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In previous work, we described conditions under which a single geometric representation $\Gamma_F \to H(\overline{\mathbb{Q}}_\ell)$ of the Galois group of a number field F lifts through a central torus quotient $\widetilde{H} \to H$ to a geometric representation. In this paper, we prove a much sharper result for systems of ℓ-adic representations, such as the ℓ-adic realizations of a motive over F, having common "good reduction" properties. Namely, such systems admit geometric lifts with good reduction outside a common finite set of primes. The method yields new proofs of theorems of Tate (the original result on lifting projective representations over number fields) and Wintenberger (an analogue of our main result in the case of a central isogeny $\widetilde{H} \to H$).

1. Introduction

Let F be a number field, and let $\Gamma_F = \mathrm{Gal}(\overline{F}/F)$ be its absolute Galois group with respect to a fixed algebraic closure $\overline{F}$. A fundamental theorem of Tate (see [Serre 1977, §6]) asserts that $H^2(\Gamma_F, \mathbb{Q}/\mathbb{Z})$ vanishes; as a result, all (continuous, ℓ-adic) projective representations of Γ_F lift to genuine representations, and more generally, whenever $\widetilde{H} \to H$ is a surjection of linear algebraic groups over $\overline{\mathbb{Q}}_\ell$ with kernel equal to a central torus in $\widetilde{H}$, all representations $\rho_\ell : \Gamma_F \to H(\overline{\mathbb{Q}}_\ell)$ lift to $\widetilde{H}(\overline{\mathbb{Q}}_\ell)$.

The ℓ-adic representations of greatest interest in number theory are those with conjectural connections to the theories of motives and automorphic forms; if the monodromy group of ρ_ℓ is semisimple, then it is expected — by conjectures of Fontaine–Mazur, Tate, Grothendieck–Serre, and Langlands — that the ρ_ℓ arising from pure motives or automorphic forms are precisely those that are geometric in the sense of Fontaine–Mazur, i.e., unramified outside a finite set of places of F, and de Rham at all places dividing ℓ. The paper [Patrikis 2016c] established a variant of Tate's lifting theorem for such geometric Galois representations. There are

obstructions when F has real embeddings, but at least for totally imaginary F, any geometric $\rho_\ell : \Gamma_F \to H(\overline{\mathbb{Q}}_\ell)$ satisfying a natural "Hodge symmetry" requirement admits a geometric lift $\tilde{\rho}_\ell : \Gamma_F \to \widetilde{H}(\overline{\mathbb{Q}}_\ell)$ [Patrikis 2016c, Theorem 3.2.10]. This geometric lifting theorem leads to a precise expectation for the corresponding lifting problem for motivic Galois representations. Namely, if $\mathcal{G}_{F,E}$ denotes the motivic Galois group for pure motives over F with coefficients in a number field E — we will make this setup precise in Section 2, but for now the reader may take homological motives under the standard conjectures — and if $\widetilde{H} \to H$ is now a surjection of groups over E with central torus kernel, then we conjecture [Patrikis 2016c, Conjecture 4.3.1] that any motivic Galois representation $\rho : \mathcal{G}_{F,E} \to H$ lifts to $\widetilde{H}$, at least after some finite extension of coefficients:

$$
\begin{array}{ccc}
 & & \widetilde{H}_{\overline{E}} \\
 & \nearrow^{\tilde{\rho}} & \big\downarrow \\
\mathcal{G}_{F,\overline{E}} & \xrightarrow[\rho \otimes_E \overline{E}]{} & H_{\overline{E}}
\end{array}
$$

There is essentially one classical example (with several variants) of this conjecture, a well-known construction of Kuga and Satake [1967], which associates to a complex, for our purposes projective, K3 surface X a complex abelian variety $\mathrm{KS}(X)$, related by an inclusion of Hodge-structures $H^2(X, \mathbb{Q}) \subset H^1(\mathrm{KS}(X), \mathbb{Q})^{\otimes 2}$. In the motivic Galois language, finding $\mathrm{KS}(X)$ amounts (when $F = \mathbb{C}$) to finding a lift $\tilde{\rho}$ of the representation $\rho_X : \mathcal{G}_{\mathbb{C},\mathbb{Q}} \to H = \mathrm{SO}(H^2(X)(1))$, through the surjection $\widetilde{H} = \mathrm{GSpin}(H^2(X)(1)) \to H$. Progress on the general conjecture, when the motives in question do not lie in the Tannakian subcategory of motives generated by abelian varieties, seems to require entirely new ideas.[1]

The aim of this paper is to establish a Galois-theoretic result which is necessary for this conjecture to hold, but considerably more delicate than the basic geometric lifting theorem of [Patrikis 2016c, Theorem 3.2.10]. Namely, any motive M over F has good reduction outside a finite set of primes: for any choice of variety X in whose cohomology M appears, X spreads out as a smooth projective scheme over $\mathcal{O}_F[1/N]$ for some integer N. In particular, by the base-change theorems of étale cohomology [Deligne 1977] and the crystalline p-adic comparison isomorphism [Faltings 1989], for any motivic Galois representation $\rho : \mathcal{G}_{F,E} \to H$, the λ-adic realizations $\rho_\lambda : \Gamma_F \to H(E_\lambda)$ have *good reduction* outside a finite set of primes S, in the sense (also see Definition 1.1) that each ρ_λ factors through $\Gamma_{F,S \cup S_\lambda}$ and is crystalline at all places of $S_\lambda \setminus (S_\lambda \cap S)$; here S_λ denotes the primes of F with the same residue characteristic as λ, and $\Gamma_{F,S \cup S_\lambda}$ is the Galois group of the maximal extension of F inside $\overline{F}$ that is unramified outside of $S \cup S_\lambda$. Certainly a necessary

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[1]For some, admittedly limited, examples, see [Patrikis 2016a; 2016b].

condition for the generalized Kuga–Satake conjecture to hold is that the realizations $\{\rho_\lambda\}_\lambda$ of ρ should lift to geometric representations $\tilde{\rho}_\lambda : \Gamma_{F,P\cup S_\lambda} \to \widetilde{H}(\bar{E}_\lambda)$ that likewise have good reduction outside a common finite set of places P. This is what we will show, as a consequence of a more general result. To state it, we first make a couple of definitions.

Definition 1.1. A collection $\{\rho_\lambda : \Gamma_F \to H(\bar{E}_\lambda)\}_\lambda$, as λ varies over finite places of E, of geometric Galois representations is *ramification-compatible* if there exist

(1) a finite set S of places of F such that each ρ_λ is unramified outside of $S \cup S_\lambda$, i.e., factors through

$$\rho_\lambda : \Gamma_{F,S\cup S_\lambda} \to H(\bar{E}_\lambda),$$

and for v in S_λ but not in S, $\rho_\lambda|_{\Gamma_{F_v}}$ is crystalline; and

(2) a central cocharacter $\omega : \boldsymbol{G}_{m,E} \to H$ and a collection of conjugacy classes

$$\{[\mu_\tau : \boldsymbol{G}_{m,\bar{E}} \to H_{\bar{E}}]\}_{\tau:F\hookrightarrow\bar{E}}$$

satisfying $[\mu_\tau] = \omega \cdot [\mu_{c\tau}^{-1}]$ for any choice of complex conjugation $c \in \mathrm{Gal}(\bar{E}/\mathbb{Q})$, such that for all E-embeddings $\iota_\lambda : \bar{E} \hookrightarrow \bar{E}_\lambda$, inducing via τ some $\tau_{\iota_\lambda} : F_v \hookrightarrow \bar{E}_\lambda$, the conjugacy class $[\mu_\tau \otimes_{\bar{E},\iota_\lambda} \bar{E}_\lambda]$ is equal to the conjugacy class of τ_{ι_λ}-labeled Hodge–Tate cocharacters associated to $\rho_\lambda|_{\Gamma_{F_v}}$.

If a single representation ρ_λ satisfies the condition in item (1), we say ρ_λ *has good reduction outside S*. If it satisfies the condition in item (2) (for some collection of cocharacters ω, $\{\mu_\tau\}$), then we say it satisfies *Hodge symmetry*.

Remark 1.2.   • Note that the ρ_λ need not be "compatible" in the usual sense (frobenii acting compatibly): if the "coefficients" of the ρ_λ are bounded in a rather strong sense — there exists a common number field over which their frobenius characteristic polynomials are defined — one expects that our collection of ρ_λ should partition (dividing up the λ's) into finitely many compatible systems.[2]

• The Hodge symmetry requirement of part (2) of Definition 1.1 is not the most general constraint that pertains to a compatible system of ℓ-adic representations. It will always hold for the λ-adic realizations of motives, as we will see in Lemma 2.3, when we take the ambient group to be the motivic Galois group G_ρ of the underlying motivic Galois representation ρ, and not some larger group. But there may be compatible systems (of motivic origin) where the criterion in part (2) of Definition 1.1 fails; for example, consider $\rho_\ell : \Gamma_\mathbb{Q} \to \mathrm{PGL}_3(\bar{\mathbb{Q}}_\ell)$ given

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[2]To see the relevance of bounding the coefficients, the reader may contrast the case of elliptic curves (over $\mathbb{Q}$, say) unramified outside S with that of all weight-two modular forms unramified outside S: of the former there are finitely many isogeny classes, since the conductor is bounded, whereas the latter can have level divisible by arbitrarily high powers of the primes in S.

by the projectivization of $\kappa_\ell \oplus 1 \oplus 1$, with κ_ℓ denoting the ℓ-adic cyclotomic character. The relation $[\mu_\tau] = \omega \cdot [\mu_{c\tau}^{-1}]$ implies in this case $[\mu_\tau] = [\mu_\tau^{-1}]$, which is false. A way around this is given in [Patrikis 2016c, §3.2], where the Hodge symmetry hypothesis is formulated in a way that conjecturally holds for any geometric representation $\rho_\ell \colon \Gamma_F \to H(\overline{\mathbb{Q}}_\ell)$, regardless of its (reductive) algebraic monodromy group. The proof of [Patrikis 2016c, Theorem 3.2.10] thus requires a slightly trickier group-theoretic argument than the one we require here. We have opted in this paper to keep the simpler condition (2) above, so as to focus on what is new in the arguments, and because of its obvious centrality from a motivic point of view (in particular, its sufficiency for Corollary 1.6).

Here is the main theorem:

Theorem 1.3. *Let E be a number field, and let $\widetilde{H} \to H$ be a surjection of linear algebraic groups over E with kernel equal to a central torus in $\widetilde{H}$. Let F be a totally imaginary number field, and let S be a finite set of places of F containing the archimedean places. Fix a set of cocharacters $\{\mu_\tau\}_{\tau: F \hookrightarrow \overline{E}}$ satisfying the "Hodge symmetry" condition of part (2) of Definition 1.1. Then there exists a finite set of places $P \supset S$ such that for any place λ of E, any embedding $\iota_\lambda \colon \overline{E} \hookrightarrow \overline{E}_\lambda$, and any geometric representation $\rho_\lambda \colon \Gamma_{F, S \cup S_\lambda} \to H(\overline{E}_\lambda)$ such that*

- *ρ_λ has good reduction outside S, and*
- *the conjugacy classes of labeled Hodge–Tate cocharacters of ρ_λ are induced via ι_λ from $\{\mu_\tau\}_{\tau: F \hookrightarrow \overline{E}}$ (again, see Definition 1.1 for details),*

the representation ρ_λ admits a geometric lift $\tilde{\rho}_\lambda \colon \Gamma_{F, P \cup S_\lambda} \to \widetilde{H}(\overline{E}_\lambda)$ having good reduction outside P.

In particular, if $\{\rho_\lambda \colon \Gamma_{F, S \cup S_\lambda} \to H(\overline{E}_\lambda)\}_\lambda$ is a ramification-compatible system, then there exist a finite set of places $P \supset S$ and lifts $\tilde{\rho}_\lambda \colon \Gamma_{F, P \cup S_\lambda} \to \widetilde{H}(\overline{E}_\lambda)$ such that $\{\tilde{\rho}_\lambda\}_\lambda$ is a ramification-compatible system.

Remark 1.4. All results of this paper, once we take into account the caveat of [Patrikis 2016c, §2.8] (see too [Patrikis 2015, Proposition 5.5]), admit straightforward variants when F has real places. Thus, for real F, the analogue of Theorem 1.3 either holds exactly as written, or after replacing F by any totally imaginary (e.g., composite with a quadratic imaginary) extension. We do not want to discuss this at any length here, but we simply remind the reader that the prototypical example in which F is totally real, and Theorem 1.3 fails as stated, is that of the projective motivic Galois representation associated to a *mixed-parity* Hilbert modular form.

The proof of this theorem is completed in Theorem 1.3. The typical application is to the collection of Galois representations $\{\rho_\lambda\}_\lambda$ associated to a motivic Galois representation; we make this precise in Corollary 1.6.

Next we describe applications of Theorem 1.3 to the more general problem of lifting through surjections $H' \to H$ with central kernel of multiplicative type. Here in general we cannot expect as strong a result as Theorem 1.3. First, the Hodge–Tate cocharacters of ρ_λ may not lift to H', in which case there can be no geometric lifts to H'. Second, even if the Hodge–Tate cocharacters lift, the Galois representations may only lift after a finite base change of F. For example, if ρ_λ is the projectivization of the Galois representation associated to a weight 3 modular form, $\det(\rho_\lambda) : \Gamma_\mathbb{Q} \to \overline{E}_\lambda^\times$ does not admit a square root. A beautiful result of Wintenberger [1995, Théorème 2.1.4, Théorème 2.1.7] shows that when $H' \to H$ is an isogeny, a result similar to Theorem 1.3 holds, as long as in the conclusion F is replaced by a suitable finite extension. Here we treat the general case of multiplicative-type quotients:

Theorem 1.5 (see Corollary 3.18). *Let $H' \to H$ be a surjection of linear algebraic groups over E whose kernel is central and of multiplicative type. Let F be a number field, and let S be a finite set of places of F containing the archimedean places. Fix a set of cocharacters $\{\mu_\tau\}_{\tau : F \hookrightarrow \overline{E}}$ as in part (2) of Definition 1.1, and moreover, assume that each μ_τ lifts to a cocharacter of H'.*

Then there exist a finite set of places $P \supset S$, and a finite extension F'/F, such that any geometric representation $\rho_\lambda : \Gamma_{F, S \cup S_\lambda} \to H(\overline{E}_\lambda)$ having good reduction outside S, and whose Hodge–Tate cocharacters arise from the set $\{\mu_\tau\}_{\tau : F \hookrightarrow \overline{E}}$ via some embedding $\overline{E} \hookrightarrow \overline{E}_\lambda$, admits a geometric lift $\tilde{\rho}_\lambda : \Gamma_{F', P \cup S_\lambda} \to H'(\overline{E}_\lambda)$ having good reduction outside P.

In particular, if $\{\rho_\lambda : \Gamma_{F, S \cup S_\lambda} \to H(\overline{E}_\lambda)\}_\lambda$ is a ramification-compatible system with Hodge cocharacters $\{\mu_\tau\}_{\tau : F \hookrightarrow \overline{E}}$, then there exist a finite set of places $P \supset S$, a finite extension F'/F, and lifts $\tilde{\rho}_\lambda : \Gamma_{F', P \cup S_\lambda} \to H'(\overline{E}_\lambda)$ such that $\{\tilde{\rho}_\lambda\}_\lambda$ is a ramification-compatible system.

We deduce Wintenberger's original result in Corollary 3.16. Our proof differs in an essential way, as it passes through Theorem 1.3, which cannot be deduced from the methods of [Wintenberger 1995]. Our problem resembles Wintenberger's in that both lead to a basic difficulty of annihilating cohomological obstruction classes in infinitely many Galois cohomology groups, one for each λ, but needing to do so in an "independent-of-λ" fashion. The arguments themselves, however, are in fact orthogonal to one another: Wintenberger always kills cohomology by making a finite base change on F, whereas that is precisely what we are forbidden from doing if we want the more precise results of Theorem 1.3. Moreover, our methods also yield a novel proof of Tate's original vanishing theorem (see Corollary 3.9). In fact, Corollary 3.9 establishes a more precise form of Tate's theorem: the latter of course shows that the image under the canonical map $H^2(\Gamma_{F,S}, \mathbb{Z}/N) \to H^2(\Gamma_F, \mathbb{Q}/\mathbb{Z})$ is zero, and our refinement quantifies how much additional ramification must be

added, and how much the coefficients must be enlarged, in order to annihilate $H^2(\Gamma_{F,S}, \mathbb{Z}/N)$. Our arguments thus achieve, from scratch, a satisfying common generalization of the theorems of Wintenberger and Tate.

In Corollaries 3.12 and 3.14, we give a couple of applications to lifting λ-adic realizations such that the associated "similitude characters" (e.g., determinant or Clifford norm) of the lifts form strongly compatible systems. Note that even in the case of the classical Kuga–Satake construction, this compatibility is only achieved as a consequence of having an arithmetic descent of the (Hodge-theoretically defined) Kuga–Satake abelian variety; such a descent depends on the deformation theory of K3 surfaces and monodromy arguments (due to Deligne [1972] and André [1996a]).

Our final result is the promised motivic application:

Corollary 1.6. *Let F be a totally imaginary number field, let E be a number field, and let $\mathcal{G}_{F,E}$ denote the motivic Galois group, defined by André's motivated cycles, of pure motives over F with coefficients in E (see Section 2). Let $\widetilde{H} \to H$ be a surjection of linear algebraic groups over E whose kernel is a central torus in $\widetilde{H}$, and let $\rho : \mathcal{G}_{F,E} \to H$ be any motivic Galois representation, with associated λ-adic realizations $\rho_\lambda : \Gamma_{F,S\cup S_\lambda} \to H(\overline{E}_\lambda)$ for some finite set S of places of F. Then there exist a finite, independent of λ, set $P \supset S$ of places of F and, for all λ, lifts*

$$
\begin{array}{ccc}
 & & \widetilde{H}(\overline{E}_\lambda) \\
 & \overset{\tilde{\rho}_\lambda}{\nearrow} & \downarrow \\
\Gamma_{F,P\cup S_\lambda} & \xrightarrow{\rho_\lambda} & H(\overline{E}_\lambda)
\end{array}
$$

such that each $\tilde{\rho}_\lambda$ is de Rham at all places in S_λ, and is moreover crystalline at all places in $S_\lambda \setminus (S_\lambda \cap P)$.

Now suppose $H' \to H$ is a surjection of linear algebraic groups whose kernel is central but of arbitrary multiplicative type, and let $\rho : \mathcal{G}_{F,E} \to H$ again be a motivic Galois representation. Assume that the labeled Hodge cocharacters of ρ (see Definition 2.1) lift to H'. Then there exist a finite set $P \supset S$ of places of F, a finite extension F'/F, and, for all λ, lifts

$$
\begin{array}{ccc}
 & & H'(\overline{E}_\lambda) \\
 & \overset{\tilde{\rho}_\lambda}{\nearrow} & \downarrow \\
\Gamma_{F',P\cup S_\lambda} & \xrightarrow{\rho_\lambda|_{\Gamma_{F'}}} & H(\overline{E}_\lambda)
\end{array}
$$

such that $\tilde{\rho}_\lambda$ is de Rham at all places above S_λ, and is moreover crystalline at all places above $S_\lambda \setminus (S_\lambda \cap P)$.

Remark 1.7. Even admitting a strong finiteness conjecture, that there are finitely many isomorphism classes of ρ (as in Corollary 1.6), having coefficients in E, prescribed Hodge–Tate cocharacters, and good reduction outside a fixed finite set S, Theorem 1.3 still says rather more than Corollary 1.6, since even for fixed λ it applies to infinitely many distinct ρ_λ simultaneously (because we have not bounded the coefficients: recall the example of modular forms of weight two whose nebentypus characters have unbounded conductor, even though supported on the fixed finite set S of primes).

We close this introduction by emphasizing what we do *not* prove. The realizations $\{\rho_\lambda\}_\lambda$ of ρ should, moreover, form a weakly compatible system of Galois representations in the sense that the conjugacy class of $\rho_\lambda(fr_v)$ is defined over $\bar{E}$ and is suitably independent of λ (for v outside $S \cup S_\lambda$), and in turn one would hope to construct lifts $\tilde{\rho}_\lambda$ with the same frobenius compatibility. This problem seems to be out of reach: I know of no way to establish such results using only Galois-theoretic techniques, although indeed they would follow (assuming the standard conjectures) from the generalized Kuga–Satake conjecture. Alternatively, it is possible to establish results of this nature in settings where ρ_λ and $\tilde{\rho}_\lambda$ are constructed as automorphic Galois representations: the most significant example of this is the recent work of Kret and Shin [2016] associating GSpin-value Galois representations to certain (discrete series at infinity, Steinberg at some finite place) cuspidal automorphic representations of $\mathrm{GSp}_{2n}(A_F)$, for F totally real. Crucially, their construction uses the already-known construction of (SO_{2n+1}-valued) Galois representations for the restrictions of such automorphic representations to $\mathrm{Sp}_{2n}(A_F)$, so it is very much in the spirit of the generalized Kuga–Satake lifting problem.

2. Hodge symmetry

In this section we establish a motivic setting in which our general Galois-theoretic results apply; this setting will both serve as motivation for subsequent sections and allow us to deduce Corollary 1.6 from Theorem 1.3 (and Corollary 3.18). The reader who does not find the motivic language illuminating can safely skip this section.

Rather than working with (pure) homological motives and assuming the standard conjectures, we work with a category of motives that is unconditionally semisimple and Tannakian — and in which we can prove unconditional results — but that would, under the standard conjectures, turn out to be equivalent to the category of homological motives. Namely, let $\mathcal{M}_{F,E}$ denote André's category of motivated motives over F with coefficients in E; see [André 1996b]. We begin by elaborating on the consequences of Hodge symmetry in $\mathcal{M}_{F,E}$. Throughout this discussion, it will be convenient to fix embeddings $\tau_0 : F \hookrightarrow \bar{E}$ and $\iota_\infty : \bar{E} \hookrightarrow \mathbb{C}$. The composite

$\iota_\infty \tau_0 : F \hookrightarrow \mathbb{C}$ yields a Betti fiber functor $H_{\iota_\infty \tau_0} : \mathcal{M}_{F,E} \to \mathrm{Vect}_E$, making $\mathcal{M}_{F,E}$ into a neutral Tannakian category over E. We denote by $\mathcal{G} = \mathcal{G}_B(\iota_\infty \tau_0)$ the associated Tannakian group (tensor automorphisms of the fiber functor), so that $H_{\iota_\infty \tau_0}$ induces an equivalence of tensor categories $\mathcal{M}_{F,E} \xrightarrow{\sim} \mathrm{Rep}(\mathcal{G})$.

We will consider other cohomological realizations on $\mathcal{M}_{F,E}$, and their comparisons with the Betti fiber functor. Let $H_{dR} : \mathcal{M}_{F,E} \to \mathrm{Fil}_{F \otimes_{\mathbb{Q}} E}$ denote the de Rham realization, taking values in filtered $F \otimes_{\mathbb{Q}} E$-modules, and for each place λ of E, let H_λ denote the λ-adic realization, which takes values in finite E_λ-modules with a continuous action of Γ_F. For all embeddings $\tau : F \hookrightarrow \overline{E}$, we obtain an $\overline{E}$-valued fiber functor

$$\omega_{dR,\tau} : M \mapsto \mathrm{gr}^\bullet(e_\tau H_{dR}(M)),$$

where e_τ is the idempotent induced by $\tau \otimes 1 : F \otimes_{\mathbb{Q}} E \to \overline{E}$. Let $\mathcal{G}_{dR}(\tau) = \mathrm{Aut}^\otimes(\omega_{dR,\tau})$ be the associated Tannakian group over $\overline{E}$. Of course this fiber functor factors through the category $\mathrm{Gr}_{\overline{E}}$ of graded $\overline{E}$-vector spaces

$$\mathcal{M}_{F,\overline{E}} \longrightarrow \mathrm{Gr}_{\overline{E}}$$
$$\searrow \qquad \swarrow$$
$$\mathrm{Vect}_{\overline{E}}$$
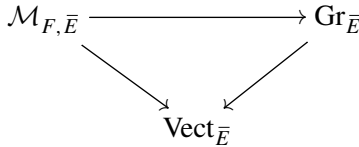
so we obtain a corresponding homomorphism $\mu_\tau : \boldsymbol{G}_{m,\overline{E}} \to \mathcal{G}_{dR}(\tau)$. Without specifying τ, we obtain a fiber functor (see [Deligne and Milne 1982, §3]) $\omega_{dR} = \mathrm{gr}^\bullet H_{dR}$ on $\mathcal{M}_{F,E}$ valued in projective $F \otimes_{\mathbb{Q}} E$-modules. By [loc. cit., Theorem 3.2], the functor $\underline{\mathrm{Hom}}^\otimes(H_{\iota_\infty \tau_0}, \omega_{dR})$ is a $\mathcal{G}$-torsor over $F \otimes_{\mathbb{Q}} E$. In particular, for all $\tau : F \hookrightarrow \overline{E}$, we can choose a point of $\underline{\mathrm{Hom}}^\otimes(H_{\iota_\infty \tau_0} \otimes_E \overline{E}, \omega_{dR,\tau})$ to induce a cocharacter μ_τ of $\mathcal{G}_{\overline{E}}$, and the conjugacy class $[\mu_\tau]$ of μ_τ is independent of this choice.

Definition 2.1. For each $\tau : F \hookrightarrow \overline{E}$, we call any $\mu_\tau : \boldsymbol{G}_{m,\overline{E}} \to \mathcal{G}_{\overline{E}}$ as above a τ-labeled Hodge cocharacter; it is a representative of the conjugacy class of cocharacters $[\mu_\tau]$, the latter being canonically independent of any of the above choices of isomorphisms of fiber functors.

Lemma 2.2. *For all $\sigma \in \mathrm{Gal}(\overline{E}/E)$, $[\mu_\tau] = [\mu_{\sigma \tau}]$. In particular, $[\mu_\tau]$ only depends on the restriction of τ to the maximal CM (or totally real) subfield F_{cm} of F.*

Proof. We decompose $F \otimes_{\mathbb{Q}} E = \prod_i E_i$ into a product of fields, writing p_i for the projection onto E_i. Any E-algebra homomorphism $\tau : F \otimes_{\mathbb{Q}} E \to \overline{E}$ factors through $p_{i(\tau)}$ for a unique $i(\tau)$, and then the $\mathrm{Gal}(\overline{E}/E)$-orbit of τ is precisely those E-algebra homomorphisms (i.e., embeddings $F \hookrightarrow \overline{E}$) $\tau' : F \otimes_{\mathbb{Q}} E \to \overline{E}$ such that $i(\tau) = i(\tau')$. The first claim follows, since both $\omega_{dR,\tau}$ and $\omega_{dR,\sigma \tau}$ can be factored through $p_{i(\tau)} \circ \omega_{dR}$. The second claim follows from the first, and the fact that all motives arise by scalar extension from motives with coefficients in CM (or totally real) fields [Patrikis 2016c, Lemma 4.1.22]. $\square$

Next note that the canonical weight-grading on $\mathcal{M}_{F,E}$ induces a *central* weight homomorphism

$$\omega : \boldsymbol{G}_{m,E} \to \mathcal{G},$$

and likewise for any other choice of fiber functor and Tannakian group (because ω is central, it is in fact canonically independent of any choice of isomorphism between fiber functors). Hodge symmetry then results from the complex conjugation action on Betti cohomology, interpreted via the Betti–de Rham comparison isomorphism, which is a distinguished $\mathbb{C}$-point of $\underline{\mathrm{Hom}}^{\otimes}(\omega_{dR,\tau} \otimes_{\overline{E},\iota_{\infty}} \mathbb{C}, H_{\iota_{\infty}\tau} \otimes_{E,\iota_{\infty}} \mathbb{C})$. Namely, complex conjugation on complex-analytic spaces induces (see [Patrikis 2016c, Lemma 4.1.24]) natural isomorphisms (without restricting to particular graded pieces for the weight and Hodge filtrations, these are isomorphisms of fiber functors over $\mathbb{C}$)

$$\mathrm{gr}^p(e_\tau H_{dR}^w(M)) \otimes_{\overline{E},\iota_{\infty}} \mathbb{C} \xrightarrow{\sim} \mathrm{gr}^{w-p}(e_{c\tau} H_{dR}^w(M) \otimes_{\overline{E},\iota_{\infty}} \mathbb{C}), \tag{1}$$

where $c \in \mathrm{Aut}(\overline{E})$ is the choice of complex conjugation for which $\overline{\iota_{\infty}}\tau = \iota_{\infty}c\tau$. We deduce the following relation:

Lemma 2.3. *For any embedding $\tau : F \hookrightarrow \overline{E}$, and any choice of complex conjugation $c \in \mathrm{Aut}(\overline{E})$, the conjugacy classes of cocharacters $[\mu_\tau]$ and $[\mu_{c\tau}]$ satisfy*

$$[\mu_\tau] = \omega \cdot [\mu_{c\tau}^{-1}],$$

where ω is the weight cocharacter.

Proof. For the choice of complex conjugation specified by $\overline{\iota_{\infty}}\tau = \iota_{\infty}c\tau$, the relation $[\mu_\tau] = \omega \cdot [\mu_{c\tau}^{-1}]$ follows, after base extension $\iota_{\infty} : \overline{E} \to \mathbb{C}$, from (1) above; but this relation necessarily descends to $\overline{E}$, since the conjugacy classes of cocharacters are defined over any algebraically closed subfield of $\mathbb{C}$. It only remains to observe that $[\mu_{c\tau}]$ is independent of the choice of complex conjugation on $\overline{E}$. This follows from the second assertion of Lemma 2.2. $\square$

The comparison isomorphisms of p-adic Hodge theory then imply that the analogue of Lemma 2.3 also holds for the associated Hodge–Tate cocharacters. For any place λ of E, fix an algebraic closure $\overline{E}_\lambda$. Embeddings $\tau : F \hookrightarrow \overline{E}$ and $\iota_\lambda : \overline{E} \hookrightarrow \overline{E}_\lambda$ then induce $\tau_{\iota_\lambda} : F_v \hookrightarrow \overline{E}_\lambda$ for a suitable place v of F of the same residue characteristic p as λ. Meanwhile, the restriction to Γ_{F_v} of the λ-adic realization induces

$$\mathcal{M}_{F,E} \xrightarrow{H_\lambda|_{\Gamma_{F_v}}} \mathrm{Rep}_{E_\lambda}^{dR}(\Gamma_{F_v}) \xrightarrow{D_{dR}} \mathrm{Fil}_{F_v \otimes_{\mathbb{Q}_p} E_\lambda} \xrightarrow{e_{\tau_{\iota_\lambda}}} \mathrm{Fil}_{\overline{E}_\lambda} \xrightarrow{\mathrm{gr}} \mathrm{Gr}_{\overline{E}_\lambda} \to \mathrm{Vect}_{\overline{E}_\lambda}, \tag{2}$$

where $D_{dR} : \mathrm{Rep}_{E_\lambda}^{dR}(\Gamma_{F_v}) \to \mathrm{Fil}_{F_v \otimes_{\mathbb{Q}_p} E_\lambda}$ denotes Fontaine's functor restricted to the category of de Rham representations. (Here we have invoked Faltings' p-adic de Rham comparison isomorphism from [Faltings 1989], and the fact — already noted by André [1996b] — that it extends to a comparison isomorphism on all of $\mathcal{M}_{F,E}$;

for details of the latter point, see [Patrikis 2016c, Lemma 4.1.25].) Of course, $\text{Rep}_{E_\lambda}^{dR}(\Gamma_{F_v})$ also has its standard forgetful fiber functor (let us say $\bar{E}_\lambda$-valued), yielding a Tannakian group $\Gamma_{v,\lambda}^{dR}$ for de Rham Γ_{F_v}-representations over $\bar{E}_\lambda$; by choosing an isomorphism between the two $\bar{E}_\lambda$-valued fiber functors on $\text{Rep}_{E_\lambda}^{dR}(\Gamma_{F_v})$, we obtain a canonical conjugacy class (recalling [Deligne and Milne 1982, Theorem 3.2]) of "τ_{ι_λ}-labeled Hodge–Tate cocharacters" $[\mu_{\tau_{\iota_\lambda}}]$ of $\Gamma_{v,\lambda}^{dR}$. Specializing, this construction defines the labeled Hodge–Tate cocharacters of any de Rham Galois representation $\rho : \Gamma_{F_v} \to H(E_\lambda)$, for any affine algebraic group H over E_λ.

To relate the τ_{ι_λ}-labeled Hodge–Tate cocharacters in the motivic setting to the Hodge cocharacters previously discussed, note that the de Rham comparison isomorphism [Faltings 1989] yields a natural isomorphism of tensor functors $\mathcal{M}_{F,E} \to \text{Gr}_{\bar{E}_\lambda}$:

$$\text{gr}(e_\tau(H_{dR}(M) \otimes_E \bar{E}) \otimes_{\bar{E},\iota_\lambda} \bar{E}_\lambda) \cong \text{gr}(e_{\tau_{\iota_\lambda}}(D_{dR}(H_\lambda(M)|_{\Gamma_{F_v}}) \otimes_{E_\lambda} \bar{E}_\lambda)).$$

We deduce the following corollary.

Corollary 2.4. *For any embedding $\tau : F \hookrightarrow \bar{E}$, and any embedding $\iota_\lambda : \bar{E} \hookrightarrow \bar{E}_\lambda$, there is an equality of conjugacy classes*

$$[\mu_\tau \otimes_{\bar{E},\iota_\lambda} \bar{E}_\lambda] = [\mu_{\tau_{\iota_\lambda}}].$$

In particular, for all λ, and for all E-embeddings $\iota_\lambda : \bar{E} \hookrightarrow \bar{E}_\lambda$, the conjugacy classes $[\mu_{\tau_{\iota_\lambda}}]$ are independent of (λ, ι_λ) when regarded as valued in the common group $\mathcal{G}_{\bar{E}}$.

3. Lifting

In this section we prove our main results. First we recall the setting and some notational conventions that will be in effect for the rest of the paper.

Let F be a totally imaginary field (see Remark 1.4), let S be a finite set of places of F containing the infinite places, and let E be any number field. Fix an algebraic closure $\bar{F}$, and set $\Gamma_F = \text{Gal}(\bar{F}/F)$. We write $F(S)$ for the maximal extension of F inside $\bar{F}$ that is unramified outside of S, and we set $\Gamma_{F,S} = \text{Gal}(F(S)/F)$. We denote the ring of S-integers in F by $\mathcal{O}_F[1/S]$. We also set $F_S = \prod_{v \in S} F_v$. If L is a finite extension of F (inside $\bar{F}$), we then abusively continue to write S for the set of all places of L above those in S, with corresponding notation $L(S)$, $\Gamma_{L,S}$, etc. For a place λ of E, let S_λ denote the set of places of F with the same residue characteristic as λ. We freely use the terminology established in Definition 1.1.

Torus quotients. In this subsection we prove Theorem 1.3. We begin in the next few paragraphs by gathering together all of the "independent of λ and ρ_λ" data, and the auxiliary constructions we make on top of this data. Continuing with

F, S, and E as above, also fix a surjection $\widetilde{H} \to H$ of linear algebraic groups over E whose kernel is a central torus, which we denote by C. (We will without comment also write C for the base change to various algebraically closed fields containing E.) Next fix an isogeny complement H_1 of C in $\widetilde{H}$ (for existence of such H_1, see [Conrad 2011, Proposition 5.3, Step 1]); thus, $H_1 \cdot C = \widetilde{H}$, and $H_1 \cap C$ is finite. For technical reasons, we will later want to include all primes dividing $\#(H_1 \cap C)(\overline{E})$ in the set of bad primes S; this will be indicated at the necessary point (see the discussion following Lemma 3.5), but it does no harm simply to add these primes to S from now on. Consider the quotient map $\xi : \widetilde{H} \to \widetilde{H}/H_1$; $\widetilde{Z}^\vee = \widetilde{H}/H_1$ is a torus, and there is an isogeny $C \to \widetilde{Z}^\vee$, with kernel $C \cap H_1$. Fix a split torus $\widetilde{Z}$ over F whose dual group (constructed over $\overline{E}$) is isomorphic to $\widetilde{Z}^\vee \otimes_E \overline{E}$, and fix such an identification (implicit from now on).

Fix a set of cocharacters $\{\mu_\tau : \boldsymbol{G}_{m,\overline{E}} \to H_{\overline{E}}\}_{\tau : F \hookrightarrow \overline{E}}$, and a central cocharacter $\omega : \boldsymbol{G}_{m,\overline{E}} \to H_{\overline{E}}$, satisfying the Hodge symmetry requirement of item (2) of Definition 1.1. Denote by F_{cm} the maximal CM subfield of F. The condition in Definition 1.1 implies that the cocharacter μ_τ depends only on the restriction of τ to F_{cm}; we denote this restriction by $\tau_{cm} : F_{cm} \hookrightarrow \overline{E}$. We fix a set of representatives I of $\mathrm{Hom}(F_{cm}, \overline{E})$ modulo complex conjugation, and for each $\sigma \in I$, we fix a lift $\widetilde{\mu}_\sigma$ to $\widetilde{H}$ of μ_σ, as well as a (central) lift $\widetilde{\omega}$ of ω. Note that this is possible, because C is a torus. If $\tau : F \hookrightarrow \overline{E}$ restricts to a $\sigma \in I$, we then set $\widetilde{\mu}_\tau = \widetilde{\mu}_\sigma$; if not, then $c\tau : F \hookrightarrow \overline{E}$ restricts to a $\sigma \in I$, and we then set $\widetilde{\mu}_\tau = \widetilde{\omega}\widetilde{\mu}_\sigma{}^{-1}$.

Lemma 3.1. *Fix once and for all an embedding $\iota_\infty : \overline{E} \hookrightarrow \mathbb{C}$. There exists an algebraic automorphic representation ψ of $\widetilde{Z}(\boldsymbol{A}_F)$ such that for all $\tau : F \hookrightarrow \overline{E}$, inducing $\tau_{\iota_\infty} : F_v \hookrightarrow \mathbb{C}$ by composition with ι_∞, the local component $\psi_v : F_v^\times \to \mathbb{C}^\times$ is given by*

$$\psi_v(z) = \tau_{\iota_\infty}(z)^{\xi(\widetilde{\mu}_\tau)} \overline{\tau_{\iota_\infty}}(z)^{\xi(\widetilde{\mu}_{c\tau})}.$$

(Recall that ξ is the quotient $\widetilde{H} \to \widetilde{Z}^\vee$.)

Proof. We readily reduce to the case $\widetilde{Z} = \boldsymbol{G}_m$, where it follows from the description, due to Weil [1956], of the possible archimedean components of algebraic Hecke characters. (This is where Hodge-symmetry is required.) □

From now on we fix such a ψ, and we let T denote the finite set of places of F such that ψ is unramified outside T. For any embedding $\iota_\lambda : \overline{E} \hookrightarrow \overline{E}_\lambda$, we can then consider the λ-adic realization[3]

$$\psi_{\iota_\lambda} : \Gamma_{F, T \cup S_\lambda} \to \widetilde{Z}^\vee(\overline{E}_\lambda).$$

Each ψ_{ι_λ} is a geometric Galois representation, with good reduction outside T, and for any $\tau : F \hookrightarrow \overline{E}$, inducing $\tau_{\iota_\lambda} : F_v \hookrightarrow \overline{E}_\lambda$, the Hodge–Tate cocharacter of ψ_{ι_λ} associated to τ_{ι_λ} is $\xi(\widetilde{\mu}_\tau) \otimes_{\overline{E}, \iota_\lambda} \overline{E}_\lambda$.

---

[3]To be precise, this depends on ι_∞ and ι_λ; but ι_∞ is fixed throughout the paper.

Now we consider any geometric representation

$$\rho_\lambda : \Gamma_{F, S \cup S_\lambda} \to H(\bar{E}_\lambda)$$

having good reduction outside S, along with an embedding $\iota_\lambda : \bar{E} \hookrightarrow \bar{E}_\lambda$ such that the Hodge–Tate cocharacters of ρ_λ arise from the collection $\{\mu_\tau : \boldsymbol{G}_{m,\bar{E}} \to H_{\bar{E}}\}_{\tau : F \hookrightarrow \bar{E}}$ via ι_λ. Because the kernel of $\widetilde{H} \to H$ is a central torus, a fundamental theorem of Tate (see [Serre 1977, §6]) ensures in this case that ρ_λ, *as a representation of Γ_F, lifts to $\widetilde{H}$.* As we will see, our arguments in fact imply Tate's theorem (Corollary 3.9), so we do not need to assume it in what follows.

We can define an obstruction class $\mathcal{O}(\rho_\lambda)$ to lifting ρ_λ to a continuous representation $\Gamma_{F, S \cup S_\lambda} \to H_1(\bar{E}_\lambda)$ in the usual way: choose a topological (but not group-theoretic) lift ρ'_λ, and then form the 2-cocycle $(g, h) \mapsto \rho'_\lambda(gh)\rho'_\lambda(h)^{-1}\rho'_\lambda(g)^{-1}$, defining

$$\mathcal{O}(\rho_\lambda) \in H^2(\Gamma_{F, S \cup S_\lambda}, H_1 \cap C).$$

Here and in what follows, we simply write $H_1 \cap C$ for the $\bar{E}_\lambda$-points of this finite group scheme.

Remark 3.2. Here lies the essential difficulty to be overcome: while Tate's theorem allows us to annihilate the cohomology classes $\mathcal{O}(\rho_\lambda)$ — after allowing some additional ramification and enlarging the subgroup $H_1 \cap C$ of C — we have to carry out this annihilation in a way that is independent of λ, and moreover, for fixed λ independent of ρ_λ. Simultaneous annihilation of the $\mathcal{O}(\rho_\lambda)$ using only a uniform, finite enlargement of the allowable ramification set and of the subgroup of C in fact does not seem to be possible; we will as a first step have to define modified versions of these obstruction classes that take into account the Hodge numbers of ψ.

Before proceeding, we reinterpret the obstruction $\mathcal{O}(\rho_\lambda)$ (we will only use the local version of what follows; in particular, the arguments of the present section depend only on the local version of Tate's theorem, which is an almost immediate consequence of local duality).

Lemma 3.3. *Let v be a finite place of F, and suppose that $\tilde{\rho}_\lambda : \Gamma_{F_v} \to \widetilde{H}(\bar{E}_\lambda)$ is any continuous homomorphism lifting $\rho_\lambda|_{\Gamma_{F_v}}$. Then $\mathcal{O}(\rho_\lambda)|_{\Gamma_{F_v}}$ is equal to the inverse of $\mathcal{O}(\xi(\tilde{\rho}_\lambda))$, the obstruction associated to lifting $\xi(\tilde{\rho}_\lambda) : \Gamma_{F_v} \to \widetilde{Z}^\vee(\bar{E}_\lambda)$ to C. (The same holds if we replace Γ_{F_v} by Γ_F, but we do not require this.)*

Proof. Before beginning the proof proper, we make precise our convention for coboundary maps: the *inverse* appearing in the conclusion of the lemma is crucial, and it is easy to get confused if one is not careful with the definitions. Let Γ be a group and M a (for simplicity) trivial Γ-module. For a function $\alpha : \Gamma^n \to M$, set

$$\delta(\alpha)(g_1, \ldots, g_{n+1}) = \alpha(g_2, \ldots, g_n) + \sum_{i=1}^n (-1)^i \alpha(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1})$$
$$+ (-1)^{n+1} \alpha(g_1, \ldots, g_n).$$

For $n = 1$, this says $\delta(\alpha)(g, h) = \alpha(h)\alpha(gh)^{-1}\alpha(g)$, and in the situation considered above $(g, h) \mapsto \rho_\lambda'(gh)\rho_\lambda'(h)^{-1}\rho_\lambda'(g)^{-1}$ is in fact a 2-cocycle.

Tate's theorem implies that, for sufficiently large m, the image of $\mathcal{O}(\rho_\lambda)$ in $H^2(\Gamma_{F_v}, C[m])$ vanishes, i.e., $\mathcal{O}(\rho_\lambda) = \delta(\phi)$ for some $\phi : \Gamma_{F_v} \to C[m]$. The product $\rho_\lambda' \cdot \phi$ is then a homomorphism $\Gamma_{F_v} \to (H_1 \cdot C[m])(\overline{E}_\lambda)$ lifting ρ_λ; we set $\tilde{\rho}_\lambda = \rho_\lambda' \cdot \phi$. Clearly $\xi(\tilde{\rho}_\lambda) = \xi(\phi)$, and then $\mathcal{O}(\xi(\tilde{\rho}_\lambda))$ is (tautologically) represented by the cocycle $(g, h) \mapsto \phi(gh)\phi(h)^{-1}\phi(g)^{-1}$, i.e., by $\delta(\phi)^{-1} = \mathcal{O}(\rho_\lambda)^{-1} \in Z^2(\Gamma_F, H_1 \cap C)$.[4] This proves the claim for our particular lift $\tilde{\rho}_\lambda$, but any other lift $\tilde{\rho}_\lambda^1$ gives rise to the same obstruction $\mathcal{O}(\xi(\tilde{\rho}_\lambda^1))$. (The global claim holds for the same reasons, if we admit the global version of Tate's theorem.) $\qquad\square$

To address the difficulty indicated in Remark 3.2, we begin by using the abelian representations coming from ψ to construct a second obstruction class. Namely, consider the realization ψ_{ι_λ}, which, for notational simplicity, from now on we simply denote by ψ_λ. The automorphic representation ψ is unramified outside the finite set of places T of F, so ψ_λ is a geometric representation $\Gamma_{F, T \cup S_\lambda} \to \widetilde{Z}^\vee(\overline{E}_\lambda)$, which has good reduction outside T (i.e., is crystalline at primes of S_λ not in T). Via the isogeny $C \to \widetilde{Z}^\vee$, we can then form a cohomology class measuring the obstruction to lifting ψ_λ to C: let ψ_λ' denote a topological lift $\Gamma_{F, T \cup S_\lambda} \to C(\overline{E}_\lambda)$, defining as before a cohomology class

$$\mathcal{O}(\psi_\lambda) \in H^2(\Gamma_{F, T \cup S_\lambda}, H_1 \cap C).$$

We can in turn define (via inflation) a cohomology class

$$\mathcal{O}(\rho_\lambda, \psi_\lambda) = \mathcal{O}(\rho_\lambda) \cdot \mathcal{O}(\psi_\lambda) \in H^2(\Gamma_{F, S \cup T \cup S_\lambda}, H_1 \cap C),$$

which is represented by the 2-cocycle (recall that C is central in $\widetilde{H}$)

$$(g, h) \mapsto (\rho_\lambda' \cdot \psi_\lambda')(gh)(\rho_\lambda' \cdot \psi_\lambda')(h)^{-1}(\rho_\lambda' \cdot \psi_\lambda')(g)^{-1}.$$

(Note, however, that the function $g \mapsto (\rho_\lambda' \cdot \psi_\lambda')(g)$ is valued in $\widetilde{H}$, not in H_1.)

We need one more lemma before getting to the crucial local result (Lemma 3.5).

Lemma 3.4. *For all places $v \in S_\lambda$, and for any choice of embedding $\iota_\lambda : \overline{E} \hookrightarrow \overline{E}_\lambda$, there exists a de Rham lift*

$$
\begin{array}{ccc}
& & \widetilde{H}(\overline{E}_\lambda) \\
& \overset{\tilde{\rho}_\lambda}{\nearrow} & \downarrow \\
\Gamma_{F_v} & \underset{\rho_\lambda}{\longrightarrow} & H(\overline{E}_\lambda)
\end{array}
$$

of $\rho_\lambda|_{\Gamma_{F_v}}$ such that for all embeddings $\tau_\lambda : F_v \hookrightarrow \overline{E}_\lambda$, the τ_λ-labeled Hodge–Tate

---

[4]Note that ϕ is valued in $C[m]$, not $H_1 \cap C$, so $\delta(\phi)$ need not be a coboundary in $Z^2(\Gamma_{F_v}, H_1 \cap C)$.

cocharacter of $\tilde{\rho}_\lambda$ is (conjugate to) $\widetilde{\mu_\tau} \otimes_{\bar{E},\iota_\lambda} \bar{E}_\lambda$, where $\tau : F \hookrightarrow \bar{E}$ is defined by the diagram

$$
\begin{array}{ccc}
F_v & \xrightarrow{\tau_\lambda} & \bar{E}_\lambda \\
\uparrow & \iota_\lambda \uparrow & \\
F & \xrightarrow{\tau} & \bar{E}
\end{array}
$$

Moreover, if ρ_λ is crystalline, then $\tilde{\rho}_\lambda$ may be taken to be crystalline.

Proof. For each $\tau_\lambda : F_v \hookrightarrow \bar{E}_\lambda$, set for notational simplicity $\widetilde{\mu_{\tau_\lambda}} = \widetilde{\mu_\tau} \otimes_{\bar{E},\iota_\lambda} \bar{E}_\lambda$, where τ is determined as in the diagram, and where $\widetilde{\mu_\tau}$ is the lift of μ_τ we have fixed above. The proof of [Patrikis 2016c, Corollary 3.2.12] shows that for any collection of cocharacters lifting the Hodge cocharacters of ρ_λ, and in particular for our $\widetilde{\mu_{\tau_\lambda}}$, there exists a Hodge–Tate lift $\tilde{\rho}_\lambda : \Gamma_{F_v} \to \widetilde{H}(\bar{E}_\lambda)$ whose τ_λ-labeled Hodge–Tate cocharacter is $\widetilde{\mu_{\tau_\lambda}}$. Now consider the isogeny lifting problem

$$
\begin{array}{ccc}
 & & \widetilde{H}(\bar{E}_\lambda) \\
 & \nearrow & \downarrow \\
\Gamma_{F_v} & \xrightarrow[(\rho_\lambda, \xi(\tilde{\rho}_\lambda))]{} & H(\bar{E}_\lambda) \times \widetilde{Z}^\vee(\bar{E}_\lambda)
\end{array}
$$

Since $(\rho_\lambda, \xi(\tilde{\rho}_\lambda))$ admits a Hodge–Tate lift (namely, $\tilde{\rho}_\lambda$), and is itself de Rham (ρ_λ is de Rham by assumption, and any abelian Hodge–Tate representation is de Rham), we can apply [Conrad 2011, Corollary 6.7] to deduce the existence of a de Rham lift $\tilde{\rho}'_\lambda$, which clearly has the same Hodge–Tate cocharacters as $\tilde{\rho}_\lambda$, since they differ by a finite-order twist. If we further assume ρ_λ is crystalline, then we need only a minor modification to this argument: some power $\xi(\tilde{\rho}_\lambda)^d$ is crystalline, so if we instead consider the problem of lifting the crystalline representation $(\rho_\lambda, [d]\xi(\tilde{\rho}_\lambda))$ through the composite isogeny

$$
\widetilde{H} \to H \times \widetilde{Z}^\vee \xrightarrow{\mathrm{id} \times [d]} H \times \widetilde{Z}^\vee,
$$

then again [Conrad 2011, Corollary 6.7] applies to produce a crystalline lift of ρ_λ with the desired Hodge–Tate cocharacters. $\qquad\square$

Here is the key lemma:

Lemma 3.5. *For any place $v \in S_\lambda$ not belonging to the finite set $S \cup T$, the restriction $\mathcal{O}(\rho_\lambda, \psi_\lambda)|_{\Gamma_{F_v}}$ is trivial.*

Proof. Under the assumption on v, both ρ_λ and ψ_λ are crystalline at v. Lemma 3.4 above shows that $\rho_\lambda|_{\Gamma_{F_v}}$ admits a crystalline lift $\tilde{\rho}_\lambda : \Gamma_{F_v} \to \widetilde{H}(\bar{E}_\lambda)$ such that $\xi(\tilde{\rho}_\lambda)$ has the same (labeled) Hodge–Tate cocharacters as $\psi_\lambda|_{\Gamma_{F_v}}$. Since they are both crystalline, it follows (see [Chai et al. 2014, 3.9.7 Corollary]) that $\xi(\tilde{\rho}_\lambda) \cdot \psi_\lambda^{-1}|_{\Gamma_{F_v}}$

is unramified; this is an elaboration of the familiar fact that a crystalline character whose Hodge–Tate weights are zero must be unramified. In particular, replacing the initial lift $\tilde{\rho}_\lambda|_{\Gamma_{F_v}}$ by an unramified twist, we may assume $\xi(\tilde{\rho}_\lambda) = \psi_\lambda$ as homomorphisms $\Gamma_{F_v} \to \widetilde{Z}^\vee(\overline{E}_\lambda)$. But recall that Lemma 3.3 implies that $\mathcal{O}(\rho_\lambda) = \mathcal{O}(\xi(\tilde{\rho}_\lambda))^{-1}$, so we deduce that $\mathcal{O}(\rho_\lambda) \cdot \mathcal{O}(\psi_\lambda)|_{\Gamma_{F_v}}$ is trivial.   □

Since the set of places $S \cup T$ is finite, by the local version of Tate's theorem, the vanishing of $H^2(\Gamma_{F_v}, \mathbb{Q}/\mathbb{Z})$ for all places v of F, we may enlarge $H_1 \cup C$ to some $C[m]$ inside the torus C so as to kill the image of $H^2(\Gamma_{F_v}, H_1 \cap C) \to H^2(\Gamma_{F_v}, C[m])$ for all $v \in S \cup T$. (We emphasize that m only depends on the set of places $S \cup T$ of F and the finite group $H_1 \cap C$.) It follows then from Lemma 3.5 that if λ does not belong to $S \cup T$, then $\mathcal{O}(\rho_\lambda, \psi_\lambda)$ in fact belongs to

$$\mathrussian{Ш}^2_{S \cup T \cup S_\lambda}(F, C[m]) = \ker\left(H^2(\Gamma_{F, S \cup T \cup S_\lambda}, C[m]) \to \bigoplus_{v \in S \cup T \cup S_\lambda} H^2(\Gamma_{F_v}, C[m])\right).$$

We can moreover guarantee that this holds regardless of λ by an additional finite enlargement of m (since the number of exceptional λ is finite). Furthermore, by including the primes dividing $\#(H_1 \cap C)$ in $S \cup T$ (if necessary), we can assume that m is divisible only by primes in $S \cup T$. (Note that inflation to allow additional primes of ramification still has image in the corresponding Shafarevich–Tate group, since Γ_{F_v}/I_{F_v} has cohomological dimension one for all finite places v.) Thus, after these uniform enlargements of m and $S \cup T$ (which we do not reflect in the notation), we have $\mathcal{O}(\rho_\lambda, \psi_\lambda) \in \mathrussian{Ш}^2_{S \cup T \cup S_\lambda}(F, C[m])$.

We are now in a position to apply global duality to analyze the cohomology group $\mathrussian{Ш}^2_{S \cup T \cup S_\lambda}(F, C[m])$. We will need, however, to allow still more primes of ramification in order to kill the class $\mathcal{O}(\rho_\lambda, \psi_\lambda)$; the following crucial lemma allows us to do this in a way that does not depend on λ, but before stating the lemma, we have to recall the Grunwald–Wang theorem (in a somewhat specialized form).

Theorem 3.6 (Grunwald–Wang; see Theorem X.1 of [Artin and Tate 1968]). *Let F be a number field, and let m be a positive integer. Then an element $x \in F^\times$ belongs to $(F^\times)^m$ if and only if x is in $(F_v^\times)^m$ for all places v of F, except when all three of the following conditions, referred to as the special case, hold for the pair (F, m):*

- *Let s_F denote the largest integer r such that $\eta_r = \zeta_{2^r} + \zeta_{2^r}^{-1}$ is an element of F (here ζ_{2^r} denotes a primitive 2^r-th root of unity). Then -1, $2 + \eta_{s_F}$, and $-(2 + \eta_{s_F})$ are nonsquares in F.*

- $\mathrm{ord}_2(m) > s_F$.

- *The set of 2-adic places of F at which -1, $2 + \eta_{s_F}$, and $-(2 + \eta_{s_F})$ are nonsquares in F is empty.*

In the special case, the element $(2 + \eta_{s_F})^{m/2}$ is the unique (up to $(F^\times)^m$-multiple) counterexample to the local-global principle for m-th powers in $F^\times$.

Here is the lemma:

Lemma 3.7. *Recall that $S \cup T$ is a fixed finite set of places of the number field F, and that m is a fixed integer. Let V be a finite set of finite places of F such that*

- *all elements of V are unramified in $F(\mu_m)$,*
- *the places of $F(\mu_m)$ lying above V generate the class group of $F(\mu_m)$, and*
- *every element of $\mathrm{Gal}(F(\mu_m)/F)$ is equal to a (geometric, say) frobenius element at v for some $v \in V$.*

Then for all places λ of E we can deduce:

(1) *If (F, m) is not in the Grunwald–Wang special case, $\mathrm{III}^2_{S\cup T\cup V\cup S_\lambda}(F, C[m])$ is trivial.*

(2) *If (F, m) is in the Grunwald–Wang special case, then the image of the canonical map*
$$\mathrm{III}^2_{S\cup T\cup V\cup S_\lambda}(F, C[m]) \to \mathrm{III}^2_{S\cup T\cup V\cup S_\lambda}(F, C[2m])$$
is trivial.

Proof. First note that such sets V exist, by finiteness of the class number and the Čebotarev density theorem. Since (all places of F above) the primes dividing m are contained in $S \cup T$, an application of Poitou–Tate duality immediately reduces us to showing (as a Galois module, $C[m]$ is $\dim(C)$ copies of $\mathbb{Z}/m$) the following cases:

(1) If (F, m) is not in the Grunwald–Wang special case, then
$$\mathrm{III}^1_{S\cup T\cup V\cup S_\lambda}(F, \mu_m) = 0.$$

(2) If (F, m) is in the special case, then the map
$$\mathrm{III}^1_{S\cup T\cup V\cup S_\lambda}(F, \mu_{2m}) \to \mathrm{III}^1_{S\cup T\cup V\cup S_\lambda}(F, \mu_m)$$
induced by $\mu_{2m} \xrightarrow{2} \mu_m$ is zero.

We first restrict to $\Gamma_{F(\mu_m), S\cup T\cup V\cup S_\lambda}$ (note that this is actually restriction to a subgroup, since $F(\mu_m)/F$ is ramified only at primes in $S \cup T$), obtaining an element of $\mathrm{III}^1_{S\cup T\cup V\cup S_\lambda}(F(\mu_m), \mu_m)$. After this restriction, as we will see, the Grunwald–Wang theorem does not intervene.

To lighten the notation in the rest of the proof, we define $L = F(\mu_m)$ and $Q_\lambda = S \cup T \cup V \cup S_\lambda$. We also refer the reader to the notation established at the beginning of Section 3. Recall that $F(Q_\lambda)$ denotes the maximal extension of F inside $\bar{F}$ that is unramified outside Q_λ; it contains L. Let $\mathcal{O}_{F(Q_\lambda)}$ denote the ring

of Q_λ-integers in $F(Q_\lambda)$ (i.e., the elements of $F(Q_\lambda)$ that are integral outside of places above Q_λ). We then have an exact (Kummer theory) sequence

$$1 \to \mu_m \to \mathcal{O}^\times_{F(Q_\lambda)} \xrightarrow{m} \mathcal{O}^\times_{F(Q_\lambda)} \to 1,$$

and the corresponding long exact sequence in Γ_{L,Q_λ}-cohomology yields an isomorphism

$$\mathcal{O}_L\Big[\frac{1}{Q_\lambda}\Big]^\times / \Big(\mathcal{O}_L\Big[\frac{1}{Q_\lambda}\Big]^\times\Big)^m \xrightarrow{\sim} H^1(\Gamma_{L,Q_\lambda}, \mu_m);$$

critically, surjectivity here follows from the vanishing of $H^1(\Gamma_{L,Q_\lambda}, \mathcal{O}^\times_{F(Q_\lambda)})$, which itself is a consequence of the natural isomorphism $\mathrm{Cl}_{Q_\lambda}(L) \cong H^1(\Gamma_{L,Q_\lambda}, \mathcal{O}^\times_{F(Q_\lambda)})$ [Neukirch et al. 2000, Proposition 8.3.11(ii)] and our assumption that V (and hence Q_λ) generates the class group of L. Restricting the Kummer theory isomorphism to classes that are locally trivial at each place of Q_λ, we also obtain the isomorphism

$$\Big(\mathcal{O}_L\Big[\frac{1}{Q_\lambda}\Big]^\times \cap (L^\times_{Q_\lambda})^m\Big) / \Big(\mathcal{O}_L\Big[\frac{1}{Q_\lambda}\Big]^\times\Big)^m \xrightarrow{\sim} \Sha^1_{Q_\lambda}(L, \mu_m).$$

We claim these groups are trivial. Indeed, take $\alpha \in \mathcal{O}_L[1/Q_\lambda]^\times \cap (L^\times_{Q_\lambda})^m$, and consider the (abelian) extension $L(\alpha^{1/m})/L$. Global class field theory yields the reciprocity isomorphism

$$\mathbb{A}^\times_L / \big(L^\times N_{L(\alpha^{1/m})/L}(\mathbb{A}^\times_{L(\alpha^{1/m})})\big) \xrightarrow{\sim} \mathrm{Gal}(L(\alpha^{1/m})/L),$$

but by assumption the source of this map admits a surjection

$$\mathbb{A}^\times_L / \Big(L^\times L^\times_\infty L^\times_{Q_\lambda} \prod_{w \notin Q_\lambda} \mathcal{O}^\times_{L_w}\Big) \twoheadrightarrow \mathbb{A}^\times_L / \big(L^\times N_{L(\alpha^{1/m})/L}(\mathbb{A}^\times_{L(\alpha^{1/m})})\big).$$

(At unramified places, the image of the norm map contains the local units; and at places in Q_λ, $L(\alpha^{1/m})/L$ is split.) By assumption ($\mathrm{Cl}_{Q_\lambda}(L) = 0$), the source of this surjection is trivial, so $L(\alpha^{1/m}) = L$, and we deduce that $\Sha^1_{Q_\lambda}(L, \mu_m) = 0$.

It follows that inflation identifies the group $\Sha^1_{Q_\lambda}(F, \mu_m)$ with the classes in $H^1(\mathrm{Gal}(L/F), \mu_m)$ that are trivial upon restriction to Q_λ. Since every element of $\mathrm{Gal}(L/F)$ is a frobenius element at some prime in $V \subset Q_\lambda$, $\Sha^1_{Q_\lambda}(F, \mu_m)$ is actually equal to the set of *everywhere* locally trivial classes

$$\Sha^1_{|F|}(F, \mu_m) := \ker\Big(H^1(\Gamma_F, \mu_m) \to \prod_{v \in |F|} H^1(\Gamma_{F_v}, \mu_m)\Big),$$

where $|F|$ denotes the set of all places of F. This is precisely the subject of the Grunwald–Wang theorem, and it is zero if (F, m) is not in the special case. Thus, we need only consider the possibility that (F, m) is in the special case, where $\Sha^1_{|F|}(F, \mu_m)$ has order two, and a representative of the nontrivial class is the (image under the Kummer map of the) element $(2 + \eta_{s_F})^{m/2}$ of $(F^\times)^{m/2}$. This description

holds regardless of m, so in particular the nontrivial class of $\text{III}^1_{|F|}(F, \mu_{2m})$ is represented by $(2 + \eta_{s_F})^m$. Its image under $\mu_{2m} \xrightarrow{2} \mu_m$, which via Kummer theory is induced by the identity map $F^\times \to F^\times$, is again $(2 + \eta_{s_F})^m$, which is now visibly an m-th power, completing the proof.[5] $\qquad\square$

We summarize our conclusion, noting that the value of m in the following corollary may be $2m$ in the earlier notation:

Corollary 3.8. *There is an integer m and a finite set of places $Q \supset S \cup T$, both independent of λ and of the choice of ρ_λ having good reduction outside S and the prescribed Hodge–Tate cocharacters $\{\mu_\tau\}_{\tau: F \hookrightarrow \bar{E}}$, such that the image of $\mathcal{O}(\rho_\lambda, \psi_\lambda)$ in $H^2(\Gamma_{F, Q \cup S_\lambda}, C[m])$ is zero.*

Before proceeding, it is worth noting that the argument just given yields a novel proof of the global version of Tate's vanishing theorem (taking as input the much easier local theorem); it is also a stronger proof, yielding an explicit upper bound on how much ramification has to be allowed, and how much the coefficients need to be enlarged, in order to kill a cohomology class in $H^2(\Gamma_{F,V}, \mathbb{Z}/N)$ for some finite set of places V and integer N.

Corollary 3.9. *Let V be a finite set of places of F, and let N be an integer. Then the image of $H^2(\Gamma_{F,V}, \mathbb{Z}/N)$ in $H^2(\Gamma_{F, V \cup W}, \mathbb{Z}/2NM)$ is trivial, where*

- *M is large enough that for all $v \in V$, the image of*

$$H^2(\Gamma_{F_v}, \mathbb{Z}/N) \to H^2(\Gamma_{F_v}, \mathbb{Z}/NM)$$

  is zero,[6] and

- *once M is fixed as above, W is large enough that*
  - *$V \cup W$ contains (all places above) $2NM$,*
  - *$\text{Cl}_{V \cup W}(F(\mu_{NM})) = 0$, and*
  - *each element of $\text{Gal}(F(\mu_{NM})/F)$ is equal to a frobenius element at w for some $w \in W$.*

(The factors of two are only necessary in the Grunwald–Wang special case.) In particular, $H^2(\Gamma_F, \mathbb{Q}/\mathbb{Z}) = 0$.

Remark 3.10. A different proof of Tate's theorem (without arithmetic duality theorems, but instead relying on a finer study of Hecke characters of F) is given in [Serre 1977, §6.5]. There Serre remarks that Tate originally proved the vanishing theorem using global duality, but further assuming Leopoldt's conjecture; we have of course circumvented Leopoldt here.

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[5]In concrete terms, this says that if an element of $F^\times$ is everywhere locally a $(2m)$-th power, then it is globally an m-th power.

[6]This is easy to make explicit, using local duality, in terms of $\mu_\infty(F_v)$.

Now we return to the conclusion of Corollary 3.8. Let $b_\lambda : \Gamma_{F,Q \cup S_\lambda} \to C[m]$ be a cochain trivializing $\mathcal{O}(\rho_\lambda, \psi_\lambda)$. Then

$$\tilde{\rho}_\lambda = \rho'_\lambda \cdot \psi'_\lambda \cdot b_\lambda : \Gamma_{F,Q \cup S_\lambda} \to \widetilde{H}(\overline{E}_\lambda)$$

is a homomorphism lifting ρ_λ. We claim that $\tilde{\rho}_\lambda$ is, moreover, de Rham at all places in S_λ. To see this, note that under the isogeny $\widetilde{H} \to H \times \widetilde{Z}^\vee$, $\tilde{\rho}_\lambda$ pushes forward to $(\rho_\lambda, \psi_\lambda \xi(b_\lambda))$, the second coordinate being a finite-order twist of ψ_λ (and in particular, de Rham). But now we can invoke the local results of Wintenberger [1995, §1] and Conrad [2011, Theorem 6.2], asserting that a lift of a de Rham representation through an isogeny is de Rham if and only if the Hodge–Tate cocharacter lifts through the isogeny (which is obviously the case here, as ψ was constructed to ensure this).

Finally, we can refine this to the statement that ρ_λ admits a geometric lift that is moreover crystalline at all places of S_λ, provided S_λ does not intersect a certain finite set of primes that is independent of λ and ρ_λ (but somewhat larger than the set Q we have thus far constructed). This will complete the proof of Theorem 1.3.

Proof of Theorem 1.3. We resume the above discussion. So far we have a constructed geometric lifts $\tilde{\rho}_\lambda : \Gamma_{F,Q \cup S_\lambda} \to \widetilde{H}(\overline{E}_\lambda)$, where Q contains $S \cup T$ and whatever other additional primes are needed for the conclusion of Corollary 3.8. The only remaining task is to show that for some (independent of ρ_λ) set P, we can modify the initial lift (by a finite-order twist) to guarantee that it has good reduction outside P. Under the isogeny $\widetilde{H} \to H \times \widetilde{Z}^\vee / \xi(C[m])$, $\tilde{\rho}_\lambda$ pushes forward to

$$\tau_\lambda := (\rho_\lambda, \psi_\lambda \mod \xi(C[m])),$$

which is crystalline for all v in S_λ but not in $S \cup T$. For all $v \in S_\lambda \setminus (S_\lambda \cap (S \cup T))$, $\tilde{\rho}_\lambda|_{\Gamma_{F_v}}$ is of course a de Rham lift of τ_λ, so [Conrad 2011, Theorem 6.2 and Corollary 6.7] (building on [Wintenberger 1995]) shows that $\tau_\lambda|_{\Gamma_{F_v}}$ admits *some* crystalline lift $\tilde{\tau}_{\lambda,v} : \Gamma_{F_v} \to \widetilde{H}(\overline{E}_\lambda)$, and therefore there are finite-order characters $\chi_{\lambda,v} : \Gamma_{F_v} \to C[m]$ such that each $\tilde{\rho}_\lambda|_{\Gamma_{F_v}} \cdot \chi_{\lambda,v}$ is crystalline. We wish to glue the inertial restrictions $\chi_{\lambda,v}|_{I_{F_v}}$ together into a global character, with an independent-of-λ control on the ramification. The cokernel of the restriction map

$$\mathrm{Hom}(\Gamma_{F,Q \cup S_\lambda}, C[m]) \to \bigoplus_{v \in S_\lambda} \mathrm{Hom}(I_{F_v}, C[m])^{\Gamma_{F_v}/I_{F_v}} \qquad (3)$$

may be nontrivial;[7] but we will show that any element of the cokernel is annihilated by appropriate enlargements of Q and m.

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[7]Of course, we only need to consider the cokernel of the map to the direct sum over $v \in S_\lambda \setminus (S_\lambda \cap (S \cup T))$; to lighten the notation we will work with all $v \in S_\lambda$, taking some arbitrary (e.g., trivial) choice of $\chi_{\lambda,v}$ at any places in $(S \cup T) \cap S_\lambda$.

By the congruence subgroup property for GL_1 (a theorem of Chevalley [1951]), there is an ideal $\mathfrak{n}$ of $\mathcal{O}_F$ such that

$$\{x \in \mathcal{O}_F^\times : x \equiv 1 \pmod{\mathfrak{n}}\} \subseteq (\mathcal{O}_F^\times)^m.$$

Let R be the set of primes supporting $\mathfrak{n}$ (note that $\mathfrak{n}$ and R are independent of ρ_λ!), and set

$$U_R = \left\{ (x_v)_{v \in R} \in \prod_{v \in R} \mathcal{O}_{F_v}^\times : x_v \equiv 1 \pmod{\mathfrak{n}} \text{ for all } v \in R \right\}.$$

Then whenever $S_\lambda \cap R = \varnothing$ (so, excluding a finite number of bad λ), consider the character (here and in what follows, we suppress the class field theory identifications)

$$(\chi_{\lambda,v})_{v \in S_\lambda} \times 1 \times 1 \times 1 : \prod_{v \in S_\lambda} \mathcal{O}_{F_v}^\times \times \prod_{v \in R} U_R \times \prod_{v \notin R \cup S_\lambda} \mathcal{O}_{F_v}^\times \times F_\infty^\times \to C[m],$$

which extends by 1 to a character

$$\left(\prod_{v \in S_\lambda} \mathcal{O}_{F_v}^\times \times \prod_{v \in R} U_R \times \prod_{v \notin R \cup S_\lambda} \mathcal{O}_{F_v}^\times \times F_\infty^\times \right) \cdot F^\times \to C[m]$$

(an element of the intersection is a global unit congruent to 1 modulo $\mathfrak{n}$, hence is contained in $(\mathcal{O}_F^\times)^m$, where $\chi_{\lambda,v}$ is obviously trivial). We can then extend from this finite-index subgroup of $\mathbb{A}_F^\times$ to a character $\chi_\lambda : \mathbb{A}_F^\times / F^\times \to \mu_\infty(C)$. In fact, we see that χ_λ can be chosen to be valued in $C[M]$ for m sufficiently large but independent of λ: m can be quantified in terms of the generalized class group of level U_R, but the details do not concern us.

Replacing $\tilde{\rho}_\lambda$ by its finite-order twist

$$\tilde{\rho}_\lambda \cdot \chi_\lambda : \Gamma_{F,Q \cup R \cup S_\lambda} \to \widetilde{H}(\overline{E}_\lambda),$$

we have achieved geometric lifts of ρ_λ with compatible Hodge–Tate cocharacters, and which are crystalline at all places in S_λ outside of $R \cup S \cup T$. $\square$

Remark 3.11. Contrast the final step [Wintenberger 1995, §2.3.5] of Wintenberger's main theorem, where to ensure crystallinity of the lifts he makes a further finite base change on F (having already made several such in order to show lifts exist, as is necessary in his isogeny setup), adding appropriate roots of unity and then passing to a Hilbert class field to kill a cokernel analogous to that of (3). As elsewhere, our argument is orthogonal to Wintenberger's, in allowing additional ramification and larger coefficients, rather than passing to a finite extension of F.

We now deduce some corollaries on finding lifts of ramification-compatible systems whose "similitude characters" (determinant, Clifford norm, etc.) form strongly compatible systems, in the sense that at all finite places their associated Weil group representations are isomorphic (see, e.g., [Barnet-Lamb et al. 2014,

§5.1], where these are called *strictly* compatible). As with Theorem 1.3 and the preceding results, we show a somewhat stronger finiteness result, which applies to all representations with good reduction outside a fixed finite set S. These corollaries will follow from the above results and the Hermite–Minkowski finiteness theorem.

Corollary 3.12. *Let F, S, and $\{\mu_\tau\}$ be as in the statement of Theorem 1.3, except now F may be any number field. Then there exist a finite set of places $P \supset S$ and a finite extension F'/F such that any geometric ρ_λ with good reduction outside S, and with Hodge–Tate cocharacters arising from $\{\mu_\tau\}$ via an embedding $\iota_\lambda : \bar{E} \hookrightarrow \bar{E}_\lambda$, admits a geometric lift $\tilde{\rho}_\lambda : \Gamma_{F, P \cup S_\lambda} \to \widetilde{H}(\bar{E}_\lambda)$ such that the restrictions*

$$\xi(\tilde{\rho}_\lambda) : \Gamma_{F', P \cup S_\lambda} \to \widetilde{Z}^\vee(\bar{E}_\lambda)$$

are equal to the ι_λ-adic realizations of the single (independent of λ and ρ_λ) algebraic Hecke character ψ of $\widetilde{Z}(\mathbb{A}_F)$.

In particular, let $\{\rho_\lambda : \Gamma_{F, S \cup S_\lambda} \to H(\bar{E}_\lambda)\}_\lambda$ be a ramification-compatible system. Then there exist a ramification-compatible system of lifts $\{\tilde{\rho}_\lambda : \Gamma_{F, P \cup S_\lambda} \to \widetilde{H}(\bar{E}_\lambda)\}_\lambda$, and a finite, independent-of-λ extension F'/F such that the restrictions

$$\xi(\tilde{\rho}_\lambda) : \Gamma_{F', P \cup S_\lambda} \to \widetilde{Z}^\vee(\bar{E}_\lambda)$$

form a strongly compatible system.

Proof. We may assume that the number field F is totally imaginary. Consider the lifts $\tilde{\rho}_\lambda : \Gamma_{F, P \cup S_\lambda} \to \widetilde{H}(\bar{E}_\lambda)$ produced by Theorem 1.3. We write $\xi(\tilde{\rho}_\lambda) = \psi_\lambda \cdot \eta_\lambda$, where $\eta_\lambda : \Gamma_{F, P \cup S_\lambda} \to \widetilde{Z}^\vee[M]$ is a finite-order character; the independent-of-λ bound on the order was established within the proof of Theorem 1.3. Moreover, for all $v \in S_\lambda \setminus (S_\lambda \cap P)$, $\tilde{\rho}_\lambda$ and ψ_λ are crystalline at v, so as long as $S_\lambda \cap P$ is empty, η_λ factors through $\Gamma_{F, P} \to \widetilde{Z}^\vee[M]$ (we again use that a finite-order crystalline character is unramified). By the Hermite–Minkowski theorem, there are a finite number of such characters η_λ. For the finite number of bad λ (at which $S_\lambda \cap P \neq \varnothing$), the same finiteness assertion holds. Thus, after a finite base change F'/F, trivializing this finite collection of possible characters η_λ, we see that $\xi(\tilde{\rho}_\lambda)|_{\Gamma_{F', P \cup S_\lambda}} = \psi_\lambda|_{\Gamma_{F', P \cup S_\lambda}}$ for all λ. The second part of the corollary follows since the λ-adic realizations of an abelian L-algebraic representation form a strongly compatible system, as is evident from the construction of ψ_λ, as in, e.g., [Serre 1968]. □

We would like to upgrade this to a compatibility statement not just for the push-forwards $\xi(\tilde{\rho}_\lambda)$, but for the full abelianizations $\tilde{\rho}^{ab} : \Gamma_{F, P \cup S_\lambda} \to \widetilde{H}^{ab}(\bar{E}_\lambda)$. Of course, such a result requires first (taking $\widetilde{H} = H$) having the corresponding assertion for the abelianizations $\rho_\lambda^{ab} : \Gamma_{F, S \cup S_\lambda} \to H^{ab}(\bar{E}_\lambda)$. Here, however, it is of course false without imposing further conditions on the system $\{\rho_\lambda\}_\lambda$ (see Remark 3.15). There are various conditions we might impose on the ρ_λ to ensure (potential) compatibility

of the $\rho_\lambda^{\mathrm{ab}}$. Perhaps most interesting is to restrict the coefficients of $\rho_\lambda^{\mathrm{ab}}$. To that end, we first prove a finiteness result for Galois characters:

Lemma 3.13. *Let F be a number field, and let S be a finite set of places of F. Fix a finite extension E'/E (inside $\bar{E}$), a set $\{m_\tau\}_{\tau:F\hookrightarrow\bar{E}}$ of integers satisfying the Hodge-symmetry condition of Definition 1.1, and an embedding $\iota_\infty : \bar{E} \hookrightarrow \mathbb{C}$. Then there exist a finite extension F'/F, and an algebraic Hecke character α of $\mathbb{A}_{F'}$, such that any geometric character $\omega_\lambda : \Gamma_{F,S\cup S_\lambda} \to \bar{E}_\lambda^\times$*

- *having good reduction outside S;*
- *having labeled Hodge–Tate weights corresponding to $\{m_\tau\}$ via some embedding $\iota_\lambda : \bar{E} \hookrightarrow \bar{E}_\lambda$; and*
- *for which $\omega_\lambda(fr_v)$ belongs to $(E')^\times$ for a density-one set of places v of F;*

will upon restriction $\omega_\lambda|_{\Gamma_{F'}}$ become isomorphic to the ι_λ-adic realization of α.

Proof. We may assume F is totally imaginary. Invoking the Hodge symmetry hypothesis, we apply Lemma 3.1 to produce an algebraic Hecke character α of F whose archimedean components are given in terms of the m_τ, exactly as in Lemma 3.1 (with $\xi(\widetilde{\mu}_\tau) = m_\tau$). Let T denote the finite set of ramified places of α, and let $\mathbb{Q}(\alpha)$ denote the field of coefficients of α (by definition the fixed field of all automorphisms of $\mathbb{C}$ that preserve the nonarchimedean component of α; we will regard $\mathbb{Q}(\alpha)$ as a subfield of $\bar{E}$ via our fixed ι_∞). Thus the ι_λ-adic realizations $\alpha_\lambda : \Gamma_{F,T\cup S_\lambda} \to \bar{E}_\lambda^\times$ have labeled Hodge–Tate weights matching those of ω_λ. Since $\mathbb{Q}(\alpha)$ contains the values $\alpha_\lambda(fr_v)$ for all $v \notin T \cup S_\lambda$, and $\omega_\lambda\alpha_\lambda^{-1}: \Gamma_{F,S\cup T\cup S_\lambda} \to \bar{E}_\lambda^\times$ is finite-order (all of its Hodge–Tate weights are zero), we see that $(\omega_\lambda\alpha_\lambda^{-1})(fr_v)$ belongs to the finite (independent of λ) set $\mu_\infty(E'\mathbb{Q}(\alpha))$ for a density-one set of v. By Čebotarev, the character $\omega_\lambda\alpha_\lambda^{-1}$ takes all of its values in $\mu_\infty(E'\mathbb{Q}(\alpha))$. As long as $S_\lambda \cap (S\cup T)$ is empty, $\omega_\lambda\alpha_\lambda^{-1}$ is moreover unramified at S_λ (because it is crystalline of finite order), so as in Corollary 3.12, there are (again by Hermite–Minkowski) a finite number of such characters $\omega_\lambda\alpha_\lambda^{-1}$. We deduce the existence of a single number field F' over which $\omega_\lambda|_{\Gamma_{F'}} = \alpha_\lambda|_{\Gamma_{F'}}$, for any λ and any ω_λ as in the statement of the lemma. □

We deduce a potential compatibility statement for the full abelianizations $\tilde{\rho}_\lambda^{\mathrm{ab}}$:

Corollary 3.14. *For simplicity, assume that H^{ab} is of multiplicative type. Let F, S, and $\{\mu_\tau\}$ be as in the statement of Theorem 1.3, except with F now allowed to be any number field. Also fix a finite extension E' of E. Then there exist a finite set of places $P \supset S$, a finite extension F'/F, and an algebraic Hecke character β of the split group $\widetilde{D}$ over F' whose dual group over $\bar{E}$ is isomorphic to $(H^{\mathrm{ab}})^0 \otimes_E \bar{E}$ (and we fix such an isomorphism), satisfying the following: if a geometric $\rho_\lambda : \Gamma_{F,S\cup S_\lambda} \to H(\bar{E}_\lambda)$*

- *has good reduction outside S;*
- *has Hodge–Tate cocharacters arising from $\{\mu_\tau\}$ via $\iota_\lambda : \overline{E} \hookrightarrow \overline{E}_\lambda$;*
- *and admits, for some faithful representation r of H^{ab}, a density-one set of places v of F such that the characteristic polynomial $\mathrm{ch}(r \circ \tilde{\rho}_\lambda^{\mathrm{ab}})(fr_v)$ has coefficients in E';*

then there is a geometric lift $\tilde{\rho}_\lambda : \Gamma_{F, P \cup S_\lambda} \to \widetilde{H}(\overline{E}_\lambda)$ having good reduction outside P, such that the restriction $\tilde{\rho}_\lambda^{\mathrm{ab}} : \Gamma_{F', P \cup S_\lambda} \to \widetilde{H}^{\mathrm{ab}}(\overline{E}_\lambda)$ is equal to the ι_λ-adic realization β_λ of β.

In particular, let $\{\rho_\lambda : \Gamma_{F, S \cup S_\lambda} \to H(\overline{E}_\lambda)\}_\lambda$ be a ramification-compatible system, and assume that for some faithful representation r of H^{ab}, some number field E', and for almost all λ, there is a density-one set of places v of F such that the characteristic polynomial $\mathrm{ch}(r \circ \tilde{\rho}_\lambda^{\mathrm{ab}})(fr_v)$ has coefficients in E'. Then there is a ramification-compatible system $\tilde{\rho}_\lambda : \Gamma_{F, P \cup S_\lambda} \to \widetilde{H}(\overline{E}_\lambda)$ lifting ρ_λ, and a finite extension F'/F such that

$$\tilde{\rho}_\lambda^{\mathrm{ab}}|_{\Gamma_{F'}} : \Gamma_{F', P \cup S_\lambda} \to \widetilde{H}^{\mathrm{ab}}(\overline{E}_\lambda)$$

forms a strongly compatible system.

Proof. The proof follows familiar lines. Since C is central, the abelianization $\widetilde{H}^{\mathrm{ab}}$ is simply $\widetilde{H}/H_1^{\mathrm{der}}$, so there is a natural map

$$f : \widetilde{H}^{\mathrm{ab}} \to \widetilde{H}/H_1 \times H/\mathrm{im}(H_1^{\mathrm{der}}) = \widetilde{Z}^\vee \times H^{\mathrm{ab}}$$

under which $\tilde{\rho}_\lambda^{\mathrm{ab}}$ pushes forward to $(\xi(\tilde{\rho}_\lambda), \rho_\lambda^{\mathrm{ab}})$. (We have chosen $\{\tilde{\rho}_\lambda\}_\lambda$ as in Theorem 1.3 and Corollary 3.12, of course.) First we claim that a conclusion analogous to that of the corollary holds for the pair $(\xi(\tilde{\rho}_\lambda), \rho_\lambda^{\mathrm{ab}})$, and certainly it suffices to check this independently for the two components. The assertion for $\xi(\tilde{\rho}_\lambda)$ is Corollary 3.12, and for $\rho_\lambda^{\mathrm{ab}}$ it follows easily from Lemma 3.13 (first reduce, by a finite base change, to the case where H^{ab} is connected, using the fact that $\pi_0(H^{\mathrm{ab}})$ is of course finite and independent of λ). Thus, letting D denote a split torus whose dual group is identified with $(H^{\mathrm{ab}})^0$, there exists a finite extension F_1/F such that $f(\tilde{\rho}_\lambda^{\mathrm{ab}})|_{\Gamma_{F_1, P \cup S_\lambda}}$ is the ι_λ-adic realization of a Hecke character (which does not depend on λ or ρ_λ) of $\widetilde{Z} \times D$.

Now suppose that β is a Hecke character of $\widetilde{D}$ for which the ι_λ-adic realization $\beta_\lambda : \Gamma_{F, T \cup S_\lambda} \to (\widetilde{H}^{\mathrm{ab}})^0(\overline{E}_\lambda)$ has labeled Hodge–Tate cocharacters matching those of $\tilde{\rho}_\lambda^{\mathrm{ab}}$. Since

$$f(\tilde{\rho}_\lambda^{\mathrm{ab}} \cdot \beta_\lambda^{-1})|_{\Gamma_{F_1, P \cup T \cup S_\lambda}}$$

is automorphic (independently of λ, ρ_λ) of finite order, it is trivial after a finite base change F_2/F_1. Now observe that the kernel of f is finite, so $(\tilde{\rho}_\lambda^{\mathrm{ab}} \cdot \beta_\lambda^{-1})|_{\Gamma_{F_2, P \cup T \cup S_\lambda}}$ has finite order, bounded only in terms of $\#\ker(f)$, and is crystalline away from $P \cup T$;

as before, we find a further finite extension F_3/F_2 such that $(\tilde{\rho}_\lambda^{\mathrm{ab}} \cdot \beta_\lambda^{-1})|_{\Gamma_{F_3}} = 1$. The conclusion of the corollary then holds with $F' = F_3$. $\square$

Remark 3.15.   • It does not suffice to ask for a fixed number field E such that all $\rho_\lambda^{\mathrm{ab}}$ are valued in $H^{\mathrm{ab}}(E_\lambda)$. For instance, taking $F = \mathbb{Q}$ and $S = \{p\}$, and for all n choosing a prime $\ell_n \equiv 1 \pmod{\varphi(p^n)}$, we can define $\rho_{\ell_n} : \Gamma_{\mathbb{Q},\{p\}} \to \mathbb{Q}_{\ell_n}^\times$ as the composition of the mod p^n cyclotomic character with an inclusion $(\mathbb{Z}/p^n\mathbb{Z})^\times \hookrightarrow \mu_{\ell_n-1} \hookrightarrow \mathbb{Q}_{\ell_n}^\times$, and for all $\ell \notin \{\ell_n\}_n$ we can take ρ_ℓ to be the trivial character. Then $\{\rho_\ell : \Gamma_{\mathbb{Q},\{p\}} \to \mathbb{Q}_\ell^\times\}_\ell$ is an abelian, ramification-compatible system that does not become a strongly compatible system after any finite base change.

• Having only hypothesized ramification-compatibility for the $\{\rho_\lambda\}_\lambda$, we cannot hope for the stronger conclusion that the $\{\tilde{\rho}_\lambda^{\mathrm{ab}}\}_\lambda$ form a strongly compatible system over F itself.

General multiplicative-type quotients. In fact, the argument of Theorem 1.3 directly implies the main theorem of [Wintenberger 1995], as well as a generalization to lifting through quotients where the kernel is central of multiplicative type. We thus obtain an essentially different proof (and generalization) of Wintenberger's result. In this section, we briefly describe how this works.

Corollary 3.16 (Wintenberger). *Let $H_1 \to H$ be a central isogeny of linear algebraic groups over E, and let S be a finite set of places of F. Then there exist a finite extension F'/F and a finite set of places $P \supset S$ of F such that any geometric representation*

$$\rho_\lambda : \Gamma_{F,S\cup S_\lambda} \to H(E_\lambda)$$

having

   • *good reduction outside S, and*

   • *labeled Hodge–Tate cocharacters that lift to H_1,*

lifts to a geometric representation $\rho_\lambda' : \Gamma_{F',P\cup S_\lambda} \to H_1(E_\lambda)$, which moreover has good reduction outside P.

Proof. We begin by replacing F by a finite extension F_0 such that image of $\rho_\lambda|_{\Gamma_{F_0}}$ is contained in the image of $H_1(E_\lambda) \to H(E_\lambda)$. That such an extension, depending only on $H_1 \to H$, S, and F, exists follows as in [Wintenberger 1995, 2.3.2], and we do not repeat the argument. We note, though, that making this construction in an independent-of-λ fashion already uses liftability of the Hodge–Tate cocharacters. (If we were not concerned with preserving E_λ-rationality of the lift, then we could skip this step.) It is then possible to build an obstruction class $\mathcal{O}(\rho_\lambda) \in H^2(\Gamma_{F_0,S\cup S_\lambda}, \ker(H_1 \to H)(E_\lambda))$ via a topological lift ρ_λ' to $H_1(E_\lambda)$.

Embed $\ker(H_1 \to H) \otimes_E \bar{E}$ into a torus C, and with the kernel embedded antidiagonally, form the new group $\widetilde{H} = (H_1 \times C)/\ker(H_1 \to H)$. The surjection $\widetilde{H} \to H$ now has kernel equal to a central torus C, and as before we let $\widetilde{Z}^\vee$ be the (torus) quotient $\widetilde{H}/H_1$. By hypothesis, we can lift the Hodge–Tate cocharacters of ρ_λ to H_1; when pushed forward to $\widetilde{Z}^\vee$, these lifts are of course trivial. Thus, in the notation of Lemma 3.1, we may take the trivial Hecke character $\psi = 1$ of $\widetilde{Z}(\mathbb{A}_{F_0})$. For topological lifts ψ'_λ to $C(E_\lambda)$ (as in Lemma 3.3) of the (trivial) λ-adic realizations ψ_λ, we may of course also take $\psi'_\lambda = 1$. Theorem 1.3 then produces a finite set of primes $P \supset S$ and an integer M, both only depending on $H_1 \to H$, S, and F, and a geometric lift $\tilde{\rho}_\lambda : \Gamma_{F_0, P \cup S_\lambda} \to H_1(E_\lambda) \cdot C[M]$ such that $\tilde{\rho}_\lambda$ has good reduction outside P. (The assertion that $\tilde{\rho}_\lambda$ is valued in the subset $H_1(E_\lambda) \cdot C[M]$ of $\widetilde{H}(\bar{E}_\lambda)$ follows from the explicit description of $\tilde{\rho}_\lambda$, since ρ'_λ lands in $H_1(E_\lambda)$, and ψ'_λ is trivial.) For all λ for which $S_\lambda \cap P = \varnothing$, $\xi(\tilde{\rho}_\lambda) : \Gamma_{F_0, P \cup S_\lambda} \to \widetilde{Z}^\vee[M]$ is also unramified at S_λ, and all such characters are trivialized by a common finite extension F_1/F_0. For the finite number of λ such that $S_\lambda \cap P$ is nonempty, we can again trivialize the possible $\xi(\tilde{\rho}_\lambda)$ by restricting to a common finite extension F_2/F_0. Taking $F' = F_1 F_2$, all $\tilde{\rho}_\lambda|_{\Gamma_{F', P \cup S_\lambda}}$ land in $H_1(E_\lambda)$, proving the corollary. $\square$

Remark 3.17. For Wintenberger's result, take $E = \mathbb{Q}$. He also shows [1995, 2.3.6] that there is a second finite extension F''/F' (only depending on $H_1 \to H$, F, and S) such that any two lifts ρ'_λ as in the corollary become equal after restriction to $\Gamma_{F''}$. This refinement similarly follows in our setup, but there is no need to repeat Wintenberger's argument.

Here is the more general version with multiplicative-type kernels. Note that, as with Theorem 1.3, but unlike Corollary 3.16, it makes use of a "Hodge symmetry" hypothesis.

Corollary 3.18. *Let $H' \to H$ be a surjection of linear algebraic groups over E whose kernel is central and of multiplicative type. Let F be a number field, and let S be a finite set of places of F containing the archimedean places. Fix a set of cocharacters $\{\mu_\tau\}_{\tau : F \hookrightarrow \bar{E}}$ as in part (2) of Definition 1.1, and moreover, assume that each μ_τ lifts to a cocharacter of H'.*

Then there exist a finite set of places $P \supset S$, and a finite extension F'/F, such that any geometric representation $\rho_\lambda : \Gamma_{F, S \cup S_\lambda} \to H(\bar{E}_\lambda)$ having good reduction outside S, and whose Hodge–Tate cocharacters arise from the set $\{\mu_\tau\}_{\tau : F \hookrightarrow \bar{E}}$ via some embedding $\bar{E} \hookrightarrow \bar{E}_\lambda$, admits a geometric lift $\tilde{\rho}_\lambda : \Gamma_{F', P \cup S_\lambda} \to H'(\bar{E}_\lambda)$ having good reduction outside P.

In particular, if $\{\rho_\lambda : \Gamma_{F, S \cup S_\lambda} \to H(\bar{E}_\lambda)\}_\lambda$ is a ramification-compatible system with Hodge cocharacter $\{\mu_\tau\}_{\tau : F \hookrightarrow \bar{E}}$, then there exist a finite set of places $P \supset S$, a finite extension F'/F, and lifts $\tilde{\rho}_\lambda : \Gamma_{F', P \cup S_\lambda} \to H'(\bar{E}_\lambda)$ such that $\{\tilde{\rho}_\lambda\}_\lambda$ is a ramification-compatible system.

Proof. As in the proofs of Theorem 1.3 and Corollary 3.16, we construct an isogeny complement $H_1 \subset H$ to $\ker(H' \to H)$, as well as an enlargement $\widetilde{H} \supset H'$ surjecting onto H with a central torus kernel. We then run the argument of Theorem 1.3, starting from lifts $\{\mu'_\tau\}$ to H' of the Hodge cocharacters: the Hecke character ψ (in the notation of that proof) then constructed has λ-adic realizations that push-forward to finite-order characters $\psi_\lambda : \Gamma_{F, T \cup S_\lambda} \to \widetilde{H}/H'(\overline{E}_\lambda)$, and from here it is easy to proceed; we omit the details, since the argument will by now be familiar. □

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