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Nonvanishing of Dirichlet L-functions

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We show that for at least $\frac{3}{8}$ of the primitive Dirichlet characters χ of large prime modulus, the central value $L\left(\frac{1}{2}, \chi\right)$ does not vanish.

1. Introduction

The zeros of L-functions on the critical line are as important in number theory as they are mysterious. At the real point on the critical line (the central point), an L-function is expected to vanish only for either a good reason or a trivial reason. A good reason is when the central value has some arithmetic significance which explains why it may vanish. For example, the central value of the L-function attached to an elliptic curve over a number field is expected to vanish if and only if the elliptic curve has positive rank (according to the Birch and Swinnerton-Dyer conjecture). A trivial reason is when the functional equation implies that the central value is zero. For instance, the L-function of any odd Hecke–Maass form f has functional equation $L\left(\frac{1}{2}, f\right) = -L\left(\frac{1}{2}, f\right)$ at the central point. In all other cases, the most extensive success in proving the nonvanishing of L-functions has been achieved through the use of mollifiers. For notable examples of the mollifier method, see [Kowalski et al. 2000a; 2000b; Iwaniec and Sarnak 2000; Soundararajan 2000] as well as the works discussed below.

In this paper, we study the classical nonvanishing problem of primitive Dirichlet L-functions. It is conjectured that $L\left(\frac{1}{2}, \chi\right) \neq 0$ for every primitive Dirichlet character χ. Consider for each odd prime p the family of L-functions

$$\{L(s, \chi) : \chi \text{ is primitive modulo } p\};$$

this family has size $p-2$. Viewing $L\left(\frac{1}{2}, \chi\right)$ as a statistical object, we would like to understand its distribution as $p \to \infty$. One way to get a handle on the distribution is through understanding the moments of $L\left(\frac{1}{2}, \chi\right)$, but currently only moments of small order are known. Nevertheless, this is enough to make some progress in proving that a positive proportion of the family is nonvanishing.

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Asymptotic expressions for the first and second moments of $L\left(\frac{1}{2}, \chi\right)$ are well known. By a classical result of Paley [1931], we have

$$\frac{1}{p} \sum_{\chi \bmod p}^{\star} L\left(\tfrac{1}{2}, \chi\right) \sim 1,$$
$$\frac{1}{p} \sum_{\chi \bmod p}^{\star} \left|L\left(\tfrac{1}{2}, \chi\right)\right|^2 \sim \log p,$$

where $\sum^{\star}$ restricts the summation to the primitive characters. The discrepancy between the first and second moments indicates fluctuations in the sizes of the central values. Using these moments and the Cauchy–Schwarz inequality, one can only infer that at least 0% of the family is nonvanishing, since

$$\frac{1}{p} \sum_{\substack{\chi \bmod p \\ L(1/2,\chi)\neq 0}}^{\star} 1 \geq \frac{\left|\frac{1}{p}\sum_{\chi \bmod p}^{*} L\left(\frac{1}{2}, \chi\right)\right|^2}{\frac{1}{p}\sum_{\chi \bmod p}^{*}\left|L\left(\frac{1}{2}, \chi\right)\right|^2} \gg \frac{1}{\log p}.$$

The mollifier method is used to remedy this situation. The origin of the method traces back to the works of Bohr and Landau [1914] and Selberg [1942] on zeros of the Riemann zeta function. The starting idea is to introduce a quantity $M(\chi)$, called the "mollifier", which, on average, approximates the inverses of the supposedly nonvanishing values $L\left(\frac{1}{2}, \chi\right)$. The goal is to choose a mollifier such that the mollified first and second moments are comparable; that is,

$$\frac{1}{p} \sum_{\chi \bmod p}^{\star} L\left(\tfrac{1}{2}, \chi\right) M(\chi) \asymp 1,$$
$$\frac{1}{p} \sum_{\chi \bmod p}^{\star} \left|L\left(\tfrac{1}{2}, \chi\right) M(\chi)\right|^2 \asymp 1.$$

From this, a positive nonvanishing proportion can be inferred:

$$\frac{1}{p} \sum_{\substack{\chi \bmod p \\ L(1/2,\chi)\neq 0}}^{\star} 1 \geq \frac{1}{p} \sum_{\substack{\chi \bmod p \\ L(1/2,\chi)M(\chi)\neq 0}}^{\star} 1 \geq \frac{\left|\frac{1}{p}\sum_{\chi \bmod p}^{*} L\left(\frac{1}{2}, \chi\right)M(\chi)\right|^2}{\frac{1}{p}\sum_{\chi \bmod p}^{*}\left|L\left(\frac{1}{2}, \chi\right)M(\chi)\right|^2} \gg 1. \tag{1-1}$$

Balasubramanian and Murty [1992] were the first to do this; however, their mollifier was inefficient and they obtained only a very small positive proportion of nonvanishing.

Next came the work of Iwaniec and Sarnak [1999], who introduced a systematic technique that has since served as a model for other families of L-functions. Iwaniec and Sarnak took the mollifier

$$M(\chi)=\sum_{m\le M}\frac{y_m\chi(m)}{m^{1/2}}, \tag{1-2}$$

where $M=p^\theta$ is the mollifier length and (y_m) is a sequence of real numbers satisfying $y_m\ll p^\epsilon$. They established the asymptotics of the mollified first and second moments for $\theta<\frac{1}{2}$ and found that the choice of coefficients which maximizes the ratio in (1-1) is essentially

$$y_m=\mu(m)\frac{\log(M/m)}{\log M}, \tag{1-3}$$

yielding a nonvanishing proportion of

$$\frac{1}{p}\sideset{}{^\star}\sum_{\substack{\chi \bmod p\\ L(1/2,\chi)\ne 0}} 1\ge\frac{\theta}{1+\theta}.$$

This can be taken as close to $\frac{1}{3}$ as possible on letting θ approach $\frac{1}{2}$. Computing the mollified moments for larger values of θ would result in a higher proportion of nonvanishing, but this appears to be very difficult to do. The problem seems to have been attempted by Bettin, Chandee, and Radziwiłł. In [Bettin et al. 2015], these authors solved the parallel problem for the Riemann zeta function, by obtaining the asymptotics as $T\to\infty$ of

$$\int_T^{2T}\left|\zeta\left(\tfrac{1}{2}+it\right)\right|^2\left|\sum_{m\le M}\frac{y_m}{m^{(1/2)+it}}\right|^2 dt,$$

where $M=T^\theta$, for values of θ slightly larger than $\frac{1}{2}$. However with regard to the problem for Dirichlet L-functions, the authors remarked, “Our proof would not extend to give an asymptotic formula in this case, and additional input is needed.”

Shortly after the work of Iwaniec and Sarnak, in their study of the nonvanishing of high derivatives of Dirichlet L-functions, Michel and VanderKam [2000] used the “twisted” mollifier

$$M(\chi)=\sum_{m\le M}\frac{y_m\chi(m)}{m^{1/2}}+\frac{\bar\tau_\chi}{p^{1/2}}\sum_{m\le M}\frac{y_m\bar\chi(m)}{m^{1/2}}, \tag{1-4}$$

where $M=p^\theta$, y_m is as in (1-3), and τ_χ is the Gauss sum as defined in their paper. Heuristically, this is a better mimic of $L\left(\frac{1}{2},\chi\right)^{-1}$ because the approximate functional equation of $L\left(\frac{1}{2},\chi\right)$ essentially consists of a sum of two Dirichlet polynomials, one multiplied by a Gauss sum. A similar two-piece mollifier was first used by Soundararajan [1995] in the context of the Riemann zeta function. Michel and

VanderKam [2000] proved for $\theta < \frac{1}{4}$ a nonvanishing proportion of

$$\frac{1}{p} \sideset{}{^\star}\sum_{\substack{\chi \bmod p \\ L(1/2,\chi)\neq 0}} 1 \geq \frac{2\theta}{1+2\theta},$$

recovering the $\frac{1}{3}$ proportion of Iwaniec and Sarnak [1999]. For this method too, computing the mollified moments for larger θ would result in a higher proportion of nonvanishing.

The nonvanishing problem was stuck at the proportion $\frac{1}{3}$ for ten years until Bui [2012] dexterously proved a nonvanishing proportion of 0.3411. His breakthrough was not to increase the length of any existing mollifier but to use an ingenious new two-piece mollifier. Bui [2012, page 1857] commented that "There are two different approaches to improve the results in this and other problems involving mollifiers. One can either extend the length of the Dirichlet polynomial or use some 'better' mollifiers. The former is certainly much more difficult." We take the former, more difficult approach.

Our first idea to attack the nonvanishing problem is to increase the length of the Michel–VanderKam mollifier. This may be a somewhat unexpected avenue because previous attempts at lengthening mollifiers have, as far as we are aware, been directed at the Iwaniec–Sarnak mollifier. Our second idea is to establish an estimate for a trilinear sum of Kloosterman sums with general coefficients (Lemma 3.2). To prove this, we appeal to some work of Fouvry, Ganguly, Kowalski and Michel [Fouvry et al. 2014]. These authors proved best possible estimates for sums of products of Kloosterman sums to prime moduli by using powerful algebro-geometric methods (this work built on [Fouvry et al. 2004] and was later generalized in [Fouvry et al. 2015]). We stress that although the deepest part of our proof comes from [Fouvry et al. 2014], it is not clear how this work is related to the nonvanishing problem. We figure out this relationship.

Before stating our result, it should be said that the works [Iwaniec and Sarnak 1999; Michel and VanderKam 2000; Bui 2012] actually treat general moduli, while we are restricting to prime moduli, which is arguably the most interesting case.

Theorem 1.1. *Let $\epsilon > 0$ be arbitrary. For all primes p large enough in terms of ϵ, there are at least $\left(\frac{3}{8} - \epsilon\right)$ of the primitive Dirichlet characters $\chi \pmod p$ for which $L\left(\frac{1}{2}, \chi\right) \neq 0$.*

The significance of our work is that we show for the first time how to increase the length of a classical mollifier in this context. An interesting open problem that remains is to increase the length of the Iwaniec–Sarnak mollifier. Our nonvanishing proportion $\frac{3}{8}$ improves upon that of Bui for prime moduli. For general moduli, Bui's nonvanishing proportion 0.3411 is still the best known.

Throughout the paper, we use the standard convention that ϵ denotes an arbitrarily small positive constant which may differ from one occurrence to the next, and that the implied constants in the various estimates depend on ϵ.

2. The work of Michel and VanderKam

We briefly summarize the mollifier method of Michel and VanderKam [2000], setting the ground for our further discussion.

Let the mollifier $M(\chi)$ be given by (1-4), where the mollifier length is $M = p^\theta$ and the real mollifying coefficients y_m are given by (1-3). Michel and VanderKam asymptotically evaluated the mollified first moment

$$\frac{2}{p} \sum_{\chi \bmod p}^{+} L\left(\tfrac{1}{2}, \chi\right) M(\chi)$$

for $\theta < \frac{1}{2}$, where $\sum^{+}$ restricts the summation to the even primitive characters, of which there are about $\frac{p}{2}$. The evaluation for the odd primitive characters is entirely similar. They evaluated the mollified second moment

$$\begin{aligned} \frac{2}{p} \sum_{\chi \bmod p}^{+} \left|L\left(\tfrac{1}{2}, \chi\right) M(\chi)\right|^2 &= \frac{4}{p} \sum_{\chi \bmod p}^{+} \left|L\left(\tfrac{1}{2}, \chi\right)\right|^2 \left|\sum_{m \le M} \frac{y_m \chi(m)}{m^{1/2}}\right|^2 \\ &\quad + \frac{4}{p} \sum_{\chi \bmod p}^{+} \left|L\left(\tfrac{1}{2}, \chi\right)\right|^2 \frac{\tau_\chi}{p^{1/2}} \left(\sum_{m \le M} \frac{y_m \chi(m)}{m^{1/2}}\right)^2 \end{aligned} \tag{2-1}$$

for $\theta < \frac{1}{4}$; see [Michel and VanderKam 2000, Equation (10)] for the above identity. An asymptotic for the first sum on the right-hand side of (2-1) is derived for $\theta < \frac{1}{2}$, as was done by Iwaniec and Sarnak [1999], but the second sum is more difficult and could only be handled for $\theta < \frac{1}{4}$. In the end, the main terms of the mollified moments of Michel and VanderKam yield a nonvanishing proportion of $2\theta/(1+2\theta)$, by taking $P_0(t) = t$ in [Michel and VanderKam 2000, Section 7].

Let us concentrate on the second sum on the right-hand side of (2-1). Recall the standard approximate functional equation (see for example [Michel and VanderKam 2000, Equation (3)]):

$$\left|L\left(\tfrac{1}{2}, \chi\right)\right|^2 = 2 \sum_{n_1, n_2 \ge 1} \frac{\chi(n_1)\bar{\chi}(n_2)}{(n_1 n_2)^{1/2}} V\left(\frac{n_1 n_2}{p}\right), \tag{2-2}$$

where

$$V(x) = \frac{1}{2\pi i} \int_{(2)} \frac{\Gamma\left(\frac{s}{2} + \frac{1}{4}\right)^2}{\Gamma\left(\frac{1}{4}\right)^2} (\pi x)^{-s} \frac{ds}{s}.$$

By moving the line of integration, one shows that $V(x) \ll_c x^{-c}$ for any $c > 0$, whence the sum in (2-2) is essentially supported on $n_1 n_2 \le p^{1+\epsilon}$. Therefore,

$$\frac{4}{p} \sideset{}{^+}\sum_{\chi \bmod p} \left|L\left(\tfrac{1}{2}, \chi\right)\right|^2 \frac{\tau_\chi}{p^{1/2}} \left(\sum_{m \le M} \frac{y_m \chi(m)}{m^{1/2}} \right)^2$$
$$= \sum_{\substack{n_1, n_2 \ge 1 \\ m_1, m_2 \le M}} \frac{y_{m_1} y_{m_2}}{(n_1 n_2 m_1 m_2)^{1/2}} V\left(\frac{n_1 n_2}{p}\right) \frac{4}{p} \sideset{}{^+}\sum_{\chi \bmod p} \frac{\tau_\chi}{p^{1/2}} \chi(n_1 m_1 m_2) \bar{\chi}(n_2). \tag{2-3}$$

By [Michel and VanderKam 2000, Equation (17)] or [Iwaniec and Sarnak 1999, Equation (3.4)], for $(n, p) = 1$ we have

$$\sideset{}{^+}\sum_{\chi \bmod p} \tau_\chi \chi(n) = p \cos\left(\frac{2\pi \bar{n}}{p}\right) + O(1),$$

so that (2-3) equals

$$\frac{4}{p^{1/2}} \operatorname{Re} \sum_{\substack{n_1, n_2 \ge 1 \\ m_1, m_2 \le M \\ (n_1 n_2 m_1 m_2, p) = 1}} \frac{y_{m_1} y_{m_2}}{(n_1 n_2 m_1 m_2)^{1/2}} V\left(\frac{n_1 n_2}{p}\right) e\left(\frac{n_2 \overline{n_1 m_1 m_2}}{p}\right) + O\left(\frac{M}{p^{1-\epsilon}}\right) \tag{2-4}$$

for any $\epsilon > 0$, where $e(x) = e^{2\pi i x}$ and $\bar{n}$ denotes the multiplicative inverse of n modulo p for $(n, p) = 1$. The terms with $m_1 m_2 = 1$ contain a main term of (2-3); see [Michel and VanderKam 2000, Section 6]. Consider the rest of the terms in dyadic intervals. Let

$$\mathcal{B}(M_1, M_2, N_1, N_2) = \frac{1}{(p M_1 M_2 N_1 N_2)^{1/2}}$$
$$\times \sum_{\substack{n_1, n_2 \ge 1 \\ M_1 \le m_1 \le 2M_1 \\ M_2 \le m_2 \le 2M_2 \\ (n_1 n_2 m_1 m_2, p) = 1}} y_{m_1} y_{m_2} e\left(\frac{n_2 \overline{n_1 m_1 m_2}}{p}\right) V\left(\frac{n_1 n_2}{p}\right) f_1\left(\frac{n_1}{N_1}\right) f_2\left(\frac{n_2}{N_2}\right) \tag{2-5}$$

for $1 \le M_1, M_2 \le \frac{1}{2} M$, $M_1 M_2 \ge 2$, $1 \le N_1 N_2 \le p^{1+\epsilon}$, and any fixed smooth functions f_1, f_2 compactly supported on the positive reals. Michel and VanderKam [2000, Equations (24) and (27)] proved the bounds

$$\mathcal{B}(M_1, M_2, N_1, N_2) \ll p^\epsilon \left(\frac{M^2 N_1}{p N_2}\right)^{1/2} \tag{2-6}$$

and

$$\mathcal{B}(M_1, M_2, N_1, N_2) \ll p^\epsilon \left(\frac{M^2 N_2}{N_1}\right)^{1/2} + \frac{M}{p^{1-\epsilon}}. \tag{2-7}$$

These bounds together yield $\mathcal{B}(M_1, M_2, N_1, N_2) \ll p^{-\epsilon}$, provided that $M \le p^{(1/4)-\epsilon}$. Thus the contribution to (2-4) of the terms with $m_1 m_2 \ge 2$ is $O(p^{-\epsilon})$ for $\theta < \frac{1}{4}$.

In the next section we will show how to improve the bound (2-7), in the ranges where (2-6) is not useful. This together with (2-6) will imply that

$$\mathcal{B}(M_1, M_2, N_1, N_2) \ll p^{-\epsilon}$$

for larger values of θ, thereby extending the asymptotics of Michel and VanderKam.

3. Proof of Theorem 1.1

To get the bounds (2-6) and (2-7), Michel and VanderKam obtained cancellation in only the (n_1, n_2)-sums of $\mathcal{B}(M_1, M_2, N_1, N_2)$. On the other hand, we use the (m_1, m_2)-sums to our advantage. To set this up, we first prove some estimates for averages of products of Kloosterman sums. Let

$$S(a, b; c) = \sum_{\substack{x \bmod c \\ x\bar{x} \equiv 1 \bmod c}} e\Big(\frac{ax + b\bar{x}}{c}\Big)$$

denote the Kloosterman sum. The following lemma is a consequence of a result from [Fouvry et al. 2014].

Lemma 3.1. *For $B < p$, we have*

$$\sum_{1 \le b_1, b_2, b_3, b_4 \le B} \Big| \sum_{h \bmod p} S(h, \bar{b}_1; p) S(h, \bar{b}_2; p) S(h, \bar{b}_3; p) S(h, \bar{b}_4; p) \Big| \ll B^4 p^{5/2} + B^2 p^3. \quad (3\text{-}1)$$

Proof. Write the left-hand side of (3-1) as

$$\sum_{b_1, b_2, b_3, b_4 \le B} = \sum_{\substack{b_1, b_2, b_3, b_4 \le B \\ (b_1, b_2, b_3, b_4) \in \mathfrak{D}}} + \sum_{\substack{b_1, b_2, b_3, b_4 \le B \\ (b_1, b_2, b_3, b_4) \notin \mathfrak{D}}}$$

where $\mathfrak{D}$ is the set of tuples (b_1, b_2, b_3, b_4) such that no component b_i is distinct from the others. Note that $|\mathfrak{D}| \ll B^2$.

On the one hand, it follows from the Weil bound for Kloosterman sums that

$$\sum_{\substack{b_1, b_2, b_3, b_4 \le B \\ (b_1, b_2, b_3, b_4) \in \mathfrak{D}}} \Big| \sum_{h \bmod p} S(h, \bar{b}_1; p) S(h, \bar{b}_2; p) S(h, \bar{b}_3; p) S(h, \bar{b}_4; p) \Big| \ll B^2 p^3.$$

On the other hand, if $(b_1, b_2, b_3, b_4) \notin \mathfrak{D}$, then in the language of [Fouvry et al. 2014, Definition 3.1], $(\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4)$ is not in "mirror configuration". Thus [Fouvry et al. 2014, Proposition 3.2] asserts that

$$\sum_{h \bmod p} S(h, \bar{b}_1;\, p) S(h, \bar{b}_2;\, p) S(h, \bar{b}_3;\, p) S(h, \bar{b}_4;\, p) \ll p^{5/2},$$

saving a factor of $p^{1/2}$ over Weil's bound. So

$$\sum_{\substack{b_1, b_2, b_3, b_4 \le B \\ (b_1, b_2, b_3, b_4) \notin \mathfrak{D}}} \left| \sum_{h \bmod p} S(h, \bar{b}_1;\, p) S(h, \bar{b}_2;\, p) S(h, \bar{b}_3;\, p) S(h, \bar{b}_4;\, p) \right| \ll B^4 p^{5/2}.$$

The lemma follows. □

Let now

$$\mathcal{S} = \sum_{\substack{1 \le |n| \le N \\ 1 \le a \le A \\ 1 \le b \le B}} x_n y_a z_b S(n, \overline{ab};\, p),$$

where the coefficients satisfy $x_n, y_a, z_b \ll p^{\epsilon}$, $y_a = 0$ for $p|a$, and $z_b = 0$ for $p|b$.

Lemma 3.2. *For $NA \le \frac{p}{2}$ and $B < p$, we have*

$$\mathcal{S} \ll p^{\epsilon} N^{3/4} A^{3/4} (B p^{5/8} + B^{1/2} p^{3/4}).$$

Proof. On applying the Cauchy–Schwarz inequality, we infer

$$|\mathcal{S}|^2 \ll p^{\epsilon} NA \sum_{\substack{|n| \le N \\ a \le A}} \left| \sum_{b \le B} z_b S(n\bar{a}, \bar{b};\, p) \right|^2.$$

Hence

$$|\mathcal{S}|^2 \ll p^{\epsilon} NA \sum_{h \bmod p} \nu(h) \left| \sum_{b \le B} z_b S(h, \bar{b};\, p) \right|^2, \tag{3-2}$$

where

$$\nu(h) = \sum_{\substack{|n| \le N \\ a \le A \\ n\bar{a} \equiv h \bmod p}} 1.$$

On applying Cauchy–Schwarz to (3-2), we find that

$$|\mathcal{S}|^4 \ll p^{\epsilon} N^2 A^2 \Bigl(\sum_{h \bmod p} \nu(h)^2 \Bigr) \Bigl(\sum_{h \bmod p} \left| \sum_{b \le B} z_b S(h, \bar{b};\, p) \right|^4 \Bigr). \tag{3-3}$$

Observe that

$$\sum_{h \bmod p} \nu(h)^2 = \sum_{\substack{|n_1|, |n_2| \le N \\ a_1, a_2 \le A \\ n_1 \bar{a}_1 \equiv n_2 \bar{a}_2 \bmod p}} 1 = \sum_{\substack{|n_1|, |n_2| \le N \\ a_1, a_2 \le A \\ n_1 a_2 \equiv n_2 a_1 \bmod p}} 1.$$

Since $NA \leq \frac{p}{2}$ by assumption, it follows that

$$\sum_{h \bmod p} \nu(h)^2 = \sum_{\substack{n_1 a_2 = n_2 a_1 \\ |n_1|, |n_2| \leq N \\ a_1, a_2 \leq A}} 1 \ll p^{\epsilon} NA.$$

Therefore, (3-3) becomes

$$|\mathcal{S}|^4 \ll p^{\epsilon} N^3 A^3 \sum_{b_1, b_2, b_3, b_4 \leq B} \left| \sum_{h \bmod p} S(h, \bar{b}_1; p) S(h, \bar{b}_2; p) S(h, \bar{b}_3; p) S(h, \bar{b}_4; p) \right|.$$

Finally, we apply Lemma 3.1 to conclude that

$$|\mathcal{S}|^4 \ll p^{\epsilon} N^3 A^3 (B^4 p^{5/2} + B^2 p^3),$$

and the lemma is proved. □

We are now in a position to prove a new bound for our nonvanishing problem.

Lemma 3.3. *For $N_1/N_2 > p^{\epsilon} M$ and $M < p^{1-\epsilon}$, we have*

$$\mathcal{B}(M_1, M_2, N_1, N_2) \ll p^{\epsilon} \left(\frac{N_2 M^3}{N_1 p^3} \right)^{1/4} \left(p^{5/8} + \frac{p^{3/4}}{M^{1/2}} \right) + \frac{M}{p^{1-\epsilon}}. \tag{3-4}$$

Proof. In (2-5), separate n_1 into residue classes modulo p and apply the Poisson summation formula to get

$$\mathcal{B}(M_1, M_2, N_1, N_2) = \frac{1}{(pM_1 M_2 N_1 N_2)^{1/2}} \frac{N_1}{p} \sum_{\substack{k \in \mathbb{Z} \\ n_2 \geq 1, (n_2, p) = 1 \\ M_1 \leq m_1 \leq 2M_1 \\ M_2 \leq m_2 \leq 2M_2}} y_{m_1} y_{m_2} S(kn_2, \overline{m_1 m_2}; p) f_2\left(\frac{n_2}{N_2} \right) F(k), \tag{3-5}$$

where

$$F(k) = \int_{-\infty}^{\infty} f_1(x) V\left(\frac{x N_1 n_2}{p} \right) e\left(\frac{-xkN_1}{p} \right) dx.$$

Repeatedly integrating by parts, we find that $F(k) \ll_c (kN_1/p)^{-c}$ for any $c > 0$. Thus, the k-sum may be restricted to $|k| \leq p^{1+\epsilon}/N_1$.

The contribution to (3-5) of the terms with $k = 0$ is

$$\frac{1}{(pM_1 M_2 N_1 N_2)^{1/2}} \frac{N_1}{p} \sum_{\substack{n_2 \geq 1, (n_2, p) = 1 \\ M_1 \leq m_1 \leq 2M_1 \\ M_2 \leq m_2 \leq 2M_2}} y_{m_1} y_{m_2} S(0, \overline{m_1 m_2}; p) f_2\left(\frac{n_2}{N_2} \right) F(0)$$

$$\ll \frac{(N_1 N_2 M_1 M_2)^{1/2}}{p^{(3/2)-\epsilon}} \ll \frac{M}{p^{1-\epsilon}},$$

on using that the Ramanujan sum $S(0, \overline{m_1 m_2}; p)$ equals -1. This is the last term in (3-4). The contribution of the terms with $|k| > 0$ is bounded using Lemma 3.2, by putting

$$n = kn_2, \quad x_n = f_2(n_2/N_2)F(k) \text{ if } (n_2, p) = 1, \quad x_n = 0 \text{ if } p|n_2, \quad N = N_2 p^{1+\epsilon}/N_1,$$
$$a = m_1, \quad y_a = y_{m_1}, \quad A = 2M_1,$$
$$b = m_2, \quad z_b = y_{m_2}, \quad B = 2M_2.$$

Note that the conditions of Lemma 3.2, namely $B < p$ and $NA \le \frac{p}{2}$, are satisfied by the assumptions that $M < p^{1-\epsilon}$ and that $N_1/N_2 > p^\epsilon M$. The bound (3-4) follows. $\square$

Finally, we sum up the work done to arrive at the following power-saving result.

Lemma 3.4. *We have $\mathcal{B}(M_1, M_2, N_1, N_2) \ll p^{-\epsilon}$ for $M < p^{(3/10)-\epsilon}$.*

Proof. Assume first that $M < p^{(1/3)-\epsilon}$. If $N_1/N_2 \le p^\epsilon M$, it follows from (2-6) that $\mathcal{B}(M_1, M_2, N_1, N_2) \ll p^{-\epsilon}$, whence the lemma follows.

We therefore suppose that $N_1/N_2 > p^\epsilon M$. Now since the conditions of Lemma 3.3 are met, we have the bound (3-4). In this bound, we may suppose $N_2/N_1 < M^2/p^{1-\epsilon}$, since otherwise by (2-6), we have $\mathcal{B}(M_1, M_2, N_1, N_2) \ll p^{-\epsilon}$. Thus, (3-4) becomes

$$\mathcal{B}(M_1, M_2, N_1, N_2) \ll \frac{M^{5/4}}{p^{1-\epsilon}}\left(p^{5/8} + \frac{p^{3/4}}{M^{1/2}}\right) + p^{-(1/6)+\epsilon}.$$

The bound is $O(p^{-\epsilon})$ precisely when $M \ll p^{(3/10)-\epsilon}$. The lemma follows. $\square$

Proof of Theorem 1.1. By Lemma 3.4, the nonvanishing proportion $2\theta/(1+2\theta)$ of Michel and VanderKam is valid for any $\theta < \frac{3}{10}$. On letting θ approach $\frac{3}{10}$, we infer that the nonvanishing proportion is at least $\frac{3}{8} - \epsilon$ for any $\epsilon > 0$. $\square$

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