On an analytic version of
Lazard's isomorphism

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We prove a comparison theorem between locally analytic group cohomology and Lie algebra cohomology for locally analytic representations of a Lie group over a nonarchimedean field of characteristic 0. The proof is similar to that of van Est's isomorphism and uses only a minimum of functional analysis.

Introduction

In a seminal paper, Lazard [1965] established two basic theorems concerning the cohomology of a compact $\mathbb{Q}_p$-analytic Lie group G with Lie algebra $\mathfrak{g}$. Firstly, if V is a finite-dimensional $\mathbb{Q}_p$-vector space with continuous G-action, the natural map from locally analytic group cohomology $H^*_{\mathrm{an}}(G, V)$, defined in terms of locally analytic cochains, to continuous group cohomology $H^*_{\mathrm{cont}}(G, V)$ is an isomorphism. Secondly, there is a natural isomorphism between the direct limit $\mathrm{colim}_{G' \subset G} H^*_{\mathrm{cont}}(G', V)$, where G' runs through the system of open subgroups of G, and the Lie algebra cohomology $H^*(\mathfrak{g}, V)$. Hence, combining both, there is a natural isomorphism

$$\operatorname*{colim}_{G' \subset G \text{ open}} H^*_{\mathrm{an}}(G', V) \cong H^*(\mathfrak{g}, V). \tag{1}$$

These results play an important role in arithmetic geometry, in particular in the theory of Galois representations, or in the study of p-adic regulators [Huber and Kings 2011].

At least for certain Lie groups, integral and K-analytic versions have been obtained by Huber, Kings, and Naumann [Huber et al. 2011] when K is a finite extension of $\mathbb{Q}_p$. The proofs are based on Lazard's original argument via continuous group cohomology, and are not easily accessible. A somewhat simplified proof has been given by Lechner [2012] using formal group cohomology.

On the other hand, the situation for a real Lie group G is much more transparent. The analogous result is van Est's isomorphism $H_d^*(G, V) \cong H^*(\mathfrak{g}, K; V)$, which relates differentiable group cohomology with relative Lie algebra cohomology for a maximal compact subgroup $K \subseteq G$. Its proof is based on the following observations: The quotient G/K is contractible, hence the de Rham complex $\Omega^*(G/K, V)$ with coefficients in a G-representation V is a resolution of V. Moreover, for any k, the space $\Omega_d^k(G/K, V)$ is G-acyclic. Hence, $H_d^*(G, V)$ is computed by the G-invariants of the complex $\Omega^*(G/K, V)$, which is precisely the Chevalley–Eilenberg complex computing relative Lie algebra cohomology $H^*(\mathfrak{g}, K; V)$.

It is a natural question whether a similar argument works in the nonarchimedean situation. In this note, we show that this is indeed the case. This gives a direct proof of the isomorphism (1) and generalizes it with respect to the ground field and the coefficients:

Theorem. *Let K be a nonarchimedean field of characteristic 0. Let G be a locally K-analytic Lie group and $G \to \mathrm{Aut}(V)$ a locally analytic representation on a barreled locally convex K-vector space. Denote by $\mathfrak{g}$ the K-Lie algebra of G. Then there are natural isomorphisms*

$$\operatorname*{colim}_{G' \subset G \text{ open}} H_{\mathrm{an}}^*(G', V) \cong H^*(\mathfrak{g}, V),$$

where the colimit is taken over the system of open subgroups of G.

The rough argument is as follows: The de Rham complex $\Omega^*(G, V)$ is a resolution of the locally constant V-valued functions on G. As in the real case, each $\Omega^k(G, V)$ is G-acyclic, hence the cohomology of the locally constant V-valued functions on G is isomorphic to the Lie algebra cohomology $H^*(\mathfrak{g}, V)$ (see Sections 3 and 4 for precise results). The theorem then follows by taking the direct limit over the open subgroups of G.

The proof also shows that, for compact G, one can recover the locally analytic group cohomology from the Lie algebra cohomology as the invariants under the natural G-action: $H_{\mathrm{an}}^*(G, V) \cong H^*(\mathfrak{g}, V)^G$ (see Corollary 21).

Moreover, we describe the comparison map between locally analytic group cohomology and Lie algebra cohomology explicitly on the level of complexes: It is

given by differentiating locally analytic cocycles at 1 (see Section 5). As pointed out by the referee, one can use the methods of [Huber and Kings 2011] to show that, on cohomology groups, this comparison map agrees with the one studied by Lazard in the case that K is $\mathbb{Q}_p$ and V is finite-dimensional.

In order to apply usual arguments from homological algebra, we show, following [Flach 2008], that the locally analytic cochain cohomology groups can be interpreted as derived functors of the global section functor on a topos BG (Sections 1 and 2). The nice feature of this is that it gives a quick proof of the main results which requires only a minimum of functional analysis.

An alternative approach to the cohomology of locally analytic representations of Lie groups over finite extensions of $\mathbb{Q}_p$ is due to Kohlhaase [2011]. It is based on relative homological algebra. He obtains similar results under an additional assumption on the group, which, as he proves, is fulfilled in many cases. The cohomology groups he defines are finer than ours in the sense that they themselves carry a locally convex topology. In contrast to the groups we use, they do not always coincide with the cohomology groups defined in terms of locally analytic cochains.

I would like to thank the referee for useful remarks, in particular concerning the comparison of our isomorphism with Lazard's original one.

Notations and conventions. For the whole paper, we let K be a nonarchimedean field of characteristic 0, i.e., K is equipped with a nontrivial nonarchimedean absolute value $|\cdot|$ such that K is complete for the topology defined by $|\cdot|$. By a manifold we will always mean a paracompact, finite-dimensional locally K-analytic manifold. Note that, by [Schneider 2011, Corollary 18.8], any locally K-analytic Lie group is paracompact. For manifolds X, Y, we denote by $C^{\mathrm{an}}(X, Y)$ the set of locally K-analytic maps from X to Y. We will refer to them simply as analytic maps.

1. Locally analytic group cohomology

In this section, we describe the topos-theoretic approach to locally analytic group cohomology. We refer to [Flach 2008] for the case of continuous cohomology.

Denote by La the category of manifolds. We let Sh(La) be the category of sheaves on La with respect to the topology generated by open coverings. For this topology, every representable presheaf is a sheaf, hence we have the Yoneda embedding $y : \mathrm{La} \to \mathrm{Sh}(\mathrm{La})$.

Let G be a Lie group. Then $y(G)$ is a group object in Sh(La). The category of sheaves with a $y(G)$-action is a topos [SGA 4_1 1972, Exposé IV, 2.4], called the classifying topos of $y(G)$. It will be denoted by BG.[1] We denote its global section functor by $\Gamma : BG \to \mathrm{Set}, \Gamma(\mathscr{F}) = \mathrm{Hom}_{BG}(*, \mathscr{F}) = \mathscr{F}(*)^G$. Similarly, if X is an

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[1]More precisely, we assume the existence of universes and only consider manifolds which are elements of a given universe $\mathcal{U}$. Then Sh(La) and BG are $\mathcal{V}$-topoi for a universe $\mathcal{V}$ with $\mathcal{U} \in \mathcal{V}$.

object of BG, we denote by $\Gamma(X,-) = \mathrm{Hom}_{BG}(X,-)$ the functor of sections over X. As usual, we define cohomology groups via the derived functors of the global section functor:

Definition 2. Let $\mathscr{A}$ be an abelian group object of BG. Then we set

$$H^i(G,\mathscr{A}) := (R^i\Gamma)(\mathscr{A}).$$

Example 3. Let V be a finite-dimensional K-vector space with a linear G-action such that the map $G \times V \to V$ defining the action is analytic. This induces an action $y(G) \times y(V) \to y(V)$, and hence $y(V)$ can naturally be considered as an element of BG. We have $\Gamma(y(V)) = V^G$. In the next section, we will show that the higher cohomology groups $H^i(G, y(V))$ coincide with the cohomology groups defined in terms of locally analytic cochains with coefficients in V.

As another example, let M be a continuous G-module, i.e., a topological abelian group equipped with a linear G-action such that $G \times M \to M$ is continuous. Then we have the sheaf of continuous M-valued functions $C(-, M)$ on La. It also carries a natural action by $y(G)$. It follows from Proposition 8 below that the groups $H^i(G, C(-, M))$ can be identified with the continuous cochain cohomology groups of M.

We want to describe the cohomology groups defined in Definition 2 in terms of a concrete complex. We begin with some general considerations.

Let $\mathscr{T}$ be a topos, and let $\mathscr{G}$ be a group object in $\mathscr{T}$. For objects $\mathscr{A}, \mathscr{B}$ of $B\mathscr{G}$ the internal hom $\mathscr{H}om_{B\mathscr{G}}(\mathscr{A}, \mathscr{B})$ is given as follows: The underlying object of $\mathscr{T}$ is $\mathscr{H}om_{\mathscr{T}}(\mathscr{A}, \mathscr{B})$ and the action of $\mathscr{G}$ is given by the formula

$$(g\phi)(a) = g(\phi(g^{-1}a)).$$

Denote by $i : * \to \mathscr{G}$ the morphism from the trivial group in $\mathscr{T}$ to $\mathscr{G}$. It induces a geometric morphism of topoi (see [SGA 4₁ 1972, Exposé IV, 4.5])

$$i : \mathscr{T} \cong B* \to B\mathscr{G}.$$

The left adjoint i^* simply forgets the $\mathscr{G}$-action. The right adjoint is given by $i_*(\mathscr{F}) = \mathscr{H}om_{B\mathscr{G}}({}_l\mathscr{G}, \mathscr{F})$, where ${}_l\mathscr{G}$ is $\mathscr{G}$ with its natural left action, viewed as an object ob $B\mathscr{G}$, and $\mathscr{F}$ is viewed as object of $B\mathscr{G}$ with trivial $\mathscr{G}$-action. The functor i^* also has a left adjoint $i_!$ given by $\mathscr{F} \mapsto {}_l\mathscr{G} \times \mathscr{F}$ with $\mathscr{G}$-action via the first factor.

For an object $\mathscr{A} \in B\mathscr{G}$, we denote by $\mathscr{A}^\natural$ the object of $B\mathscr{G}$ with the same underlying object in $\mathscr{T}$ and trivial $\mathscr{G}$-action.

Lemma 4. *For $\mathscr{A}, \mathscr{B} \in B\mathscr{G}$ we have*

$$\mathscr{H}om_{B\mathscr{G}}({}_l\mathscr{G} \times \mathscr{A}, \mathscr{B}) \cong i_* \mathscr{H}om_{\mathscr{T}}(i^*\mathscr{A}, i^*\mathscr{B}).$$

Proof. Let $\mathscr{X}$ be an object of $B\mathscr{G}$. Then we have natural isomorphisms

$$\operatorname{Hom}_{B\mathscr{G}}(\mathscr{X}, i_* \mathscr{H}om_{\mathscr{T}}(i^*\mathscr{A}, i^*\mathscr{B})) \cong \operatorname{Hom}_{\mathscr{T}}(i^*\mathscr{X}, \mathscr{H}om_{\mathscr{T}}(i^*\mathscr{A}, i^*\mathscr{B}))$$

$$\cong \operatorname{Hom}_{\mathscr{T}}(i^*(\mathscr{X} \times \mathscr{A}), i^*\mathscr{B})$$

$$\cong \operatorname{Hom}_{B\mathscr{G}}(i_! i^*(\mathscr{X} \times \mathscr{A}), \mathscr{B})$$

$$\cong \operatorname{Hom}_{B\mathscr{G}}(\iota^{\mathscr{G}} \times (\mathscr{X} \times \mathscr{A})^{\natural}, \mathscr{B})$$

$$\cong \operatorname{Hom}_{B\mathscr{G}}(\iota^{\mathscr{G}} \times \mathscr{X} \times \mathscr{A}, \mathscr{B})$$

$$\cong \operatorname{Hom}_{B\mathscr{G}}(\mathscr{X}, \mathscr{H}om_{B\mathscr{G}}(\iota^{\mathscr{G}} \times \mathscr{A}, \mathscr{B})),$$

where we used the isomorphism $\iota^{\mathscr{G}} \times (\mathscr{X} \times \mathscr{A})^{\natural} \cong \iota^{\mathscr{G}} \times \mathscr{X} \times \mathscr{A}$ given by $(\mathrm{pr}_1, \text{action})$. This implies the lemma. $\square$

We now consider the case $\mathscr{T} = \operatorname{Sh}(\operatorname{La}), \mathscr{G} = y(G)$. For a sheaf $\mathscr{F}$ on La, the sheaf underlying $i_*\mathscr{F}$ is, by the above, given by $X \mapsto \mathscr{H}om_{\operatorname{Sh}(\operatorname{La})}(y(G), \mathscr{F})(X) \cong \mathscr{F}(G \times X)$ (Yoneda lemma).

Remark 5. By our general assumption, every manifold X in La is paracompact. By [Schneider 2011, Proposition 8.7], it is even *strictly paracompact*, i.e., every open covering of X can be refined by a covering by pairwise-disjoint open subsets. This implies in particular that the functor of sections over X is exact on the category of abelian sheaves on La.

Lemma 6. *The functor i_* from abelian sheaves on* La *to abelian group objects in BG is exact.*

Proof. Since i_* is a right adjoint, it is left exact. Consider an epimorphism $\mathscr{A} \to \mathscr{B}$ of abelian sheaves on La. By the above remark, the functor of sections over $G \times X$ is exact, and hence $\mathscr{A}(G \times X) \to \mathscr{B}(G \times X)$ is an epimorphism of abelian groups. From this we deduce that $i_*\mathscr{A} \to i_*\mathscr{B}$ is an epimorphism. $\square$

Corollary 7. *For any abelian sheaf $\mathscr{A}$ on* La, *we have*

$$H^i(G, i_*\mathscr{A}) \cong \begin{cases} \mathscr{A}(*) & \text{if } i = 0, \\ 0 & \text{else.} \end{cases}$$

Proof. Since the left adjoint i^* is exact, i_* sends injectives to injectives. Since i_* is exact and $\Gamma \circ i_*(-) = \operatorname{Hom}_{BG}(*, i_*(-)) \cong \operatorname{Hom}_{\operatorname{Sh}(\operatorname{La})}(*, -)$ we see that

$$H^i(G, i_*\mathscr{A}) \cong H^i_{\operatorname{Sh}(\operatorname{La})}(*, \mathscr{A}) \cong \begin{cases} \mathscr{A}(*) & \text{if } i = 0, \\ 0 & \text{else.} \end{cases} \quad \square$$

We let $E_\bullet G$ be the simplicial manifold given in degree p by $E_p G := G^{p+1}$, and $\phi^* : E_q G \to E_p G$, for $\phi : \{0 < \cdots < p\} \to \{0 < \cdots < q\}$, given by

$$(g_0, \ldots, g_q) \mapsto (g_{\phi(0)}, \ldots, g_{\phi(p)}).$$

The group G acts on $E_\bullet G$ via diagonal left multiplication. We denote the simplicial object of BG given by $y(E_\bullet G)$ equipped with diagonal $y(G)$-action by $_lE_\bullet G$.

For an abelian group object $\mathscr{A}$ in BG, the degree-wise sections over $_lE_\bullet G$ form a cosimplicial abelian group $\Gamma(_lE_\bullet G, \mathscr{A})$.

Proposition 8. *Let $\mathscr{A}$ be an abelian group object of BG. Then*

$$H^*(G, \mathscr{A}) \cong H^*(\Gamma(_lE_\bullet G, \mathscr{A})).$$

Proof. The projection $_ly(G) \to *$ is an epimorphism in BG. The Čech nerve of this morphism is precisely $_lE_\bullet G$. We thus have a quasi-isomorphism

$$\mathscr{A} \cong \mathscr{H}om_{BG}(*, \mathscr{A}) \xrightarrow{\sim} \mathscr{H}om_{BG}(_lE_\bullet G, \mathscr{A}).$$

Using Lemma 4 and Corollary 7 we see that the complex on the right-hand side consists of Γ-acyclic objects. We conclude using $\Gamma(\mathscr{H}om_{BG}(_lE_\bullet G, \mathscr{A})) = \mathrm{Hom}_{BG}(*, \mathscr{H}om_{BG}(_lE_\bullet G, \mathscr{A})) \cong \mathrm{Hom}_{BG}(_lE_\bullet G, \mathscr{A}) = \Gamma(_lE_\bullet G, \mathscr{A})$. $\qquad\square$

2. Locally analytic representations

In Example 3, we saw how to associate an abelian group object of BG to any finite-dimensional locally analytic representation of G. The goal of this section is to extend this to arbitrary locally analytic representations in the sense of [Schneider and Teitelbaum 2002], and to relate the cohomology groups thus defined to the cohomology groups defined in terms of analytic cochains (Proposition 12).

We begin by recalling some basic notions about analytic functions and locally analytic representations. References are [Féaux de Lacroix 1998] or [Schneider and Teitelbaum 2002, §§2,3].

If W is a Banach space, a function $f : X \to W$ from a manifold X to W is called analytic, if, in local charts, it is given by convergent power series with coefficients in W. The reader who is only interested in representations on Banach spaces can skip the following general definition and also all "admissibility" considerations later on. Let V be a locally convex separated K-vector space. A *BH-space*[2] for V is a continuous inclusion of a separated Banach space $W \hookrightarrow V$ [Féaux de Lacroix 1998, §1.2]. Let X be a manifold. A function $f : X \to V$ is called *analytic* if every $x \in X$ admits a neighborhood U and a BH-space $W \hookrightarrow V$ such that $f|_U$ factors through an analytic map $U \to W$. We denote the set of all analytic functions $X \to V$ by $C^{\mathrm{an}}(X, V)$. By [Féaux de Lacroix 1998, Korollar 2.4.4], $C^{\mathrm{an}}(X, V)$ is a module over the algebra of analytic functions $C^{\mathrm{an}}(X, K)$. For varying X, this is a sheaf on La denoted by $C^{\mathrm{an}}(-, V)$.

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[2]Banach–Hausdorff

A *topological representation* of the Lie group G on V is an action of G on V by continuous automorphisms, i.e., a homomorphism $G \to \mathrm{Aut}(V)$ to the abstract group of continuous automorphisms $\mathrm{Aut}(V)$ of V. The topological representation is called *locally analytic* if all orbit maps $G \to V, g \mapsto gv$, are analytic (this is called a weakly analytic representation in [Féaux de Lacroix 1998, Definition 3.1.5]).

Example 9. Let W be a Banach space. Then $\mathrm{Aut}(W)$ is an open subset of the Banach space of continuous endomorphisms $\mathrm{End}(W)$. Féaux de Lacroix [1998, Korollar 3.1.9] showed that a topological representation of G on W is locally analytic if and only if the corresponding homomorphism $r : G \to \mathrm{Aut}(W) \subset \mathrm{End}(W)$ is analytic. Assume that this is the case. Let X be a manifold, and let $\rho : X \to G$ and $f : X \to W$ be analytic maps. Then the point-wise product $\rho f : X \to W$, $x \mapsto \rho(x) f(x)$, is again analytic. Indeed, ρf equals the composition

$$X \xrightarrow{(\rho, \mathrm{id}_X)} G \times X \xrightarrow{r \times f} \mathrm{Aut}(W) \times W \subseteq \mathrm{End}(W) \times W \xrightarrow{\mathrm{ev}} W,$$

where the first two maps are analytic by assumption and the evaluation map ev is continuous and bilinear. It follows that we get an action of $y(G)$ on the sheaf $C^{\mathrm{an}}(-, W)$, and $C^{\mathrm{an}}(-, W)$ can naturally be viewed as an object of BG. This generalizes Example 3.

For a general locally analytic representation of G on V, this need no longer be true. Instead, we have to consider the subsheaf of $C^{\mathrm{an}}(-, V)$ of admissible functions, as we explain in the next paragraph. The example above shows that for representations on Banach spaces every analytic function is admissible.

Let $G \to \mathrm{Aut}(V)$ be a topological representation. We call an analytic function $f : X \to V$ *admissible* if the map $\hat{f} : G \times X \to V$, $(g, x) \mapsto gf(x)$ is analytic. Note that $\hat{f}$ is analytic if and only if its restriction $\hat{f}|_{U \times X}$ for some open subset $U \subset G$ is analytic. Indeed, if this is the case, then for any $h \in G$ the restriction $\hat{f}|_{hU \times X}$ is equal to the composition

$$(hU \times X) \xrightarrow{(g,x) \mapsto (h^{-1}g, x)} U \times X \xrightarrow{\hat{f}|_{U \times X}} V \xrightarrow{h \cdot} V,$$

where the first two maps are analytic and the last is continuous and linear. We define

$$\underline{V}(X) := C^{\mathrm{ad}}(X, V) := \{f \in C^{\mathrm{an}}(X, V) \mid f \text{ is admissible}\}.$$

This is a $C^{\mathrm{an}}(X, K)$-submodule of $C^{\mathrm{an}}(X, V)$. We claim that $\underline{V}$ is a subsheaf of $C^{\mathrm{an}}(-, V)$ and that the point-wise multiplication by G-valued analytic maps defines an action of $y(G)$ on $\underline{V}$. We will henceforth view $\underline{V}$ as an abelian group object of BG.

Proof of the claim. If $f \in \underline{V}(X)$ and $\phi : Y \to X$ is an analytic map between manifolds, then $f \circ \phi$ is analytic. Moreover, $\widehat{f \circ \phi} = \hat{f} \circ (\mathrm{id}_G \times \phi)$ is analytic as well, hence $f \circ \phi$ is admissible. Thus $\underline{V}$ is a presheaf. Since admissibility is a local property, $\underline{V}$ is a sheaf.

Now let $\rho : X \to G$ be an analytic map. We define ρf by $(\rho f)(x) := \rho(x) f(x)$. We have to show that ρf is analytic and admissible. But this is clear since ρf equals the composition

$$X \xrightarrow{(\rho, \,\mathrm{id}_X)} G \times X \xrightarrow{\hat{f}} V$$

and $\widehat{\rho f}$ equals the composition

$$G \times X \xrightarrow{(g,x) \mapsto (g\rho(x),x)} G \times X \xrightarrow{\hat{f}} V. \qquad \square$$

Example 10. A topological representation of G on V is locally analytic if and only if $\underline{V}(*) = V$.
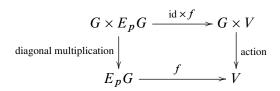
Definition 11. For a locally analytic representation of G on V and $i \geq 0$ we define the locally analytic group cohomology of G with coefficients in V as

$$H^i_{\mathrm{an}}(G, V) := H^i(G, \underline{V}).$$

Recall that a homogeneous analytic p-cochain of G with coefficients in V is an analytic function $f : E_p G \to V$ which is G-equivariant, i.e., satisfies $f(gg_0, \ldots, gg_p) = gf(g_0, \ldots g_p)$. We denote the complex of homogeneous analytic cochains by $C^{\mathrm{an}}_G(E_\bullet G, V)$. Its differential is induced by the simplicial structure of $E_\bullet G$.

Proposition 12. *The cohomology $H^*_{\mathrm{an}}(G, V)$ is isomorphic to the cohomology of the complex of homogeneous analytic cochains $C^{\mathrm{an}}_G(E_\bullet G, V)$.*

Proof. By Proposition 8 we have $H^*(G, \underline{V}) \cong H^*(\Gamma(_l E_\bullet G, \underline{V}))$. Using the Yoneda lemma we see that a section in $\Gamma(_l E_p G, \underline{V}) = \mathrm{Hom}_{BG}(_l E_p G, \underline{V})$ is just an admissible function $f : E_p G \to V$ such that

$$
\begin{array}{ccc}
G \times E_p G & \xrightarrow{\mathrm{id} \times f} & G \times V \\
{\scriptstyle \text{diagonal multiplication}} \downarrow & & \downarrow {\scriptstyle \text{action}} \\
E_p G & \xrightarrow{\quad f \quad} & V
\end{array}
$$

commutes, i.e., a G-equivariant admissible function $E_p G \to V$.

To prove the claim, it suffices to show that, vice versa, every G-equivariant analytic function $f : E_p G \to V$ is admissible. But, by the G-equivariance, $\hat{f}$ is equal to the composition

$$G \times E_p G \xrightarrow{(g,(g_0,\ldots,g_p)) \mapsto (gg_0,\ldots,gg_p)} E_p G \xrightarrow{f} V$$

and thus analytic. Thus f is admissible.   □

3. Differential forms and Lie algebra cohomology

In this section, we introduce sheaves of differential forms with coefficients in a locally analytic representation V of G as objects of BG. Again, unless V is a Banach space, we have to restrict to admissible forms. We show that the Lie algebra cohomology of the K-Lie algebra $\mathfrak{g}$ of G with coefficients in V can be computed as the cohomology in BG of the complex of forms on G with coefficients in V.

Let V be a separated locally convex K-vector space. For a submersion $Y \to X$ we denote by $\Omega^k(Y/X, V)$ the vector space of relative analytic k-forms with values in V. Here, a k-form ω is called analytic if for any k-tuple $\phi_1, \ldots, \phi_k$ of local sections of the vertical tangent bundle $T(Y/X)$ the function $Y \to V$, $y \mapsto \omega(y)(\phi_1(y), \ldots, \phi_k(y))$ is analytic. It suffices to check this for the local sections given by a local frame of $T(Y/X)$. In particular, every $y \in Y$ admits a neighborhood U and a BH-space $W \hookrightarrow V$ such that $\omega|_U$ is in the image of $\Omega^k(U/X, W) \hookrightarrow \Omega^k(U/X, V)$. It follows that the exterior derivative $d\omega$ is a well-defined form in $\Omega^{k+1}(Y/X, V)$. If V is finite-dimensional, this is the usual notion of analytic forms.

For a fixed manifold Y, we have a complex of sheaves $\underline{\Omega}^*(Y, V)$ on La defined by

$$\underline{\Omega}^*(Y, V)(X) := \Omega^*(X \times Y/X, V).$$

Let V be a locally analytic representation of G. We would like to equip this complex with a $y(G)$-action. As for functions, we have to restrict to a subcomplex of admissible forms in order to do this. Again, one can ignore this, if one is only interested in the case that V is a Banach space.

A form $\omega \in \Omega^k(Y/X, V)$ is called *admissible* if the form $\hat{\omega}$ on $G \times Y/G \times X$ given by

$$\hat{\omega}(g, y)(\mathsf{x}_1, \ldots, \mathsf{x}_k) := g \cdot (\omega(y)(\mathsf{x}_1, \ldots, \mathsf{x}_k)),$$

where $g \in G$, $y \in Y$ and $\mathsf{x}_i \in T_{(g,y)}(G \times Y/G \times X) \cong T_y(Y/X)$, is analytic. Equivalently, ω is admissible if and only if for any k-tuple of local sections $\phi_1, \ldots, \phi_k$ of $T(Y/K)$ as above the function $\omega(\phi_1, \ldots, \phi_k)$ is admissible. As in the case of functions, this is the case if and only if $\hat{\omega}|_{U \times Y}$ is analytic for some open subset $U \subseteq G$. The admissible k-forms form a $C^{\mathrm{an}}(Y, K)$-submodule of $\Omega^k(Y/X, V)$, which we denote by $\Omega^k_{\mathrm{ad}}(Y/X, V)$. They are also stable under

the exterior derivative: Let ω be an admissible form. Since G acts on V by continuous linear automorphisms we have $\widehat{d\omega} = d\hat{\omega}$, and this form is analytic. Thus, the admissible forms $\Omega^*_{\mathrm{ad}}(Y/X, V)$ form a subcomplex of the de Rham complex $\Omega^*(Y/X, V)$.

Example 13. If V is a Banach space, it follows from Example 9 that any V-valued analytic k-form is admissible.

We now fix a manifold Y. For an analytic map between manifolds $X' \to X$, the pullback map $\Omega^k(X \times Y/X) \to \Omega^k(X' \times Y/X')$ preserves admissible forms. Since admissibility is a local condition, $\underline{\Omega}^k_{\mathrm{ad}}(Y, V)$, defined by

$$\underline{\Omega}^k_{\mathrm{ad}}(Y, V)(X) := \Omega^k_{\mathrm{ad}}(X \times Y/X, V),$$

is a subsheaf of $\underline{\Omega}^k(Y, V)$, and $\underline{\Omega}^*_{\mathrm{ad}}(Y, V) \subseteq \underline{\Omega}^*(Y, V)$ is a subcomplex.

We define an action of $y(G)$ on $\underline{\Omega}^k_{\mathrm{ad}}(G, V)$ as follows: Let $\rho : X \to G$ be an analytic map and $\omega \in \Omega^k_{\mathrm{ad}}(X \times G/X, V)$ an admissible form. For any $h \in G$, left translation by h^{-1} induces a map $T_{(x,g)}(X \times G/X) \to T_{(x,h^{-1}g)}(X \times G/X)$, written $\mathsf{x} \mapsto h^{-1}\mathsf{x}$. Using this notation, we define $\rho\omega$ by the formula

$$(\rho\omega)(x, g)(\mathsf{x}_1, \ldots, \mathsf{x}_k) := \rho(x) \cdot \big(\omega(x, \rho(x)^{-1}g)(\rho(x)^{-1}\mathsf{x}_1, \ldots, \rho(x)^{-1}\mathsf{x}_k)\big).$$

Lemma 14. *This is a well-defined $y(G)$-action.*

Proof. We have to show that $\rho\omega$ is analytic and admissible. Consider the analytic maps $\hat{\rho} : G \times X \times G \to G \times X \times G, (h, x, g) \mapsto (h\rho(x), x, \rho(x)^{-1}g)$ and $i_1 : X \times G \hookrightarrow G \times X \times G, (x, g) \mapsto (1, x, g)$. Then $\rho\omega = i_1^* \hat{\rho}^* \hat{\omega}$; hence $\rho\omega$ is analytic. Similarly, we have $\widehat{\rho\omega} = \hat{\rho}^* \hat{\omega}$, and hence $\rho\omega$ is admissible. $\square$

We thus consider $\underline{\Omega}^k_{\mathrm{ad}}(G, V)$ as an abelian group object in BG. We want to show that it is acyclic. Write $\widetilde{V} := \mathrm{Hom}(\bigwedge^k \mathfrak{g}, V)$. The adjoint action of G on $\mathfrak{g}$ and the given action of G on V induce a natural G-action on $\widetilde{V}$.

Lemma 15. *This representation of G on $\widetilde{V}$ is locally analytic. We have an isomorphism*

$$\underline{\Omega}^k_{\mathrm{ad}}(G, V) \cong \mathcal{H}om_{BG}(\iota y(G), \underline{\widetilde{V}}) \cong i_* i^* \underline{\widetilde{V}}.$$

Proof. Let Y be a manifold. We claim that a function $f : Y \to \widetilde{V}$ is admissible if and only if the function $f_\mathsf{x} : Y \to V, y \mapsto f(y)(\mathsf{x})$ is admissible for every $\mathsf{x} \in \bigwedge^k \mathfrak{g}$. Taking Y to be a point this implies the first assertion of the lemma.

To prove the claim, assume first that f is admissible. We have $\widehat{f_\mathsf{x}}(g, y) = g(f(y)(\mathsf{x})) = (gf(y))(g\mathsf{x}) = \hat{f}(g, y)(g\mathsf{x})$. The function $\hat{f}$ is analytic by assumption and so is $g \mapsto g\mathsf{x}$. Since the evaluation $\widetilde{V} \times \bigwedge^k \mathfrak{g} \to V$ is continuous and bilinear, and since $\bigwedge^k \mathfrak{g}$ is finite dimensional, [Féaux de Lacroix 1998, Satz 2.4.3] implies that $\widehat{f_\mathsf{x}}$ is analytic.

To see the converse, let $x_1, \ldots, x_N$ be a basis of $\bigwedge^k \mathfrak{g}$ and $x_1^*, \ldots, x_N^*$ the dual basis of $(\bigwedge^k \mathfrak{g})^\vee$. We can write f as a sum $f = \sum_{i=1}^N f_{x_i} x_i^*$ with f_{x_i} admissible. Then $\hat{f}(g, y) = \sum_i \widehat{f_{x_i}}(g, y) g(x_i^*)$ and, by [Féaux de Lacroix 1998, Satz 2.4.3] again, $\hat{f}$ is analytic.

We now prove the second assertion of the lemma. For any manifold X, right translations by elements of G induce a trivialization of the vertical tangent bundle $T(X \times G/X) \cong (X \times G) \times \mathfrak{g}$. This gives a natural isomorphism of vector spaces

$$\Omega^k(X \times G/X, V) \cong C^{\mathrm{an}}(X \times G, \widetilde{V}).$$

Using the above claim one sees that this isomorphism restricts to an isomorphism

$$\Omega^k_{\mathrm{ad}}(X \times G/X, V) \cong C^{\mathrm{ad}}(X \times G, \widetilde{V}).$$

Under this isomorphism, the $y(G)(X)$-action on the left-hand side corresponds to the action on the right-hand side induced by left translations on $X \times G$ and the action on $\widetilde{V}$ mentioned above. Using the isomorphism $C^{\mathrm{ad}}(X \times G, \widetilde{V}) \cong \mathcal{H}om_{BG}({}_l y(G), \underline{\widetilde{V}})(X)$, this gives the first isomorphism stated in the Lemma. The second follows immediately from Lemma 4. $\square$

Corollary 16. *We have*

$$H^i(G, \underline{\Omega}^k_{\mathrm{ad}}(G, V)) \cong \begin{cases} \mathrm{Hom}_K(\bigwedge^k \mathfrak{g}, V) & \text{if } i = 0, \\ 0 & \text{else}. \end{cases}$$

Proof. By Lemma 15 and Corollary 7 the higher cohomology groups vanish, and

$$H^0(G, \underline{\Omega}^k_{\mathrm{ad}}(G, V)) \cong \underline{\widetilde{V}}(*) = \mathrm{Hom}_K(\bigwedge^k \mathfrak{g}, V), \qquad (17)$$

concluding the proof. $\square$

Explicitly, this isomorphism is given by evaluating a form at $1 \in G$.

The differential d of the complex $\underline{\Omega}^*_{\mathrm{ad}}(G, V)$ is compatible with the $y(G)$-action. Hence we can view $\underline{\Omega}^*_{\mathrm{ad}}(G, V)$ as a complex in BG and we can compute its hypercohomology.

We now assume that V is barreled, i.e., that every closed convex absorbing subset is open (see [Schneider 2002, §6]). For example, any complete metrizable locally convex space, in particular any Banach space, is barreled. Differentiating the orbit maps $g \mapsto g v$ then induces an action of the Lie algebra $\mathfrak{g}$ on V [Féaux de Lacroix 1998, Sätze 3.1.3, 3.1.7].

Corollary 18. *We have natural isomorphisms*

$$H^i(G, \underline{\Omega}^*_{\mathrm{ad}}(G, V)) \cong H^i(\mathfrak{g}, V),$$

where the right-hand side is Lie algebra cohomology.

Proof. Corollary 16 gives an isomorphism

$$H^i(G, \underline{\Omega}^*_{\mathrm{ad}}(G, V)) \cong H^i(\mathrm{Hom}_K(\textstyle\bigwedge^* \mathfrak{g}, V)),$$

where the differential on $\mathrm{Hom}_K(\bigwedge^* \mathfrak{g}, V)$ is induced from the de Rham differential via (17). This is precisely the Chevalley–Eilenberg complex computing Lie algebra cohomology. $\qquad\square$

4. Differential forms and locally analytic group cohomology

As before, we fix a locally analytic representation $G \to \mathrm{Aut}(V)$. In this section we use the Poincaré lemma to compare the hypercohomology of the complex of V-valued admissible forms with locally analytic group cohomology, and we give the proof of the theorem announced on page 938 in the introduction.

Fix a manifold Y. A function $f : Y \times X \to V$ will be called locally constant along Y if, for every $(y, x) \in Y \times X$, there exist open neighborhoods $Y' \subseteq Y$ of y and $X' \subseteq X$ of x such that $f|_{Y' \times X'}$ factors through the projection $Y' \times X' \to X'$. We define

$$\underline{C}^{\mathrm{lc}}_{\mathrm{ad}}(Y, V)(X) := \{ f \in C^{\mathrm{ad}}(X \times Y, V) \mid f \text{ is locally constant along } Y \}.$$

It is easy to see that $X \mapsto \underline{C}^{\mathrm{lc}}_{\mathrm{ad}}(Y, V)(X)$ defines a sheaf on La.

Proposition 19. *The inclusion in degree* 0

$$\underline{C}^{\mathrm{lc}}_{\mathrm{ad}}(Y, V) \to \underline{\Omega}^*_{\mathrm{ad}}(Y, V)$$

is a quasi-isomorphism.

If V is a Banach space, this is just the Poincaré lemma, and its usual proof works. For general locally convex V, it is a little bit more complicated, since we have to prove admissibility of primitives.

Proof. The map clearly induces an isomorphism on H^0, and it remains to show that $H^k(\underline{\Omega}^*_{\mathrm{ad}}(Y, V)) = 0$ for $k > 0$.

Let X be a manifold, and let ω be a closed form in $\Omega^k_{\mathrm{ad}}(X \times Y/X, V)$. We will show that there is an $\eta \in \Omega^{k-1}_{\mathrm{ad}}(X \times Y/X, V)$ such that $d\eta = \omega$. Since all manifolds are strictly paracompact, it is enough to construct such an η locally on X and Y (see Remark 5).

The rest of the proof uses some results and notations from the Appendix. It can be skipped on first reading. Since $d\hat{\omega} = \widehat{d\omega} = 0$, the form $\hat{\omega} \in \Omega^k(G \times X \times Y/G \times X, V)$ is closed. Replacing G be a small open neighborhood of 1 and using local charts, we may assume that there are multiradii $\delta \in \mathbb{R}^m_+$, $\epsilon \in \mathbb{R}^n_+$ such that $G \times X \cong B_\delta(0) \subset K^m$, $Y \cong B_\epsilon(0) \subset K^n$, and a BH-space $W \hookrightarrow V$ such that $\hat{\omega}$ is given by a power series in $F_\delta(\Omega^k_\epsilon(W))$ (see (25)). Choose a multiradius $\epsilon' < \epsilon$. The homotopy operator $h : \Omega^k_\epsilon(W) \to \Omega^{k-1}_{\epsilon'}(W)$ given by Lemma 26 induces an operator

$h : F_\delta(\Omega^k_\epsilon(W)) \to F_\delta(\Omega^{k-1}_{\epsilon'}(W))$. We set $\widetilde{\eta} := h(\hat{\omega})$. Hence $\widetilde{\eta}$ represents a relative analytic $(k-1)$-form on $G \times X \times Y'/G \times X$ with an open subset $Y' \subset Y$. Since $\hat{\omega}$ is closed, we have $d\widetilde{\eta} = \hat{\omega}|_{G \times X \times Y'}$.

For $g \in G$, let $i_g : X \times Y \to G \times X \times Y$ (and similarly with Y replaced by Y') be the inclusion $(x, y) \mapsto (g, x, y)$. We set $\eta := i_1^* \widetilde{\eta}$. Clearly, $d\eta = i_1^* d\widetilde{\eta} = i_1^* \hat{\omega} = \omega$. To prove that η is admissible, we show that $\hat{\eta} = \widetilde{\eta}$. Let $\Phi_g : V \to V$ be the continuous automorphism given by the action of g. We have to check that $i_g^* \widetilde{\eta} = \Phi_g \circ \eta$. By restriction, Φ_g induces a continuous isomorphism of BH-spaces $W \to g(W)$ (more precisely, we view W as a linear subspace of V and let $g(W)$ be its image under the action of $g \in G$ with Banach space structure induced from W via the linear isomorphism $\Phi_g|_W : W \xrightarrow{\sim} g(W)$). We have

$$
\begin{aligned}
i_g^* \widetilde{\eta} = i_g^*(h(\hat{\omega})) \quad & \text{(by definition)} \\
= h(i_g^* \hat{\omega}) \quad & \text{(using (24) with } \Phi = h : \Omega^q_\epsilon(W) \to \Omega^{q-1}_{\epsilon'}(W)) \\
= h(\Phi_g \circ \omega) \quad & \text{(definition of } \hat{\omega}) \\
= \Phi_g \circ h(\omega) \quad & \text{(Lemma 27 for } \Phi = \Phi_g : W \to g(W)) \\
= \Phi_g \circ \eta \quad & \text{(since } h(\omega) = h(i_1^* \hat{\omega}) = i_1^* h(\hat{\omega}) = i_1^* \hat{\eta} = \eta). \qquad \square
\end{aligned}
$$

Proof of the Theorem. The sheaf $\underline{C}^{\text{lc}}_{\text{ad}}(G, V)$ carries a natural $y(G)$-action induced by left translations on G and the given action on V. By Proposition 19 and Corollary 18 we have isomorphisms

$$
H^*(G, \underline{C}^{\text{lc}}_{\text{ad}}(G, V)) \cong H^*(G, \underline{\Omega}^*_{\text{ad}}(G, V)) \cong H^*(\mathfrak{g}, V). \tag{20}
$$

As in the proof of Proposition 12, Proposition 8 implies that $H^*(G, \underline{C}^{\text{lc}}_{\text{ad}}(G, V))$ is the cohomology of the complex $C^{\text{lc}}_G(G \times E_\bullet G, V)$ of G-equivariant analytic functions $G \times E_p G \to V$ that are locally constant along the first factor.

Since the open subgroups $G' \subseteq G$ form a fundamental system of neighborhoods of $1 \in G$ (see [Schneider 2011, Lemma 18.7]), we have an isomorphism

$$
\operatornamewithlimits{colim}_{G' \subset G \text{ open}} C^{\text{an}}_{G'}(E_\bullet G', V) \cong \operatornamewithlimits{colim}_{G' \subset G \text{ open}} C^{\text{lc}}_{G'}(G' \times E_\bullet G', V).
$$

Because taking the colimit over a directed system is exact, we see that

$$
\operatornamewithlimits{colim}_{G' \subset G} H^*_{\text{an}}(G', V) \to \operatornamewithlimits{colim}_{G' \subset G} H^*(G', \underline{C}^{\text{lc}}(G', V))
$$

is an isomorphism. Since the isomorphisms (20) are compatible with the restriction to open subgroups, the claim follows. $\qquad \square$

There is an additional action of G on $\underline{C}_{\mathrm{ad}}^{\mathrm{lc}}(G, V)$ and on $\underline{\Omega}_{\mathrm{ad}}^{*}(G, V)$ induced by right translations on G. This action is compatible with the given $y(G)$-action. It induces a G-action on the cohomology groups. Via the isomorphism (20) this corresponds to the G-action on $H^{*}(\mathfrak{g}, V)$ induced by the adjoint action on $\mathfrak{g}$ and left multiplication on V.

Corollary 21. *If G is compact, there is a natural isomorphism*

$$H_{\mathrm{an}}^{*}(G, V) \cong H^{*}(\mathfrak{g}, V)^{G}.$$

Proof. Since G is compact, every open subgroup is of finite index and contains an open normal subgroup. If X is a compact manifold, every function in $\underline{C}_{\mathrm{ad}}^{\mathrm{lc}}(G, V)(X)$ factors through $G/H \times X$ for some open normal subgroup $H \trianglelefteq G$. Thus — using the notation from the previous proof —

$$\underline{C}_{G}^{\mathrm{lc}}(G \times E_{\bullet}G, V) = \operatorname*{colim}_{H \trianglelefteq G \text{ open}} C_{G}^{\mathrm{an}}(G/H \times E_{\bullet}G, V).$$

Since the colimit over a directed system is exact, this induces an isomorphism $H^{*}(\mathfrak{g}, V) \cong \operatorname{colim}_{H \trianglelefteq G} H^{*}(C_{G}^{\mathrm{an}}(G/H \times E_{\bullet}G, V))$. Since each quotient G/H is finite, and taking invariants under a finite group is an exact functor on K-vector spaces with an action by that group, we get

$$
\begin{aligned}
H^{*}(\mathfrak{g}, V)^{G} &\cong \operatorname*{colim}_{H \trianglelefteq G} H^{*}(C_{G}^{\mathrm{an}}(G/H \times E_{\bullet}G, V))^{G/H} \\
&\cong \operatorname*{colim}_{H \trianglelefteq G} H^{*}(C_{G}^{\mathrm{an}}(G/H \times E_{\bullet}G, V)^{G/H}) \\
&\cong \operatorname*{colim}_{H \trianglelefteq G} H^{*}(C_{G}^{\mathrm{an}}(E_{\bullet}G, V)) \cong H_{\mathrm{an}}^{*}(G, V). \qquad \square
\end{aligned}
$$

5. Explicit description of the comparison map

We want to describe an explicit map of complexes which induces the comparison map $H_{\mathrm{an}}^{*}(G, V) \to H^{*}(\mathfrak{g}, V)$. Recall that $H_{\mathrm{an}}^{*}(G, V)$ is computed by the complex of homogeneous locally analytic cochains $C_{G}^{\mathrm{an}}(E_{\bullet}G, V)$, and that $H^{*}(\mathfrak{g}, V)$ is computed by the complex of G-invariant admissible differential forms $\Omega_{\mathrm{ad}}^{*}(G, V)^{G}$.

For integers $p \geq 0$ and $0 \leq i \leq p$, we denote by d_{i} the partial exterior derivative in the direction of the $(i+1)$-th factor of the product $E_{p}G = G^{p+1}$. We denote by $\Delta_{p} : G \to E_{p}G$ the diagonal map. For $f \in C^{\mathrm{an}}(E_{p}G, V)$ we set

$$\Psi(f) := \Delta_{p}^{*}(d_{1}d_{2} \ldots d_{p}f) \in \Omega^{p}(G, V).$$

Proposition 22. *The map Ψ induces a morphism of complexes $C_{G}^{\mathrm{an}}(E_{\bullet}G, V) \to \Omega_{\mathrm{ad}}^{*}(G; V)^{G}$, which agrees with the comparison map $H_{\mathrm{an}}^{*}(G, V) \to H^{*}(\mathfrak{g}, V)$ on cohomology groups.*

Remark 23. Let us consider the special case that K is $\mathbb{Q}_p$ and V is finite dimensional. We want to indicate how the method of [Huber and Kings 2011] allows one to compare our map with Lazard's one. The space of functions $C^{\mathrm{an}}(E_pG, V)$ has topological generators of the form $f_0 \otimes \cdots \otimes f_p \otimes v$ with $f_i \in C^{\mathrm{an}}(G, K)$ and $v \in V$. For such a function we have

$$\Psi(f_0 \otimes \cdots \otimes f_p \otimes v) = f_0 \, df_1 \wedge \cdots \wedge df_p \otimes v,$$

and its image in $\mathrm{Hom}(\bigwedge^p \mathfrak{g}, V)$ is given by $f_0(1) \, df_1(1) \wedge \cdots \wedge df_p(1) \otimes v$.

There is another simplicial model $\widetilde{E}_\bullet G$ for the universal G-bundle (see [Huber and Kings 2011, §4.4]), given by $\widetilde{E}_p G = E_p G$, but with face maps

$$\tilde{\partial}_i(g_0, \ldots, g_n) = \begin{cases} (g_0, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_p) & \text{if } i = 0, \ldots p-1, \\ (g_0, \ldots, g_{p-1}) & \text{if } i = p. \end{cases}$$

The G-action on $\widetilde{E}_\bullet G$ is given by left multiplication on the first factor. There is a natural G-equivariant isomorphism $\widetilde{E}_\bullet G \cong E_\bullet G$. Huber and Kings show that Lazard's isomorphism (for G small enough) is induced by the map

$$\Phi : C_G^{\mathrm{an}}(\widetilde{E}_\bullet G, V) \to \mathrm{Hom}(\bigwedge^\bullet \mathfrak{g}, V),$$

$$\Phi(f_0 \otimes \cdots \otimes f_p \otimes v) = f_0(1) df_1(1) \wedge \cdots \wedge df_p(1) \otimes v$$

(see [Huber and Kings 2011, Proposition 4.6.1]; this is formulated in the case of trivial coefficients, but can easily be adapted to our setting). The argument of [Huber and Kings 2011, Theorem 4.7.1] shows that the composition of Φ with the isomorphism $C_G^{\mathrm{an}}(E_\bullet G, V) \cong C_G^{\mathrm{an}}(E_\bullet G, V)$ is homotopic to Ψ, hence both maps agree on cohomology groups.

Proof of Proposition 22. From the proof of Proposition 8 we have the acyclic resolution $\underline{V} \xrightarrow{\sim} \mathscr{H}om_{BG}(_!E_\bullet G, \underline{V})$. For a manifold X we have

$$\mathscr{H}om_{BG}(_!E_\bullet G, \underline{V})(X) = C^{\mathrm{ad}}(X \times E_\bullet G, V)$$

with $y(G)$-action induced from left translations on $E_\bullet G$ and the action on V. We define $\Psi : C^{\mathrm{ad}}(X \times E_p G, V) \to \Omega_{\mathrm{ad}}^p(X \times G/X, V)$ by the same formula as above. We claim that this gives a morphism of complexes $\Psi : \mathscr{H}om_{BG}(_!E_\bullet G, \underline{V}) \to \underline{\Omega}_{\mathrm{ad}}^*(G, V)$ in BG.

Proof of the claim. One checks without difficulty that Ψ is equivariant for the $y(G)$-action. Now consider $f \in C^{\mathrm{ad}}(E_p G \times X, V)$. Recall the face maps

$$\partial_i : E_{p+1}G \to E_p G, \quad (g_0, \ldots, g_{p+1}) \mapsto (g_0, \ldots, \widehat{g_i}, \ldots, g_{p+1}).$$

The differential of the complex $C^{\mathrm{ad}}(E_\bullet G \times X, V)$ maps f to

$$\sum_{i=0}^{p+1} (-1)^i \partial_i^* f.$$

Since $\partial_i^* f$ is constant along the $(i+1)$-th factor G, we have $d_i(\partial_i^* f) = 0$. Since the partial derivatives commute up to sign, it follows that

$$
\begin{aligned}
\Psi\left(\sum_{i=0}^{p+1} (-1)^i \partial_i^* f\right) &= \Psi(\partial_0^* f) \\
&= \Delta_{p+1}^*(d_1 \dots d_{p+1}(\partial_0^* f)) \\
&= \Delta_{p+1}^*(\partial_0^*(d_0 \dots d_p f)) \\
&= \Delta_p^*(d_0 \dots d_p f) \\
&= \Delta_p^*(d(d_1 \dots d_p f)) \\
&= d(\Delta_p^*(d_1 \dots d_p f)) \\
&= d(\Psi(f)). \qquad\qquad \square
\end{aligned}
$$

We thus have a commutative diagram

$$
\begin{array}{ccc}
\underline{V} & \xrightarrow{\;\simeq\;} & \mathcal{H}om_{BG}({}_l E_\bullet G, \underline{V}) \\
& & \downarrow{\scriptstyle \Psi} \\
& & \underline{\Omega}_{\mathrm{ad}}^*(G, V)
\end{array}
$$

where the complexes on the right-hand side consist of acyclic sheaves. The proposition now follows by taking global sections. $\qquad \square$

Appendix: The Poincaré lemma

Let W be a K-Banach space with norm $\|\cdot\|$. For a multiradius $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \mathbb{R}_+^n$ we denote the space of ϵ-convergent power series in n variables $x = (x_1, \dots, x_n)$ with coefficients in W by $F_\epsilon(W)$:

$$F_\epsilon(W) := \left\{ \sum_{I \in \mathbb{N}_0^n} a_I x^I \mid a_I \in W, \|a_I\|\epsilon^I \xrightarrow{\;I \to \infty\;} 0 \right\}.$$

Equipped with the norm $\left\| \sum_I a_I x^I \right\|_\epsilon := \max_I \|a_I\|\epsilon^I$, this is again a Banach space.

Let $\Phi : W \to W'$ be a continuous linear map between Banach spaces. It induces a continuous linear map $F_\epsilon(W) \to F_\epsilon(W')$. Let $B_\epsilon(0) \subset K^n$ be the closed ball of radius ϵ around 0. For any $x \in B_\epsilon(0)$ we have the evaluation at x, written

$i_x^* : F_\epsilon(W) \to W$, and similarly for W'. Since Φ is continuous the diagram

$$
\begin{array}{ccc}
F_\epsilon(W) & \xrightarrow{\ \Phi\ } & F_\epsilon(W') \\
{\scriptstyle i_x^*}\downarrow & & \downarrow{\scriptstyle i_x^*} \\
W & \xrightarrow{\ \Phi\ } & W'
\end{array}
\tag{24}
$$

commutes.

For $q \geq 0$ we denote by $\Omega_\epsilon^q(W)$ the space of ϵ-convergent W-valued q-forms in n variables:

$$
\Omega_\epsilon^q(W) := \textstyle\bigwedge_K^q (K^n)^\vee \otimes_K F_\epsilon(W).
$$

Since $\bigwedge_K^q (K^n)^\vee$ is a finite-dimensional K-vector space, this is again a Banach space. The usual differential defines a continuous linear map $d : \Omega_\epsilon^q(W) \to \Omega_\epsilon^{q+1}(W)$.

There is a natural injection $\Omega_\epsilon^q(W) \hookrightarrow \Omega^q(B_\epsilon(0), W)$ into the space of locally analytic W-valued q-forms. It is compatible with the differential. More generally, if $\delta \in \mathbb{R}_+^m$ is a second multiradius, we can identify δ-convergent power series with coefficients in $\Omega_\epsilon^q(W)$ with relative W-valued forms:

$$
F_\delta(\Omega_\epsilon^q(W)) \hookrightarrow \Omega^q(B_\delta(0) \times B_\epsilon(0)/B_\delta(0), W).
\tag{25}
$$

On the other hand, every relative q-form is in the image of (25) after shrinking δ and ϵ appropriately.

Let $\epsilon' \in \mathbb{R}_+^n$ be a multiradius which is component-wise strictly smaller than ϵ, written $\epsilon' < \epsilon$. There is a continuous inclusion $i : \Omega_\epsilon^q(W) \hookrightarrow \Omega_{\epsilon'}^q(W)$.

Lemma 26 (Poincaré lemma). *Let $\epsilon' < \epsilon$ and $q > 0$. Then there exists a bounded linear map*

$$
h : \Omega_\epsilon^q(W) \to \Omega_{\epsilon'}^{q-1}(W)
$$

such that $d \circ h + h \circ d = i$.

Proof. We have

$$
\Omega_\epsilon^q(W) = \bigoplus_{1 \leq k_1 < \cdots < k_q \leq n} F_\epsilon(W)\, dx_{k_1} \ldots dx_{k_q}.
$$

Set $C := \max_i (\epsilon_i / \epsilon_i')$. By assumption we have $C > 1$. Hence, for integers $N \gg 0$ we have $|1/(N + q)| \leq C^N$. We define

$$
h(x^I dx_{k_1} \ldots dx_{k_q}) := \frac{1}{|I| + q} \sum_{\alpha=1}^{q} (-1)^{\alpha-1} x^{I+e_{k_\alpha}}\, dx_{k_1} \ldots \widehat{dx_{k_\alpha}} \ldots dx_{k_q}.
$$

and

$$
h\left(\sum a_I x^I dx_{k_1} \ldots dx_{k_q} \right) := \sum a_I h(x^I dx_{k_1} \ldots dx_{k_q}).
$$

Since

$$\left\|\frac{a_I}{|I|+q}\right\| \epsilon'^I \le \|a_I\| C^{|I|} \epsilon'^I \le \|a_I\| \epsilon^I \quad \text{for } |I| \gg 0,$$

it follows that the power series $\sum_I a_I/(|I|+q)x^{I+e_{k_\alpha}}$ is ϵ'-convergent, whence that h is well-defined, and also that h is a bounded linear operator.

By continuity, it is now enough to check the equality $dh + hd = i$ on monomials $x^I dx_{k_1} \ldots dx_{k_q}$. Relabeling the coordinates, we may moreover assume that $(k_1, \ldots, k_q) = (1, \ldots, q)$. We have

$$dh(x^I dx_1 \ldots dx_q)$$

$$= d\left(\frac{1}{|I|+q}\sum_{\alpha=1}^{q}(-1)^{\alpha-1}x^{I+e_\alpha}dx_1 \ldots \widehat{dx_\alpha} \ldots dx_q\right)$$

$$= \left(\frac{1}{|I|+q}\sum_{\alpha=1}^{q}(i_\alpha+1)x^I dx_1 \ldots dx_q\right)$$

$$+ \frac{1}{|I|+q}\sum_{\alpha=1}^{q}\sum_{\beta=q+1}^{n}(-1)^{\alpha-1}(-1)^{q-1}i_\beta x^{I+e_\alpha-e_\beta}dx_1 \ldots \widehat{dx_\alpha} \ldots dx_q dx_\beta$$

$$= \frac{(\sum_{\alpha=1}^{q} i_\alpha)+q}{|I|+q}x^I dx_1 \ldots dx_q$$

$$+ \frac{1}{|I|+q}\sum_{\alpha=1}^{q}\sum_{\beta=q+1}^{n}(-1)^{\alpha+q}i_\beta x^{I+e_\alpha-e_\beta}dx_1 \ldots \widehat{dx_\alpha} \ldots dx_q dx_\beta$$

and

$$hd(x^I dx_1 \ldots dx_q)$$

$$= h\left((-1)^q\sum_{\beta=q+1}^{n}i_\beta x^{I-e_\beta}dx_1 \ldots dx_q dx_\beta\right)$$

$$= \frac{(-1)^q}{|I|+q}\sum_{\alpha=1}^{q}\sum_{\beta=q+1}^{n}(-1)^{\alpha-1}i_\beta x^{I+e_\alpha-e_\beta}dx_1 \ldots \widehat{dx_\alpha} \ldots dx_q dx_\beta$$

$$+ \frac{(-1)^q}{|I|+q}\sum_{\beta=q+1}^{n}(-1)^q i_\beta x^I dx_1 \ldots dx_q$$

$$= \frac{1}{|I|+q}\sum_{\alpha=1}^{q}\sum_{\beta=q+1}^{n}(-1)^{\alpha+q-1}i_\beta x^{I+e_\alpha-e_\beta}dx_1 \ldots \widehat{dx_\alpha} \ldots dx_q dx_\beta$$

$$+ \frac{(\sum_{\beta=q+1}^{n} i_\beta)}{|I|+q}x^I dx_1 \ldots dx_q$$

Thus, $(dh + hd)(x^I dx_1 \ldots dx_k) = x^I dx_1 \ldots dx_k$. This finishes the proof. $\square$

Lemma 27. *Let* $\Phi : W \to W'$ *be a bounded linear map between Banach spaces. It induces a map* $\Omega_\epsilon^q(W) \to \Omega_\epsilon^q(W')$, *denoted by the same symbol. For* $q > 0$ *and* $\epsilon' < \epsilon$, *the diagram*

$$
\begin{array}{ccc}
\Omega_\epsilon^q(W) & \xrightarrow{\ h\ } & \Omega_{\epsilon'}^{q-1}(W) \\
\Phi \downarrow & & \downarrow \Phi \\
\Omega_\epsilon^q(W') & \xrightarrow{\ h\ } & \Omega_{\epsilon'}^{q-1}(W')
\end{array}
$$

commutes.

Proof. This follows directly from the definitions. $\square$

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