A note on secondary K-theory

Gonçalo Tabuada

A note on secondary K-theory

Gonçalo Tabuada

We prove that Toën's secondary Grothendieck ring is isomorphic to the Grothendieck ring of smooth proper pretriangulated dg categories previously introduced by Bondal, Larsen, and Lunts. Along the way, we show that those short exact sequences of dg categories in which the first term is smooth proper and the second term is proper are necessarily split. As an application, we prove that the canonical map from the derived Brauer group to the secondary Grothendieck ring has the following injectivity properties: in the case of a commutative ring of characteristic zero, it distinguishes between dg Azumaya algebras associated to nontorsion cohomology classes and dg Azumaya algebras associated to torsion cohomology classes (= ordinary Azumaya algebras); in the case of a field of characteristic zero, it is injective; in the case of a field of positive characteristic p, it restricts to an injective map on the p-primary component of the Brauer group.

1. Introduction and statement of results

A *dg category* $\mathcal{A}$, over a base commutative ring k, is a category enriched over complexes of k-modules; see Section 3. Every (dg) k-algebra A gives rise naturally to a dg category with a single object. Another source of examples is provided by schemes since the category of perfect complexes of every quasicompact quasiseparated k-scheme admits a canonical dg enhancement; see [Lunts and Orlov 2010]. Following [Kontsevich 1998], a dg category $\mathcal{A}$ is called *smooth* if it is compact as a bimodule over itself and *proper* if the complexes of k-modules $\mathcal{A}(x, y)$ are compact. Examples include the finite dimensional k-algebras of finite global dimension (when k is a perfect field) and the dg categories of perfect complexes associated to smooth proper k-schemes. Following [Bondal and Kapranov 1990], a dg category $\mathcal{A}$ is called *pretriangulated* if the Yoneda functor $\mathrm{H}^0(\mathcal{A}) \to \mathcal{D}_c(\mathcal{A})$, $x \mapsto \hat{x}$, is an equivalence of categories. As explained in Section 3, every dg category $\mathcal{A}$ admits a pretriangulated "envelope" $\mathrm{perf}_{\mathrm{dg}}(\mathcal{A})$.

Bondal, Larsen, and Lunts [Bondal et al. 2004, §5] introduced the *Grothendieck ring of smooth proper pretriangulated dg categories* $\mathcal{PT}(k)$. This ring is defined by generators and relations. The generators are the quasiequivalence classes of smooth proper pretriangulated dg categories.[1] The relations $[\mathcal{B}] = [\mathcal{A}] + [\mathcal{C}]$ arise from the dg categories $\mathcal{A}, \mathcal{C} \subseteq \mathcal{B}$ for which the triangulated subcategories $\mathrm{H}^0(\mathcal{A}), \mathrm{H}^0(\mathcal{C}) \subseteq \mathrm{H}^0(\mathcal{B})$ are admissible and induce a semiorthogonal decomposition $\mathrm{H}^0(\mathcal{B}) = \langle \mathrm{H}^0(\mathcal{A}), \mathrm{H}^0(\mathcal{C}) \rangle$. The multiplication law is given by $\mathcal{A} \bullet \mathcal{B} := \mathrm{perf}_{\mathrm{dg}}(\mathcal{A} \otimes^L \mathcal{B})$, where $- \otimes^L -$ stands for the derived tensor product of dg categories. Among other applications, Bondal, Larsen, and Lunts constructed an interesting motivic measure with values in $\mathcal{PT}(k)$.

Toën [2009; 2011, §5.4] introduced a "categorified" version of the classical Grothendieck ring named the *secondary Grothendieck ring* $K_0^{(2)}(k)$. By definition, $K_0^{(2)}(k)$ is the quotient of the free abelian group on the Morita equivalence classes of smooth proper dg categories by the relations $[\mathcal{B}] = [\mathcal{A}] + [\mathcal{C}]$ arising from short exact sequences of dg categories $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$. Thanks to [Drinfeld 2004, Proposition 1.6.3], the derived tensor product of dg categories endows $K_0^{(2)}(k)$ with a commutative ring structure. Among other applications, the ring $K_0^{(2)}(k)$ was used in the study of derived loop spaces; see [Ben-Zvi and Nadler 2012; Toën and Vezzosi 2015; 2009].

Theorem 1.1. *The rings $\mathcal{PT}(k)$ and $K_0^{(2)}(k)$ are isomorphic.*

The proof of Theorem 1.1 is based on the fact that those short exact sequences of dg categories $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ in which $\mathcal{A}$ is smooth proper and $\mathcal{B}$ is proper are necessarily split; see Theorem 4.4. This result is of independent interest. Intuitively speaking, it shows us that the smooth proper dg categories behave as "injective" objects. In the setting of triangulated categories, this idea of "injectivity" goes back to the pioneering work [Bondal and Kapranov 1989].

2. Applications

Following [Toën 2012], a dg k-algebra A is called a *dg Azumaya algebra* if the underlying complex of k-modules is a compact generator of the derived category $\mathcal{D}(k)$ and the canonical morphism $A^{\mathrm{op}} \otimes^L A \to R\,\mathrm{Hom}(A, A)$ in $\mathcal{D}(k)$ is invertible. The ordinary Azumaya algebras (see [Grothendieck 1995a]) are the dg Azumaya algebras whose underlying complex is k-flat and concentrated in degree zero. For every nontorsion étale cohomology class $\alpha \in H^2_{\mathrm{et}}(\mathrm{Spec}(k), \mathbb{G}_m)$ there exists a dg Azumaya algebra A_α, representing α, which is *not* Morita equivalent to an ordinary Azumaya algebra; see [Toën 2012, page 584]. Unfortunately, the

---

[1]Bondal, Larsen, and Lunts worked originally with pretriangulated dg categories. In this case the classical Eilenberg's swindle argument implies that the associated Grothendieck ring is trivial. In order to obtain a nontrivial Grothendieck ring, we need to restrict ourselves to smooth proper dg categories; consult [Tabuada 2005, §7] for further details.

construction of A_α is highly inexplicit; consult [Tabuada and Van den Bergh 2014, Appendix B] for some properties of these mysterious dg algebras. In the case where k is a field, every dg Azumaya algebra is Morita equivalent to an ordinary Azumaya algebra; see [Toën 2012, Proposition 2.12].

The *derived Brauer group* $\mathrm{dBr}(k)$ *of* k is the set of Morita equivalence classes of dg Azumaya algebras. The (multiplicative) group structure is induced by the derived tensor product of dg categories and the inverse of A is given by A^{op}. Since every dg Azumaya algebra is smooth proper, we have a canonical map

$$\mathrm{dBr}(k) \to K_0^{(2)}(k). \tag{2.1}$$

By analogy with the canonical map from the Picard group to the Grothendieck ring

$$\mathrm{Pic}(k) \to K_0(k), \tag{2.2}$$

it is natural to ask[2] if (2.1) is injective. Note that, in contrast with (2.2), the canonical map (2.1) does not seem to admit a "determinant" map in the converse direction. In this note, making use of Theorem 1.1 and of the recent theory of noncommutative motives (see Section 5), we establish several injectivity properties of (2.1).

Recall that k has characteristic zero (resp. positive prime characteristic p) if the kernel of the unique ring homomorphism $\mathbb{Z} \to k$ is $\{0\}$ (resp. $p\mathbb{Z}$).

Theorem 2.3. *Let k be a noetherian[3] commutative ring of characteristic zero (resp. positive prime characteristic p) and A a dg Azumaya algebra which is not Morita equivalent to an ordinary Azumaya algebra. If $K_0(k)_\mathbb{Q} \simeq \mathbb{Q}$ (resp. $K_0(k)_{\mathbb{F}_p} \simeq \mathbb{F}_p$), then the image of $[A]$ under the canonical map (2.1) is nontrivial. Moreover, when k is of characteristic zero (resp. positive prime characteristic p), this nontrivial image is different from the images of the ordinary Azumaya algebras (resp. of the ordinary Azumaya algebras whose index is not a multiple of p).*

As proved in [Gabber 1981, Theorem II.1], every torsion étale cohomology class $\alpha \in H^2_{\mathrm{et}}(\mathrm{Spec}(k), \mathbb{G}_m)_{\mathrm{tor}}$ can be represented by an ordinary Azumaya algebra A_α. Therefore, Theorem 2.3 shows us that in some cases the canonical map (2.1) distinguishes between torsion and nontorsion classes.

Example 2.4. Let k be the noetherian local ring of the singular point of the normal complex algebraic surface constructed in [Mumford 1961, page 16]. As explained in [Grothendieck 1995b, page 75], k is a local $\mathbb{C}$-algebra of dimension 2 whose étale cohomology group $H^2_{\mathrm{et}}(\mathrm{Spec}(k), \mathbb{G}_m)$ contains nontorsion classes α. Therefore,

---

[2]In the case where k is a field, Toën [2011, §5.4] asked if the canonical map (2.1) is nonzero. This now follows automatically from Theorems 2.5 and 2.7.

[3]As pointed out by the anonymous referee, this assumption can be removed using absolute noetherian approximation. We leave the details to the reader.

since $K_0(k) \simeq \mathbb{Z}$, Theorem 2.3 can be applied to the associated dg Azumaya algebras A_α.

Theorem 2.5. *Let k be a field of characteristic zero. In this case, the canonical map* (2.1) *is injective.*

Example 2.6. (i) When k is the field of real numbers $\mathbb{R}$, we have $\mathrm{Br}(\mathbb{R}) \simeq \mathbb{Z}/2\mathbb{Z}$.

(ii) When k is the field of p-adic numbers $\mathbb{Q}_p$, we have $\mathrm{Br}(\mathbb{Q}_p) \simeq \mathbb{Q}/\mathbb{Z}$.

Theorem 2.7. *Let k be a field of positive characteristic p and A, B two central simple k-algebras. If $p \mid \mathrm{ind}(A^{\mathrm{op}} \otimes B)$, where* ind *stands for index, then the images of $[A]$ and $[B]$ under the canonical map* (2.1) *are different. This holds in particular when* $\mathrm{ind}(A)$ *and* $\mathrm{ind}(B)$ *are coprime[4] and p divides $\mathrm{ind}(A)$ or $\mathrm{ind}(B)$.*

Corollary 2.8. *When k is a field of positive characteristic p, the restriction of the canonical map* (2.1) *to the p-primary torsion subgroup $\mathrm{Br}(k)\{p\}$ is injective. Moreover, the image of $\mathrm{Br}(k)\{p\} - 0$ is disjoint from the image of $\bigoplus_{q \neq p} \mathrm{Br}(k)\{q\}$.*

Proof. The index and the period of a central simple algebra have the same prime factors. Therefore, the proof of the first claim follows from the divisibility relation $\mathrm{ind}(A^{\mathrm{op}} \otimes B) \mid \mathrm{ind}(A) \cdot \mathrm{ind}(B)$. The proof of the second claim is now clear. $\square$

Example 2.9. Let k be a field of characteristic $p > 0$. Given a character χ and an invertible element $b \in k^\times$, the associated cyclic algebra (χ, b) belongs to the p-primary torsion subgroup $\mathrm{Br}(k)\{p\}$. Moreover, thanks to the work of Albert (see [Gille and Szamuely 2006, Theorem 9.1.8]), every element of $\mathrm{Br}(k)\{p\}$ is of this form. Making use of Corollary 2.8, we hence conclude that the canonical map (2.1) distinguishes all these cyclic algebras. Furthermore, the image of $\mathrm{Br}(k)\{p\} - 0$ in the secondary Grothendieck ring $K_0^{(2)}(k)$ is disjoint from the image of $\bigoplus_{q \neq p} \mathrm{Br}(k)\{q\}$.

Every ring homomorphism $k \to k'$ gives rise to the following commutative square:

$$
\begin{array}{ccc}
\mathrm{dBr}(k) & \xrightarrow{\ (2.1)\ } & K_0^{(2)}(k) \\
{\scriptstyle -\otimes_k^L k'} \downarrow & & \downarrow {\scriptstyle -\otimes_k^L k'} \\
\mathrm{dBr}(k') & \xrightarrow[\ (2.1)\ ]{} & K_0^{(2)}(k')
\end{array}
$$

By combining it with Theorems 2.5 and 2.7, we hence obtain the following result:

Corollary 2.10. *Let A and B be dg Azumaya k-algebras. If there exists a ring homomorphism $k \to k'$, with k' a field of characteristic zero (resp. positive characteristic p) such that $[A \otimes_k^L k'] \neq [B \otimes_k^L k']$ in $\mathrm{Br}(k')$ (resp. $p \mid \mathrm{ind}((A^{\mathrm{op}} \otimes^L B) \otimes_k^L k')$), then the images of $[A]$ and $[B]$ under the canonical map* (2.1) *are different.*

---

[4]When $\mathrm{ind}(A)$ and $\mathrm{ind}(B)$ are coprime we have $\mathrm{ind}(A^{\mathrm{op}} \otimes B) = \mathrm{ind}(A) \cdot \mathrm{ind}(B)$.

Example 2.11 (local rings). Let k be a complete local ring with residue field k' of characteristic zero (resp. positive characteristic p). As proved in [Auslander and Goldman 1961, Theorem 6.5], the assignment $A \mapsto A \otimes_k^L k'$ gives rise to a group isomorphism $\mathrm{Br}(k) \simeq \mathrm{Br}(k')$. Therefore, by combining Corollary 2.10 with Theorem 2.5 (resp. Corollary 2.8), we conclude that the restriction of the canonical map (2.1) to the subgroup $\mathrm{Br}(k) \subset \mathrm{dBr}(k)$ (resp. $\mathrm{Br}(k)\{p\} \subset \mathrm{dBr}(k)$) is injective.

Example 2.12 (domains). Let k be a regular noetherian domain of characteristic zero (resp. positive prime characteristic p) with field of fractions k'. Since $H_{\mathrm{et}}^1(\mathrm{Spec}(k), \mathbb{Z}) = 0$ and all étale cohomology classes of $H_{\mathrm{et}}^2(\mathrm{Spec}(k), \mathbb{G}_m)$ are torsion (see [Grothendieck 1995b, Proposition 1.4]), Theorem II.1 of [Gabber 1981] implies that the derived Brauer group $\mathrm{dBr}(k) \simeq H_{\mathrm{et}}^1(\mathrm{Spec}(k), \mathbb{Z}) \times H_{\mathrm{et}}^2(\mathrm{Spec}(k), \mathbb{G}_m)$ agrees with $\mathrm{Br}(k)$. As proved in [Auslander and Goldman 1961, Theorem 7.2], the assignment $A \mapsto A \otimes_k^L k'$ gives rise to an injective group homomorphism $\mathrm{Br}(k) \to \mathrm{Br}(k')$. Therefore, by combining Corollary 2.10 with Theorem 2.5 (resp. Corollary 2.8), we conclude that the canonical map (2.1) (resp. the restriction of (2.1) to $\mathrm{Br}(k)\{p\} \subset \mathrm{Br}(k)$) is injective.

Example 2.13 (Weyl algebras). Let F be a field of positive characteristic p. Thanks to [Revoy 1973], the classical Weyl algebra $W_n(F)$, $n \geq 1$, defined as the quotient of $F\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$ by the relations $[\partial_i, x_j] = \delta_{ij}$, can be considered as an (ordinary) Azumaya algebra over the ring of polynomials $k := F[x_1^p, \ldots, x_n^p, \partial_1^p, \ldots, \partial_n^p]$, $n \geq 1$. Consider the composition

$$k := F[x_1^p, \ldots, x_n^p, \partial_1^p, \ldots, \partial_n^p] \to F[x_1^p, \partial_1^p] \to \mathrm{Frac}(F[x_1^p, \partial_1^p]) =: k',$$

where the first homomorphism sends x_i^p, ∂_i^p, $i > 1$, to zero and $\mathrm{Frac}(F[x_1^p, \partial_1^p])$ denotes the field of fractions of the integral domain $F[x_1^p, \partial_1^p]$. As explained in [Wodzicki 2011, §4], we have $\mathrm{ind}(W_n(F) \otimes_k^L k') = p$. Therefore, thanks to Corollary 2.10, we conclude that the image of $W_n(F)$ under (2.1) is nontrivial.

Example 2.14 (algebras of p-symbols). Let F be a field of positive characteristic p, $k := F[x_1^p, \partial_1^p]$ the algebra of polynomials, and $k' := \mathrm{Frac}(F[x_1^p, \partial_1^p])$ the field of fractions. Following [Wodzicki 2011, §1], given elements $a, b \in k$, let us denote by $\mathcal{S}_{ab}(k) \in {}_p\mathrm{Br}(k)$ the associated (ordinary) Azumaya k-algebra of p-symbols. For example, when $a = x_1^p$ and $b = \partial_1^p$, we have $\mathcal{S}_{ab}(k) = W_1(F)$. As proved in [Wodzicki 2011, §6], we have $\mathrm{ind}(\mathcal{S}_{ab}(k) \otimes_k^L k') = p$ if and only if

$$b \neq c_0^p + c_1^p a + \cdots + c_{p-1}^p a^{p-1} - c_{p-1}$$
$$\text{for every } c_0 + c_1 t + \cdots + c_{p-1} t^{p-1} \in k'[t]. \quad (2.15)$$

Therefore, thanks to Corollary 2.10, we conclude that whenever a and b satisfy condition (2.15) the image of $\mathcal{S}_{ab}(k)$ under the canonical map (2.1) is nontrivial.

Remark 2.16 (stronger results). As explained in Section 6, Theorems 2.3, 2.5 and 2.7, and Corollaries 2.8 and 2.10, follow from stronger analogous results where instead of $K_0^{(2)}(k)$ we consider the Grothendieck ring of the category of noncommutative Chow motives; consult Theorems 6.5, 6.12 and 6.20, Corollary 6.22, and Remark 6.23.

3. Background on dg categories

Let $(\mathcal{C}(k), \otimes, k)$ be the symmetric monoidal category of cochain complexes of k-modules. A *dg category* $\mathcal{A}$ is a category enriched over $\mathcal{C}(k)$ and a *dg functor* $F : \mathcal{A} \to \mathcal{B}$ is a functor enriched over $\mathcal{C}(k)$; consult Keller's ICM survey [2006]. Let us denote by $\mathrm{dgcat}(k)$ the category of (small) dg categories and dg functors.

Let $\mathcal{A}$ be a dg category. The opposite dg category $\mathcal{A}^{\mathrm{op}}$ has the same objects and is defined as $\mathcal{A}^{\mathrm{op}}(x, y) := \mathcal{A}(y, x)$. The category $\mathrm{H}^0(\mathcal{A})$ has the same objects as $\mathcal{A}$ and morphisms $\mathrm{H}^0(\mathcal{A})(x, y) := H^0(\mathcal{A}(x, y))$, where $H^0(-)$ stands for the zeroth cohomology.

A *right dg $\mathcal{A}$-module* is a dg functor $M : \mathcal{A}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{dg}}(k)$ with values in the dg category $\mathcal{C}_{\mathrm{dg}}(k)$ of complexes of k-modules. Given $x \in \mathcal{A}$, let us write $\hat{x}$ for the Yoneda right dg $\mathcal{A}$-module defined by $y \mapsto \mathcal{A}(x, y)$. Let $\mathcal{C}(\mathcal{A})$ be the category of right dg $\mathcal{A}$-modules. As explained in [Keller 2006, §3.2], $\mathcal{C}(\mathcal{A})$ carries a Quillen model structure whose weak equivalences (resp. fibrations) are the objectwise quasi-isomorphisms (resp. surjections). The *derived category* $\mathcal{D}(\mathcal{A})$ *of* $\mathcal{A}$ is the associated homotopy category. Let $\mathcal{D}_c(\mathcal{A})$ be the full triangulated subcategory of compact objects. The dg structure of $\mathcal{C}_{\mathrm{dg}}(k)$ makes $\mathcal{C}(\mathcal{A})$ naturally into a dg category $\mathcal{C}_{\mathrm{dg}}(\mathcal{A})$. Let us write $\mathrm{perf}_{\mathrm{dg}}(\mathcal{A})$ for the full dg subcategory of $\mathcal{C}_{\mathrm{dg}}(\mathcal{A})$ consisting of those cofibrant right dg $\mathcal{A}$-modules which belong to $\mathcal{D}_c(\mathcal{A})$. Note that we have the Yoneda dg functor $\mathcal{A} \to \mathrm{perf}_{\mathrm{dg}}(\mathcal{A}) \subset \mathcal{C}_{\mathrm{dg}}(\mathcal{A})$, $x \mapsto \hat{x}$, and that $\mathrm{H}^0(\mathrm{perf}_{\mathrm{dg}}(\mathcal{A})) \simeq \mathcal{D}_c(\mathcal{A})$.

A dg functor $F : \mathcal{A} \to \mathcal{B}$ is called a *quasiequivalence* if the morphisms of k-modules $F(x, y) : \mathcal{A}(x, y) \to \mathcal{B}(F(x), F(y))$ are quasi-isomorphisms and the induced functor $\mathrm{H}^0(F) : \mathrm{H}^0(\mathcal{A}) \to \mathrm{H}^0(\mathcal{B})$ is an equivalence of categories. More generally, F is called a *Morita equivalence* if it induces an equivalence of derived categories $\mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$; see [Keller 2006, §4.6]. As proved in [Tabuada 2005, Theorem 5.3], $\mathrm{dgcat}(k)$ carries a Quillen model structure whose weak equivalences are the Morita equivalences. Let us denote by $\mathrm{Hmo}(k)$ the associated homotopy category.

The tensor product $\mathcal{A} \otimes \mathcal{B}$ of two dg categories $\mathcal{A}$ and $\mathcal{B}$ is defined as follows: the set of objects is the cartesian product and $(\mathcal{A} \otimes \mathcal{B})((x, w), (y, z)) := \mathcal{A}(x, y) \otimes \mathcal{B}(w, z)$. As explained in [Keller 2006, §4.3], this construction can be derived — and denoted by $\otimes^L$ — thus giving rise to a symmetric monoidal structure on $\mathrm{Hmo}(k)$ with $\otimes$-unit the dg category k.

4. Proof of Theorem 1.1

The smooth proper dg categories can be characterized as the dualizable objects of the symmetric monoidal category Hmo(k); see [Cisinski and Tabuada 2012, §5]. Consequently, Kontsevich's notions of smoothness and properness are invariant under Morita equivalence.

Recall from [Keller 2006, §4.6] that a short exact sequence of dg categories is a sequence of morphisms $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ in the homotopy category Hmo(k) inducing an exact sequence of triangulated categories $0 \to \mathcal{D}_c(\mathcal{A}) \to \mathcal{D}_c(\mathcal{B}) \to \mathcal{D}_c(\mathcal{C}) \to 0$ in the sense of Verdier. As proved in [Tabuada 2008, Lemma 10.3], the morphism $\mathcal{A} \to \mathcal{B}$ is isomorphic to an inclusion of dg categories $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{C}$ identifies with Drinfeld's dg quotient $\mathcal{B}/\mathcal{A}$.

Definition 4.1. A short exact sequence of dg categories $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ is called *split* if the triangulated subcategory $\mathcal{D}_c(\mathcal{A}) \subseteq \mathcal{D}_c(\mathcal{B})$ is admissible.

Remark 4.2. In the case of a split short exact sequence of dg categories we have an induced equivalence between $\mathcal{D}_c(\mathcal{C})$ and the right orthogonal $\mathcal{D}_c(\mathcal{A})^{\perp} \subseteq \mathcal{D}_c(\mathcal{B})$. Consequently, we obtain a semiorthogonal decomposition $\mathcal{D}_c(\mathcal{B}) = \langle \mathcal{D}_c(\mathcal{A}), \mathcal{D}_c(\mathcal{C}) \rangle$.

Let us write $K_0^{(2)}(k)^s$ for the ring defined similarly to $K_0^{(2)}(k)$ but with *split* short exact sequences of dg categories instead of short exact sequences of dg categories.

Proposition 4.3. *The rings $\mathcal{PT}(k)$ and $K_0^{(2)}(k)^s$ are isomorphic.*

Proof. The assignment $\mathcal{A} \mapsto \mathcal{A}$ clearly sends quasiequivalence classes of smooth proper pretriangulated dg categories to Morita equivalence classes of smooth proper dg categories. Let $\mathcal{A}, \mathcal{C} \subseteq \mathcal{B}$ be smooth proper pretriangulated dg categories for which the triangulated subcategories $\mathrm{H}^0(\mathcal{A}), \mathrm{H}^0(\mathcal{C}) \subseteq \mathrm{H}^0(\mathcal{B})$ are admissible and induce a semiorthogonal decomposition $\mathrm{H}^0(\mathcal{B}) = \langle \mathrm{H}^0(\mathcal{A}), \mathrm{H}^0(\mathcal{C}) \rangle$. Consider the full dg subcategory $\mathcal{B}'$ of $\mathcal{B}$ consisting of those objects which belong to $\mathrm{H}^0(\mathcal{A})$ or to $\mathrm{H}^0(\mathcal{C})$. Thanks to the preceding semiorthogonal decomposition, the inclusion dg functor $\mathcal{B}' \subseteq \mathcal{B}$ is a Morita equivalence. Consider also the dg functor $\pi : \mathcal{B}' \to \mathcal{C}$ which is the identity on $\mathcal{C}$ and which sends all the remaining objects to a fixed zero object 0 of $\mathcal{C}$. Under this notation, we have the split short exact sequence of dg categories $0 \to \mathcal{A} \hookrightarrow \mathcal{B}' \xrightarrow{\pi} \mathcal{C} \to 0$. We hence conclude that the assignment $\mathcal{A} \mapsto \mathcal{A}$ gives rise to a group homomorphism $\mathcal{PT}(k) \to K_0^{(2)}(k)^s$.

As explained in [Tabuada 2005, §5], the pretriangulated dg categories can be (conceptually) characterized as the fibrant objects of the Quillen model structure on dgcat(k); see Section 3. Moreover, given a dg category $\mathcal{A}$, the Yoneda dg functor $\mathcal{A} \to \mathrm{perf}_{\mathrm{dg}}(\mathcal{A})$, $x \mapsto \hat{x}$, is a fibrant resolution. This implies that $\mathcal{PT}(k) \to K_0^{(2)}(k)^s$ is moreover a surjective ring homomorphism. It remains then only to show its injectivity. Given a split short exact sequence of smooth proper dg categories $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$, which we can assume pretriangulated, recall from Remark 4.2

that we have an associated semiorthogonal decomposition $H^0(\mathcal{B}) = \langle H^0(\mathcal{A}), H^0(\mathcal{C}) \rangle$. Let us write $\mathcal{C}'$ for the full dg subcategory of $\mathcal{B}$ consisting of those objects which belong to $H^0(\mathcal{C})$. Note that $\mathcal{C}'$ is pretriangulated and quasiequivalent to $\mathcal{C}$. Note also that since the triangulated subcategory $H^0(\mathcal{A}) \subseteq H^0(\mathcal{B})$ is admissible, the triangulated subcategory $H^0(\mathcal{C}') \simeq H^0(\mathcal{A})^\perp \subseteq H^0(\mathcal{B})$ is also admissible. We hence conclude that the relation $[\mathcal{B}] = [\mathcal{A}] + [\mathcal{C}] \Leftrightarrow [\mathcal{B}] = [\mathcal{A}] + [\mathcal{C}']$ holds in $\mathcal{PT}(k)$, and consequently that the surjective ring homomorphism $\mathcal{PT}(k) \to K_0^{(2)}(k)^s$ is moreover injective. $\square$

Thanks to Proposition 4.3, the proof of Theorem 1.1 now follows automatically from the following result of independent interest:

Theorem 4.4. *Let* $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ *be a short exact sequence of dg categories. If* $\mathcal{A}$ *is smooth proper and* $\mathcal{B}$ *is proper, then the sequence is split.*

Proof. Without loss of generality, we can assume that the dg categories $\mathcal{A}$ and $\mathcal{B}$ are pretriangulated. Let us prove first that the triangulated subcategory $H^0(\mathcal{A}) \subseteq H^0(\mathcal{B})$ is right admissible, i.e., that the inclusion functor admits a right adjoint. Given an object $z \in \mathcal{B}$, consider the composition

$$H^0(\mathcal{A})^{\mathrm{op}} \xrightarrow{H^0(\mathcal{B}(-,z))} H^0(\mathrm{perf}_{\mathrm{dg}}(k)) \simeq \mathcal{D}_c(k) \xrightarrow{H^0(-)} \mathrm{mod}(k) \qquad (4.5)$$

with values in the category of finitely generated k-modules. Thanks to Proposition 4.8 (with $F = \mathcal{B}(-, z)$), the functor (4.5) is representable. Let us denote by x the representing object. Since the composition (4.5) is naturally isomorphic to the (contravariant) functor $\mathrm{Hom}_{H^0(\mathcal{B})}(-, z) : H^0(\mathcal{A})^{\mathrm{op}} \to \mathrm{mod}(k)$, we have $\mathrm{Hom}_{H^0(\mathcal{A})}(y, x) \simeq \mathrm{Hom}_{H^0(\mathcal{B})}(y, z)$ for every $y \in \mathcal{A}$. By taking $y = x$, we hence obtain a canonical morphism $\eta : x \to z$ and consequently a distinguished triangle $x \xrightarrow{\eta} z \to \mathrm{cone}(\eta) \to \Sigma(x)$ in the triangulated category $H^0(\mathcal{B})$. The associated long exact sequences allow us then to conclude that $\mathrm{cone}(\eta)$ belongs to the right orthogonal $H^0(\mathcal{A})^\perp \subseteq H^0(\mathcal{B})$. This implies that the triangulated subcategory $H^0(\mathcal{A}) \subseteq H^0(\mathcal{B})$ is right admissible. The proof of left admissibility is similar: simply replace $\mathcal{B}(-, z)$ by the covariant dg functor $\mathcal{B}(z, -)$; see Remark 4.9. $\square$

Notation 4.6 (bimodules). Let $\mathcal{A}$ and $\mathcal{B}$ be two dg categories. A *dg $\mathcal{A}$-$\mathcal{B}$-bimodule* is a dg functor $\mathrm{B} : \mathcal{A} \otimes^{\mathbf{L}} \mathcal{B}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{dg}}(k)$, i.e., a right dg $(\mathcal{A}^{\mathrm{op}} \otimes^{\mathbf{L}} \mathcal{B})$-module. Associated to a dg functor $F : \mathcal{A} \to \mathcal{B}$, we have the dg $\mathcal{A}$-$\mathcal{B}$-bimodule

$$_F\mathrm{B} : \mathcal{A} \otimes^{\mathbf{L}} \mathcal{B}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{dg}}(k), \quad (x, z) \mapsto \mathcal{B}(z, F(x)). \qquad (4.7)$$

Let us write $\mathrm{rep}(\mathcal{A}, \mathcal{B})$ for the full triangulated subcategory of $\mathcal{D}(\mathcal{A}^{\mathrm{op}} \otimes^{\mathbf{L}} \mathcal{B})$ consisting of those dg $\mathcal{A}$-$\mathcal{B}$-bimodules B such that for every object $x \in \mathcal{A}$ the associated right dg $\mathcal{B}$-module $\mathrm{B}(x, -)$ belongs to $\mathcal{D}_c(\mathcal{B})$. Similarly, let $\mathrm{rep}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})$ be the full dg subcategory of $\mathcal{C}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{op}} \otimes^{\mathbf{L}} \mathcal{B})$ consisting of those cofibrant right dg $\mathcal{A}$-$\mathcal{B}$-bimodules which belong to $\mathrm{rep}(\mathcal{A}, \mathcal{B})$. By construction, $H^0(\mathrm{rep}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})) \simeq \mathrm{rep}(\mathcal{A}, \mathcal{B})$.

Proposition 4.8 (representability). *Let $\mathcal{A}$ be a smooth proper pretriangulated dg category and $G : \mathrm{H}^0(\mathcal{A})^{\mathrm{op}} \to \mathrm{mod}(k)$ be a (contravariant) functor with values in the category of finitely generated k-modules. Assume that there exists a dg functor $F : \mathcal{A}^{\mathrm{op}} \to \mathrm{perf}_{\mathrm{dg}}(k)$ and a natural isomorphism between G and the composition*

$$\mathrm{H}^0(\mathcal{A})^{\mathrm{op}} \xrightarrow{\mathrm{H}^0(F)} \mathrm{H}^0(\mathrm{perf}_{\mathrm{dg}}(k)) \simeq \mathcal{D}_c(k) \xrightarrow{\mathrm{H}^0(-)} \mathrm{mod}(k).$$

Under these assumptions, the functor G is representable.

Remark 4.9. Given a smooth proper pretriangulated dg category $\mathcal{A}$, the opposite dg category $\mathcal{A}^{\mathrm{op}}$ is also smooth, proper, and pretriangulated. Therefore, Proposition 4.8 (with $\mathcal{A}$ replaced by $\mathcal{A}^{\mathrm{op}}$) is also a corepresentability result.

Proof. Following Notation 4.6, let $_F\mathrm{B} \in \mathrm{rep}(\mathcal{A}^{\mathrm{op}}, \mathrm{perf}_{\mathrm{dg}}(k))$ be the dg $\mathcal{A}^{\mathrm{op}}$-$\mathrm{perf}_{\mathrm{dg}}(k)$-bimodule associated to the dg functor F. Thanks to Lemma 4.11 below, there exists an object $x \in \mathcal{A}$ and an isomorphism in the triangulated category $\mathrm{rep}(\mathcal{A}^{\mathrm{op}}, \mathrm{perf}_{\mathrm{dg}}(k))$ between the dg $\mathcal{A}^{\mathrm{op}}$-$\mathrm{perf}_{\mathrm{dg}}(k)$-bimodules $_F\mathrm{B}$ and $_{\hat{x}}\mathrm{B}$. Making use of the functor

$$\mathrm{rep}(\mathcal{A}^{\mathrm{op}}, \mathrm{perf}_{\mathrm{dg}}(k)) \to \mathrm{Fun}_\Delta(\mathrm{H}^0(\mathcal{A})^{\mathrm{op}}, \mathcal{D}_c(k)), \quad \mathrm{B} \mapsto - \otimes^{\boldsymbol{L}}_{\mathcal{A}^{\mathrm{op}}} \mathrm{B}, \qquad (4.10)$$

where $\mathrm{Fun}_\Delta(-, -)$ stands for the category of triangulated functors, we obtain an isomorphism between the functors $- \otimes^{\boldsymbol{L}}_{\mathcal{A}^{\mathrm{op}}} {}_F\mathrm{B} \simeq \mathrm{H}^0(F)$ and $- \otimes^{\boldsymbol{L}}_{\mathcal{A}^{\mathrm{op}}} {}_{\hat{x}}\mathrm{B} \simeq \mathrm{H}^0(\hat{x})$. By composing them with $H^0(-) : \mathcal{D}_c(k) \to \mathrm{mod}(k)$, we hence conclude that G is naturally isomorphic to the representable functor $\mathrm{Hom}_{\mathrm{H}^0(\mathcal{A})}(-, x)$. $\qquad\square$

Lemma 4.11. *Given a smooth proper pretriangulated dg category $\mathcal{A}$, the dg functor*

$$\mathcal{A} \to \mathrm{rep}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{op}}, \mathrm{perf}_{\mathrm{dg}}(k)), \quad x \mapsto {}_{\hat{x}}\mathrm{B} \qquad (4.12)$$

is a quasiequivalence.

Proof. As proved in [Cisinski and Tabuada 2012, Theorem 5.8], the dualizable objects of the symmetric monoidal category $\mathrm{Hmo}(k)$ are the smooth proper dg categories. Moreover, the dual of a smooth proper dg category $\mathcal{A}$ is the opposite dg category $\mathcal{A}^{\mathrm{op}}$ and the evaluation morphism is given by the dg functor

$$\mathcal{A} \otimes^{\boldsymbol{L}} \mathcal{A}^{\mathrm{op}} \to \mathrm{perf}_{\mathrm{dg}}(k), \quad (x, y) \mapsto \mathcal{A}(y, x). \qquad (4.13)$$

The symmetric monoidal category $\mathrm{Hmo}(k)$ is closed; see [Keller 2006, §4.3]. Given dg categories $\mathcal{A}$ and $\mathcal{B}$, their internal Hom is given by $\mathrm{rep}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})$. Therefore, by adjunction, (4.13) corresponds to the dg functor (4.12). Thanks to the unicity of dualizable objects, we hence conclude that (4.12) is a Morita equivalence. The proof now follows from the fact that a Morita equivalence between pretriangulated dg categories is necessarily a quasiequivalence; see [Tabuada 2005, §5]. $\qquad\square$

Alternative proof of Theorem 4.4 when k is a field. Without loss of generality, we can assume that the dg categories $\mathcal{A}$ and $\mathcal{B}$ are pretriangulated. Note first that $\mathcal{B}$ is proper if and only if we have $\sum_n \dim \mathrm{Hom}_{\mathrm{H}^0(\mathcal{B})}(w, z[n]) < \infty$ for any two objects w and z. Since the dg category $\mathcal{A}$ is smooth, the triangulated category $\mathrm{H}^0(\mathcal{A})$ admits a strong generator in the sense of [Bondal and van den Bergh 2003]; see [Lunts 2010, Lemmas 3.5 and 3.6]. Using the fact that the (contravariant) functor (4.5) is cohomological and that the triangulated category $\mathrm{H}^0(\mathcal{A})$ is idempotent complete, we hence conclude from Bondal and Van den Bergh's powerful (co)representability result [2003, Theorem 1.3] that the functor $\mathrm{Hom}_{\mathrm{H}^0(\mathcal{B})}(-, z) : \mathrm{H}^0(\mathcal{A})^{\mathrm{op}} \to \mathrm{vect}(k)$ is representable. The remainder of the proof is now similar.

Remark 4.14 (Orlov's regularity). Following [Orlov 2015, Definition 3.13], a dg category $\mathcal{A}$ is called *regular* if the triangulated category $\mathcal{D}_c(\mathcal{A})$ admits a strong generator in the sense of [Bondal and van den Bergh 2003]. Examples include the dg categories of perfect complexes associated to regular separated noetherian k-schemes. Smoothness implies regularity but the converse does not hold. The preceding proof shows us that Theorem 4.4 holds more generally when $\mathcal{A}$ is a regular proper dg category.

5. Noncommutative motives

For a survey or book on noncommutative motives, we invite the reader to consult [Tabuada 2012] or [Tabuada 2015], respectively. Let $\mathcal{A}$ and $\mathcal{B}$ be two dg categories. As proved in [Tabuada 2005, Corollary 5.10], we have an identification between $\mathrm{Hom}_{\mathrm{Hmo}(k)}(\mathcal{A}, \mathcal{B})$ and the isomorphism classes of the category $\mathrm{rep}(\mathcal{A}, \mathcal{B})$, under which the composition law of $\mathrm{Hmo}(k)$ corresponds to the derived tensor product of bimodules. Since the dg $\mathcal{A}$-$\mathcal{B}$-bimodules (4.7) belong to $\mathrm{rep}(\mathcal{A}, \mathcal{B})$, we hence obtain a symmetric monoidal functor

$$\mathrm{dgcat}(k) \to \mathrm{Hmo}(k), \quad \mathcal{A} \mapsto \mathcal{A}, \quad F \mapsto {}_F\mathrm{B}. \tag{5.1}$$

The *additivization* of $\mathrm{Hmo}(k)$ is the additive category $\mathrm{Hmo}_0(k)$ with the same objects as $\mathrm{Hmo}(k)$ and with morphisms given by $\mathrm{Hom}_{\mathrm{Hmo}_0(k)}(\mathcal{A}, \mathcal{B}) := K_0 \mathrm{rep}(\mathcal{A}, \mathcal{B})$, where $K_0 \mathrm{rep}(\mathcal{A}, \mathcal{B})$ stands for the Grothendieck group of the triangulated category $\mathrm{rep}(\mathcal{A}, \mathcal{B})$. The composition law is induced by the derived tensor product of bimodules and the symmetric monoidal structure extends by bilinearity from $\mathrm{Hmo}(k)$ to $\mathrm{Hmo}_0(k)$. Note that we have a symmetric monoidal functor

$$\mathrm{Hmo}(k) \to \mathrm{Hmo}_0(k), \quad \mathcal{A} \mapsto \mathcal{A}, \quad \mathrm{B} \mapsto [\mathrm{B}]. \tag{5.2}$$

Given a commutative ring of coefficients R, the *R-linearization* of $\mathrm{Hmo}_0(k)$ is the R-linear category $\mathrm{Hmo}_0(k)_R$ obtained by tensoring the morphisms of $\mathrm{Hmo}_0(k)$ with R. Note that $\mathrm{Hmo}_0(k)_R$ inherits an R-linear symmetric monoidal structure

and that we have the symmetric monoidal functor

$$\mathrm{Hmo}_0(k) \to \mathrm{Hmo}_0(k)_R, \quad \mathcal{A} \mapsto \mathcal{A}, \quad [B] \mapsto [B]_R. \tag{5.3}$$

Let us denote by $U(-)_R : \mathrm{dgcat}(k) \to \mathrm{Hmo}_0(k)_R$ the composition (5.3)∘(5.2)∘(5.1).

Noncommutative Chow motives. The category of *noncommutative Chow motives* $\mathrm{NChow}(k)_R$ is defined as the idempotent completion of the full subcategory of $\mathrm{Hmo}_0(k)_R$ consisting of the objects $U(\mathcal{A})_R$ with $\mathcal{A}$ a smooth proper dg category. This category is not only R-linear and idempotent complete, but also additive and rigid[5] symmetric monoidal; see [Tabuada 2012, §4]. Given dg categories $\mathcal{A}$ and $\mathcal{B}$, with $\mathcal{A}$ smooth proper, we have $\mathrm{rep}(\mathcal{A}, \mathcal{B}) \simeq \mathcal{D}_c(\mathcal{A}^{\mathrm{op}} \otimes^L \mathcal{B})$. Hence, we obtain isomorphisms

$$\mathrm{Hom}_{\mathrm{NChow}(k)_R}(U(\mathcal{A})_R, U(\mathcal{B})_R) := K_0(\mathrm{rep}(\mathcal{A}, \mathcal{B}))_R \simeq K_0(\mathcal{A}^{\mathrm{op}} \otimes^L \mathcal{B})_R.$$

When $R = \mathbb{Z}$, we write $\mathrm{NChow}(k)$ instead of $\mathrm{NChow}(k)_\mathbb{Z}$ and U instead of $U(-)_\mathbb{Z}$.

Noncommutative numerical motives. Given an R-linear, additive, rigid symmetric monoidal category $\mathcal{C}$, its $\mathcal{N}$-*ideal* is defined as

$$\mathcal{N}(a, b) := \{f \in \mathrm{Hom}_\mathcal{C}(a, b) \mid \text{for all } g \in \mathrm{Hom}_\mathcal{C}(b, a) \text{ we have } \mathrm{tr}(g \circ f) = 0\},$$

where $\mathrm{tr}(g \circ f)$ stands for the categorical trace of the endomorphism $g \circ f$. The category of *noncommutative numerical motives* $\mathrm{NNum}(k)_R$ is defined as the idempotent completion of the quotient of $\mathrm{NChow}(k)_R$ by the $\otimes$-ideal $\mathcal{N}$. By construction, this category is R-linear, additive, rigid symmetric monoidal, and idempotent complete.

Notation 5.4. In the case where k is a field, we write $\mathrm{CSA}(k)_R$ for the full subcategory of $\mathrm{NNum}(k)_R$ consisting of the objects $U(A)_R$ with A a central simple k-algebra, and $\mathrm{CSA}(k)_R^\oplus$ for the closure of $\mathrm{CSA}(k)_R$ under finite direct sums.

The next result is a slight variant of [Marcolli and Tabuada 2014b, Theorem 1.10].

Theorem 5.5 (semisimplicity). *Let k be a commutative ring of characteristic zero (resp. positive prime characteristic p) and R a field with the same characteristic. If $K_0(k)_\mathbb{Q} \simeq \mathbb{Q}$ (resp. $K_0(k)_{\mathbb{F}_p} \simeq \mathbb{F}_p$), then $\mathrm{NNum}(k)_R$ is abelian semisimple.*

Proof. As explained in [Cisinski and Tabuada 2012, Example 8.9], Hochschild homology gives rise to an additive symmetric monoidal functor $HH : \mathrm{Hmo}_0(k) \to \mathcal{D}(k)$. The dualizable objects of the derived category $\mathcal{D}(k)$ are the compact ones. Therefore, since the symmetric monoidal subcategory $\mathrm{NChow}(k)$ of $\mathrm{Hmo}_0(k)$ is rigid and every symmetric monoidal functor preserves dualizable objects, the preceding functor restricts to an additive symmetric monoidal functor

$$\mathrm{NChow}(k) \to \mathcal{D}_c(k), \quad U(\mathcal{A}) \mapsto HH(\mathcal{A}). \tag{5.6}$$

---

[5]Recall that a symmetric monoidal category is called *rigid* if all its objects are dualizable.

Assume first that k is of characteristic zero. Let us write $k_\mathbb{Q}$ for the localization of k at the multiplicative set $\mathbb{Z} \setminus \{0\}$. Choose a maximal ideal $\mathfrak{m}$ of $k_\mathbb{Q}$ and consider the associated residue field $F := k_\mathbb{Q}/\mathfrak{m}$. By composing (5.6) with the base-change (derived) symmetric monoidal functors $\mathcal{D}_c(k) \to \mathcal{D}_c(k_\mathbb{Q}) \to \mathcal{D}_c(F)$, we hence obtain an induced $\mathbb{Q}$-linear symmetric monoidal functor

$$\mathrm{NChow}(k)_\mathbb{Q} \to \mathcal{D}_c(F), \quad U(\mathcal{A})_\mathbb{Q} \mapsto HH(\mathcal{A}) \otimes_k^L F. \tag{5.7}$$

Since by assumption we have $\mathrm{End}_{\mathrm{NChow}(k)_\mathbb{Q}}(U(k)_\mathbb{Q}) = K_0(k)_\mathbb{Q} \simeq \mathbb{Q}$, we conclude from André and Kahn's general results [2005, Theorem 1a; 2002, Theorem A.2.10], applied to the functor (5.7), that the category $\mathrm{NNum}(k)_\mathbb{Q}$ is abelian semisimple. The proof now follows from the fact that the $\otimes$-ideal $\mathcal{N}$ is compatible with change of coefficients along the field extension $R/\mathbb{Q}$; consult [Bruguières 2000, Proposition 1.4.1] for further details.

Assume now that k is of positive prime characteristic p. Choose a maximal ideal $\mathfrak{m}$ of k and consider the associated residue field $F := k/\mathfrak{m}$. As in the characteristic zero case, we obtain an induced $\mathbb{F}_p$-linear symmetric monoidal functor

$$\mathrm{NChow}(k)_{\mathbb{F}_p} \to \mathcal{D}_c(F), \quad U(\mathcal{A})_{\mathbb{F}_p} \mapsto HH(\mathcal{A}) \otimes_k^L F,$$

which allows us to conclude that the category $\mathrm{NNum}(k)_R$ is abelian semisimple. $\square$

6. Proof of Theorems 2.3, 2.5, and 2.7

We start by studying the noncommutative Chow motives of dg Azumaya algebras. These results are of independent interest.

Proposition 6.1. *Let k be a commutative ring and A a dg Azumaya k-algebra which is not Morita equivalent to an ordinary Azumaya algebra. If k is noetherian, then we have $U(A)_\mathbb{Q} \not\simeq U(k)_\mathbb{Q}$ in $\mathrm{NChow}(k)_\mathbb{Q}$ and $U(A)_{\mathbb{F}_q} \not\simeq U(k)_{\mathbb{F}_q}$ in $\mathrm{NChow}(k)_{\mathbb{F}_q}$ for every prime number q.*

Proof. As proved in [Tabuada and Van den Bergh 2014, Theorem B.15], we have $U(A)_\mathbb{Q} \not\simeq U(k)_\mathbb{Q}$ in $\mathrm{NChow}(k)_\mathbb{Q}$. The proof that $U(A)_{\mathbb{F}_q} \not\simeq U(k)_{\mathbb{F}_q}$ in $\mathrm{NChow}(k)_{\mathbb{F}_q}$ is similar: simply further assume that q does not divides the positive integers m, n used in [loc. cit.]. $\square$

Proposition 6.2. *Let k be a field, A and B two central simple k-algebras, and R a commutative ring of positive prime characteristic p.*

(i) *If $p \mid \mathrm{ind}(A^{\mathrm{op}} \otimes B)$, then $U(A)_R \not\simeq U(B)_R$ in $\mathrm{NChow}(k)_R$. Moreover, we have $\mathrm{Hom}_{\mathrm{NNum}(k)_R}(U(A)_R, U(B)_R) = \mathrm{Hom}_{\mathrm{NNum}(k)_R}(U(B)_R, U(A)_R) = 0$.*

(ii) *If $p \nmid \mathrm{ind}(A^{\mathrm{op}} \otimes B)$ and R is a field, then $U(A)_R \simeq U(B)_R$ in $\mathrm{NChow}(k)_R$.*

Proof. As explained in the proof of [Tabuada and Van den Bergh 2014, Proposition 2.25], we have natural identifications $\mathrm{Hom}_{\mathrm{NChow}(k)}(U(A), U(B)) \simeq \mathbb{Z}$, under which the composition law (in $\mathrm{NChow}(k)$)

$$\mathrm{Hom}(U(A), U(B)) \times \mathrm{Hom}(U(B), U(C)) \to \mathrm{Hom}(U(A), U(C))$$

corresponds to the bilinear pairing

$$\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad (n, m) \mapsto n \cdot \mathrm{ind}(A^{\mathrm{op}} \otimes B) \cdot \mathrm{ind}(B^{\mathrm{op}} \otimes C) \cdot m.$$

Hence, we obtain natural identifications $\mathrm{Hom}_{\mathrm{NChow}(k)_R}(U(A)_R, U(B)_R) \simeq R$. Moreover, since $\mathrm{ind}(A^{\mathrm{op}} \otimes B) = \mathrm{ind}(B^{\mathrm{op}} \otimes A)$, the composition law (in $\mathrm{NChow}(k)_R$)

$$\mathrm{Hom}(U(A)_R, U(B)_R) \times \mathrm{Hom}(U(B)_R, U(A)_R) \to \mathrm{Hom}(U(A)_R, U(A)_R)$$

corresponds to the bilinear pairing

$$R \times R \to R, \quad (n, m) \mapsto n \cdot \mathrm{ind}(A^{\mathrm{op}} \otimes B)^2 \cdot m; \tag{6.3}$$

similarly with A and B replaced by B and A, respectively.

If $p \mid \mathrm{ind}(A^{\mathrm{op}} \otimes B)$, then the bilinear pairing (6.3) is zero. This implies that $U(A)_R \not\simeq U(B)_R$ in $\mathrm{NChow}(k)_R$. Moreover, since the categorical trace of the zero endomorphism is zero, we conclude that all the elements of the R-modules $\mathrm{Hom}_{\mathrm{NChow}(k)_R}(U(A)_R, U(B)_R)$ and $\mathrm{Hom}_{\mathrm{NChow}(k)_R}(U(B)_R, U(A)_R)$ belong to the $\mathcal{N}$-ideal. In other words, we have $\mathrm{Hom}_{\mathrm{NNum}(k)_R}(U(A)_R, U(B)_R) = 0$ and also $\mathrm{Hom}_{\mathrm{NNum}(k)_R}(U(B)_R, U(A)_R) = 0$. This proves item (i).

If $p \nmid \mathrm{ind}(A^{\mathrm{op}} \otimes B)$ and R is a field, then $\mathrm{ind}(A^{\mathrm{op}} \otimes B)$ is invertible in R. It follows then from the bilinear pairing (6.3) that $U(A)_R \simeq U(B)_R$ in $\mathrm{NChow}(k)_R$. This proves item (ii). $\qquad \square$

Let $\mathcal{A}, \mathcal{C} \subseteq \mathcal{B}$ be smooth proper pretriangulated dg categories for which the triangulated subcategories $\mathrm{H}^0(\mathcal{A}), \mathrm{H}^0(\mathcal{C}) \subseteq \mathrm{H}^0(\mathcal{B})$ are admissible and induce a semiorthogonal decomposition $\mathrm{H}^0(\mathcal{B}) = \langle \mathrm{H}^0(\mathcal{A}), \mathrm{H}^0(\mathcal{C}) \rangle$. As proved in [Tabuada 2005, Theorem 6.3], the inclusion dg functors $\mathcal{A}, \mathcal{C} \subseteq \mathcal{B}$ induce an isomorphism $U(\mathcal{A}) \oplus U(\mathcal{C}) \simeq U(\mathcal{B})$ in the additive category $\mathrm{NChow}(k)$. Consequently, if we denote by $K_0(\mathrm{NChow}(k))$ the Grothendieck ring of the symmetric monoidal additive $\mathrm{NChow}(k)$, we obtain a well-defined ring homomorphism

$$\mathcal{PT}(k) \to K_0(\mathrm{NChow}(k)), \quad [\mathcal{A}] \mapsto [U(\mathcal{A})].$$

By precomposing it with the isomorphism $K_0^{(2)}(k) \simeq \mathcal{PT}(k)$ of Theorem 1.1 and with (2.1), we hence obtain the canonical map

$$\mathrm{dBr}(k) \to K_0(\mathrm{NChow}(k)), \quad [A] \mapsto [U(A)]. \tag{6.4}$$

The proof of Theorem 2.3 now follows from the next result.

Theorem 6.5. *Let k be a noetherian commutative ring of characteristic zero* (*resp. positive prime characteristic p*) *and A a dg Azumaya algebra which is not Morita equivalent to an ordinary Azumaya algebra. If $K_0(k)_{\mathbb{Q}} \simeq \mathbb{Q}$* (*resp. $K_0(k)_{\mathbb{F}_p} \simeq \mathbb{F}_p$*), *then the image of* [A] *under the canonical map* (6.4) *is nontrivial. Moreover, when k is of characteristic zero* (*resp. positive prime characteristic p*), *this nontrivial image is different from the images of the ordinary Azumaya algebras* (*resp. of the ordinary Azumaya algebras whose index is not a multiple of p*).

Proof. Similarly to ordinary Azumaya algebras (see [Tabuada and Van den Bergh 2015, Lemma 8.10]), we have the equivalence of symmetric monoidal triangulated categories

$$\mathcal{D}_c(k) \xrightarrow{\simeq} \mathcal{D}_c(A^{\mathrm{op}} \otimes^L A), \quad M \mapsto M \otimes^L A,$$

where the symmetric monoidal structure on $\mathcal{D}_c(k)$ (resp. $\mathcal{D}_c(A^{\mathrm{op}} \otimes^L A)$) is induced by $-\otimes^L -$ (resp. $-\otimes^L_A -$). Consequently, we obtain an induced ring isomorphism

$$\mathrm{End}_{\mathrm{NChow}(k)}(U(k)) \xrightarrow{\simeq} \mathrm{End}_{\mathrm{NChow}(k)}(U(A)). \tag{6.6}$$

Let us prove the first claim. Assume that k is of characteristic zero; the proof of the cases where k is of positive prime characteristic p is similar. By definition of the category of noncommutative Chow motives, the left-hand side of (6.6) is given by the Grothendieck ring $K_0(k)$. Therefore, the assumption $K_0(k)_{\mathbb{Q}} \simeq \mathbb{Q}$ combined with the isomorphism (6.6) implies that $\mathrm{End}_{\mathrm{NChow}(k)_{\mathbb{Q}}}(U(k)_{\mathbb{Q}}) \simeq \mathbb{Q}$ and $\mathrm{End}_{\mathrm{NChow}(k)_{\mathbb{Q}}}(U(A)_{\mathbb{Q}}) \simeq \mathbb{Q}$. By construction of the category of noncommutative numerical motives, we have $\mathrm{End}_{\mathrm{NNum}(k)_{\mathbb{Q}}}(U(k)_{\mathbb{Q}}) \simeq \mathbb{Q}$. Using the fact that $U(A)_{\mathbb{Q}} \in \mathrm{NChow}(k)_{\mathbb{Q}}$ is a $\otimes$-invertible object and that the $\mathbb{Q}$-linear quotient functor $\mathrm{NChow}(k)_{\mathbb{Q}} \to \mathrm{NNum}(k)_{\mathbb{Q}}$ is symmetric monoidal, we hence conclude that $\mathrm{End}_{\mathrm{NNum}(k)_{\mathbb{Q}}}(U(A)_{\mathbb{Q}})$ is also isomorphic to $\mathbb{Q}$. This gives rise to the implication

$$U(A)_{\mathbb{Q}} \not\simeq U(k)_{\mathbb{Q}} \text{ in } \mathrm{NChow}(k)_{\mathbb{Q}} \implies U(A)_{\mathbb{Q}} \not\simeq U(k)_{\mathbb{Q}} \text{ in } \mathrm{NNum}(k)_{\mathbb{Q}}. \tag{6.7}$$

Note that since the quotient functor $\mathrm{NChow}(k)_{\mathbb{Q}} \to \mathrm{NNum}(k)_{\mathbb{Q}}$ is full, implication (6.7) is equivalent to the fact that every morphism $U(A)_{\mathbb{Q}} \to U(k)_{\mathbb{Q}}$ in $\mathrm{NChow}(k)_{\mathbb{Q}}$ which becomes invertible in $\mathrm{NNum}(k)_{\mathbb{Q}}$ is already invertible in $\mathrm{NChow}(k)_{\mathbb{Q}}$.

Recall from Proposition 6.1 that $U(A)_{\mathbb{Q}} \not\simeq U(k)_{\mathbb{Q}}$ in $\mathrm{NChow}(k)_{\mathbb{Q}}$ when k is noetherian. Making use of (6.7), we hence conclude that $U(A)_{\mathbb{Q}} \not\simeq U(k)_{\mathbb{Q}}$ in $\mathrm{NNum}(k)_{\mathbb{Q}}$. By definition, we have $[U(A)] = [U(k)]$ in the Grothendieck ring $K_0(\mathrm{NChow}(k))$ if and only if the following condition holds:

there exists an $NM \in \mathrm{NChow}(k)$ such that $U(A) \oplus NM \simeq U(k) \oplus NM$. (6.8)

Thanks to Theorem 5.5, the category $\mathrm{NNum}(k)_{\mathbb{Q}}$ is abelian semisimple. Consequently, it satisfies the cancellation property with respect to direct sums. Therefore,

if condition (6.8) holds, one would conclude that $U(A)_{\mathbb{Q}} \simeq U(k)_{\mathbb{Q}}$ in $\mathrm{NNum}(k)_{\mathbb{Q}}$, which is a contradiction. This finishes the proof of the first claim.

Let us now prove the second claim. Note first that by combining Proposition 6.1 with implication (6.7), we conclude that

$$U(A)_{\mathbb{Q}} \not\simeq U(k)_{\mathbb{Q}} \text{ in } \mathrm{NNum}(k)_{\mathbb{Q}} \quad \left(\text{resp. } U(A)_{\mathbb{F}_p} \not\simeq U(k)_{\mathbb{F}_p} \text{ in } \mathrm{NNum}(k)_{\mathbb{F}_p} \right). \quad (6.9)$$

Let B be an ordinary Azumaya k-algebra (resp. an ordinary Azumaya k-algebra whose index is not a multiple of p). In the latter case, by definition of index, we can assume without loss of generality that the rank of B is not a multiple of p. We have $[U(A)] = [U(B)]$ in the Grothendieck ring $K_0(\mathrm{NChow}(k))$ if and only if

$$\text{there exists an } NM \in \mathrm{NChow}(k) \text{ such that } U(A) \oplus NM \simeq U(B) \oplus NM. \quad (6.10)$$

Thanks to Theorem 5.5, the category $\mathrm{NNum}(k)_{\mathbb{Q}}$ (resp. $\mathrm{NNum}(k)_{\mathbb{F}_p}$) is abelian semisimple. Consequently, it satisfies the cancellation property with respect to direct sums. Therefore, if (6.10) holds, one would conclude that $U(A)_{\mathbb{Q}} \simeq U(B)_{\mathbb{Q}}$ in $\mathrm{NNum}(k)_{\mathbb{Q}}$ (resp. $U(A)_{\mathbb{F}_p} \simeq U(B)_{\mathbb{F}_p}$ in $\mathrm{NNum}(k)_{\mathbb{F}_p}$). On one hand, Corollary B.14 of [Tabuada and Van den Bergh 2014] implies that $U(B)_{\mathbb{Q}} \simeq U(k)_{\mathbb{Q}}$ in $\mathrm{NNum}(k)_{\mathbb{Q}}$. This contradicts the left-hand side of (6.9). On the other hand, since the rank of B is invertible in $\mathbb{F}_p$, the corollary implies that $U(B)_{\mathbb{F}_p} \simeq U(k)_{\mathbb{F}_p}$. This contradicts the right-hand side of (6.9). The proof of the second claim is then finished. $\square$

Proposition 6.11. *Let k be a field and R a field of positive characteristic p. In this case, the category $\mathrm{CSA}(k)_R^{\oplus}$ (see Notation 5.4) is equivalent to the category of $\mathrm{Br}(k)\{p\}$-graded finite dimensional R-vector spaces.*

Proof. Let A be a central simple k-algebra. Similarly to the proof of Theorem 6.5, we have a ring isomorphism $\mathrm{End}_{\mathrm{CSA}(k)_R}(U(A)_R) \simeq R$.

Let A, B be central simple k-algebras such that $[A], [B] \in \mathrm{Br}(k)\{p\}$ and $[A] \neq [B]$. Since $\mathrm{ind}(A^{\mathrm{op}} \otimes B) \mid \mathrm{ind}(A^{\mathrm{op}}) \cdot \mathrm{ind}(B)$ and $[A] \neq [B]$, we have $p \mid \mathrm{ind}(A^{\mathrm{op}} \otimes B)$. Therefore, Proposition 6.2(i) implies that $\mathrm{Hom}_{\mathrm{CSA}(k)_R}(U(A)_R, U(B)_R) = 0$ and also that $\mathrm{Hom}_{\mathrm{CSA}(k)_R}(U(B)_R, U(A)_R) = 0$.

Let A be a central simple k-algebra such that $[A] \in \bigoplus_{q \neq p} \mathrm{Br}(k)\{q\}$. Then, Proposition 6.2(ii) implies that $U(A)_R \simeq U(k)_R$ in $\mathrm{CSA}(k)_R$.

The proof now follows automatically from the combination of the above facts. $\square$

The proof of Theorem 2.5 now follows from the next result.

Theorem 6.12. *Let k be a field of characteristic zero. In this case, the canonical map (6.4) is injective.*

Proof. Let A and B be two central simple k-algebras such that $[A] \neq [B]$ in $\mathrm{Br}(k)$. Recall that $\mathrm{ind}(A^{\mathrm{op}} \otimes B) = 1$ if and only if $[A] = [B]$. Therefore, let us choose a prime number p such that $p \mid \mathrm{ind}(A^{\mathrm{op}} \otimes B)$. Thanks to Proposition 6.2(i), we have

$U(A)_{\mathbb{F}_p} \not\simeq U(B)_{\mathbb{F}_p}$ in $\mathrm{NChow}(k)_{\mathbb{F}_p}$. Consequently, similarly to implication (6.7), we have $U(A)_{\mathbb{F}_p} \not\simeq U(B)_{\mathbb{F}_p}$ in $\mathrm{NNum}(k)_{\mathbb{F}_p}$. By definition, we have $[U(A)] = [U(B)]$ in the Grothendieck ring $K_0(\mathrm{NChow}(k))$ if and only if the following condition holds:

there exists an $NM \in \mathrm{NChow}(k)$ such that $U(A) \oplus NM \simeq U(B) \oplus NM$. (6.13)

Thanks to Lemma 6.17 below, if condition (6.13) holds, then there exist nonnegative integers $n, m \geq 0$ and a noncommutative numerical motive NM' such that

$$\bigoplus_{i=1}^{n+1} U(A)_{\mathbb{F}_p} \oplus \bigoplus_{j=1}^{m} U(B)_{\mathbb{F}_p} \oplus NM' \simeq \bigoplus_{i=1}^{n} U(A)_{\mathbb{F}_p} \oplus \bigoplus_{j=1}^{m+1} U(B)_{\mathbb{F}_p} \oplus NM' \quad (6.14)$$

in $\mathrm{NNum}(k)_{\mathbb{F}_p}$. Note that the composition bilinear pairing (in $\mathrm{NNum}(k)_{\mathbb{F}_p}$)

$$\mathrm{Hom}(U(A)_{\mathbb{F}_p}, NM') \times \mathrm{Hom}(NM', U(A)_{\mathbb{F}_p}) \to \mathrm{Hom}(U(A)_{\mathbb{F}_p}, U(A)_{\mathbb{F}_p}) \quad (6.15)$$

is zero; similarly for $U(B)_{\mathbb{F}_p}$. This follows from the fact that the right-hand side of (6.15) identifies with $\mathbb{F}_p$, from the fact that the category $\mathrm{NNum}(k)_{\mathbb{F}_p}$ is $\mathbb{F}_p$-linear, and from the fact that the noncommutative numerical motive NM' does not contain $U(A)_{\mathbb{F}_p}$ as a direct summand. The composition bilinear pairing (in $\mathrm{NNum}(k)_{\mathbb{F}_p}$)

$$\mathrm{Hom}(U(A)_{\mathbb{F}_p}, NM') \times \mathrm{Hom}(NM', U(B)_{\mathbb{F}_p}) \to \mathrm{Hom}(U(A)_{\mathbb{F}_p}, U(B)_{\mathbb{F}_p}) \quad (6.16)$$

is also zero; similarly with A and B replaced by B and A, respectively. This follows automatically from the fact that the right-hand side of (6.16) is zero; see Proposition 6.2(i). Now, note that the triviality of the pairings (6.15)–(6.16) implies that the isomorphism (6.14) restricts to an isomorphism

$$U(A)_{\mathbb{F}_p} \oplus \bigoplus_{i=1}^{n} U(A)_{\mathbb{F}_p} \oplus \bigoplus_{j=1}^{m} U(B)_{\mathbb{F}_p} \simeq U(B)_{\mathbb{F}_p} \oplus \bigoplus_{i=1}^{n} U(A)_{\mathbb{F}_p} \oplus \bigoplus_{j=1}^{m} U(B)_{\mathbb{F}_p}$$

in the category $\mathrm{CSA}(k)_{\mathbb{F}_p}^{\oplus} \subset \mathrm{NNum}(k)_{\mathbb{F}_p}$. Since $\mathrm{CSA}(k)_{\mathbb{F}_p}^{\oplus}$ is equivalent to the category of $\mathrm{Br}(k)\{p\}$-graded finite dimensional $\mathbb{F}_p$-vector spaces (see Proposition 6.11), it satisfies the cancellation property with respect to direct sums. Consequently, we conclude from the preceding isomorphism that $U(A)_{\mathbb{F}_p} \simeq U(B)_{\mathbb{F}_p}$ in $\mathrm{NNum}(k)_{\mathbb{F}_p}$, which is a contradiction. This finishes the proof. □

Lemma 6.17. *There exist nonnegative integers $n, m \geq 0$ and a noncommutative numerical motive $NM' \in \mathrm{NNum}(k)_{\mathbb{F}_p}$ such that*:

(i) *We have $NM_{\mathbb{F}_p} \simeq \bigoplus_{i=1}^{n} U(A)_{\mathbb{F}_p} \oplus \bigoplus_{j=1}^{m} U(B)_{\mathbb{F}_p} \oplus NM'$ in $\mathrm{NNum}(k)_{\mathbb{F}_p}$.*

(ii) *The noncommutative numerical motive NM' does not contain $U(A)_{\mathbb{F}_p}$ or $U(B)_{\mathbb{F}_p}$ as a direct summand.*

Proof. Recall that the category $\mathrm{NNum}(k)_{\mathbb{F}_p}$ is idempotent complete. Therefore, by inductively splitting the (possible) direct summands $U(A)_{\mathbb{F}_p}$ and $U(B)_{\mathbb{F}_p}$ of the noncommutative numerical motive $NM_{\mathbb{F}_p}$, we obtain an isomorphism

$$NM_{\mathbb{F}_p} \simeq U(A)_{\mathbb{F}_p} \oplus \cdots \oplus U(A)_{\mathbb{F}_p} \oplus U(B)_{\mathbb{F}_p} \oplus \cdots \oplus U(B)_{\mathbb{F}_p} \oplus NM'$$

in $\mathrm{NNum}(k)_{\mathbb{F}_p}$, with NM' satisfying condition (ii). We claim that the number of copies of $U(A)_{\mathbb{F}_p}$ and $U(B)_{\mathbb{F}_p}$ is finite; note that this concludes the proof. We will focus ourselves on the case $U(A)_{\mathbb{F}_p}$; the proof of the case $U(B)_{\mathbb{F}_p}$ is similar. Suppose that the number of copies of $U(A)_{\mathbb{F}_p}$ is infinite. Since we have natural isomorphisms

$$\mathrm{Hom}_{\mathrm{NNum}(k)_{\mathbb{F}_p}}(U(A)_{\mathbb{F}_p}, U(A)_{\mathbb{F}_p}) \simeq \mathbb{F}_p, \tag{6.18}$$

this would allow us to construct an infinite sequence $f_1, f_2, \dots$ of vectors in the $\mathbb{F}_p$-vector space $\mathrm{Hom}_{\mathrm{NNum}(k)_{\mathbb{F}_p}}(U(A)_{\mathbb{F}_p}, NM_{\mathbb{F}_p})$, with f_i corresponding to the element $1 \in \mathbb{F}_p$ of (6.18), such that $f_1, \dots, f_r$ is linearly independent for every positive integer r. In other words, this would allow us to conclude that the $\mathbb{F}_p$-vector space $\mathrm{Hom}_{\mathrm{NNum}(k)_{\mathbb{F}_p}}(U(A)_{\mathbb{F}_p}, NM_{\mathbb{F}_p})$ is infinite dimensional. Recall from the proof of [Bruguières 2000, Proposition 1.4.1] that the map $\mathbb{Z} \to \mathbb{F}_p$ gives rise to a surjective homomorphism

$$\mathrm{Hom}_{\mathrm{NNum}(k)}(U(A), NM) \otimes_{\mathbb{Z}} \mathbb{F}_p \twoheadrightarrow \mathrm{Hom}_{\mathrm{NNum}(k)_{\mathbb{F}_p}}(U(A)_{\mathbb{F}_p}, NM_{\mathbb{F}_p}). \tag{6.19}$$

Since, by assumption, the base field k is of characteristic zero, the abelian group $\mathrm{Hom}_{\mathrm{NNum}(k)}(U(A), NM)$ is finitely generated; see [Tabuada and Van den Bergh 2014, Theorem 1.2]. Therefore, we conclude that the right-hand side of (6.19) is a finite dimensional $\mathbb{F}_p$-vector space, which is a contradiction. This finishes the proof. $\qquad\square$

The proof of Theorem 2.7 now follows from the next result.

Theorem 6.20. *Let k be a field of positive characteristic p and A, B two central simple k-algebras. If $p \mid \mathrm{ind}(A^{\mathrm{op}} \otimes B)$, then the images of $[A]$ and $[B]$ under the canonical map* (6.4) *are different. This holds in particular when* $\mathrm{ind}(A)$ *and* $\mathrm{ind}(B)$ *are coprime and p divides* $\mathrm{ind}(A)$ *or* $\mathrm{ind}(B)$.

Proof. If $p \mid \mathrm{ind}(A^{\mathrm{op}} \otimes B)$, then Proposition 6.2(i) implies that $U(A)_{\mathbb{F}_p} \not\simeq U(B)_{\mathbb{F}_p}$ in $\mathrm{NChow}(k)_{\mathbb{F}_p}$. Consequently, similarly to implication (6.7), we conclude that $U(A)_{\mathbb{F}_p} \not\simeq U(B)_{\mathbb{F}_p}$ in $\mathrm{NNum}(k)_{\mathbb{F}_p}$. By definition, we have $[U(A)] = [U(B)]$ in the Grothendieck ring $K_0(\mathrm{NChow}(k))$ if and only if the following condition holds:

there exists an $NM \in \mathrm{NChow}(k)$ such that $U(A) \oplus NM \simeq U(B) \oplus NM$. (6.21)

Thanks to Theorem 5.5, the category $\mathrm{NNum}(k)_{\mathbb{F}_p}$ is abelian semisimple. Consequently, it satisfies the cancellation property with respect to direct sums. Therefore,

if condition (6.21) holds, one would conclude that $U(A)_{\mathbb{F}_p} \simeq U(B)_{\mathbb{F}_p}$ in $\mathrm{NNum}(k)_{\mathbb{F}_p}$, which is a contradiction. This finishes the proof. □

Corollary 6.22. *When k is a field of positive characteristic p, the restriction of the canonical map (6.4) to the p-primary torsion subgroup $\mathrm{Br}(k)\{p\}$ is injective. Moreover, the image of $\mathrm{Br}(k)\{p\} - 0$ is disjoint from the image of $\bigoplus_{q \neq p} \mathrm{Br}(k)\{q\}$.*

Remark 6.23. As proved in [Marcolli and Tabuada 2014a, Theorem 7.1], every ring homomorphism $k \to k'$ gives rise to the following commutative square:

$$
\begin{array}{ccc}
\mathrm{dBr}(k) & \xrightarrow{\;(6.4)\;} & K_0(\mathrm{NChow}(k)) \\
{\scriptstyle -\otimes_k^L k'} \downarrow & & \downarrow {\scriptstyle -\otimes_k^L k'} \\
\mathrm{dBr}(k') & \xrightarrow[\;(6.4)\;]{} & K_0(\mathrm{NChow}(k'))
\end{array}
$$

Therefore, by combining it with Theorems 6.12 and 6.20, we conclude that Corollary 2.10 also holds with (2.1) replaced by (6.4).

Acknowledgements

References

[André and Kahn 2002] Y. André and B. Kahn, "Nilpotence, radicaux et structures monoïdales", *Rend. Sem. Mat. Univ. Padova* **108** (2002), 107–291. MR 1956434 Zbl 1165.18300

[André and Kahn 2005] Y. André and B. Kahn, "Erratum: "Nilpotency, radicals and monoidal structures" [Rend. Sem. Mat. Univ. Padova **108** (2002), 107–291; MR1956434]", *Rend. Sem. Mat. Univ. Padova* **113** (2005), 125–128. MR 2168984 Zbl 1167.18302

[Auslander and Goldman 1961] M. Auslander and O. Goldman, "The Brauer group of a commutative ring", *Trans. Amer. Math. Soc.* **97** (1961), 367–409. MR 0121392 Zbl 0100.26304

[Ben-Zvi and Nadler 2012] D. Ben-Zvi and D. Nadler, "Loop spaces and connections", *J. Topol.* **5**:2 (2012), 377–430. MR 2928082 Zbl 1246.14027

[Bondal and Kapranov 1989] A. I. Bondal and M. M. Kapranov, "Representable functors, Serre functors, and reconstructions", *Izv. Akad. Nauk SSSR Ser. Mat.* **53**:6 (1989), 1183–1205, 1337. In Russian; translated in *Math. USSR-Izv.* **35**:3 (1990), 519–541. MR 1039961

[Bondal and Kapranov 1990] A. I. Bondal and M. M. Kapranov, "Framed triangulated categories", *Mat. Sb.* **181**:5 (1990), 669–683. In Russian; translated in *Math. USSR-Sb.* **70**:1 (1991), 93–107. MR 1055981

[Bondal and van den Bergh 2003] A. Bondal and M. van den Bergh, "Generators and representability of functors in commutative and noncommutative geometry", *Mosc. Math. J.* **3**:1 (2003), 1–36, 258. MR 1996800 Zbl 1135.18302

[Bondal et al. 2004] A. I. Bondal, M. Larsen, and V. A. Lunts, "Grothendieck ring of pretriangulated categories", *Int. Math. Res. Not.* **2004**:29 (2004), 1461–1495. MR 2051435 Zbl 1079.18008

[Bruguières 2000] A. Bruguières, "Tresses et structure entière sur la catégorie des représentations de SL_N quantique", *Comm. Algebra* **28**:4 (2000), 1989–2028. MR 1747368 Zbl 0951.18003

[Cisinski and Tabuada 2012] D.-C. Cisinski and G. Tabuada, "Symmetric monoidal structure on non-commutative motives", *J. K-Theory* **9**:2 (2012), 201–268. MR 2922389 Zbl 1256.19002

[Drinfeld 2004] V. Drinfeld, "DG quotients of DG categories", *J. Algebra* **272**:2 (2004), 643–691. MR 2028075 Zbl 1064.18009

[Gabber 1981] O. Gabber, "Some theorems on Azumaya algebras", pp. 129–209 in *The Brauer group* (Sem., Les Plans-sur-Bex, 1980), edited by M. Kervaire and M. Ojanguren, Lecture Notes in Math. **844**, Springer, Berlin-New York, 1981. MR 611868 Zbl 0472.14013

[Gille and Szamuely 2006] P. Gille and T. Szamuely, *Central simple algebras and Galois cohomology*, Cambridge Studies in Advanced Mathematics **101**, Cambridge University Press, 2006. MR 2266528 Zbl 1137.12001

[Grothendieck 1995a] A. Grothendieck, "Le groupe de Brauer, I: Algèbres d'Azumaya et interprétations diverses", exposé no. 290, 199–219 in *Séminaire Bourbaki,* 1964/65, Soc. Math. France, Paris, 1995. MR 1608798

[Grothendieck 1995b] A. Grothendieck, "Le groupe de Brauer, II: Théorie cohomologique", exposé no. 297, 287–307 in *Séminaire Bourbaki,* 1965/66, Soc. Math. France, Paris, 1995. MR 1608805

[Keller 2006] B. Keller, "On differential graded categories", pp. 151–190 in *International Congress of Mathematicians*, vol. II, edited by M. Sanz-Solé et al., Eur. Math. Soc., Zürich, 2006. MR 2275593 Zbl 1140.18008

[Kontsevich 1998] M. Kontsevich, "Triangulated categories and geometry", lecture notes, ENS, Paris, 1998, available at http://www.math.uchicago.edu/~mitya/langlands/kontsevich.ps.

[Lunts 2010] V. A. Lunts, "Categorical resolution of singularities", *J. Algebra* **323**:10 (2010), 2977–3003. MR 2609187 Zbl 1202.18006

[Lunts and Orlov 2010] V. A. Lunts and D. O. Orlov, "Uniqueness of enhancement for triangulated categories", *J. Amer. Math. Soc.* **23**:3 (2010), 853–908. MR 2629991 Zbl 1197.14014

[Marcolli and Tabuada 2014a] M. Marcolli and G. Tabuada, "Noncommutative Artin motives", *Selecta Math. (N.S.)* **20**:1 (2014), 315–358. MR 3147418 Zbl 1296.14002

[Marcolli and Tabuada 2014b] M. Marcolli and G. Tabuada, "Noncommutative motives, numerical equivalence, and semi-simplicity", *Amer. J. Math.* **136**:1 (2014), 59–75. MR 3163353 Zbl 1315.18020

[Mumford 1961] D. Mumford, "The topology of normal singularities of an algebraic surface and a criterion for simplicity", *Inst. Hautes Études Sci. Publ. Math.* **9** (1961), 5–22. MR 0153682 Zbl 0108.16801

[Orlov 2015] D. Orlov, "Smooth and proper noncommutative schemes and gluing of DG categories", preprint, 2015. arXiv 1402.7364

[Revoy 1973] P. Revoy, "Algèbres de Weyl en caractéristique *p*", *C. R. Acad. Sci. Paris Sér. A-B* **276** (1973), A225–A228. MR 0335564 Zbl 0265.16007

[Tabuada 2005] G. Tabuada, "Invariants additifs de DG-catégories", *Int. Math. Res. Not.* **2005**:53 (2005), 3309–3339. MR 2196100 Zbl 1094.18006

[Tabuada 2008] G. Tabuada, "Higher *K*-theory via universal invariants", *Duke Math. J.* **145**:1 (2008), 121–206. MR 2451292 Zbl 1166.18007

[Tabuada 2012] G. Tabuada, "A guided tour through the garden of noncommutative motives", pp. 259–276 in *Topics in noncommutative geometry*, edited by G. Cortiñas, Clay Math. Proc. **16**, American Mathematical Society, Providence, RI, 2012. MR 2986869 Zbl 1316.19001

[Tabuada 2015] G. Tabuada, *Noncommutative motives*, University Lecture Series **63**, American Mathematical Society, Providence, RI, 2015. MR 3379910 Zbl 06501755

[Tabuada and Van den Bergh 2014] G. Tabuada and M. Van den Bergh, "Noncommutative motives of separable algebras", preprint, 2014. arXiv 1411.7970

[Tabuada and Van den Bergh 2015] G. Tabuada and M. Van den Bergh, "Noncommutative motives of Azumaya algebras", *J. Inst. Math. Jussieu* **14**:2 (2015), 379–403. MR 3315059 Zbl 06420271

[Toën 2009] B. Toën, "Secondary K-theory", Talk at the Workshop on Topological Field Theories, Northwestern University, May 2009, available at https://www.ma.utexas.edu/users/benzvi/GRASP/lectures/NWTFT/toennw.pdf.

[Toën 2011] B. Toën, "Lectures on dg-categories", pp. 243–302 in *Topics in algebraic and topological K-theory*, edited by G. Cortiñas, Lecture Notes in Math. **2008**, Springer, Berlin, 2011. MR 2762557 Zbl 1216.18013

[Toën 2012] B. Toën, "Derived Azumaya algebras and generators for twisted derived categories", *Invent. Math.* **189**:3 (2012), 581–652. MR 2957304 Zbl 1275.14017

[Toën and Vezzosi 2009] B. Toën and G. Vezzosi, "Chern character, loop spaces and derived algebraic geometry", pp. 331–354 in *Algebraic topology*, edited by N. A. Baas et al., Abel Symp. **4**, Springer, Berlin, 2009. MR 2597742 Zbl 1177.14042

[Toën and Vezzosi 2015] B. Toën and G. Vezzosi, "Caractères de Chern, traces équivariantes et géométrie algébrique dérivée", *Selecta Math.* (*N.S.*) **21**:2 (2015), 449–554. MR 3338682 Zbl 06441726

[Wodzicki 2011] M. Wodzicki, "Algebras of p-symbols, noncommutative p-residue, and the Brauer group", pp. 283–304 in *Noncommutative geometry and global analysis*, edited by A. Connes et al., Contemp. Math. **546**, American Mathematical Society, Providence, RI, 2011. MR 2815141 Zbl 1246.58024

tabuada@math.mit.edu   *Departamento de Matemática e Centro de Matemática e Aplicações, Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, Quinta da Torre, 2829-516 Caparica, Portugal*

Current address: *Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139, United States*

Algebra & Number Theory

msp.org/ant

See inside back cover or msp.org/ant for submission instructions.

Algebra & Number Theory