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Discriminant formulas and applications

Kenneth Chan, Alexander A. Young and James J. Zhang

The discriminant is a classical invariant associated to algebras which are finite over their centers. It was shown recently by several authors that if the discriminant of A is "sufficiently nontrivial" then it can be used to answer some difficult questions about A. Two such questions are: What is the automorphism group of A? Is A Zariski cancellative?

We use the discriminant to study these questions for a class of (generalized) quantum Weyl algebras. Along the way, we give criteria for when such an algebra is finite over its center and prove two conjectures of Ceken, Wang, Palmieri and Zhang.

Introduction

In algebraic number theory, the discriminant takes on a familiar form: let L be a Galois extension of the field $\mathbb{Q}$ and write $\mathcal{O}_L = \mathbb{Z}[\alpha] \cong \mathbb{Z}[x]/(f)$, where f is the minimal polynomial (or the characteristic polynomial) of α. Then we have

$$\Delta_{L/\mathbb{Q}} = \prod_{i \neq j} (r_i - r_j),$$

where $r_1, \ldots, r_n$ are the roots of f. In noncommutative algebra, the discriminant has long been used to study orders and lattices in a central simple algebra [Reiner 1975]. Recently, it has been shown that the discriminant plays a remarkable role in solving some classical and notoriously difficult questions:

(1) *Automorphism problem*: determining the full automorphism groups of noncommutative Artin–Schelter regular algebras [CPWZ 2015a; 2016].

(2) *Zariski cancellation problem*: concerning the cancellative property of noncommutative algebras such as skew polynomial rings [Bell and Zhang 2016].

(3) *Isomorphism problem*: finding a criterion for when two algebras are isomorphic, within certain classes of noncommutative algebras [CPWZ 2015b].

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Despite the usefulness of the discriminant in algebraic number theory, algebraic geometry and noncommutative algebra, it is extremely hard to compute, especially in high dimensional and high rank cases. In [CPWZ 2015a; 2016], the authors made two conjectures on discriminant formulas for some classes of noncommutative algebras. Our main aim is to prove these two conjectures.

Let k be a base commutative domain and let $k^\times$ be the set of invertible elements in k. The discriminant of a noncommutative algebra A over a central subalgebra $Z \subseteq A$, denoted by $d(A/Z)$, will be reviewed in Section 1. Let $q \in k^\times$ be an invertible element in k and let A_q be the q-quantum Weyl algebra generated by x and y and subject to the relation $yx = qxy + 1$. Our first result is:

Theorem 0.1. *Let q be a primitive n-th root of unity for some $n \geq 2$. Then the discriminant of A_q over its center $Z(A_q)$ is*

$$d(A_q/Z(A_q)) = c(nm)^{n^2}((1-q)^n x^n y^n - 1)^{n(n-1)},$$

where c is some element in $k^\times$ and $m = \prod_{i=2}^{n-1}(1+q+\cdots+q^{i-1})$. By convention, $m = 1$ when $n = 2$.

Theorem 0.1 answers [CPWZ 2016, Conjecture 5.3] affirmatively.

For $n \geq 2$, let W_n be the k-algebra generated by $x_1, \ldots, x_n$ and subject to the relations $x_i x_j + x_j x_i = 1$ for all $i \neq j$ [CPWZ 2015a, Introduction]. This algebra is called a (-1)-*quantum Weyl algebra* [CPWZ 2015b, Introduction]. Let

$$M := \begin{pmatrix} 2x_1^2 & 1 & \cdots & 1 \\ 1 & 2x_2^2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 2x_n^2 \end{pmatrix}.$$

Let Z denote the central subalgebra $k\left[x_1^2, \ldots, x_n^2\right] \subseteq W_n$. Our second result is:

Theorem 0.2. *Suppose 2 is invertible in k. Then the discriminant of W_n over the subalgebra Z is*

$$d(W_n/Z) = c(\det M)^{2^{n-1}},$$

where c is an element in $k^\times$.

Theorem 0.2 answers [CPWZ 2015a, Question 4.12(2)] affirmatively.

These results suggest that the discriminant has elegant expressions in some situations. Because of its usefulness, more discriminant formulas should be established; see Lemma 6.4.

This paper contains other related results which we now describe. Let T be a commutative algebra over k and let $\boldsymbol{q} := \{q_{ij} \in T^\times \mid 1 \leq i < j \leq n\}$ and

$\mathcal{A} := \{a_{ij} \in T \mid 1 \le i < j \le n\}$ be sets of elements in T. The skew polynomial ring $T_{\boldsymbol{q}}[x_1, \dots, x_n]$ is a T-algebra generated by $x_1, \dots, x_n$ and subject to the relations

$$x_j x_i = q_{ij} x_i x_j \quad \text{for all } 1 \le i < j \le n. \tag{E0.2.1}$$

A generalized quantum Weyl algebra associated to $(\boldsymbol{q}, \mathcal{A})$ is a T-central filtered algebra of the form

$$V_n(\boldsymbol{q}, \mathcal{A}) = \frac{T\langle x_1, \dots, x_n\rangle}{(x_j x_i - q_{ij} x_i x_j - a_{ij} \mid i < j)} \tag{E0.2.2}$$

such that the associated graded ring $\operatorname{gr} V_n(\boldsymbol{q}, \mathcal{A})$ is naturally isomorphic to the skew polynomial ring $T_{\boldsymbol{q}}[x_1, \dots, x_n]$. Another way of constructing $V_n(\boldsymbol{q}, \mathcal{A})$ is to use an iterated Ore extension starting with T. To calculate the discriminant of $V_n(\boldsymbol{q}, \mathcal{A})$ over its center, one needs to determine the center of $V_n(\boldsymbol{q}, \mathcal{A})$. The discriminant is defined whenever $V_n(\boldsymbol{q}, \mathcal{A})$ is a finite module over a central subring Z [CPWZ 2016], and it is most useful when $V_n(\boldsymbol{q}, \mathcal{A})$ is a free module over Z [CPWZ 2015a]. Since $\operatorname{gr} V_n(\boldsymbol{q}, \mathcal{A})$ is isomorphic to $T_{\boldsymbol{q}}[x_1, \dots, x_n]$, it is a finite module over its center if and only if each q_{ij} is a root of unity. Using this, we can show that the algebra $V_n(\boldsymbol{q}, \mathcal{A})$ is a finite module over its center if and only if the parameters q_{ij} are all nontrivial roots of unity. Also, when the center of $V_n(\boldsymbol{q}, \mathcal{A})$ is a polynomial ring, $V_n(\boldsymbol{q}, \mathcal{A})$ is a finitely generated free module over its center. The following useful result concerns the centers of $V_n(\boldsymbol{q}, \mathcal{A})$ and $T_{\boldsymbol{q}}[x_1, \dots, x_n]$.

To state it, we need some notation. When q_{ij} is a root of unity, there are two integers k_{ij} and d_{ij} such that

$$q_{ij} = \exp\bigl(2\pi\sqrt{-1}k_{ij}/d_{ij}\bigr),$$

where $d_{ij} := o(q_{ij}) < \infty$, $|k_{ij}| < d_{ij}$ and $(k_{ij}, d_{ij}) = 1$. Further, we can choose k_{ij} so that $k_{ij} = -k_{ji}$, since $q_{ji} = q_{ij}^{-1}$. Let $L_i = \operatorname{lcm}\{d_{ij} \mid j = 1, \dots, n\}$. Let $\overline{Y}$ be the $n \times n$ matrix $(k_{ij} L_i / d_{ij})_{n\times n}$. For each prime p, define $\overline{Y}_p = \overline{Y} \otimes \mathbb{F}_p$. Let m be any natural number. Let $I_{p,m}$ be the set containing i such that $L_i \in p^m\mathbb{Z} - p^{m+1}\mathbb{Z}$. Finally, let $\overline{Y}_{p,m}$ be the submatrix of $\overline{Y}_p$ taken from the rows and columns with indices $i \in I_{p,m}$.

Theorem 0.3. *Suppose q_{ij} is a root of unity and not 1 for all $i < j$.*

(1) *The center of $T_{\boldsymbol{q}}[x_1, \dots, x_n]$ is a polynomial ring if and only if it is of the form $T\bigl[x_1^{L_1}, \dots, x_n^{L_n}\bigr]$, if and only if $\det(\overline{Y}_{p,m}) \neq 0$ in $\mathbb{F}_p$ for all primes p and all integers $m > 0$ such that $I_{p,m} \neq \varnothing$.*

(2) *If the center of $T_{\boldsymbol{q}}[x_1, \dots, x_n]$ is the subalgebra $T\bigl[x_1^{L_1}, \dots, x_n^{L_n}\bigr]$, then the center of $V_n(\boldsymbol{q}, \mathcal{A})$ is the same subalgebra and $V_n(\boldsymbol{q}, \mathcal{A})$ is finitely generated and free over it.*

The above criterion can be simplified when $n = 3$ or 4 [Corollaries 5.4 and 5.5]. The point of Theorem 0.3 is that it provides an explicit linear algebra criterion for when the center of $T_{\boldsymbol{q}}[x_1, \dots, x_n]$ is isomorphic to a polynomial ring.

Question 0.4. Suppose that $A := V_n(\boldsymbol{q}, \mathcal{A})$ is finitely generated and free over its center Z. What is the discriminant $d(A/Z)$?

Theorems 0.1 and 0.2 answer this question for two special cases.

A secondary goal of this paper is to provide some quick applications. These discriminant formulas have potential applications in algebraic geometry, number theory and the study of Clifford algebras. In Section 8 (the final section), we give some immediate applications of discriminants to the cancellation problem and the automorphism problem for several classes of noncommutative algebras.

Let us briefly review some definitions. An algebra A is called *cancellative* if $A[t] \cong B[t]$ for some algebra B implies $A \cong B$. Let $\operatorname{Aut}(A)$ be the group of all algebra automorphisms of A. Let A be connected graded. An algebra automorphism g of A is called *unipotent* if

$$g(v) = v + (\text{higher degree terms})$$

for all homogeneous elements $v \in A$. Let $\operatorname{Aut}_{\mathrm{uni}}(A)$ denote the subgroup of $\operatorname{Aut}(A)$ consisting of all unipotent automorphisms [CPWZ 2016, after Theorem 3.1]. When $\operatorname{Aut}_{\mathrm{uni}}(A)$ is trivial, $\operatorname{Aut}(A)$ is usually small and easy to handle. We will give a criterion on when $\operatorname{Aut}_{\mathrm{uni}}(A)$ is trivial.

Let A be a domain and let F be a subset of A. Let $\operatorname{Sw}(F)$ be the set of $g \in A$ such that $f = agb$ for some $a, b \in A$ and $0 \neq f \in F$. Let $D_1(F)$ be the k-subalgebra of A generated by $\operatorname{Sw}(F)$. For $n > 2$, we define $D_n(F) = D_1(D_{n-1}(F))$ inductively, and define $D(F) = \bigcup_{n \geq 1} D_n(F)$. This algebra is called the *F-divisor subalgebra of A*. When $F = \{d(A/Z)\}$, $D(F)$ is called the *discriminant-divisor subalgebra of A* and is denoted by $\mathbb{D}(A)$. The main result in Section 8 is the following.

Theorem 0.5. *Suppose k is a field of characteristic zero. Let A be a connected graded domain of finite Gelfand–Kirillov dimension. Assume that A is finitely generated and free over its center. If $\mathbb{D}(A) = A$, then A is cancellative and* $\operatorname{Aut}_{\mathrm{uni}}(A) = \{1\}$.

The above theorem can be applied to some Artin–Schelter regular algebras of global dimension 4 in Examples 6.3 and 8.4. Further applications are certainly expected.

This paper is organized as follows. Background material about discriminants is provided in Section 1. We prove Theorem 0.1 in Section 2 and Theorem 0.2 in Section 3. Sections 4–6 concern the question of when $T_{\boldsymbol{q}}[x_1, \dots, x_n]$ and $V_n(\boldsymbol{q}, \mathcal{A})$ are finitely generated and free over their centers and contain the proof of Theorem 0.3. In Section 7, we review and introduce some invariants related to discriminants, locally nilpotent derivations, and automorphisms, which will be used in Section 8. In Section 8, some applications are provided and Theorem 0.5 is proven.

1. Preliminaries

In this section we recall some definitions and basic properties of the discriminant. A basic reference is [CPWZ 2015a, Section 1].

Throughout, let k be a base commutative domain and let everything be over k. Let A be an algebra and let Z be a central subalgebra of A such that A is finitely generated and free over Z. A modified version of the discriminant was introduced in [CPWZ 2016] when A is not free over Z; however, in this paper, we only consider the case when A is finitely generated and free over Z. Let r be the rank of A over Z.

We embed A in the endomorphism ring $\mathrm{End}(A_Z)$ by sending $a \in A$ to the left multiplication $l_a : A \to A$. Since A is free over Z of rank r, $\mathrm{End}(A_Z) \cong M_{r\times r}(Z)$. Define the trace function

$$\mathrm{tr} : A \longrightarrow \mathrm{End}(A_Z) \cong M_{r\times r}(Z) \xrightarrow{\mathrm{tr}_m} Z, \tag{E1.0.1}$$

where tr_m is the usual matrix trace. The trace function tr is independent of the choice of basis of A over Z.

Definition 1.1. [CPWZ 2015a, Definition 1.3(3)] Retain the above notation. Suppose that A is a free module over a central subalgebra Z with a Z-basis $\{z_1, \ldots, z_r\}$. The discriminant of A over Z is

$$d(A/Z) = \det(\mathrm{tr}(z_i z_j))_{r\times r} \in Z.$$

By [CPWZ 2015a, Proposition 1.4(2)], $d(A/Z)$ is unique up to a scalar in $Z^\times$. For $x, y \in Z$, we use the notation $x =_{Z^\times} y$ to indicate that $x = cy$ for some $c \in Z^\times$. So $d(A/Z) =_{Z^\times} \det(\mathrm{tr}(z_i z_j))_{r\times r}$ as in [CPWZ 2015a, Definition 1.3(3)]. The following lemma is easy.

Lemma 1.2. *Retain the notation of Definition 1.1. Let (A', Z') be another pair of algebras such that Z' is a central subalgebra of A' and A' is a free Z'-module of rank r. Let $g : A \to A'$ be an algebra homomorphism such that*:

(a) $g(Z) \subseteq Z'$.

(b) $\{g(z_1), \ldots, g(z_r)\}$ *is a Z'-basis of A'.*

Then $g(d(A/Z)) =_{(Z')^\times} d(A'/Z')$.

Proof. For any $a \in A$, we define $a' = g(a)$. Write $az_i = \sum_{j=1}^r a_{ij} z_j$ for all i. By applying g to the last equation, we have $a'z'_i = \sum_{j=1}^r a'_{ij} z'_j$. By definition (E1.0.1), $\mathrm{tr}(a) = \sum_i a_{ii}$ and

$$\mathrm{tr}(g(a)) = \mathrm{tr}(a') = \sum_i a'_{ii} = g\Big(\sum_i a_{ii}\Big) = g(\mathrm{tr}(a))$$

for all $a \in A$. By Definition 1.1 and the above equation,

$$g(d(A/Z)) = g(\det(\operatorname{tr}(z_i z_j))_{r\times r}) = \det(\operatorname{tr}(z'_i z'_j))_{r\times r} =_{(Z')^\times} d(A'/Z'). \qquad \square$$

Let Z be a central subalgebra of A and consider an Ore set $C \subset Z$. Then the localization ZC^{-1} is central in AC^{-1}.

Lemma 1.3. *Let Z be a central subalgebra of A. Suppose A is free over Z of rank r. Then AC^{-1} is free over ZC^{-1} of rank r. As a consequence,*

$$d\big(AC^{-1}/ZC^{-1}\big) =_{(ZC^{-1})^\times} d(A/Z).$$

Proof. Let $\{z_1, \dots, z_r\}$ be a Z-basis of A. Then it is also a ZC^{-1}-basis of AC^{-1}. The consequence follows from Lemma 1.2. $\square$

We will need the following result from [CPWZ 2016]. We use T in place of k to denote a commutative domain.

Proposition 1.4. *Let T be a commutative domain and let $A = T_{\boldsymbol{q}}[x_1, \dots, x_n]$. Suppose $Z := T\big[x_1^{\alpha_1}, \dots, x_n^{\alpha_n}\big]$ is a central subalgebra of A, where the α_i are positive integers.*

(1) [CPWZ 2016, Proposition 2.8] *Let $r = \prod_{i=1}^n \alpha_i$. Then*

$$d(A/Z) =_{T^\times} r^r \left(\prod_{i=1}^n x_i^{\alpha_i - 1}\right)^r.$$

(2) *If $n = 2$, $Z = T\big[x_1^m, x_2^m\big]$, and q_{12} is a primitive m-th root of unity, then*

$$d(A/Z) =_{T^\times} m^{2m^2} \big(x_1^m x_2^m\big)^{m(m-1)}.$$

(3) *If $q_{ij} = -1$ for all $i < j$ and $\alpha_i = 2$ for all i, then*

$$d(A/Z) =_{T^\times} 2^{n2^n} \left(\prod_{i=1}^n x_i^2\right)^{2^{n-1}}.$$

Proof. Parts (2) and (3) are special cases of part (1). $\square$

The next lemma is a special case of [CPWZ 2016, Proposition 4.10]. Suppose Z is a central subalgebra of A and A is free over Z of rank $r < \infty$. We fix a Z-basis of A, say $b := \{b_1 = 1, b_2, \dots, b_r\}$. Suppose A is an $\mathbb{N}$-filtered algebra such that the associated graded ring $\operatorname{gr} A$ is a domain. For any element $f \in A$, let $\operatorname{gr} f$ denote the associated element in $\operatorname{gr} A$. Let $\operatorname{gr} b$ denote the set $\{\operatorname{gr} b_1, \dots, \operatorname{gr} b_r\}$, which is a subset of $\operatorname{gr} A$.

Lemma 1.5. *Retain the above notation. Suppose that* $\operatorname{gr} A$ *is finitely generated and free over* $\operatorname{gr} Z$ *with basis* $\operatorname{gr} b$. *Then*

$$\operatorname{gr}(d(A/Z)) =_{(\operatorname{gr} Z)^\times} d(\operatorname{gr} A/\operatorname{gr} Z).$$

2. Discriminant of A_q over its center

Let T be a commutative domain and let $q \in T^{\times}$ be a primitive n-th root of unity for some $n \geq 2$. Let A_q be the q-quantum Weyl algebra over T generated by x and y and subject to the relation $yx = qxy + a$ for some $a \in T$. This agrees with the definition of A_q given in the Introduction when $T = k$ and $a = 1$. It is easy to check that the center of A_q, denoted by $Z(A_q)$, is $T[x^n, y^n]$, and that A_q is free over $Z(A_q)$ of rank n^2. A $Z(A_q)$-basis of A_q is $\mathcal{B} := \{x^i y^j \mid 0 \leq i,\ j \leq n-1\}$. The aim of this section is to compute the discriminant $d(A_q/Z(A_q))$.

Let A' be the T-subalgebra of A_q generated by $x' := (1-q)x$ and y. Since $yx' = qx'y + (1-q)a$ and $(1-q)$ may not be invertible, there is no obvious algebra homomorphism from A_q to A'. Let Z' be the subalgebra $T[(x')^n, y^n]$ which is the center of A'.

Lemma 2.1. *Retain the above notation. Then*

$$d(A'/Z') = (1-q)^{n^2(n-1)} d(A_q/Z(A_q)).$$

Proof. Let $\mathrm{tr}' : A' \to Z'$ be the trace function defined in (E1.0.1). We use this trace function to compute the discriminant $d(A'/Z')$.

Let $\mathcal{B}' := \{(x')^i y^j\}_{0 \leq i,j \leq n-1}$. Then $\mathcal{B}'$ is a Z'-basis of A'. Note that A' and A_q have the same ring of fractions and $Z(A_q)$ and Z' have the same fraction field. Since the trace function is independent of the choice of basis, we have $\mathrm{tr}'(a) = \mathrm{tr}(a)$ for all $a \in A'$.

Picking any two elements $b_s = x^{i_s} y^{j_s}$ and $b_t = x^{i_t} y^{j_t}$ in $\mathcal{B}$, we have corresponding elements $b'_s = (x')^{i_s} y^{j_s}$ and $b'_t = (x')^{i_t} y^{j_t}$ in $\mathcal{B}$. Hence

$$\mathrm{tr}'(b'_s b'_t) = \mathrm{tr}\big((1-q)^{i_s+i_t} b_s b_t\big) = (1-q)^{i_s+i_t}\,\mathrm{tr}(b_s b_t).$$

By definition, $d(A'/Z') = \det\big[\mathrm{tr}'(b'_s b'_t)_{b'_s, b'_t \in \mathcal{B}'}\big]$. Hence we have

$$\begin{aligned} d(A'/Z') &= \det\big[(\mathrm{tr}'(b'_s b'_t))_{b'_s, b'_t \in \mathcal{B}'}\big] = \det\big[\big((1-q)^{i_s+i_t}\,\mathrm{tr}(b_s b_t)\big)_{b_s, b_t \in \mathcal{B}}\big] \\ &= (1-q)^N \det\big[(\mathrm{tr}(b_s b_t))_{b_s, b_t \in \mathcal{B}}\big] = (1-q)^N d(A_q/Z(A_q)), \end{aligned}$$

where

$$N = \sum_{\text{all } i_s, i_t} (i_s + i_t) = 2 \sum_{\text{all } i_s} i_s = 2n(0+1+2+\cdots+(n-1)) = n^2(n-1).$$

The assertion follows. □

Following the above lemma, we first compute $d(A'/Z')$. We can rewrite A' as $T\langle x', y\rangle/(yx' - qx'y - (1-q)a)$ so that the positions of x' and y are more symmetrical.

Let $C=\{(y^n)^i \mid i\geq 1\}$. Consider the localizations $Z'':=Z'C^{-1}$ and $A'':=A'C^{-1}$. Let

$$x'':=x'-ay^{-1}=(1-q)x-(ay^{-n})y^{n-1}\in A''.$$

Lemma 2.2. *Retain the above notation. The following hold*:

(1) $yx''-qx''y=0$.

(2) $A'':=A'C^{-1}$ *is generated by* T, $(y^n)^{-1}$, x'' *and* y.

(3) $(x'')^n$ *is central and* $d(A''/Z'')=_{(Z'')^\times} n^{2n^2}((x'')^n y^n)^{n(n-1)}$.

(4) $d(A''/Z'')=_{(Z'')^\times} n^{2n^2}((1-q)^n x^n y^n - a^n)^{n(n-1)}$.

Proof. (1) We have $yx''-qx''y=y((1-q)x-ay^{-1})-q((1-q)x-ay^{-1})y=0$.

(2) This is clear.

(3) Since $q^n=1$, $(x'')^n$ commutes with y by part (1). By part (2), $(x'')^n$ commutes with every element in A''.

Consider an algebra homomorphism $g:T_q[x_1,x_2]\to A''$ determined by $g(x_1)=x''$ and $g(x_2)=y$. Then the center of $B:=T_q[x_1,x_2]$ is $R:=T\left[x_1^n,x_2^n\right]$ and $\left\{x_1^i x_2^j \mid 0\leq i,\ j\leq n-1\right\}$ is an R-basis of B. It is clear that A'' is free of rank n^2 and that $A''=\sum_{0\leq i,j\leq n-1}(x')^i y^j Z''$. Hence $\{(x'')^i y^j \mid 0\leq i,\ j\leq n-1\}$ is a Z''-basis of A''. Then the hypotheses of Lemma 1.2 hold. Applying Lemma 1.2 to g, we have $g(d(B/R))=_{(Z')^\times} d(A''/Z'')$. By Proposition 1.4(2), $d(B/R)=n^{2n^2}\left(x_1^n x_2^n\right)^{n(n-1)}$. Therefore, $d(A''/Z'')=_{(Z')^\times} n^{2n^2}((x'')^n y^n)^{n(n-1)}$.

(4) In the following, we will let $\psi=y^{-1}$, $z=x''$ and $p=q^{-1}$. The commutation relation between x' and ψ is

$$\psi x'=(1-q)\psi x=(1-q)(px\psi-pa\psi^2)=px'\psi-(p-1)a\psi^2. \quad \text{(E2.2.1)}$$

Recall that $z=x''=x'-a\psi$. Write $z^n=\sum_{i=0}^n c_i(x')^i\psi^{n-i}$. Since z^n is central (see part (3)), we have $c_i=0$ unless $i=0,n$. It is clear that $c_n=1$. Next we determine c_0. Since A'' is a free module over Z'' with basis $\{(x')^i\psi^j \mid 0\leq i,\ j\leq n-1\}$, we can work modulo the right Z''-submodule W generated by $(x')^i\psi^j$, where $0<i<n$ and $0\leq j<n$. Let $\equiv$ denote equivalence mod W.

By induction, for $i=1,\dots,n-1$, we have

$$\psi^i x'=p^i x'\psi^i-(p^i-1)(a\psi^{i+1}). \quad \text{(E2.2.2)}$$

Then $\psi^i x'\equiv -(p^i-1)(a\psi^{i+1})$. For each $1\leq j\leq n-1$, write

$$z^j=\sum_{i=0}^{j} c_i^j (x')^i\psi^{j-i}.$$

Then $x'z^j \in W$ for all $j < n-1$ and $x'z^{n-1} \equiv (x')^n$. For each j, we have $\psi^{j-1}z^{n-j} = \sum_{i=0}^{n-j} d_i^j (x')^i \psi^{n-1-i}$ for some $d_i^j \in Z'$, so

$$x'\psi^{j-1}z^{n-j} \in W \tag{E2.2.3}$$

for all $j \geq 2$. By the above computation and (E2.2.1)–(E2.2.3), we have

$$\begin{aligned}
z^n - (x')^n &= (x' - a\psi)z^{n-1} - (x')^n \\
&= x'z^{n-1} - (x')^n - a\psi z^{n-1} \\
&\equiv -a\psi(x' - a\psi)z^{n-2} \\
&\equiv -a(px'\psi - (p-1)a\psi^2 - a\psi^2)z^{n-2} \\
&\equiv -a(-pa)\psi^2 z^{n-2} - apx'\psi z^{n-2} \\
&\equiv -a(-pa)\psi^2 z^{n-2} \\
&\equiv -a(-pa)(\psi^2 x - a\psi^3)z^{n-3} \\
&\equiv -a(-pa)(-p^2a)\psi^3 z^{n-3} \\
&\vdots \\
&\equiv -a(-pa)(-p^2a)\cdots(-p^{n-1}a)\psi^n \\
&= (-a)^n p^{(n-1)n/2}\psi^n = -a^n\psi^n.
\end{aligned}$$

Therefore,

$$z^n \equiv -a^n\psi^n + (x')^n.$$

Hence $c_0 = -a^n$ and $z^n = (x')^n - a^n\psi^n$. Combining all of the above, we have

$$(x'')^n y^n = ((x')^n - a^n\psi^n)y^n = (x')^n y^n - a^n = (1-q)^n x^n y^n - a^n.$$

Part (4) follows from part (3) and the above formula. □

Lemma 2.3. *The discriminant of A' over its center Z' is*

$$d(A'/Z') =_{T^\times} n^{2n^2}((1-q)^n x^n y^n - a^n)^{n(n-1)}.$$

Proof. Let g be the embedding of A' into $A'' = A'C^{-1}$, viewed as an inclusion. By Lemma 1.2, g sends $d(A'/Z')$ to $d(A''/Z'')$. Combining this fact with Lemma 2.2(4), we have

$$\begin{aligned}
d(A'/Z') &=_{(Z'')^\times} g(d(A'/Z(A'))) =_{(Z'')^\times} d(A''/Z'') \\
&=_{(Z'')^\times} n^{2n^2}((1-q)^n x^n y^n - a^n)^{n(n-1)}.
\end{aligned}$$

Let Φ be the element $d(A'/Z')\{n^{2n^2}((1-q)^n x^n y^n - a^n)^{n(n-1)}\}^{-1}$, which can be viewed as an element in the quotient ring of A'. By the above equation, Φ is in $(Z'')^\times$. Since $Z'' = T[(x')^n, y^{\pm n}]$, Φ is of the form αy^{sn} for some $\alpha \in T^\times$

and some s. By symmetry, Φ is also of the form $\beta(x')^{tn}$ for some $\beta \in T^\times$ and some t. Hence $s = t = 0$, $\alpha = \beta \in T^\times$ and $\Phi = \alpha \in T^\times$. Therefore, $d(A'/Z') = \alpha n^{2n^2}((1-q)^n x^n y^n - a^n)^{n(n-1)}$ and the assertion follows. $\square$

Now let

$$m := \prod_{i=2}^{n-1}(1+q+\cdots+q^{i-1}). \tag{E2.3.1}$$

We can show that $n = (1-q)^{n-1}m$ by first factoring the polynomial $x^n - 1 \in T[x]$ and dividing by $(x-1)$:

$$x^n - 1 = \prod_{i=0}^{n-1}(x - q^i) \quad \Longrightarrow \quad \sum_{i=0}^{n-1} x^i = \frac{x^n - 1}{x - 1} = \prod_{i=1}^{n-1}(x - q^i).$$

We then substitute 1 for x as follows:

$$n = \prod_{i=1}^{n-1}(1 - q^i) = (1-q)^{n-1}\prod_{i=2}^{n-1}(1+q+\cdots+q^{i-1}) = (1-q)^{n-1}m. \tag{E2.3.2}$$

Now we are ready to prove the main result of this section, which also recovers Theorem 0.1.

Theorem 2.4. *Retain the above notation. The discriminant of A_q over its center $Z(A_q)$ is*

$$d(A_q/Z(A_q)) =_{T^\times} (nm)^{n^2}((1-q)^n x^n y^n - a^n)^{n(n-1)}.$$

Proof. Using Lemmas 2.1 and 2.3 and equation (E2.3.2), we have

$$(1-q)^{n^2(n-1)} d(A_q/Z(A_q)) =_{T^\times} (nm(1-q)^{n-1})^{n^2}((1-q)^n x^n y^n - a^n)^{n(n-1)}.$$

Since A_q is a domain, we obtain

$$d(A_q/Z(A_q)) =_{T^\times} (nm)^{n^2}((1-q)^n x^n y^n - a^n)^{n(n-1)}. \qquad \square$$

Remark 2.5. (1) By [CPWZ 2016, Lemma 2.7(7)], the integer n in Theorem 2.4 is nonzero in T. However, n and m may not be invertible in general.

(2) Theorem 0.1 is clearly a consequence of Theorem 2.4.

A slight generalization of Theorem 2.4 is the following.

Theorem 2.6. *Let T be a commutative domain and $q \in T^\times$ be a primitive n-th root of unity. Let B be the T-algebra of the form*

$$\frac{T\langle x, y\rangle}{(yx - qxy = a,\ x^n = b,\ y^n = c)},$$

where $a, b, c \in T$. Suppose that B is a free module over T with basis $\{x^i y^j \mid 0 \le i,\ j \le n-1\}$. Then $d(B/T) =_{T^\times} (nm)^{n^2}((1-q)^n x^n y^n - a^n)^{n(n-1)}$, where m is given in (E2.3.1).

Proof. First note that it is well-known and easy to check that T is the center of B.

Recall that A_q is the algebra of the form $T\langle x, y\rangle/(yx - qxy = a)$. There is a natural algebra homomorphism g from A_q to B sending x to x and y to y and $t \in T$ to $t \in T$. Then the hypotheses in Lemma 1.2 hold. By Lemma 1.2, $g(d(A_q/Z(A_q))) = d(B/T)$. Now the assertion follows from Theorem 2.4. $\square$

3. Discriminant of Clifford algebras

In this section we assume that $2^{-1} \in k$. We fix an integer $n \ge 2$.

Let T be a commutative domain and let $\mathcal{A} := \{a_{ij} \mid 1 \le i < j \le n\}$ be a set of scalars in T. We write $a_{ji} = a_{ij}$ if $i < j$. Let $V_n(\mathcal{A})$ be the T-algebra generated by $x_1, \dots, x_n$ and subject to the relations

$$x_i x_j + x_j x_i = a_{ij} \quad \text{for all } i \neq j.$$

This algebra was studied in [CPWZ 2015a; 2015b]. Some basic properties of $V_n(\mathcal{A})$ are given in [CPWZ 2015a, Section 4]. Let M_1 be the matrix

$$M_1 := \begin{pmatrix} 2x_1^2 & a_{12} & \cdots & a_{1n} \\ a_{21} & 2x_2^2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & 2x_n^2 \end{pmatrix}. \tag{E3.0.1}$$

This is a symmetric matrix with entries in $Z := T\left[x_1^2, \dots, x_n^2\right]$. We will define a sequence of matrices M_i later. Note that Z is a central subalgebra of $V_n(\mathcal{A})$. If we write $M_1 = (m_{ij,1})_{n\times n}$, then $m_{ij,1} = x_j x_i + x_i x_j$ for all i, j.

The algebra $V_n(\mathcal{A})$ is a Clifford algebra over Z. We will recall the definition of the Clifford algebra associated to a quadratic form in the second half of this section. In the next few lemmas, we are basically diagonalizing the quadratic form, which is elementary and well-known in the classical case; see [Lam 2005, Chapter I, Corollary 2.4] for some related material. Since we need an explicit construction to complete the proof of our main result, details will be provided below.

We will introduce a sequence of new variables starting with

$$x_{i,1} = x_i \quad \text{for all } i = 1, \dots, n,$$

and

$$a_{ij,1} = a_{ij} \quad \text{for all } i \neq j, \quad \text{and} \quad a_{ii,1} = 2x_i^2 \quad \text{for all } i.$$

So we have $x_{j,1}x_{i,1}+x_{i,1}x_{j,1}=a_{ij,1}$ for all $i,\ j$. Let

$$x_{1,2}:=x_{1,1} \quad \text{and} \quad x_{i,2}:=x_{i,1}-\tfrac{1}{2}a_{1i,1}x_{1,1}^{-2}x_{1,1} \quad \text{for all } i\geq 2. \tag{E3.0.2}$$

Lemma 3.1. *Retain the above notation.*

(1) $x_{i,2}x_{1,2}+x_{1,2}x_{i,2}=0$ *for all* $i\geq 2$.

(2) $x_{i,2}^2=x_{i,1}^2-\frac{1}{4}a_{1i,1}^2x_{1,1}^{-2}$ *for all* $i\geq 2$.

(3) $x_{i,2}x_{j,2}+x_{j,2}x_{i,2}=a_{ij,1}-\frac{1}{2}a_{1i,1}a_{1j,1}x_{1,1}^{-2}$ *for all* $2\leq i<j\leq n$.

(4) *Let* M_2 *be the matrix* $(x_{i,2}x_{j,2}+x_{j,2}x_{i,2})_{1\leq i,j\leq n}$. *Then* $\det M_2=\det M_1$.

(5) *Let*

$$C_1=\{x_{1,1}^{2i}\}_{i\geq 1}.$$

Then the localization $V_n(\mathcal{A})[C_1^{-1}]$ *is free over* $Z[C_1^{-1}]$ *with basis* $\{x_{1,2}^{d_1}\cdots x_{n,2}^{d_n}\mid d_s=0,1\}$.

Proof. (1)–(3) These follow by direct computation.

(4) Let N be the matrix

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0\\ -\frac{1}{2}a_{12,1}x_{1,1}^{-2} & 1 & 0 & \cdots & 0\\ -\frac{1}{2}a_{13,1}x_{1,1}^{-2} & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -\frac{1}{2}a_{1n,1}x_{1,1}^{-2} & 0 & 0 & \cdots & 1\end{pmatrix}.$$

By linear algebra and part (3), one can check that $NM_1N^T=M_2$. Since $\det N=1$, we have $\det M_2=\det M_1$.

(5) First of all, $V_n(\mathcal{A})$ is free over Z with basis $\{x_{1,1}^{d_1}\cdots x_{n,1}^{d_n}\mid d_s=0,1\}$. In the localization $V_n(\mathcal{A})[C_1^{-1}]$, this basis can be transformed to a basis $\{x_{1,2}^{d_1}\cdots x_{n,2}^{d_n}\mid d_s=0,1\}$ by using (E3.0.2). □

After we have $x_{i,2}$, define $a_{ij,2}$ to be $x_{i,2}x_{j,2}+x_{j,2}x_{i,2}$ for all $i,\ j$. Now we define $x_{i,s}$ and $a_{ij,s}$ inductively.

Definition 3.2. Let $s\geq 3$ and suppose that $x_{i,s-1}$ and $a_{ij,s-1}$ are defined inductively. Define

$$\begin{aligned} x_{i,s}&:=x_{i,s-1} && \text{for all } i<s,\\ x_{i,s}&:=x_{i,s-1}-\tfrac{1}{2}a_{s-1i,s-1}x_{s-1,s-1}^{-1} && \text{for all } i\geq s. \end{aligned} \tag{E3.2.1}$$

Define $a_{ij,s}:=x_{i,s}x_{j,s}+x_{j,s}x_{i,s}$ for all $i,\ j$.

Similar to Lemma 3.1, we have the following lemma. Its proof is also similar to the proof of Lemma 3.1, so it is omitted.

Lemma 3.3. *Retain the above notation. Let $2 \le s \le n$.*

(1) $x_{i,s}x_{j,s} + x_{j,s}x_{i,s} = 0$ *for all* $i < j$ *and* $i < s$.

(2) $x_{i,s} = x_{i,s-1}$ *if* $i < s$ *and* $x_{i,s}^2 = x_{i,s-1}^2 - \frac{1}{4}a_{s-1i,s-1}^2 x_{s-1,s-1}^{-2}$ *for all* $i \ge s$.

(3) $x_{i,s}x_{j,s} + x_{j,s}x_{i,s} = a_{ij,s-1} - \frac{1}{2}a_{s-1i,s-1}a_{s-1j,s-1}x_{s-1,s-1}^{-2}$ *for all* $s \le i < j \le n$.

(4) *Let M_s be the matrix $(x_{i,s}x_{j,s} + x_{j,s}x_{i,s})_{1\le i,j\le n}$. Then* $\det M_s = \det M_1$.

(5) *Let C_{s-1} be the Ore set*

$$\left\{x_{1,1}^{2i_1}x_{2,2}^{2i_1}\cdots x_{s-1,s-1}^{2i_{s-1}}\right\}_{i_1,\dots,i_{s-1}\ge 1}.$$

Then the localization $V_n(\mathcal{A})\left[C_{s-1}^{-1}\right]$ is free over $Z\left[C_{s-1}^{-1}\right]$ with basis $\left\{x_{1,s}^{d_1}\cdots x_{n,s}^{d_n} \mid d_s = 0, 1\right\}$.

We need two more lemmas before we prove the main result.

Lemma 3.4. *Let T be a commutative domain. Let A be a T-algebra containing T as a subalgebra, generated by $x_1, \dots, x_n$ and satisfying the relations $x_jx_i + x_ix_j = 0$ for all $i < j$ and $x_i^2 = a_i \in T$. Suppose that A is a free module over T with basis $\left\{x_1^{d_1}\cdots x_n^{d_n} \mid d_s = 0, 1\right\}$. Then*

$$d(A/T) =_{T^\times} \left(\prod_{i=1}^n 2x_i^2\right)^{2^{n-1}} =_{T^\times} \left(\prod_{i=1}^n x_i^2\right)^{2^{n-1}}.$$

Proof. Let $B = T_{-1}[x_1, \dots, x_n]$ and $Z = T\left[x_1^2, \dots, x_n^2\right]$. Then B is a free module over Z with basis $\left\{x_1^{d_1}\cdots x_n^{d_n} \mid d_s = 0, 1\right\}$. Let g be the algebra map from B to A sending T to T, x_i to x_i. Then the hypotheses in Lemma 1.2 holds. By Lemma 1.2, $g(d(B/Z)) =_{T^\times} d(A/T)$. Note that $d(B/Z)$ was computed in Proposition 1.4(3) to be $\left(\prod_{i=1}^n 2x_i^2\right)^{2^{n-1}}$, as we assume that 2 is invertible. Now the assertion follows. □

Let A be an Ore domain and let $Q(A)$ denote the skew field of fractions of A. Let Z be the commutative subalgebra $T\left[x_1^2, \dots, x_n^2\right] \subset V_n(\mathcal{A})$. For each $1 \le 1 \le n$, let Z_i be the subring of $Q(Z)$ of the form

$$Q\left(T\left[x_1^2, \dots, \widehat{x_i^2}, \dots, x_n^2\right]\right)\left[x_i^2\right].$$

Lemma 3.5. *Retain the above notation.*

(1) $\bigcap_{i=1}^n Z_i = Q(T)\left[x_1^2, \dots, x_n^2\right]$.

(2) $Z\left[C_{n-1}^{-1}\right] \subseteq Z_n$, *where $Z\left[C_{n-1}^{-1}\right]$ is defined in Lemma 3.3(5).*

Proof. (1) This is an easy commutative algebra fact.

(2) By Lemma 3.3(2) and induction, each $x_{i,s}^2$, for all $1 \le i < n$ and all $1 \le s \le n$, is in $Q\left(T\left[x_1^2, \dots, x_{n-1}^2\right]\right)$. So $Z\left[C_{n-1}^{-1}\right] \subseteq Z_n$. □

Theorem 3.6. *Suppose 2 is invertible. Let $Z = T[x_1^2, \dots, x_n^2]$. Then*

$$d(V_n(\mathcal{A})/Z) =_{T^\times} (\det M_1)^{2^{n-1}},$$

where M_1 is given in (E3.0.1).

Proof. Consider the variables $\{x_{i,n}\}_{i=1}^n$ defined in Lemma 3.3. By Lemma 3.3(5), $V_n(\mathcal{A})[C_{n-1}^{-1}]$ is free over $Z[C_{n-1}^{-1}]$ with basis $\{x_{1,s}^{d_1} \cdots x_{n,s}^{d_n} \mid d_s = 0, 1\}$. By Lemma 3.4, the discriminant

$$d\big(V_n(\mathcal{A})[C_{n-1}^{-1}]/Z[C_{n-1}^{-1}]\big)$$

is of the form $\big(\prod_{i=1}^n x_i^2\big)^{2^{n-1}}$ up to a unit in $Z[C_{n-1}^{-1}]$. By Lemma 3.3(4), we have

$$d\big(V_n(\mathcal{A})[C_{n-1}^{-1}]/Z[C_{n-1}^{-1}]\big) = \left(\prod_{i=1}^n x_i^2\right)^{2^{n-1}} = (\det M_n)^{2^{n-1}} = (\det M_1)^{2^{n-1}}.$$

By Lemma 1.3,

$$d(V_n(\mathcal{A})/Z) =_{(Z[C_{n-1}^{-1}])^\times} d\big(V_n(\mathcal{A})[C_{n-1}^{-1}]/Z[C_{n-1}^{-1}]\big) =_{(Z[C_{n-1}^{-1}])^\times} (\det M_1)^{2^{n-1}}.$$

Let Φ be the element $d(V_n(\mathcal{A})/Z)^{-1}(\det M_1)^{2^{n-1}}$. Then $\Phi \in \big(Z[C_{n-1}^{-1}]\big)^\times$. This means that both Φ and Φ^{-1} are in $Z[C_{n-1}^{-1}] \subseteq Z_n$. By symmetry, Φ is Z_i for all i. Thus Φ is in $\bigcap_{i=1}^n Z_i = Q(T)[x_1^2, \dots, x_n^2]$. Similarly, Φ^{-1} is in $Q(T)[x_1^2, \dots, x_n^2]$. Therefore, $\Phi, \Phi^{-1} \in Q(T)$.

Write $d(V_n(\mathcal{A})/Z) = c(\det M_1)^{2^{n-1}}$, where $c = \Phi^{-1} \in Q(T)$. It remains to show $c \in Z^\times$. Note that $V_n(\mathcal{A})$ is a filtered algebra such that $\operatorname{gr} V_n(\mathcal{A}) \cong T_{-1}[x_1, \dots, x_n]$. By Lemma 1.5,

$$\operatorname{gr} d(V_n(\mathcal{A})/Z) =_{Z^\times} d(\operatorname{gr} V_n(\mathcal{A})/\operatorname{gr} Z).$$

The left-hand side of the above is $c\big(\prod_{i=1} x_i^2\big)^{2^{n-1}}$ and the right-hand side of the above is $\big(\prod_{i=1} x_i^2\big)^{2^{n-1}}$ by Proposition 1.4(3) (assuming 2 is invertible). Thus $c \in Z^\times$, as required. □

Theorem 0.2 is a special case of Theorem 3.6 by taking $a_{ij} = 1$ for all $i < j$.

The algebras $V_n(\mathcal{A})$ and W_n are special Clifford algebras. Now we consider a Clifford algebra in a more general setting. Let T be a commutative domain and let V be a free T-module of rank n. Given a quadratic form $q : V \to T$, we can associate to this data the Clifford algebra

$$C(V, q) = \frac{T\langle V\rangle}{(x^2 - q(x) \mid x \in V)}.$$

Note that this q is different from the parameter q in the definition of the q-quantum Weyl algebra A_q and the parameter set $\boldsymbol{q}$ in the $V_n(\boldsymbol{q}, \mathcal{A})$ and $T_{\boldsymbol{q}}[x_1, \dots, x_n]$. Consider the bilinear form associated to q,

$$b(x, y) = \tfrac{1}{2}(q(x+y) - q(x) - q(y)) \tag{E3.6.1}$$

for all $x, y \in V$. If we choose a T-basis $x_1, \dots, x_n$ for V and let

$$\mathfrak{B} := (b_{ij}) = (b(x_i, x_j))_{n\times n} \in T^{n\times n} \tag{E3.6.2}$$

be the symmetric matrix which represents b with respect to this basis, then the relations of $C(V, q)$ are

$$x_i x_j + x_j x_i = 2b_{ij} \quad \text{for all } i, j. \tag{E3.6.3}$$

Define $\det(q)$ to be $\det(\mathfrak{B})$.

The following main result is a consequence of Theorem 3.6 and Lemma 1.2.

Theorem 3.7. *Let $A := C(V, q)$ be a Clifford algebra over a commutative domain T defined by a quadratic form $q : V \to T$. Pick a T-basis of V, say $\{x_i\}_{i=1}^n$. Then*

$$d(A/T) =_{T^\times} (\det(x_i x_j + x_j x_i)_{n\times n})^{2^{n-1}} =_{T^\times} \det(q)^{2^{n-1}}. \tag{E3.7.1}$$

Proof. Let $b : V^{\otimes 2} \to T$ be the symmetric bilinear form associated to the quadratic form q. Let $a_{ij} = 2b(x_i, x_j)$ for all $i < j$ and $\mathcal{A} = \{a_{ij}\}_{1\le i<j\le n}$. Then there is a canonical algebra surjection $\pi : V_n(\mathcal{A}) \to C(V, q)$ sending $x_i \to x_i$ for all $i = 1, \dots, n$ and $t \to t$ for all $t \in T$, and the kernel of π is the ideal generated by $\{x_i^2 - b_{ii}\}_{i=1}^n$. Clearly, $\pi\left(T\left[x_1^2, \dots, x_n^2\right]\right) = T$ and the matrix $(x_i x_j + x_j x_i)_{n\times n}$ equals M_1. It is easy to check that $\left\{x_1^{d_1} \cdots x_n^{d_n} \mid d_i = 0, 1\right\}$ is a basis of $V_n(\mathcal{A})$ over $T\left[x_1^2, \dots, x_n^2\right]$ and a basis of $C(V, q)$ over T. The first equation of (E3.7.1) follows from Theorem 3.6 and Lemma 1.2 and the second equation follows from the fact that $2\mathfrak{B} = (x_i x_j + x_j x_i)_{n\times n}$ and 2 is invertible. $\square$

In the rest of this section we briefly discuss "generic Clifford algebras", which will appear again in Section 8. (This generic Clifford algebra should be called a "universal Clifford algebra", but the term "universal Clifford algebra" has already been used).

Fix an integer n. Let I be the set $\{(i, j) \mid 1 \le i \le j \le n\}$ that can be thought of as the quotient set $\{(i, j) \mid 1 \le i,\ j \le n\}/((i, j) \sim (j, i))$. Let w denote the integer $\frac{1}{2}n(n+1)$. There is a bijection between I and the set of the first w integers $\{1, 2, \dots, w\}$. Let T_g be the commutative domain $k[t_{(i,j)} \mid (i, j) \in I]$, which is isomorphic to $k[t_1, \dots, t_w]$. Define a T_g-algebra A_g generated by $x_1, \dots, x_n$ and subject to the relations

$$x_i x_j + x_j x_i = 2t_{(i,j)} \quad \text{for all } 1 \le i \le j \le n. \tag{E3.7.2}$$

Let $V_g=\bigoplus_{i=1}^n T_g x_i$. Define a bilinear form $b_g : V_g\otimes V_g\to T_g$ by $b_g(x_i,x_j)=t_{(i,j)}$ and the associated quadratic form by $q_g(x)=b_g(x,x)$ for all $x\in V_g$. The "generic Clifford algebra" A_g is defined to be the Clifford algebra associated to (V_g,q_g). For any Clifford algebra $C(V,q)$ over a commutative ring T, by comparing (E3.6.3) with (E3.7.2), one sees that there is an algebra map $A_g\to C(V,q)$ sending $x_i\to x_i$ and $t_{(i,j)}\to b_{ij}$. Define $\deg x_i=1$ for all i and $\deg t_{(i,j)}=2$ for all $(i,j)\in I$. Then A_g is a connected graded algebra over k.

We also define some factor algebras of A_g. Let J be a subset of $\{(i,j)\mid 1\le i<j\le n\}$ and let w_J denote the integer $w-|J|$. Let $T_{g,J}$ be the commutative polynomial ring $k[t_{i,j}\mid (i,j)\in I\setminus J]$, which is isomorphic to $k[t_1,\dots,t_{w_J}]$. Define a $T_{g,J}$-algebra $A_{g,J}$ generated by $x_1,\dots,x_n$ and subject to the relations

$$x_ix_j+x_jx_i=\begin{cases}2t_{(i,j)}, & (i,j)\in I\setminus J,\\ 0, & (i,j)\in J.\end{cases} \tag{E3.7.3}$$

Let $V_{g,J}=\bigoplus_{i=1}^n T_{g,J}x_i$. Define a bilinear form $b_{g,J}:V_{g,J}\otimes V_{g,J}\to T_{g,J}$ by

$$b_{g,J}(x_i,x_j)=\begin{cases}t_{(i,j)}, & (i,j)\in I\setminus J,\\ 0, & (i,j)\in J,\end{cases}$$

and the associated quadratic form by $q_{g,J}(x)=b_g(x,x)$ for all $x\in V_{g,J}$. Then $A_{g,J}$ is the Clifford algebra associated to $(V_{g,J},q_{g,J})$. If $J\subseteq J'\subseteq\{(i,j)\mid 1\le i<j\le n\}$, there is an algebra map $A_{g,J}\to A_{g,J'}$ sending $x_i\to x_i$ and

$$t_{(i,j)}\to\begin{cases}t_{(i,j)}, & (i,j)\notin J',\\ 0, & (i,j)\in J'\setminus J.\end{cases}$$

In particular, $A_{g,J}$ is a connected graded factor ring of A_g.

In part (4) of the next lemma, we will use a few undefined concepts that are related to the homological properties of an algebra. We refer to [Levasseur 1992; Lu et al. 2007; Rogalski and Zhang 2012] for definitions.

Lemma 3.8. *Retain the above notation. Assume that k is a field of characteristic not 2. Let J' be subset of $\{(i,j)\mid 1\le i<j\le n\}$ and let $J=J'\setminus\{(i_0,j_0)\}$ for some $(i_0,j_0)\in J'$.*

(1) *The Hilbert series of A_g is*

$$H_{A_g}(t)=\frac{(1+t)^n}{(1-t^2)^w},\quad \text{where } w=\tfrac12 n(n+1).$$

(2) *The Hilbert series of $A_{g,J}$ is*

$$H_{A_{g,J}}(t)=\frac{(1+t)^n}{(1-t^2)^{w_J}},\quad \text{where } w_J=w-|J|.$$

(3) *$t_{(i_0,j_0)}$ is a central regular element in $A_{g,J'}$, and $A_{g,J}=A_{g,J'}/(t_{(i_0,j_0)})$.*

(4) *A_g and $A_{g,J}$ are connected graded Artin–Schelter regular, Auslander regular, Cohen–Macaulay noetherian domains.*

Proof. (1) Note that A_g is a free module over T_g with basis $\{x_1^{d_1} \cdots x_n^{d_n} \mid d_s = 0, 1\}$. Recall that $\deg x_i = 1$ and $\deg t_{(i,j)} = 2$. We have

$$H_{A_g}(t) = (1+t)^n H_{T_g}(t) = \frac{(1+t)^n}{(1-t^2)^w}.$$

(2) The proof is similar. Use the fact that $H_{T_{g,J}}(t) = 1/(1-t^2)^{w_J}$.

(3) It is clear that $t_{(i_0,j_0)}$ is central in $A_{g,J'}$ and that $A_{g,J} = A_{g,J'}/(t_{(i_0,j_0)})$. So the ideal $(t_{(i_0,j_0)})$ is the left ideal $t_{(i_0,j_0)}A_{g,J'}$ and the right ideal $A_{g,J'}t_{(i_0,j_0)}$. By parts (1) and (2), the Hilbert series of $(t_{(i_0,j_0)})$ is $t^2 H_{A_{g,J'}}(t)$. So $t_{(i_0,j_0)}$ is regular.

(4) We only provide a proof for A_g. The proof for $A_{g,J}$ is similar.

From part (3), $J_M := \{t_{(i,j)} \mid 1 \le i < j \le n\}$ is a sequence of regular central elements in A_g of positive degree. It is easy to see that $A_{g,J_M} (= A_g/(J_M))$ is isomorphic to the skew polynomial ring $k_{-1}[x_1, \ldots, x_n]$, which is an Artin–Schelter regular, Auslander regular, Cohen–Macaulay noetherian domain. Applying [Lu et al. 2007, Lemma 7.6] repeatedly, A_g has finite global dimension. Applying [Levasseur 1992, Proposition 3.5, Theorem 5.10] repeatedly, A_g is a noetherian Auslander Gorenstein and Cohen–Macaulay domain. By [Levasseur 1992, Theorem 6.3], A_g is Artin–Schelter Gorenstein. Since A_g has finite global dimension, it is Auslander regular and Artin–Schelter regular. □

Remark 3.9. Retain the above notation. (1) Some homological properties of the algebra A_g are given in Lemma 3.8. It would be interesting to work out combinatorial and geometric invariants (and properties) of A_g. For example, what are the point-module and line-module schemes of A_g? Definitions of these schemes can be found in [Vancliff and Van Rompay 2000; Vancliff et al. 1998].

(2) Another way of presenting A_g is the following. Let S be a k-vector space of dimension n. Define A_g to be $k\langle S\rangle/([x^2, y] = 0 \mid \text{for all } x, y, \in S)$. By using this new expression, one can easily see that the group of graded algebra automorphisms of A_g, denoted by $\mathrm{Aut}_{\mathrm{gr}}(A_g)$, is isomorphic to $\mathrm{GL}_n(k)$.

(3) Suppose $n \ge 2$. The full automorphism group $\mathrm{Aut}(A_g)$ has not been determined. It is known that $\mathrm{Aut}(A_g)$ is not affine. For example, if $f(t)$ is a polynomial in t, then

$$x_i \to \begin{cases} x_i, & i > 1, \\ x_1 + f([x_1, x_2]^2)x_2, & i = 1, \end{cases}$$

extends to an algebra automorphism of A_g.

(4) It seems interesting to study the "cubic algebra" $k\langle S\rangle/([x^3, y] = 0 \mid \text{for all } x, y \in S)$ and higher-degree analogues.

(5) The quotient division ring of A_g, denoted by D_g, is called the "generic Clifford division algebra of rank n". It would be interesting to study algebraic properties or invariants of D_g.

4. Center of skew polynomial rings

To use the discriminant most effectively, one needs to first understand the center of an algebra. In this section we give a criterion for when $T_{\boldsymbol{q}}[x_1, \dots, x_n]$ is free over its center and when the center of $T_{\boldsymbol{q}}[x_1, \dots, x_n]$ is a polynomial ring.

Recall that T is a commutative domain and $\boldsymbol{q} := \{q_{ij} \in T^\times \mid 1 \le i < j \le n\}$ is a set of invertible scalars. Let $P := T_{\boldsymbol{q}}[x_1, \dots, x_n]$ be the skew polynomial ring over T and subject to the relations (E0.2.1). We assume that $d_{ij} := o(q_{ij}) < \infty$ and write

$$q_{ij} = \exp\bigl(2\pi\sqrt{-1}k_{ij}/d_{ij}\bigr), \tag{E4.0.1}$$

where $|k_{ij}| < d_{ij}$ and $(k_{ij}, d_{ij}) = 1$. Note that, by our convention, $q_{ij} = q_{ji}^{-1}$ for all i, j. Hence, we choose $k_{ij} = -k_{ji}$ and $d_{ij} = d_{ji}$. We also adopt the convention that if $q_{ij} = 1$ then $k_{ij} = 0$ and $d_{ij} = 1$. In particular, $k_{ii} = 0$ and $d_{ii} = 1$. We can extend P to $P\bigl[x_1^{-1}, \dots, x_n^{-1}\bigr]$, with an inverse for each x_i, with the expected relations

$$x_i x_i^{-1} = x_i^{-1} x_i = 1, \quad x_j x_i^{-1} = q_{ij}^{-1} x_i^{-1} x_j, \quad \text{and} \quad x_j^{-1} x_i^{-1} = q_{ij} x_i^{-1} x_j^{-1}.$$

We need to do some analysis to understand the center of P. Let η_i denote conjugation by x_i, sending $f \mapsto x_i^{-1} f x_i$, and let $\xi = x_1^{s_1} \cdots x_n^{s_n}$. Then

$$\eta_i(\xi) = \exp\bigl(2\pi\sqrt{-1}\boldsymbol{e}_i^T Y \boldsymbol{s}\bigr)\xi,$$

where $Y \in \mathfrak{so}_n(\mathbb{Q})$ has (i, j)-th entry k_{ij}/d_{ij}, $\boldsymbol{s}$ is the column vector whose i-th entry is s_i appearing in the powers of ξ, and $\boldsymbol{e}_i$ is the i-th standard basis vector in $\mathbb{Q}^n$.

Lemma 4.1. *Retain the above notation. Then ξ is in the center $Z(P)$ of P if and only if $Y\boldsymbol{s} \in \mathbb{Z}^n$.*

Proof. Since P is generated by $\{x_i\}$, we have $\xi \in Z(P)$ if and only if $\eta_i(\xi) = \xi$ for all i, if and only if $\exp\bigl(2\pi\sqrt{-1}\boldsymbol{e}_i^T Y\boldsymbol{s}\bigr) = 1$, if and only if $\boldsymbol{e}_i^T Y\boldsymbol{s} \in \mathbb{Z}$ for all i, and finally, if and only if $Y\boldsymbol{s} \in \mathbb{Z}^n$. □

By choosing the standard basis for $\mathbb{Q}^n$, we can consider Y as a linear transformation $\mathbb{Q}^n \to \mathbb{Q}^n$ by sending $\boldsymbol{s} \mapsto Y\boldsymbol{s}$. Here we view $\mathbb{Q}^n$ as column vectors and Y as a left multiplication. We can restrict this map to $\mathbb{Z}^n \subset \mathbb{Q}^n$ (embedded via the standard basis) and compose with the quotient $\mathbb{Q}^n \to \mathbb{Q}^n/\mathbb{Z}^n$ to obtain a $\mathbb{Z}$-module homomorphism $Y' : \mathbb{Z}^n \to \mathbb{Q}^n/\mathbb{Z}^n$.

Lemma 4.2. *Retain the above notation. Then $\xi \in Z(P)$ if and only if $\boldsymbol{s} \in \ker(Y')$.*

Proof. By Lemma 4.1, $\xi \in Z(P)$ if and only if $Y\boldsymbol{s} \in \mathbb{Z}^n$, which is equivalent to $Y'(\boldsymbol{s}) = 0$ by the definition of Y'. □

Let D be the matrix $(d_{ij})_{n\times n}$ and let L_i be the lcm of the entries in the i-th row of D, namely, $L_i = \operatorname{lcm}\{d_{ij} \mid j = 1, \ldots, n\}$. Since D is a symmetric matrix, L_i is also the lcm of the entries in i-th column. Observe that $Z(P)$ contains the central subring $P' := k\big[x_1^{L_1}, \ldots, x_n^{L_n}\big]$. In other words, $\ker(Y')$ contains the $\mathbb{Z}$-lattice Λ spanned by $L_i \boldsymbol{e}_i$ for $i = 1, \ldots, n$. Therefore, Y' factors through

$$\mathbb{Z}^n \to M := \mathbb{Z}^n/\Lambda = \bigoplus_{i=1}^{n} \mathbb{Z}/L_i\mathbb{Z}.$$

For each $\boldsymbol{s} \in \mathbb{Z}^n$, the i-th entry of $Y'(\boldsymbol{s})$ is $\sum_j k_{ij}s_j/d_{ij} \in \mathbb{Q}/\mathbb{Z}$, which is L_i-torsion, or equivalently, in $L_i^{-1}\mathbb{Z}/\mathbb{Z}$. Therefore, Y' induces a map

$$M \to M' := \bigoplus_{i=1}^{n} L_i^{-1}\mathbb{Z}/\mathbb{Z}.$$

Since M' is naturally isomorphic to M, we can define an endomorphism

$$\overline{Y} : M \to M$$

by setting

$$\overline{Y}\boldsymbol{s} = \left(\sum_{j=1}^{n} L_i(k_{ij}s_j/d_{ij})\right)_{i=1}^{n}.$$

In particular, $\overline{Y}\boldsymbol{e}_j = \sum_{i=1}^{n}(k_{ij}L_i/d_{ij})\boldsymbol{e}_i$. Sometimes we think of $\overline{Y}$ as a matrix:

$$\overline{Y} = (k_{ij}L_i/d_{ij})_{n\times n} = \operatorname{diag}(L_1, \ldots, L_n)Y.$$

The following lemma is a reinterpretation of [CPWZ 2016, Lemma 2.3].

Lemma 4.3. *Retain the above notation. The following are equivalent.*

(1) *The center $Z(P)$ of P is a polynomial ring.*

(2) *$Z(P) = P'$.*

(3) $\ker(\overline{Y}) = 0$.

(4) *$\overline{Y}$ is an isomorphism.*

Proof. (1) $\Leftrightarrow$ (2): One implication is clear. For the other implication, we assume that the center $Z(P)$ is a polynomial ring. By [CPWZ 2016, Lemma 2.3], $Z(P)$ is of the form $T\big[x_1^{a_i}, \ldots, x_n^{a_i}\big]$. It is easy to check that $L_i \mid a_i$ for all i. Since $Z(P) \supseteq P'$, $a_i = L_i$ for all i. The assertion follows.

(3) $\Rightarrow$ (2): Let $\xi := x_1^{s_1} \cdots x_n^{s_n} \in Z(P)$ and $\boldsymbol{s} = (s_i)_{i=1}^{n}$. By Lemma 4.2, $\boldsymbol{s} \in \ker(Y')$. Since $\overline{Y}$ is induced by Y', we have $\overline{Y}(\boldsymbol{s}) = 0$. By part (3), $\boldsymbol{s} = 0$ in $M = \mathbb{Z}^n/\Lambda$. So $\boldsymbol{s} \in \Lambda$, which is equivalent to $\xi \in P'$. Therefore, $Z(P) = P'$, as desired.

(2) $\Rightarrow$ (3): Let $\xi := x_1^{s_1} \cdots x_n^{s_n} \in P$, where $\boldsymbol{s} := (s_i)_{i=1}^{n} \in \ker(\overline{Y})$, viewed as a vector in M. By the definition of M, we might assume that each s_i is nonnegative and less

than L_i. Since $\overline{Y}$ is induced by Y', we have $\boldsymbol{s} \in \ker(Y')$. By Lemma 4.2, $\xi \in Z(P)$. By part (2) and our choice of $0 \le s_i < L_i$, we have $\xi = 1$ or $\boldsymbol{s} = 0$, as desired.

(3) $\Leftrightarrow$ (4): This is clear since M is finite. □

The advantage of working with $\overline{Y}$ is that $\ker(\overline{Y}) = 0$ is equivalent to $\overline{Y}$ being an isomorphism. Next we need to understand when $\overline{Y}$ is an isomorphism. For the rest of this section we use $\otimes$ for $\otimes_{\mathbb{Z}}$ and $\mathbb{F}_p$ for $\mathbb{Z}/p\mathbb{Z}$.

Lemma 4.4. *The morphism $\overline{Y}$ is an isomorphism if and only if $\overline{Y} \otimes \mathbb{F}_p$ is an isomorphism for all primes p.*

Proof. As a $\mathbb{Z}$-module, M is finite, and it suffices to show that $\overline{Y}$ is surjective if and only if $\overline{Y} \otimes \mathbb{F}_p$ is surjective for each prime p. This is clear since $- \otimes \mathbb{F}_p$ is right exact, so surjectivity of a map can be checked on closed fibers. □

Fix any prime p. Let $M_p = M \otimes \mathbb{F}_p$ and $\overline{Y}_p = \overline{Y} \otimes \mathbb{F}_p$. For any $\boldsymbol{e}_i$, if $L_i \notin p\mathbb{Z}$, then the image of $\boldsymbol{e}_i$ is zero in M_p. We can therefore use $\{\boldsymbol{e}_i \mid L_i \in p\mathbb{Z}\}$ as a basis of M_p. Consequently, M_p is a vector space over $\mathbb{F}_p$ of dimension at most n, and we can write $\overline{Y}_p$ as a matrix over $\mathbb{F}_p$. Next we will decompose the vector space M_p and the matrix $\overline{Y}_p$.

For each positive integer m, let $M_{p,m}$ denote the subspace of M_p generated by $\{\boldsymbol{e}_i \mid L_i \in p^m\mathbb{Z} - p^{m+1}\mathbb{Z}\}$. Let $\overline{Y}_{p,m}$ be the endomorphism

$$M_{p,m} \longrightarrow M_p \xrightarrow{\overline{Y}_p} M_p \longrightarrow M_{p,m},$$

where the first map is the inclusion and the last map is the natural projection using the given basis $\{\boldsymbol{e}_i \mid L_i \in p\mathbb{Z}\}$. Then $\overline{Y}_{p,m}$ can be expressed as the submatrix of $\overline{Y}$ taken from the rows and columns with indices i such that $\boldsymbol{e}_i \in M_{p,m}$. For all but finitely many values of m, we have $M_{p,m} = 0$, and in this case, $\overline{Y}_{p,m}$ is a 0×0 matrix. We adopt the convention that the determinant of a 0×0 matrix is 1. In general, $\det(\overline{Y}_{p,m})$ is in $\mathbb{F}_p$.

Lemma 4.5. *The following are equivalent.*

(1) *The map $\overline{Y}_p$ is an isomorphism.*

(2) *For all positive integers m, $\overline{Y}_{p,m}$ is an isomorphism.*

(3) $\det(\overline{Y}_{p,m}) \neq 0$ *for all positive integers m.*

Proof. It is clear that (2) and (3) are equivalent, so we need only show that (1) and (2) are equivalent.

Let $m > 0$, and let i, j be such that $L_i \in p^m\mathbb{Z} - p^{m+1}\mathbb{Z}$ and $L_j \notin p^m\mathbb{Z}$. Since $L_j = \operatorname{lcm}\{d_{kj} \mid k = 1, \ldots, n\}$, we have $d_{ij} \notin p^m\mathbb{Z}$ and $k_{ij}L_i/d_{ij} \in p\mathbb{Z}$. Therefore, the $\boldsymbol{e}_i$-component of $\overline{Y}_p\boldsymbol{e}_j$ is zero. We can extend this to show that, for any $m > m' > 0$, the $M_{p,m'}$-component of $\overline{Y}_p(M_{p,m})$ is zero, or equivalently,

$$\overline{Y}_p(M_{p,m}) \subseteq \bigoplus_{n\geq m} M_{p,n} =: N_m.$$

This implies that, for any $m > 0$, $\overline{Y}_p$ acts as an endomorphism on N_m. Since each M_p is finite dimensional, $\overline{Y}_p$ is an isomorphism if and only if it acts as an isomorphism on each subquotient $N_m/N_{m+1} \cong M_{p,m}$. This action is already given by $\overline{Y}_{p,m}$, so the assertion follows. □

Combining all the lemmas in this section we have:

Theorem 4.6. *The center of the skew polynomial ring $T_{\boldsymbol{q}}[x_1, \dots, x_n]$ is a polynomial ring if and only if $\det(\overline{Y}_{p,m}) \neq 0$ for all primes p and all integers $m > 0$.*

Theorem 4.6 is a slight generalization of Theorem 0.3(a) without the hypothesis that $q_{ij} \neq 1$ for all $i \neq j$. The definition of the matrices $\overline{Y}_{p,m}$ is not straightforward, so we give an example below. Hopefully, the example will show that this matrix is not hard to understand.

Example 4.7. We start with the following skew-symmetric matrix with entries in $\mathbb{Q}$:

$$Y := \begin{pmatrix} 0 & \frac{4}{27} & \frac{2}{9} & 0 & \frac{2}{3} & \frac{3}{5} \\ -\frac{4}{27} & 0 & \frac{1}{3} & \frac{7}{9} & \frac{1}{3} & \frac{1}{5} \\ -\frac{2}{9} & -\frac{1}{3} & 0 & \frac{1}{6} & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{7}{9} & -\frac{1}{6} & 0 & \frac{2}{3} & 0 \\ -\frac{2}{3} & -\frac{1}{3} & -\frac{1}{2} & -\frac{2}{3} & 0 & \frac{5}{8} \\ -\frac{3}{5} & -\frac{1}{5} & -\frac{1}{2} & 0 & -\frac{5}{8} & 0 \end{pmatrix}.$$

One can easily construct q_{ij} by (E4.0.1) and the skew polynomial ring $T_{\boldsymbol{q}}[x_1, \dots, x_6]$ by (E0.2.1), but the point of this example is to work out the matrices $\overline{Y}_{p,m}$ for all primes p and all $m > 0$. By considering the denominators of the entries of Y, one sees that

$$(L_1, L_2, L_3, L_4, L_5, L_6) = (3^3 \cdot 5, 3^3 \cdot 5, 2 \cdot 3^2, 2 \cdot 3^2, 2^3 \cdot 3, 2^3 \cdot 5).$$

This implies that $\overline{Y}_{p,m}$ is a trivial matrix (or a 0×0 matrix) except for $p = 2, 3, 5$. Next we consider

$$\overline{Y} = \operatorname{diag}(L_1, \dots, L_6) Y = \begin{pmatrix} 0 & 20 & 30 & 0 & 90 & 81 \\ -20 & 0 & 45 & 105 & 45 & 27 \\ -4 & -6 & 0 & 3 & 9 & 9 \\ 0 & -14 & -3 & 0 & 12 & 0 \\ -16 & -8 & -12 & -16 & 0 & 15 \\ -24 & -8 & -20 & 0 & -25 & 0 \end{pmatrix}.$$

Recall that $M_{p,m}$ has a basis $\{\boldsymbol{e}_i \mid L_i \in p^m\mathbb{Z} - p^{m+1}\mathbb{Z}\}$ and $\overline{Y}_{p,m}$ is the square submatrix of $\overline{Y}$ with indices $\{i \mid L_i \in p^m\mathbb{Z} - p^{m+1}\mathbb{Z}\}$ and with entries evaluated in $\mathbb{F}_p$.

For $p = 2$, $\overline{Y}_{2,m}$ are the following:

- $\overline{Y}_{2,1}$ is the principle (3, 4)-submatrix of Y, and is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.
- $\overline{Y}_{2,3}$ uses indices 5, 6, and is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.
- For all $m = 2$ or $m > 3$, $\overline{Y}_{2,m}$ is trivial.

Therefore, $\overline{Y}_2$ is an isomorphism by Lemma 4.5.

For $p = 3$, $\overline{Y}_{3,m}$ are the following:

- $\overline{Y}_{3,1}$ uses only index 5, and is the 1×1 zero matrix.
- $\overline{Y}_{3,2}$ uses indices 3, 4, and is the 2×2 zero matrix.
- $\overline{Y}_{3,3}$ uses indices 1, 2, and is $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.
- For all $m > 3$, $\overline{Y}_{3,m}$ is trivial.

Since $\det(\overline{Y}_{3,1}) = \det(\overline{Y}_{3,2}) = 0$, $\overline{Y}_3$ is not an isomorphism by Lemma 4.5. Consequently, the center of $T_{\boldsymbol{q}}[x_1, \ldots, x_6]$ is not a polynomial ring by Theorem 4.6.

For $p = 5$, $\overline{Y}_{5,m}$ are the following:

- $\overline{Y}_{5,1}$ uses indices 1, 2, 6, and
$$\overline{Y}_{5,1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ -1 & -2 & 0 \end{pmatrix}.$$
- For all $m > 1$, $\overline{Y}_{5,m}$ is trivial.

It is easy to check that $\det(\overline{Y}_{5,1}) = 0$. Therefore, $\overline{Y}_5$ is not an isomorphism.

For $p > 5$, $\overline{Y}_{p,m}$ is trivial for all $m > 0$.

5. Low dimensional cases

We start with some easy consequences of Theorem 4.6 and then discuss the case when n is 3 or 4.

Corollary 5.1. *Suppose there are a prime p and an $m > 0$ such that $M_{p,m}$ is odd dimensional. Then $\overline{Y}_p$ is not an isomorphism. As a consequence, the center of $T_{\boldsymbol{q}}[x_1, \ldots, x_n]$ is not a polynomial ring.*

Proof. If $\overline{Y}_{p,m}$ is a skew-symmetric matrix of odd size, its determinant is zero (this is true even when $p = 2$). The rest follows from Lemma 4.5 and Theorem 4.6. $\square$

Corollary 5.2. *Suppose there is a prime p such that M_p is odd dimensional. Then $\overline{Y}_p$ is not an isomorphism. As a consequence, the center of $T_{\boldsymbol{q}}[x_1, \ldots, x_n]$ is not a polynomial ring.*

Proof. Since $M_p = \bigoplus_{m=1}^{\infty} M_{p,m}$, if it is odd dimensional, at least one $M_{p,m}$ must be odd dimensional. The assertion follows from Corollary 5.1. □

Corollary 5.3. *Suppose, for each prime p, that $p \mid d_{ij}$ for at most one pair (i, j), $1 \le i < j \le n$. Then $\overline{Y}_p$ is an isomorphism for each p. As a consequence, the center of $T_{\boldsymbol{q}}[x_1, \dots, x_n]$ is a polynomial ring.*

Proof. If $d_{ij} \notin p\mathbb{Z}$ for all i, j, then $L_i \notin p\mathbb{Z}$ for all i, $M_p = 0$ and $\overline{Y}_p$ is trivially an isomorphism.

If $d_{ij} \in p^m\mathbb{Z} - p^{m+1}\mathbb{Z}$ for some i, j and some positive integer m, and each of every other term $d_{k\ell}$ is not in $p\mathbb{Z}$, then $L_i, L_j \in p^m\mathbb{Z} - p^{m+1}\mathbb{Z}$, and each of every other L_k is not in $p\mathbb{Z}$. This shows that $\overline{Y}_{p,m}$ is a nonzero 2×2 skew-symmetric matrix (i.e., $\det(\overline{Y}_{p,m}) \neq 0$) and $M_{p,m'} = 0$ for each $m' \neq m$. The rest follows from Lemma 4.5 and Theorem 4.6. □

Next we give simple criteria for $\overline{Y}$ to be an isomorphism in the cases $n = 3, 4$.

Corollary 5.4. *The center of $T_{\boldsymbol{q}}[x_1, x_2, x_3]$ is a polynomial ring if and only if $(d_{ij}, d_{ik}) = 1$ for all different i, j, k.*

Proof. There are only three d terms — d_{12}, d_{13}, and d_{23}. If each (d_{ij}, d_{ik}) equals 1, then no prime is a factor of more than one term in $\{d_{ij}\}$. By Corollary 5.3, the center of $T_{\boldsymbol{q}}[x_1, x_2, x_3]$ is a polynomial ring.

Conversely, suppose that p is a prime such that $d_{ij}, d_{ik} \in p\mathbb{Z}$ for some i, j, k. Then $L_1, L_2, L_3 \in p\mathbb{Z}$. This implies that M_p has dimension 3. Hence, by Corollary 5.2, $\overline{Y}_p$ is not an isomorphism. So $\overline{Y}$ is not an isomorphism. Therefore, the center of $T_{\boldsymbol{q}}[x_1, x_2, x_3]$ is not a polynomial ring by Lemma 4.3. □

Corollary 5.5. *The center of $T_{\boldsymbol{q}}[x_1, x_2, x_3, x_4]$ is a polynomial ring if and only if, for each prime p, one of the following holds*:

(a) *$L_i \notin p\mathbb{Z}$ for all i.*

(b) *For some positive integer m, $\overline{Y}_{p,m}$ is 4×4 with nonzero determinant.*

(c) *There are distinct indices $i, j, k, \ell \in \{1, 2, 3, 4\}$ and a nonnegative integer m such that $d_{ij} \in p^{m+1}\mathbb{Z}$, $d_{k\ell} \in p^m\mathbb{Z} - p^{m+1}\mathbb{Z}$, and every other d term is not in $p^{m+1}\mathbb{Z}$.*

Proof. Let $P = T_{\boldsymbol{q}}[x_1, x_2, x_3, x_4]$. By Lemmas 4.3 and 4.4, $Z(P)$ is a polynomial ring if and only if $\overline{Y}_p$ is an isomorphism for all p. It remains to show that, for each p, $\overline{Y}_p$ is an isomorphism if and only if one of (a), (b), or (c) holds. Now we fix p and prove the assertion in three cases according to the shape of M_p.

First we prove the "if" part.

(a) If $L_i \notin p\mathbb{Z}$ for all i, then $M_p = 0$ and $\overline{Y}_p$ is trivially an isomorphism. This handles the case when $M_p = 0$.

(b) If for some $m > 0$, $\overline{Y}_{p,m}$ is 4×4 with nonzero determinant, then every other $\overline{Y}_{p,r}$ (for all $r \neq m$) is a 0×0 matrix and, consequently, $\overline{Y}_p$ is an isomorphism. This is the case when $M_p = M_{p,m}$ is 4-dimensional for one m.

(c) Assume the hypotheses in part (c). Let $m' > m$ be the integer such that $d_{ij} \in p^{m'}\mathbb{Z} - p^{m'+1}\mathbb{Z}$. If $m = 0$, then d_{ij} is the only d term divisible by p. Hence $\overline{Y}_{p,m'}$ is a skew-symmetric 2×2 nonzero matrix and $\overline{Y}_{p,r}$ is trivial for all $r \neq m'$. Therefore, $\overline{Y}_p$ is an isomorphism. If $m > 0$, then $\overline{Y}_{p,m}$ and $\overline{Y}_{p,m'}$ are both skew-symmetric and 2×2, and (because $k_{k\ell}L_k/d_{k\ell} \notin p\mathbb{Z}$) nonzero. Furthermore, every other $\overline{Y}_{p,r}$ is 0×0 for all $r \neq m, m'$. Therefore, $\overline{Y}_p$ is an isomorphism.

For the rest we prove the "only if" part.

Suppose that $\overline{Y}_p$ is an isomorphism. By Corollary 5.2, M_p is even dimensional, that is, $\dim M_p = 0$, 2 or 4.

The $\dim M_p = 0$ case coincides with the case when $L_i \notin p\mathbb{Z}$ for all i, so we obtain case (a).

For the $\dim M_p = 2$ case, at least one d_{ij} lies in $p\mathbb{Z}$ and L_i, L_j lie in $p\mathbb{Z}$, and no other d term is a multiple of p, so $\overline{Y}_p$ is necessarily an isomorphism. We can set $m = 0$, so that $d_{ij} \in p^{m+1}\mathbb{Z}$, and all other d_{ab} are not in $p^{m+1}\mathbb{Z}$. So we obtain (c).

All that remains is the $\dim M_p = 4$ case. Each $M_{p,m}$ is even dimensional by Corollary 5.1. If $\dim M_{p,m} = 4$ for some m, then $\overline{Y}_{p,m}$ is 4×4 and $\overline{Y}_p$ is an isomorphism if and only if $\det(\overline{Y}_{p,m}) \neq 0$. So we obtain case (b).

Finally, suppose there exist $m' > m > 0$ such that $\dim M_{p,m} = \dim M_{p,m'} = 2$. Let i, j, k, ℓ be distinct such that $L_i, L_j \in p^{m'}\mathbb{Z} - p^{m'+1}\mathbb{Z}$ and $L_k, L_\ell \in p^m\mathbb{Z} - p^{m+1}\mathbb{Z}$. We must have that $d_{ij} \in p^{m'}\mathbb{Z} \subseteq p^{m+1}\mathbb{Z}$ and every other d term is not in $p^{m+1}\mathbb{Z}$. If $d_{k\ell} \notin p^m\mathbb{Z}$, then $k_{k\ell}L_k/d_{k\ell}, k_{\ell k}L_\ell/d_{\ell k} \in p\mathbb{Z}$ and $\overline{Y}_{p,m}$ is the 2×2 zero matrix, yielding a contradiction. Therefore, $d_{k\ell}$ must be in $p^m\mathbb{Z}$. So we obtain case (c) again. □

6. Center of generalized Weyl algebras

Let T be a commutative k-domain. In this section we assume that $\boldsymbol{q} := \{q_{ij}\}$ is a set of roots of unity in T and let $\mathcal{A} := \{a_{ij} \mid 1 \leq i < j \leq j\}$ be a subset of T. Define the generalized Weyl algebra associated to $(\boldsymbol{q}, \mathcal{A})$ to be the central T-algebra

$$V(\boldsymbol{q}, \mathcal{A}) := \frac{T\langle x_1, \ldots, x_n\rangle}{(x_j x_i - q_{ij} x_i x_j - a_{ij} \mid i \neq j)}.$$

Consider a filtration on $V(\boldsymbol{q}, \mathcal{A})$ with $\deg x_i = 1$ and $\det t = 0$ for all $t \in T$. Suppose

$$\operatorname{gr} V(\boldsymbol{q}, \mathcal{A}) \text{ is naturally isomorphic to } T_{\boldsymbol{q}}[x_1, \ldots, x_n]. \tag{E6.0.1}$$

Consider the hypothesis that,

$$\text{for any pair } (i, j),\ a_{ij} = 0 \text{ whenever } q_{ij} = 1. \tag{E6.0.2}$$

Proposition 6.1. *Suppose (E6.0.1) and (E6.0.2) and let $A = V(\boldsymbol{q}, \mathcal{A})$. If the center $Z(\operatorname{gr} A)$ is a polynomial ring, then so is $Z(A)$, and $Z(A) \cong Z(\operatorname{gr} A)$.*

Proof. If $Z(\operatorname{gr} A)$ is a polynomial ring, then $Z(\operatorname{gr} A) = T\big[x_1^{L_1}, \dots, x_n^{L_n}\big]$, where $L_i = \operatorname{lcm}\{d_{ij} \mid j = 1, \dots, n\}$ (Lemma 4.3). Recall that d_{ij} is the order of q_{ij}.

First we claim that $x_i^{L_i}$ is in the center of A. For each j, we have the equation $x_j x_i = q_{ij} x_i x_j + a_{ij}$. If $q_{ij} = 1$, then x_j commutes with x_i by hypothesis (E6.0.2), so x_j commutes with $x_i^{L_i}$. If $q_{ij} \neq 1$, then the order of q_{ij} is d_{ij}. The equation $x_j x_i = q_{ij} x_i x_j + a_{ij}$ implies that x_j commutes with $x_i^{d_{ij}}$, as each $x_j x_i^k$ is equal to $q_{ij}^k x_i^k x_j + \big(1 + q_{ij} + \cdots + q_{ij}^{k-1}\big) a_{ij}$. Since d_{ij} divides L_i, x_j commutes with $x_i^{L_i}$ for all $j \neq i$. This shows that $x_i^{L_i}$ is central.

Since $\operatorname{gr} A$ is the skew polynomial ring $T_{\boldsymbol{q}}[x_1, \dots, x_n]$, it is easy to check that $\operatorname{gr} Z(A) \subset Z(\operatorname{gr} A)$. Since $Z(\operatorname{gr} A)$ is generated by $\big\{x_i^{L_i}\big\}_{i=1}^n$, induction on the degree of element $f \in Z(A)$ shows that f is generated by $x_i^{L_i}$. Therefore, the assertion follows. $\square$

Proposition 6.2. *Retain the above notation and suppose (E6.0.1). If $a_{ij} \neq 0$ for some $i \neq j$, then $q_{ik} q_{jk} = 1$ for all $k \neq i$ or j.*

Proof. We resolve $x_k x_j x_i$ in two different ways:

$$\begin{aligned}
(x_k x_j) x_i &= (q_{jk} x_j x_k + a_{jk}) x_i \\
&= q_{jk} x_j (x_k x_i) + a_{jk} x_i \\
&= q_{jk} x_j (q_{ik} x_i x_k + a_{ik}) + a_{jk} x_i \\
&= q_{jk} q_{ik} (x_j x_i) x_k + q_{jk} a_{ik} x_j + a_{jk} x_i \\
&= q_{jk} q_{ik} (q_{ij} x_i x_j + a_{ij}) x_k + q_{jk} a_{ik} x_j + a_{jk} x_i \\
&= q_{jk} q_{ik} q_{ij} x_i x_j x_k + q_{jk} q_{ik} a_{ij} x_k + q_{jk} a_{ik} x_j + a_{jk} x_i,
\end{aligned}$$

and similarly

$$\begin{aligned}
x_k (x_j x_i) &= x_k (q_{ij} x_i x_j + a_{ij}) \\
&= q_{ij} (x_k x_i) x_j + a_{ij} x_k \\
&= q_{ij} (q_{ik} x_i x_k + a_{ik}) x_j + a_{ij} x_k \\
&= q_{ij} q_{ik} x_i (x_k x_j) + q_{ij} a_{ik} x_j + a_{ij} x_k \\
&= q_{ij} q_{ik} q_{jk} x_i x_j x_k + q_{ij} q_{ik} a_{jk} x_i + q_{ij} a_{ik} x_j + a_{ij} x_k.
\end{aligned}$$

Comparing the coefficients of x_k gives the result. $\square$

When an algebra A is finitely generated and free over its center (as in the situation of Proposition 6.1), one should be able to compute the discriminant of A over its center. We give an example here.

Example 6.3. Let A be generated by x_1, x_2, x_3, x_4 and subject to the relations

$$\begin{aligned} x_3x_1 - x_1x_2 &= 0, & x_4x_2 + x_2x_4 &= 0, \\ x_3x_2 - x_2x_3 &= 0, & x_3x_4 + x_4x_3 &= 0, \\ x_4x_1 + x_1x_4 &= 0, & x_1x_2 + x_2x_1 &= x_3^2 + x_4^2. \end{aligned} \tag{E6.3.1}$$

This is the example in [Vancliff and Van Rompay 2000, Lemma 1.1] (with $\lambda = 0$). It is an iterated Ore extension, and therefore Artin–Schelter regular of global dimension 4.

It is not hard to check that the center of A is generated by x_i^2. This algebra is a factor ring of the algebra B over $T := k[t]$ generated by x_1, x_2, x_3, x_4 and subject to the relations

$$\begin{aligned} x_3x_1 - x_1x_2 &= 0, & x_4x_2 + x_2x_4 &= 0, \\ x_3x_2 - x_2x_3 &= 0, & x_3x_4 + x_4x_3 &= 0, \\ x_4x_1 + x_1x_4 &= 0, & x_1x_2 + x_2x_1 &= t. \end{aligned} \tag{E6.3.2}$$

Note that $\operatorname{gr} B$ is a skew polynomial ring over T with the above relations by setting $t = 0$. The Y-matrix is

$$\begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}.$$

By Corollary 5.5(b), B has center $T\big[x_1^2, x_2^2, x_3^2, x_4^2\big]$. The discriminant of B over its center is $2^{48}\big(4x_1^2x_2^2 - t^2\big)^8 x_3^{16}x_4^{16}$, by the next lemma. By Lemma 1.2, the discriminant of A over its center is $2^{48}\big(4x_1^2x_2^2 - \big(x_3^2+x_4^2\big)^2\big)^8 x_3^{16}x_4^{16}$. We will see in the next sections that $\mathbb{D}(A) = A$. As a consequence of Theorem 0.5, A is cancellative and the automorphism group of A is affine.

Lemma 6.4. *Suppose the $k[t]$-algebra B is generated by x_1, x_2, x_3, x_4 and subject to the six relations given (E6.3.2). Then the discriminant of B over its center is $2^{48}\big(4x_1^2x_2^2 - t^2\big)^8 x_3^{16}x_4^{16}$.*

Sketch of the proof. It is routine to check that the center of B is

$$Z(B) = k[t]\big[x_1^2, x_2^2, x_3^2, x_4^2\big].$$

The algebra B is a free module over $Z(B)$ of rank 16 with a $Z(B)$-basis $\big\{x_1^a x_2^b x_3^c x_4^d \mid a, b, c, d = 0, 1\big\}$. Let $\{z_1, \dots z_{16}\}$ be the above $Z(B)$-basis. Then we can compute the matrix $(\mathrm{tr}(z_i z_j))_{16\times 16}$:

$$
\begin{pmatrix}
16 & 0 & 0 & 0 & 0 & 8t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 16a & 8t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 8t & 16b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 16c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8ct & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 16d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8dt & 0 & 0 & 0\\
8t & 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 16ac & 0 & 8ct & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -16ad & 0 & -8dt & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 8ct & 0 & 16bc & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -8dt & 0 & -16bd & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -16cd & 0 & 0 & 0 & 0 & -8cdt\\
0 & 0 & 0 & 8ct & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 8dt & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16acd & 8cdt & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8cdt & 16bcd & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8cdt & 0 & 0 & 0 & 0 & \delta
\end{pmatrix}
$$

Here $\alpha=-16ab+8t^2$, $\beta=-16abc+8ct^2$, $\gamma=-16abd+8dt^2$, $\delta=16abcd-8cdt^2$, and $a=x_1^2$, $b=x_2^2$, $c=x_3^2$, $d=x_4^2$. We skip the details in computing the above traces. By using Maple, its determinant is $2^{48}\bigl(4x_1^2x_2^2-t^2\bigr)^8x_3^{16}x_4^{16}$. $\square$

7. Three subalgebras

In this section we discuss three (possibly different) subalgebras of A, all of which are helpful for the applications in the next section.

Makar-Limanov invariants. The first subalgebra is the Makar-Limanov invariant of A [Makar-Limanov 1996]. This invariant has been very useful in commutative algebra. For any k-algebra A, let $\mathrm{Der}(A)$ denote the set of all k-derivations of A and let $\mathrm{LND}(A)$ denote the set of locally nilpotent k-derivations of A.

Definition 7.1. Let A be an algebra over k.

(1) The *Makar-Limanov invariant* of A is

$$\mathrm{ML}(A)=\bigcap_{\delta\in\mathrm{LND}(A)}\ker(\delta). \tag{E7.1.1}$$

(2) We say that A is LND-*rigid* if $\mathrm{ML}(A)=A$, or $\mathrm{LND}(A)=\{0\}$.

(3) We say that A is *strongly* LND-*rigid* if $\mathrm{ML}(A[t_1,\dots,t_d])=A$ for all $d\ge 0$.

The following lemma is clear. Part (2) follows from the fact that $\partial\in\mathrm{LND}(A)$ if and only if $g^{-1}\partial g\in\mathrm{LND}(A)$.

Lemma 7.2. *Let A be an algebra.*

(1) $\mathrm{ML}(A)$ *is a subalgebra of A.*

(2) *For any $g \in \mathrm{Aut}(A)$, we have $g(\mathrm{ML}(A)) = \mathrm{ML}(A)$.*

Divisor subalgebras. Throughout this subsection let A be a domain containing $\mathbb{Z}$. Let F be a subset of A. Let $\mathrm{Sw}(F)$ be the set of $g \in A$ such that $f = agb$ for some $a, b \in A$ and $0 \neq f \in F$. Here Sw stands for "subword", which can be viewed as a divisor.

Definition 7.3. Let F be a subset of A.

(1) Let $D_0(F) = F$. Inductively define $D_n(F)$ as the k-subalgebra of A generated by $\mathrm{Sw}(D_{n-1}(F))$. The subalgebra $D(F) = \bigcup_{n \geq 0} D_n(F)$ is called the *F-divisor subalgebra of A*. If F is the singleton $\{f\}$, we simply write $D(\{f\})$ as $D(f)$.

(2) If $f = d(A/Z)$ (if it exists), we call $D(f)$ *the discriminant-divisor subalgebra of A*, or *DDS of A*, and write it as $\mathbb{D}(A)$.

The following lemma is well-known [Makar-Limanov 2008, p. 4].

Lemma 7.4. *Let x, y be nonzero elements in A and let $\partial \in \mathrm{LND}(A)$. If $\partial(xy) = 0$, then $\partial(x) = \partial(y) = 0$.*

Proof. Let m and n be the largest integers such that $\partial^m(x) \neq 0$ and $\partial^n(y) \neq 0$. Then the product rule and the choice of m, n imply that

$$\partial^{m+n}(xy) = \sum_{i=0}^{m+n} \binom{n+m}{i} \partial^i(x) \partial^{m+n-i}(y) = \binom{n+m}{m} \partial^m(x) \partial^n(y) \neq 0.$$

So $m + n = 0$. The assertion follows. □

Lemma 7.5. *Let F be a subset of* $\mathrm{ML}(A)$. *Then* $D(F) \subseteq \mathrm{ML}(A)$.

Proof. Let ∂ be any element in $\mathrm{LND}(A)$. By hypothesis, $\partial(f) = 0$ for all $f \in F$. By Lemma 7.4, $\partial(x) = 0$ for all $x \in \mathrm{Sw}(F)$. So $\partial = 0$ when restricted to $D_1(F)$. By induction, $\partial = 0$ when restricted to $D(F)$. The assertion follows by taking arbitrary $\partial \in \mathrm{LND}(A)$. □

Lemma 7.6. *Suppose $d(A/Z)$ is defined. Then the DDS $\mathbb{D}(A)$ is preserved by all $g \in \mathrm{Aut}(A)$.*

Proof. By [CPWZ 2015a, Lemma 1.8(6)] or [CPWZ 2016, Lemma 1.4(4)], $d(A/Z)$ is g-invariant up to a unit. So, if $g \in \mathrm{Aut}(A)$, then g maps $\mathrm{Sw}(d(A/Z))$ to $\mathrm{Sw}(d(A/Z))$ and $D_1(d(A/Z))$ to $D_1(d(A/Z))$. By induction, one sees that g maps $D_n(d(A/Z))$ to $D_n(d(A/Z))$. So the assertion follows. □

We need to find some elements $f \in A$ so that $\partial(f) = 0$ for all $\partial \in \mathrm{LND}(A)$. The next lemma was proven in [CPWZ 2016, Proposition 1.5].

Lemma 7.7. *Let Z be the center of A and let $d \geq 0$. Suppose $A^\times = k^\times$. Assume that A is finitely generated and free over Z. Then we have $\partial(d(A/Z)) = 0$ for all $\partial \in \mathrm{LND}(A[t_1, \dots, t_d])$.*

Proof. Let f denote the element $d(A[t_1, \dots, t_d]/Z[t_1, \dots, t_d])$ in $Z[t_1, \dots, t_d]$. By [CPWZ 2016, Proposition 1.5], $\partial(f) = 0$. By [CPWZ 2015a, Lemma 5.4],

$$f =_{k^\times} d(A/Z).$$

The assertion follows. □

Here is the first relationship between the two subalgebras.

Proposition 7.8. *Retain the hypothesis of Lemma 7.7. Let $d \geq 0$. Then*

$$\mathbb{D}(A) \subseteq \mathrm{ML}(A[t_1, \dots, t_d]) \subseteq A.$$

Proof. It is clear that $\mathrm{ML}(A[t_1, \dots, t_d]) \subseteq A$ by [Bell and Zhang 2016]. Let f equal $d(A/Z)$, which is in $A \subseteq A[t_1, \dots, t_d]$. By Lemma 7.7, $f \in \mathrm{ML}(A[t_1, \dots, t_d])$. Let $D'(f)$ be the discriminant-divisor subalgebra of f in $A[t_1, \dots, t_d]$. By Lemma 7.5, $D'(f) \subseteq \mathrm{ML}(A[t_1, \dots, t_d])$. It is clear from the definition that $D(f) \subseteq D'(f)$. Therefore, the assertion follows. □

In particular, by taking $d = 0$, we have $\mathbb{D}(A) \subseteq \mathrm{ML}(A)$.

Aut-*bounded subalgebra.* In this subsection we assume that A is filtered such that the associated graded ring $\operatorname{gr} A$ is a connected graded domain. Later we further assume that A is connected graded. Since $\operatorname{gr} A$ is a connected graded domain, we can define $\deg f$ to be the degree of $\operatorname{gr} f$, and the degree satisfies the equation

$$\deg(xy) = \deg x + \deg y$$

for all $x, y \in A$.

Definition 7.9. Retain the above hypotheses. Let G be a subgroup of $\mathrm{Aut}(A)$ and let V be a subset of A.

(1) Let x be an element in A. The *G-bound* of x is

$$\deg_G(x) := \sup\{\deg(g(x)) \mid g \in G\}.$$

(2) Let g be in $\mathrm{Aut}(A)$. The *V-bound* of g is

$$\deg_g(V) := \sup\{\deg(g(x)) \mid x \in V\}.$$

(3) The *G-bounded subalgebra* of A, denoted by $\beta_G(A)$, is the set of elements x in A with finite G-bound. It is clear that $\beta_G(A)$ is a subalgebra of A (Lemma 7.10(1)). In particular, the Aut-*bounded subalgebra* of A, denoted by $\beta(A)$, is the set of elements x in A with finite $\mathrm{Aut}(A)$-bound.

The following lemma is easy, so we omit the proof.

Lemma 7.10. *Retain the above notation. Let G be a subgroup of* $\mathrm{Aut}(A)$.

(1) *The set $\beta_G(A)$ is a subalgebra of A.*

(2) *$g(\beta_G(A)) = \beta_G(A)$ for all $g \in G$.*

Here is the relation between the two subalgebras $\mathbb{D}(A)$ and $\beta(A)$. Let V be a subset of A. We say V is of *bounded degree* if there is an N such that $\deg(v) < N$ for all $v \in V$.

Proposition 7.11. *Let A be a filtered algebra such that* $\mathrm{gr}\, A$ *is a connected graded domain. Suppose that $G \subseteq$* $\mathrm{Aut}(A)$ *and $F \subseteq A$.*

(1) *If $G(F)$ has bounded degree, then $D(F) \subseteq \beta_G(A)$.*

(2) *If $f \in A$ is such that $g(f) =_{Z(A)^\times} f$ for all $g \in G$, then $D(f) \subseteq \beta_G(A)$.*

(3) *Assume that A is finitely generated and free over its center Z. Let $f = d(A/Z)$. Then $\mathbb{D}(A) = D(f) \subseteq \beta(A)$.*

Proof. (1) We have $D_0(F) = F \subseteq \beta_G(A)$ by assumption and use induction on n. Suppose that $D_{n-1}(F) \subseteq \beta_G(A)$. Assume that $D_n(F)$ is not contained in $\beta_G(A)$. Then there exists an $x \in D_n(A)$ such that $G(x)$ does not have bounded degree. Since $D_n(A)$ is generated by $\mathrm{Sw}(D_{n-1}(A))$ as an algebra, there is an $f \in \mathrm{Sw}(D_{n-1}(A))$ such that $G(f)$ does not have bounded degree. By definition of $\mathrm{Sw}(D_{n-1}(A))$, there exists a nonzero $f' \in D_{n-1}(A)$ and $a, b \in A$ such that $f' = afb$. Since $\mathrm{gr}\, A$ is a domain, we have $\deg(g(f')) = \deg(g(a)) + \deg(g(f)) + \deg(g(b))$ for all $g \in G$. Hence $G(f')$ does not have bounded degree, which is a contradiction. Hence $D_n(F) \subseteq \beta_G(A)$ for all $n \geq 1$. Therefore, $D(F) \subseteq \beta_G(A)$.

(2) Since $Z(A)^\times \subseteq A_0$, we see that $G(f)$ has bounded degree, hence part (2) follows from part (1).

(3) The third assertion is a special case of part (2) by Lemma 1.2. $\square$

Under the hypotheses of Propositions 7.8 and 7.11 (and assuming that A is finitely generated and free over its center Z), we have:

$$\begin{tikzcd}[column sep=small, row sep=small] & \mathbb{D}(A) & \\ \mathrm{ML}(A) \arrow[ur, phantom, sloped, "\subseteq"] & & \beta(A) \arrow[ul, phantom, sloped, "\supseteq"] \\ & A \arrow[ul, phantom, sloped, "\subseteq"] \arrow[ur, phantom, sloped, "\supseteq"] & \end{tikzcd}$$

For the rest of this section, we assume that A is a connected graded domain and that k contains the field $\mathbb{Q}$. An automorphism g of A is called *unipotent* if

$$g(v) = v + (\text{higher degree terms}) \qquad \text{(E7.11.1)}$$

for all homogeneous elements $v \in A$. Let $\mathrm{Aut}_{\mathrm{uni}}(A)$ denote the subgroup of $\mathrm{Aut}(A)$ consisting of unipotent automorphisms [CPWZ 2016, after Theorem 3.1]. If $g \in \mathrm{Aut}_{\mathrm{uni}}(A)$, we can define

$$\log g := -\sum_{i=1}^{\infty} \frac{1}{i}(1-g)^i. \tag{E7.11.2}$$

Let C be the completion of A with respect to the graded maximal ideal $\mathfrak{m} := A_{\geq 1}$. Then C is a local ring containing A as a subalgebra. We can define $\deg_l : C \to \mathbb{Z}$ by setting $\deg_l(v)$ to be the lowest degree of the nonzero homogeneous components of $v \in C$. We define *a unipotent automorphism of* C in a similar way to (E7.11.1) by using $\deg_l$. It is clear that if $g \in \mathrm{Aut}_{\mathrm{uni}}(A)$, then it induces a unipotent automorphism of C, which is still denoted by g.

Lemma 7.12. *Let A be a connected graded domain. Let $g \in \mathrm{Aut}_{\mathrm{uni}}(A)$ and let G be any subgroup of $\mathrm{Aut}(A)$ containing g. Let B denote $\beta_G(A)$. Then $(\log g)|_B$ is a locally nilpotent derivation of B. Further, $g|_B$ is the identity if and only if $(\log g)|_B$ is zero.*

Proof. Let C be the completion of A with respect to the graded maximal ideal $\mathfrak{m} := A_{\geq 1}$. Let g also denote the algebra automorphism of C induced by g. Then g is also a unipotent automorphism of C.

Since g is unipotent, $\deg_l(1-g)(v) > \deg_l v$ for any $0 \neq v \in C$. By induction, one has $\deg(1-g)^n(v) \geq n + \deg v$ for all $n \geq 1$. Thus $(\log g)(v)$ converges and therefore is well-defined. It follows from a standard argument that $\log g$ is a derivation of C (this is also a consequence of [Freudenburg 2006, Proposition 2.17(b)]).

Let v be an element in $B := \beta_G(A)$. Note that $g^n(v) \in B$ for all n by Lemma 7.10. Since $v \in B$, there is an N_0 such that $\deg g^n(v) < N_0$ for all n. If $(1-g)^n(v) \neq 0$, then

$$\deg(1-g)^n(v) = \deg\left(\sum_{i=0}^{n} \binom{n}{i} g^i(v)\right) < N_0 \quad \text{for all } n. \tag{E7.12.1}$$

When $n \geq N_0$, the inequalities from the previous paragraph imply that

$$\deg_l(1-g)^n(v) \geq n + \deg v \geq N_0, \tag{E7.12.2}$$

which contradicts (E7.12.1) unless $(1-g)^n(v) = 0$. Therefore,

$$(1-g)^n(v) = 0 \quad \text{for all } n > N_0. \tag{E7.12.3}$$

By (E7.12.3), the infinite sum of $\log g$ in (E7.11.2) terminates when applied to $v \in B$, and $(\log g)(v) \in A$. By Lemma 7.10, $(\log g)(v) \in B$. Since $\log g$ is a derivation of C, it is a derivation when restricted to B.

Next we need to show that it is a locally nilpotent derivation when restricted to B. It suffices to verify that, for any $v \in B$, $(\log g)^N(v) = 0$ for $N \gg 0$, which follows from (E7.11.2) and (E7.12.3).

The final assertion follows from the fact that g is the exponential function of $\log g$ and $\log g$ is locally nilpotent. □

Now we are ready to prove the second part of Theorem 0.5 without the finite GK-dimension hypothesis.

Theorem 7.13. *Let k be a field of characteristic zero and let A be a connected graded domain over k. Assume that A is finitely generated and free over its center Z in part* (2).

(1) *If* $\mathrm{ML}(A) = \beta(A) = A$, *then* $\mathrm{Aut}_{\mathrm{uni}}(A) = \{1\}$.

(2) *If* $\mathbb{D}(A) = A$, *then* $\mathrm{Aut}_{\mathrm{uni}}(A) = \{1\}$.

Proof. (1) By hypothesis, $B := \beta(A)$ equals A. Let $g \in \mathrm{Aut}_{\mathrm{uni}}(A)$. Then $(\log g)|_B$ is a locally nilpotent derivation of B by Lemma 7.12. Hence $\log g \in \mathrm{LND}(A)$. Since $\mathrm{ML}(A) = A$, we have $\mathrm{LND}(A) = \{0\}$. So $\log g = 0$. By Lemma 7.12, g is the identity.

(2) Combining the hypothesis $\mathbb{D}(A) = A$ with Propositions 7.8 and 7.11, we have $\mathrm{ML}(A) = \beta(A) = A$. The assertion follows from part (1). □

8. Applications

In this section we assume that k is a field of characteristic zero.

Zariski cancellation problem. The Zariski cancellation problem for noncommutative algebras was studied in [Bell and Zhang 2016]. We recall some definitions and results.

Definition 8.1. [Bell and Zhang 2016, Definition 1.1] Let A be an algebra.

(1) We call A *cancellative* if $A[t] \cong B[t]$ for some algebra B implies that $A \cong B$.

(2) We call A *strongly cancellative* if, for any $d \geq 1$, $A[t_1, \dots, t_d] \cong B[t_1, \dots, t_d]$ for some algebra B implies that $A \cong B$.

The original Zariski cancellation problem, or ZCP, asks if the polynomial ring $k[t_1, \dots, t_n]$, where k is a field, is cancellative. A recent result of Gupta [2014a; 2014b] settled the question negatively in positive characteristic for $n \geq 3$. The ZCP in characteristic zero remains open for $n \geq 3$. Some history and partial results can be found in [Bell and Zhang 2016], where the authors used discriminants and locally nilpotent derivations to study the ZCP for noncommutative rings.

One of their main results is the following.

Theorem 8.2 [Bell and Zhang 2016, Theorems 0.4 and 3.3]. *Let A be a finitely generated domain of finite Gelfand–Kirillov dimension. If A is strongly* LND-*rigid* (*respectively*, LND-*rigid*), *then A is strongly cancellative* (*respectively*, *cancellative*).

Now we have an immediate consequence, which is the first part of Theorem 0.5. Combining it with Theorem 7.13, we have finished the proof of Theorem 0.5.

Theorem 8.3. *Let A be a finitely generated domain of finite GK-dimension. Let Z be the center of A and suppose $A^{\times} = k^{\times}$. Assume that A is finitely generated and free over Z. If $A = \mathbb{D}(A)$, then A is strongly cancellative.*

Proof. Combining the hypothesis $A = \mathbb{D}(A)$ with Proposition 7.8, we have

$$A = \mathbb{D}(A) \subseteq \mathrm{ML}(A[t_1, \dots, t_d]) \subseteq A.$$

So $\mathrm{ML}(A[t_1, \dots, t_d]) = A$, or A is strongly LND-rigid. The assertion follows from Theorem 8.2. □

Next we give two examples.

Example 8.4. Let A be generated by x_1, x_2, x_3, x_4 and subject to the relations

$$\begin{aligned} x_1x_2 + x_2x_1 &= 0, & x_2x_3 + x_3x_2 &= 0, \\ x_1x_3 + x_3x_1 &= 0, & x_3x_4 + x_4x_3 &= 0, \\ x_1x_4 + x_4x_1 &= x_3^2, & x_2x_4 + x_4x_2 &= 0. \end{aligned}$$

This is an iterated Ore extension, so it is Artin–Schelter regular of global dimension 4. This is a special case of the algebra in [Vancliff et al. 1998, Definition 3.1]. Set $x_i^2 = y_i$ for $i = 1, \dots, 4$. Then $Z(A) = k[y_1, y_2, y_3, y_4]$. The M_1-matrix of (E3.0.1) is

$$(a_{ij})_{4\times 4} = \begin{pmatrix} 2y_1 & 0 & 0 & y_3 \\ 0 & 2y_2 & 0 & 0 \\ 0 & 0 & 2y_3 & 0 \\ y_3 & 0 & 0 & 2y_4 \end{pmatrix}.$$

The determinant $\det(a_{ij})$ is $f_0 := 4y_2y_3(4y_1y_4 - y_3^2)$. By Theorem 3.7, the discriminant $f := d(A/Z)$ is $f_0^{2^3}$. It is clear that $y_2, y_3 \in \mathrm{Sw}(f)$ and $y_1, y_4 \in \mathrm{Sw}(D_1(f))$. Thus $x_i \in \mathrm{Sw}(D_2(f))$ for all i. Consequently, $A = \mathbb{D}(A)$. By Theorem 8.3, A is strongly cancellative.

The next example is somewhat generic.

Example 8.5. Let T be a commutative domain and let $A = C(V, q)$ be the Clifford algebra associated to a quadratic form $q : V \to T$ where V is a free T-module of rank n. Suppose that n is even. Then the center of A is T [Lam 2005, Chapter 5, Theorem 2.5(a)]. We assume that A is a domain with $A^{\times} = k^{\times}$. Let $t_1, \dots, t_w$ be a set of generators of T, and suppose that $q(V) \subseteq (t_1 \cdots t_w)T$ or $\det(q) \in (t_1 \cdots t_w)T$.

Then by Theorem 3.7 we have $f := d(A/T) \in (t_1 \cdots t_w)^{2^{n-1}}$. So $t_s \in \mathrm{Sw}(f)$ for all s. This shows that $T \subseteq \mathbb{D}(A)$ and then $A = \mathbb{D}(A)$ (as $x_i^2 \in T$). By Theorem 8.3, A is strongly cancellative.

Remark 8.6. Let A be the algebra in Example 6.3. Using the formula for $d(A/Z)$ given in Lemma 6.4, it is easy to see that $A = \mathbb{D}(A)$. So A is cancellative by Theorem 8.3.

Automorphism problem. By [CPWZ 2015a; 2016], the discriminant controls the automorphism group of some noncommutative algebras. In this section we compute some automorphism groups by using the discriminants computed in previous sections. We first recall some definitions and results.

We modify the definitions in [CPWZ 2015a; 2016] slightly. Let A be an $\mathbb{N}$-filtered algebra such that $\mathrm{gr}\, A$ is a connected graded domain. Let $X := \{x_1, \ldots, x_n\}$ be a set of elements in A such that it generates A and $\mathrm{gr}\, X$ generates $\mathrm{gr}\, A$. We do not require $\deg x_i = 1$ for all i.

Definition 8.7. Let f be an element in A and let $X' = \{x_1, \ldots, x_m\}$ be a subset of X. We say f is *dominating* over X' if, for any subset $\{y_1, \ldots, y_n\} \subseteq A$ that is linearly independent in the quotient k-space A/k, there is a lift of f, say $F(X_1, \ldots, X_n)$, in the free algebra $k\langle X_1, \ldots, X_n\rangle$, such that $\deg F(y_1, \ldots, y_n) > \deg f$ whenever $\deg y_i > \deg x_i$ for some $x_i \in X'$.

The following lemma is easy.

Lemma 8.8. *Retain the above notation. Suppose $f := d(A/Z)$ is dominating over X'. Then for every automorphism $g \in \mathrm{Aut}(A)$, we have $\deg g(x_i) \le \deg x_i$ for all $x_i \in X'$.*

Proof. Let $y_i = g(x_i)$. Then $\{y_1, \ldots, y_n\}$ is linearly independent in A/k (as $\{x_1, \ldots, x_n\}$ is linearly independent in A/k). If $\deg y_i > \deg x_i$ for some i, by the dominating property, there is a lift of f in the free algebra, say $F(X_1, \ldots, X_n)$, such that $\deg F(y_1, \ldots, y_n) > \deg f$. Since g is an algebra automorphism,

$$F(y_1, \ldots, y_n) = F(g(x_1), \ldots, g(x_n)) = g(F(x_1, \ldots, x_n)) = g(f).$$

By [CPWZ 2015a, Lemma 1.8(6)], $g(f) = f$ (up to a unit in Z). Hence

$$\deg F(y_1, \ldots, y_n) = \deg g(f) = \deg f,$$

yielding a contradiction. Therefore, $\deg g(x_i) = \deg y_i \le \deg x_i$ for all i. □

We will study the automorphism group of a class of Clifford algebras; see Example 8.5.

Example 8.9. Let A be the Clifford algebra over a commutative k-domain T as in Example 8.5 and assume that n is even. Let $\{z_1, \ldots, z_n\}$ denote a set of generators for A. We will use $\{x_1, \dots, x_n\}$ for the generators of the generic Clifford algebra A_g defined in Section 3. Then there is an algebra homomorphism from A_g to A sending x_i to z_i for all i. Since n is even, T is the center of A. Assume that A is a filtered algebra such that $\operatorname{gr} A$ is a connected graded domain, so we can define the degree of any nonzero element in A. Further assume that $\deg t_i = 2$ (not 1) for all $i = 1, \dots, w$ and $\deg z_i > 2$ for all $i = 1, 2, \dots, n$. In particular, there is no element of degree 1. Some explicit examples are given later in this example.

Recall that we assumed $q(V) \subseteq (t_1 \cdots t_w)T$. Let $2b_{ij} = z_j z_i + z_i z_j$. Then we can write $b_{ij} = (t_1 \cdots t_w)^N b'_{ij}$ for some $N > 0$. By Theorem 3.7, the discriminant is $f := d(A/T) = \big[\big(\prod_{s=1}^{w} t_s\big)^N d'\big]^{2^{n-1}}$, where $d' = \det(2b'_{ij})_{n\times n}$. We need another hypothesis, which is that

$$\deg d' < N. \tag{E8.9.1}$$

Let $X' = \{t_i\}_{i=1}^{w}$ and $X = \{z_i\}_{i=1}^{n} \cup X'$. Then f is a noncommutative polynomial over X'. We first claim that f is dominating over X'. Let $\{y_i\}_{i=1}^{w}$ be a set of elements in $A \setminus k$. If $\deg y_i > 2$ for some i, then $\deg\big[\big(\prod_{s=1}^{w} y_s\big)^N d'(y_1, \dots, y_w)\big]^{2^{n-1}}$ is strictly larger than the degree of f, as we assume that $\deg d' < N$. This shows the claim.

Now let g be any algebra automorphism of A and let y_i be $g(t_i)$ for all i. Then, by Lemma 8.8, $\deg y_i = 2$. It follows from the relations $z_i z_i = b_{ii}$ that $\deg z_i > 3$. Hence $(\operatorname{gr} A)_2$ is generated by the t_i. This implies that y_i is in the span of X' and k. In some sense, every automorphism of A is affine (with respect to X'). It is a big step in understanding the automorphism group of A.

Below we study the automorphism group of a family of subalgebras of the generic Clifford algebra A_g of rank n that is defined in Section 3. As before, we assume n is even. We have two different sets of variables t, one for A_g and the other for general A. It would be convenient to unify these in the following discussion. So we identify $\{t_{(i,j)} \mid 1 \le i \le j \le n\}$ with $\{t_i\}_{i=1}^{w}$ via a bijection ϕ. Here $w = \frac{1}{2}n(n+1)$ as in the definition of A_g (Section 3).

Let r be any positive integer and let $B_{g,r}$ be the graded subalgebra of A_g generated by $\{t_{(i,j)}\}$ for all $1 \le i \le j \le n$ (or $\{t_i\}_{i=1}^{w}$) and $z_i := x_i\big(\prod_{k=1}^{w} t_k\big)^r$ for all $i = 1, 2, \dots, n$. Since $B_{g,r}$ is a graded subalgebra of A_g, it is a connected graded domain. This is also a Clifford algebra over $T_g := k[t_{(i,j)}]$ generated by $z_1, \dots, z_n$ and subject to the relations

$$z_j z_i + z_i z_j = 2\left(\prod_{k=1}^{w} t_k\right)^{2r} t_{(i,j)} =: 2b_{ij}$$

from which the bilinear form b and associated quadratic form q can easily be recovered. In particular, $q(V) \subseteq \big(\prod_{k=1}^{w} t_k\big)^{2r} T_g$, where $V = \bigoplus_{i=1}^{n} T_g z_i$. By the definition of A_g, we have $\deg t_i = 2$. Then $\deg z_i = 1 + 4rw > 3$. Now we assume that $N := 2r$ is bigger than $2n$, which is the degree of $d' := \det(t_{(i,j)})$. So we have

$$n < r, \quad \text{or equivalently} \quad \deg d' < N,$$

as required by (E8.9.1). See also Remark 8.10.

Let g be an algebra automorphism of $B_{g,d}$. By the above discussion, $g(t_i)$, for each i, is a linear combination of $\{t_j\}_{j=1}^{w}$ and 1. Using the relations $z_i^2 = b_{ii}$, we see that $\deg g(z_i) = \deg(z_i)$ for all i. Thus g must be a filtered automorphism of $B_{g,d}$.

Since g preserves the discriminant f and f is homogeneous in t_i, we have $\deg g(t_i) = 2$. Further, by using the expression of f and the fact that T_g is a UFD, $g(t_i)$ can not be a linear combination of the t_j of more than one term. Thus $g(t_i) = c_i t_j$ for some j and some $c_i \in k^{\times}$. This implies that there is a permutation $\sigma \in S_w$ and a collection of units $\{c_i\}_{i=1}^{w}$ such that $g(t_i) = c_i t_{\sigma(i)}$ for all i. Since g is filtered (by the last paragraph), $g(z_i) = \sum_{h=1}^{n} d_{ih} z_h + e_i$, where $d_{ih}, e_i \in k$. Applying g to the relation

$$z_i^2 = b_{ii} = \left(\prod_{i=1}^{w} t_i\right)^N t_{\phi(i,i)}, \quad \text{where } N := 2r,$$

we obtain that

$$\left(\sum_h d_{ih} z_h\right)^2 + 2e_i \left(\sum_h d_{ih} z_h\right) + e_i^2 = \left(\prod_{i=1}^{w} c_i t_i\right)^N g(t_{\phi(i,i)}).$$

Since $\left(\sum_h d_{ih} z_h\right)^2 \in T$, we have $e_i \left(\sum_h d_{ih} z_h\right) = 0$. Consequently, $e_i = 0$ and $g(z_i) = \sum_{h=1}^{n} d_{ih} z_h$. Applying g to the relations

$$z_i z_j + z_j z_i = 2b_{ij} = 2\left(\prod_{i=1}^{w} t_i\right)^N t_{\phi(i,j)}$$

and expanding the left-hand side, we obtain

$$\sum_{h,l} d_{ih} d_{jl} (z_h z_l + z_l z_h) = 2\left(\prod_{i=1}^{w} c_i t_i\right)^N g(t_{\phi(i,j)}).$$

Hence $d_{ih} d_{jl}$ is nonzero for only one pair (h, l). Thus there is a set of units $\{d_i\}_{i=1}^{n}$ and a permutation $\psi \in S_n$ such that $g(z_i) = d_i z_{\psi(i)}$ for all $i = 1, \ldots, n$. Then the above equation implies that

$$d_i d_j \left(\prod_{i=1}^{w} t_i\right)^N t_{\phi(\psi(i),\psi(j))} = \left(\prod_{i=1}^{w} c_i\right)^N \left(\prod_{i=1}^{w} t_i\right)^N c_{\phi(i,j)} t_{\sigma(\phi(i,j))}$$

for all i, j. Therefore,

$$\phi(\psi(i), \psi(j)) = \sigma(\phi(i, j)) \tag{E8.9.2}$$

and

$$d_i d_j = \left(\prod_{i=1}^{w} c_i\right)^N c_{\phi(i,j)} \tag{E8.9.3}$$

for all i, j.

By (E8.9.2), σ is completely determined by $\psi \in S_n$. Let $\bar{d}_i = d_i\left(\prod_{i=1}^{w} c_i\right)^{-r}$. Then (E8.9.3) says that $\bar{d}_i\bar{d}_j = c_{\phi(i,j)}$. So $\prod_{i=1}^{w} c_i = \prod_{1\le i\le j\le n} \bar{d}_i\bar{d}_j$. This means the $c_{\phi(i,j)}$ and d_i are completely determined by the $\bar{d}_i$. In conclusion,

$$\operatorname{Aut}(B_{g,r}) \cong \{\psi \in S_n\} \ltimes \left\{\bar{d}_i \in k^\times \mid i = 1, \dots, n\right\} \cong S_n \ltimes (k^\times)^n.$$

In particular, every algebra automorphism of $B_{g,r}$ is a graded algebra automorphism.

Remark 8.10. As a consequence of the computation in Example 8.9, $\operatorname{Aut}(B_{g,r})$ is independent of the parameter r when $r > n$. In fact, this assertion holds for all $r > 0$, but its proof requires a different and longer analysis, so it is omitted. On the other hand, $\operatorname{Aut}(B_{g,0}) = \operatorname{Aut}(A_g)$ is very different; see Remark 3.9(3).

We will work out one more automorphism group below.

Example 8.11. We continue to study Example 8.4 and prove that every algebra automorphism of A in Example 8.4 is graded. Some unimportant details are omitted due to the length.

Claim 1: $\mathfrak{m} := A_{\ge 1}$ is the only ideal of codimension 1 satisfying $\dim \mathfrak{m}/\mathfrak{m}^2 = 4$. Suppose $I = (x_1 - a_1, x_2 - a_2, x_3 - a_3, x_4 - a_4)$ is an ideal of A of codimension 1 such that $\dim_k I/I^2 = 4$. Then the map $\pi : x_i \to a_i$ for all i extends to an algebra homomorphism $A \to k$. Applying π to the relations of A in (E8.4.1), we obtain

$$a_1a_2 = 0, \quad a_1a_3 = 0, \quad 2a_1a_4 = a_3^2, \quad a_2a_3 = 0, \quad a_3a_4 = 0, \quad a_2a_4 = 0.$$

Therefore, (a_i) is either $(a_1, 0, 0, 0)$, or $(0, a_2, 0, 0)$, or $(0, 0, 0, a_4)$. By symmetry, we consider the first case and the details of the other cases are omitted. Let $z_i = x_i - a_i$ for all i. Then the first relation of (E8.4.1) becomes

$$z_1z_2 + z_2z_1 = (x_1 - a_1)x_2 + x_2(x_1 - a_1) = -2a_1x_2 = -2a_1z_2.$$

So $2a_1z_2 \in I^2$. Since $\dim I/I^2 = 4$, we have $a_1 = 0$. Thus we have proved Claim 1.

One of the consequences of Claim 1 is that any algebra automorphism of A preserves $\mathfrak{m}$. So we have a short exact sequence

$$1 \to \operatorname{Aut}_{\mathrm{uni}}(A) \to \operatorname{Aut}(A) \to \operatorname{Aut}_{\mathrm{gr}}(A) \to 1,$$

where $\operatorname{Aut}_{\mathrm{gr}}(A)$ is the group of graded algebra automorphisms of A and $\operatorname{Aut}_{\mathrm{uni}}(A)$ is the group of unipotent algebra automorphisms of A.

Claim 2: If f is a nonzero normal element in degree 1, then $B := A/(f)$ is an Artin–Schelter regular domain of global dimension 3. By [Rogalski and Zhang 2012, Lemma 1.1], B has global dimension 3. Since A satisfies the χ-condition [Artin and Zhang 1994], so does B. As a consequence, B is AS regular of global dimension 3 [Artin and Schelter 1987]. It is well-known that every Artin–Schelter regular algebra of global dimension 3 is a domain (following by the Artin–Schelter–Tate–Van den Bergh classification [Artin and Schelter 1987; Artin et al. 1991; 1990]).

Claim 3: If $f \in A_1$ is a normal element, then $f \in kx_2$ or $f \in kx_3$. First of all, both x_2 and x_3 are normal elements by the relations (E8.4.1). Note that $x_i g = \eta_{-1}(g) x_i$ for $i = 2, 3$, where η_{-1} is the algebra automorphism of A sending x_i to $-x_i$ for all i.

Suppose that f is nonzero normal and $f \notin kx_3 \cup kx_4$. Then the image $\bar{f}$ of f is normal in $A/(x_3)$. Since $A/(x_3)$ is a skew polynomial ring, by [Kirkman et al. 2010, Lemma 3.5(d)], $\bar{f}$ is a scalar multiple of x_i for some $i = 1, 2$, or 4. This implies that f is either $ax_1 + bx_3$, or $ax_2 + bx_3$, or $ax_4 + bx_3$ for some $a, b \in k$. If $b = 0$, then $f = x_1$ or x_4. The relation $x_1x_4 + x_4x_1 = x_3^2$ implies that $A/(f)$ is not a domain (as $x_3^2 = 0$ in $A/(f)$). This contradicts Claim 2. So the only possible case is $f = x_2$ (again yielding a contradiction). Now assume that $b \neq 0$ (and $a \neq 0$ because $f \notin kx_3 \cup kx_4$). We consider the first case and the details of the other cases are similar and omitted. Since $f = ax_1 + bx_3$, the relation $x_1x_3 + x_3x_1 = 0$ implies that $x_1^2 = 0$ in $A/(f)$, which contradicts Claim 2. In all these cases, we obtain a contradiction, and therefore $f \in kx_2$ or $f \in kx_3$.

Since $A/(x_2)$ is not isomorphic to $A/(x_3)$, there is no algebra automorphism sending x_2 to x_3. As a consequence, any graded automorphism ψ of A maps $x_2 \to c_2x_2$ and $x_3 \to c_3x_3$. Let g be any graded algebra automorphism of A. Let $\bar{g}$ be the induced algebra automorphism of $A/(x_3)$. By [Kirkman et al. 2010, Lemma 3.5(e)], $\bar{g}$ sends $x_1 \to c_1x_1$ and $x_4 \to c_4x_4$, or $x_1 \to c_1x_4$ and $x_4 \to c_4x_1$. Then, by using the original relations in (E8.4.1), one can check that g is of the form

$$x_1 \to c_1x_1, \quad x_2 \to c_2x_2, \quad x_3 \to c_3x_3, \quad x_4 \to c_4x_4,$$

where $c_1c_2 = c_3^2 = c_4^2$, or

$$x_1 \to c_1x_4, \quad x_2 \to c_2x_2, \quad x_3 \to c_3x_3, \quad x_4 \to c_4x_1,$$

where $c_1c_2 = c_3^2 = c_4^2$. So

$$\mathrm{Aut_{gr}}(A) \cong \big\{(c_1, c_2, c_3, c_4) \in (k^\times)^4 \mid c_1c_2 = c_3^2 = c_4^2\big\},$$

which is completely determined.

Claim 4: $\mathrm{Aut_{uni}}(A)$ is trivial. Recall that the discriminant of A over its center is

$$d := \big(x_2^2x_3^2\big(4x_1^2x_4^2 - x_3^4\big)\big)^8.$$

By Example 8.4, the DDS subalgebra $\mathbb{D}(A)$ is the whole algebra A. The assertion follows from Theorem 0.5.

Combining all these claims, one sees that $\mathrm{Aut}(A) = \mathrm{Aut}_{\mathrm{gr}}(A)$, which is described in Claim 3.

Remark 8.12. Ideas as in Remark 8.10 also apply to Example 6.3 and a similar conclusion holds. The interested reader can fill out the details.

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