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Horrocks correspondence on arithmetically Cohen–Macaulay varieties

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We describe a vector bundle $\mathscr{E}$ on a smooth n-dimensional arithmetically Cohen–Macaulay variety in terms of its cohomological invariants $H^i_*(\mathscr{E})$, $1 \le i \le n-1$, and certain graded modules of "socle elements" built from $\mathscr{E}$. In this way we give a generalization of the Horrocks correspondence. We prove existence theorems, where we construct vector bundles from these invariants, and uniqueness theorems, where we show that these data determine a bundle up to isomorphism. The cases of the quadric hypersurface in $\mathbb{P}^{n+1}$ and the Veronese surface in $\mathbb{P}^5$ are considered in more detail.

Introduction

In a fundamental paper, Horrocks [1964] described all vector bundles on projective space $\mathbb{P}^n$ in terms of their intermediate cohomology modules. He described these cohomology modules using what he called a $\mathfrak{Z}$-complex, and showed that the category of vector bundles modulo stable equivalence was equivalent to the category of all $\mathfrak{Z}$-complexes modulo exact free complexes. In particular, this gives the well-known Horrocks criterion for a vector bundle to be a sum of line bundles in terms of the vanishing of its intermediate cohomology. His results were reformulated by Walters [1996] in the language of derived categories, and extended to sheaves by Coandă [2010]. Beĭlinson [1978] described the derived category of sheaves on a projective space using complexes built from an "exceptional sequence" $\{\mathscr{O}_{\mathbb{P}^n}(-n), \dots, \mathscr{O}_{\mathbb{P}^n}(-1), \mathscr{O}_{\mathbb{P}^n}\}$ of line bundles on $\mathbb{P}^n$, and Kapranov [1988] gave a similar description for smooth quadric hypersurfaces by enlarging the sequence to include the spinor bundles Σ of the quadric. Ancona and Ottaviani [1991] used these methods to extend the Horrocks splitting criterion to quadrics, with a theorem that a vector bundle $\mathscr{E}$ on a quadric $\mathscr{Q}_n$ (of dimension n) is a sum of line bundles if and only if $\mathscr{E}$ has its intermediate cohomology modules $H^i_*(\mathscr{E})$ all zero for $1 \le i \le n-1$ and also $H^{n-1}_*(\mathscr{E} \otimes \Sigma) = 0$ for the spinor bundles Σ.

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In this paper, we copy Horrocks' method on a smooth arithmetically Cohen–Macaulay (ACM) subvariety X of projective space. Given a vector bundle $\mathscr{E}$ on X, we construct a $\mathfrak{Z}$-complex of free A-modules (where A is the coordinate ring of X). This complex, when sheafified, gives rise to a vector bundle $\mathscr{F}$ on X which we call a Horrocks data bundle for $\mathscr{E}$, since it comes with a map $\beta : \mathscr{F} \to \mathscr{E}$ which is an isomorphism on intermediate cohomology modules. When $H^0_*(\beta)$ is surjective, the kernel of β is some ACM bundle on X.

These methods of Horrocks provide for ACM varieties a vector-bundle version of results of Auslander and Bridger [1969, Proposition 4.26, Corollary 4.27], who gave a structure theorem for a module M of finite Gorenstein dimension n over a commutative ring, showing that $M \oplus P$ for some projective module P can be expressed as an extension of a module $H_n(M)$ of projective dimension n by a module of zero Gorenstein dimension, where the map $M \to H_n(M)$ satisfies a universal property. In an unpublished preprint, Buchweitz [1986] proved a similar result for finitely generated modules over strongly Gorenstein (noncommutative) rings. We will see that the graded A-module F of global sections of the Horrocks data bundle $\mathscr{F}$ will have $F^\vee$ of finite projective dimension.

With this natural extension of Horrocks' arguments to an ACM variety, we give a generalization of the Horrocks correspondence in Section 1. Our goal in looking at a Horrocks correspondence on X is to look for cohomological invariants that determine $\mathscr{E}$. We will take the Horrocks data bundle as encoding all the intermediate cohomology for $\mathscr{E}$, and view it as one of the invariants. So we will study the bundles $\mathscr{E}$ with a fixed (minimal) Horrocks data bundle $\mathscr{F}$. While for the map $\mathscr{F} \to \mathscr{E}$ the induced map of first cohomology modules $H^1_*(\mathscr{F}) \to H^1_*(\mathscr{E})$ is an isomorphism, for various irreducible ACM bundles $\mathscr{B}$ on X, the map $H^1_*(\mathscr{F} \otimes \mathscr{B}^\vee) \to H^1_*(\mathscr{E} \otimes \mathscr{B}^\vee)$ may have a kernel. These kernels will give more cohomological invariants and we will call them modules of $\mathscr{B}$-socle elements. In Theorems 1.10 and 1.11, we see how these invariants determine $\mathscr{E}$ up to direct sums of ACM bundles. We also give a splitting criterion for the bundle $\mathscr{E}$ to be a sum of line bundles restricted from projective space. What is lacking in Section 1 is an understanding of which modules of socle elements are obtained from a vector bundle for a general ACM variety.

In Section 2 we describe the case of quadrics, on which ACM bundles are well understood due to Knörrer [1987]. In particular, for the spinor bundles Σ_i on a quadric $\mathscr{Q}_n$, modules of Σ_i-socle elements of a Horrocks data bundle $\mathscr{F}$ are just graded vector spaces. We show that a vector bundle $\mathscr{E}$ exists for each choice of Horrocks data bundle $\mathscr{F}$ and vector spaces V_i of Σ_i-socle elements of $\mathscr{F}$, and that two vector bundles with the same data of $\mathscr{F}$, V_i (up to obvious isomorphisms) are isomorphic up to direct sums of ACM bundles. In this way we generalize the results obtained in [Malaspina and Rao 2014] on $\mathscr{Q}_2$.

In Section 3 we deal with the Veronese surface $\mathcal{V} \subset \mathbb{P}^5$. The study of vector bundles on $\mathcal{V}$ is trivial by Horrocks if we view $\mathcal{V}$ as $\mathbb{P}^2$. But, as another illustration of the methods, it is an interesting example of an arithmetically Cohen–Macaulay embedding which is not arithmetically Gorenstein and for which the ACM bundles are easy to handle.

1. Horrocks data bundles on ACM varieties

Let X be a smooth ACM variety of dimension n in $\mathbb{P}^{n+r}$ over a field k. For any sheaf $\mathcal{B}$ on X, $H^i_*(\mathcal{B})$ will denote $\bigoplus_{l\in\mathbb{Z}} H^i(X,\mathcal{B}(l))$. The coordinate ring of X, $A = H^0_*(\mathcal{O}_X)$, is a noetherian Cohen–Macaulay graded k-algebra. $H^i_*(\mathcal{B})$ is a graded module over A. Let M be the category of graded, finitely generated A-modules and graded homomorphisms. Any finitely generated projective graded A module has the form $\bigoplus_i A(a_i)$ for some shifts $a_i \in \mathbb{Z}$ in grading, and will be called a free A-module. Let $\mathfrak{P} \subset$ M be the full subcategory of finitely generated free A-modules. $\mathcal{C}^-(\mathrm{M})$ and $\mathcal{C}^-(\mathfrak{P})$ will denote the categories of all complexes, bounded above, of objects in M and $\mathfrak{P}$ respectively, where morphisms are maps between two complexes. Since M has enough projectives, given a complex $C^\bullet$ of objects in M, bounded above, one can find a free resolution, i.e., a complex $P^\bullet$ in $\mathcal{C}^-(\mathfrak{P})$ with a quasi-isomorphism $P^\bullet \to C^\bullet$.

Let $\mathcal{E} \in$ VB be an object in the category of finite-rank vector bundles on X. $H^i_*(\mathcal{E})$ is an A-module of finite length for $1 \le i \le n-1$. A vector bundle will be called free if it has the form $\bigoplus_i \mathcal{O}_X(a_i)$. A vector bundle $\mathcal{E}$ will be called ACM (arithmetically Cohen–Macaulay) if $H^i_*(\mathcal{E}) = 0$ for all $1 \le i \le n-1$. Since X is ACM, every free bundle is ACM. By Serre duality, the line bundle ω_X is an ACM line bundle.

Given $\mathcal{E}$, let E denote the graded A-module $H^0_*(\mathcal{E})$. Denoting duals by $^\vee$ in the categories VB and M, we have $H^0_*(\mathcal{E}^\vee) \cong (H^0_*(\mathcal{E}))^\vee$. Following Horrocks, we choose a resolution of $H^0_*(\mathcal{E}^\vee)$ by finitely generated free modules

$$\ldots \longrightarrow C^{3\vee} \longrightarrow C^{2\vee} \longrightarrow C^{1\vee} \longrightarrow C^{0\vee} \longrightarrow H^0_*(\mathcal{E}^\vee) \longrightarrow 0. \tag{1}$$

In [Horrocks 1964], this could be chosen as a finite resolution, but in our case it may be infinite. However, if $K = \ker(C^{n-2\vee} \to C^{n-3\vee})$, then $\mathcal{K}$ is an ACM vector bundle on X, where $\mathcal{K} = \widetilde{K}$ is the sheaf obtained from K. Replacing the terms up to and including $C^{n-1\vee}$ by K and dualizing, we get the complex

$$C^\bullet_{\{0,n\}} : 0 \longrightarrow C^0 \xrightarrow{\delta^1_{C^\bullet}} C^1 \xrightarrow{\delta^2_{C^\bullet}} C^2 \xrightarrow{\delta^3_{C^\bullet}} \cdots \xrightarrow{\delta^{n-2}_{C^\bullet}} C^{n-2} \longrightarrow K^\vee \longrightarrow 0. \tag{2}$$

The exact sequence (1), when sheafified, gives an exact sequence of vector bundles, and its dual gives the exact sequence of vector bundles

$$0 \longrightarrow \mathcal{E} \longrightarrow \widetilde{C}^0 \xrightarrow{\delta^1_{C^\bullet}} \widetilde{C}^1 \xrightarrow{\delta^2_{C^\bullet}} \widetilde{C}^2 \xrightarrow{\delta^3_{C^\bullet}} \cdots \xrightarrow{\delta^{n-2}_{C^\bullet}} \widetilde{C}^{n-2} \longrightarrow \mathcal{K}^\vee \longrightarrow 0. \tag{3}$$

From this it becomes evident that $E = H^0_*(\mathscr{E})$ is given as $H^0(C^\bullet_{\{0,n\}})$, and $H^i_*(\mathscr{E}) = H^i(C^\bullet_{\{0,n\}})$ for $i = 1, \dots n-1$ (where $C^{n-1}_{\{0,n\}}$ is understood to refer to $K^\vee$).

E itself has a free resolution (again possibly infinite). Splice $C^\bullet_{\{0,n\}}$ with a free resolution $L^\bullet$ of E and call the resulting complex $C^\bullet$. The complex $C^\bullet$ is bounded above and has the property that $H^i(C^\bullet) = H^i_*(\mathscr{E})$ for $i = 1, \dots n-1$ and equals 0 for other values of i.

Choose a free resolution $P^\bullet$ in $\mathscr{C}^-(\mathfrak{P})$ of $C^\bullet$:

$$\begin{array}{ccccccccccccccc}
P^\bullet: \cdots & \longrightarrow & P^{-2} & \longrightarrow & P^{-1} & \longrightarrow & P^0 & \xrightarrow{\delta^1_{P^\bullet}} & P^1 & \xrightarrow{\delta^2_{P^\bullet}} & \cdots & \xrightarrow{\delta^{n-2}_{P^\bullet}} & P^{n-2} & \longrightarrow & P^{n-1} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
C^\bullet: \cdots & \longrightarrow & L^{-2} & \longrightarrow & L^{-1} & \longrightarrow & C^0 & \xrightarrow{\delta^1_{C^\bullet}} & C^1 & \xrightarrow{\delta^2_{C^\bullet}} & \cdots & \xrightarrow{\delta^{n-2}_{C^\bullet}} & C^{n-2} & \longrightarrow & K^\vee \longrightarrow 0
\end{array}$$

Then $P^\bullet$ is an element in $\mathscr{C}^-(\mathfrak{P})$ with the property that $H^i(P^\bullet)$ is an A-module of at most finite length for $1 \le i \le n-1$ and is zero for other i. In [Horrocks 1964] the bounded version of such a free complex was called a $\mathfrak{Z}$-complex, while Walters [1996] called the category of such complexes $\mathrm{FinL}(\mathfrak{P})$. In our setting, we will call it a Horrocks data complex and use the notation of [Walter 1996]. We also define a "Horrocks data bundle" for each such Horrocks data complex:

Definition 1.1. $\mathrm{FinL}^-(\mathfrak{P})$ is the full subcategory of all complexes $P^\bullet$ in $\mathscr{C}^-(\mathfrak{P})$ with the property that $H^i(P^\bullet)$ is an A-module of at most finite length for $1 \le i \le n-1$ and is zero for other i. A complex $P^\bullet$ in $\mathrm{FinL}^-(\mathfrak{P})$ will be called a Horrocks data complex. For such a complex, let $F = \ker(\delta^1_{P^\bullet} : P^0 \to P^1)$. Then the sheaf $\mathscr{F} = \widetilde{F}$ will be called a Horrocks data bundle on X.

It should be clear that the above $\mathscr{F}$ is a vector bundle on X with the property that $H^i_*(\mathscr{F}) = H^i(P^\bullet)$ for $1 \le i \le n-1$.

Lemma 1.2 [Horrocks 1964, Theorem 7.2]. *$F^\vee$ has a finite free resolution.*

Proof. Horrocks' proof cited above is when A is a regular ring, but remains valid when A is Cohen–Macaulay. Another proof (indicated by the referee) is: $0 \to (P^{n-1})^\vee \to (P^{n-2})^\vee \to \cdots \to (P^0)^\vee \to F^\vee \to 0$ is a complex in M, locally free and exact away from the maximal ideal for the vertex of the cone over X, and hence is exact by the Peskine–Szpiro acyclicity lemma. □

Since the modules of global sections of a nonfree ACM bundle and of its dual bundle on X have infinite projective dimension over A, it follows that a Horrocks data bundle $\mathscr{F}$ can have no nonfree ACM bundle or its dual as a summand.

Since any $P^\bullet$ in $\mathscr{C}^-(\mathfrak{P})$ decomposes as $M^\bullet \oplus L^\bullet$, where $M^\bullet$ is a minimal free complex and $L^\bullet$ is an acyclic free complex, we get $\mathscr{F} = \mathscr{F}_{\min} \oplus \mathscr{L}$, where $\mathscr{F}$, $\mathscr{F}_{\min}$, and $\mathscr{L}$ are the Horrocks data bundles corresponding to $P^\bullet$, $M^\bullet$, and $L^\bullet$ respectively.

$\mathcal{L}$ is a free bundle and $\mathcal{F}_{\min}$ will be called a "minimal" Horrocks data bundle. The projective space version of the following isomorphism theorem can be found in [Horrocks 1964, Theorem 7.5, Proposition 9.5] or [Walter 1996, Lemma 2.11].

Proposition 1.3. *Let $\sigma : \mathcal{F} \to \mathcal{F}'$ be a homomorphism between two minimal Horrocks data bundles on X such that σ induces isomorphisms $H^i_*(\mathcal{F}) \to H^i_*(\mathcal{F}')$ for $1 \le i \le n-1$. Then σ is an isomorphism.*

Proof. The proofs of the results cited above work in our ACM setting as well. □

Returning to the vector bundle $\mathcal{E}$, let $P^\bullet$ be a free resolution of $C^\bullet$ as described above. Let $P^\bullet_{\ge 0}$ denote the naive truncation of $P^\bullet$ at the zeroth term. We get the induced homomorphism of complexes

$$P^\bullet_{\ge 0} \to C^\bullet_{\{0,n\}}.$$

For F defined as $\ker \delta^1_{P^\bullet}$, there is an induced homomorphism $F \to E$. For the Horrocks data bundle $\mathcal{F} = \widetilde{F}$, we get a homomorphism $\beta : \mathcal{F} \to \mathcal{E}$ which induces isomorphisms $H^i_*(\mathcal{F}) \to H^i_*(\mathcal{E})$ for $1 \le i \le n-1$. Hence any vector bundle $\mathcal{E}$ has a "Horrocks datum", as we now define:

Definition 1.4. Let $\mathcal{E}$ be a vector bundle on X. A pair $(\mathcal{F}, \beta)$ will be called a Horrocks datum for $\mathcal{E}$ if $\mathcal{F}$ is a Horrocks data bundle and β is a homomorphism $\beta : \mathcal{F} \to \mathcal{E}$ which induces isomorphisms $H^i_*(\mathcal{F}) \to H^i_*(\mathcal{E})$ for $1 \le i \le n-1$.

A point on terminology: Auslander's approximation theorem [Auslander and Bridger 1969, Proposition 4.26, Corollary 4.27] quoted in the introduction states that, given a module M of finite Gorenstein dimension n, there exist a projective module P, a module $H_n(M)$ of projective dimension n, a module M_n of zero Gorenstein dimension and an exact sequence $0 \to M_n \to M \oplus P \to H_n(M) \to 0$. Following Auslander's suggestion, Buchweitz [1986, Corollary 5.3.3] called $H_n(M)$ (with the map $M \to H_n(M)$) a "hull of finite projective dimension" for M, and M_n the maximal Cohen–Macaulay approximation to M.

In the case where our variety X is arithmetically Gorenstein, Auslander's sequence can be seen as coming from the dual of the η-sequence of Theorem 1.7 below: given E, the η-sequence $0 \to K \to F \to E \to 0$ dualized gives $0 \to E^\vee \to F^\vee \to K^\vee$, where $F^\vee$ has finite projective dimension. When X is arithmetically Gorenstein, $K^\vee$ is a maximal Cohen–Macaulay module and $F^\vee \to K^\vee$ is surjective. Pull back the exact sequence by a surjection $L \to K^\vee \to 0$, with L projective. It splits. This induces an exact sequence $0 \to N \to E^\vee \oplus L \to F^\vee \to 0$, where N (the kernel of $L \to K^\vee$) is a maximal Cohen–Macaulay module. This fits the above approximation theorem for $E^\vee$.

However, we have chosen the notation “$\mathcal{F}$ is the Horrocks data bundle for $\mathcal{E}$” since $\mathcal{F}$ encodes all the intermediate cohomology data of $\mathcal{E}$.

Theorem 1.5. *Let $\mathcal{E}_1, \mathcal{E}_2$ be vector bundles on X with Horrocks data $(\mathcal{F}_1, \beta_1)$, $(\mathcal{F}_2, \beta_2)$ respectively. Let $\sigma : \mathcal{E}_1 \to \mathcal{E}_2$ be a homomorphism.*

(1) *There is a free bundle $\mathcal{L}$ and a commuting square*

$$\begin{array}{ccc} \mathcal{F}_1 & \longrightarrow & \mathcal{F}_2 \oplus \mathcal{L} \\ \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle (\beta_2,*)} \\ \mathcal{E}_1 & \xrightarrow[\sigma]{} & \mathcal{E}_2 \end{array}$$

(2) *If $H^0_*(\beta_2) : H^0_*(\mathcal{F}_2) \to H^0_*(\mathcal{E}_2)$ is surjective, the free bundle $\mathcal{L}$ can be chosen to be zero.*

Proof. It is straightforward to see that the construction of the complex $C^\bullet$ out of the vector bundle $\mathcal{E}$ is functorial in the sense that, given $\sigma : \mathcal{E}_1 \to \mathcal{E}_2$, there is an induced morphism from $C_1^\bullet \to C_2^\bullet$ with the property that the homomorphisms $H^i(C_1^\bullet) \to H^i(C_2^\bullet)$ coincide with $H^i(\sigma) : H^i_*(\mathcal{E}_1) \to H^i_*(\mathcal{E}_2)$ for $1 \le i \le n-1$. In the special case of $\beta_k : \mathcal{F}_k \to \mathcal{E}_k$, a Horrocks datum, we get a quasi-isomorphism $P_k^\bullet \to \mathcal{C}_k^\bullet$, where $P_k^\bullet$ is the Horrocks data complex associated to $\mathcal{F}_k$, so that $P_k^\bullet \to \mathcal{C}_k^\bullet$ is a free resolution of $\mathcal{C}_k^\bullet$. Now given a morphism of complexes $C_1^\bullet \to C_2^\bullet$, we can lift the morphism to their free resolutions, after adding a free acyclic complex to $P_2^\bullet$. This gives the commuting square of part (1). The proof of part (2) is elementary. ☐

The following theorems (Theorems 1.6 and 1.7) are to be found in more general form in [Buchweitz 1986] as the “syzygy theorem for Gorenstein rings”. The diagram in Theorem 1.8 below is Buchweitz’s octahedron [1986, (5.3.1)].

Theorem 1.6 (γ sequence for $\mathcal{E}$). *Let $\mathcal{E}$ be a vector bundle on X and $(\mathcal{F}, \beta)$ a Horrocks datum for $\mathcal{E}$. From the Horrocks data complex $P^\bullet$ for $\mathcal{F}$, consider the exact sequence $\Psi : 0 \to \mathcal{F} \to \mathcal{P}^0 \to \mathcal{G} \to 0$, where $\mathcal{P}^0 = \widetilde{P}^0$ and $\mathcal{G} = \widetilde{G}$ with $G = \ker \delta^2_{P^\bullet}$. We define γ as the pushout of Ψ by β*

$$\begin{array}{ccccccccc} \Psi : 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{P}^0 & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \beta} & & \downarrow & & \| & & \\ \gamma : 0 & \longrightarrow & \mathcal{E} & \xrightarrow[f]{} & \mathcal{A} & \xrightarrow[g]{} & \mathcal{G} & \longrightarrow & 0 \end{array}$$

(1) *Given two bundles $\mathcal{E}_1, \mathcal{E}_2$, a morphism $\sigma : \mathcal{E}_1 \to \mathcal{E}_2$, and Horrocks data $(\mathcal{F}_1, \beta_1)$, $(\mathcal{F}_2, \beta_2)$ for each bundle, we obtain a commuting box of short exact*

sequences (using obvious notation)

$$\begin{array}{ccc} \Psi_1 & \longrightarrow & \Psi_2 \oplus \lambda \\ \downarrow{\beta_1} & & \downarrow{(\beta_2,*)} \\ \gamma_1 & \xrightarrow[\sigma]{} & \gamma_2 \end{array}$$

where λ *is a short exact sequence* $0 \to \mathcal{L} \to \mathcal{L} \to 0 \to 0$ *of free bundles. If* $H^0_*(\beta_2)$ *is surjective onto* $H^0_*(\mathcal{E}_2)$, λ *may be taken to be zero.*

(2) $H^{n-1}_*(\mathcal{G}) = 0$, *and* $\mathcal{A}$ *is an ACM bundle on* X.

(3) *Up to a short exact sequence* $0 \to 0 \to \mathcal{L} \to \mathcal{L} \to 0$ *of free bundles, the sequence* γ *depends only on* $\mathcal{E}$ *and not on the choice of Horrocks datum.*

Proof. (1) σ lifts to a map $\mathcal{F}_1 \to \mathcal{F}_2 \oplus \mathcal{L}$ to give a commuting square, by Theorem 1.5. $\mathcal{F}_2 \oplus \mathcal{L}$ is a Horrocks data bundle for the Horrocks data complex, where P^0 is replaced by $P^0 \oplus \mathcal{L}$ but with the same bundle $\mathcal{G}_2$. It is easy to see that the map $\mathcal{F}_1 \to \mathcal{F}_2 \oplus \mathcal{L}$ extends to a map of sequences $\Psi_1 \to \Psi_2 \oplus \lambda$. The pushouts of Ψ_2 and $\Psi_2 \oplus \lambda$ give the same sequence γ_2. Lastly, since we have a commuting square from the first line of the proof, the pushouts of Ψ_1 and $\Psi_2 \oplus \lambda$ give a commuting box of exact sequences.

(2) By construction, $H^{n-1}_*(\mathcal{G}) = H^n(P^\bullet) = 0$. Since we have isomorphisms $H^i_*(\mathcal{G}) \cong H^{i+1}_*(\mathcal{F}) \cong H^{i+1}_*(\mathcal{E})$ for $1 \le i \le n-2$ and $H^0_*(\mathcal{G}) \twoheadrightarrow H^1_*(\mathcal{F}) \cong H^1_*(\mathcal{E})$, we conclude that $\mathcal{A}$ is ACM.

(3) This follows from the first part when we apply the previous theorem to the identity morphism from $\mathcal{E}$ to $\mathcal{E}$. Indeed, the theorem, together with Proposition 1.3, shows that any two Horrocks data bundles for $\mathcal{E}$ are stably equivalent. ☐

Theorem 1.7 (η sequence for $\mathcal{E}$). *Let* $(\mathcal{F}, \beta)$ *be a Horrocks datum for the bundle* $\mathcal{E}$ *such that* $H^0_*(\beta)$ *is surjective. We define the* η *sequence for* $\mathcal{E}$ *to be*

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{F} \xrightarrow{\beta} \mathcal{E} \longrightarrow 0,$$

where $\mathcal{K}$ *is the kernel bundle.*

(1) $\mathcal{K}$ *is an ACM bundle.*

(2) η *is determined by* $\mathcal{E}$ *up to a short exact sequence* $0 \to \mathcal{L} \to \mathcal{L} \to 0 \to 0$ *of free bundles.*

(3) *Given a morphism* $\sigma : \mathcal{E}_1 \to \mathcal{E}_2$, *there is an induced morphism of short exact sequences* $\eta_1 \to \eta_2$.

Proof. The proof is easy. We just mention that the induced map $\eta_1 \to \eta_2$ depends on the choice of a map from $\mathcal{F}_1$ to $\mathcal{F}_2$ that lifts σ (as obtained from Theorem 1.5). □

Theorem 1.8 (diagram of $\mathcal{E}$). *Let $(\mathcal{F}, \beta)$ be a Horrocks datum for the bundle $\mathcal{E}$ such that $H^0_*(\beta)$ is surjective. The γ and η sequences of $\mathcal{E}$ fit into a diagram for $\mathcal{E}$*

$$
\begin{array}{cccccccccc}
 & & & 0 & & 0 & & & & \\
 & & & \downarrow & & \downarrow & & & & \\
 & & & \mathcal{K} & = & \mathcal{K} & & & & \\
 & & & \downarrow \alpha & & \downarrow & & & & \\
\Psi: & 0 & \longrightarrow & \mathcal{F} & \xrightarrow[g]{} & \mathcal{P}^0 & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \\
 & & & \downarrow \beta & & \downarrow & & \| & & \\
\gamma: & 0 & \longrightarrow & \mathcal{E} & \xrightarrow[f]{} & \mathcal{A} & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \\
 & & & \downarrow & & \downarrow & & & & \\
 & & & 0 & & 0 & & & & \\
 & & & \eta & & \Delta & & & &
\end{array}
$$

Given a morphism $\sigma : \mathcal{E}_1 \to \mathcal{E}_2$, there is an induced map from the diagram of $\mathcal{E}_1$ to the diagram of $\mathcal{E}_2$.

Proof. While the existence of the diagram is clear, the map from diagram of $\mathcal{E}_1$ to the diagram of $\mathcal{E}_2$ with appropriate commuting boxes exists because the choice of a map from $\mathcal{F}_1$ to $\mathcal{F}_2$ that lifts σ will determine $\eta_1 \to \eta_2$ and then allows a choice of a map $\Psi_1 \to \Psi_2$. This now gives the commuting box of short exact sequences of Theorem 1.6. □

The following is a criterion for obtaining a map between two γ-sequences:

Proposition 1.9. *Let $\mathcal{E}, \mathcal{E}'$ be two vector bundles with the same (minimal) Horrocks data bundle $\mathcal{F}_{\min}$ and Horrocks data $(\mathcal{F}_{\min}, \beta)$, $(\mathcal{F}_{\min}, \beta')$. Let $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k$ be the distinct nonfree irreducible ACM bundles (up to twists by $\mathcal{O}_X(a)$) that appear as summands in the middle term $\mathcal{A}_{\mathcal{E}}$ of the γ-sequence of $\mathcal{E}$. For each $\mathcal{B}_i$, let V_i be the kernel of the map $H^1_*(\beta \otimes 1_{\mathcal{B}_i^\vee})$ from $H^1_*(\mathcal{F}_{\min} \otimes \mathcal{B}_i^\vee)$ to $H^1_*(\mathcal{E} \otimes \mathcal{B}_i^\vee)$, and let V'_i be the same with β replaced by β'. If $V_i \subseteq V'_i$ for all i, then there exists a map $\phi : \mathcal{E} \to \mathcal{E}'$ such that the γ-sequence of $\mathcal{E}'$ is the pushout by ϕ of the γ-sequence for $\mathcal{E}$.*

Proof. Since the γ-sequences γ, γ' are pushouts by β, β' of the Ψ-sequence for $\mathcal{F}_{\min}$

$$\Psi : 0 \longrightarrow \mathcal{F}_{\min} \longrightarrow \mathcal{P}^0 \longrightarrow \mathcal{G}_{\min} \longrightarrow 0$$

in the commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Hom}(\mathcal{P}^0, \mathcal{E}') & \longrightarrow & \mathrm{Hom}(\mathcal{F}_{\min}, \mathcal{E}') & \xrightarrow[\delta(\Psi)]{} & \mathrm{Ext}^1(\mathcal{G}_{\min}, \mathcal{E}') & \longrightarrow & \mathrm{Ext}^1(\mathcal{P}^0, \mathcal{E}') \\
 & & \uparrow \beta & & \| & & \uparrow \\
 & & \mathrm{Hom}(\mathcal{E}, \mathcal{E}') & \xrightarrow[\delta(\gamma)]{} & \mathrm{Ext}^1(\mathcal{G}_{\min}, \mathcal{E}') & \longrightarrow & \mathrm{Ext}^1(\mathcal{A}_{\mathcal{E}}, \mathcal{E}')
\end{array}$$

it suffices to show that $\gamma' \in \mathrm{Ext}^1(\mathcal{G}_{\min}, \mathcal{E}')$ maps to zero in $\mathrm{Ext}^1(\mathcal{A}_{\mathcal{E}}, \mathcal{E}')$, for then there is an element $\sigma \in \mathrm{Hom}(\mathcal{E}, \mathcal{E}')$ such that $\sigma \circ \beta$ differs from β' by a map that factors through $\mathcal{P}^0$.

Let $\rho : \mathcal{A}_{\mathcal{E}} \to \mathcal{G}_{\min}$ be the map occurring in the γ-sequence of $\mathcal{E}$. Then under the connecting homomorphism for $\gamma \otimes \mathcal{A}_{\mathcal{E}}^{\vee}$, ρ maps to zero under $H^0_*(\mathcal{G}_{\min} \otimes \mathcal{A}_{\mathcal{E}}^{\vee}) \to H^1_*(\mathcal{E} \otimes \mathcal{A}_{\mathcal{E}}^{\vee})$. Hence, under the connecting homomorphism of $\Psi \otimes \mathcal{A}_{\mathcal{E}}^{\vee}$, ρ maps to the kernel of $H^1_*(\mathcal{F}_{\min} \otimes \mathcal{A}_{\mathcal{E}}^{\vee}) \to H^1_*(\mathcal{E} \otimes \mathcal{A}_{\mathcal{E}}^{\vee})$. By the assumption $V_i \subseteq V_i'$ for all i, ρ also maps to the kernel of $H^1_*(\mathcal{F}_{\min} \otimes \mathcal{A}_{\mathcal{E}}^{\vee}) \to H^1_*(\mathcal{E}' \otimes \mathcal{A}_{\mathcal{E}}^{\vee})$. It follows that the pullback of γ' by ρ splits, which was the desired result. □

This criterion leads to an isomorphism theorem on X:

Theorem 1.10 (isomorphism theorem). *Let $\mathcal{E}, \mathcal{E}'$ be two vector bundles on X, with the same minimal Horrocks data bundle $\mathcal{F}_{\min}$ and Horrocks data $(\mathcal{F}_{\min}, \beta)$, $(\mathcal{F}_{\min}, \beta')$. Let $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_k$ be the distinct nonfree irreducible ACM bundles (up to twists by $\mathcal{O}_X(a)$) that appear as summands in either of the middle terms $\mathcal{A}_{\mathcal{E}}, \mathcal{A}_{\mathcal{E}'}$ of the γ-sequences of $\mathcal{E}, \mathcal{E}'$. If for each i the kernel of $H^1_*(\beta \otimes 1_{\mathcal{B}_i^{\vee}})$ equals the kernel of $H^1_*(\beta' \otimes 1_{\mathcal{B}_i^{\vee}})$ and if $\mathcal{E}$ and $\mathcal{E}'$ have no ACM summands, then $\mathcal{E} \cong \mathcal{E}'$.*

Proof. If $\mathcal{F}$ is free, $\mathcal{E}, \mathcal{E}'$ are ACM and the theorem does not apply. So we will assume that $\mathcal{F}_{\min}$ is a nonfree minimal Horrocks data bundle. By applying Proposition 1.9, there exists a homomorphism $\sigma : \mathcal{E} \to \mathcal{E}'$ and a commutative diagram of γ-sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{A}_{\mathcal{E}} & \longrightarrow & \mathcal{G}_{\min} & \longrightarrow & 0 \\
 & & \downarrow \sigma & & \downarrow \sigma_1 & & \| & & \\
0 & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{A}_{\mathcal{E}'} & \longrightarrow & \mathcal{G}_{\min} & \longrightarrow & 0
\end{array}$$

Tensor the diagram by $\mathcal{B}^{\vee}$, where $\mathcal{B}$ will stand for any of the distinct irreducible ACM bundles (up to twists by $\mathcal{O}_X(a)$) that appear as summands in $\mathcal{A}_{\mathcal{E}'}$, including the possible free line bundle $\mathscr{O}_X$. In the induced diagram of cohomology, we get

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
H^0_*(\mathscr{E}\otimes\mathscr{B}^\vee) & \xrightarrow[\sigma]{} & H^0_*(\mathscr{E}'\otimes\mathscr{B}^\vee) \\
\downarrow & & \downarrow \\
H^0_*(\mathscr{A}_{\mathscr{E}}\otimes\mathscr{B}^\vee) & \xrightarrow[\sigma_1]{} & H^0_*(\mathscr{A}_{\mathscr{E}'}\otimes\mathscr{B}^\vee) \\
\downarrow & & \downarrow \\
H^0_*(\mathscr{G}_{\min}\otimes\mathscr{B}^\vee) & = & H^0_*(\mathscr{G}_{\min}\otimes\mathscr{B}^\vee) \\
\downarrow & & \downarrow \\
H^1_*(\mathscr{E}\otimes\mathscr{B}^\vee) & \xrightarrow[\sigma]{} & H^1_*(\mathscr{E}'\otimes\mathscr{B}^\vee) \\
\downarrow & & \downarrow \\
H^1_*(\mathscr{A}_{\mathscr{E}}\otimes\mathscr{B}^\vee) & \xrightarrow[\sigma_1]{} & H^1_*(\mathscr{A}_{\mathscr{E}'}\otimes\mathscr{B}^\vee)
\end{array}$$

The map $H^0_*(\mathscr{G}_{\min}\otimes\mathscr{B}^\vee)\to H^1_*(\mathscr{E}\otimes\mathscr{B}^\vee)$ factors through $H^1_*(\mathscr{F}\otimes\mathscr{B}^\vee)$, since γ is the pushout of Ψ by β. The condition of equality of kernels for $H^1_*(\beta\otimes 1_{\mathscr{B}^\vee})$ and $H^1_*(\beta'\otimes 1_{\mathscr{B}^\vee})$ implies that the kernel in $H^0_*(\mathscr{G}_{\min}\otimes\mathscr{B}^\vee)$ is the same for $\mathscr{E}$ and $\mathscr{E}'$. Therefore the mapping cone map $H^0_*(\mathscr{E}'\otimes\mathscr{B}^\vee)\oplus H^0_*(\mathscr{A}_{\mathscr{E}}\otimes\mathscr{B}^\vee)\to H^0_*(\mathscr{A}_{\mathscr{E}'}\otimes\mathscr{B}^\vee)$ is surjective. Viewing each summand $\mathscr{B}$ of $\mathscr{A}_{\mathscr{E}'}$, the identity global section in $H^0(\mathscr{B}\otimes\mathscr{B}^\vee)$ is in the image of this surjection. It cannot be in the image of $H^0_*(\mathscr{E}'\otimes\mathscr{B}^\vee)$ since $\mathscr{E}'$ does not have $\mathscr{B}$ as a summand. Hence it is in the image of some $\mathscr{B}'$ term in $\mathscr{A}_{\mathscr{E}}$. This forces $\mathscr{B}'$ to equal $\mathscr{B}$, and the map $\sigma_1:\mathscr{A}_{\mathscr{E}}\to\mathscr{A}_{\mathscr{E}'}$ has to split over this $\mathscr{B}$ term in $\mathscr{A}_{\mathscr{E}'}$.

It follows that σ_1 is a (split) surjection. Hence $\sigma:\mathscr{E}\to\mathscr{E}'$ is onto. The roles of $\mathscr{E},\mathscr{E}'$ can be interchanged, showing that they are bundles of the same rank. Hence $\sigma:\mathscr{E}\cong\mathscr{E}'$. □

The following theorem is in the same vein, and extends Proposition 1.3:

Theorem 1.11. *Let $\sigma:\mathscr{E}\to\mathscr{E}'$ be a sheaf homomorphism between two vector bundles on X, where $\mathscr{E}'$ has no ACM summands. Suppose that σ induces isomorphisms $H^i_*(\mathscr{E})\to H^i_*(\mathscr{E}')$ for $1\le i\le n-1$, and also, for each nonfree irreducible ACM bundle $\mathscr{B}$ appearing in $\mathscr{A}_{\mathscr{E}'}$, suppose that the induced map $H^1_*(\mathscr{E}\otimes\mathscr{B}^\vee)\to H^1_*(\mathscr{E}'\otimes\mathscr{B}^\vee)$ is an isomorphism. Then σ is a split surjection decomposing $\mathscr{E}$ into $\mathscr{E}'\oplus\mathscr{C}$, where $\mathscr{C}$ is an ACM bundle.*

Proof. By Theorem 1.5, σ can be lifted to a map $\tilde{\sigma} : \mathcal{F}_{\min} \to \mathcal{F}'_{\min}$ of minimal Horrocks data bundles. Since $H^i_*(\tilde{\sigma})$ is an isomorphism for $1 \le i \le n-1$, $\tilde{\sigma}$ is an isomorphism. So, for convenience, we may assume that $\mathcal{F}_{\min} = \mathcal{F}'_{\min}$, and, according to Theorem 1.6, σ induces a map of γ-sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{A}_{\mathcal{E}} & \longrightarrow & \mathcal{G}_{\min} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma_1} & & \| & & \\ 0 & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{A}_{\mathcal{E}'} & \longrightarrow & \mathcal{G}_{\min} & \longrightarrow & 0 \end{array}$$

For each $\mathcal{B}$ appearing in $\mathcal{A}_{\mathcal{E}'}$, as in the proof of the previous theorem after tensoring by $\mathcal{B}^\vee$ we can look at the diagram of cohomology. Since $H^1_*(\mathcal{E} \otimes \mathcal{B}^\vee) \to H^1_*(\mathcal{E}' \otimes \mathcal{B}^\vee)$ is an isomorphism, the kernel in $H^0_*(\mathcal{G}_{\min} \otimes \mathcal{B}^\vee)$ is the same for $\mathcal{E}$ and $\mathcal{E}'$. The previous argument repeats to show that the homomorphism $\sigma_1 : \mathcal{A}_{\mathcal{E}} \to \mathcal{A}_{\mathcal{E}'}$ is a split surjection, with a kernel $\mathcal{C}$ which is ACM. Hence $\sigma : \mathcal{E} \to \mathcal{E}'$ is also a split surjection with kernel equal to $\mathcal{C}$. □

Since the A-submodules $V_i = \ker(H^1_*(\mathcal{F}_{\min} \otimes \mathcal{B}_i^\vee) \to H^1_*(\mathcal{E} \otimes \mathcal{B}_i^\vee))$ play such an important role in the above description of a bundle $\mathcal{E}$, it is worthwhile to make the following definition describing its properties:

Definition 1.12. Let $\mathcal{F}$ be a sheaf on X and $\mathcal{B}$ an ACM bundle on X with a minimal set of generators for $H^0_*(\mathcal{B})$ given by $\bigoplus_j \mathcal{O}_X(a_j) \to \mathcal{B} \to 0$. The kernel of $H^1_*(\mathcal{F} \otimes \mathcal{B}^\vee) \to H^1_*(\mathcal{F} \otimes \bigoplus_j \mathcal{O}_X(-a_j))$ will be called the A-module of $\mathcal{B}$-socle elements for $\mathcal{F}$ and denoted by $H^1_*(\mathcal{F} \otimes \mathcal{B}^\vee)_{\text{soc}}$. A homogeneous element in this kernel in degree d will be a $\mathcal{B}$-socle element in $H^1(\mathcal{F}(d) \otimes \mathcal{B}^\vee)$.

Remark 1.13. (1) For a vector bundle $\mathcal{F}$, the module of $\mathcal{B}$-socle elements for $\mathcal{F}$ has finite length over the field k.

(2) Suppose $\mathcal{B}^\vee \to \mathcal{O}_X(b)$ is any map. Then, for any sheaf $\mathcal{F}$, a $\mathcal{B}$-socle element in $H^1_*(\mathcal{F} \otimes \mathcal{B}^\vee)$ maps to zero in $H^1_*(\mathcal{F}(b))$, since $\mathcal{B}^\vee \to \mathcal{O}_X(b)$ factors through $\bigoplus_j \mathcal{O}_X(-a_j)$.

(3) Suppose $\mathcal{E}$ is a bundle on X with Horrocks datum $(\mathcal{F}_{\min}, \beta)$. Then, for any ACM bundle $\mathcal{B}$, the module $V = \ker(H^1_*(\mathcal{F}_{\min} \otimes \mathcal{B}^\vee) \to H^1_*(\mathcal{E} \otimes \mathcal{B}^\vee))$ consists of $\mathcal{B}$-socle elements for $\mathcal{F}_{\min}$. Indeed, the map $H^1_*\big(\mathcal{F}_{\min} \otimes \bigoplus_j \mathcal{O}_X(-a_j)\big) \to H^1_*\big(\mathcal{E} \otimes \bigoplus_j \mathcal{O}_X(-a_j)\big)$ is an isomorphism.

Example 1.14. Any ACM variety X with a nondegenerate embedding into $\mathbb{P}^N$ has a Horrocks data bundle given by $\Omega^1_{\mathbb{P}}|_X$ with $H^1_*(\Omega^1_{\mathbb{P}}|_X) = k$ and with an exact sequence

$$0 \longrightarrow \Omega^1_{\mathbb{P}}|_X \longrightarrow \mathcal{O}_X(-1)^{\oplus N+1} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

For any ACM bundle $\mathcal{B}$ on X, without free summands and with $\mathcal{B}^\vee \hookrightarrow \bigoplus_j \mathcal{O}_X(-a_j)$, consider the diagram

$$\begin{array}{ccc} H^0_*(\mathcal{O}_X \otimes \mathcal{B}^\vee) & \longrightarrow & H^1_*(\Omega^1_{\mathbb{P}}|_X \otimes \mathcal{B}^\vee) \\ \downarrow & & \downarrow \\ H^0_*\big(\mathcal{O}_X \otimes \bigoplus_j \mathcal{O}_X(-a_j)\big) & \longrightarrow & H^1_*\big(\Omega^1_{\mathbb{P}}|_X \otimes \bigoplus_j \mathcal{O}_X(-a_j)\big) \end{array}$$

Then any minimal generator of the module $H^0_*(\mathcal{O}_X \otimes \mathcal{B}^\vee)$ maps to a nongenerator in $H^0_*\big(\mathcal{O}_X \otimes \bigoplus_j \mathcal{O}_X(-a_j)\big)$, and hence maps to zero in $H^1_*\big(\Omega^1_{\mathbb{P}}|_X \otimes \bigoplus_j \mathcal{O}_X(-a_j)\big) = \bigoplus_j k(-a_j)$. Thus the image of $H^0_*(\mathcal{O}_X \otimes \mathcal{B}^\vee)$ in $H^1_*(\Omega^1_{\mathbb{P}}|_X \otimes \mathcal{B}^\vee)$ is nonzero and consists of $\mathcal{B}$-socle elements for $\Omega^1_{\mathbb{P}}|_X$. So, for any ACM bundle $\mathcal{B}$ on X, without free summands, the Horrocks data bundle $\Omega^1_{\mathbb{P}}|_X$ will have $\mathcal{B}$-socle elements.

For a general ACM variety X, one would expect infinitely many families of nonisomorphic and irreducible ACM bundles; hence this shows that even for a fixed Horrocks data bundle $\mathcal{F}_{\min}$, the number of bundles $\mathcal{E}$ with Horrocks datum $(\mathcal{F}_{\min}, \beta_{\mathcal{E}})$ would get out of control, especially with the construction given below. In later sections, we will limit our attention to the quadric hypersurface and the Veronese surface, where there are only finitely many ACM bundles. In these sections, we will also be able to deal with arbitrary submodules of $\mathcal{B}$-socle elements, instead of the entire $\mathcal{B}$-socle module of the rather crude theorem below.

Theorem 1.15 (existence). *Let $\mathcal{F}_{\min}$ be a minimal Horrocks data bundle on X, and let $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_k$ be a finite collection of irreducible, nonfree ACM bundles on X. Then there is a vector bundle $\mathcal{E}$ on X with Horrocks datum $(\mathcal{F}_{\min}, \beta)$ and with* $\ker H^1_*(\beta \otimes 1_{\mathcal{B}_i^\vee}) = H^1_*(\mathcal{F}_{\min} \otimes \mathcal{B}_i^\vee)_{\mathrm{soc}}$ *for $1 \le i \le k$.*

Proof. Each $H^1_*(\mathcal{F}_{\min} \otimes \mathcal{B}_i^\vee)_{\mathrm{soc}}$ is an A-module, and we can pick a collection of minimal generators for the module. Let K_i be the vector subspace spanned by this collection inside $H^1_*(\mathcal{F}_{\min} \otimes \mathcal{B}_i^\vee)_{\mathrm{soc}}$. Let $\mathcal{B} = \bigoplus(K_i \otimes_k \mathcal{B}_i)$. The data $K_i, 1 \le i \le k$, can be viewed as a $\mathcal{B}$-socle element in $H^1_*(\mathcal{F}_{\min} \otimes \mathcal{B}^\vee)$, and hence gives an extension (that defines a bundle $\mathcal{E}$)

$$0 \longrightarrow \mathcal{F}_{\min} \xrightarrow{\beta} \mathcal{E} \xrightarrow{\rho} \mathcal{B} \longrightarrow 0.$$

Since the element is a socle element, the pullback of the sequence under any map $\mathcal{O}_X(b) \to \mathcal{B}$ will split. Hence $H^0_*(\rho)$ is surjective, giving $(\mathcal{F}_{\min}, \beta)$ the Horrocks datum for $\mathcal{E}$.

By construction, the subspace $K_i \cdot I_{\mathcal{B}_i}$ in $H^0_*(\mathcal{B} \otimes \mathcal{B}_i^\vee)$ maps isomorphically to $K_i \subseteq H^1_*(\mathcal{F}_{\min} \otimes \mathcal{B}_i^\vee)_{\mathrm{soc}}$. Hence the image of the map of A-modules $H^0_*(\mathcal{B} \otimes \mathcal{B}_i^\vee) \to H^1_*(\mathcal{F}_{\min} \otimes \mathcal{B}_i^\vee)_{\mathrm{soc}}$ is onto. □

Remark 1.16. (1) The same construction can be done for an arbitrary A-submodule V_i of $H^1_*(\mathcal{F}_{\min} \otimes \mathcal{B}_i^\vee)_{\mathrm{soc}}$. We would choose K_i to be the subspace spanned by a set of minimal generators for V_i. In the last step of the above proof, we find that image of the map of A-modules $H^0_*(\mathcal{B} \otimes \mathcal{B}_i^\vee) \to H^1_*(\mathcal{F}_{\min} \otimes \mathcal{B}_i^\vee)_{\mathrm{soc}}$ contains V_i, and could possibly be larger. Hence the Horrocks invariants of $\mathcal{E}$, $\ker H^1_*(\beta \otimes 1_{\mathcal{B}_i^\vee})$, may not be precisely recognizable in this case.

(2) In the above theorem, for the $\mathcal{E}$ so constructed, it is possible to identify $A_{\mathcal{E}}$ in the case when X is arithmetically Gorenstein, or when the dual of each of the ACM bundles $\mathcal{B}_i$, $1 \le i \le k$, is also ACM: since the γ-sequence of $\mathcal{E}$ is the pushforward of the Ψ-sequence for $\mathcal{F}_{\min}$, we get the exact sequence $0 \to \mathcal{P}^0 \to \mathcal{A}_{\mathcal{E}} \to \mathcal{B} \to 0$, which is forced to split by the extra hypotheses. Once the ACM bundles in $\mathcal{A}_{\mathcal{E}}$ are identified, it is possible to compare $\mathcal{E}$ with other bundles via the uniqueness theorems (Theorems 1.10, 1.11).

(3) However, in the non-arithmetically Gorenstein case, a clear description of $\mathcal{A}_{\mathcal{E}}$ may not be apparent at the end of the construction of the theorem. We will give an example later (Example 3.3) where an identification of $\mathcal{A}_{\mathcal{E}}$ requires more work.

It is easy to obtain a splitting criterion for a vector bundle $\mathcal{E}$ on X to be free, which gives for example the criterion for quadrics in [Ancona and Ottaviani 1991] that was cited in the introduction. Once again, in the theorem below, note that the condition invoking any ACM bundle is not very useful when there are too many ACM bundles on X. It is more interesting (see the proof below) in the case where the choices for $\mathcal{B}$ are limited, for example, if one could limit the possible ACM bundles that might appear as a summand in the diagram of $\mathcal{E}$.

Theorem 1.17 (a splitting criterion). *Let $\mathcal{E}$ be a vector bundle of rank $\le r$ on X, a smooth ACM variety of dimension n, such that $H^i_*(\mathcal{E}^\vee) = 0$ for $1 \le i \le \min\{r-1, n-1\}$ and also $H^1_*(\mathcal{E}^\vee \otimes \mathcal{B}) = 0$ for any ACM bundle $\mathcal{B}$ on X. Then $\mathcal{E}$ is free.*

Proof. Now the η-sequence (Theorem 1.7) of $\mathcal{E}$, $0 \to \mathcal{K} \to \mathcal{F} \to \mathcal{E} \to 0$, gives an element in $H^1(\mathcal{E}^\vee \otimes \mathcal{K})$ which is zero by hypothesis. Hence $\mathcal{K}$ and $\mathcal{E}$ are summands of $\mathcal{F}$. Since $\mathcal{F}$ is a Horrocks data bundle, it can have no nonfree ACM summand, so $\mathcal{K}$ must be free. Thus $\mathcal{E}$ itself is a Horrocks data bundle.

If $r \ge n$, $\mathcal{E}^\vee$ is ACM. But the dual of a Horrocks data bundle has finite resolution, so $\mathcal{E}^\vee$ must be free.

If $r < n$, consider the sequence (3) with $\mathcal{E}$ replaced by $\mathcal{E}^\vee$. From the vanishing of cohomologies of $\mathcal{E}^\vee$, when we look at the complex of global sections of the sequence, we conclude that the module $E^\vee$ is an $(r+1)$-th syzygy, and $E^\vee$ has finite projective dimension since $\mathcal{E}$ is a Horrocks data bundle. By the Evans–Griffith syzygy theorem [1981], $\mathcal{E}^\vee$ is free. □

Remark 1.18. If X is a smooth quadric hypersurface, the above splitting criterion is also equivalent to Corollary 4.3 of [Ballico and Malaspina 2009]. Splitting criteria have been established on other varieties. For a Grassmannian of lines $G(1,n)$, which supports infinitely many irreducible ACM bundles when $n \geq 4$, it is possible to prove a splitting criterion (see Theorem 2.6 of [Arrondo and Malaspina 2010]) with a finite number of cohomological vanishing conditions involving only the ACM bundles $S^i Q$, where $i = 1, \ldots, n-2$ and Q is the tautological rank-two bundle. Similarly, on multiprojective spaces $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$, there is a splitting criterion (see Theorem 3.9 of [Ballico and Malaspina 2011]) with a finite number of cohomological vanishing conditions involving only the ACM bundles $\mathbb{O}(k_1, \ldots, k_s)$, where $-n_j \leq k_j \leq 0$. These results are much stronger than Theorem 1.17. Due to the generality of our setting, we are unable to prove a splitting criterion with conditions involving only a finite number of ACM bundles.

However, when there is additional analysis of the ACM bundles, more can be said. For example, Arrondo and Graña [1999] identified a list of six specific ACM bundles on $G(1,4)$, and showed that any other ACM bundle $\mathcal{B}$ is a summand of a bundle that appears in the middle of a short exact sequence of bundles, where the bundles on either side are built from direct sums of twists of these six bundles. Hence in our Theorem 1.17, applied to $G(1,4)$, it suffices to consider only these six specific bundles for $\mathcal{B}$. It is now straightforward to check that Ottaviani's splitting criterion on $G(1,4)$ (which is just one case of [Ottaviani 1987, Théorème 1]) follows from Theorem 1.17. (He assumed that $H^i_*(\mathcal{E}^\vee) = 0$ for $1 \leq i \leq 5$ and his other hypotheses imply that $H^1_*(\mathcal{E}^\vee \otimes \mathcal{B}) = 0$ for these six ACM bundles.)

2. Quadric hypersurfaces

Let $\mathcal{Q}_n \subset \mathbb{P}^{n+1}$ be a smooth quadric hypersurface. We will work over a field of characteristic not two. The quadratic form defining $\mathcal{Q}_n$ descends to a quadratic form on the tangent bundle of $\mathcal{Q}_n$. Hence one can define spinor bundles on $\mathcal{Q}_n$ [Karrer 1973]. Set $l := \lfloor (n+1)/2 \rfloor$. If n is even, then $\mathcal{Q}_n$ has two distinct spinor bundles Σ_1 and Σ_2 of rank 2^{l-1}. If n is odd, then $\mathcal{Q}_n$ has a unique spinor bundle, which we denote Σ_1, of rank 2^{l-1}. Algebraic properties of these bundles were studied by Ottaviani [1988], who obtained them using the geometry of the variety of all maximal linear subspaces of $\mathcal{Q}_n$ to construct morphisms from $\mathcal{Q}_n$ to $G(2^{l-1}, 2^l)$. He shows that these spinor bundles on $\mathcal{Q}_n$ are ACM bundles. Kapranov [1988] showed how these bundles were crucial in describing the derived category of sheaves on the quadric. Meanwhile, Knörrer [1987], classifying maximal Cohen–Macaulay modules over isolated quadratic hypersurface singularities, described these bundles as the fundamental ACM bundles on $\mathcal{Q}_n$ (see [Buchweitz et al. 1987] for the interpretation of Knörrer's results in terms of bundles). Knörrer's classification of ACM bundles on $\mathcal{Q}_n$ was proved also in [Ancona and Ottaviani 1991].

We use a unified notation Σ_i for spinor bundles on $\mathcal{Q}_n$, where for even n, i can take on the values $1, 2$, while if n is odd, i can be only 1. We follow the notation of [Kapranov 1988], whose spinor bundles differ from those in [Ottaviani 1988] by a twist of 1. Hence Σ_i is generated by its global sections and $\Sigma_i(-1)$ has no sections.

We will call a bundle of the form $\Sigma_i(a)$ a twisted spinor bundle on $\mathcal{Q}_n$. The fundamental theorem of [Knörrer 1987] is:

Theorem 2.1. *Any ACM bundle on $\mathcal{Q}_n$ is a direct sum of line bundles and twisted spinor bundles.*

The spinor bundles on $\mathcal{Q}_n$ satisfy some dualities [Ottaviani 1988]: when n is odd or $n \equiv 0 \pmod 4$, $\Sigma_i^\vee \cong \Sigma_i(-1)$, while if $n \equiv 2 \pmod 4$, $\Sigma_i^\vee \cong \Sigma_j(-1)$, where $j \neq i$.

In addition, the spinor bundles on $\mathcal{Q}_n$ satisfy canonical sequences. To further unify the notation, when n is odd or when $n \equiv 2 \pmod 4$, define $i \mapsto \bar{i}$ to be the identity on indices, and when $n \equiv 0 \pmod 4$, define $i \mapsto \bar{i}$ to be the transposition of the indices 1 and 2. With this notation, we have the canonical sequences

$$0 \longrightarrow \Sigma_{\bar{i}}^\vee \xrightarrow{u_i} \mathcal{O}^{\oplus 2^l} \xrightarrow{v_i} \Sigma_i \longrightarrow 0 \tag{4}$$

(see [Ottaviani 1988, Theorem 2.8]).

Ottaviani [1988, Lemma 2.7] proved that, for any spinor bundle Σ_i, $\mathrm{End}(\Sigma_i) = H^0(\Sigma_i \otimes \Sigma_i^\vee) = k$ and $\mathrm{Hom}(\Sigma_i, \Sigma_j) = 0$ for $i \neq j$. Using this, and tensoring the sequence above with $\Sigma_i^\vee$, we get $H^1(\Sigma_{\bar{i}}^\vee \otimes \Sigma_i^\vee) = k$, where Id_{Σ_i} maps to a generator of $H^1(\Sigma_{\bar{i}}^\vee \otimes \Sigma_i^\vee)$. For completeness, the following lemma is also easy to prove:

Lemma 2.2.
$$H^1_*(\Sigma_{\bar{i}}^\vee \otimes \Sigma_i^\vee) = k, \tag{5}$$
$$H^1_*(\Sigma_j^\vee \otimes \Sigma_i^\vee) = 0 \quad \textit{if } j \neq \bar{i}. \tag{6}$$

Recall the definition of socle elements.

Definition 2.3. Let $\mathcal{F}$ be a sheaf on $\mathcal{Q}_n$. The sequence dual to (4) tensored by $\mathcal{F}$ gives

$$0 \longrightarrow \mathcal{F} \otimes \Sigma_i^\vee \longrightarrow \mathcal{F} \otimes \mathcal{O}^{\oplus 2^l} \longrightarrow \mathcal{F} \otimes \Sigma_{\bar{i}} \longrightarrow 0$$

and a natural map $H^1_*(\mathcal{F} \otimes \Sigma_i^\vee) \to H^1_*(\mathcal{F} \otimes \mathcal{O}^{\oplus 2^l})$.

An element in $H^1(\mathcal{F}(d) \otimes \Sigma_i^\vee)$ will be called a Σ_i-socle element for $\mathcal{F}$ in degree d if it is annihilated by the map $H^1(\mathcal{F}(d) \otimes \Sigma_i^\vee) \to H^1_*(\mathcal{F} \otimes \mathcal{O}^{\oplus 2^l})$.

The terminology "socle" comes from the case of a quadric surface studied in [Malaspina and Rao 2014], where socle elements were annihilated by multiplication by the forms lifted from one of the $\mathbb{P}^1$ factors of $\mathcal{Q}_2$. We have extended this terminology to all ACM bundles in Section 1.

Lemma 2.4. *Let $\mathcal{F}$ be a sheaf on $\mathcal{Q}_n$. Let V be a finite-dimensional graded subspace consisting of Σ_i-socle elements in $H^1_*(\mathcal{F}\otimes\Sigma_i^\vee)$. Then there is a homomorphism $\alpha : V\otimes\Sigma_{\bar{i}}^\vee \to \mathcal{F}$ such that $H^1_*(\alpha\otimes 1_{\Sigma_i^\vee})$ has image V.*

Proof. Consider the dual canonical sequence (4) tensored by $\mathcal{F}$

$$0 \longrightarrow \mathcal{F}\otimes\Sigma_i^\vee \longrightarrow \mathcal{F}\otimes\mathcal{O}^{\oplus 2^l} \longrightarrow \mathcal{F}\otimes\Sigma_{\bar{i}} \longrightarrow 0.$$

We get

$$H^0(\mathcal{F}\otimes\Sigma_{\bar{i}}) \to H^1(\mathcal{F}\otimes\Sigma_i^\vee) \to H^1(\mathcal{F}\otimes\mathcal{O}^{\oplus 2^l}).$$

There is a graded subspace V' of $H^0_*(\mathcal{F}\otimes\Sigma_{\bar{i}})$ which is mapped isomorphically to $V\subset H^1_*(\mathcal{F}\otimes\Sigma_i^\vee)$. This induces a map $\alpha : V'\otimes_k\Sigma_{\bar{i}}^\vee \to \mathcal{F}$. Thus we can construct the commuting diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathcal{F}\otimes\Sigma_i^\vee & \xleftarrow[\alpha\otimes 1]{} & \big(V'\otimes_k\Sigma_{\bar{i}}^\vee\big)\otimes\Sigma_i^\vee \\
\downarrow{\scriptstyle 1\otimes v_i^\vee} & & \downarrow{\scriptstyle 1\otimes v_i^\vee} \\
\mathcal{F}\otimes\mathcal{O}^{\oplus 2^l} & \xleftarrow[\alpha\otimes 1]{} & \big(V'\otimes_k\Sigma_{\bar{i}}^\vee\big)\otimes\mathcal{O}^{\oplus 2^l} \\
\downarrow{\scriptstyle 1\otimes u_i^\vee} & & \downarrow{\scriptstyle 1\otimes u_i^\vee} \\
\mathcal{F}\otimes\Sigma_{\bar{i}} & \xleftarrow[\alpha\otimes 1]{} & \big(V'\otimes_k\Sigma_{\bar{i}}^\vee\big)\otimes\Sigma_{\bar{i}} \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

Then $H^1_*(\alpha\otimes 1) : H^1_*((V'\otimes_k\Sigma_{\bar{i}}^\vee)\otimes\Sigma_i^\vee) \to H^1_*(\mathcal{F}\otimes\Sigma_i^\vee)$ gives $V'\cong V$. □

Corollary 2.5. *Let $\mathcal{F}$ be a vector bundle on $\mathcal{Q}_n$. Then any graded vector subspace V of Σ_i-socle elements in $H^1_*(\mathcal{F}\otimes\Sigma_i^\vee)_{\text{soc}}$ is an A-submodule of $H^1_*(\mathcal{F}\otimes\Sigma_i^\vee)_{\text{soc}}$.*

Proof. In the proof above $H^1_*(\alpha\otimes 1_{\Sigma_i^\vee})$ is an A-module homomorphism, and by Lemma 2.2 the A-module $H^1_*((V'\otimes_k\Sigma_{\bar{i}}^\vee)\otimes\Sigma_i^\vee)$ has the trivial A-module structure, where multiplication by graded elements in A of positive degree is zero. □

For any vector bundle $\mathcal{E}$ on $\mathcal{Q}_n$, we will define invariants as follows:

Definition 2.6 (Horrocks invariants of $\mathscr{E}$). Let $\mathscr{E}$ be a vector bundle on $\mathscr{Q}_n$. It has a minimal associated Horrocks datum $(\mathscr{F}_{\min}, \beta)$. Let

$$V_i = \ker H^1(\beta \otimes \mathrm{Id}_{\Sigma_i^\vee}) : H^1_*(\mathscr{F}_{\min} \otimes \Sigma_i^\vee) \to H^1_*(\mathscr{E} \otimes \Sigma_i^\vee).$$

Then V_i is a graded subspace of $H^1_*(\mathscr{F}_{\min} \otimes \Sigma_i^\vee)_{\mathrm{soc}}$. The collection $(\mathscr{F}_{\min}, V_i)$ will be called Horrocks invariants for $\mathscr{E}$. (As usual, when n is even, this means $(\mathscr{F}_{\min}, V_1, V_2)$ and when n is odd, it means $(\mathscr{F}_{\min}, V_1)$.)

Remark 2.7. (1) $\mathscr{E}$ is ACM if and only if $\mathscr{F}_{\min}$ is the zero bundle. $V_i = 0$ as well.

(2) In general, $V_i = 0$ for all i if and only if $\mathscr{E}$ is a direct sum of a Horrocks data bundle and an ACM bundle.

(3) If $\mathscr{B}$ is an ACM bundle, then $\mathscr{E}$ and $\mathscr{E} \oplus \mathscr{B}$ will have the same Horrocks invariants.

(4) If $(\mathscr{F}_{\min}, \beta, V_i)$ is a collection of Horrocks invariants for $\mathscr{E}$ and ϕ is an automorphism of $\mathscr{F}_{\min}$, then ϕ can be used to change $\beta : \mathscr{F}_{\min} \to \mathscr{E}$ and hence also V_i to get a new collection of Horrocks invariants for $\mathscr{E}$.

(5) The definition could have used an arbitrary Horrocks data bundle $\mathscr{F}$ for $\mathscr{E}$ instead of the minimal one $\mathscr{F}_{\min}$, since $H^1_*(\Sigma_i^\vee) = 0$ and hence the description of V_i would not change.

A stronger existence theorem for quadrics can now be stated than was proved in Theorem 1.15. Below we have a statement that deals with arbitrary subspaces of socle elements:

Theorem 2.8 (existence). *Let $\mathscr{F}_{\min}$ be a minimal Horrocks data bundle on $\mathscr{Q}_n$ and let V_i be a graded vector subspace of $H^1_*(\mathscr{F}_{\min} \otimes \Sigma_i^\vee)_{\mathrm{soc}}$. Then there exists a vector bundle $\mathscr{E}$ with the Horrocks invariants $(\mathscr{F}_{\min}, V_1, V_2)$ (when n is even) and invariants $(\mathscr{F}_{\min}, V_1)$ (when n is odd).*

Proof. We follow the approach in Theorem 1.15. For notational convenience, assume n is even, so $i = 1, 2$. Let $\mathscr{B} = (V_1 \otimes_k \Sigma_1) \oplus (V_2 \otimes_k \Sigma_2)$. As in the earlier proof, we obtain a short exact sequence (defining $\mathscr{E}$):

$$0 \longrightarrow \mathscr{F}_{\min} \xrightarrow{\beta} \mathscr{E} \xrightarrow{\rho} (V_1 \otimes_k \Sigma_1) \oplus (V_2 \otimes_k \Sigma_2) \longrightarrow 0,$$

where $(\mathscr{F}_{\min}, \beta)$ is a Horrocks datum for the bundle $\mathscr{E}$ so obtained. Our goal is now to show that the image of $H^0_*(\mathscr{B} \otimes \Sigma_i^\vee) \to H^1_*(\mathscr{F}_{\min} \otimes \Sigma_i^\vee)$ is V_i, whereas in the earlier proof we showed that it contained V_i. Let $\Sigma_j(a)$ be any summand in $\mathscr{B}$, and pick a nonzero section $s \in H^0(\Sigma_j(a) \otimes \Sigma_i^\vee(b))$, or a map $s : \Sigma_i(-b) \to \Sigma_j(a)$. Then $a + b \geq 0$. The section $s \in H^0(\mathscr{B} \otimes \Sigma_i^\vee(b))$ maps to zero in $H^1_*(\mathscr{F}_{\min} \otimes \Sigma_i^\vee)$ if and only if the pullback of the short exact sequence by the map $s : \Sigma_i(-b) \to \mathscr{B}$ is a split sequence. If $a + b > 0$, by Lemma 2.2 the map $s : \Sigma_i(-b) \to \Sigma_j(a)$

factors through $\mathcal{O}^{\oplus 2^l}(a)$. The pullback of the short exact sequence by the map $\mathcal{O}^{\oplus 2^l}(a) \to \Sigma_j(a) \subseteq \mathcal{B}$ splits since the extension is defined by socle elements. Hence so does the pullback by the map $\Sigma_i(-b) \to \Sigma_j(a) \subseteq \mathcal{B}$.

It follows that the only nonzero contribution from this summand $\Sigma_j(a)$ to the image of $H^0(\mathcal{B} \otimes \Sigma_i^\vee(b))$ occurs when $a + b = 0$. If $i \neq j$, $\mathrm{Hom}(\Sigma_i, \Sigma_j) = 0$ and so no section s can be found. If $i = j$, $\mathrm{End}(\Sigma_i) = k$ and it follows that the image of s lies in V_i. Thus the image of $H^0_*(\mathcal{B} \otimes \Sigma_i^\vee)$ is exactly V_i. □

As pointed out after Theorem 1.15, if $\mathcal{F}_{\min}$ has a Ψ-sequence $0 \to \mathcal{F}_{\min} \to \mathcal{P}^0 \to \mathcal{G}_{\min} \to 0$, then the $\mathcal{E}$ constructed in the above theorem has γ-sequence given as

$$0 \longrightarrow \mathcal{E} \longrightarrow \bigoplus_i (V_i \otimes_k \Sigma_i) \oplus \mathcal{P}^0 \longrightarrow \mathcal{G}_{\min} \longrightarrow 0.$$

It is also easy to see that since $\mathcal{F}_{\min}$ has no summands of type Σ_i, neither does $\mathcal{E}$. Conversely, suppose $\mathcal{E}$ is a vector bundle on $\mathcal{Q}_n$ with Horrocks invariants $(\mathcal{F}_{\min}, V_i)$ and with no summands of type Σ_i. It will follow from the next theorems that $\mathcal{E}$ has a γ-sequence with $\mathcal{A}_{\mathcal{E}} = \bigoplus_i (V_i \otimes_k \Sigma_i) \oplus \mathcal{P}'$, where $\mathcal{P}'$ is free.

The following two uniqueness results follow easily from the general theorems of Section 1.

Theorem 2.9 (uniqueness). *Given two bundles $\mathcal{E}, \mathcal{E}'$ on $\mathcal{Q}_n$ without ACM summands and with Horrocks invariants $(\mathcal{F}_{\min}, V_i)$, $(\mathcal{F}'_{\min}, V_i')$, suppose that there exists $\phi : \mathcal{F}_{\min} \xrightarrow{\sim} \mathcal{F}'_{\min}$ such that the induced isomorphisms $H^1_*(\mathcal{F}_{\min} \otimes \Sigma_i^\vee) \cong H^1_*(\mathcal{F}'_{\min} \otimes \Sigma_i^\vee)$ carry V_i to V_i' for each i. Then $\mathcal{E}$ and $\mathcal{E}'$ are isomorphic.*

Proof. We may assume that $\mathcal{E}$ and $\mathcal{E}'$ have the same minimal Horrocks data bundle $\mathcal{F}_{\min}$. If $\mathcal{F}_{\min}$ is zero, $\mathcal{E}, \mathcal{E}'$ are ACM and the theorem does not apply. So we will assume that $\mathcal{F}_{\min}$ is a nonfree minimal Horrocks data bundle. If V_i is 0 for $i = 1, 2$, then $\mathcal{E}$ is stably equivalent to $\mathcal{F}_{\min}$, and, being without ACM summands, it must be isomorphic to $\mathcal{F}_{\min}$. Since V_i' will also be zero, the same is true for $\mathcal{E}'$ and we conclude that $\mathcal{E} \cong \mathcal{E}'$. So assume V_i is nonzero for some i. If there is an automorphism ϕ of $\mathcal{F}_{\min}$ which carries V_i to V_i', in the diagram of Theorem 1.8 for $\mathcal{E}'$, we may replace $\beta' : \mathcal{F}_{\min} \to \mathcal{E}'$ by $\beta' \circ \phi^{-1}$ and so on, and assume that β and β' give the same kernel V_i in $H^1_*(F_{\min} \otimes \Sigma_i^\vee)$.

We can now apply Theorem 1.10 to conclude the result. □

Theorem 2.10. *Let $\mathcal{E}, \mathcal{E}'$ be vector bundles on $\mathcal{Q}_n$ with no ACM summands. Suppose $\sigma : \mathcal{E} \to \mathcal{E}'$ is a homomorphism such that σ induces $H^j_*(\mathcal{E}) \cong H^j_*(\mathcal{E}')$ for $1 \leq j \leq n-1$ and also isomorphisms $H^1_*(\mathcal{E} \otimes \Sigma_i^\vee) \cong H^1_*(\mathcal{E}' \otimes \Sigma_i^\vee)$ for all i. Then σ is an isomorphism.*

Proof. This is just Theorem 1.11 with the additional condition that $\mathcal{E}$ has no ACM summands. □

3. The Veronese surface

The Veronese surface $\mathcal{V} \subset \mathbb{P}^5$ is an arithmetically Cohen–Macaulay embedding which is not arithmetically Gorenstein. The study of vector bundles on $\mathcal{V}$ is trivial if we view $\mathcal{V}$ as $\mathbb{P}^2$. Below we discuss how the techniques of Section 1 apply to the embedded variety $\mathcal{V}$. With its polarization from the embedding, $\mathcal{V}$ has two irreducible, nonfree ACM bundles (up to twists). Hence, as in the case of quadric hypersurfaces of even dimension, we can define Horrocks invariants $(\mathcal{F}_{\min}, V, W)$ for any vector bundle $\mathcal{E}$ on $\mathcal{V}$. But unlike in the case of the quadric, where V, W were independent of each other, here there is a dependency between them.

In the following discussion, we will write $\mathcal{O}_{\mathcal{V}}(1)$ for $\mathcal{O}_{\mathbb{P}^5}(1)|_{\mathcal{V}}$ and $\mathcal{O}_{\mathcal{V}}(n)$ for $\mathcal{O}_{\mathcal{V}}(1)^{\otimes n}$. We will write $\mathcal{L}$ for $\mathcal{O}_{\mathbb{P}^2}(1)$ and $\mathcal{U}$ for $\Omega^1_{\mathcal{V}} \otimes \mathcal{L}$. Then the only irreducible ACM bundles on $\mathcal{V}$ (with respect to the polarization $\mathcal{O}_{\mathcal{V}}(1)$) are $\mathcal{O}_{\mathcal{V}}(n)$, $\mathcal{L}(n)$ and $\mathcal{U}(n)$. In the diagram of a bundle $\mathcal{E}$ on $\mathcal{V}$ in Theorem 1.8, the terms $\mathcal{A}_{\mathcal{E}}$ and $\mathcal{K}_{\mathcal{E}}$ are built out of these three types of irreducible ACM bundles. The vector bundle $\mathcal{G}$ is a free bundle and the Ψ-sequence is the sheafification of a free presentation of the A-module $H^1_*(\mathcal{E})$. The connection between $\mathcal{A}_{\mathcal{E}}$ and $\mathcal{K}_{\mathcal{E}}$, given by the Δ-sequence in the diagram of $\mathcal{E}$, is controlled by the canonical sequences

$$0 \longrightarrow \mathcal{U} \xrightarrow{\ u\ } 3\mathcal{O}_{\mathcal{V}} \xrightarrow{\ v\ } \mathcal{L} \longrightarrow 0 \tag{7}$$

and

$$0 \longrightarrow 3\mathcal{U}(-1) \oplus \mathcal{O}_{\mathcal{V}}(-1) \longrightarrow 9\mathcal{O}_{\mathcal{V}}(-1) \longrightarrow \mathcal{U} \longrightarrow 0, \tag{8}$$

where the second can be simplified noncanonically to

$$0 \longrightarrow 3\mathcal{U}(-1) \xrightarrow{\ u'\ } 8\mathcal{O}_{\mathcal{V}}(-1) \xrightarrow{\ v'\ } \mathcal{U} \longrightarrow 0. \tag{9}$$

In addition, there is the canonical sequence

$$0 \longrightarrow \mathcal{O}_{\mathcal{V}}(-1) \longrightarrow 3\mathcal{L}(-1) \longrightarrow \mathcal{U} \longrightarrow 0. \tag{10}$$

The two uniqueness theorems of Section 1 apply in this setting, where given a bundle $\mathcal{E}$ on $\mathcal{V}$ we can construct Horrocks invariants for $\mathcal{E}$ as $(\mathcal{F}_{\min}, V, W)$, where $(\mathcal{F}_{\min}, \beta)$ is a Horrocks datum for $\mathcal{E}$, $V = \ker(H^1_*(\mathcal{F}_{\min} \otimes \mathcal{L}^\vee) \to H^1_*(\mathcal{E} \otimes \mathcal{L}^\vee))$ and $W = \ker(H^1_*(\mathcal{F}_{\min} \otimes \mathcal{U}^\vee) \to H^1_*(\mathcal{E} \otimes \mathcal{U}^\vee))$. Thus to complete the classification of bundles on $\mathcal{V}$ by this method it remains to get a description of any constraints on $V \subseteq H^1_*(\mathcal{F} \otimes \mathcal{L}^\vee)$ and $W \subseteq H^1_*(\mathcal{F}_{\min} \otimes \mathcal{U}^\vee)$, and to finally show that given $(\mathcal{F}_{\min}, V, W)$ with these constraints, there exists a bundle $\mathcal{E}$ with those invariants.

By Remark 1.13, V is an A-submodule of $\mathcal{L}$-socle elements in $H^1_*(\mathcal{F}_{\min} \otimes \mathcal{L}^\vee)_{\mathrm{soc}}$ and W is an A-submodule of $\mathcal{U}$-socle elements in $H^1_*(\mathcal{F}_{\min} \otimes \mathcal{U}^\vee)_{\mathrm{soc}}$. By the next lemma, there is no distinction between the concepts of graded A-submodules and graded vector subspaces of socle elements:

Lemma 3.1. *For any vector bundle $\mathcal{F}$ on $\mathcal{V}$, in the A-module structures of both $H^1_*(\mathcal{F}\otimes\mathcal{L}^\vee)_{\mathrm{soc}}$ and $H^1_*(\mathcal{F}\otimes\mathcal{U}^\vee)_{\mathrm{soc}}$, multiplication by graded elements of positive degree in A is zero.*

Proof. Let $\eta\in H^1(\mathcal{F}(d)\otimes\mathcal{L}^\vee)_{\mathrm{soc}}$, giving a short exact sequence $0\to\mathcal{F}(d)\to\mathcal{A}\to\mathcal{L}\to 0$. Consider multiplication by $x\in A$ of degree one, $\cdot x:\mathcal{L}(-1)\to\mathcal{L}$. The pullback by this map of the short exact sequence (7) is split since $H^1(\mathcal{U}\otimes\mathcal{L}^\vee(1))=0$. So $\cdot x:\mathcal{L}(-1)\to\mathcal{L}$ factors through $3\mathcal{O}_\mathcal{V}$. By the definition of $\mathcal{L}$-socle elements, the pullback of η by $3\mathcal{O}_\mathcal{V}\to\mathcal{L}$ splits, hence also the pullback of η by $\cdot x:\mathcal{L}(-1)\to\mathcal{L}$. Thus $x\cdot\eta=0$.

A similar proof works for an element $\eta\in H^1(\mathcal{F}(d)\otimes\mathcal{U}^\vee)_{\mathrm{soc}}$. One notices that the pullback by $\cdot x:\mathcal{U}(-1)\to\mathcal{U}$ of the short exact sequence (9) is split because $H^1_*(\mathcal{U}\otimes\mathcal{U}^\vee)=3k$ supported in $H^1(\mathcal{U}\otimes\mathcal{U}^\vee(-1))$. ☐

In the definition of $\mathcal{U}$-socle elements for $\mathcal{F}$, the noncanonical inclusion $\mathcal{U}^\vee\hookrightarrow 8\mathcal{O}_\mathcal{V}(1)$ can be replaced by a canonical composite inclusion $\mathcal{U}^\vee\hookrightarrow 3\mathcal{L}^\vee(1)\hookrightarrow 9\mathcal{O}_\mathcal{V}(1)$. For any bundle $\mathcal{F}$, this gives a canonical map

$$\phi_\mathcal{F}: H^1_*(\mathcal{F}\otimes\mathcal{U}^\vee)_{\mathrm{soc}}\to 3H^1_*(\mathcal{F}(1)\otimes\mathcal{L}^\vee)_{\mathrm{soc}}.$$

When $\mathcal{E}$ is a vector bundle with Horrocks invariants $(\mathcal{F}_{\min}, V, W)$, it is immediate to see that V and W are related by $\phi_{\mathcal{F}_{\min}}(W)\subseteq 3V(1)$. This is a dependency between V and W. In fact, this is the only requirement on the pair (V,W) for proving an existence theorem on the Veronese surface:

Theorem 3.2. *Let $\mathcal{F}_{\min}$ be a minimal Horrocks data bundle on $\mathcal{V}$, and let V, W be graded vector subspaces of $H^1_*(\mathcal{F}_{\min}\otimes\mathcal{L}^\vee)_{\mathrm{soc}}$, $H^1_*(\mathcal{F}_{\min}\otimes\mathcal{U}^\vee)_{\mathrm{soc}}$ with the property that $\phi_{\mathcal{F}_{\min}}(W)\subseteq 3V(1)$. Then there is a vector bundle $\mathcal{E}$ on $\mathcal{V}$ with Horrocks invariants $(\mathcal{F}_{\min}, V, W)$.*

Proof. Construct $\mathcal{E}$ as an extension of $\mathcal{F}_{\min}$ by $\mathcal{B}=(V\otimes_k\mathcal{L})\oplus(W\otimes_k\mathcal{U})$:

$$0\longrightarrow\mathcal{F}_{\min}\xrightarrow{\ \beta\ }\mathcal{E}\longrightarrow\mathcal{B}\longrightarrow 0. \tag{$*$}$$

Since V, W are subspaces of socle elements, $\mathcal{E}$ has $(\mathcal{F}_{\min},\beta)$ as its Horrocks datum. We wish to understand the images of $H^0_*(\mathcal{B}\otimes\mathcal{L}^\vee)\to H^1_*(\mathcal{F}_{\min}\otimes\mathcal{L}^\vee)$ and $H^0_*(\mathcal{B}\otimes\mathcal{U}^\vee)\to H^1_*(\mathcal{F}_{\min}\otimes\mathcal{U}^\vee)$. $\mathrm{End}(\mathcal{L})=\mathrm{End}(\mathcal{U})=k$ and the image of $V\cdot I_\mathcal{L}\subseteq H^0(V\otimes\mathcal{L}\otimes\mathcal{L}^\vee)$ and $W\cdot I_\mathcal{U}\subseteq H^0(W\otimes\mathcal{U}\otimes\mathcal{U}^\vee)$ give V and W in $H^1_*(\mathcal{F}_{\min}\otimes\mathcal{L}^\vee)_{\mathrm{soc}}$ and $H^1_*(\mathcal{F}_{\min}\otimes\mathcal{U}^\vee)_{\mathrm{soc}}$. It remains to analyze any other contributions to the two images inside $H^1_*(\mathcal{F}_{\min}\otimes\mathcal{L}^\vee)_{\mathrm{soc}}$ and $H^1_*(\mathcal{F}_{\min}\otimes\mathcal{U}^\vee)_{\mathrm{soc}}$, and prove that the images are just V and W respectively.

Let $\mathcal{L}(b), \mathcal{U}(b)$ be any summands in $(V\otimes_k\mathcal{L})\oplus(W\otimes_k\mathcal{U})$. Consider maps $\sigma_1:\mathcal{L}(a)\to\mathcal{L}(b)$, $\sigma_2:\mathcal{L}(a)\to\mathcal{U}(b)$, $\sigma_3:\mathcal{U}(a)\to\mathcal{U}(b)$, $\sigma_4:\mathcal{U}(a)\to\mathcal{L}(b)$. For σ_1, assume $a<b$ since we wish to omit endomorphisms of $\mathcal{L}$. Likewise for σ_3. In

the sequence (7) tensored by $\mathcal{L}^\vee(b-a)$ we have $H^1(\mathcal{U}\otimes\mathcal{L}^\vee(b-a))=0$, and in the sequence (9) tensored by $\mathcal{U}^\vee(b-a)$ we have $H^1(3\mathcal{U}(-1)\otimes\mathcal{U}^\vee(b-a))=0$. Hence σ_1 factors through $3\mathcal{O}_{\mathcal{V}}(b)$ and σ_3 factors through $8\mathcal{O}_{\mathcal{V}}(b-1)$. By the socle nature of the extension $(*)$, pullbacks of $(*)$ by σ_1 and σ_3 split; hence the element $\sigma_1\in H^0(\mathcal{L}(b)\otimes\mathcal{L}^\vee(-a))$ maps to zero in $H^1_*(\mathcal{F}_{\min}\otimes\mathcal{L}^\vee)$, and likewise σ_3 maps to zero in $H^1_*(\mathcal{F}_{\min}\otimes\mathcal{U}^\vee)$.

For σ_4 to be nonzero, we require that $a<b+1$. We know $H^1(\mathcal{U}\otimes\mathcal{U}^\vee(b-a))=0$. Hence the same argument applies to show that σ_4 factors through $3\mathcal{O}_{\mathcal{V}}(b)$, and we are done. The arguments for σ_3,σ_4 show that the image of $H^0_*(\mathcal{B}\otimes\mathcal{U}^\vee)\to H^1_*(\mathcal{F}_{\min}\otimes\mathcal{U}^\vee)$ equals W.

For σ_2 to be nonzero we require that $a<b$, and we know that

$$H^1(3\mathcal{U}(-1)\otimes\mathcal{L}^\vee(b-a))=0$$

except when $b-a=1$. Hence the only situation of difficulty is when we have $\sigma_2:\mathcal{L}(b-1)\to\mathcal{U}(b)$. Suppose the pullback of our short exact sequence $(*)$ by $\mathcal{L}(b-1)\xrightarrow{\sigma_2}\mathcal{U}(b)\hookrightarrow\mathcal{B}$ is nonsplit. The pullback of $(*)$ by $\mathcal{U}(b)\hookrightarrow\mathcal{B}$ gives a nonzero element w of degree $-b$ in $W\subseteq H^1_*(\mathcal{F}_{\min}\otimes\mathcal{U}^\vee)_{\text{soc}}$. The nonsplit pullback by $\mathcal{L}(b-1)\to\mathcal{B}$ gives a nonzero element v in $H^1(\mathcal{F}_{\min}\otimes\mathcal{L}^\vee(-b+1))_{\text{soc}}$ which is the image of w under $\sigma_2^\vee$. Since $\sigma_2^\vee$ is one component in $\mathcal{U}^\vee(-b)\hookrightarrow 3\mathcal{L}^\vee(-b+1)$, the assumption that $\phi_{\mathcal{F}_{\min}}(W)\subseteq 3V(1)$ tells us that $v\in V$. Thus, the image of $H^0_*(\mathcal{B}\otimes\mathcal{L}^\vee)\to H^1_*(\mathcal{F}_{\min}\otimes\mathcal{L}^\vee)$ equals V. □

We conclude with an example:

Example 3.3. The simplest non-ACM bundle on $\mathcal{V}$ is $\mathcal{E}=\Omega^1_{\mathcal{V}}=\mathcal{U}\otimes\mathcal{L}^\vee$, with $H^1_*(\mathcal{E})=k$ and γ-sequence $0\to\mathcal{E}\to 3\mathcal{L}^\vee\to\mathcal{O}_{\mathcal{V}}\to 0$, while its minimal Horrocks data bundle is $\mathcal{F}=\mathcal{F}_{\min}=\Omega^1_{\mathbb{P}^5}|_{\mathcal{V}}$, with Ψ sequence $0\to\mathcal{F}\to 6\mathcal{O}_{\mathcal{V}}(-1)\to\mathcal{O}_{\mathcal{V}}\to 0$. The map $\beta:\mathcal{F}\to\mathcal{E}$ is the standard map $\Omega^1_{\mathbb{P}^5}|_{\mathcal{V}}\to\Omega^1_{\mathcal{V}}$, which is a surjective map of vector bundles but not surjective on the module of global sections. The Horrocks invariants $(\mathcal{F},V,W)$ of $\mathcal{E}$ are easy to work out and are described below.

$H^1_*(\mathcal{F}\otimes\mathcal{L}^\vee)=H^1(\mathcal{F}(1)\otimes\mathcal{L}^\vee)=3k$, and $H^1_*(\mathcal{E}\otimes\mathcal{L}^\vee)=0$, hence $V=3k=H^1(\mathcal{F}(1)\otimes\mathcal{L}^\vee)$, where all elements in $H^1_*(\mathcal{F}\otimes\mathcal{L}^\vee)$ are $\mathcal{L}$-socle.

There is a commutative diagram that shows the only nonzero parts of $H^1_*(\mathcal{F}\otimes\mathcal{U}^\vee)$ and $H^1_*(\mathcal{E}\otimes\mathcal{U}^\vee)$

$$\begin{array}{ccccccc}
H^0(\mathcal{U}^\vee) & \hookrightarrow & H^1(\mathcal{F}\otimes\mathcal{U}^\vee) & \longrightarrow & H^1(6\mathcal{U}^\vee(-1)) & \longrightarrow & 0\\
\| & & \downarrow{\scriptstyle \beta\otimes I_{\mathcal{U}^\vee}} & & \downarrow & & \\
H^0(\mathcal{U}^\vee) & \cong & H^1(\mathcal{E}\otimes\mathcal{U}^\vee) & \longrightarrow & 0 & &
\end{array}$$

Hence $H^1_*(\mathcal{F}\otimes\mathcal{U}^\vee)=H^1(\mathcal{F}\otimes\mathcal{U}^\vee)$ is nine-dimensional, and the kernel W of $H^1_*(\beta\otimes I_{\mathcal{U}^\vee})$ is a six-dimensional subspace (of $\mathcal{U}$-socle elements) that maps isomorphically to $H^1(6\mathcal{U}^\vee(-1))$.

When we apply the construction of the existence theorems (Theorems 1.15, 3.2) to the data $(\mathscr{F}, V, W)$, we obtain a vector bundle $\widetilde{\mathscr{E}}$ and a pushout diagram (refer to the discussion after Theorem 1.15)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathscr{F} & \longrightarrow & 6\mathscr{O}_{\mathscr{V}}(-1) & \longrightarrow & \mathscr{O}_{\mathscr{V}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\tilde{\beta}} & & \downarrow & & \| & & \\
0 & \longrightarrow & \widetilde{\mathscr{E}} & \longrightarrow & \mathscr{A}_{\widetilde{\mathscr{E}}} & \longrightarrow & \mathscr{O}_{\mathscr{V}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathscr{B} & \cong & \mathscr{B} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where $\mathscr{B} = (V \otimes_k \mathscr{L}) \oplus (W \otimes_k \mathscr{U})$.

According to the uniqueness theorems, $\mathscr{E}$ is a rank-two summand of the rank-20 bundle $\widetilde{\mathscr{E}}$, with the remaining summand of $\widetilde{\mathscr{E}}$ consisting of ACM bundles. In this example, even $\mathscr{A}_{\widetilde{\mathscr{E}}}$ is not obvious because the middle short exact sequence is not split. Indeed, the middle sequence is the pushout of the left sequence, hence it is split if and only if, under $\mathscr{F} \to 6\mathscr{O}_{\mathscr{V}}(-1)$, the image of the element $\tau \in H^1(\mathscr{F} \otimes \mathscr{B}^\vee)$ is zero in $H^1(6\mathscr{O}_{\mathscr{V}}(-1) \otimes \mathscr{B}^\vee)$. However, the components of τ in each of the $\mathscr{U}$-summands of $\mathscr{B}$ generate the vector space $W \subset H^1(\mathscr{F} \otimes \mathscr{U}^\vee)$, and W maps isomorphically to $H^1(6\mathscr{U}^\vee(-1))$. Hence the image of τ is nonzero.

To understand $\widetilde{\mathscr{E}}$ and $\mathscr{A}_{\widetilde{\mathscr{E}}}$, a little more work is needed. The fact that W maps isomorphically to $H^1(6\mathscr{U}^\vee(-1))$ tells us that the middle short exact sequence contains six copies of the canonical sequence (10). Hence $\mathscr{A}_{\widetilde{\mathscr{E}}} = 21\mathscr{L}^\vee$. The map $\mathscr{A}_{\widetilde{\mathscr{E}}} \to \mathscr{O}_{\mathscr{V}}$ is now easy to understand and shows that $\widetilde{\mathscr{E}} = \mathscr{E} \oplus 18\mathscr{L}^\vee$.

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