Semiample invertible sheaves with semipositive continuous hermitian metrics

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Let \((L, h)\) be a pair of a semiample invertible sheaf and a semipositive continuous hermitian metric on a proper algebraic variety over \(\mathbb{C}\). In this paper, we prove that \((L, h)\) is semiample metrized, answering a generalization of a question of S. Zhang.

Introduction

Let \(X\) be a proper algebraic variety over \(\mathbb{C}\). Let \(L\) be an invertible sheaf on \(X\), and let \(h\) be a continuous hermitian metric of \(L\). We say that \((L, h)\) is semiample metrized if, for any \(\epsilon > 0\), there is \(n > 0\) such that, for any \(x \in X(\mathbb{C})\), we can find \(l \in H^0(X, L^\otimes n) \setminus \{0\}\) with

\[
\sup \{h^\otimes n(l, l)(w) \mid w \in X(\mathbb{C})\} \leq e^{\epsilon n} h^\otimes n(l, l)(x).
\]

Shouwu Zhang proposed the following question:

**Question 0.1** [Zhang 1995, Question 3.6]. If \(L\) is ample and \(h\) is smooth and semipositive, does it follow that \((L, h)\) is semiample metrized?

Theorem 3.5 of the same reference gives an affirmative answer in the case where \(X\) is smooth over \(\mathbb{C}\). The purpose of this paper is to give an answer for a generalization of the above question. First of all, we fix some notation: We say that \(L\) is semiample if there is a positive integer \(n_0\) such that \(L^\otimes n_0\) is generated by global sections. Moreover, \(h\) is said to be semipositive (or we say that \((L, h)\) is semipositive) if, for any point \(x \in X(\mathbb{C})\) and a local basis \(s\) of \(L\) on a neighborhood of \(x\), 

\[ -\log h(s, s) \text{ is plurisubharmonic around } x \]

(for the definition of plurisubharmonicity on a singular variety, see Section 1). Note that \(h\) is not necessarily smooth. By using the recent work of Coman, Guedj and Zeriahi [Coman et al. 2013], we have the following answer:

**Theorem 0.2.** If \(L\) is semiample and \(h\) is continuous and semipositive, then \((L, h)\) is semiample metrized.

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1. Plurisubharmonic functions on singular complex analytic spaces

Let $T$ be a reduced complex analytic space. An upper-semicontinuous function
\[ \varphi : T \to \mathbb{R} \cup \{-\infty\} \]
is said to be plurisubharmonic if $\varphi \not\equiv -\infty$ and, for each $x \in T$, there is an analytic closed embedding $\iota_x : U_x \hookrightarrow W_x$ of an open neighborhood $U_x$ of $x$ into an open set $W_x$ of $\mathbb{C}^{n_x}$ together with a plurisubharmonic function $\Phi_x$ on $W_x$ such that $\varphi|_{U_x} = \iota_x^*(\Phi_x)$. For an analytic map $f : T' \to T$ of reduced complex analytic spaces and a plurisubharmonic function $\varphi$ on $T$, it is easy to see that $\varphi \circ f$ is either identically $-\infty$ or plurisubharmonic on $T'$. By [Fornæss and Narasimhan 1980, Theorem 5.3.1], an upper-semicontinuous function $\varphi : T \to \mathbb{R} \cup \{-\infty\}$ is plurisubharmonic if and only if, for any analytic map $\varphi : \mathbb{D} \to T$, $\varphi \circ \varrho$ is either identically $-\infty$ or subharmonic on $\mathbb{D}$, where $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$. Moreover, if $T$ is compact and $\varphi$ is plurisubharmonic on $T$, then $\varphi$ is locally constant.

Let $\omega$ be a smooth $(1, 1)$-form on $T$, that is, in the same way as in the definition of plurisubharmonic functions, $\omega$ is a smooth $(1, 1)$-form on the regular part of $T$ and, for each $x \in T$, there is an analytic closed embedding $\iota_x : U_x \hookrightarrow W_x$ of an open neighborhood $U_x$ of $x$ into an open set $W_x$ of $\mathbb{C}^{n_x}$ together with a smooth $(1, 1)$-form $\Omega_x$ on $W_x$ such that $\omega|_{U_x} = \iota_x^*(\Omega_x)$. We assume that $\omega$ is locally given by $dd^c(u)$ for some smooth function $u$ on a neighborhood of $x$. Let $\phi$ be a quasip plurisubharmonic function on $T$; that is, for each $x \in T$, $\phi$ can be locally written as the sum of a smooth function and a plurisubharmonic function around $x$. We say that $\phi$ is $\omega$-plurisubharmonic if there is an open covering $T = \bigcup \lambda U_{\lambda}$, together with a smooth function $u_{\lambda}$ on $U_{\lambda}$ for each $\lambda$, such that $\omega|_{U_{\lambda}} = dd^c(u_{\lambda})$ and $\phi|_{U_{\lambda}} + u_{\lambda}$ is plurisubharmonic on $U_{\lambda}$. The condition for $\omega$-plurisubharmonicity is often denoted by $dd^c(|\phi|) + \omega \geq 0$.

Here we consider the following lemma:

**Lemma 1.1.** Let $f : X \to Y$ be a surjective and proper morphism of algebraic varieties over $\mathbb{C}$. Let $\varphi$ be a real-valued function on $Y(\mathbb{C})$.

1. $\varphi$ is continuous if and only if $\varphi \circ f$ is continuous.

2. Assume that $\varphi$ is continuous. Then $\varphi$ is plurisubharmonic if and only if $\varphi \circ f$ is plurisubharmonic.

**Proof.** (1) It is sufficient to see that if $\varphi \circ f$ is continuous, then $\varphi$ is continuous. Otherwise, there are $y \in Y(\mathbb{C})$, $\epsilon_0 > 0$ and a sequence $\{y_n\}$ on $Y(\mathbb{C})$ such that $\lim_{n \to \infty} y_n = y$ and $|\varphi(y_n) - \varphi(y)| \geq \epsilon_0$ for all $n$. We choose $x_n \in X(\mathbb{C})$ such that $f(x_n) = y_n$. As $f : X \to Y$ is proper, we can find a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x := \lim_{i \to \infty} x_{n_i}$ exists in $X(\mathbb{C})$. Note that

\[ f(x) = \lim_{i \to \infty} f(x_{n_i}) = \lim_{i \to \infty} y_{n_i} = y. \]
so that, as \( \varphi \circ f \) is continuous,
\[
\varphi(y) = (\varphi \circ f)(x) = \lim_{i \to \infty} (\varphi \circ f)(x_{n_i}) = \lim_{i \to \infty} \varphi(f(x_{n_i})) = \lim_{i \to \infty} \varphi(y_{n_i}),
\]
which is a contradiction, so that \( \varphi \) is continuous.

(2) We need to check that if \( \varphi \circ f \) is plurisubharmonic, then \( \varphi \) is plurisubharmonic. By using Chow’s lemma, we may assume that \( f : X \to Y \) is projective. Moreover, since the assertion is local with respect to \( Y \), we may further assume that there is a closed embedding \( \iota : X \hookrightarrow Y \times \mathbb{P}^N \) such that \( p \circ \iota = f \), where \( p : Y \times \mathbb{P}^n \to Y \) is the projection to the first factor. The remaining proof is same as the last part of the proof of [Demailly 1985, Theorem 1.7]. Let \( g : (\mathbb{D}, 0) \to (Y, y) \) be a germ of an analytic map. By the theorem of Fornæss and Narasimhan, it is sufficient to show that \( \varphi \circ g \) is subharmonic. Clearly we may assume that \( g \) is given by the normalization of a 1-dimensional irreducible germ \((C, y)\) in \((Y, y)\). Using hyperplanes in \( \mathbb{P}^N \), we can find \( x \in X \) and a 1-dimensional irreducible germ \((C', x)\) in \((X, x)\) such that \((C', x)\) lies over \((C, y)\). Let \( g' : (\mathbb{D}, 0) \to (X, x) \) be the germ of an analytic map given by the normalization of \((C', x)\). Then we have an analytic map \( \sigma : (\mathbb{D}, 0) \to (\mathbb{D}, 0) \) with \( g \circ \sigma = f \circ g' : \)
\[
\begin{array}{ccc}
(\mathbb{D}, 0) & \xrightarrow{g'} & (X, x) \\
\downarrow \sigma & & \downarrow f \\
(\mathbb{D}, 0) & \xrightarrow{g} & (Y, y)
\end{array}
\]
Changing a variable of \((\mathbb{D}, 0)\), we may assume that \( \sigma \) is given by \( \sigma(z) = z^m \) for some positive integer \( m \). Then \( \varphi \circ g \circ \sigma \) is subharmonic because \( \varphi \circ f \) is plurisubharmonic. Therefore, as \( \sigma \) is étale over the outside of 0, \( \varphi \circ g \) is subharmonic on the outside of 0, and hence \( \varphi \circ g \) is subharmonic on \((\mathbb{D}, 0)\) by the removability of singularities of subharmonic functions. \( \square \)

2. Descent of a semipositive continuous hermitian metric

Here, we consider a descent problem of a semipositive continuous hermitian metric.

**Theorem 2.1.** Let \( f : X \to Y \) be a surjective and proper morphism of algebraic varieties over \( \mathbb{C} \) with \( f_*\mathcal{O}_X = \mathcal{O}_Y \). Let \( L \) be an invertible sheaf on \( Y \). If \( h' \) is a semipositive continuous hermitian metric of \( f^*(L) \), then there is a semipositive continuous hermitian metric \( h \) of \( L \) such that \( h' = f^*(h) \).

**Proof.** Let \( h_0 \) be a continuous hermitian metric of \( L \) on \( Y \). There is a continuous function \( \phi \) on \( X(\mathbb{C}) \) such that \( h' = \exp(\phi) f^*(h_0) \). Let \( F \) be a subvariety of \( X \) such that \( F \) is an irreducible component of a fiber of \( f : X \to Y \). Then, as
\[
(f^*(L), h')|_F \simeq (\mathcal{O}_F, \exp(\phi|_F)),
\]
we can see that $-\phi|_F$ is plurisubharmonic, so that $\phi|_F$ is constant. Therefore, for any point $y \in Y(\mathbb{C})$, $\phi|_{\mu^{-1}(y)}$ is constant because $\mu^{-1}(y)$ is connected, and hence there is a function $\psi$ on $Y(\mathbb{C})$ such that $\psi \circ f = \phi$. By Lemma 1.1(1), $\psi$ is continuous, so that, if we set $h := \exp(\psi)h_0$, then $h$ is continuous on $Y(\mathbb{C})$ and $h' = f^*(h)$.

Finally, let us see that $h$ is semipositive. As this is a local question on $Y$, we may assume that there is a local basis $s$ of $L$ over $Y$. If we set $\varphi = -\log h(s, s)$, then $\varphi \circ f$ is plurisubharmonic because $h'$ is semipositive. Therefore, by Lemma 1.1(2), $\varphi$ is plurisubharmonic, as required.

3. The proof of Theorem 0.2

In the case where $X$ is smooth over $\mathbb{C}$, $L$ is ample and $h$ is smooth, this theorem was proved by Zhang [1995, Theorem 3.5]. First we assume that $L$ is ample. Then there are a positive integer $n_0$ and a closed embedding $X \hookrightarrow \mathbb{P}^N$ such that $\mathcal{O}_{\mathbb{P}^N}(1)|_X \cong L^\otimes n_0$. Let $h_{FS}$ be the Fubini–Study metric of $\mathcal{O}_{\mathbb{P}^n}(1)$. Let $\phi$ be the continuous function on $X(\mathbb{C})$ given by $h^\otimes n_0 = \exp(-\phi)h_{FS}|_X$. We set $\omega = c_1(\mathcal{O}_{\mathbb{P}^N}(1), h_{FS})$. Then $\phi$ is $(\omega|_X)$-plurisubharmonic. Therefore, by [Coman et al. 2013, Corollary C], there is a sequence $\{\varphi_i\}$ of smooth functions on $\mathbb{P}^N(\mathbb{C})$ with the following properties:

1. $\varphi_i$ is $\omega$-plurisubharmonic for all $i$.
2. $\varphi_i \geq \varphi_{i+1}$ for all $i$.
3. For $x \in X(\mathbb{C})$, $\lim_{i \to \infty} \varphi_i(x) = \phi(x)$.

Since $X$ is compact and $\phi$ is continuous, (3) implies that the sequence $\{\varphi_i\}$ converges to $\phi$ uniformly on $X(\mathbb{C})$. We choose $i$ such that $|\phi(x) - \varphi_i(x)| \leq \epsilon n_0/2$ for all $x \in X$. We set $h_i = \exp(-\varphi_i)h_{FS}$. Then $h_i$ is a semipositive smooth hermitian metric of $\mathcal{O}_{\mathbb{P}^N}(1)$. Therefore, there is a positive integer $n_1$ such that, for $x \in \mathbb{P}^N(\mathbb{C})$, we can find $l \in H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(n_1)) \setminus \{0\}$ with

$$\sup\{h_i^\otimes n_1(l, l)(w) \mid w \in \mathbb{P}^N(\mathbb{C})\} \leq e^{n_1(\epsilon n_0/2)} h_i^\otimes n_1(l, l)(x).$$

In particular, if $x \in X(\mathbb{C})$, then $l(x) \neq 0$ (so that $l|_X \neq 0$) and

$$\sup\{h_i^\otimes n_1(l, l)(w) \mid w \in X(\mathbb{C})\} \leq e^{\epsilon n_0 n_1/2} h_i^\otimes n_1(l, l)(x).$$

Note that

$$h_i^\otimes n_0 e^{-\epsilon n_0/2} \leq h_i \leq h_i^\otimes n_0$$

(3-1)

on $X(\mathbb{C})$, because $h_i = h_i^\otimes n_0 \exp(\phi - \varphi_i)$ and $-\epsilon n_0/2 \leq \phi - \varphi_i \leq 0$ on $X(\mathbb{C})$. Therefore,

$$\sup\{h_i^\otimes n_0 n_1(l, l)(w) \mid w \in X(\mathbb{C})\} e^{-n_0 n_1 \epsilon/2} \leq \sup\{h_i^\otimes n_1(l, l)(w) \mid w \in X(\mathbb{C})\}$$
Then the following theorem is a consequence of Theorem 0.2 together with the arguments used in the proof of Theorem 0.2.

Therefore, the assertion of the theorem follows from the previous observation.

In general, as $L$ is semiample, there are a positive integer $n_2$, a projective algebraic variety $Y$ over $\mathbb{C}$, a morphism $f : X \to Y$ and an ample invertible sheaf $A$ on $Y$ such that $f_*\mathcal{O}_X = \mathcal{O}_Y$ and $f^*(A) \simeq L^\otimes n_2$. Thus, by Theorem 2.1, there is a semipositive continuous hermitian metric $k$ of $A$ such that $(f^*(A), f^*(k)) \simeq (L^\otimes n_2, h^\otimes n_2)$. Therefore, the assertion of the theorem follows from [Zhang 1995, Theorem 2.2], in which the assumption that $Y$ actually assumed to be a subvariety of $X$. However, we can give a direct proof using ideas in the proof of Theorem 0.2.

4. A variant of Theorem 0.2

The following theorem is a consequence of Theorem 0.2 together with the arguments in [Zhang 1995, Theorem 3.3]. However, we can give a direct proof using ideas in the proof of Theorem 0.2.

**Theorem 4.1.** Let $X$ be a projective algebraic variety over $\mathbb{C}$. Let $L$ be an ample invertible sheaf on $X$ and let $h$ be a semipositive continuous hermitian metric of $L$. Let us fix a reduced subscheme $Y$ of $X$, $l \in H^0(Y, L|_Y)$ and a positive number $\epsilon$. Then, for the given $X$, $L$, $h$, $Y$, $l$ and $\epsilon$, there is a positive integer $n_1$ such that, for all $n \geq n_1$, we can find $l' \in H^0(X, L^\otimes n)$ with $l'|_Y = l^\otimes n$ and

$$\sup\{h^\otimes n(l', l')(w) \mid w \in X(\mathbb{C})\} \leq e^{n\epsilon} \sup\{h(l, l)(w) \mid w \in Y(\mathbb{C})\}^n.$$

**Proof.** In the case where $X$ is smooth over $\mathbb{C}$ and $h$ is smooth and positive, the assertion of the theorem follows from [Zhang 1995, Theorem 2.2], in which $Y$ is actually assumed to be a subvariety of $X$. However, the proof works well under the assumption that $Y$ is a reduced subscheme. First of all, let us see the theorem in the case where $X$ is smooth over $\mathbb{C}$ and $h$ is smooth and semipositive. As $L$ is ample, there is a positive smooth hermitian metric $t$ of $L$ with $t \leq h$. Let us choose a positive integer $m$ such that $e^{-\epsilon/2} \leq (t/h)^{1/m} \leq 1$ on $X(\mathbb{C})$. If we set $t_m = h^{1-1/m}t^{1/m}$, then $t_m$ is smooth and positive, so that, for a sufficiently large integer $n$, there is $l' \in H^0(X, L^\otimes n)$ such that $l'|_Y = l^\otimes n$ and

$$\sup\{t_m^\otimes n(l', l')(w) \mid w \in X(\mathbb{C})\} \leq e^{n\epsilon/2} \sup\{t_m(l, l)(w) \mid w \in Y(\mathbb{C})\}^n,$$

and hence the assertion follows because $e^{-\epsilon/2}h \leq t_m \leq h$ on $X(\mathbb{C})$.

For a general case, we use the same symbols $n_0$, $X \hookrightarrow \mathbb{P}^N$, $h_{FS}$, $\phi$, $\omega$ and $\{\varphi_i\}$ as in the proof of Theorem 0.2. Clearly we may assume that $l \neq 0$. Since $L$ is ample, if $a_0$ is a sufficiently large integer, then, for each $j = 0, \ldots, n_0 - 1$, there is
Let us fix a positive number $A$ such that
\[
\sup\{h^\otimes n_0a_0+j(l_j, l_j)(w) \mid w \in X(\mathbb{C})\} \leq e^A \sup\{h(l, l)(w) \mid w \in Y(\mathbb{C})\}^{n_0a_0+j} \quad (4-1)
\]
for $j = 0, \ldots, n_0 - 1$. We choose $i$ with $|\phi(x) - \varphi_i(x)| \leq \epsilon n_0/2$ for all $x \in X$, and we set $h_i = \exp(-\varphi_i)h_{FS}$. As $h_i$ is smooth and semipositive, for the given $\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1), h_i, Y, L^\otimes n_0$ (as an element of $H^0(Y, \mathcal{O}_{\mathbb{P}^N}(1)|_Y)$) and $n_0\epsilon/4$, there is a positive integer $a_1$ such that the assertion of the theorem holds for all $a \geq a_1$. We put
\[
n_1 := n_0 \max\left\{a_1 + a_0 + 1, \frac{4A}{n_0\epsilon} - 3a_0 + 1\right\}.
\]
Let $n$ be an integer with $n \geq n_1$. If we set $n = n_0(a + a_0) + j$ ($0 \leq j \leq n_0 - 1$), then
\[
a \geq a_1 \quad \text{and} \quad a \geq \frac{4A}{n_0\epsilon} - 4a_0,
\]
so that we can find $l'' \in H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(a))$ with $l''|_Y = L^\otimes n_0a$ and
\[
\sup\{h_i^\otimes a(l'', l'')(w) \mid w \in \mathbb{P}^N(\mathbb{C})\} \leq e^{a(n_0\epsilon/4)} \sup\{h_i(l^\otimes n_0, l^\otimes n_0)(w) \mid w \in Y(\mathbb{C})\}^a,
\]
which implies that
\[
\sup\{h^\otimes n_0a(l'', l'')(w) \mid w \in X(\mathbb{C})\} \leq e^{(3/4)n_0a\epsilon} \sup\{h(l, l)(w) \mid w \in Y(\mathbb{C})\}^{n_0a} \quad (4-2)
\]
because of (3-1). Here we set $l' = l'' \otimes l_j$. Then, $l'|_Y = L^\otimes n$ and, using (4-1) and (4-2), we have
\[
\sup\{h^\otimes n(l', l')(w) \mid w \in X(\mathbb{C})\} \\
\leq \sup\{h^\otimes n_0a(l'', l'')(w) \mid w \in X(\mathbb{C})\} \sup\{h^\otimes n_0a_0+j(l_j, l_j)(w) \mid w \in X(\mathbb{C})\} \\
\leq e^{(3/4)n_0a\epsilon + A} \sup\{h(l, l)(w) \mid w \in Y(\mathbb{C})\}^n,
\]
which implies the assertion because $(3/4)n_0a\epsilon + A \leq \epsilon n$. \hfill \Box

5. Arithmetic application

As an application of Theorem 0.2, we have the following generalization of the arithmetic Nakai–Moishezon criterion (see [Zhang 1995, Corollary 4.8]).

Corollary 5.1. Let $\mathcal{X}$ be a projective and flat integral scheme over $\mathbb{Z}$. Let $\mathcal{L}$ be an invertible sheaf on $\mathcal{X}$ such that $\mathcal{L}$ is nef on every fiber of $\mathcal{X} \to \mathbb{Z}$. Let $h$ be an $F_\infty$-invariant semipositive continuous hermitian metric of $\mathcal{L}$, where $F_\infty$ is the complex conjugation map $\mathcal{X}(\mathbb{C}) \to \mathcal{X}(\mathbb{C})$. If $\deg(\hat{\Delta}(\mathcal{L}, h)|_Y)^{\dim \mathcal{X}} > 0$ for all horizontal integral subschemes $Y$ of $\mathcal{X}$, then, for an $F_\infty$-invariant continuous hermitian invertible sheaf $(\mathcal{M}, k)$ on $\mathcal{X}$, $H^0(\mathcal{X}, \mathcal{L}^\otimes n \otimes \mathcal{M})$ has a basis consisting of strictly small sections for a sufficiently large integer $n$.
Proof. Let $X$ be the generic fiber of $\mathcal{X} \to \text{Spec}(\mathbb{Z})$ and let $Y$ be a subvariety of $X$. Let $\mathcal{Y}$ be the Zariski closure of $Y$ in $\mathcal{X}$. As
\[ \widehat{\deg}(\hat{c}_1((\mathcal{L}, h)|_{\mathcal{Y}})^{\dim \mathcal{Y}}) > 0, \]
$(\mathcal{L}, h)|_{\mathcal{Y}}$ is big by [Moriwaki 2012, Theorem 6.6.1], so that $H^0(\mathcal{Y}, \mathcal{L}^{\otimes n_0}|_{\mathcal{Y}}) \setminus \{0\}$ has a strictly small section for a sufficiently large integer $n_0$. Moreover, if we set $L = \mathcal{L}|_X$, then $L|_Y$ is big, and hence $\deg(L^{\dim Y} \cdot Y) > 0$ because $L$ is nef. Therefore, $L$ is ample by the Nakai–Moishezon criterion for ampleness. In particular, by Theorem 0.2, $h$ is semiample metrized. Thus the assertion follows from the arguments in [Zhang 1995, Theorem 4.2].

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