists a $k_0 \in \mathbb{N}$ such that

$$\begin{aligned}&\limsup_{D\to\infty} \frac{(n+1)!}{D^{n+1}}\mathcal{H}_{\mathrm{norm}}(X; D) \leq h_{\overline{\mathcal{O}(1)}_k}(X) \\ &\quad \leq \liminf_{D\to\infty} \frac{(n+1)!}{D^{n+1}}\mathcal{H}_{\mathrm{norm}}(X; D) + O(\varepsilon) + \frac{\log\delta}{k}O(1), \quad \forall k \geq k_0.\end{aligned} \tag{20}$$

Since $\lim_{k\to\infty} h_{\overline{\mathcal{O}(1)}_k}(X) = h_{\overline{\mathcal{O}(1)}_\infty}(X)$ (see for instance [Zhang 1995]) and since $h_{\overline{\mathcal{O}(1)}_\infty}(X) = \hat{h}(X)$ (see [Philippon and Sombra 2008, p. 342]), we get

$$\liminf_{D\to\infty} \frac{(n+1)!}{D^{n+1}} \mathcal{H}_{\mathrm{norm}}(X;\, D) = \limsup_{D\to\infty} \frac{(n+1)!}{D^{n+1}} \mathcal{H}_{\mathrm{norm}}(X;\, D) = \hat{h}(X). \tag{21}$$

Thus, we have proved Theorem 1.1. □

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