Algebra & Number Theory

Volume 11
2017
No. 4

msp

Algebra & Number Theory

msp.org/ant

See inside back cover or msp.org/ant for submission instructions.

The subscription price for 2017 is US $325/year for the electronic version, and $520/year (+$55, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ALGEBRA AND NUMBER THEORY 11:4 (2017)
dx.doi.org/10.2140/ant.2017.11.767

Cup products of line bundles on homogeneous varieties and generalized PRV components of multiplicity one

Ivan Dimitrov and Mike Roth

Let $X = G/B$ and let L_1 and L_2 be two line bundles on X. Consider the cup-product map

$$H^{d_1}(X, L_1) \otimes H^{d_2}(X, L_2) \xrightarrow{\cup} H^{d}(X, L),$$

where $L = L_1 \otimes L_2$ and $d = d_1 + d_2$. We answer two natural questions about the map above: When is it a nonzero homomorphism of representations of G? Conversely, given generic irreducible representations V_1 and V_2, which irreducible components of $V_1 \otimes V_2$ may appear in the right hand side of the equation above? For the first question we find a combinatorial condition expressed in terms of inversion sets of Weyl group elements. The answer to the second question is especially elegant: the representations V appearing in the right hand side of the equation above are exactly the generalized PRV components of $V_1 \otimes V_2$ of stable multiplicity one. Furthermore, the highest weights $(\lambda_1, \lambda_2, \lambda)$ corresponding to the representations (V_1, V_2, V) fill up the generic faces of the Littlewood–Richardson cone of G of codimension equal to the rank of G. In particular, we conclude that the corresponding Littlewood–Richardson coefficients equal one.

Research of I. Dimitrov and M. Roth was partially supported by NSERC grants.
MSC2010: primary 14F25; secondary 17B10.
Keywords: Homogeneous variety, Littlewood–Richardson coefficient, Borel–Weil–Bott theorem, PRV component.

1. Introduction

1.1. ***Main problems.*** The main object of study of this paper is the cup-product map

$$\mathrm{H}^{d_1}(\mathrm{X},\mathrm{L}_1)\otimes\cdots\otimes\mathrm{H}^{d_k}(\mathrm{X},\mathrm{L}_k)\xrightarrow{\cup}\mathrm{H}^{d}(\mathrm{X},\mathrm{L}),\tag{1.1.1}$$

where $\mathrm{X}=\mathrm{G}/\mathrm{B}$, G is a semisimple algebraic group over an algebraically closed field of characteristic zero, B is a Borel subgroup of G, $\mathrm{L}_1,\ldots,\mathrm{L}_k$ are arbitrary line bundles on X, $\mathrm{L}=\mathrm{L}_1\otimes\cdots\otimes\mathrm{L}_k$, $d_1,\ldots,d_k$ are nonnegative integers, and $d=d_1+\cdots+d_k$.

We assume that both sides of (1.1.1) are nonzero for otherwise the cup-product map is the zero map. Without loss of generality we may also assume that the line bundles $\mathrm{L}_1,\ldots,\mathrm{L}_k$, and L are G-equivariant; then both sides of (1.1.1) carry a natural G-module structure and the cup-product map is G-equivariant. Furthermore by the Borel–Weil–Bott theorem there are irreducible representations $\mathrm{V}_{\mu_1},\ldots,\mathrm{V}_{\mu_k}$, and V_μ so that $\mathrm{H}^{d_i}(\mathrm{X},\mathrm{L}_i)=\mathrm{V}^*_{\mu_i}$ for $i=1,\ldots,k$, and $\mathrm{H}^d(\mathrm{X},\mathrm{L})=\mathrm{V}^*_\mu$ as representations of G. The dual of (1.1.1) is thus a G-homomorphism

$$\mathrm{V}_\mu\to\mathrm{V}_{\mu_1}\otimes\cdots\otimes\mathrm{V}_{\mu_k}.\tag{1.1.2}$$

Since $\mathrm{V}_{\mu_1},\ldots,\mathrm{V}_{\mu_k}$, and V_μ are irreducible representations, (1.1.1) is either surjective or zero; respectively, (1.1.2) is either injective or zero. This leads us naturally to the two main problems of this paper.

Problem I. When is (1.1.1) a surjection of nontrivial representations?

Problem II. For which $(k+1)$-tuples $(\mathrm{V}_{\mu_1},\ldots,\mathrm{V}_{\mu_k},\mathrm{V}_\mu)$ of irreducible representations of G can V_μ be realized as a component of $\mathrm{V}_{\mu_1}\otimes\cdots\otimes\mathrm{V}_{\mu_k}$ via (1.1.2) for appropriate line bundles $\mathrm{L}_1,\ldots,\mathrm{L}_k$ on X?

We call an irreducible representation V_μ that can be embedded into $\mathrm{V}_{\mu_1}\otimes\cdots\otimes\mathrm{V}_{\mu_k}$ via (1.1.2) a *cohomological component of* $\mathrm{V}_{\mu_1}\otimes\cdots\otimes\mathrm{V}_{\mu_k}$. A variation of Problem II is to determine the cohomological components of $\mathrm{V}_{\mu_1}\otimes\cdots\otimes\mathrm{V}_{\mu_k}$, for $\mathrm{V}_{\mu_1},\ldots,\mathrm{V}_{\mu_k}$ fixed.

With the exception of some quite degenerate cases for Problem II, we provide a complete solution to both problems.

1.2. ***Solution of Problem I.*** Fix a maximal torus $\mathrm{T}\subseteq\mathrm{B}$. The G-equivariant line bundles on X are in one-to-one correspondence with the characters of T. For a character λ of T, we denote by L_λ the line bundle on X corresponding to the one dimensional representation of B on which T acts via $-\lambda$.

The affine action of the Weyl group $\mathcal{W}$ of G on the lattice of T-characters Λ is defined as

$$w\cdot\lambda=w(\lambda+\rho)-\rho,$$

where ρ, as usual, denotes the half-sum of the roots of B. A character $\lambda \in \Lambda$ is *regular* if there exists a (necessarily unique) element $w \in \mathcal{W}$ such that $w \cdot \lambda$ is a dominant character. Following Kostant [1961, Definition 5.10], we define *the inversion set* Φ_w *of* $w \in \mathcal{W}$ as the set $\Phi_w = w^{-1}\Delta^- \cap \Delta^+$, where $\Delta^- = -\Delta^+$ is the set of negative roots of G.

Let $\lambda_1, \ldots, \lambda_k \in \Lambda$ be the (regular) characters such that $L_i = L_{\lambda_i}$ for $1 \leqslant i \leqslant k$. Then $L = L_\lambda$, where $\lambda = \sum_{i=1}^k \lambda_i$. Assume that λ is also regular and denote by w and $w_1, \ldots, w_k$ the Weyl group elements for which $w \cdot \lambda$ and $w_i \cdot \lambda_i$, for $1 \leqslant i \leqslant k$, are dominant. With this notation we prove the following criterion for surjectivity of (1.1.1).

Theorem I. *For any semisimple* G, *if* $H^d(X, L_\lambda) \neq 0$, *then the cup-product map* (1.1.1) *is surjective if and only if*

$$\Phi_w = \bigsqcup_{i=1}^{k} \Phi_{w_i}. \tag{1.2.1}$$

Studying the structure of $(k+1)$-tuples $(w_1, \ldots, w_k, w)$ satisfying (1.2.1) is an interesting combinatorial problem which we do not address here. A recursive description of such $(k+1)$-tuples in types A, B, and C is given in [Dewji et al. 2017]. For some open questions concerning (1.2.1) see the expository article [Dimitrov and Roth 2009].

1.3. ***Solution of Problem II.*** We say that a component V_μ of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ has *stable multiplicity one* if the multiplicity of $V_{m\mu}$ in $V_{m\mu_1} \otimes \cdots \otimes V_{m\mu_k}$ is one for all $m \gg 0$. We say that V_μ is a *generalized PRV component of* $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ if there exist $w_1, \ldots, w_k$, and $w \in \mathcal{W}$ such that $w^{-1}\mu = w_1^{-1}\mu_1 + \cdots + w_k^{-1}\mu_k$. (See Sections 2.3 and 6.1 for further discussion of these conditions.)

Theorem II. (a) *Let* V_μ *be a cohomological component of* $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$. *Then* V_μ *is a generalized PRV component of* $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ *of stable multiplicity one.*

(b) *Conversely, assume that* V_μ *is a generalized PRV component of* $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ *of stable multiplicity one. If, in addition, one of the following holds*:

(i) *at least one of* $\mu_1, \ldots, \mu_k$ *or* μ *is strictly dominant*,

(ii) G *is a simple classical group or a product of simple classical groups*,

then V_μ *is a cohomological component of* $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$.

It is unfortunate that in part (b) above we require condition (i) or (ii). Indeed, we believe that we do not need these conditions but we impose them due to our inability to overcome a combinatorial problem.

Remark. In type A a conjecture of Fulton, proved by Knutson, Tao, and Woodward [Knutson et al. 2004, §6.1, §7] states that if V_μ is a component of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$

of multiplicity one, then $V_{m\mu}$ has multiplicity one in $V_{m\mu_1} \otimes \cdots \otimes V_{m\mu_k}$ for all $m \geqslant 1$. Together with Theorem II, this means that in type A a component V_μ is a cohomological component of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ if and only if V_μ is a PRV component of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ of multiplicity one.[1]

1.4. ***Representation-theoretic implications of Theorem II.*** The representation-theoretic significance of Theorem II is twofold: it provides both a geometric construction of special components of a tensor product via the Bott theorem, and a new way of generalizing the classical PRV component.

The Borel–Weil–Bott theorem provides a geometric realization of every irreducible representation of G as the cohomology (in any degree) of an appropriate line bundle on X. In particular, every irreducible representation equals the space of global sections of a unique line bundle on X. In this sense the Borel–Weil theorem (the statement about cohomology in degree zero) suffices since the Bott theorem (the statement about higher cohomology) yields the same representations. However, in addition to being representations, the cohomology groups carry a ring structure induced from the cup product. Theorem II employs this structure to give a geometric realization of certain components of a tensor product of representations. As far as we know this is the first use of the Bott theorem for a geometric construction of representations in the case when G is a semisimple algebraic group over a field of characteristic zero.

We are borrowing the term "generalized PRV component" from the case when $k = 2$. Parthasarathy, Ranga Rao, and Varadarajan [Parthasarathy et al. 1967] established that if μ is in the $\mathcal{W}$-orbit of $\mu_1 + w_0\mu_2$ (where w_0 denotes the longest element of $\mathcal{W}$), then V_μ is a component of $V_{\mu_1} \otimes V_{\mu_2}$. Moreover, they proved that V_μ has multiplicity one in, and is the smallest component of, $V_{\mu_1} \otimes V_{\mu_2}$. It is true more generally that if μ is in the $\mathcal{W}$-orbit of $\mu_1 + v\mu_2$ (where v is now an arbitrary element of $\mathcal{W}$) then V_μ is again a component of $V_{\mu_1} \otimes V_{\mu_2}$; this was established independently by Kumar [1988] and Mathieu [1989].

Unlike the original PRV component, a generalized PRV component V_μ may have multiplicity greater than one in $V_{\mu_1} \otimes V_{\mu_2}$. However, by Theorem II, every cohomological component is a generalized PRV component of stable multiplicity one. The cohomological components also retain an aspect of the minimality of the original PRV component: every cohomological component of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ is extreme among all components of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$. These properties of cohomological components suggest that they may be viewed as the "true" analog of the original PRV component.

Figure 1 illustrates Theorem II in the case $k = 2$.

[1]This is not true in other types; see the component $V_{1,0}$ of $V_{1,1} \otimes V_{1,1}$ in the middle example of Figure 1.

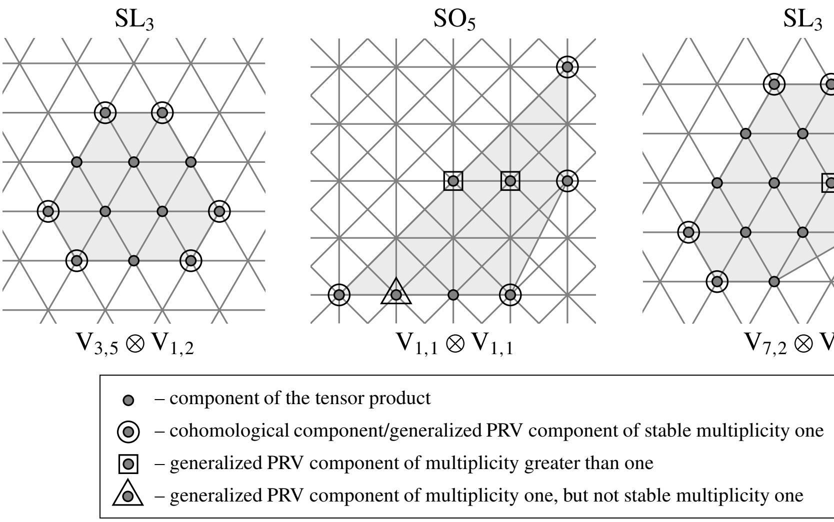

Figure 1. Illustration of Theorem II when $k = 2$.

1.5. ***Other results.*** In conclusion we mention several other results which may be of independent interest.

The cup product and Schubert calculus. Recall that a basis for the cohomology ring $\mathrm{H}^*(\mathrm{X}, \mathbb{Z})$ of $\mathrm{X} = \mathrm{G}/\mathrm{B}$ is given by the classes of the Schubert cycles $\{[\mathrm{X}_w]\}_{w \in \mathcal{W}}$ indexed by the elements of the Weyl group $\mathcal{W}$. The dual basis $\{[\Omega_w]\}_{w \in \mathcal{W}}$, is given by $\Omega_w := \mathrm{X}_{w_0 w}$. With the notation of Section 1.2 we prove the following:

Theorem III. *For any semisimple algebraic group* G,

(a) *if* $\bigcap_{i=1}^k [\Omega_{w_i}] \cdot [\mathrm{X}_w] = 1$ *then the cup-product map* (1.1.1) *is surjective*;

(b) *if* $\bigcap_{i=1}^k [\Omega_{w_i}] \cdot [\mathrm{X}_w] = 0$ *then the cup-product map* (1.1.1) *is zero.*

We use Theorem III as stated above and a variation of its proof to prove Theorem I. In general it is not known if condition (1.2.1) implies that $\bigcap_{i=1}^k [\Omega_{w_i}] \cdot [\mathrm{X}_w] = 1$. In [Dimitrov and Roth ≥ 2017] we show that this is the case when G is a classical group or G_2; We do not know if condition (1.2.1) implies that the intersection number is one in the other exceptional cases.

Diagonal Bott–Samelson–Demazure–Hansen varieties. We construct a class of varieties which generalize the Bott–Samelson–Demazure–Hansen varieties. One way to understand these varieties is a resolution of singularities of the total space of intersections of translates of Schubert varieties; see Theorem 3.7.4. Other notable results related to this construction include Lemma 3.8.1, which controls the multiplicity of cohomological components, and Theorem 3.9.1, which provides a new proof of the necessity of the inequalities determining the Littlewood–Richardson cone. These varieties have applications outside this paper. For instance, in a future paper we use them to establish multiplicity bounds for the Littlewood–Richardson coefficients generalizing the Klymik bound, each of which has the same asymptotic order of growth as the multiplicity function, with each "centered" around a particular cohomological component (in a way that the Klymik bound appears as the version for the highest weight component). These varieties are also used in [Roth 2011] to prove reduction rules for Littlewood–Richardson coefficients.

1.6. ***Related Work.*** After the initial version of this paper appeared on the arXiv, other authors have worked on related ideas. V. Tsanov [2013], considers the more general situation of an embedding $\mathrm{G}_1 \hookrightarrow \mathrm{G}_2$ of complex semisimple Lie groups, inducing an embedding

$$\mathrm{X}_1 := \mathrm{G}_1/\mathrm{B}_1 \hookrightarrow \mathrm{X}_2 := \mathrm{G}_2/\mathrm{B}_2,$$

where B_1 and B_2 are nested Borel subgroups. The main result of [Tsanov 2013] extends Theorem I to this setting, giving necessary and sufficient conditions for the pullback map $\mathrm{H}^d(\mathrm{X}_1, \mathrm{L}|_\mathrm{X}) \leftarrow \mathrm{H}^d(\mathrm{X}_2, \mathrm{L})$ to be nonzero, when L is an equivariant bundle on X_2; see [Tsanov 2013, Theorem 2.2]. The arguments in [Tsanov 2013] use Lie algebra cohomology, and are quite different in character from the arguments of this paper.

In a preprint, N. Ressayre [2009, Theorem 1] states that every generalized PRV component of stable multiplicity one is a cohomological component. That is, this result states that part (b) of Theorem II holds without requiring either of the conditions (i) or (ii) of (b).

Finally, the varieties $\mathrm{X} = \mathrm{G}/\mathrm{B}$ considered in this paper have the property that they are projective varieties acted on transitively by an algebraic group. There is another natural class of varieties also fitting this description, namely Abelian varieties. Here Mumford's index theorem and the theorem on irreducibility of the theta-group representation take the place of the Borel–Weil–Bott theorem. N. Grieve [2014] proves results on the surjectivity of cup-product maps between cohomology of line bundles on Abelian varieties, again subject to certain combinatorial restrictions.

2. Notation and background results

2.1. ***Notation and conventions.*** The ground field is algebraically closed of characteristic zero. Throughout the paper we fix a semisimple connected algebraic group G, a Borel subgroup $\mathrm{B} \subset \mathrm{G}$, and a maximal torus $\mathrm{T} \subset \mathrm{B}$. All parabolic subgroups we consider contain T. The Lie algebras of algebraic groups are denoted by Fraktur letters, e.g., $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{t}$, etc. We use the term "G-module" instead of "representation of G" to avoid differentiating between representations of algebraic groups and modules over the respective Lie algebras; likewise, since T is fixed, we use the term "weight" both for characters of T and weights of $\mathfrak{t}$; in particular we only consider integral weights of $\mathfrak{t}$.

The point

$$wB/B \in X_w \subseteq X = G/B,$$

where $w \in \mathcal{W}$ and X_w is the corresponding Schubert variety, is denoted by w for short. If $M = G/P$ for some parabolic P we similarly use w to indicate the point $wP/P \in M$.

If Λ is the lattice of weights of T we denote the group ring of Λ by $\mathbb{Z}[\Lambda]$, i.e.,

$$\mathbb{Z}[\Lambda] = \left\{ \sum_{i=1}^{k} c_i e^{\lambda_i} \,\middle|\, c_i \in \mathbb{Z}, \lambda_i \in \Lambda \right\}.$$

For a T-module $\mathcal{M}$, the *formal character of* $\mathcal{M}$ is

$$\operatorname{Ch} \mathcal{M} = \sum_{\lambda \in \Lambda} \dim \mathcal{M}^{\lambda} e^{\lambda} \in \mathbb{Z}[\Lambda],$$

where

$$\mathcal{M}^{\lambda} = \{ x \in \mathcal{M} \mid t \cdot x = \lambda(t) x \text{ for every } t \in \mathfrak{t} \}.$$

All formal characters discussed in this paper are contained in $\mathbb{Z}[\Delta]$. For a subset $\Phi \subseteq \Delta$, the formal character of $\bigoplus_{\alpha \in \Phi} \mathfrak{g}^{\alpha}$ is denoted by $\langle \Phi \rangle$, i.e.,

$$\langle \Phi \rangle = \sum_{\alpha \in \Phi} e^{\alpha}.$$

If w is an element of the Weyl group $\mathcal{W}$, then $\ell(w)$ means the length of any minimal expression giving w as a product of simple reflections. If $\underline{v}$ is a word in the simple reflections, then $\ell(\underline{v})$ is the number of reflections in the word. Note that, if $\underline{v}$ is a word in simple reflections, and $v \in \mathcal{W}$ is the corresponding element of the Weyl group, then $\ell(\underline{v}) = \ell(v)$ if and only if $\underline{v}$ is a reduced word. If $\underline{v} = s_{i_1} \cdots s_{i_m}$ is a nonempty word, we denote by $\underline{v}_R$ the word $s_{i_1} \cdots s_{i_{m-1}}$ obtained from $\underline{v}$ by dropping the rightmost reflection in $\underline{v}$. If $\underline{\boldsymbol{v}} = (\underline{v}_1, \ldots, \underline{v}_k)$ is a sequence of words then we set $\ell(\underline{\boldsymbol{v}}) = \sum_{i=1}^{k} \ell(\underline{v}_i)$.

A list of symbols used in the paper can be found on page 812.

2.2. ***Inversion sets.*** Let Δ^+ be the set of positive roots of $\mathfrak{g}$ (with respect to B). Following Kostant [1961, Definition 5.10], for any element w of the Weyl group $\mathcal{W}$ we define Φ_w, the *inversion set* of w, to be the set of positive roots sent to negative roots by w, i.e.,

$$\Phi_w := w^{-1} \Delta^- \cap \Delta^+. \tag{2.2.1}$$

For a subset Φ of Δ^+, we set $\Phi^{\mathrm{c}} := \Delta^+ \setminus \Phi$. We will need the following formulas, which follow easily from the definition:

$$\Phi_{w_0 w} = \Phi_w^{\mathrm{c}}, \tag{2.2.2}$$

$$w^{-1}\Delta^+ = \Phi_w^{\mathrm{c}} \sqcup -\Phi_w, \tag{2.2.3}$$

$$w^{-1}\cdot 0 = w^{-1}\rho - \rho = -\sum_{\alpha\in\Phi_w} \alpha. \tag{2.2.4}$$

2.3. ***Generalized PRV components.*** For fixed dominant weights μ_1, μ_2, and μ it is clear that the two conditions,

(a) there exist w_1, w_2, and w in $\mathcal{W}$ such that $w^{-1}\mu = w_1^{-1}\mu_1 + w_2^{-1}\mu_2$,

(b) there exists v in $\mathcal{W}$ such that μ is in the $\mathcal{W}$-orbit of $\mu_1 + v\mu_2$,

are equivalent. If these conditions are satisfied we call V_μ a *generalized PRV component of* $\mathrm{V}_{\mu_1} \otimes \mathrm{V}_{\mu_2}$.

As is suggested by the name, but is far from obvious from the definition, every generalized PRV component of $\mathrm{V}_{\mu_1} \otimes \mathrm{V}_{\mu_2}$ is in fact a component of the tensor product $\mathrm{V}_{\mu_1} \otimes \mathrm{V}_{\mu_2}$ of G-modules. This was first proved when $v = w_0$ (i.e., when μ is in the $\mathcal{W}$-orbit of $\mu_1 + w_0\mu_2$) in [Parthasarathy et al. 1967]. In the literature this component is referred to simply as the *PRV component*. The general case, that V_μ is a component of $\mathrm{V}_{\mu_1} \otimes \mathrm{V}_{\mu_2}$ for an arbitrary v, became known as the PRV conjecture, and was established independently by Kumar [1988] and Mathieu [1989].

In the present paper we extend the notion of generalized PRV component to components of the tensor product of k irreducible G-modules for $k \geqslant 2$. We call V_μ a *generalized PRV component of* $\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}$ if there exist $w_1, \ldots, w_k$, and w in $\mathcal{W}$ such that $w^{-1}\mu = \sum_{i=1}^k w_i^{-1}\mu_i$. A straightforward induction from the case $k = 2$ implies that every generalized PRV component of $\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}$ is a component of the tensor product $\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}$ of G-modules. We record the special case when $\mu = 0$ for use in the proof of Theorem I.

Lemma 2.3.1. *For any dominant weights $\mu_1, \ldots, \mu_k$, and Weyl group elements $w_1, \ldots, w_k$, if $\sum_{i=1}^k w_i^{-1}\mu_i = 0$ then $(\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k})^{\mathrm{G}} \neq 0$.*

2.4. ***Borel–Weil–Bott theorem.*** Suppose that λ is a regular weight, so there is a unique $w \in \mathcal{W}$ with $w \cdot \lambda \in \Lambda^+$. The Borel–Weil–Bott theorem identifies the cohomology of the line bundle L_λ on X as G-modules:

$$\mathrm{H}^d(\mathrm{X}, \mathrm{L}_\lambda) = \begin{cases} \mathrm{V}^*_{w\cdot\lambda} & \text{if } d = \ell(w), \\ 0 & \text{otherwise.} \end{cases}$$

If λ is not a regular weight then the cohomology of L_λ is zero in all degrees.

2.5. ***Serre duality on*** X**.** For any weight λ set $\mathrm{S}(\lambda) = -\lambda - 2\rho$. Since the canonical bundle K_X of X is equal to $\mathrm{L}_{-2\rho}$ we see that $\mathrm{L}_{\mathrm{S}(\lambda)} = \mathrm{K}_\mathrm{X} \otimes \mathrm{L}_\lambda^*$. In other words, S is the function that for each weight λ returns the weight $\mathrm{S}(\lambda)$ of the line bundle Serre dual to L_λ; the map S is clearly an involution. Let w be any element of the Weyl group and λ any weight. A straightforward computation shows that S commutes with the affine action of the Weyl group, i.e., that $w \cdot \mathrm{S}(\lambda) = \mathrm{S}(w \cdot \lambda)$.

Lemma 2.5.1. *If λ is a regular weight and w the unique element of the Weyl group with $w \cdot \lambda \in \Lambda^+$ then $(w_0 w) \cdot \mathrm{S}(\lambda) \in \Lambda^+$.*

Proof. If μ is a dominant weight then $\mathrm{V}_\mu^* = \mathrm{V}_{-w_0\mu}$. Therefore if $w \cdot \lambda = \mu \in \Lambda^+$ then

$$(w_0 w) \cdot \mathrm{S}(\lambda) = w_0 \cdot \mathrm{S}(w \cdot \lambda) = w_0 \cdot \mathrm{S}(\mu) = -w_0\mu \in \Lambda^+, \tag{2.5.2}$$

concluding the proof. □

Since $\ell(w_0 w) = \mathrm{N} - \ell(w)$, the calculation above fits in neatly with the Borel–Weil–Bott theorem (BWB) and Serre duality. If λ is a regular weight and w an element of the Weyl group with $w \cdot \lambda = \mu \in \Lambda^+$ then we have

$$\begin{aligned}
\mathrm{V}_\mu &= \big(\mathrm{H}^{\ell(w)}(\mathrm{X}, \mathrm{L}_\lambda)\big)^* && \text{(by BWB)}\\
&= \mathrm{H}^{\mathrm{N}-\ell(w)}(\mathrm{X}, \mathrm{K}_\mathrm{X} \otimes \mathrm{L}_\lambda^*) && \text{(by Serre duality)}\\
&= \mathrm{H}^{\ell(w_0 w)}(\mathrm{X}, \mathrm{L}_{\mathrm{S}(\lambda)}) && \text{(see Section 2.5)}\\
&= \mathrm{V}_{-w_0\mu}^*, && \text{(by BWB and (2.5.2)).}
\end{aligned}$$

2.6. ***Schubert varieties.*** For an element $w \in \mathcal{W}$ of the Weyl group the *Schubert variety* X_w is defined by

$$\mathrm{X}_w := \overline{\mathrm{B}w\mathrm{B}/\mathrm{B}} \subseteq \mathrm{G}/\mathrm{B} = \mathrm{X}.$$

Recall that the classes of the Schubert cycles $\{[\mathrm{X}_w]\}_{w \in \mathcal{W}}$ give a basis for the cohomology ring $\mathrm{H}^*(\mathrm{X}, \mathbb{Z})$ of X. Each $[\mathrm{X}_w]$ is a cycle of complex dimension $\ell(w)$. The dual Schubert cycles $\{[\Omega_w]\}_{w \in \mathcal{W}}$, given by $\Omega_w := \mathrm{X}_{w_0 w}$, also form a basis. Each $[\Omega_w]$ is a cycle of complex codimension $\ell(w)$. The work of Demazure [1974], Kempf [1976], Ramanathan [1985], and Seshadri [1987] shows that each Schubert variety X_w is normal with rational singularities.

Remark. If $w_1, \ldots, w_k$, and $w \in \mathcal{W}$ are such that $\ell(w) = \sum \ell(w_i)$, then the intersection $\bigcap_{i=1}^k [\Omega_{w_i}] \cdot [\mathrm{X}_w]$ is a number. The number is the coefficient of $[\Omega_w]$ when writing the product $\bigcap_{i=1}^k [\Omega_{w_i}]$ in terms of the basis $\{[\Omega_v]\}_{v \in \mathcal{W}}$.

To reduce notation we use w to also refer to the point $w\mathrm{B}/\mathrm{B} \in \mathrm{X}_w \subseteq \mathrm{X}$. In particular, for the identity $e \in \mathcal{W}$, $\mathrm{X}_e = \{e\}$. Note that $e \in \mathrm{X}$ is also the image of 1_G under the projection from G onto X.

Definition. The *Bruhat order* on the Weyl group $\mathcal{W}$ is the partial order given by the relation $v \leqslant w$ if and only if $\mathrm{X}_v \subseteq \mathrm{X}_w$. The minimum element in this order is e and the maximum element is w_0, corresponding to the subvarieties $\mathrm{X}_e = \{e\}$ and $\mathrm{X}_{w_0} = \mathrm{X}$ respectively.

The following result will be used several times throughout the paper.

Lemma 2.6.1. *Suppose that $w_1, \ldots, w_k$ are elements of the Weyl group such that $\Delta^+ = \bigsqcup_{i=1}^k \Phi_{w_i}$. Then $\bigcap_{i=1}^k [\Omega_{w_i}] \neq 0$.*

Proof. Each class $[\Omega_{w_i}]$ is represented by any translation of the cycle Ω_{w_i}, so to understand $\bigcap_{i=1}^k [\Omega_{w_i}]$ we can study the intersection of schemes

$$\bigcap_{i=1}^{k} (w_0 w_i)^{-1} \Omega_{w_i}. \tag{2.6.2}$$

Each of the schemes $(w_0 w_i)^{-1} \Omega_{w_i}$ passes through $e \in \mathrm{X}$. The tangent space to $(w_0 w)^{-1} \Omega_w$ at e is

$$\mathrm{Lie}\big((w_0 w)^{-1} \mathrm{B} (w_0 w)\big) / \mathrm{Lie}(\mathrm{B}) = \bigoplus_{\alpha \in -\Phi_{w_0 w}} \mathfrak{g}^{\alpha} \overset{(2.2.2)}{=} \bigoplus_{\alpha \in -\Phi_w^{\mathrm{c}}} \mathfrak{g}^{\alpha} \subseteq \mathfrak{b}^- = \mathrm{T}_e \mathrm{X},$$

where we have identified $\mathrm{T}_e\mathrm{X}$ with $\mathfrak{b}^-$ via the projection $\mathrm{G} \to \mathrm{X}$. Noting that

$$\bigcap_{i=1}^{k} \Phi_w^{\mathrm{c}} = \Big(\bigcup_{i=1}^{k} \Phi_w \Big)^{\mathrm{c}} = (\Delta^+)^{\mathrm{c}} = \varnothing,$$

we conclude that the intersection of the tangent spaces of the varieties $(w_0 w_i)^{-1} \Omega_{w_i}$ at $e \in \mathrm{X}$ is 0. Hence the intersection (2.6.2) is transverse at the identity. By Kleiman's transversality theorem [1974, Corollary 4(ii)], small translations of each of the varieties $(w_0 w_i)^{-1} \Omega_{w_i}$ will intersect properly and compute the intersection number. Small translations of varieties cannot remove transverse points of intersection and thus $\bigcap_{i=1}^k [\Omega_{w_i}] \neq 0$. □

2.7. *Symmetric and nonsymmetric forms.* Most questions we consider, including Problem I and Problem II, can be stated in nonsymmetric and symmetric forms and it is frequently convenient to switch from one to the other. We illustrate this procedure by showing how to switch from the nonsymmetric to the symmetric form of Problem I.

In the nonsymmetric form we are given $w_1, \ldots, w_k$, and w, such that $\ell(w) = \sum \ell(w_i)$, and $\lambda_1, \ldots, \lambda_k$, and λ, such that $\lambda = \sum \lambda_i$, satisfying the additional conditions that $w_i \cdot \lambda_i \in \Lambda^+$ for $i = 1, \ldots, k$, and $w \cdot \lambda \in \Lambda^+$. This corresponds to the data of a cup-product problem:

$$\mathrm{H}^{\ell(w_1)}(\mathrm{X}, \mathrm{L}_{\lambda_1}) \otimes \cdots \otimes \mathrm{H}^{\ell(w_k)}(\mathrm{X}, \mathrm{L}_{\lambda_k}) \xrightarrow{\cup} \mathrm{H}^{\ell(w)}(\mathrm{X}, \mathrm{L}_{\lambda}). \tag{2.7.1}$$

Set $\mu_i = w_i \cdot \lambda_i$ for $i = 1, \ldots, k$, and $\mu = w \cdot \lambda$ to keep track of the modules which appear as cohomology groups. By the Borel–Weil–Bott theorem the map (2.7.1) corresponds to a G-equivariant map

$$\mathrm{V}^*_{\mu_1} \otimes \mathrm{V}^*_{\mu_2} \otimes \cdots \otimes \mathrm{V}^*_{\mu_k} \to \mathrm{V}^*_{\mu}.$$

By Serre duality $\mathrm{H}^{\mathrm{N}-\ell(w)}(\mathrm{X}, \mathrm{K_X} \otimes \mathrm{L}^*_\lambda) \neq 0$ and the cup-product map

$$\mathrm{H}^{\ell(w)}(\mathrm{X}, \mathrm{L}_\lambda) \otimes \mathrm{H}^{\mathrm{N}-\ell(w)}(\mathrm{X}, \mathrm{K_X} \otimes \mathrm{L}^*_\lambda) \xrightarrow{\cup} \mathrm{H}^{\mathrm{N}}(\mathrm{X}, \mathrm{K_X})$$

is a perfect pairing. Since $\mathrm{H}^{\ell(w)}(\mathrm{X}, \mathrm{L}_\lambda)$ is an irreducible G-module, the surjectivity of (2.7.1) is equivalent to the surjectivity of the cup-product map

$$\mathrm{H}^{\ell(w_1)}(\mathrm{X}, \mathrm{L}_{\lambda_1}) \otimes \cdots \otimes \mathrm{H}^{\ell(w_k)}(\mathrm{X}, \mathrm{L}_{\lambda_k}) \otimes \mathrm{H}^{\mathrm{N}-\ell(w)}(\mathrm{X}, \mathrm{K_X} \otimes \mathrm{L}^*_\lambda) \xrightarrow{\cup} \mathrm{H}^{\mathrm{N}}(\mathrm{X}, \mathrm{K_X}). \quad (2.7.2)$$

To get the symmetric form of this problem, we set $w_{k+1} = w_0 w$, $\lambda_{k+1} = \mathrm{S}(\lambda) = -\lambda - 2\rho$, and $\mu_{k+1} = -w_0\mu = w_{k+1} \cdot \lambda_{k+1}$. Then $\mathrm{L}_{\lambda_{k+1}} = \mathrm{K_X} \otimes \mathrm{L}^*_\lambda$ by Section 2.5, $w_{k+1} \cdot \lambda_{k+1} \in \Lambda^+$ by Lemma 2.5.1, and $\ell(w_{k+1}) = \mathrm{N} - \ell(w)$, so that (2.7.2) becomes

$$\mathrm{H}^{\ell(w_1)}(\mathrm{X}, \mathrm{L}_{\lambda_1}) \otimes \cdots \otimes \mathrm{H}^{\ell(w_k)}(\mathrm{X}, \mathrm{L}_{\lambda_k}) \otimes \mathrm{H}^{\ell(w_{k+1})}(\mathrm{X}, \mathrm{L}_{\lambda_{k+1}}) \xrightarrow{\cup} \mathrm{H}^{\mathrm{N}}(\mathrm{X}, \mathrm{K_X}).$$

Since

$$\sum_{i=1}^{k+1} \lambda_i = \lambda + (-\lambda - 2\rho) = -2\rho$$

and $\mathrm{L}_{-2\rho} = \mathrm{K_X}$, this is again a cup-product problem of the type we consider, but now all weights $\lambda_1, \ldots, \lambda_{k+1}$ and Weyl group elements $w_1, \ldots, w_{k+1}$ play equal roles.

By (2.2.2) $\Phi_{w_{k+1}} = \Phi^c_w$ and therefore the condition that $\Phi_w = \bigsqcup_{i=1}^k \Phi_{w_i}$ is equivalent to the condition $\Delta^+ = \bigsqcup_{i=1}^{k+1} \Phi_{w_i}$. Since $[\Omega_{w_{k+1}}] = [\mathrm{X}_{w_0 w_{k+1}}] = [\mathrm{X}_w]$, the intersection numbers $\bigcap_{i=1}^k [\Omega_{w_i}] \cdot [\mathrm{X}_w]$ and $\bigcap_{i=1}^{k+1} [\Omega_{w_i}]$ are the same. Finally, the multiplicity of V_μ in $\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}$ is the same as the multiplicity of the trivial module in $\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k} \otimes \mathrm{V}_{\mu_{k+1}}$ because $\mathrm{V}_{\mu_{k+1}} = \mathrm{V}_{-w_0\mu} = \mathrm{V}^*_\mu$.

To go from the symmetric form to the nonsymmetric form we simply reverse the above procedure, although of course we are free to desymmetrize with respect to any of the indices $i = 1, \ldots, k+1$, and not just the last one.

For convenience we list in Table 1 the symmetric and nonsymmetric forms of some formulas and expressions we are interested in. Since k is an arbitrary positive integer, after switching to the symmetric form we often use k in place of $k+1$ to reduce notation.

2.8. ***Demazure reflections.*** Suppose that W and M are varieties and $\pi : \mathrm{W} \to \mathrm{M}$ is a $\mathbb{P}^1$-fibration, i.e., a smooth morphism with fibers isomorphic to $\mathbb{P}^1$. Let L be a line bundle on W and b be the degree of L on the fibers of π. Demazure [1976,

Nonsymmetric	Symmetric
$\bigotimes_{i=1}^{k} \mathrm{H}^{\ell(w_i)}(\mathrm{X}, \mathrm{L}_{\lambda_i}) \to \mathrm{H}^{\ell(w)}(\mathrm{X}, \mathrm{L}_{\lambda})$	$\bigotimes_{i=1}^{k+1} \mathrm{H}^{\ell(w_i)}(\mathrm{X}, \mathrm{L}_{\lambda_i}) \to \mathrm{H}^{\mathrm{N}}(\mathrm{X}, \mathrm{K}_{\mathrm{X}})$
$\sum_{i=1}^{k} \ell(w_i) = \ell(w)$	$\sum_{i=1}^{k+1} \ell(w_i) = \mathrm{N}$
$\sum_{i=1}^{k} \lambda_i = \lambda$	$\sum_{i=1}^{k+1} \lambda_i = -2\rho$
$\sum_{i=1}^{k} w_i^{-1}\mu_i - w^{-1}\mu$	$\sum_{i=1}^{k+1} w_i^{-1}\mu_i$
$\sum_{i=1}^{k} w_i^{-1}\cdot 0 - w^{-1}\cdot 0$	$\sum_{i=1}^{k+1} w_i^{-1}\cdot 0 + 2\rho$
$\Phi_w = \bigsqcup_{i=1}^{k} \Phi_{w_i}$	$\Delta^+ = \bigsqcup_{i=1}^{k+1} \Phi_{w_i}$
$\bigcap_{i=1}^{k} [\Omega_{w_i}] \cdot [\mathrm{X}_w]$	$\bigcap_{i=1}^{k+1} [\Omega_{w_i}]$

Table 1. Nonsymmetric and symmetric forms of some formulas.

Theorem 1] proves the following isomorphism of vector bundles on M:

$$\mathrm{R}^i \pi_* \mathrm{L} \cong \mathrm{R}^{1-i}\pi_*(\mathrm{L} \otimes \omega_\pi^{b+1}) \quad \text{for } i = 0, 1, \tag{2.8.1}$$

where ω_π is the relative cotangent bundle of π. The line bundle $\mathrm{L} \otimes \omega_\pi^{b+1}$ is called the *Demazure reflection of* L *with respect to* π.

Note that there is at most one value of i for which the resulting vector bundles are nonzero: $i = 0$ if $b \geqslant 0$, $i = 1$ if $b \leqslant -2$, and neither if $b = -1$. Equation (2.8.1) and the corresponding Leray spectral sequence give the isomorphisms

$$\mathrm{H}^j(\mathrm{W}, \mathrm{L}) \cong \begin{cases} \mathrm{H}^{j+1}(\mathrm{W}, \mathrm{L} \otimes \omega_\pi^{b+1}) & \text{if } b \geqslant 0, \\ \mathrm{H}^{j-1}(\mathrm{W}, \mathrm{L} \otimes \omega_\pi^{b+1}) & \text{if } b \leqslant -2, \end{cases} \quad \text{for all} \quad j.$$

Link between Demazure reflections and the affine action. Let α_i be any simple root, P_{α_i} the parabolic associated to α_i, and $\pi_i : \mathrm{X} \to \mathrm{M}_i := \mathrm{G}/\mathrm{P}_{\alpha_i}$ the corresponding $\mathbb{P}^1$-fibration. The relative cotangent bundle ω_{π_i} of π_i is the line bundle $\mathrm{L}_{-\alpha_i}$. Given any $\lambda \in \Lambda$, the degree of the line bundle L_λ on the fibers of π_i is $\lambda(\alpha_i^\vee)$, where $\alpha_i^\vee$ is the coroot corresponding to α_i. We thus obtain that the Demazure reflection of L_λ with respect to the fibration π_i is the line bundle

$$\mathrm{L}_\lambda \otimes \mathrm{L}_{-\alpha_i}^{\lambda(\alpha_i^\vee)+1} = \mathrm{L}_{\lambda-(\lambda(\alpha_i^\vee)+1)\alpha_i} = \mathrm{L}_{s_i\lambda-\alpha_i} = \mathrm{L}_{s_i\cdot\lambda},$$

where s_i is the simple reflection corresponding to α_i. The combinatorics of performing Demazure reflections with respect to the various $\mathbb{P}^1$-fibrations of X is therefore kept track of by the affine action of the Weyl group on Λ. In particular, if $v = s_{i_1}\cdots s_{i_m} \in \mathcal{W}$ and $\lambda \in \Lambda$, the result of applying the Demazure reflections with respect to the fibrations $\pi_{i_m}, \pi_{i_{m-1}}, \ldots, \pi_{i_1}$ in that order to L_λ is $\mathrm{L}_{v\cdot\lambda}$.

Demazure reflections and base change. Given any morphism $h : \mathrm{Y}_2 \to \mathrm{M}$ we can form the fiber product diagram

$$\begin{tikzcd} \mathrm{Y}_1 \arrow[r, "f"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\square"] & \mathrm{W} \arrow[d, "\pi"] \\ \mathrm{Y}_2 \arrow[r, "h"] & \mathrm{M} \end{tikzcd}$$

If π is a $\mathbb{P}^1$-fibration then so is π_1, and $\omega_{\pi_1} = f^*\omega_\pi$. Therefore, for any line bundle L on V, we have

$$f^*(\mathrm{L} \otimes \omega_\pi^{b+1}) = (f^*\mathrm{L}) \otimes \omega_{\pi_1}^{b+1},$$

where b is the degree of L on the fibers of π. The degree of $f^*\mathrm{L}$ on π_1 is also b and therefore the formula above shows that the pullback of the Demazure reflection of L with respect to π is the Demazure reflection of the pullback of L with respect to π_1. Furthermore, by the theorem on cohomology and base change, the natural morphisms

$$\mathrm{R}^i\pi_{1*}(f^*\mathrm{L}) \xleftarrow{\simeq} h^*(\mathrm{R}^i\pi_*\mathrm{L}),$$
$$\mathrm{R}^{1-i}\pi_{1*}((f^*\mathrm{L}) \otimes \omega_{\pi_1}^{b+1}) \xleftarrow{\simeq} h^*(\mathrm{R}^{1-i}\pi_*(\mathrm{L} \otimes \omega_\pi^{b+1}))$$

are isomorphisms for $i = 0, 1$.

2.9. E$_2$*-terms and computation of maps on cohomology.* Suppose that we have a commutative diagram of varieties

$$\begin{tikzcd} \mathrm{W}' \arrow[r, hook, "\gamma"] \arrow[d, "\pi'"] & \mathrm{W} \arrow[d, "\pi"] \\ \mathrm{M}' \arrow[r, hook] & \mathrm{M} \end{tikzcd}$$

where the vertical maps are proper and the horizontal maps are closed immersions. Suppose further that we have coherent sheaves $\mathcal{F}$ on W and $\mathcal{F}'$ on W$'$, and a map $\varphi : \gamma^*\mathcal{F} \to \mathcal{F}'$ of sheaves on W$'$. The map φ induces maps $\varphi_d : \mathrm{H}^d(\mathrm{W}, \mathcal{F}) \to \mathrm{H}^d(\mathrm{W}', \mathcal{F}')$ on cohomology and maps $\varphi_{d,k} : \mathrm{H}^{d-k}(\mathrm{M}, \mathrm{R}^k_{\pi*}\mathcal{F}) \to \mathrm{H}^{d-k}(\mathrm{M}', \mathrm{R}^k_{\pi*}\mathcal{F}')$ on the E$_2$-terms of the Leray spectral sequences for $\mathcal{F}$ and $\mathcal{F}'$ with respect to π and π'. Assume that both spectral sequences degenerate at the E$_2$-term. In Section 5.4 we will need to know when we can compute φ_d by knowing the maps $\varphi_{d,k}$.

By the definition of convergence of a spectral sequence there are increasing filtrations

$$0 = \mathrm{U}_{-1} \subseteq \mathrm{U}_0 \subseteq \cdots \subseteq \mathrm{U}_d = \mathrm{H}^d(\mathrm{W}, \mathcal{F}),$$
$$0 = \mathrm{U}'_{-1} \subseteq \mathrm{U}'_0 \subseteq \cdots \subseteq \mathrm{U}'_d = \mathrm{H}^d(\mathrm{W}', \mathcal{F}'),$$

such that $\mathrm{U}_k/\mathrm{U}_{k-1} = \mathrm{H}^{d-k}(\mathrm{M}, \mathrm{R}^k_{\pi*}\mathcal{F})$ and $\mathrm{U}'_k/\mathrm{U}'_{k-1} = \mathrm{H}^{d-k}(\mathrm{M}', \mathrm{R}^k_{\pi*}\mathcal{F}')$ for $k = 0, \ldots, d$. Since the map φ_d on the cohomology groups is compatible with the filtrations (in the sense that $\varphi_d(\mathrm{U}_k) \subseteq \mathrm{U}'_k$ for $k = -1, \ldots, d$), φ_d induces maps between the associated graded pieces of the filtrations; these maps are exactly the maps $\varphi_{d,k}$.

We will need to know that φ_d can be computed from the maps $\varphi_{d,k}$ in an elementary case. Suppose there is a unique k such that $\mathrm{U}_k/\mathrm{U}_{k-1}$ is nonzero (and so $\mathrm{U}_k/\mathrm{U}_{k-1} = \mathrm{H}^d(\mathrm{W}, \mathcal{F})$), and a unique k' such that $\mathrm{U}'_{k'}/\mathrm{U}'_{k'-1}$ is nonzero (and so $\mathrm{U}'_{k'}/\mathrm{U}'_{k-1} = \mathrm{H}^d(\mathrm{W}', \mathcal{F}')$). Then we can compute φ_d from the maps $\varphi_{d,k}$ if and only if $k = k'$; if this occurs then $\varphi_d = \varphi_{d,k}$.

In order to show that we must check the condition $k = k'$ above, i.e., that the map on E_2-terms does not always determine the map φ_d, we give the following example of a nonzero map between cohomology groups of sheaves where the induced map on E_2-terms is zero. This example is also a cup-product map.

Example 2.9.1. Let $\mathrm{W} = \mathbb{P}^m \times \mathbb{P}^m$ for some $m \geqslant 1$, $\mathcal{F} = \mathcal{O}_{\mathbb{P}^m}(1) \boxtimes \mathcal{O}_{\mathbb{P}^m}(-r)$ with $r \geqslant m+2$, and let $\mathcal{G} = \mathcal{O}_\Delta(1-r)$ be the restriction of $\mathcal{F}$ to the diagonal of W. We have $\mathrm{H}^m(\mathrm{W}, \mathcal{F}) = \mathrm{H}^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1)) \otimes \mathrm{H}^m(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(-r))$ and $\mathrm{H}^m(\mathrm{W}, \mathcal{G}) = \mathrm{H}^m(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1-r))$. The natural restriction map $\varphi : \mathcal{F} \to \mathcal{G}$ induces the cup-product map

$$\varphi_m : \mathrm{H}^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1)) \otimes \mathrm{H}^m(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(-r)) \xrightarrow{\cup} \mathrm{H}^m(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1-r)),$$

which is a surjective map of nonzero groups.

If $\pi : \mathrm{W} \to \mathrm{M} = \mathbb{P}^m$ is the projection onto the first factor then both of the Leray spectral sequences degenerate at the E_2 term with only one nonzero entry in each sequence. We have

$$\mathrm{H}^m(\mathrm{W}, \mathcal{F}) = \mathrm{H}^0(\mathrm{M}, \mathrm{R}^m_{\pi*}\mathcal{F}) \quad \text{(i.e., } k = m\text{)},$$
$$\mathrm{H}^m(\mathrm{W}, \mathcal{G}) = \mathrm{H}^m(\mathrm{M}, \pi_*\mathcal{G}) \quad \text{(i.e., } k' = 0\text{)}.$$

The maps $\varphi_{m,k}$ on the E_2-terms are clearly zero, even though φ_m is nonzero.

2.10. ***Bott–Samelson–Demazure–Hansen varieties.*** Let $\underline{v} = s_{i_1} \cdots s_{i_m}$ be a word, not necessarily reduced, of simple reflections. Associated to $\underline{v}$ is a variety $\mathrm{Z}_{\underline{v}}$, a left action of B on $\mathrm{Z}_{\underline{v}}$, and a B-equivariant map $f_{\underline{v}} : \mathrm{Z}_{\underline{v}} \to \mathrm{X}$. If $\underline{v}$ is nonempty there is also a B-equivariant map $\pi_{\underline{v}} : \mathrm{Z}_{\underline{v}} \to \mathrm{Z}_{\underline{v}_R}$ expressing $\mathrm{Z}_{\underline{v}}$ as a $\mathbb{P}^1$-bundle over $\mathrm{Z}_{\underline{v}_R}$ together with a B-equivariant $\sigma_{\underline{v}} : \mathrm{Z}_{\underline{v}_R} \to \mathrm{Z}_{\underline{v}}$ section such that $f_{\underline{v}_R} = f_{\underline{v}} \circ \sigma_{\underline{v}}$.

These varieties were originally constructed by Demazure [1974] and Hansen [1973] following an analogous construction by Bott and Samelson [1958] in the compact case. In this subsection we recall their construction and several related facts. We give two different descriptions of the construction; both will be used in the constructions in Section 3.

Recursive construction. Recall that e is unique point of X fixed by B. If the word $\underline{v}$ is empty we define $Z_{\underline{v}}$ to be e, the map $f_{\underline{v}}$ to be the inclusion $e \hookrightarrow X$, and the B-action on $Z_{\underline{v}}$ to be trivial.

If $\underline{v} = s_{i_1} \cdots s_{i_m}$ is nonempty, let $\underline{u} = \underline{v}_R = s_{i_1} \cdots s_{i_{m-1}}$ be the word obtained by dropping the rightmost reflection of $\underline{v}$. By induction we have already constructed $Z_{\underline{u}}$ and the map $f_{\underline{u}} : Z_{\underline{u}} \to X$. Set $h = \pi_{i_m} \circ f_{\underline{u}}$, where π_{i_m} is the G-equivariant projection (and $\mathbb{P}^1$-fibration) $X \to M_{i_m} = G/P_{\alpha_{i_m}}$. We then define $Z_{\underline{v}}$ to be the fiber product $Z_{\underline{u}} \times_{M_{i_m}} X$, and $f_{\underline{v}}$ and $\pi_{\underline{v}}$ to be the maps from the fiber product to X and to $Z_{\underline{u}}$ respectively. Since $h = \pi_{i_m} \circ f_{\underline{u}}$, by the universal property of the fiber product there exists a unique map $\sigma_{\underline{v}} : Z_{\underline{u}} \to Z_{\underline{v}}$ such that $f_{\underline{u}} = f_{\underline{v}} \circ \sigma_{\underline{v}}$ and $\mathrm{id}_{Z_{\underline{u}}} = \pi_{\underline{v}} \circ \sigma_{\underline{v}}$. These maps are summarized in the following diagram, where the square is a fiber product:

$$
\begin{tikzcd}
Z_{\underline{v}} \arrow[rr, "f_{\underline{v}}"] \arrow[dd, "\pi_{\underline{v}}"] \arrow[ddrr, phantom, "\square"] & & X \arrow[dd, "\pi_{i_m}"] \\
& & \\
Z_{\underline{u}} \arrow[rr, "h"] \arrow[uurr, "f_{\underline{u}}"] \arrow[uu, bend left, "\sigma_{\underline{v}}"] & & M_{i_m}
\end{tikzcd}
\tag{2.10.1}
$$

Since B acts on $Z_{\underline{u}}$ and on X, and the maps $f_{\underline{u}}$, π_{i_m}, and h are B-equivariant, by the universal property of the fiber product, the diagram (2.10.1) induces a B-action on $Z_{\underline{v}}$ such that $f_{\underline{v}}$ and $\sigma_{\underline{v}}$ are B-equivariant maps. Since each morphism $\sigma_{\underline{v}}$ is a $\mathbb{P}^1$-fibration it follows immediately that each $Z_{\underline{v}}$ is a smooth proper variety of dimension $\ell(\underline{v})$.

Direct construction. For any word $\underline{v}$ set

$$
P_{\underline{v}} := \begin{cases} e & \text{if } \underline{v} \text{ is empty,} \\ P_{\alpha_{i_1}} \times \cdots \times P_{\alpha_{i_m}} & \text{if } \underline{v} = s_{i_1} \cdots s_{i_m} \text{ is nonempty.} \end{cases}
$$

If $\underline{v}$ is empty we define $Z_{\underline{v}}$, $f_{\underline{v}}$, and the B-action as in the direct construction.

If $\underline{v} = s_{i_1} \cdots s_{i_m}$ is nonempty then $Z_{\underline{v}}$ is the quotient of $P_{\underline{v}}$ by B^m, where an element $(b_1, \ldots, b_m)$ of B^m acts on the right on $(p_1, \ldots, p_m)$ by

$$
(p_1, \ldots, p_m) \cdot (b_1, \ldots, b_m) = \big(p_1 b_1,\ b_1^{-1} p_2 b_2,\ b_2^{-1} p_3 b_3, \ldots, b_{m-1}^{-1} p_m b_m\big).
$$

The left action of B on $P_{\underline{v}}$ given by

$$
b \cdot (p_1, p_2, \ldots, p_m) = (b p_1, p_2, \ldots, p_m)
$$

commutes with the right action of B^m and therefore descends to a left action of B on $Z_{\underline{v}}$. We denote the corresponding B-equivariant quotient map by $\psi_{\underline{v}} : P_{\underline{v}} \to Z_{\underline{v}}$.

The product map $\mathrm{P}_{\underline{v}} \xrightarrow{\phi_{\underline{v}}} \mathrm{G}$ given by $(p_1, \ldots, p_m) \mapsto p_1 \cdots p_m$ is equivariant for the left B-action described above and left multiplication of G by B. Under the homomorphism of groups $\mathrm{B}^m \to \mathrm{B}$ given by the projection $(b_1, \ldots, b_m) \mapsto b_m$ the product map $\phi_{\underline{v}}$ is also equivariant for the right action of B^m on $\mathrm{P}_{\underline{v}}$ and the right multiplication of G by B. The product map therefore descends to a left B-equivariant morphism $f_{\underline{v}} : \mathrm{Z}_{\underline{v}} \to \mathrm{X}$.

Let $\underline{u} = \underline{v}_R = s_{i_1} \cdots s_{i_{m-1}}$ be the word obtained by dropping the rightmost reflection in $\underline{v}$. The projection map $\mathrm{pr}_{\underline{v}} : \mathrm{P}_{\underline{v}} \to \mathrm{P}_{\underline{u}}$ sending $(p_1, \ldots, p_m)$ to $(p_1, \ldots, p_{m-1})$ is equivariant with respect to the projection $\mathrm{B}^m \to \mathrm{B}^{m-1}$ sending $(b_1, \ldots, b_m)$ to $(b_1, \ldots, b_{m-1})$. Similarly the inclusion map $j_{\underline{v}} : \mathrm{P}_{\underline{u}} \hookrightarrow \mathrm{P}_{\underline{v}}$ sending $(p_1, \ldots, p_{m-1})$ to $(p_1, \ldots, p_{m-1}, 1_{\mathrm{G}})$ is equivariant with respect to the inclusion $\mathrm{B}^{m-1} \hookrightarrow \mathrm{B}^m$ sending $(b_1, \ldots, b_{m-1})$ to $(b_1, \ldots, b_{m-1}, b_{m-1})$. The maps $\mathrm{pr}_{\underline{v}}$ and $j_{\underline{v}}$ respect the left B-action on $\mathrm{P}_{\underline{v}}$ and $\mathrm{P}_{\underline{u}}$, and therefore descend to B-equivariant maps $\pi_{\underline{v}} : \mathrm{Z}_{\underline{v}} \to \mathrm{Z}_{\underline{u}}$ and $\sigma_{\underline{v}} : \mathrm{Z}_{\underline{u}} \to \mathrm{Z}_{\underline{v}}$. Since $\mathrm{pr}_{\underline{v}} \circ j_{\underline{v}} = \mathrm{id}_{\mathrm{P}_{\underline{u}}}$ and $\phi_{\underline{v}} \circ j_{\underline{v}} = \phi_{\underline{u}}$, taking quotients we obtain $\pi_{\underline{v}} \circ \sigma_{\underline{v}} = \mathrm{id}_{\mathrm{Z}_{\underline{u}}}$ and $f_{\underline{v}} \circ \sigma_{\underline{v}} = f_{\underline{u}}$. Finally, the fibers of $\pi_{\underline{v}}$ are isomorphic to $\mathrm{P}_{\alpha_{i_m}}/\mathrm{B} \cong \mathbb{P}^1$.

We record the following well-known facts about the construction above.

Proposition 2.10.2. (a) *The varieties* $\mathrm{Z}_{\underline{v}}$ *produced by the recursive and direct constructions above are isomorphic over* X.

(b) *If* $\underline{v} = s_{i_1} \cdots s_{i_m}$ *is a reduced word with product* v *then the image of* $f_{\underline{v}} : \mathrm{Z}_{\underline{v}} \to \mathrm{X}$ *is* X_v *and* $f_{\underline{v}}$ *is a resolution of singularities of* X_v.

Proof. Part (b) is proved in [Demazure 1974] and [Hansen 1973]. To show (a) it is enough to show that the varieties produced by the direct construction satisfy the fiber product diagram (2.10.1). This is most easily checked after pulling back (2.10.1) via the maps $\mathrm{G} \to \mathrm{X}$ and $\mathrm{P}_{\underline{u}} = \mathrm{P}_{i_1} \times \cdots \times \mathrm{P}_{i_{m-1}} \to \mathrm{Z}_{\underline{u}}$; the details are omitted here. □

Maximum points. Let $\underline{v} = s_{i_1} \cdots s_{i_m}$ be a reduced word with product v. The image of $\mathrm{Z}_{\underline{v}}$ under $f_{\underline{v}}$ is X_v, by Proposition 2.10.2(b), and one can check that there is a unique point $p_{\underline{v}}$ of $\mathrm{Z}_{\underline{v}}$ which maps to $v \in \mathrm{X}_v$. More specifically, from the point of view of the direct construction, the point $(s_{i_1}, \ldots, s_{i_m})$ is a point of $\mathrm{P}_{\underline{v}}$ and its image under the quotient map $\mathrm{P}_{\underline{v}} \to \mathrm{Z}_{\underline{v}}$ is $p_{\underline{v}}$. From the point of view of the recursive construction one starts with $p_{\varnothing} = e$, and recursively defines $p_{\underline{v}}$ to be unique torus fixed point in the $\mathbb{P}^1$-fiber of $\pi_{\underline{v}} : \mathrm{Z}_{\underline{v}} \to \mathrm{Z}_{\underline{u}}$ over $p_{\underline{u}}$ which is not equal to $\sigma_{\underline{v}}(p_{\underline{u}})$, where $\underline{u} = \underline{v}_R = s_{i_1} \cdots s_{i_{m-1}}$. Note that $p_{\underline{v}}$ is the unique torus fixed point of $\mathrm{Z}_{\underline{v}}$ whose image in X_v is the largest in the Bruhat order among torus-fixed points of X_v. We call $p_{\underline{v}}$ the *maximum point* of $\mathrm{Z}_{\underline{v}}$.

Since $p_{\underline{v}}$ is a torus fixed point, the torus acts on the tangent space $\mathrm{T}_{p_{\underline{v}}}\mathrm{Z}_{\underline{v}}$ and it will be important for us to know the formal character of $\mathrm{T}_{p_{\underline{v}}}\mathrm{Z}_{\underline{v}}$. It follows inductively from the recursive construction that

$$\mathrm{Ch}(\mathrm{T}_{p_{\underline{v}}}\mathrm{Z}_{\underline{v}}) = \langle \Phi_{\underline{v}^{-1}} \rangle. \tag{2.10.3}$$

2.11. ***Semistability of torus fixed points.*** The following lemma is due to Kostant.

Lemma 2.11.1. *Let* W *be a projective variety with a* G*-action and* L *a* G*-equivariant ample line bundle on* W*. A torus fixed point* $q \in \mathrm{W}$ *is semistable with respect to* L *if and only if the weight of* L_q *is zero. In this case the orbit of* q *is closed in the semistable locus.*

Proof. If the action of the torus on the fiber L_q is nontrivial then it is easy to see (for instance using the Hilbert–Mumford criterion for semistability, [Mumford et al. 1994, Theorem 2.1, p. 49]) that q is not a stable point.

Conversely, suppose that the weight of L_q is zero. Replacing L by a multiple we may assume that L is very ample and gives an embedding $\mathrm{W} \hookrightarrow \mathbb{P}^r$ for some r. Let $\mathbb{A}^{r+1}$ be the affine space corresponding to $\mathbb{P}^r$ and $\mathbb{A}^{r+1} \setminus \{0\} \to \mathbb{P}^r$ be the quotient map. Then G acts linearly on $\mathbb{A}^{r+1}$ inducing an action on $\mathbb{P}^r$ compatible with the action on W. Let $\tilde{q}$ be any lift to $\mathbb{A}^{r+1}$ of the image of q in $\mathbb{P}^r$. The condition that the torus act trivially on L_q is equivalent to the condition that $\tilde{q}$ be fixed by T under the G-action on $\mathbb{A}^{r+1}$. Kostant ([1963, p. 354, Remark 11]) proves that for any finite dimensional module of a reductive group G and any point $\tilde{q}$ fixed by T, the G-orbit of $\tilde{q}$ is closed; this result was also later generalized by Luna [1975, Theorem (∗∗)]. Since G is reductive and the orbit of $\tilde{q}$ does not meet zero, there is a G-invariant homogeneous form of some degree m which is nonzero on $\tilde{q}$. This corresponds to a G-invariant section $s \in \mathrm{H}^0(\mathrm{W}, \mathrm{L}^m)^{\mathrm{G}}$ such that $s(q) \neq 0$. We thus see that if the weight of L_q is zero then q is a semistable point, and the orbit of q is closed in the semistable locus. □

3. Diagonal Bott–Samelson–Demazure–Hansen–Kumar varieties

In this section we give a generalization of the varieties from Section 2.10. The construction is a variation of a construction of Kumar [1988]; see Section 3.10 for a comparison. These varieties are obtained by applying the idea of the Bott–Samelson resolution to the diagonal inclusion $\mathrm{X} \hookrightarrow \mathrm{X}^k$. They can also be thought of as a desingularization of the total space of the variety of intersections of translates of Schubert cycles. This alternate description is established in Theorem 3.7.4.

More specifically, for each sequence $\underline{\boldsymbol{v}} = (\underline{v}_1, \dots, \underline{v}_k)$ of words we construct a smooth variety $\mathrm{Y}_{\underline{\boldsymbol{v}}}$ of dimension $\mathrm{N} + \ell(\underline{\boldsymbol{v}})$ with a G-action together with a proper map $f_{\underline{\boldsymbol{v}}} : \mathrm{Y}_{\underline{\boldsymbol{v}}} \to \mathrm{X}^k$ which is G-equivariant for the diagonal action of G on X^k. If $\underline{\boldsymbol{u}}$ is the sequence obtained by dropping a single simple reflection from the right of one of the $\underline{v}_j$'s then $\mathrm{Y}_{\underline{\boldsymbol{v}}}$ is a $\mathbb{P}^1$-fibration over $\mathrm{Y}_{\underline{\boldsymbol{u}}}$, and there is a section $\mathrm{Y}_{\underline{\boldsymbol{u}}} \hookrightarrow \mathrm{Y}_{\underline{\boldsymbol{v}}}$ compatible with the maps $f_{\underline{\boldsymbol{v}}}$ and $f_{\underline{\boldsymbol{u}}}$ to X^k. The fibration and section maps are

G-equivariant; moreover they are compatible with the $\mathbb{P}^1$-fibrations on factors of X^k. These relationships are summarized in diagram (3.1.2).

3.1. ***Recursive construction.*** Let $\underline{\boldsymbol{v}} = (\underline{v}_1, \dots, \underline{v}_k)$ be a sequence of words. If all $\underline{v}_j$ are empty, i.e., if $\underline{\boldsymbol{v}} = (\varnothing, \dots, \varnothing)$, we set $\mathrm{Y}_{\underline{\boldsymbol{v}}} = \mathrm{X}$ and let $f_{\underline{\boldsymbol{v}}} : \mathrm{Y}_{\underline{\boldsymbol{v}}} \to \mathrm{X}^k$ be the diagonal embedding.

Otherwise suppose that $\underline{v}_j$ is nonempty. Let

$$\underline{u}_l := \begin{cases} \underline{v}_l & \text{if } l \neq j, \\ (\underline{v}_j)_R & \text{if } l = j, \end{cases} \qquad l = 1, \dots, k, \tag{3.1.1}$$

and set $\underline{\boldsymbol{u}} = (\underline{u}_1, \dots, \underline{u}_k)$. By induction on $\ell(\underline{\boldsymbol{v}})$ we may assume that $\mathrm{Y}_{\underline{\boldsymbol{u}}}$ and the map $f_{\underline{\boldsymbol{u}}} : \mathrm{Y}_{\underline{\boldsymbol{u}}} \to \mathrm{X}^k$ have been constructed. If $\underline{v}_j = s_{i_1} \cdots s_{i_m}$, so that $\underline{u}_j = s_{i_1} \cdots s_{i_{m-1}}$ then we define $\mathrm{Y}_{\underline{\boldsymbol{v}}}$, the map $f_{\underline{\boldsymbol{v}}}$, the projection $\pi_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}}$, and the section $\sigma_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}}$ by the following fiber product square:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathrm{Y}_{\underline{\boldsymbol{v}}} \arrow[r, "f_{\underline{\boldsymbol{v}}}"] \arrow[d, "\pi_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}}"] & \mathrm{X}^k \arrow[d, "(\mathrm{id}_{\mathrm{X}})^{j-1} \times \pi_{i_m} \times (\mathrm{id}_{\mathrm{X}})^{k-j}"] \\
\mathrm{Y}_{\underline{\boldsymbol{u}}} \arrow[u, bend left, "\sigma_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}}"] \arrow[ur, "f_{\underline{\boldsymbol{u}}}", "\square" description] \arrow[r] & \mathrm{X}^{j-1} \times \mathrm{M}_{i_m} \times \mathrm{X}^{k-j}
\end{tikzcd} \tag{3.1.2}$$

Here $\pi_{i_m} : \mathrm{X} \to \mathrm{M}_{i_m} := \mathrm{G}/\mathrm{P}_{\alpha_{i_m}}$ is the natural projection, and $\mathrm{X}^k \to \mathrm{X}^{j-1} \times \mathrm{M}_{i_m} \times \mathrm{X}^{k-j}$ is the projection π_{i_m} on the j-th factor and the identity on all others. The bottom map

$$\mathrm{Y}_{\underline{\boldsymbol{u}}} \to \mathrm{X}^{j-1} \times \mathrm{M}_{i_m} \times \mathrm{X}^{k-j}$$

is the map $f_{\underline{\boldsymbol{u}}}$ to X^k followed by the map $\mathrm{X}^k \to \mathrm{X}^{j-1} \times \mathrm{M}_{i_m} \times \mathrm{X}^{k-j}$ above.

Since $\mathrm{X}^k \to \mathrm{X}^{j-1} \times \pi_{i_m} \times \mathrm{X}^{k-j}$ is a $\mathbb{P}^1$-fibration the same is true of $\pi_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}}$. We conclude by induction that the variety $\mathrm{Y}_{\underline{\boldsymbol{v}}}$ is smooth, proper, and irreducible of dimension $\mathrm{N} + \ell(\underline{\boldsymbol{v}})$. The maps $f_{\underline{\boldsymbol{u}}}$ and $\mathrm{id}_{\mathrm{Y}_{\underline{\boldsymbol{u}}}}$ from $\mathrm{Y}_{\underline{\boldsymbol{u}}}$ to X^k and $\mathrm{Y}_{\underline{\boldsymbol{u}}}$ respectively give rise to the section $\sigma_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}}$. By construction we have

$$f_{\underline{\boldsymbol{u}}} = f_{\underline{\boldsymbol{v}}} \circ \sigma_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}} \quad \text{and} \quad \mathrm{id}_{\mathrm{Y}_{\underline{\boldsymbol{u}}}} = \pi_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}} \circ \sigma_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}}.$$

This construction is well defined. Indeed, assume that we had dropped a simple reflection from the right of $\underline{v}_{j'}$, $j' \neq j$ to obtain a sequence of words $\underline{\boldsymbol{u}}'$ and used $\mathrm{Y}_{\underline{\boldsymbol{u}}'}$ instead of $\mathrm{Y}_{\underline{\boldsymbol{u}}}$ to construct $\mathrm{Y}_{\underline{\boldsymbol{v}}}$. We claim that the resulting variety $\mathrm{Y}_{\underline{\boldsymbol{v}}}$ is the same. This follows easily by induction on $\ell(\underline{\boldsymbol{v}})$ and the fact that the diagram expressing the commutativity of the projections on the different factors is a fiber square:

$$\begin{tikzcd}
X^k \arrow[rr, "(\mathrm{id}_X)^{j'-1}\times\pi_{i_m}\times(\mathrm{id}_X)^{k-j'}"] \arrow[d, "(\mathrm{id}_X)^{j-1}\times\pi_{i_m}\times(\mathrm{id}_X)^{k-j}"'] \arrow[drr, phantom, "\square"] & & X^{j'-1}\times M_{i'_m}\times X^{k-j'} \arrow[d] \\
X^{j-1}\times M_{i_m}\times X^{k-j} \arrow[rr] & & X^{j'-1}\times M_{i_{m'}}\times X^{j-j'-1}\times M_{i_m}\times X^{k-j}
\end{tikzcd}$$

Here, by symmetry, we have assumed that $j' < j$.

3.2. ***Direct construction.*** Let $\underline{\boldsymbol{v}} = (\underline{v}_1, \dots, \underline{v}_k)$ be a sequence of words. The group B acts diagonally on $Z_{\underline{v}_1} \times \cdots \times Z_{\underline{v}_k}$ on the left. We define $Y_{\underline{\boldsymbol{v}}}$ to be the quotient of $G \times (Z_{\underline{v}_1} \times \cdots \times Z_{\underline{v}_k})$ by the left B-action

$$b \cdot (g, z_1, \dots, z_k) = (gb^{-1}, b \cdot z_1, \dots, b \cdot z_k). \tag{3.2.1}$$

Since $G \times (Z_{\underline{v}_1} \times \cdots \times Z_{\underline{v}_k})$ is smooth and B acts without fixed points, the quotient $Y_{\underline{\boldsymbol{v}}}$ is smooth.

The group G acts on $G \times (Z_{\underline{v}_1} \times \cdots \times Z_{\underline{v}_k})$ by left multiplication on the first factor. Since this action commutes with the action of B, it descends to an action of G on $Y_{\underline{\boldsymbol{v}}}$. The map from $G \times (Z_{\underline{v}_1} \times \cdots \times Z_{\underline{v}_k})$ to X^k given by

$$(g, z_1, \dots, z_k) \mapsto \big(g\cdot f_{\underline{v}_1}(z_1),\ g\cdot f_{\underline{v}_2}(z_2), \dots, g\cdot f_{\underline{v}_k}(z_k)\big) \tag{3.2.2}$$

is invariant under the B-action. If we let G act on X^k diagonally then (3.2.2) is also G-equivariant and hence descends to a G-equivariant morphism $f_{\underline{\boldsymbol{v}}} : Y_{\underline{\boldsymbol{v}}} \to X^k$.

As in the direct construction, we suppose that $\underline{v}_j$ is nonempty, define $\underline{u}_l$ by (3.1.1) and set $\underline{\boldsymbol{u}} = (\underline{u}_1, \dots, \underline{u}_k)$. The B-equivariant morphisms $\pi_{\underline{v}} : Z_{\underline{v}_j} \to Z_{\underline{u}_j}$ and $\sigma_{\underline{v}_j} : Z_{\underline{u}_j} \to Z_{\underline{v}_j}$ from Section 2.10 give rise to B-equivariant morphisms between $G \times (Z_{\underline{v}_1} \times \cdots \times Z_{\underline{v}_k})$ and $G \times (Z_{\underline{u}_1} \times \cdots \times Z_{\underline{u}_k})$ and hence to a G-equivariant $\mathbb{P}^1$-fibration $\pi_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}} : Y_{\underline{\boldsymbol{v}}} \to Y_{\underline{\boldsymbol{u}}}$ and a G-equivariant section $\sigma_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}} : Y_{\underline{\boldsymbol{u}}} \to Y_{\underline{\boldsymbol{v}}}$. These maps fit together to give diagram (3.1.2).

3.3. ***Expanded version of the direct construction.*** Combining the formulas for $P_{\underline{v}}$ from Section 2.10 with the direct construction above we obtain a more explicit expression for $Y_{\underline{\boldsymbol{v}}}$. If $\underline{\boldsymbol{v}} = (\underline{v}_1, \dots, \underline{v}_k)$ with $\underline{v}_j = s_{i_{1,j}} \cdots s_{i_{m_j,j}}$ for $j = 1, \dots, k$ then we define $Y_{\underline{\boldsymbol{v}}}$ to be the quotient of

$$G \times P_{\underline{v}_1} \times \cdots \times P_{\underline{v}_k} = G \times (P_{i_{1,1}} \times \cdots \times P_{i_{m_1,1}}) \times \cdots \times (P_{i_{1,k}} \times \cdots \times P_{i_{m_k,k}})$$

by the right action of $B \times B^{m_1} \times \cdots \times B^{m_k}$, where an element

$$(b_0 \mid b_{1,1}, \dots, b_{m_1,1} \mid b_{1,2}, \dots, b_{m_2,2} \mid \cdots \mid b_{1,k}, \dots, b_{m_k,k})$$

acts from the right on

$$(g \mid p_{i_{1,1}}, p_{i_{2,1}}, \dots, p_{i_{m_1,1}} \mid p_{i_{1,2}}, p_{i_{2,2}}, \dots, p_{i_{m_2,2}} \mid \cdots \mid p_{i_{1,k}}, \dots, p_{i_{m_k,k}})$$

to give

$$\big(gb_0 \mid b_0^{-1} p_{i_{1,1}} b_{1,1},\ b_{1,1}^{-1} p_{i_{2,1}} b_{2,1}, \dots, b_{m_1-1,1}^{-1} p_{i_{m_1,1}} b_{m_1,1} \mid \cdots$$
$$\cdots \mid b_0^{-1} p_{i_{1,k}} b_{1,k}, \dots, b_{m_k-1,k}^{-1} p_{i_{m_k,k}} b_{m_k,k}\big).$$

(In the expressions above the vertical lines "|" are used to indicate logical groupings, but otherwise have no significance.) The group G acts on $\mathrm{G} \times \mathrm{P}_{\underline{v}_1} \times \cdots \times \mathrm{P}_{\underline{v}_k}$ by left multiplication on the G factor, this action descends to a left action on $\mathrm{Y}_{\underline{\boldsymbol{v}}}$.

The map $f_{\underline{\boldsymbol{v}}}$ is induced by the map sending an element

$$(g \mid p_{i_{1,1}}, p_{i_{2,1}}, \dots, p_{i_{m_1,1}} \mid p_{i_{1,2}}, p_{i_{2,2}}, \dots, p_{i_{m_2,2}} \mid \cdots \mid p_{i_{1,k}}, \dots, p_{i_{m_k,k}})$$

of $\mathrm{G} \times \mathrm{P}_{\underline{v}_1} \times \cdots \times \mathrm{P}_{\underline{v}_k}$ to

$$(gp_{i_{1,1}} p_{i_{2,1}} \cdots p_{i_{m_1,1}} \mid gp_{i_{1,2}} p_{i_{2,2}} \cdots p_{i_{m_2,2}} \mid \cdots \mid gp_{i_{1,k}} \cdots p_{i_{m_k,k}}) \tag{3.3.1}$$

in X^k. From the explicit formulas this is clearly a G-equivariant map.

Finally, if $\underline{\boldsymbol{v}}$ is a sequence of words, and $\underline{\boldsymbol{u}}$ is a sequence obtained by dropping the rightmost reflection of a single word in $\underline{\boldsymbol{v}}$ (as in Section 3.2) then the G-equivariant $\mathbb{P}^1$-fibration $\pi_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}} : \mathrm{Y}_{\underline{\boldsymbol{v}}} \to \mathrm{Y}_{\underline{\boldsymbol{u}}}$ and the G-equivariant section $\sigma_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}} : \mathrm{Y}_{\underline{\boldsymbol{u}}} \to \mathrm{Y}_{\underline{\boldsymbol{v}}}$ are constructed using the obvious formulas analogous to those in Section 2.10. It again follows easily from these formulas that $f_{\underline{\boldsymbol{u}}} = f_{\underline{\boldsymbol{v}}} \circ \sigma_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}}$.

Remark. Note that the variety $\mathrm{Y}_{\underline{\boldsymbol{v}}}$ depends on the sequence of words $\underline{\boldsymbol{v}} = (\underline{v}_1, \dots, \underline{v}_k)$ and not just on the corresponding sequence $(v_1, \dots, v_k)$ of Weyl group elements. If we choose a different reduced factorization of each v_i the resulting variety is birational to $\mathrm{Y}_{\underline{\boldsymbol{v}}}$ over X^k. The proof is omitted because we do not need this fact.

3.4. ***The map*** $f_\circ$**.** As before, let $\underline{\boldsymbol{v}} = (\underline{v}_1, \dots, \underline{v}_k)$ be a sequence of words. Besides the map $f_{\underline{\boldsymbol{v}}}$ to X^k, each $\mathrm{Y}_{\underline{\boldsymbol{v}}}$ comes with a G-equivariant map $f_\circ$ to X expressing $\mathrm{Y}_{\underline{\boldsymbol{v}}}$ as a $\mathrm{Z}_{\underline{v}_1} \times \cdots \times \mathrm{Z}_{\underline{v}_k}$-bundle over X.

From the point of view of the construction in Section 3.1 $f_\circ$ is the composite map

$$\mathrm{Y}_{\underline{\boldsymbol{v}}} \xrightarrow{\pi_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}}} \mathrm{Y}_{\underline{\boldsymbol{u}}} \to \cdots \to \mathrm{Y}_{\underline{\varnothing}} = \mathrm{X}$$

obtained by dropping the elements in the entries of $\underline{\boldsymbol{v}}$ one at a time. The fiber over e in X is then the result of applying the recursive construction in Section 2.10 separately for each $\underline{v}_i$, $i = 1, \dots, k$, and so the fiber is $\mathrm{Z}_{\underline{v}_1} \times \cdots \times \mathrm{Z}_{\underline{v}_k}$.

From the point of view of the construction in Section 3.2 one starts with the projection $\mathrm{G} \times (\mathrm{Z}_{\underline{v}_1} \times \cdots \times \mathrm{Z}_{\underline{v}_k}) \to \mathrm{G}$ onto the first factor. This is B-equivariant for the right action of B on G and hence descends to a morphism $f_\circ : Y_{\underline{\boldsymbol{v}}} \to X$ expressing $Y_{\underline{\boldsymbol{v}}}$ as a $Z_{\underline{v}_1} \times \cdots \times Z_{\underline{v}_k}$-bundle over X.

Let $\underline{\boldsymbol{u}} = (\underline{\varnothing}, \underline{v}_1, \dots, \underline{v}_k)$. Since the action of B on the point $Z_{\underline{\varnothing}} = e$ is trivial, we have an isomorphism

$$G \times Z_{\underline{v}_1} \times \cdots \times Z_{\underline{v}_k} \simeq G \times Z_{\underline{\varnothing}} \times Z_{\underline{v}_1} \times \cdots \times Z_{\underline{v}_k}$$

of B-varieties and hence a G-isomorphism $\phi : Y_{\underline{\boldsymbol{v}}} \to Y_{\underline{\boldsymbol{u}}}$. From the explicit description in (3.2.2) we see that the composite map $f_{\underline{\boldsymbol{u}}} \circ \phi : Y_{\underline{\boldsymbol{v}}} \to X^{k+1}$ followed by projection onto the first factor is $f_\circ$, and that $f_{\underline{\boldsymbol{u}}} \circ \phi$ followed by projection onto the last k factors is $f_{\underline{\boldsymbol{v}}}$.

Thus the map $f_\circ \times f_{\underline{\boldsymbol{v}}} : Y_{\underline{\boldsymbol{v}}} \to X \times X^k$ is equal to the map

$$f_{(\underline{\varnothing}, \underline{v}_1, \dots, \underline{v}_k)} : Y_{(\underline{\varnothing}, \underline{v}_1, \dots, \underline{v}_k)} \to X^{k+1}$$

under the isomorphism ϕ. This will be used in the proof of Theorem 3.7.4.

3.5. *Maximum point.* Let $\underline{\boldsymbol{v}} = (\underline{v}_1, \dots, \underline{v}_k)$ be a sequence of words. We define the *maximum point* $p_{\underline{\boldsymbol{v}}}$ of $Y_{\underline{\boldsymbol{v}}}$ to be the product maximum point (Section 2.10) $p_{\underline{v}_1} \times \cdots \times p_{\underline{v}_k}$ in the fiber $Z_{\underline{v}_1} \times \cdots \times Z_{\underline{v}_k}$ of $f_\circ$ over e in X. Alternatively, if $\underline{v}_j = (s_{i_{1,j}}, \dots, s_{i_{m_j,j}})$ for $j = 1, \dots, k$ then (in the notation of Section 3.3) the point

$$(e \mid s_{i_{1,1}}, s_{i_{2,1}}, \dots, s_{i_{m_1,1}} \mid \cdots \mid s_{i_{1,k}}, \dots, s_{i_{m_k,k}})$$

is a point of

$$G \times (P_{i_{1,1}} \times \cdots \times P_{i_{m_1,1}}) \times \cdots \times (P_{i_{1,k}} \times \cdots \times P_{i_{m_k,k}})$$

and its image in $Y_{\underline{\boldsymbol{v}}}$ under the quotient map by $B \times B^{m_1} \times \cdots \times B^{m_k}$ is the maximum point $p_{\underline{\boldsymbol{v}}}$. If each $\underline{v}_j$ is a factorization of some $v_j \in \mathcal{W}$, then the image $f_{\underline{\boldsymbol{v}}}(p_{\underline{\boldsymbol{v}}})$ of the maximum point in X^k is the point $q_{\underline{\boldsymbol{v}}} := (v_1, \dots, v_k)$.

3.6. *Tangent space formulas.* We will need to know the formal character (see Section 2.1) of the tangent space of $Y_{\underline{\boldsymbol{v}}}$ at the maximum point $p_{\underline{\boldsymbol{v}}}$. If each $\underline{v}_j$ is a reduced word with product v_j, then the formal character of the tangent space to $Z_{\underline{v}_j}$ at $p_{\underline{v}_j}$ is $\langle \Phi_{v_j^{-1}} \rangle$ and the formal character of the tangent space of X at e is $\langle \Delta^- \rangle$.

Since the fibration $f_\circ$ is smooth, the formal character of $T_{p_{\underline{\boldsymbol{v}}}} Y_{\underline{\boldsymbol{v}}}$ is the sum of these formal characters, i.e.,

$$\mathrm{Ch}(T_{p_{\underline{\boldsymbol{v}}}} Y_{\underline{\boldsymbol{v}}}) = \langle \Delta^- \rangle + \sum_{i=1}^{k} \langle \Phi_{v_i^{-1}} \rangle.$$

If $v_j = w_j^{-1} w_0$ for $j = 1, \dots, k$, then by (2.2.2) this is the same as

$$\mathrm{Ch}(T_{p_{\underline{\boldsymbol{v}}}} Y_{\underline{\boldsymbol{v}}}) = \Delta^- + \sum_{i=1}^{k} \langle \Phi^{c}_{w_i} \rangle. \tag{3.6.1}$$

3.7. *Fibers and images of $f_{\underline{\boldsymbol{v}}}$.*

Lemma 3.7.1. *Let $\underline{\boldsymbol{v}} = (\varnothing, \underline{v}_2, \ldots, \underline{v}_k)$ be a sequence of words, with each $\underline{v}_i$ a reduced factorization of v_i, and let $\mathrm{X}_{\underline{\boldsymbol{v}}}$ be the (reduced) image of $f_{\underline{\boldsymbol{v}}}$ in X^k. Then:*

(a) *Projection onto the first factor of X^k endows $\mathrm{X}_{\underline{\boldsymbol{v}}}$ with the structure of a fiber bundle over X with fiber isomorphic to $\mathrm{X}_{v_2} \times \cdots \times \mathrm{X}_{v_k}$.*

(b) *The variety $\mathrm{X}_{\underline{\boldsymbol{v}}}$ is normal with rational singularities of dimension $\mathrm{N} + \ell(\underline{\boldsymbol{v}})$, and the induced map $\mathrm{Y}_{\underline{\boldsymbol{v}}} \to \mathrm{X}_{\underline{\boldsymbol{v}}}$ is birational with connected fibers.*

Proof. Projection on the first factor of X^k gives a G-equivariant morphism $\mathrm{X}_{\underline{\boldsymbol{v}}} \xrightarrow{\eta} \mathrm{X}$. Since G acts transitively on X this morphism is surjective and all fibers are isomorphic, i.e., this expresses $\mathrm{X}_{\underline{\boldsymbol{v}}}$ as a fiber bundle over X. To study the fibers we look at the fiber $\eta^{-1}(e)$ over the B-fixed point e of X.

Consider the diagram

$$\begin{tikzcd}
\mathrm{G} \times e \times \mathrm{Z}_{\underline{v}_2} \times \cdots \times \mathrm{Z}_{\underline{v}_k} \arrow[r, "\psi_{\underline{\boldsymbol{v}}}"] \arrow[d, "\mathrm{id}_{\mathrm{G}} \times \mathrm{id}_{x_o} \times f_{\underline{v}_1} \times \cdots \times f_{\underline{v}_k}"'] & \mathrm{Y}_{\underline{\boldsymbol{v}}} \arrow[d, "f_{\underline{\boldsymbol{v}}}"] \\
\mathrm{G} \times e \times \mathrm{X}_{v_2} \times \cdots \times \mathrm{X}_{v_k} \arrow[r, "\phi"] & \mathrm{X}^k
\end{tikzcd} \tag{3.7.2}$$

where ϕ is given by $\phi(g, e, x_2, \ldots, x_k) = (g \cdot e, g \cdot x_2, \ldots, g \cdot x_k) \in \mathrm{X}^k$. Since $\psi_{\underline{\boldsymbol{v}}}$ and the leftmost vertical map are surjective, the image of $f_{\underline{\boldsymbol{v}}}$ is the same as the image of ϕ. Since B is the stabilizer of e, the fiber $\eta^{-1}(e)$ is the image of $\mathrm{B} \times e \times \mathrm{X}_{v_2} \times \cdots \times \mathrm{X}_{v_k}$ under ϕ. But each Schubert variety X_w is stable under the action of B and therefore the image above is just $e \times \mathrm{X}_{v_2} \times \cdots \times \mathrm{X}_{v_k}$, proving (a).

From the fibration η it is clear that

$$\dim(\mathrm{X}_{\underline{\boldsymbol{v}}}) = \dim(\mathrm{X}) + \sum_{i=2}^{k} \dim(\mathrm{X}_{v_i}) = \mathrm{N} + \sum_{i=2}^{k} \ell(v_i) = \mathrm{N} + \ell(\underline{\boldsymbol{v}}),$$

because each $\underline{v}_i$ is reduced and hence $\ell(v_i) = \ell(\underline{v}_i)$ for $i \geqslant 2$.

The product of normal varieties is again normal, and the product of varieties with rational singularities also has rational singularities. Since each X_w is normal with rational singularities (Section 2.6), the fibers also have this property, and therefore so does $\mathrm{X}_{\underline{\boldsymbol{v}}}$ (since the properties of being normal or having rational singularities are local, and $\mathrm{X}_{\underline{\boldsymbol{v}}}$ is locally the product of the fiber and a smooth variety).

Since each map $f_{\underline{v}_i} : \mathrm{Z}_{\underline{v}_i} \to \mathrm{X}_{v_i}$ is a resolution of singularities of a normal variety, each $f_{\underline{v}_i}$ is birational with connected fibers. It follows that the map $\mathrm{Y}_{\underline{\boldsymbol{v}}} \to \mathrm{X}_{\underline{\boldsymbol{v}}}$, which is the quotient of the leftmost vertical map in (3.7.2) by the action of B, is also birational with connected fibers. This proves (b). □

Definition 3.7.3. If X_w is any Schubert subvariety of X and q is any point of X, we define the subvariety $q\mathrm{X}_w$ of X to be the result of translating X_w by any element in the B-coset corresponding to q. Since X_w is B-stable the result is independent of the choice of representative for q.

The following theorem gives more precise information about the image and fibers of $f_{\underline{\boldsymbol{v}}}$.

Theorem 3.7.4. *Let* $\underline{\boldsymbol{v}} = (\underline{v}_1, \ldots, \underline{v}_k)$ *be a sequence of reduced words with corresponding Weyl group elements* $(v_1, \ldots, v_k)$. *Then there exists a factorization* $f_{\underline{\boldsymbol{v}}} : Y_{\underline{\boldsymbol{v}}} \xrightarrow{\tau} Q_{\underline{\boldsymbol{v}}} \xrightarrow{h} X^k$ *such that*

(a) $Q_{\underline{\boldsymbol{v}}}$ *is normal with rational singularities*;

(b) *the map* $\tau : Y_{\underline{\boldsymbol{v}}} \to Q_{\underline{\boldsymbol{v}}}$ *is proper and birational with connected fibers*;

(c) *for each point* $(q_1, \ldots, q_k)$ *of* X^k *there is a natural inclusion*

$$h^{-1}(q_1, \ldots, q_k) \hookrightarrow \bigcap_{i=1}^{k} q_i X_{v_i^{-1}}$$

of the scheme-theoretic fiber $h^{-1}(q_1, \ldots, q_k)$ *into the scheme-theoretic intersection* $\bigcap_{i=1}^{k} q_i X_{v_i^{-1}}$;

(d) *the inclusion of schemes in* (c) *induces an isomorphism at the level of reduced schemes*, *or in other words*, *the set-theoretic fiber* $h^{-1}(q_1, \ldots, q_k)$ *is equal to the set-theoretic intersection* $\bigcap_{i=1}^{k} q_i X_{v_i^{-1}}$.

Proof. Let $f_\circ \times f_{\underline{\boldsymbol{v}}} : Y_{\underline{\boldsymbol{v}}} \to X \times X^k$ be the product of $f_{\underline{\boldsymbol{v}}}$ and the map $f_\circ : Y_{\underline{\boldsymbol{v}}} \to X$ from Section 3.4 expressing $Y_{\underline{\boldsymbol{v}}}$ as a $Z_{\underline{\boldsymbol{v}}_1} \times \cdots \times Z_{\underline{\boldsymbol{v}}_k}$-bundle over X. We define $Q_{\underline{\boldsymbol{v}}}$ to be the image of $f_\circ \times f_{\underline{\boldsymbol{v}}}$ with the reduced scheme structure, τ to be the map from $Y_{\underline{\boldsymbol{v}}}$ onto $Q_{\underline{\boldsymbol{v}}}$, and h to be the map from $Q_{\underline{\boldsymbol{v}}}$ to X^k induced by the projection $X \times X^k \to X^k$. By construction $f_{\underline{\boldsymbol{v}}} = h \circ \tau$.

Letting $\psi_{\underline{\boldsymbol{v}}}$ be the map (from Section 3.2) defining $Y_{\underline{\boldsymbol{v}}}$ as a quotient of $B \times Z_{\underline{v}_1} \times \cdots \times Z_{\underline{v}_k}$ and $\phi : G \times X_{v_1} \times \cdots \times X_{v_k} \to Q_{\underline{\boldsymbol{v}}} \subseteq X \times X^k$ as the map sending $(g, x_1, \ldots, x_k)$ to $(gB/B, g \cdot x_1, \ldots, g \cdot x_k)$ in $X \times X^k$, we obtain a refinement of diagram (3.7.2):

$$\begin{tikzcd}
G \times Z_{\underline{v}_1} \times \cdots \times Z_{\underline{v}_k} \arrow[r, "\psi_{\underline{\boldsymbol{v}}}"] \arrow[d] \arrow[dr, phantom, "\square"] & Y_{\underline{\boldsymbol{v}}} \arrow[d, "\tau"'] \arrow[dr, "f_\circ \times f_{\underline{\boldsymbol{v}}}"] & \\
G \times X_{v_1} \times \cdots \times X_{v_k} \arrow[r, "\phi"] & Q_{\underline{\boldsymbol{v}}} \arrow[r, hook] \arrow[d, "h"'] & X \times X^k \arrow[d] \\
 & X^k \arrow[r, equal] & X^k
\end{tikzcd}$$

Since $Y_{\underline{\boldsymbol{v}}} \simeq Y_{(\varnothing, \underline{v}_1, \ldots, \underline{v}_k)}$ (see Section 3.4) and under this isomorphism the map $f_\circ \times f_{\underline{\boldsymbol{v}}}$ is the map $f_{(\varnothing, \underline{v}_1, \ldots, \underline{v}_k)}$, it follows from Lemma 3.7.1(b) that $Q_{\underline{\boldsymbol{v}}}$ is normal with rational singularities and that $\tau : Y_{\underline{\boldsymbol{v}}} \to Q_{\underline{\boldsymbol{v}}}$ is birational with connected fibers, proving (a) and (b).

The composite map

$$G \times P_{\underline{v}_1} \times \cdots \times P_{\underline{v}_k} \to G \times Z_{\underline{v}_1} \times \cdots \times Z_{\underline{v}_k} \xrightarrow{\psi_{\underline{\boldsymbol{v}}}} Y_{\underline{\boldsymbol{v}}} \xrightarrow{\tau} Q_{\underline{\boldsymbol{v}}}$$

is given (in the notation of Section 3.3) by sending

$$(g \mid p_{i_{1,1}}, p_{i_{2,1}}, \ldots, p_{i_{m_1,1}} \mid p_{i_{1,2}}, p_{i_{2,2}}, \ldots, p_{i_{m_2,2}} \mid \cdots \mid p_{i_{1,k}}, \ldots, p_{i_{m_k,k}})$$

to

$$(g \mid g p_{i_{1,1}} p_{i_{2,1}} \cdots p_{i_{m_1,1}} \mid g p_{i_{1,2}} p_{i_{2,2}} \cdots p_{i_{m_2,2}} \mid \cdots \mid g p_{i_{1,k}} \cdots p_{i_{m_k,k}})$$

in $X \times X^k$. A point q of X is therefore in the fiber

$$h^{-1}(q_1, \ldots, q_k) \subseteq X \times q_1 \times \cdots \times q_k = X$$

if for any B-coset representatives $g, g_1, \ldots, g_k$ of $q, q_1, \ldots, q_k$, there exist elements $\{p_{i,j}\}$ in the respective parabolic subgroups such that we can solve the equations

$$\begin{aligned} g p_{i_{1,1}} p_{i_{2,1}} \cdots p_{i_{m_1,1}} &= g_1, \\ \vdots \qquad\qquad &\vdots \;\; \vdots \\ g p_{i_{1,k}} p_{i_{2,k}} \cdots p_{i_{m_k,k}} &= g_k. \end{aligned}$$

Moving the $p_{i,j}$'s to the right hand side, the system above becomes

$$\begin{aligned} g &= g_1 p_{i_{m_1,1}}^{-1} \cdots p_{i_{2,1}}^{-1} p_{i_{1,1}}^{-1}, \\ \vdots \; &\vdots \qquad\qquad \vdots \\ g &= g_k p_{i_{m_k,k}}^{-1} \cdots p_{i_{2,k}}^{-1} p_{i_{1,k}}^{-1}, \end{aligned}$$

which is equivalent to q belonging in the intersection $\bigcap_{i=1}^k q_i X_{v_i^{-1}}$, proving (d).

Let $\underline{v}$ be a reduced word with product v. By part (d) the set

$$Q'_{\underline{v}} := \big\{(q, p) \in X \times X \mid p \in q X_v\big\}$$

is the image of $f_{(\varnothing, \underline{v})} : Y_{(\varnothing, \underline{v})} \to X \times X$ and is therefore a closed subvariety of $X \times X$. Alternatively $Q'_{\underline{v}}$ is the Zariski closure of the set $\{(g, g \cdot v) \mid g \in G\} \subseteq X \times X$.

For $i = 1, \ldots, k$, let $p_i : X \times X^k \to X \times X$ be the map which is the product of id_X with projection $X^k \to X$ onto the i-th factor. The intersection

$$Q'_{\underline{\boldsymbol{v}}} := \bigcap_{i=1}^{k} p_i^{-1}(Q'_{\underline{v}_i})$$

is a closed subscheme of $X \times X^k$ which, by (d), agrees set theoretically with $Q_{\underline{v}}$. Since $Q_{\underline{v}}$ is reduced, we have the inclusion of schemes $Q_{\underline{v}} \subseteq Q'_{\underline{v}}$. If h' is the map $h' : Q'_{\underline{v}} \to X^k$ induced by projection, then the scheme-theoretic fibers of h are naturally a subscheme of the scheme-theoretic fibers of h' (and both are naturally subschemes of X). The scheme-theoretic fiber of h' is the scheme-theoretic intersection $\bigcap_{i=1}^k q_i X_{v_i^{-1}}$, proving (c). □

The image $X_{\underline{v}}$ is therefore the set of translations $(q_1, \ldots, q_k)$ in X^k for which the intersection $\bigcap_{i=1}^k q_i X_{v_i^{-1}}$ of translated Schubert varieties is nonempty, and the set-theoretic fibers of h are the intersections themselves. Moreover, $Q_{\underline{v}}$ is the incidence correspondence of intersections of translates of Schubert varieties (the first coordinate in $X \times X^k$ is the intersection, the remaining k coordinates are the parameters $(q_1, \ldots, q_k)$ controlling the translates). Theorem 3.7.4 shows that $Y_{\underline{v}}$ is a resolution of singularities of $Q_{\underline{v}}$.

Corollary 3.7.5. *Let $\underline{v} = (\underline{v}_1, \ldots, \underline{v}_k)$ be a sequence of reduced words with corresponding Weyl group elements $(v_1, \ldots, v_k)$ such that $\sum_{i=1}^k \ell(\underline{v}_i) = (k-1)N$. Then the degree of the map $f_{\underline{v}} : Y_{\underline{v}} \to X^k$ is given by the intersection number $\bigcap_{i=1}^k [\Omega_{w_0 v_i^{-1}}] = \bigcap_{i=1}^k [X_{v_i^{-1}}]$.*

Remark. The dimension of $Y_{\underline{v}}$ in this case is $N + \sum \ell(\underline{v}_i) = kN = \dim(X^k)$ so it is reasonable to ask for the degree of the map.

Proof. Since we are working in characteristic zero, the degree of $f_{\underline{v}}$ is given by the number of points in a generic fiber. By Theorem 3.7.4 the map $p : Y_{\underline{v}} \to Q_{\underline{v}}$ is birational, and so the generic fiber of $f_{\underline{v}}$ is the same as the generic fiber of $h : Q_{\underline{v}} \to X^k$. By the Kleiman transversality theorem, if $q_1, \ldots, q_k$ are generic, the scheme-theoretic intersection $\bigcap_{i=1}^k q_i X_{v_i^{-1}}$ is reduced and finite, and the number of points is equal to the intersection number $\bigcap_{i=1}^k [X_{v_i^{-1}}] = \bigcap_{i=1}^k [\Omega_{w_0 v_i^{-1}}]$ in $H^*(X, \mathbb{Z})$. By Theorem 3.7.4(c–d) if the scheme-theoretic intersection $\bigcap_{i=1}^k q_i X_{v_i^{-1}}$ is reduced it is equal to the scheme-theoretic fiber $h^{-1}(q_1, \ldots, q_k)$, proving the corollary. □

3.8. *Key lemma.* We now prove an important lemma which will allow us to derive several results necessary for the proofs of Theorems I and II. The lemma itself will also be used in the proof of Theorem I.

Lemma 3.8.1. *Let $\underline{v}$ be a sequence of reduced words, L be a G-equivariant line bundle on $Y_{\underline{v}}$ and $s \in H^0(Y_{\underline{v}}, L)^G$ be a nonzero G-invariant section. Then:*

(a) *the weight of L at the T-fixed maximum point (Section 3.5) $p = p_{\underline{v}} \in Y_{\underline{v}}$ belongs to $\operatorname{span}_{\mathbb{Z}_{\geqslant 0}} \Delta^+$;*

(b) *the weight of L at p is zero if and only if s does not vanish at p;*

(c) *without supposing that L has a G-invariant section, if L is an equivariant bundle on $Y_{\underline{v}}$ and the weight of L at p is zero, then $\dim H^0(Y_{\underline{v}}, L)^G \leqslant 1$.*

Remark. Part (c) will be used often to control the size of the G-invariant sections.

Proof. Let $f_\circ : \mathrm{Y}_{\underline{\boldsymbol{v}}} \to \mathrm{X}$ be the map from Section 3.4 expressing $\mathrm{Y}_{\underline{\boldsymbol{v}}}$ as a $\mathrm{Z}_{\underline{v}_1} \times \cdots \times \mathrm{Z}_{\underline{v}_k}$-bundle over X. The section s cannot vanish on any fiber of $f_\circ$ since (by G-invariance and transitivity of G-action on X) s would vanish on all of $\mathrm{Y}_{\underline{\boldsymbol{v}}}$. We can thus restrict s to get a nonzero section on the fiber $\mathrm{Z}_{\underline{v}_1} \times \cdots \times \mathrm{Z}_{\underline{v}_k}$ of $f_\circ$ over $e \in \mathrm{X}$; this fiber contains the maximum point p.

The formal character of the tangent space at the maximum point p_i of $\mathrm{Z}_{\underline{v}_i}$ is $\langle \Phi_{v_i^{-1}} \rangle$; i.e., all the weights of this space are positive roots. Since the maximum point $p = p_1 \times \cdots \times p_k \in \mathrm{Z} := \mathrm{Z}_{\underline{v}_1} \times \cdots \times \mathrm{Z}_{\underline{v}_k}$ is the product of the maximum points of the factors, each of the weights on the tangent space of p in Z is also a positive root.

Let $\mathfrak{m}_p$ be the maximal ideal of p in $\mathcal{O}_{\mathrm{Z},p}$. For every $r \geqslant 0$ we get a T-equivariant restriction map

$$\mathrm{H}^0(\mathrm{Z}, \mathrm{L}|_\mathrm{Z}) \to \mathrm{L} \otimes_{\mathcal{O}_\mathrm{Z}} (\mathcal{O}_{\mathrm{Z},p}/\mathfrak{m}_p^{r+1}) = \mathrm{L} \otimes \left(\mathcal{O}_\mathrm{Z}/\mathfrak{m}_p \oplus \mathfrak{m}_p/\mathfrak{m}_p^2 \oplus \cdots \oplus \mathfrak{m}_p^r/\mathfrak{m}_p^{r+1}\right),$$

which is an injection for r sufficiently large. In particular, for sufficiently large r, the section s restricts to a nonzero element of $\mathrm{L} \otimes_{\mathcal{O}_\mathrm{Z}} \mathcal{O}_{\mathrm{Z},p}/\mathfrak{m}_p^{r+1}$. Since s is an invariant section, this means that the zero weight is a weight of $\mathrm{L} \otimes_{\mathcal{O}_\mathrm{Z}} (\mathcal{O}_{\mathrm{Z},p}/\mathfrak{m}_p^{r+1})$, and so must appear in one of the factors $\mathrm{L} \otimes_{\mathcal{O}_\mathrm{Z}} (\mathfrak{m}_p^i/\mathfrak{m}_p^{i+1}) = \mathrm{L} \otimes_{\mathcal{O}_\mathrm{Z}} \mathrm{Sym}^i(\mathfrak{m}_p/\mathfrak{m}_p^2)$ for $i = 0, \ldots, r$.

Since $\mathfrak{m}_p/\mathfrak{m}_p^2$ is dual to the tangent space at p, all weights of $\mathfrak{m}_p/\mathfrak{m}_p^2$ are negative roots, and therefore the weights of $\mathrm{Sym}^i(\mathfrak{m}_p/\mathfrak{m}_p^2)$ belong to $\mathrm{span}_{\mathbb{Z}_{\leqslant 0}} \Delta^+$. Tensoring with L multiplies the formal character of $\mathrm{Sym}^i(\mathfrak{m}_p/\mathfrak{m}_p^2)$ by the weight of L at p. Thus the zero weight is a weight of $\mathrm{L} \otimes_{\mathcal{O}_\mathrm{Z}} \mathrm{Sym}^i(\mathfrak{m}_p/\mathfrak{m}_p^2)$ only if the weight of L at p belongs to $\mathrm{span}_{\mathbb{Z}_{\geqslant 0}} \Delta^+$. This proves (a).

The value of s at p is the restriction of s to the factor $\mathrm{L} \otimes_{\mathcal{O}_\mathrm{Z}} (\mathcal{O}_{\mathrm{Z},p}/\mathfrak{m}_p) = \mathrm{L}_p$. If s does not vanish at p the weight of L_p is therefore zero. Conversely, if the weight of L_p is zero then the weights of $\mathrm{L} \otimes_{\mathcal{O}_\mathrm{Z}} \mathrm{Sym}^i(\mathfrak{m}_p/\mathfrak{m}_p^2)$ are nonzero for $i \geqslant 1$. Hence the only possibility for the invariant section s under the restriction map is to have nonzero restriction to $\mathrm{L} \otimes_{\mathcal{O}_\mathrm{Z}} (\mathcal{O}_{\mathrm{Z},p}/\mathfrak{m}_p) = \mathrm{L}_p$, proving (b).

Suppose that the weight of L at p is zero. If there were two linearly independent sections $s_1, s_2 \in \mathrm{H}^0(\mathrm{Y}_{\underline{\boldsymbol{v}}}, \mathrm{L})^\mathrm{G}$ then some nonzero linear combination would vanish at p contradicting (b). Hence if the weight is zero we must have $\dim \mathrm{H}^0(\mathrm{Y}_{\underline{\boldsymbol{v}}}, \mathrm{L}) \leqslant 1$, giving (c). □

3.9. *Applications of Lemma 3.8.1.*

Theorem 3.9.1. *Suppose that $w_1, \ldots, w_k$, and w are elements of the Weyl group such that*

$$\ell(w) = \sum_{i=1}^{k} \ell(w_i) \quad \textit{and} \quad \bigcap_{i=1}^{k} [\Omega_{w_i}] \cdot [\mathrm{X}_w] \neq 0 \textit{ in } \mathrm{H}^*(\mathrm{X}, \mathbb{Z}).$$

Then:

(a) *For any dominant weights* $\mu_1, \ldots, \mu_k$, *and* μ *such that the irreducible module* V_μ *is a component of* $\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}$, *the weight* $\sum_{i=1}^k w_i^{-1}\mu_i - w^{-1}\mu$ *belongs to* $\mathrm{span}_{\mathbb{Z}_{\geqslant 0}} \Delta^+$.

(b) *If* $\sum_{i=1}^k w_i^{-1}\mu_i - w^{-1}\mu = 0$ *then* $\mathrm{mult}(\mathrm{V}_\mu, \mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}) = 1$.

(c) $\sum_{i=1}^k w_i^{-1} \cdot 0 - w^{-1} \cdot 0 = \sum_{i=1}^k (w_k^{-1}\rho - \rho) - (w^{-1}\rho - \rho)$ *belongs to* $\mathrm{span}_{\mathbb{Z}_{\geqslant 0}} \Delta^+$.

(d) *If* $\sum_{i=1}^k w_i^{-1} \cdot 0 = w^{-1} \cdot 0$ *then* $\Phi_w = \bigsqcup_{i=1}^k \Phi_{w_i}$.

Note that the action of the Weyl group in parts (a) *and* (b) *is the homogeneous action*, *while the action in parts* (c) *and* (d) *is the affine action.*

Proof. Let $v_i = w_i^{-1}w_0$ for $i = 1, \ldots, k$, $v_{k+1} = w^{-1}$, let $\underline{v}_i$ be a reduced word with product v_i, for $i = 1, \ldots, k+1$, and set $\underline{\boldsymbol{v}} = (\underline{v}_1, \ldots, \underline{v}_{k+1})$. Then $\sum \ell(\underline{v}_i) = (k+1-1)\mathrm{N}$ and so, by Corollary 3.7.5, the degree of $f_{\underline{\boldsymbol{v}}} : \mathrm{Y}_{\underline{\boldsymbol{v}}} \to \mathrm{X}^{k+1}$ is given by the intersection number

$$\bigcap_{i=1}^{k+1} [\Omega_{w_0 v_i^{-1}}] = \bigcap_{i=1}^{k} [\Omega_{w_i}] \cdot [\mathrm{X}_w].$$

By hypothesis this intersection number is nonzero and therefore $f_{\underline{\boldsymbol{v}}}$ is surjective.

Given dominant weights $\mu_1, \ldots, \mu_k$, and μ let $\lambda_i = -w_0\mu_i$ for $i = 1, \ldots, k$ and $\lambda_{k+1} = \mu$. Set L to be the line bundle $\mathrm{L}_{\lambda_1} \boxtimes \cdots \boxtimes \mathrm{L}_{\lambda_{k+1}}$ on X^{k+1}, so that $\mathrm{H}^0(\mathrm{X}^{k+1}, \mathrm{L}) = \mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k} \otimes \mathrm{V}_\mu^*$ and $\dim \mathrm{H}^0(\mathrm{X}^{k+1}, \mathrm{L})^{\mathrm{G}}$ is the multiplicity of V_μ in the tensor product $\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}$.

Since $f_{\underline{\boldsymbol{v}}}$ is surjective, pullback induces an inclusion

$$\mathrm{H}^0(\mathrm{Y}_{\underline{\boldsymbol{v}}}, f_{\underline{\boldsymbol{v}}}^*\mathrm{L}) \xleftarrow{f_{\underline{\boldsymbol{v}}}^*} \mathrm{H}^0(\mathrm{X}^{k+1}, \mathrm{L})$$

and, in particular, $\dim \mathrm{H}^0(\mathrm{Y}_{\underline{\boldsymbol{v}}}, f_{\underline{\boldsymbol{v}}}^*\mathrm{L})^{\mathrm{G}} \geqslant \dim \mathrm{H}^0(\mathrm{X}^{k+1}, \mathrm{L})^{\mathrm{G}}$. We know, by applying Lemma 3.8.1(a), that if $f_{\underline{\boldsymbol{v}}}^*\mathrm{L}$ has a nonzero G-invariant section then the weight of $f_{\underline{\boldsymbol{v}}}^*\mathrm{L}$ at the maximum point $p_{\underline{\boldsymbol{v}}}$ belongs to $\mathrm{span}_{\mathbb{Z}_{\geqslant 0}} \Delta^+$. This weight is

$$\sum_{i=1}^{k+1} v_i(-\lambda_i) = \sum_{i=1}^{k} (w_i^{-1}w_0)(w_0\mu_i) + w^{-1}(-\mu) = \sum_{i=1}^{k} w_i^{-1}\mu_i - w^{-1}\mu, \quad (3.9.2)$$

proving (a).

If the weight in (3.9.2) is zero then $\dim \mathrm{H}^0(\mathrm{X}^{k+1}, \mathrm{L})^{\mathrm{G}} \leqslant \dim \mathrm{H}^0(\mathrm{Y}_{\underline{\boldsymbol{v}}}, f_{\underline{\boldsymbol{v}}}^*\mathrm{L})^{\mathrm{G}} \leqslant 1$ by Lemma 3.8.1(c), and so if V_μ is a component of $\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}$ then it is of multiplicity at most one. The fact that V_μ actually is a component of the tensor product is a consequence of the solution of the PRV conjecture — see Section 2.3 for a discussion. This proves (b).

The map $f_{\underline{v}} : \mathrm{Y}_{\underline{v}} \to \mathrm{X}^{k+1}$ induces a natural map $f_{\underline{v}}^{*}\mathrm{K}_{\mathrm{X}^{k+1}} \to \mathrm{K}_{\mathrm{Y}_{\underline{v}}}$ which is given by a global section s of $\mathrm{H}^0(\mathrm{Y}_{\underline{v}}, (f_{\underline{v}}^{*}\mathrm{K}_{\mathrm{X}^{k+1}})^{*} \otimes \mathrm{K}_{\mathrm{Y}_{\underline{v}}})$. Since $\mathrm{Y}_{\underline{v}}$ and X^{k+1} have the same dimension and since $f_{\underline{v}}$ is surjective, this section is nonzero. Because the pullback morphism is natural, the section s is G-invariant. By Lemma 3.8.1(a) the weight of the line bundle $\mathrm{K}_{\mathrm{Y}_{\underline{v}}/\mathrm{X}^{k+1}} := (f_{\underline{v}}^{*}\mathrm{K}_{\mathrm{X}^{k+1}})^{*} \otimes \mathrm{K}_{\mathrm{Y}_{\underline{v}}}$ at the maximum point $p_{\underline{v}}$ belongs to $\mathrm{span}_{\mathbb{Z}\geqslant 0}\,\Delta^{+}$.

By (3.6.1), (2.2.2), and (2.2.3) the formal characters of the tangent spaces at $p_{\underline{v}}$ in $\mathrm{Y}_{\underline{v}}$ and $q_{\underline{v}} := f_{\underline{v}}(p_{\underline{v}})$ in X^{k+1} are, respectively

$$\mathrm{Ch}(\mathrm{T}_{p_{\underline{v}}}\mathrm{Y}_{\underline{v}}) = \langle \Phi_w \rangle + \langle \Delta^{-} \rangle + \sum_{i=1}^{k} \langle \Phi^{\mathrm{c}}_{w_i} \rangle \tag{3.9.3}$$

and

$$\mathrm{Ch}(\mathrm{T}_{q_{\underline{v}}}\mathrm{X}^{k+1}) = \big(\langle \Phi_w \rangle + \langle -\Phi^{\mathrm{c}}_{w} \rangle\big) + \sum_{i=1}^{k} \big(\langle \Phi^{\mathrm{c}}_{w_i} \rangle + \langle -\Phi_{w_i} \rangle\big). \tag{3.9.4}$$

A short calculation using formula (2.2.4) shows that the weight of $\mathrm{K}_{\mathrm{Y}_{\underline{v}}/\mathrm{X}^{k+1}}$ at $p_{\underline{v}}$ is $\sum_{i=1}^{k}(w_k^{-1}\rho - \rho) - (w^{-1}\rho - \rho)$, proving (c).

If the weight $\sum_{i=1}^{k}(w_k^{-1}\rho - \rho) - (w^{-1}\rho - \rho)$ is zero then, by Lemma 3.8.1(b), the section s is nonzero at $p_{\underline{v}}$. This means that $f_{\underline{v}}$ is unramified at $p_{\underline{v}}$ and therefore the tangent space map $\mathrm{T}_{\mathrm{Y}_{\underline{v}},p} \xrightarrow{df_{\underline{v}}} \mathrm{T}_{\mathrm{X}^{k+1},q}$ is an isomorphism. Hence both spaces must have the same formal characters. Comparing the negative roots and their multiplicities in (3.9.3) and (3.9.4) gives $\Delta^{-} = \big(\bigsqcup_{i=1}^{k} -\Phi_{w_i}\big) \bigsqcup -\Phi^{\mathrm{c}}_{w}$ which is equivalent to $\Phi_w = \bigsqcup_{i=1}^{k} \Phi_{w_i}$, proving (d). □

3.10. ***Relation with existing results.*** Part (a) of Theorem 3.9.1 is due to Berenstein and Sjamaar [2000] . A theorem of this type was first proved by Klyachko [1998] for GL_n. This was later extended to all semisimple groups by Berenstein and Sjamaar [2000] and by Kapovich, Leeb, and Millson [Kapovich et al. 2009]. Parts (c) and (d) are due to Belkale and Kumar [2006]: part (c) is their Theorem 29 and (d) is their Theorem 15, both in the case when the parabolic group P is the Borel group B.

Part (b) is new and crucial for controlling the multiplicities of cohomological components. The remaining statements have been included because Lemma 3.8.1 allows us to give a new, short, and unified proof of these results. In particular, we obtain a new proof of the necessity of the inequalities determining the Littlewood–Richardson cone. Namely, these inequalities are obtained by requiring that the weights in Theorem 3.9.1(a) (for all $w_1, \ldots, w_k, w$ satisfying the conditions of the theorem) belong to $\mathrm{span}_{\mathbb{Z}\geqslant 0}\,\Delta^{+}$. (The proof that these inequalities are sufficient requires a separate GIT argument.)

Relation with a construction of Kumar. Given a sequence $\underline{u}$ of simple reflections, Kumar [1988, §1.1] defined a variety $\widetilde{Z}_{\underline{u}}$ along with a map $\theta_{\underline{u}}$ from $\widetilde{Z}_{\underline{u}}$ to X^2. For any pair of words $\underline{\boldsymbol{v}} = (\underline{v}_1, \underline{v}_2)$ let $\underline{u} = \underline{v}_1^{-1}\underline{v}_2$ be the word obtained by reversing $\underline{v}_1$ and concatenating it onto the left of $\underline{v}_2$. By comparing the construction of $Y_{\underline{\boldsymbol{v}}}$ and $\widetilde{Z}_{\underline{u}}$ it is not hard to find an isomorphism $\widetilde{Z}_{\underline{u}} = Y_{\underline{\boldsymbol{v}}}$ over X^2 (i.e., such that $\theta_{\underline{u}} = f_{\underline{\boldsymbol{v}}}$ under the isomorphism). Therefore when $k = 2$ the varieties produced by our construction are the same as the ones constructed in [Kumar 1988, §1.1].

4. Proof of Theorem III

4.1. We will prove Theorem III in its symmetric form. After applying the symmetrization procedure from Section 2.7 (and replacing $k+1$ by k) we obtain:

Theorem 4.1.1 (symmetric form of Theorem III). *Let $w_1, \dots, w_k$ be elements of the Weyl group $\mathcal{W}$ such that $\sum_i \ell(w_i) = \mathrm{N}$, and let $\lambda_1, \dots, \lambda_k$ be weights such that $w_i \cdot \lambda_i$ are dominant weights for $i = 1, \dots, k$, and $\sum_{i=1}^{k} \lambda_i = -2\rho$.*

(a) *If $\bigcap_{i=1}^{k} [\Omega_{w_i}] = 1$ then the cup-product map*

$$\mathrm{H}^{\ell(w_1)}(\mathrm{X}, \mathrm{L}_{\lambda_1}) \otimes \cdots \otimes \mathrm{H}^{\ell(w_k)}(\mathrm{X}, \mathrm{L}_{\lambda_k}) \xrightarrow{\cup} \mathrm{H}^{\mathrm{N}}(\mathrm{X}, \mathrm{K}_{\mathrm{X}}) \tag{4.1.2}$$

is surjective.

(b) *If $\bigcap_{i=1}^{k} [\Omega_{w_i}] = 0$ then (4.1.2) is zero.*

The proof of Theorem 4.1.1 is given in Section 4.3. We will use the following common notation. For any sequence $\underline{\lambda} = (\lambda_1, \dots, \lambda_k)$ of weights let $\mathrm{L}_{\underline{\lambda}}$ be the line bundle

$$\mathrm{L}_{\underline{\lambda}} := \mathrm{L}_{\lambda_1} \boxtimes \cdots \boxtimes \mathrm{L}_{\lambda_k} = \mathrm{pr}_1^* \mathrm{L}_{\lambda_1} \otimes \cdots \otimes \mathrm{pr}_k^* \mathrm{L}_{\lambda_k}$$

on X^k, where $\mathrm{pr}_i : \mathrm{X}^k \to \mathrm{X}$ denotes projection onto the i-th factor.

4.2. *Inductive lemma.* Let $\underline{\lambda} = (\lambda_1, \dots, \lambda_k)$ be a sequence of weights and $\underline{\boldsymbol{v}} = (\underline{v}_1, \dots, \underline{v}_k)$ a sequence of words. Let $\underline{\boldsymbol{u}}$ be a sequence of words as in (3.1.1), i.e., $\underline{\boldsymbol{u}}$ is a sequence of words obtained by dropping a simple reflection from the right of a single member of $\underline{\boldsymbol{v}}$. The following lemma lets us propagate information about the pullback map

$$\mathrm{H}^{\mathrm{N}+\ell(\underline{\boldsymbol{v}})}(\mathrm{Y}_{\underline{\boldsymbol{v}}}, f_{\underline{\boldsymbol{v}}}^* \mathrm{L}_{\underline{\lambda}}) \xleftarrow{f_{\underline{\boldsymbol{v}}}^*} \mathrm{H}^{\mathrm{N}+\ell(\underline{\boldsymbol{v}})}(\mathrm{X}^k, \mathrm{L}_{\underline{\lambda}}) \tag{4.2.1}$$

on the top degree cohomology of $\mathrm{Y}_{\underline{\boldsymbol{v}}}$ to information about an analogous pullback map to the top degree cohomology of $\mathrm{Y}_{\underline{\boldsymbol{u}}}$. If $\underline{v}_j = s_{i_1} \cdots s_{i_m}$, so that we are dropping s_{i_m} from $\underline{v}_j$ to get $\underline{u}_j$, we denote by $\underline{\mu}$ the sequence

$$\underline{\mu} := (\lambda_1, \dots, \lambda_{j-1}, s_{i_m} \cdot \lambda_j, \lambda_{j+1}, \dots, \lambda_k).$$

Finally, we assume that the degree of $\mathrm{L}_{\underline{\lambda}}$ is negative on the fibers of the $\mathbb{P}^1$-fibration $\pi_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}} : \mathrm{Y}_{\underline{\boldsymbol{v}}} \to \mathrm{Y}_{\underline{\boldsymbol{u}}}$.

Lemma 4.2.2. *Under the conditions above, the pullback map*

$$\mathrm{H}^{\mathrm{N}+\ell(\underline{\boldsymbol{u}})}(\mathrm{Y}_{\underline{\boldsymbol{u}}}, f_{\underline{\boldsymbol{u}}}^*\mathrm{L}_{\underline{\mu}}) \xleftarrow{f_{\underline{\boldsymbol{u}}}^*} \mathrm{H}^{\mathrm{N}+\ell(\underline{\boldsymbol{u}})}(\mathrm{X}^k, \mathrm{L}_{\underline{\mu}})$$

is (a) *surjective*, (b) *zero*, *or* (c) *surjective on the space of* G*-invariants*, *if the pullback map* (4.2.1) *has the corresponding property* (a), (b), *or* (c).

Here "surjective on the space of G*-invariants" means* (*in the case of* $\mathrm{Y}_{\underline{\boldsymbol{v}}}$) *that*

$$\mathrm{H}^{\mathrm{N}+\ell(\underline{\boldsymbol{v}})}(\mathrm{Y}_{\underline{\boldsymbol{v}}}, f_{\underline{\boldsymbol{v}}}^*\mathrm{L}_{\underline{\lambda}})^{\mathrm{G}} \xleftarrow{f_{\underline{\boldsymbol{v}}}^*} \mathrm{H}^{\mathrm{N}+\ell(\underline{\boldsymbol{v}})}(\mathrm{X}^k, \mathrm{L}_{\underline{\lambda}})^{\mathrm{G}}$$

is surjective.

Proof. To reduce notation set

$$\mathrm{M}_{\underline{\boldsymbol{u}}} = \mathrm{X}^{j-1} \times \mathrm{M}_{i_m} \times \mathrm{X}^{k-j}$$

and let $\pi : \mathrm{X}^k \to \mathrm{M}_{\underline{\boldsymbol{u}}}$ be the map

$$\pi = (\mathrm{id}_{\mathrm{X}})^{j-1} \times \pi_{i_m} \times (\mathrm{id}_{\mathrm{X}})^{k-j}.$$

The fiber product diagram (3.1.2) relating $\mathrm{Y}_{\underline{\boldsymbol{v}}}$, $\mathrm{Y}_{\underline{\boldsymbol{u}}}$, X^k, and $\mathrm{M}_{\underline{\boldsymbol{u}}}$ is

$$\begin{tikzcd}
\mathrm{Y}_{\underline{\boldsymbol{v}}} \arrow[r, "f_{\underline{\boldsymbol{v}}}"] \arrow[d, "\pi_{\underline{\boldsymbol{v}}}"] \arrow[dr, phantom, "\square"] & \mathrm{X}^k \arrow[d, "\pi"] \\
\mathrm{Y}_{\underline{\boldsymbol{u}}} \arrow[r, "h"] \arrow[u, bend left, "\sigma_{\underline{\boldsymbol{v}}}"] & \mathrm{M}_{\underline{\boldsymbol{u}}}
\end{tikzcd} \tag{4.2.3}$$

where $h = \pi \circ f_{\underline{\boldsymbol{u}}}$ and where we use $\pi_{\underline{\boldsymbol{v}}}$ and $\sigma_{\underline{\boldsymbol{v}}}$ in place of $\pi_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}}$ and $\sigma_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}}$ to reduce notation.

Note that $\mathrm{L}_{\underline{\mu}}$ is the Demazure reflection of $\mathrm{L}_{\underline{\lambda}}$ with respect to π. By Section 2.8 this means that we have natural isomorphisms

$$\pi_{\underline{\boldsymbol{v}}*}(f_{\underline{\boldsymbol{v}}}^*\mathrm{L}_{\underline{\mu}}) \cong \mathrm{R}^1\pi_{\underline{\boldsymbol{v}}*}(f_{\underline{\boldsymbol{v}}}^*\mathrm{L}_{\underline{\lambda}}) \quad \text{and} \quad \pi_*\mathrm{L}_{\underline{\mu}} \cong \mathrm{R}^1\pi_*\mathrm{L}_{\underline{\lambda}} \tag{4.2.4}$$

valid on $\mathrm{Y}_{\underline{\boldsymbol{u}}}$ and $\mathrm{M}_{\underline{\boldsymbol{u}}}$ respectively. Diagram (4.2.3), the Leray spectral sequences for $\mathrm{L}_{\underline{\lambda}}$ and $\mathrm{L}_{\underline{\mu}}$ relative to π and $\pi_{\underline{\boldsymbol{v}}}$, and the isomorphisms (4.2.4) then give the commutative diagram of cohomology groups:

$$\begin{array}{ccc} \mathrm{H}^{\mathrm{N}+\ell(\underline{v})}(\mathrm{Y}_{\underline{v}}, f_{\underline{v}}^*\mathrm{L}_{\underline{\lambda}}) & \xleftarrow[\text{(4.2.1)}]{f_{\underline{v}}^*} & \mathrm{H}^{\mathrm{N}+\ell(\underline{v})}(\mathrm{X}^k, \mathrm{L}_{\underline{\lambda}}) \\ \wr\Big\| \text{Leray} & & \wr\Big\| \text{Leray} \\ \mathrm{H}^{\mathrm{N}+\ell(\underline{v})-1}(\mathrm{Y}_{\underline{u}}, \mathrm{R}^1\pi_{\underline{v}*} f_{\underline{v}}^*\mathrm{L}_{\underline{\lambda}}) & \xleftarrow{h^*} & \mathrm{H}^{\mathrm{N}+\ell(\underline{v})-1}(\mathrm{M}_{\underline{u}}, \mathrm{R}^1\pi_*\mathrm{L}_{\underline{\lambda}}) \\ \wr\Big\| \text{(4.2.4)} & & \wr\Big\| \text{(4.2.4)} \\ \mathrm{H}^{\mathrm{N}+\ell(\underline{v})-1}(\mathrm{Y}_{\underline{u}}, \pi_{\underline{v}*} f_{\underline{v}}^*\mathrm{L}_{\underline{\mu}}) & \xleftarrow{h^*} & \mathrm{H}^{\mathrm{N}+\ell(\underline{v})-1}(\mathrm{M}_{\underline{u}}, \pi_*\mathrm{L}_{\underline{\mu}}) \\ \wr\Big\| \text{Leray} & & \wr\Big\| \text{Leray} \\ \mathrm{H}^{\mathrm{N}+\ell(\underline{v})-1}(\mathrm{Y}_{\underline{v}}, f_{\underline{v}}^*\mathrm{L}_{\underline{\mu}}) & \xleftarrow{f_{\underline{v}}^*} & \mathrm{H}^{\mathrm{N}+\ell(\underline{v})-1}(\mathrm{X}^k, \mathrm{L}_{\underline{\mu}}) \end{array} \tag{4.2.5}$$

We conclude that the bottom pullback map

$$\mathrm{H}^{\mathrm{N}+\ell(\underline{v})-1}(\mathrm{Y}_{\underline{v}}, f_{\underline{v}}^*\mathrm{L}_{\underline{\mu}}) \xleftarrow{f_{\underline{v}}^*} \mathrm{H}^{\mathrm{N}+\ell(\underline{v})-1}(\mathrm{X}^k, f_{\underline{v}}^*\mathrm{L}_{\underline{\mu}})$$

is surjective, zero, or surjective on the space of G-invariants if (4.2.1) is.

On $\mathrm{Y}_{\underline{v}}$ we have the exact sequence of bundles

$$0 \to f_{\underline{v}}^*\mathrm{L}_{\underline{\mu}}(-\mathrm{Y}_{\underline{u}}) \to f_{\underline{v}}^*\mathrm{L}_{\underline{\mu}} \to f_{\underline{v}}^*\mathrm{L}_{\underline{\mu}}|_{\mathrm{Y}_{\underline{u}}} \to 0, \tag{4.2.6}$$

where we consider $\mathrm{Y}_{\underline{u}}$ to be a divisor in $\mathrm{Y}_{\underline{v}}$ via the section $\sigma_{\underline{v}}$. The degree of $f_{\underline{v}}^*\mathrm{L}_{\underline{\mu}}(-\mathrm{Y}_{\underline{u}})$ is at least -1 on the fibers of $\pi_{\underline{v}}$ so the corresponding Leray spectral sequence gives

$$\mathrm{H}^{\mathrm{N}+\ell(\underline{v})}\big(\mathrm{Y}_{\underline{v}}, f_{\underline{v}}^*\mathrm{L}_{\underline{\mu}}(-\mathrm{Y}_{\underline{u}})\big) = \mathrm{H}^{\mathrm{N}+\ell(\underline{v})}\big(\mathrm{Y}_{\underline{u}}, \pi_{\underline{v}*}(f_{\underline{v}}^*\mathrm{L}_{\underline{\mu}}(-\mathrm{Y}_{\underline{u}}))\big) = 0,$$

where the second cohomology group above equals zero by reason of dimension:

$$\mathrm{N} + \ell(\underline{v}) = \mathrm{N} + \ell(\underline{u}) + 1 = \dim(\mathrm{Y}_{\underline{u}}) + 1.$$

The end of the long exact cohomology sequence associated to (4.2.6) is therefore

$$\mathrm{H}^{\mathrm{N}+\ell(\underline{v})-1}(\mathrm{Y}_{\underline{v}}, f_{\underline{v}}^*\mathrm{L}_{\underline{\mu}}) \xrightarrow{\sigma_{\underline{v}}^*} \mathrm{H}^{\mathrm{N}+\ell(\underline{v})-1}(\mathrm{Y}_{\underline{u}}, f_{\underline{v}}^*\mathrm{L}_{\underline{\mu}}|_{\mathrm{Y}_{\underline{u}}}) \to 0. \tag{4.2.7}$$

Since $\ell(\underline{u}) = \ell(\underline{v}) - 1$, $f_{\underline{u}} = f_{\underline{v}} \circ \sigma_{\underline{v}}$, and all maps are G-equivariant, we conclude that the pullback map $f_{\underline{u}}^*$, being the composite map

$$\begin{aligned} \mathrm{H}^{\mathrm{N}+\ell(\underline{u})}(\mathrm{X}^k, \mathrm{L}_{\underline{\mu}}) \xrightarrow{f_{\underline{v}}^*} \mathrm{H}^{\mathrm{N}+\ell(\underline{u})}(\mathrm{Y}_{\underline{v}}, f_{\underline{v}}^*\mathrm{L}_{\underline{\mu}}) \xrightarrow{\sigma_{\underline{v}}^*} \mathrm{H}^{\mathrm{N}+\ell(\underline{u})}&(\mathrm{Y}_{\underline{u}}, f_{\underline{v}}^*\mathrm{L}_{\underline{\mu}}|_{\mathrm{Y}_{\underline{u}}}) \\ &= \mathrm{H}^{\mathrm{N}+\ell(\underline{u})}(\mathrm{Y}_{\underline{u}}, f_{\underline{u}}^*\mathrm{L}_{\underline{\mu}}), \end{aligned}$$

is (a) surjective, (b) zero, or (c) surjective on the space of G-invariants, if the pullback map $f_{\underline{v}}^*$ in (4.2.1) has the corresponding property (a), (b), or (c). □

Remark. In part (c) of Lemma 4.2.2 we can replace the statement about G-invariants with a statement about any isotypic component; the proof above goes through without change. We will only need the case of G-invariants as part of the proof of Theorem I in Section 5 below.

4.3. ***Proof of Theorem 4.1.1 and variation.*** For the rest of this section, we fix the following notation. Let $w_1, \ldots, w_k$ and $\lambda_1, \ldots, \lambda_k$ be as in Theorem 4.1.1. For each $i = 1, \ldots, k$ set $v_i := w_i^{-1} w_0$ and $\lambda'_i := v_i^{-1} \cdot \lambda_i$. Let $\underline{v}_i$ be a reduced factorization of v_i and let $\underline{\boldsymbol{v}} = (\underline{v}_1, \ldots, \underline{v}_k)$. Finally, set $\underline{\lambda} = (\lambda_1, \ldots, \lambda_k)$ and $\underline{\lambda}' = (\lambda'_1, \ldots, \lambda'_k)$.

Proof of Theorem 4.1.1. Since

$$\dim(\mathrm{Y}_{\underline{\boldsymbol{v}}}) = \mathrm{N} + \sum_{i=1}^{k} \ell(v_i) = \mathrm{N} + \sum_{i=1}^{k} (\mathrm{N} - \ell(w_i)) = k\mathrm{N} = \dim(\mathrm{X}^k),$$

Corollary 3.7.5 implies that the degree of $f_{\underline{\boldsymbol{v}}} : \mathrm{Y}_{\underline{\boldsymbol{v}}} \to \mathrm{X}^k$ is given by the intersection number $\bigcap_{i=1}^{k} [\Omega_{w_i}]$. Therefore the pullback map

$$\mathrm{H}^{k\mathrm{N}}(\mathrm{Y}_{\underline{\boldsymbol{v}}}, f_{\underline{\boldsymbol{v}}}^{*}\mathrm{L}_{\underline{\lambda}'}) \xleftarrow{f_{\underline{\boldsymbol{v}}}^{*}} \mathrm{H}^{k\mathrm{N}}(\mathrm{X}^k, \mathrm{L}_{\underline{\lambda}'})$$

is a surjection if $\bigcap_{i=1}^{k} [\Omega_{w_i}] = 1$ and is zero if $\bigcap_{i=1}^{k} [\Omega_{w_i}] = 0$: If $\bigcap_{i=1}^{k} [\Omega_{w_i}] = 1$ then $f_{\underline{\boldsymbol{v}}}$ is a birational map between the smooth varieties $\mathrm{Y}_{\underline{\boldsymbol{v}}}$ and X^k in characteristic zero, and so the pullback map

$$\mathrm{H}^{j}(\mathrm{Y}_{\underline{\boldsymbol{v}}}, f_{\underline{\boldsymbol{v}}}^{*}\mathrm{L}_{\underline{\lambda}'}) \xleftarrow{f_{\underline{\boldsymbol{v}}}^{*}} \mathrm{H}^{j}(\mathrm{X}^k, \mathrm{L}_{\underline{\lambda}'})$$

is an isomorphism in all degrees, and in particular is a surjection in degree $j = k\mathrm{N}$. On the other hand, if $\bigcap_{i=1}^{k} [\Omega_{w_i}] = 0$ then the image $\mathrm{X}_{\underline{\boldsymbol{v}}}$ of $f_{\underline{\boldsymbol{v}}}$ is subvariety of X^k of dimension strictly less than $k\mathrm{N}$ and therefore the pullback map $f_{\underline{\boldsymbol{v}}}^{*}$ in top cohomology, which factors through $\mathrm{H}^{k\mathrm{N}}(\mathrm{X}_{\underline{\boldsymbol{v}}}, \mathrm{L}_{\underline{\lambda}}|_{\mathrm{X}_{\underline{\boldsymbol{v}}}}) = 0$, is the zero map.

Consider a sequence

$$\underline{\boldsymbol{v}} =: \underline{\boldsymbol{v}}^0, \underline{\boldsymbol{v}}^1, \ldots, \underline{\boldsymbol{v}}^{(k-1)\mathrm{N}} := \underline{\varnothing} = (\varnothing, \ldots, \varnothing)$$

of sequences of words which reduces $\underline{\boldsymbol{v}}$ to the empty sequence, and where at each step $\underline{\boldsymbol{v}}^{j+1}$ is obtained by dropping a simple reflection from the right of a single member of $\underline{\boldsymbol{v}}^{j}$. Set $\underline{\lambda}^j = (\underline{\boldsymbol{v}}^j)^{-1} \cdot \underline{\lambda}$ where (by slight abuse of notation) $\underline{\boldsymbol{v}}^j$ is considered as an element of $\mathcal{W}^k$ and the action is componentwise. Note that $\underline{\lambda}^0 = \underline{\lambda}'$ and $\underline{\lambda}^{(k-1)\mathrm{N}} = \underline{\lambda}$. The construction of $\underline{\boldsymbol{v}}^j$ and $\underline{\lambda}^j$ implies that the degree of $\mathrm{L}_{\underline{\lambda}^j}$ is negative on the fibers of the $\mathbb{P}^1$-fibration $\pi_{\underline{\boldsymbol{v}}^j, \underline{\boldsymbol{v}}^{j+1}} : \mathrm{Y}_{\underline{\boldsymbol{v}}^j} \to \mathrm{Y}_{\underline{\boldsymbol{v}}^{j+1}}$.

Applying Lemma 4.2.2 to the pairs $(\underline{\boldsymbol{v}}^j, \underline{\boldsymbol{v}}^{j+1})$ for $j = 0, \ldots, (k-1)\mathrm{N} - 1$ we conclude that

$$\mathrm{H}^{\mathrm{N}}(\mathrm{Y}_{\underline{\varnothing}}, f_{\underline{\varnothing}}^{*}\mathrm{L}_{\underline{\lambda}}) \xleftarrow{f_{\underline{\varnothing}}^{*}} \mathrm{H}^{\mathrm{N}}(\mathrm{X}^k, \mathrm{L}_{\underline{\lambda}}) \tag{4.3.1}$$

is surjective if $\bigcap_{i=1}^{k}[\Omega_{w_i}] = 1$ and zero if $\bigcap_{i=1}^{k}[\Omega_{w_i}] = 0$. By construction $f_{\underline{\varnothing}} : \mathrm{Y}_{\underline{\varnothing}} = \mathrm{X} \to \mathrm{X}^k$ is the diagonal embedding of X into X^k and the pullback map (4.3.1) is the cup-product map. This proves Theorem 4.1.1 and completes the proof of Theorem III. □

We record a statement that will be used in the proof of Theorem I below.

Proposition 4.3.2. *If the pullback map*

$$\mathrm{H}^{k\mathrm{N}}(\mathrm{Y}_{\underline{v}}, f_{\underline{v}}^{*}\mathrm{L}_{\underline{\lambda}'}) \xleftarrow{f_{\underline{v}}^{*}} \mathrm{H}^{k\mathrm{N}}(\mathrm{X}^k, \mathrm{L}_{\underline{\lambda}'})$$

is surjective on the space of G*-invariants then the cup-product map* (4.1.2) *is surjective.*

Proof. We repeat the inductive reduction in the proof of Theorem 4.1.1 above with part (c) of Lemma 4.2.2 in place of parts (a) and (b). As a result we conclude that the cup-product map (4.1.2) is surjective on the space of G-invariants. Since $\mathrm{H}^{\mathrm{N}}(\mathrm{X}, \mathrm{K}_{\mathrm{X}})$ is the trivial G-module we conclude that (4.1.2) is surjective. □

5. Proof of Theorem I and corollaries

In this section we use Theorem III and Proposition 4.3.2 to prove Theorem I. The proof that (1.2.1) is necessary for the surjectivity of the cup-product map appears in Section 5.1 and the proof that (1.2.1) is sufficient appears in Section 5.3.

5.1. ***Proof that $\Phi_w = \bigsqcup_{i=1}^{k} \Phi_{w_i}$ is a necessary condition for surjectivity.*** We assume the notation of Section 1.2, and set $\mu_i = w_i \cdot \lambda_i$ for $i = 1, \ldots, k$, and $\mu = w \cdot \lambda$. By assumption the weights $\mu_1, \ldots, \mu_k$, and μ are dominant. By the Borel–Weil–Bott theorem each $\mathrm{H}^{\ell(w_i)}(\mathrm{X}, \mathrm{L}_{\lambda_i}) = \mathrm{V}_{\mu_i}^{*}$ and $\mathrm{H}^{d}(\mathrm{X}, \mathrm{L}_{\lambda}) = \mathrm{V}_{\mu}^{*}$.

Since $w_i^{-1}\mu_i = w_i^{-1} \cdot \mu_i - w_i^{-1} \cdot 0$ and $w^{-1}\mu_i = w^{-1} \cdot \mu_i - w^{-1} \cdot 0$, we have

$$\sum_{i=1}^{k} w_i^{-1}\mu_i - w^{-1}\mu = \left(\sum_{i=1}^{k} w_i^{-1} \cdot \mu_i - w^{-1} \cdot \mu\right) - \left(\sum_{i=1}^{k} w_i^{-1} \cdot 0 - w^{-1} \cdot 0\right).$$

Furthermore $\sum_{i=1}^{k} w_i^{-1} \cdot \mu_i - w^{-1} \cdot \mu = \sum \lambda_i - \lambda = 0$ and so the equation above becomes

$$\sum_{i=1}^{k} w_i^{-1}\mu_i - w^{-1}\mu = -\left(\sum_{i=1}^{k} w_i^{-1} \cdot 0 - w^{-1} \cdot 0\right). \tag{5.1.1}$$

If the cup-product map $\mathrm{H}^{\ell(w_1)}(\mathrm{X}, \mathrm{L}_{\lambda_1}) \otimes \cdots \otimes \mathrm{H}^{\ell(w_k)}(\mathrm{X}, \mathrm{L}_{\lambda_k}) \xrightarrow{\cup} \mathrm{H}^{d}(\mathrm{X}, \mathrm{L}_{\lambda})$ is surjective, then (after dualizing) V_{μ} must be a component of the tensor product $\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}$ and by Theorem III(b), the intersection $\bigcap_{i=1}^{k}[\Omega_{w_i}] \cdot [\mathrm{X}_w] \neq 0$ in $\mathrm{H}^{*}(\mathrm{X}, \mathbb{Z})$; we may therefore apply Theorem 3.9.1.

By Theorem 3.9.1(a) the left hand side of (5.1.1) belongs to $\mathrm{span}_{\mathbb{Z}_{\geqslant 0}} \Delta^+$ and by part (c) of the same theorem the right hand side belongs to $\mathrm{span}_{\mathbb{Z}_{\leqslant 0}} \Delta^+$. We conclude that both sides are zero and so, by Theorem 3.9.1(d), $\Phi_w = \bigsqcup_{i=1}^k \Phi_{w_i}$. □

Remark. In the first half of the argument above, the hypothesis that the cup-product map is surjective was used, along with Theorem III, to conclude that V_μ is a component of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ and $\bigcap_{i=1}^k [\Omega_{w_i}] \cdot [X_w] \neq 0$. If, on the other hand, we assume the latter two conditions then the second half of the argument still applies to give $\Phi_w = \bigsqcup_{i=1}^k \Phi_{w_i}$. We will use this observation in Corollary 5.4.7 below.

5.2. ***Setup for the proof of sufficiency.*** For convenience, we collect some of the consequences of condition (1.2.1) in its symmetric form which have effectively appeared in previous arguments, and which we will use in the proof of sufficiency.

Proposition 5.2.1. *Suppose that $w_1, \ldots, w_k$ are elements of the Weyl group such that $\Delta^+ = \bigsqcup_{i=1}^k \Phi_{w_i}$.*

Combinatorial Consequences:

(a) $\sum_{i=1}^k w_i^{-1} \cdot 0 = -2\rho$.

(b) *Suppose that $\lambda_1, \ldots, \lambda_k$ are weights such that $\sum \lambda_i = -2\rho$, and set $\mu_i = w_i \cdot \lambda_i$ for $i = 1, \ldots, k$. Then $\sum_{i=1}^k w_i^{-1} \mu_i = 0$.*

Geometric Consequences: *For each $i = 1, \ldots, k$, let $v_i = w_i^{-1} w_0$ and let $\underline{v}_i$ be a word which is a reduced factorization of v_i. We set $\underline{\boldsymbol{v}} = (\underline{v}_1, \ldots, \underline{v}_k)$ and construct as usual the variety $Y_{\underline{\boldsymbol{v}}}$ and the map $f_{\underline{\boldsymbol{v}}} : Y_{\underline{\boldsymbol{v}}} \to X^k$.*

Then

(c) $\deg(f_{\underline{\boldsymbol{v}}}) \neq 0$.

(d) *The weight of the relative canonical bundle $K_{Y_{\underline{\boldsymbol{v}}}/K_{X^k}}$ at $p_{\underline{\boldsymbol{v}}}$ is zero.*

Proof. Part (a) is immediate from the condition $\Delta^+ = \bigsqcup_{i=1}^k \Phi_{w_i}$ and formula (2.2.4). Part (b) reverses the argument used to arrive at (5.1.1) in Section 5.1:

$$\sum_{i=1}^k \mu^{-1} \mu_i = \left(\sum_{i=1}^k w_i^{-1} \cdot \mu_i \right) - \left(\sum_{i=1}^k w_i^{-1} \cdot 0 \right) = \sum_{i=1}^k \lambda_i - (-2\rho) = 0.$$

Part (c) is Corollary 3.7.5 combined with Lemma 2.6.1. Part (d) is the symmetric version of the computation in the proof of Theorem 3.9.1(d): the weight of the relative canonical bundle $K_{Y_{\underline{\boldsymbol{v}}}/X^k}$ at $p_{\underline{\boldsymbol{v}}}$ is $\sum_{i=1}^k w_i^{-1} \cdot 0 + 2\rho$, which is zero by part (a). □

5.3. ***Proof that $\Phi_w = \bigsqcup_{i=1}^k \Phi_{w_i}$ is a sufficient condition for surjectivity.*** Consider the symmetric version of the problem as in Section 2.7. It suffices to show the surjectivity of a cup-product map

$$\mathrm{H}^{\ell(w_1)}(\mathrm{X}, \mathrm{L}_{\lambda_1}) \otimes \cdots \otimes \mathrm{H}^{\ell(w_k)}(\mathrm{X}, \mathrm{L}_{\lambda_k}) \xrightarrow{\cup} \mathrm{H}^{\mathrm{N}}(\mathrm{X}, \mathrm{K}_{\mathrm{X}}), \tag{5.3.1}$$

where $w_1, \ldots, w_k$ are elements of the Weyl group such that $\sum \ell(w_i) = \mathrm{N}$, $\lambda_1, \ldots, \lambda_k$ are weights such that $w_i \cdot \lambda_i \in \Lambda^+$ for $i = 1, \ldots, k$ and $\sum \lambda_i = -2\rho$. After this reduction condition (1.2.1) becomes $\Delta^+ = \bigsqcup_{i=1}^k \Phi_{w_i}$. We recall the notation from Section 4.3: $v_i := w_i^{-1} w_0$, $\lambda_i' := v_i^{-1} \cdot \lambda_i$, $\underline{v}_i$ is a reduced factorization of v_i, $\underline{\boldsymbol{v}} = (\underline{v}_1, \ldots, \underline{v}_k)$, and $\underline{\lambda}' = (\lambda_1', \ldots, \lambda_k')$.

By Proposition 4.3.2, to show the surjectivity of (5.3.1) it is enough to show that the pullback map

$$\mathrm{H}^{k\mathrm{N}}(\mathrm{Y}_{\underline{\boldsymbol{v}}}, f_{\underline{\boldsymbol{v}}}^* \mathrm{L}_{\underline{\lambda}'})^{\mathrm{G}} \xleftarrow{f_{\underline{\boldsymbol{v}}}^*} \mathrm{H}^{k\mathrm{N}}(\mathrm{X}^k, \mathrm{L}_{\underline{\lambda}'})^{\mathrm{G}} \tag{5.3.2}$$

on the space of G-invariants is surjective. We will show that both spaces of G-invariants are one-dimensional, and that the induced map is an isomorphism. Note that by Proposition 5.2.1(c) $\deg(f_{\underline{\boldsymbol{v}}}) \neq 0$ and so $f_{\underline{\boldsymbol{v}}}$ is surjective.

The pullback map on top cohomology is Serre dual to the trace map:

$$\mathrm{H}^0(\mathrm{Y}_{\underline{\boldsymbol{v}}}, (f_{\underline{\boldsymbol{v}}}^* \mathrm{L}_{\underline{\lambda}'})^* \otimes \mathrm{K}_{\mathrm{Y}_{\underline{\boldsymbol{v}}}}) = \mathrm{H}^0\big(\mathrm{Y}_{\underline{\boldsymbol{v}}}, f_{\underline{\boldsymbol{v}}}^*(\mathrm{L}_{\underline{\lambda}'}^* \otimes \mathrm{K}_{\mathrm{X}^k}) \otimes \mathrm{K}_{\mathrm{Y}_{\underline{\boldsymbol{v}}}/\mathrm{X}^k}\big) \xrightarrow{\mathrm{Tr}_{f_{\underline{\boldsymbol{v}}}}} \mathrm{H}^0(\mathrm{X}^k, \mathrm{L}_{\underline{\lambda}'}^* \otimes \mathrm{K}_{\mathrm{X}^k}).$$

Let $s \in \mathrm{H}^0(\mathrm{Y}_{\underline{\boldsymbol{v}}}, \mathrm{K}_{\mathrm{Y}_{\underline{\boldsymbol{v}}}/\mathrm{X}^k})^{\mathrm{G}}$ be the nonzero G-invariant section giving the map $f_{\underline{\boldsymbol{v}}}^* \mathrm{K}_{\mathrm{X}^k} \to \mathrm{K}_{\mathrm{Y}_{\underline{\boldsymbol{v}}}}$ induced by $f_{\underline{\boldsymbol{v}}}$. The composition

$$\begin{aligned}\mathrm{H}^0(\mathrm{X}^k, \mathrm{L}_{\underline{\lambda}'}^* \otimes \mathrm{K}_{\mathrm{X}^k}) \xrightarrow{f_{\underline{\boldsymbol{v}}}^*} \mathrm{H}^0\big(\mathrm{Y}_{\underline{\boldsymbol{v}}}, f_{\underline{\boldsymbol{v}}}^*(\mathrm{L}_{\underline{\lambda}'}^* \otimes \mathrm{K}_{\mathrm{X}^k})\big) \xrightarrow{\cdot s} \mathrm{H}^0\big(\mathrm{Y}_{\underline{\boldsymbol{v}}}, f_{\underline{\boldsymbol{v}}}^*(\mathrm{L}_{\underline{\lambda}'}^* \otimes \mathrm{K}_{\mathrm{X}^k}) \otimes \mathrm{K}_{\mathrm{Y}_{\underline{\boldsymbol{v}}}/\mathrm{X}^k}\big) \\ \xrightarrow{\mathrm{Tr}_{f_{\underline{\boldsymbol{v}}}}} \mathrm{H}^0(\mathrm{X}^k, \mathrm{L}_{\underline{\lambda}'}^* \otimes \mathrm{K}_{\mathrm{X}^k})\end{aligned}$$

of pullback, multiplication by s, and the trace map is multiplication by $\deg(f_{\underline{\boldsymbol{v}}})$, which is nonzero. This gives us the inequality

$$\dim \mathrm{H}^0(\mathrm{X}^k, \mathrm{L}_{\underline{\lambda}'}^* \otimes \mathrm{K}_{\mathrm{X}^k})^{\mathrm{G}} \leqslant \dim \mathrm{H}^0\big(\mathrm{Y}_{\underline{\boldsymbol{v}}}, f_{\underline{\boldsymbol{v}}}^*(\mathrm{L}_{\underline{\lambda}'}^* \otimes \mathrm{K}_{\mathrm{X}^k}) \otimes \mathrm{K}_{\mathrm{Y}_{\underline{\boldsymbol{v}}}/\mathrm{X}^k}\big)^{\mathrm{G}} \tag{5.3.3}$$

and shows that in order to prove that the trace map induces an isomorphism on G-invariants it is sufficient to prove that we have equality of dimensions in (5.3.3).

Set $\mu_i = w_i \cdot \lambda_i = w_0 \cdot \lambda_i'$ for $i = 1, \ldots, k$. By the Borel–Weil–Bott Theorem we have $\mathrm{H}^{k\mathrm{N}}(\mathrm{X}^k, \mathrm{L}_{\underline{\lambda}'}) = \mathrm{V}_{\mu_1}^* \otimes \cdots \otimes \mathrm{V}_{\mu_k}^*$ and so (by Serre duality) $\mathrm{H}^0(\mathrm{X}^k, \mathrm{L}_{\underline{\lambda}'}^* \otimes \mathrm{K}_{\mathrm{X}^k}) = \mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}$. Now set $\nu_i = -w_0 \mu_i$ for $i = 1, \ldots, k$ so that $\mathrm{V}_{\nu_i} = \mathrm{V}_{\mu_i}^*$ and let $\underline{\nu} = (\nu_1, \ldots, \nu_k)$. By the calculation

$$\mathrm{S}(\lambda_i') = -\lambda_i' - 2\rho = -w_0 \cdot \mu_i - 2\rho = -(w_0 \mu_i - 2\rho) - 2\rho = -w_0 \mu_i$$

in each coordinate factor (as in Section 2.5), we conclude that $L^*_{\underline{\lambda}'} \otimes K_{X^k} = L_{\underline{\nu}}$.

The weight of $L_{\underline{\nu}}$ at $q := f_{\underline{v}}(p_{\underline{v}}) = (v_1, \ldots, v_k)$ is

$$-\sum_{i=1}^{k} v_i \nu_i = \sum_{i=1}^{k} (w_i^{-1} w_0)(w_0 \mu_i) = \sum_{i=1}^{k} w_i^{-1} \mu_i = 0,$$

where the last equality is due to Proposition 5.2.1(b). Since by Proposition 5.2.1(d) the weight of $K_{Y_{\underline{v}}/X^k}$ at $p_{\underline{v}}$ is zero, the weight of $f_{\underline{v}}^* L_{\underline{\nu}} \otimes K_{Y_{\underline{v}}/X^k}$ at $p_{\underline{v}}$ in $Y_{\underline{v}}$ is also zero and hence

$$\dim H^0(Y_{\underline{v}}, f_{\underline{v}}^* L_{\underline{\nu}} \otimes K_{Y_{\underline{v}}/X^k})^G \leqslant 1$$

by Lemma 3.8.1(c). On the other hand, Lemma 2.3.1 implies that

$$(V_{\mu_1} \otimes \cdots \otimes V_{\mu_k})^G \neq 0$$

so we conclude that $\dim H^0(X^k, L_{\underline{\nu}})^G \geqslant 1$. This gives us

$$1 \leqslant \dim H^0(X^k, L_{\underline{\nu}})^G \leqslant \dim H^0(Y_{\underline{v}}, f_{\underline{v}}^* L_{\underline{\nu}} \otimes K_{Y_{\underline{v}}/X^k})^G \leqslant 1.$$

Therefore the inequality in (5.3.3) is an equality, and the cup-product map in (5.3.1) is surjective. □

5.4. *Corollaries of Theorem I and its proof.*

Corollary 5.4.1. *The cup-product map*

$$H^0(X, L_{\lambda_1}) \otimes H^d(X, L_{\lambda_2}) \to H^d(X, L_{\lambda_1} \otimes L_{\lambda_2})$$

is surjective whenever both sides are nonzero.

Proof. If w_2 is the element of the Weyl group so that $w_2 \cdot \lambda_2 \geqslant 0$ then the conditions that λ_1 is dominant and that $L_{\lambda_1+\lambda_2}$ has cohomology in the same degree d as L_{λ_2} imply that $w_2 \cdot (\lambda_1 + \lambda_2) \geqslant 0$, and so the corollary follows from Theorem I and the obvious statement that $\Phi_{w_2} = \Phi_{w_2} \sqcup \Phi_e$. □

Corollary 5.4.2 (compatibility with Leray spectral sequence). *Suppose that* λ_1, λ_2, *and* $\lambda = \lambda_1 + \lambda_2$ *are regular weights and that the cup-product map*

$$H^{d_1}(X, L_{\lambda_1}) \otimes H^{d_2}(X, L_{\lambda_2}) \xrightarrow{\cup} H^d(X, L_\lambda)$$

is nonzero. Let P *be any parabolic subgroup of* G *containing* B, *and*

$$\pi : X \to M := G/P$$

be the corresponding projection. Then the cup-product map on X *factors as a composition*

$$\begin{tikzcd}[column sep=small]
\mathrm{H}^{d_1}(\mathrm{X},\mathrm{L}_{\lambda_1})\otimes \mathrm{H}^{d_2}(\mathrm{X},\mathrm{L}_{\lambda_2}) \arrow[rr, "\cup_{\mathrm{X}}"] \arrow[d, equal] & & \mathrm{H}^{d}(\mathrm{X},\mathrm{L}_{\lambda}) \arrow[d, equal] \\
\mathrm{H}^{d_1-i}(\mathrm{M},\mathrm{R}^{i}_{\pi*}\mathrm{L}_{\lambda_1})\otimes \mathrm{H}^{d_2-j}(\mathrm{M},\mathrm{R}^{j}_{\pi*}\mathrm{L}_{\lambda_2}) \arrow[dr, "\cup_{\mathrm{M}}"] & & \mathrm{H}^{d-i-j}(\mathrm{M},\mathrm{R}^{i+j}_{\pi*}\mathrm{L}_{\lambda}) \\
& \mathrm{H}^{d-i-j}(\mathrm{M},\mathrm{R}^{i}_{\pi*}\mathrm{L}_{\lambda_1}\otimes \mathrm{R}^{j}_{\pi*}\mathrm{L}_{\lambda_2}) \arrow[ur, "\cup_{\pi}"] &
\end{tikzcd}$$

of the cup product on M *followed by the map induced on cohomology by the relative cup-product map* $\mathrm{R}^{i}_{\pi*}\mathrm{L}_{\lambda_1}\otimes\mathrm{R}^{j}_{\pi*}\mathrm{L}_{\lambda_2}\to\mathrm{R}^{i+j}_{\pi*}\mathrm{L}_{\lambda}$ *on the fibers of* π. *A similar statement holds for the cup product of an arbitrary number of factors.*

Proof. The factorization statement amounts to a numerical condition on the cohomology degrees of the line bundles on the fibers of π ensuring that the cup-product map is computed by the map on E_2-terms of the Leray spectral sequence. This numerical condition is immediately implied by (1.2.1). We explain this in more detail below.

Set $\mathrm{L}=\mathrm{L}_{\lambda_1}\boxtimes\mathrm{L}_{\lambda_2}$ on $\mathrm{X}\times\mathrm{X}$ and consider the following factorization of the diagonal map $\delta_{\mathrm{X}}:\mathrm{X}\hookrightarrow\mathrm{X}\times\mathrm{X}$:

$$\begin{tikzcd}
\mathrm{X} \arrow[r, hook, "s"] \arrow[d, "\pi"] & \mathrm{X}\times_{\mathrm{M}}\mathrm{X} \arrow[r, hook, "t"] \arrow[d, "\psi"] \arrow[dr, phantom, "\square"] & \mathrm{X}\times\mathrm{X} \arrow[d, "\pi\times\pi"] \\
\mathrm{M} \arrow[r, equal] & \mathrm{M} \arrow[r, hook, "\delta_{\mathrm{M}}"] & \mathrm{M}\times\mathrm{M}
\end{tikzcd} \tag{5.4.3}$$

The cup-product map then factors as

$$\mathrm{H}^{d}(\mathrm{X},\mathrm{L}_{\lambda})\xleftarrow{s^*}\mathrm{H}^{d}(\mathrm{X}\times_{\mathrm{M}}\mathrm{X},t^*\mathrm{L})\xleftarrow{t^*}\mathrm{H}^{d_1}(\mathrm{X},\mathrm{L}_{\lambda_1})\otimes\mathrm{H}^{d_2}(\mathrm{X},\mathrm{L}_{\lambda_2})=\mathrm{H}^{d}(\mathrm{X}\times\mathrm{X},\mathrm{L}) \tag{5.4.4}$$

and we claim that (5.4.4) induces the factorization claimed above.

By the Borel–Weil–Bott theorem applied to the fibers of π, for each of the line bundles L_{λ_1}, L_{λ_2}, and L_{λ} there is precisely one degree for which the higher direct image sheaf is nonzero. Suppose i is the degree such that $\mathrm{R}^{i}_{\pi*}\mathrm{L}_{\lambda_1}\neq 0$, j is the degree such that $\mathrm{R}^{j}_{\pi*}\mathrm{L}_{\lambda_2}\neq 0$, and k is the degree such that $\mathrm{R}^{k}_{\pi*}\mathrm{L}_{\lambda}\neq 0$. The Leray spectral sequence for the cohomology of these bundles degenerates at the E_2 term and we have the isomorphisms $\mathrm{H}^{d_1}(\mathrm{X},\mathrm{L}_{\lambda_1})=\mathrm{H}^{d_1-i}(\mathrm{M},\mathrm{R}^{i}_{\pi*}\mathrm{L}_{\lambda_1})$, $\mathrm{H}^{d_2}(\mathrm{X},\mathrm{L}_{\lambda_2})=\mathrm{H}^{d_2-j}(\mathrm{M},\mathrm{R}^{j}_{\pi*}\mathrm{L}_{\lambda_2})$, and $\mathrm{H}^{d}(\mathrm{X},\mathrm{L}_{\lambda_1})=\mathrm{H}^{d-k}(\mathrm{M},\mathrm{R}^{k}_{\pi*}\mathrm{L}_{\lambda})$.

Since $R^{i+j}_{\pi\times\pi *}L = R^i_{\pi *}L_{\lambda_1} \boxtimes R^j_{\pi *}L_{\lambda_2}$ is a vector bundle on $M \times M$, the theorem on cohomology and base change gives us

$$R^{i+j}_{\psi *} t^* L = \delta^*_M (R^i_{\pi *}L_{\lambda_1} \boxtimes R^j_{\pi *}L_{\lambda_2}) = R^i_{\pi *}L_{\lambda_1} \otimes R^j_{\pi *}L_{\lambda_2} \quad \text{on } M,$$

and therefore we have $H^d(X \times_M X, t^*L) = H^{d-i-j}(M, R^i_{\pi *}L_{\lambda_1} \otimes R^j_{\pi *}L_{\lambda_2})$. The Leray spectral sequences for L and t^*L with respect to ψ and $\pi \times \pi$ also degenerate at the E_2-terms and have nonzero terms in the same degree. The discussion in Section 2.9 implies that the map on E_2-terms computes the pullback map t^*. Therefore t^* in (5.4.4) is equal to the map

$$H^{d-i-j}(M, R^i_{\pi *}L_{\lambda_1} \otimes R^j_{\pi *}L_{\lambda_2}) \xleftarrow{\delta^*_M} H^{d_1-i}(M, R^i_{\pi *}L_{\lambda_1}) \otimes H^{d_2-j}(M, R^j_{\pi *}L_{\lambda_2}),$$

which shows that t^* is the first part of the factorization claimed.

We now study s^*. The map s includes X as the relative diagonal of $X \times_M X$ over M. It follows that s^* induces the relative cup-product map on the higher direct image sheaves of t^*L and L_λ. Therefore the map associated to s^* on the E_2-terms of the Leray spectral sequences for t^*L and L_λ is given by the relative cup-product map

$$H^{d-i-j}(M, R^{i+j}L_\lambda) = H^{d-i-j}(M, R^{i+j}(L_{\lambda_1} \otimes L_{\lambda_2})) \xleftarrow{\cup_\pi} H^{d-i-j}(M, R^i_{\pi *}L_{\lambda_1} \otimes R^j_{\pi *}L_{\lambda_2}).$$

All that is needed to demonstrate the factorization claimed is to demonstrate the condition $k = i + j$ which ensures the map on the associated graded pieces in the E_2-terms agrees with the global map on the cohomology groups (see Section 2.9).

Suppose that w_1, w_2, and w are the elements of the Weyl group such that $w_1 \cdot \lambda_1$, $w_2 \cdot \lambda_2$, and $w \cdot \lambda$ are dominant. Then

$$k = \#(\Phi_w \cap -\Delta_P), \quad i = \#(\Phi_{w_1} \cap -\Delta_P), \quad j = \#(\Phi_{w_2} \cap -\Delta_P),$$

where the symbol # indicates the cardinality of a set. The condition $k = i + j$ guaranteeing the factorization thus amounts to the condition

$$\#(\Phi_w \cap -\Delta_P) = \#(\Phi_{w_1} \cap -\Delta_P) + \#(\Phi_{w_2} \cap -\Delta_P). \tag{5.4.5}$$

Since the original cup-product map was assumed surjective we must have $\Phi_w = \Phi_{w_1} \sqcup \Phi_{w_2}$ by Theorem I; this immediately implies that (5.4.5) holds. □

Corollary 5.4.6. *Suppose that $w_1, \ldots, w_k$ are elements of the Weyl group such that $\Delta^+ = \bigsqcup_{i=1}^k \Phi_{w_i}$, and that $\mu_1, \ldots, \mu_k$ are dominant weights satisfying the condition $\sum_{i=1}^k w_i^{-1}\mu_i = 0$. Then* $\dim(V_{\mu_1} \otimes \cdots \otimes V_{\mu_k})^G = 1$.

Proof. Set $\lambda_i = w_i^{-1} \cdot \mu_i$ for $i = 1, \ldots, k$. Then $\sum \lambda_i = -2\rho$ and we have a cup-product problem as in (5.3.1). As part of the proof of Theorem I in Section 5.3 it was established that $\dim(V_{\mu_1} \otimes \cdots \otimes V_{\mu_k})^G = 1$. Alternatively, the corollary is simply Theorem 3.9.1(b) applied in symmetric form, with Lemma 2.6.1 used to ensure that the hypotheses of the theorem are satisfied. □

Corollary 5.4.7. *Suppose that we have a cup-product map*

$$\mathrm{H}^{\ell(w_1)}(\mathrm{X}, \mathrm{L}_{\lambda_1}) \otimes \cdots \otimes \mathrm{H}^{\ell(w_k)}(\mathrm{X}, \mathrm{L}_{\lambda_k}) \xrightarrow{\cup} \mathrm{H}^d(\mathrm{X}, \mathrm{L}_\lambda)$$

and, as above, Weyl group elements $w_1, \dots, w_k$, *and* w *such that* $\mu_i := w_i \cdot \lambda_i$, $i = 1, \dots, k$, *and* $\mu := w \cdot \mu$ *are dominant weights. Then if* $\bigcap_{i=1}^k [\Omega_{w_i}] \cdot [\mathrm{X}_w] \neq 0$ *the cup-product map is surjective if and only if* V_μ *is a component of* $\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}$.

Proof. If V_μ is not a component of the tensor product the map is clearly not surjective. Conversely, if V_μ is a component, the assumption on the intersection number and the argument in Section 5.1 for the necessity of condition (1.2.1) show that $\Phi_w = \bigsqcup_{i=1}^k \Phi_{w_i}$, and therefore we conclude that the map is surjective by the sufficiency of condition (1.2.1). □

The following example illustrates Corollary 5.4.7 and provides an example which shows that condition (1.2.1) is not necessary in order to have a cup-product problem for which both sides are nonzero.

Example 5.4.8. Let $\mathrm{G} = \mathrm{SL}_6$ and $w_1 = w_2 = s_2 s_4 s_3$. For any integers $a_i, b_i \geqslant 0$ $(i = 1, 2)$ set

$$\mu_i = (0, a_i, 0, b_i, 0) \quad \text{and} \quad \lambda_i = w_i^{-1} \cdot \mu_i = (a_i + 1, b_i + 1, -4 - a_i - b_i, a_i + 1, b_i + 1).$$

(The weights are written in terms of the fundamental weights of SL_6.) Finally, let $w = s_1 s_3 s_5 s_2 s_4 s_3$ and set

$$\mu = w \cdot (\lambda_1 + \lambda_2) = (0, a_1 + a_2 + 1, 0, b_1 + b_2 + 1, 0) \in \Lambda^+.$$

We therefore get a cup-product problem:

$$\mathrm{H}^3(\mathrm{X}, \mathrm{L}_{\lambda_1}) \otimes \mathrm{H}^3(\mathrm{X}, \mathrm{L}_{\lambda_2}) \xrightarrow{\cup} \mathrm{H}^6(\mathrm{X}, \mathrm{L}_{\lambda_1 + \lambda_2}).$$

By Theorem I, this cup product cannot be surjective, since $\Phi_{w_1} = \Phi_{w_2}$; alternatively, the map cannot be surjective since V_μ is clearly not a component of $\mathrm{V}_{\mu_1} \otimes \mathrm{V}_{\mu_2}$. The intersection number $([\Omega_{w_1}] \cap [\Omega_{w_2}]) \cdot \mathrm{X}_w$ is two.

Corollary 5.4.9. *If* $\Delta^+ = \bigsqcup_{i=1}^k \Phi_{w_i}$ *then for any subset* $\mathrm{I} \subseteq \{1, \dots, k\}$ *there is an element* w *of the Weyl group such that* $\Phi_w = \bigsqcup_{i \in \mathrm{I}} \Phi_{w_i}$.

Proof. Let $\lambda_i = w_i^{-1} \cdot 0$ so that we get a cup-product problem as in (5.3.1). (Here each $\mathrm{H}^{\ell(w_i)}(\mathrm{X}, \mathrm{L}_{\lambda_i})$ is the trivial G-module). By Theorem I and the assumption on $w_1, \dots, w_k$ this cup product is surjective. It can be factored by first taking the cup product of any subset $\mathrm{I} \subseteq \{1, \dots, k\}$ of the factors and the resulting cup-product problem must also be nonzero since the larger problem is. Hence by Theorem I there is a $w \in \mathcal{W}$ with $w \cdot \left(\sum_{i \in \mathrm{I}} \lambda_i\right) \in \Lambda^+$ and such that $\Phi_w = \bigsqcup_{i \in \mathrm{I}} \Phi_{w_i}$. □

5.5. ***Comments.*** (1) Corollary 5.4.9 can also be proved independently of any of the constructions in this paper by using a similar argument in nilpotent cohomology. We are grateful to Olivier Mathieu for pointing this out to us.

(2) Using the result of Corollary 5.4.9 and induction, to prove Theorem I it is sufficient to prove it in the case $k = 2$ of the cup product of two cohomology groups into a third. We have chosen to develop the description of the varieties $\mathrm{Y}_{\underline{v}}$ for arbitrary k partly since this is the natural generality of the construction, partly because it makes no difference in our proofs, but also because some of the applications (e.g., the multiplicity bounds) do not follow by induction. Note that by the methods of this paper, even to prove the case $k = 2$ of the cup product it would be necessary to consider the case of the cup product of three factors into $\mathrm{H}^{\mathrm{N}}(\mathrm{X}, \mathrm{K}_{\mathrm{X}})$, and hence we would need the construction of $\mathrm{Y}_{\underline{v}}$ for three factors.

(3) As Example 5.4.8 shows, the natural numerical condition $\ell(w_1)+\ell(w_2)=\ell(w)$ does not imply condition (1.2.1) even if there is a nontrivial cup-product problem corresponding to w_1, w_2, and w. On the other hand, condition (5.4.5) imposes further necessary numerical conditions for (1.2.1). Namely,

$$\ell(w_1^{\mathrm{P}}) + \ell(w_2^{\mathrm{P}}) = \ell(w^{\mathrm{P}}) \quad \text{for every parabolic subgroup } \mathrm{P} \supseteq \mathrm{B} \text{ of } \mathrm{G}, \tag{5.5.1}$$

where w_1^{P}, w_2^{P}, w^{P} denote the minimal length representatives in $w_1\mathcal{W}_{\mathrm{P}}$, $w_2\mathcal{W}_{\mathrm{P}}$, $w\mathcal{W}_{\mathrm{P}}$. In the case when $\mathrm{G} = \mathrm{SL}_{n+1}$ one can show that condition (5.5.1) is sufficient for (1.2.1). The simple inductive argument relies on the fact that if $\mathrm{G} = \mathrm{SL}_{n+1}$ it is possible to assign a parabolic $\mathrm{P}_\alpha \supset \mathrm{B}$ to every root $\alpha \in \Delta^+$ in such a way that $-\alpha$ is a root of P_α but not a root of any proper parabolic subgroup of P_α containing B. We do not know if (5.5.1) is sufficient to imply (1.2.1) for general G.

(4) Corollary 5.4.2 establishes the following factorization property: any nonzero cup-product map on X factors as a cup product on G/P and fibers of $\pi : \mathrm{X} \to \mathrm{G}/\mathrm{P}$ for all $\mathrm{P} \supset \mathrm{B}$. We know of no a priori reason why this should hold. The factorization property is equivalent to (5.4.5) holding for all $\mathrm{P} \supset \mathrm{B}$ which is equivalent to (5.5.1). Hence, in the case $\mathrm{G} = \mathrm{SL}_{n+1}$ the factorization property is equivalent to (1.2.1).

6. Cohomological components and proof of Theorem II

6.1. ***Conditions on components of tensor products.*** We begin by introducing two relevant conditions. We also recall the notion of generalized PRV component from Section 2.3 for convenience.

Definitions 6.1.1. Suppose that $\mu_1, \dots, \mu_k$, and μ are dominant weights.

(a) We say that V_μ is a *generalized PRV component* of $\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}$ if there exist $w_1, \dots, w_k$, and w in $\mathcal{W}$ such that $w^{-1}\mu = \sum_{i=1}^{k} w_i^{-1}\mu_i$.

(b) We say that V_μ is a component of *stable multiplicity one* of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ if we have $\dim(V_{m\mu_1} \otimes \cdots \otimes V_{m\mu_k} \otimes V_{m\mu}^*)^G = 1$ for all $m \gg 0$.

(c) We say that V_μ is a *cohomological component* of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ if there exist $w_1, \ldots, w_k$, and w in $\mathcal{W}$ such that $w^{-1}\mu = \sum_{i=1}^k w_i^{-1}\mu_i$ and such that $\Phi_w = \bigsqcup_{i=1}^k \Phi_{w_i}$.

Under the hypothesis that $\Phi_w = \bigsqcup_{i=1}^k \Phi_{w_i}$, the condition $w^{-1}\mu = \sum_{i=1}^k w_i^{-1}\mu_i$ is equivalent to the condition $w^{-1} \cdot \mu = \sum_{i=1}^k w_i^{-1} \cdot \mu_i$. Therefore by Theorem I condition (c) is equivalent to having a surjective cup-product map

$$H^{\ell(w_1)}(X, L_{w_1^{-1}\cdot\mu_1}) \otimes \cdots \otimes H^{\ell(w_k)}(X, L_{w_k^{-1}\cdot\mu_k}) \xrightarrow{\cup} H^{\ell(w)}(X, L_{w^{-1}\cdot\mu})$$

which, after dualizing, gives an injective map

$$V_\mu \to V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}.$$

In other words, we obtain a construction of V_μ as a component of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ realized through the cohomology of X.

Note that the conditions in Definitions 6.1.1 are *homogeneous*: if V_μ is a generalized PRV component, a component of stable multiplicity one, or a cohomological component of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ then the same is true of $V_{m\mu}$ as a component of $V_{m\mu_1} \otimes \cdots \otimes V_{m\mu_k}$ for all $m \geqslant 1$. This follows immediately from the definitions.

6.2. *Proof of Theorem II*(a) *and restatement of Theorem II*(b).

Proof of Theorem II(a). Every cohomological component has multiplicity one by Theorem 3.9.1(b) (the condition on nonzero intersection holds by the nonsymmetric version of Lemma 2.6.1). By homogeneity we conclude that homological components are of stable multiplicity one. From Definitions 6.1.1(a, c) it is clear that every cohomological component is a generalized PRV component. Thus every cohomological component is a generalized PRV component of stable multiplicity one. □

For the proof of part (b) it will be more convenient to work with the symmetric form of the problem. Applying the symmetrization procedure from Section 2.7 (and replacing $k+1$ by k) we obtain the following reformulation of Theorem II(b).

Proposition 6.2.1. *Let $\mu_1, \ldots, \mu_k$ be dominant weights such that*

$$\dim(V_{m\mu_1} \otimes \cdots \otimes V_{m\mu_k})^G = 1 \quad \textit{for } m \gg 0$$

and suppose that we have elements $w_1, \ldots, w_k$ such that $\sum w_i^{-1}\mu_i = 0$. Then in either of the following two cases:

(i) *at least one of $\mu_1, \ldots, \mu_k$ is strictly dominant*,

(ii) G *is a classical simple group or product of classical simple groups*, *there exist* $\bar{w}_1, \dots, \bar{w}_k \in \mathcal{W}$ *such that*

$$\sum_{i=1}^{k} \bar{w}_i^{-1}\mu_i = 0 \quad \textit{and} \quad \Delta^+ = \bigsqcup_{i=1}^{k} \Phi_{\bar{w}_i}. \tag{6.2.2}$$

The proof of Proposition 6.2.1 will be given in Section 6.8 after some preliminary reduction steps.

For the rest of this section we assume that we have fixed dominant weights $\mu_1, \dots, \mu_k$ and Weyl group elements $w_1, \dots, w_k$ satisfying the conditions of Proposition 6.2.1.

6.3. ***Outline of the proof of Proposition 6.2.1.*** For $i = 1, \dots, k$, let P_i be the parabolic subgroup of G such that L_{μ_i} is the pullback to X of an ample line bundle $\mathrm{L}_{\tilde{\mu}_i}$ on G/P_i. Set $\mathrm{M} = \mathrm{G}/\mathrm{P}_1 \times \cdots \times \mathrm{G}/\mathrm{P}_k$ and $\mathrm{L} = \mathrm{L}_{\tilde{\mu}_1} \boxtimes \cdots \boxtimes \mathrm{L}_{\tilde{\mu}_k}$. The condition that $\dim(\mathrm{V}_{m\mu_1} \otimes \cdots \otimes \mathrm{V}_{m\mu_k})^{\mathrm{G}} = 1$ for all $m \gg 1$ implies that the GIT quotient $\mathrm{M}/\!\!/\mathrm{G}$ with respect to L is a point.

If $w_1, \dots, w_k$ are elements such that $\sum_{i=1}^{k} w_i^{-1}\mu_i = 0$ then by Lemma 2.11.1, the point $q = (w_1^{-1}, \dots, w_k^{-1})$ is a semistable point of M with a closed orbit. Let $\mathrm{H} \subseteq \mathrm{G}$ be the stabilizer subgroup of q, and $\mathcal{N}_q$ be the normal space to the orbit at q. By the Luna slice theorem and the fact that the GIT quotient $\mathrm{M}/\!\!/\mathrm{G}$ is a point we conclude that $\mathrm{Sym}^{\cdot}(\mathcal{N}_q)^{\mathrm{H}}$ is one-dimensional.

The explicit combinatorial formula for the weights appearing in $\mathcal{N}_q$ shows that a necessary condition for a solution of (6.2.2) to exist is that there is $v \in \mathcal{W}$ such that the weights of $v\mathcal{N}_q$ are contained in Δ^-. In Proposition 6.6.1 below we formulate a condition which, together with the necessary condition above, guarantees the existence of a solution of (6.2.2). Together these two conditions are equivalent to the existence of a parabolic subalgebra $\mathfrak{p}$ with reductive part Lie(H) such that the weights of $\mathcal{N}_q$ are contained in $\mathfrak{p}$.

Finally, we use the restriction that $\mathrm{Sym}^{\cdot}(\mathcal{N}_q)^{\mathrm{H}}$ is one-dimensional to show the existence of such a parabolic subalgebra when G is a classical group, or for any semisimple group G under a genericity condition.

6.4. ***Stabilizer subgroup of a semistable* T*-fixed point.*** Let P_i be the parabolic with roots $\Delta_{\mathrm{P}_i} = \{\alpha \in \Delta \mid \kappa(\alpha, \mu_i) \geqslant 0\}$, and let $\mathrm{M}_i = \mathrm{G}/\mathrm{P}_i$. The stabilizer subgroup of the point w_i^{-1} in M_i is $w_i^{-1}\mathrm{P}_i w_i$, whose roots are

$$\Delta_{w_i^{-1}\mathrm{P}_i w_i} = \big\{\alpha \in \Delta \mid \kappa(w_i\alpha, \mu_i) \geqslant 0\big\} = \big\{\alpha \in \Delta \mid \kappa(\alpha, w_i^{-1}\mu_i) \geqslant 0\big\}. \tag{6.4.1}$$

Let $\mathrm{M} = \mathrm{M}_1 \times \cdots \times \mathrm{M}_k$ and let q be the point $q = (w_1^{-1}, \dots, w_k^{-1})$ of M. We set $\mathrm{H} = \bigcap_{i=1}^{k} w_i^{-1}\mathrm{P}w_i$ to be the stabilizer subgroup of q. The condition $\sum_{i=1}^{k} w_i^{-1}\mu_i = 0$ in combination with (6.4.1) shows that the roots of H are given by

$$\Delta_{\mathrm{H}} = \big\{\alpha \in \Delta \mid \kappa(\alpha, w_i^{-1}\mu_i) = 0 \text{ for } i = 1, \dots, k\big\}. \tag{6.4.2}$$

We conclude from (6.4.2) that H is a reductive subgroup of G. Noting that $\mathrm{T} \subseteq \mathrm{H}$, the following lemma is another immediate consequence of (6.4.2).

Lemma 6.4.3. *We have* $\mathrm{H} = \mathrm{T}$ *if and only if the span of* $\{w_i^{-1}\mu_i\}_{i=1}^k$ *intersects the interior of some Weyl chamber. This happens, for instance, if any one of the weights* μ_i *is strictly dominant.*

6.5. ***Torus action at fixed points of*** M ***and combinatorial deductions.*** Let $\mathcal{W}_i = \{w \in \mathcal{W} \mid w\mu_i = \mu_i\} \subseteq \mathcal{W}$ be the stabilizer subgroup of μ_i; this is the Weyl group of P_i. We will need the formula for the formal character of the tangent space of M_i at a torus fixed point. Because of the way that the inverses of group elements enter into our formulas we make the following convention: For any element w of $\mathcal{W}$ and any i we let $w_{s(i)}$ and $w_{l(i)}$ be respectively the shortest and longest elements in the coset $\mathcal{W}_i w$. Recall also that for $\Phi \subseteq \Delta$, $\langle\Phi\rangle$ denotes the formal character $\sum_{\alpha\in\Phi} e^{\alpha}$.

With this convention, if w_i is any element of $\mathcal{W}$, the formal character of the tangent space of M_i at the torus fixed point corresponding to the coset $w_i^{-1}\mathcal{W}_i$ is

$$\mathrm{Ch}(\mathrm{T}_{w_i^{-1}}\mathrm{M}_i) = \langle\Phi_{w_{i,s(i)}}\rangle + \langle -\Phi^{\mathrm{c}}_{w_{i,l(i)}}\rangle = \big\langle\{\alpha \in \Delta \mid \kappa(\alpha, w_i^{-1}\mu_i) < 0\}\big\rangle.$$

The formal character of the tangent space of M at q is therefore

$$\mathrm{Ch}(\mathrm{T}_q\mathrm{M}) = \sum_{i=1}^{k}\big(\langle\Phi_{w_{i,s(i)}}\rangle + \langle -\Phi^{\mathrm{c}}_{w_{i,l(i)}}\rangle\big) = \sum_{i=1}^{k}\big\langle\{\alpha \in \Delta \mid \kappa(\alpha, w_i^{-1}\mu_i) < 0\}\big\rangle. \quad (6.5.1)$$

Note that the multiplicity of each root α in the equations above is the number of i for which $\kappa(\alpha, w_i^{-1}\mu_i) < 0$.

If $\alpha \notin \Delta_{\mathrm{H}}$ then there is some i for which $\kappa(\alpha, w_i^{-1}\mu_i) \neq 0$ and hence, by the condition $\sum_{i=1}^k w_i^{-1}\mu_i = 0$, there is some i for which $\kappa(\alpha, w_i^{-1}\mu_i) < 0$, i.e., α must appear as a weight in $\mathrm{T}_q\mathrm{M}$. By looking at the positive roots of $\mathrm{T}_q\mathrm{M}$ we therefore conclude that

$$(\Delta^+ \setminus \Delta^+_{\mathrm{H}}) = \bigcup \Phi_{w_{i,s(i)}}. \quad (6.5.2)$$

Let O_q be the G-orbit of q in M. Since H is the stabilizer of q, the formal character of the tangent space $\mathrm{T}_q\mathrm{O}_q$ is

$$\mathrm{Ch}(\mathrm{T}_q\mathrm{O}_q) = \langle\Delta^+ \setminus \Delta^+_{\mathrm{H}}\rangle + \langle\Delta^- \setminus \Delta^-_{\mathrm{H}}\rangle. \quad (6.5.3)$$

If $\mathcal{N}_q = \mathrm{T}_q\mathrm{M}/\mathrm{T}_q\mathrm{O}_q$ is the normal space to the orbit at q, then the union in (6.5.2) is disjoint if and only if the formal character of $\mathcal{N}_q$ contains no positive root.

Let $\mathcal{M}$ be the subspace of $\mathfrak{g}$ spanned by the root spaces corresponding to the roots appearing in $\mathcal{N}_q$. Comparing the multiplicities in (6.5.1) and (6.5.3) we conclude that the roots of $\mathcal{M}$ are

$$\Delta_{\mathcal{M}} = \big\{\alpha \in \Delta \mid \kappa(\alpha, w_i^{-1}\mu_i) < 0 \text{ for at least two } i \in \{1, \ldots, k\}\big\}. \quad (6.5.4)$$

Let $\mathfrak{s} = \mathrm{Lie(H)}$; equations (6.5.4) and (6.4.2) show that $\mathcal{M}$ is an $\mathfrak{s}$-submodule of $\mathfrak{g}$.

The point q is not the only torus fixed point in its orbit; for any $v \in \mathcal{W}$ we can act on the left to get the torus fixed point $vq = (vw_1^{-1}, \dots, vw_k^{-1})$. The weights of the normal space $\mathcal{N}_{vq}$ to the G-orbit at vq are the result of acting on the weights of $\mathcal{N}_q$ by v and are hence the roots appearing in $\mathrm{Ch}(v\mathcal{M})$.

Repeating the previous arguments with the new point vq and the new stabilizer group $v\mathrm{H}v^{-1} = \mathrm{Stab}(vq)$, gives the following result.

Lemma 6.5.5. *For any $v \in \mathcal{W}$ we have*

$$(\Delta^+ \setminus \Delta^+_{v\mathrm{H}v^{-1}}) = \bigsqcup_{i=1}^{k} \Phi_{(w_i v^{-1})_{s(i)}}$$

if and only if $v\mathcal{M} \subseteq \mathfrak{b}^-$.

6.6. *Reduction to the existence of* $\mathfrak{p}_{\mathcal{M}}$.

Proposition 6.6.1. *Suppose that there exists $v \in \mathcal{W}$ satisfying the conditions*

(i) $v\mathcal{M} \subseteq \mathfrak{b}^-$,

(ii) *there is an element $w \in \mathcal{W}$ such that $\Phi_w = \Delta^+_{v\mathrm{H}v^{-1}}$.*

Then there exist $\bar{w}_1, \dots, \bar{w}_k \in \mathcal{W}$ such that $\sum_{i=1}^{k} \bar{w}_i^{-1}\mu_i = 0$ and $\Delta^+ = \bigsqcup_{i=1}^{k} \Phi_{\bar{w}_i}$.

Proof. By condition (i) and Lemma 6.5.5 we have $(\Delta^+ \setminus \Delta^+_{v\mathrm{H}v^{-1}}) = \bigsqcup_{i=1}^{k} \Phi_{(w_i v^{-1})_{s(i)}}$. Set $\widetilde{w}_{k+1} = w$, $\mu_{k+1} = 0$, and $\widetilde{w}_i = (w_i v^{-1})_{s(i)}$ for $i = 1, \dots, k$. Conditions (i) and (ii) above and the original assumption about $w_1, \dots, w_k$ imply

$$\sum_{i=1}^{k+1} \widetilde{w}_i^{-1}\mu_i = 0 \quad \text{and} \quad \Delta^+ = \bigsqcup_{i=1}^{k+1} \Phi_{\widetilde{w}_i}. \tag{6.6.2}$$

Equation (6.6.2) and Theorem I show that there is a surjective cup-product map

$$\mathrm{H}^{\ell(\tilde{w}_1)}(\mathrm{X}, \mathrm{L}_{\tilde{w}_1^{-1}\cdot\mu_1}) \otimes \cdots \otimes \mathrm{H}^{\ell(\tilde{w}_{k+1})}(\mathrm{X}, \mathrm{L}_{\tilde{w}_{k+1}^{-1}\cdot\mu_{k+1}}) \xrightarrow{\cup} \mathrm{H}^{\mathrm{N}}(\mathrm{X}, \mathrm{K}_{\mathrm{X}}).$$

Since $\mathrm{H}^{\ell(\tilde{w}_{k+1})}(\mathrm{X}, \mathrm{L}_{\tilde{w}_{k+1}^{-1}\cdot\mu_{k+1}})$ is the trivial module, if we factor the map above by cupping the k-th and $(k+1)$-st factors together first, we obtain a surjective cup-product map onto $\mathrm{H}^{\mathrm{N}}(\mathrm{X}, \mathrm{K}_{\mathrm{X}})$ only involving the modules $\mathrm{V}^*_{\mu_1}, \dots, \mathrm{V}^*_{\mu_k}$. By invoking Theorem I again we conclude that there are $\bar{w}_1, \dots, \bar{w}_k$ such that

$$\sum_{i=1}^{k} \bar{w}_i^{-1}\mu_i = 0 \quad \text{and} \quad \Delta^+ = \bigsqcup_{i=1}^{k} \Phi_{\bar{w}_i}, \tag{6.6.3}$$

proving Proposition 6.6.1. □

Remark. If there do exist $\bar{w}_1, \dots, \bar{w}_k$ satisfying the conclusion of Proposition 6.2.1 it is not hard to show that there must exist $v \in \mathcal{W}$ so that (i) of Proposition 6.6.1 holds. As a consequence of our method of proof we see *a posteriori* that there must be a v so that both (i) and (ii) hold when G is a classical group or under a genericity condition. We do not know if condition (ii) is necessary in general.

It is useful to rephrase the conditions of Proposition 6.6.1 in terms of the existence of a particular parabolic subalgebra $\mathfrak{p}_{\mathcal{M}}$.

Lemma 6.6.4. *Let* $\mathfrak{s} = \mathrm{Lie}(\mathrm{H})$. *Suppose that there exists a parabolic subalgebra* $\mathfrak{p}_{\mathcal{M}}$ *with reductive part* $\mathfrak{s}$ *such that* $\mathcal{M} \subseteq \mathfrak{p}_{\mathcal{M}}$. *Then conditions* (i) *and* (ii) *of Proposition 6.6.1 hold.*

Proof. Let $\mathfrak{p}_{\mathcal{M}}$ be such a parabolic subalgebra. Acting by an element $v \in \mathcal{W}$ we can conjugate $\mathfrak{p}_{\mathcal{M}}$ so that $\mathfrak{b}^- \subseteq v\mathfrak{p}_{\mathcal{M}}$. This implies that $v\mathcal{M} \subseteq \mathfrak{b}^-$. Since $v\mathfrak{s}$ is the radical of a parabolic subalgebra containing $\mathfrak{b}^-$, if w is the longest element of the Weyl group of $v\mathfrak{s}$ then $\Phi_w = \Delta^+_{v\mathfrak{s}} = \Delta^+_{v\mathrm{H}v^{-1}}$. □

Remark. If there exists $v \in \mathcal{W}$ such that condition (ii) of Proposition 6.6.1 holds then one can show that $\mathfrak{p} := \mathfrak{b}^- + v\mathfrak{s}$ is a parabolic subalgebra of $\mathfrak{g}$. If condition (i) also holds for this v then $\mathfrak{p}_{\mathcal{M}} := v^{-1}\mathfrak{p}$ is a parabolic subalgebra satisfying the conditions of Lemma 6.6.4. Therefore the existence of the parabolic $\mathfrak{p}_{\mathcal{M}}$ is equivalent to the conditions in Proposition 6.6.1. Since we will not need this direction of the equivalence we omit the justification of the first assertion.

6.7. ***GIT consequences of the stable multiplicity one condition.*** Let L be the line bundle on M whose pullback to X^k is $\mathrm{L}_{\mu_1} \boxtimes \cdots \boxtimes \mathrm{L}_{\mu_k}$. Then L is a G-equivariant ample line bundle on M. By the stable multiplicity one condition we have $\dim(\mathrm{M}, \mathrm{L}^m)^{\mathrm{G}} = 1$ for all $m \gg 1$, and so the GIT quotient $\mathrm{M}/\!\!/\mathrm{G}$ is a point.

The weight of L at q is $\sum_{i=1}^k w_i^{-1}\mu_i = 0$. By Lemma 2.11.1 this means that q is a semistable point with a closed orbit. By the Luna slice theorem [1973, théorèm du slice étale, p. 97], $\mathrm{Spec}(\mathrm{Sym}(\mathcal{N}_q^*)^{\mathrm{H}})$ and the image of q in the GIT quotient $\mathrm{M}/\!\!/\mathrm{G}$ have a common étale neighborhood. Hence $\dim(\mathcal{N}_q/\mathrm{H}) = \dim(\mathrm{M}/\!\!/\mathrm{G}) = 0$, i.e., $\dim \mathrm{Sym}^{\cdot}(\mathcal{N}_q^*)^{\mathrm{H}} = 1$. Passing to the level of Lie algebras and dualizing we obtain $\dim \mathrm{Sym}^{\cdot}(\mathcal{N}_q)^{\mathfrak{s}} = 1$.

Since $\mathcal{M}$ is isomorphic to an $\mathfrak{s}$-submodule of $\mathcal{N}_q$ we arrive at the following consequence of the stable multiplicity one condition:

Lemma 6.7.1. *Under the hypotheses of Proposition 6.2.1 and with the notation of Section 6.5, we have* $\dim \mathrm{Sym}^{\cdot}(\mathcal{M})^{\mathfrak{s}} = 1$, *i.e.,* $\mathrm{Sym}^{\cdot}(\mathcal{M})^{\mathfrak{s}}$ *consists of just the constants.*

6.8. ***Proof of Proposition 6.2.1.*** By Proposition 6.6.1 and Lemma 6.6.4, to prove Proposition 6.2.1 it is enough to show the existence of the parabolic subalgebra $\mathfrak{p}_{\mathcal{M}}$. By Lemma 6.7.1 we may assume that $\dim \mathrm{Sym}^{\cdot}(\mathcal{M})^{\mathfrak{s}} = 1$.

Proof of 6.2.1(i). If any one of the weights $\mu_1, \ldots, \mu_k$ is strictly dominant, or more generally, if the span of $\{w_i^{-1}\mu_i\}_{i=1}^k$ intersects the interior of some Weyl chamber, then by Lemma 6.4.3 $\mathrm{H} = \mathrm{T}$ and so $\mathfrak{s} = \mathrm{Lie}(\mathrm{T}) = \mathfrak{t}$ and $\Delta_{\mathfrak{t}}^{+} = \varnothing$. The condition that $\dim(\mathrm{Sym}^{\cdot}(\mathcal{M})^{\mathfrak{t}}) = 1$ is then equivalent to the condition that no nontrivial nonnegative combination of weights of $\mathcal{M}$ is zero. Hence by Farkas's lemma the weights of $\mathcal{M}$ all lie strictly on one side of a hyperplane and the cone dual to the cone they span is open. We may therefore pick a weight in the interior of the dual cone which is not on any hyperplane of the Weyl chambers. The roots lying on the positive side of this hyperplane give the parabolic subalgebra $\mathfrak{p}_{\mathcal{M}}$. □

Proof of Proposition 6.2.1(ii). Equation (6.4.2) shows that the roots of $\mathfrak{s}$ are given by the vanishing of linear forms and hence $\mathfrak{s}$ is the reductive part of a parabolic subalgebra. Let $\mathfrak{a}$ be the center of $\mathfrak{s}$. For any $\nu \in \mathfrak{a}^* \setminus \{0\}$ set

$$\mathfrak{g}^{\nu} = \big\{x \in \mathfrak{g} \mid [t, x] = \nu(t)x \text{ for all } t \in \mathfrak{a}\big\}.$$

Following Kostant [2010], we call $\nu \in \mathfrak{a}^*\setminus\{0\}$ an $\mathfrak{a}$-root if $\mathfrak{g}^{\nu} \neq 0$. Let $\mathcal{R}$ be the set of $\mathfrak{a}$-roots of $\mathfrak{g}$ and $\mathcal{S}$ the subset of those $\mathfrak{a}$-roots appearing in $\mathcal{M}$, so that $\mathcal{M} = \bigoplus_{\nu \in \mathcal{S}} \mathfrak{g}^{\nu}$.

A subset $\mathcal{R}'$ of $\mathcal{R}$ is called *saturated* if whenever $\nu \in \mathcal{R}'$ and $r\nu \in \mathcal{R}$ for some $r \in \mathbb{Q}_+$ then $r\nu \in \mathcal{R}'$ as well. It follows from (6.5.4) that $\mathcal{S}$ is a saturated subset of $\mathcal{R}$.

As part of the main theorem of [Dimitrov and Roth 2017] we establish the following result.[2]

Theorem. *Let $\mathfrak{g}$ be a classical Lie algebra, $\mathfrak{s}$ be a subalgebra which is the reductive part of a parabolic subalgebra of $\mathfrak{g}$, $\mathcal{S}$ be a saturated subset of the $\mathfrak{a}$-roots $\mathcal{R}$, and $\mathcal{M} = \bigoplus_{\nu \in \mathcal{S}} \mathfrak{g}^{\nu}$. If $\dim(\mathrm{Sym}^{\cdot}(\mathcal{M}))^{\mathfrak{s}} = 1$, then there exists a parabolic subalgebra $\mathfrak{p}_{\mathcal{M}} \subseteq \mathfrak{g}$, with reductive part $\mathfrak{s}$, such that $\mathcal{M} \subseteq \mathfrak{p}_{\mathcal{M}}$.*

Thus when G is a simple classical group or a product of simple classical groups, the above theorem along with the previous reductions establish Proposition 6.2.1 and finish the proof of Theorem II. □

List of symbols

$\bigsqcup_{i=1}^{k}$	disjoint union
$\kappa(\cdot\,,\cdot)$	the Killing form of G
Λ, Λ^+	weight lattice and cone of dominant weights
V_{μ}	irreducible G-module of highest weight μ
$\mathrm{mult}(\mathrm{V}_{\mu}, \mathrm{V})$	the multiplicity of V_{μ} in V
$\{\alpha_1, \dots, \alpha_n\}$	base of simple roots of B
$\mathcal{W}$	Weyl group of $\mathfrak{g}$

[2]The proof is rather technical, and involves a case-by-case analysis of the different types, a characterization of the desired parabolics in terms of certain linearly ordered data, and an argument that the hypothesis $\dim(\mathrm{Sym}^{\cdot}(\mathcal{M}))^{\mathfrak{s}} = 1$ allows one to take a partial order constructed from $\mathcal{S}$ and extend it to a linear one.

$w \cdot \lambda$	$w(\lambda+\rho)-\rho$, the result of the affine action of $w \in \mathcal{W}$ on $\lambda \in \Lambda$
s_i	simple reflection along α_i
P_{α_i}	the minimal parabolic subgroup of G associated to α_i
P_{I}	the minimal parabolic subgroup of G associated to a set I of simple roots
$\mathcal{W}_{\mathrm{P}}$	the Weyl group of a parabolic subgroup $\mathrm{P} \subseteq \mathrm{G}$
$\operatorname{span}_{\mathbb{Z}_{\geqslant 0}} \Phi$	the set of nonnegative integer combinations of elements of $\Phi \subseteq \Delta$
$\underline{u}$ or $\underline{v}$	a word $s_{i_1} \cdots s_{i_m}$ in the simple reflections of the Weyl group
u, v	the element of $\mathcal{W}$ corresponding to $\underline{u}$ or $\underline{v}$
$\underline{v}_R$	the word obtained by dropping the rightmost reflection of $\underline{v}$
$\underline{\boldsymbol{u}}$ or $\underline{\boldsymbol{v}}$	a sequence $(\underline{u}_1, \ldots, \underline{u}_k)$ or $(\underline{v}_1, \ldots, \underline{v}_k)$ of words
Φ_w	$w^{-1}\Delta^- \cap \Delta^+$, the inversion set of $w \in \mathcal{W}$
$\langle\Phi\rangle$	$\sum_{\alpha\in\Phi} e^\alpha$, the formal character of $\bigoplus_{\alpha\in\Phi} \mathfrak{g}^\alpha$, where $\Phi \subset \Delta$
$\ell(w)$	the length of $w \in \mathcal{W}$
L_λ	the line bundle on X corresponding to the B-module on which T acts via $-\lambda$
N	the dimension of X
π_i	the projection $\pi_i : \mathrm{X} \to \mathrm{G}/\mathrm{P}_{\alpha_i}$ (a $\mathbb{P}^1$-fibration)

Acknowledgments

We thank P. Belkale, W. Fulton, A. Knutson, B. Kostant, S. Kumar, O. Mathieu, K. Purbhoo, and D. Wehlau for numerous useful discussions. Ivan Dimitrov acknowledges excellent working conditions at the Max-Planck Institute. Mike Roth acknowledges the hospitality of the University of Roma III.

References

[Belkale and Kumar 2006] P. Belkale and S. Kumar, "Eigenvalue problem and a new product in cohomology of flag varieties", *Invent. Math.* **166**:1 (2006), 185–228. MR Zbl

[Berenstein and Sjamaar 2000] A. Berenstein and R. Sjamaar, "Coadjoint orbits, moment polytopes, and the Hilbert–Mumford criterion", *J. Amer. Math. Soc.* **13**:2 (2000), 433–466. MR Zbl

[Bott and Samelson 1958] R. Bott and H. Samelson, "Applications of the theory of Morse to symmetric spaces", *Amer. J. Math.* **80** (1958), 964–1029. MR Zbl

[Demazure 1974] M. Demazure, "Désingularisation des variétés de Schubert généralisées", *Ann. Sci. École Norm. Sup.* (4) **7** (1974), 53–88. MR Zbl

[Demazure 1976] M. Demazure, "A very simple proof of Bott's theorem", *Invent. Math.* **33**:3 (1976), 271–272. MR Zbl

[Dewji et al. 2017] R. Dewji, I. Dimitrov, A. McCabe, M. Roth, D. Wehlau, and J. Wilson, "Decomposing inversion sets of permutations and applications to faces of the Littlewood–Richardson cone", *J. Algebraic Combin.* **45**:4 (2017), 1173–1216. MR

[Dimitrov and Roth 2009] I. Dimitrov and M. Roth, "Geometric realization of PRV components and the Littlewood–Richardson cone", pp. 83–95 in *Symmetry in mathematics and physics*, edited by D. Babbitt et al., Contemp. Math. **490**, Amer. Math. Soc., Providence, RI, 2009. MR Zbl

[Dimitrov and Roth 2017] I. Dimitrov and M. Roth, "Positive systems of Kostant roots", *Algebr. Represent. Theory* (online publication April 2017).

[Dimitrov and Roth ≥ 2017] I. Dimitrov and M. Roth, "Regular faces of the Littlewood–Richardson cone and intersection multiplicity one for classical groups", in preparation.

[Grieve 2014] N. Grieve, "Index conditions and cup-product maps on Abelian varieties", *Internat. J. Math.* **25**:4 (2014), 1450036, 31. MR Zbl

[Hansen 1973] H. C. Hansen, "On cycles in flag manifolds", *Math. Scand.* **33** (1973), 269–274. MR Zbl

[Kapovich et al. 2009] M. Kapovich, B. Leeb, and J. Millson, "Convex functions on symmetric spaces, side lengths of polygons and the stability inequalities for weighted configurations at infinity", *J. Differential Geom.* **81**:2 (2009), 297–354. MR Zbl

[Kempf 1976] G. R. Kempf, "Vanishing theorems for flag manifolds", *Amer. J. Math.* **98**:2 (1976), 325–331. MR Zbl

[Kleiman 1974] S. L. Kleiman, "The transversality of a general translate", *Compositio Math.* **28** (1974), 287–297. MR Zbl

[Klyachko 1998] A. A. Klyachko, "Stable bundles, representation theory and Hermitian operators", *Selecta Math.* (*N.S.*) **4**:3 (1998), 419–445. MR Zbl

[Knutson et al. 2004] A. Knutson, T. Tao, and C. Woodward, "The honeycomb model of $\mathrm{GL}_n(\mathbb{C})$ tensor products, II: Puzzles determine facets of the Littlewood–Richardson cone", *J. Amer. Math. Soc.* **17**:1 (2004), 19–48. MR Zbl

[Kostant 1961] B. Kostant, "Lie algebra cohomology and the generalized Borel–Weil theorem", *Ann. of Math.* (2) **74** (1961), 329–387. MR Zbl

[Kostant 1963] B. Kostant, "Lie group representations on polynomial rings", *Amer. J. Math.* **85** (1963), 327–404. MR Zbl

[Kostant 2010] B. Kostant, "Root systems for Levi factors and Borel–de Siebenthal theory", pp. 129–152 in *Symmetry and spaces*, edited by H. E. A. Campbell et al., Progr. Math. **278**, Birkhäuser, Boston, 2010. MR Zbl

[Kumar 1988] S. Kumar, "Proof of the Parthasarathy–Ranga Rao–Varadarajan conjecture", *Invent. Math.* **93**:1 (1988), 117–130. MR Zbl

[Luna 1973] D. Luna, "Slices étales", pp. 81–105 in *Sur les groupes algébriques*, Mém. Soc. Math. France **33**, Soc. Math. France, Paris, 1973. MR Zbl

[Luna 1975] D. Luna, "Adhérences d'orbite et invariants", *Invent. Math.* **29**:3 (1975), 231–238. MR Zbl

[Mathieu 1989] O. Mathieu, "Filtrations of B-modules", *Duke Math. J.* **59**:2 (1989), 421–442. MR Zbl

[Mumford et al. 1994] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, 3rd ed., Ergebnisse der Mathematik (2) **34**, Springer, Berlin, 1994. MR Zbl

[Parthasarathy et al. 1967] K. R. Parthasarathy, R. Ranga Rao, and V. S. Varadarajan, "Representations of complex semi-simple Lie groups and Lie algebras", *Ann. of Math.* (2) **85** (1967), 383–429. MR Zbl

[Ramanathan 1985] A. Ramanathan, "Schubert varieties are arithmetically Cohen–Macaulay", *Invent. Math.* **80**:2 (1985), 283–294. MR Zbl

[Ressayre 2009] N. Ressayre, "Eigencones and the PRV conjecture", preprint, 2009. arXiv

[Roth 2011] M. Roth, "Reduction rules for Littlewood–Richardson coefficients", *Int. Math. Res. Not.* **2011**:18 (2011), 4105–4134. MR Zbl

[Seshadri 1987] C. S. Seshadri, "Line bundles on Schubert varieties", pp. 499–528 in *Vector bundles on algebraic varieties* (Bombay, 1984), Tata Inst. Fund. Res. Stud. Math. **11**, Tata Inst. Fund. Res., Bombay, 1987. MR Zbl

[Tsanov 2013] V. V. Tsanov, "Embeddings of semisimple complex Lie groups and cohomological components of modules", *J. Algebra* **373** (2013), 1–29. MR Zbl

Communicated by Georgia Benkart
Received 2013-01-28 Revised 2016-12-11 Accepted 2017-01-13

dimitrov@mast.queensu.ca *Department of Mathematics and Statistics, Queen's University, Jeffery Hall, Kingston ON K7L 3N6, Canada*

mike.roth@queensu.ca *Department of Mathematics and Statistics, Queen's University, Jeffery Hall, Kingston ON K7L 3N6, Canada*

ALGEBRA AND NUMBER THEORY 11:4 (2017)
dx.doi.org/10.2140/ant.2017.11.817

Mass formulas for local Galois representations and quotient singularities II: Dualities and resolution of singularities

Melanie Matchett Wood and Takehiko Yasuda

A total mass is the weighted count of continuous homomorphisms from the absolute Galois group of a local field to a finite group. In the preceding paper, the authors observed that in a particular example two total masses coming from two different weightings are dual to each other. We discuss the problem of how generally such a duality holds and relate it to the existence of simultaneous resolution of singularities, using the wild McKay correspondence and the Poincaré duality for stringy invariants. We also exhibit several examples.

A correction was submitted on 1 Apr 2019 and posted online on 28 May 2019.

1. Introduction

In [Wood and Yasuda 2015] we found a close relationship between mass formulas for local Galois representations, studied in the number theory, and stringy invariants of singular varieties. In the present paper we discuss dualities of mass formulas in relation with the existence of some kinds of desingularization.

For a local field K and a finite group Γ, let $G_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$ be the absolute Galois group of K and $S_{K,\Gamma}$ be the set of continuous homomorphisms $G_K \to \Gamma$. For a function $c : S_{K,\Gamma} \to \mathbb{R}$, the *total mass* of (K, Γ, c) is defined as

$$M(K, \Gamma, c) := \frac{1}{\sharp\Gamma} \sum_{\rho \in S_{K,\Gamma}} q^{-c(\rho)},$$

with q the cardinality of the residue field of K. For the symmetric group S_n, fixing the standard representation

$$\iota : S_n \hookrightarrow \mathrm{GL}_n(\mathcal{O}_K) \subset \mathrm{GL}_n(K),$$

with $\mathcal{O}_K$ the integer ring of K, we can associate the Artin conductor $\boldsymbol{a}_\iota(\rho)$ to each continuous homomorphism $\rho : G_K \to S_n$. According to Kedlaya [2007], Bhargava's

MSC2010: primary 11S15; secondary 11G25, 14E15, 14E16.
Keywords: mass formulas, local Galois representations, quotient singularities, dualities, the McKay correspondence, equisingularities, stringy invariants.

mass formula [2007] for étale extensions of a local field is expressed as

$$M(K, S_n, \boldsymbol{a}_\iota) = \sum_{m=0}^{n-1} P(n, n-m) q^{-m},$$

where $P(n, n-m)$ is the number of partitions of the integer n into exactly $n-m$ parts. It was found in [Wood and Yasuda 2015] that for another function $\rho \mapsto \boldsymbol{w}_{2\iota}(\rho)$ originating in the wild McKay correspondence [Yasuda 2013] we have

$$M(K, S_n, -\boldsymbol{w}_{2\iota}) = \sum_{m=0}^{n-1} P(n, n-m) q^{m}.$$

Here 2ι stands for the direct sum of two copies of the representation ι. The right-hand sides of the two formulas are interchanged by replacing q with q^{-1}: this is what we call a *duality*. It is then natural to ask how generally the duality holds: what about other groups and other representations? There exists yet another function $\boldsymbol{v}_\tau$ on $S_{K,\Gamma}$ for each representation τ of Γ over $\mathcal{O}_K$. It was shown in [Wood and Yasuda 2015] that if τ is a permutation representation, then we have $\boldsymbol{a}_\tau = \boldsymbol{v}_{2\tau}$. It turns out that the function $\boldsymbol{v}_\tau$ is more appropriate when discussing dualities for nonpermutation representations. The most basic question we would like to ask is: *when are the total masses $M(K, \Gamma, \boldsymbol{v}_\tau)$ and $M(K, \Gamma, -\boldsymbol{w}_\tau)$ dual to each other in the same way as $M(K, S_n, \boldsymbol{a}_\iota)$ and $M(K, S_n, -\boldsymbol{w}_{2\iota})$*? We will observe that the duality does not always hold, but is closely related to the existence of a *simultaneous resolution of singularities* or to *equisingularities*.

To discuss the duality more rigorously, we need to consider total masses for all unramified extensions of the given local field K. For each integer $r > 0$, let K_r be the unramified extension of K of degree r. For a representation $\tau : \Gamma \to \mathrm{GL}_d(\mathcal{O}_K)$, we can naturally generalize functions $\boldsymbol{v}_\tau$ and $\boldsymbol{w}_\tau$ to ones on $S_{K_r,\Gamma}$, which we continue to denote by the same symbols. We then regard total masses $M(K_r, \Gamma, \boldsymbol{v}_\tau)$ and $M(K_r, \Gamma, -\boldsymbol{w}_\tau)$ as functions in the variable $r \in \mathbb{N}$. When they belong to a certain class of nice functions (which we call *admissible*), we can define their dual functions $\mathbb{D}(M(K_r, \Gamma, \boldsymbol{v}_\tau))$ and $\mathbb{D}(M(K_r, \Gamma, -\boldsymbol{w}_\tau))$ in such a way that the dual of the function q^r is q^{-r}. A duality which we are interested in is now expressed by the equality

$$\mathbb{D}(M(K_r, \Gamma, \boldsymbol{v}_\tau)) = M(K_r, \Gamma, -\boldsymbol{w}_\tau). \tag{1-1}$$

We call it the *strong duality*.

It is the *wild McKay correspondence* which relates total masses to singularities. Given a representation $\tau : \Gamma \to \mathrm{GL}_d(\mathcal{O}_K)$, we have the associated Γ-action on the affine space $\mathbb{A}^d_{\mathcal{O}_K}$ and the quotient scheme $X := \mathbb{A}^d_{\mathcal{O}_K}/\Gamma$, which is a normal $\mathbb{Q}$-Gorenstein variety over $\mathcal{O}_K$. In general, for a normal $\mathbb{Q}$-Gorenstein variety Y over $\mathcal{O}_K$, we can define the *stringy point count* $\sharp_{\mathrm{st}}(Y)$ as a volume of the $\mathcal{O}_K$-point set $Y(\mathcal{O}_K)$ (see [Yasuda 2014b]). This is an analogue of the stringy E-function defined in [Batyrev and Dais 1996; Batyrev 1998] and is a generalization of the number of k-points on a smooth $\mathcal{O}_K$-variety. For a k-point $y \in Y(k)$, we can similarly define the *stringy point count along* $\{y\}$ (*or the stringy weight of* y), denoted by $\sharp_{\mathrm{st}}(Y)_y$, so that we have

$$\sharp_{\mathrm{st}}(Y) = \sum_{y \in Y(k)} \sharp_{\mathrm{st}}(Y)_y.$$

Yasuda [2014b] proved that if the representation τ is faithful and the morphism $\mathbb{A}^d_{\mathcal{O}_K} \to X$ is étale in codimension one, then

$$\sharp_{\mathrm{st}}(X) = M(K, \Gamma, \boldsymbol{v}_\tau) q^d \quad \text{and} \quad \sharp_{\mathrm{st}}(X)_o = M(K, \Gamma, -\boldsymbol{w}_\tau),$$

where o is the origin of $X(k)$. This is a version of the wild McKay correspondence discussed in [Yasuda 2013; 2014a; 2016; Wood and Yasuda 2015]. Special cases were previously proved in [Yasuda 2014a; Wood and Yasuda 2015]. If X has a nice resolution, then $\sharp_{\mathrm{st}}(X)$ and $\sharp_{\mathrm{st}}(X)_o$ are explicitly computed in terms of resolution data. Using it, we can deduce a few properties of the functions $M(K_r, \Gamma, \boldsymbol{v}_\tau)$ and $M(K_r, \Gamma, -\boldsymbol{w}_\tau)$: obtained properties depend on what sort of resolution exists.

Remark 1.1. The assumption that the morphism $\mathbb{A}^d_{\mathcal{O}_K} \to X$ is étale in codimension one is just for notational simplicity. We may drop this assumption by considering the pair of X and a $\mathbb{Q}$-divisor on it rather than the variety X itself.

Let us think of X as a family of singular varieties over $\operatorname{Spec} \mathcal{O}_K$. If X admits a kind of simultaneous resolution, then

$$\frac{\sharp_{\mathrm{st}}(X) - \sharp_{\mathrm{st}}(X)_o}{q^r - 1}$$

satisfies a certain self-duality, which can be understood as the Poincaré duality of stringy invariants proved in [Batyrev and Dais 1996; Batyrev 1998] over complex numbers. Using the wild McKay correspondence, we can transform this duality into the form:

$$\begin{aligned} M(K_r, \Gamma, \boldsymbol{v}_\tau) \cdot q^{rd} - M(K_r, \Gamma, -\boldsymbol{w}_\tau) \\ = \mathbb{D}(M(K_r, \Gamma, -\boldsymbol{w}_\tau)) \cdot q^{rd} - \mathbb{D}(M(K_r, \Gamma, \boldsymbol{v}_\tau)). \quad (1\text{-}2) \end{aligned}$$

We call it the *weak duality* between $M(K_r, \Gamma, \boldsymbol{v}_\tau)$ and $M(K_r, \Gamma, -\boldsymbol{w}_\tau)$, which is indeed weaker than the strong duality (1-1).

There are three possibilities: both dualities hold, both dualities fail, and the weak one holds but the strong one does not. Examples we compute show that each possibility indeed occurs. In the tame case, the strong duality holds. For permutation representations, the weak duality holds in all examples we compute, but the strong duality does not generally hold. When $\mathcal{O}_K = \mathbb{F}_q[\![t]\!]$ and the given representation τ

is defined over $\mathbb{F}_q$, the strong duality holds in all computed examples. However, when the representation is not defined over $\mathbb{F}_q$, then the weak duality tends to fail. These examples suggest that we may think of the two dualities as a test of how equisingular the family X is.

In particular, when $\mathcal{O}_K = \mathbb{F}_q[\![t]\!]$ and τ is defined over $\mathbb{F}_q$, the associated quotient scheme $X = \mathbb{A}^d_{\mathbb{F}_q[\![t]\!]}$ would be equisingular in any reasonable sense. If there exists a nice resolution of the $\mathbb{F}_q$-variety $X \otimes_{\mathcal{O}_K} \mathbb{F}_q$, we obtain an equally nice simultaneous resolution of X just by base change. Therefore, if one believes that there exists as nice a resolution in positive characteristic as in characteristic zero, then at least the weak duality (1-2) would hold for such τ. If, on the contrary, one finds an example such that weak duality fails, this would give an example of an $\mathbb{F}_q$-variety not admitting any nice resolution. However, every example we have computed satisfies strong duality.

The study in this paper thus gives rise to interesting open problems related to dualities and equisingularities and their interaction (see Questions 4.7, 4.8 and 5.2, and Section 6). Further studies are required.

The paper is organized as follows. In Section 2, we define admissible functions and their duals. In Section 3, we discuss properties of stringy point counts. In Section 4, we define total masses and discuss how the wild McKay correspondence relates dualities of total masses with equisingularities. In Section 5, we exhibit several examples. Section 6 contains concluding remarks.

Convention and notation. We denote the set of positive integers by $\mathbb{N}$. A *local field* means a finite extension of either $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$ for a prime number p. For a local field K, we denote its integer ring by $\mathcal{O}_K$ and its residue field by k. We denote the characteristic of k by p and the cardinality of k by q. For $r \in \mathbb{N}$, we denote the unramified extension of K of degree r by K_r and its residue field (that is, the extension of k of degree r) by k_r. For a variety X defined over either $\mathcal{O}_K$ or k, we put

$$\sharp X := \sharp X(k),$$

the cardinality of k-points of X. More generally, for $r \in \mathbb{N}$, we put

$$\sharp^r X := \sharp X(k_r).$$

2. Admissible functions and their duals

In this section, we set the foundation to discuss dualities of total masses and stringy point counts by introducing *admissible functions.*

Definition 2.1. We call a function $f : \mathbb{N} \to \mathbb{C}$ *admissible* if there exist numbers $0 \neq c \in \mathbb{Q}$, $n_i \in \mathbb{Z}$ and $\alpha_i \in \mathbb{C}$ $(1 \le i \le m)$ such that for every $r \in \mathbb{N}$,

$$f(r) = \frac{1}{q^{cr}-1} \sum_{i=1}^{l} n_i \alpha_i^r .$$

We denote the set of admissible functions by AF. For an admissible function $f(r)$ as above, we define its *dual function* $\mathbb{D} f : \mathbb{N} \to \mathbb{C}$ by

$$(\mathbb{D} f)(r) := \frac{1}{q^{-cr}-1} \sum_{i=1}^{l} n_i \alpha_i^{-r} .$$

It is easy to see that the set of admissible functions is closed under addition and multiplication, thus AF has a natural ring structure. For $f \in \mathrm{AF}$, the dual $\mathbb{D} f$ is also an admissible function. Therefore $\mathbb{D}$ gives an involution of the set AF. Moreover it is easily checked that $\mathbb{D} : \mathrm{AF} \to \mathrm{AF}$ is a ring isomorphism.

A typical admissible function is a rational function in $q^{1/n}$ for some $n \in \mathbb{N}$ with a denominator of the form $q^{cr} - 1$. In that case, the dual function is obtained just by substituting q^{-1} for q. Another example is given as follows: let $k = \mathbb{F}_q$ and $k_r := \mathbb{F}_{q^r}$. For a k-variety X the function

$$r \mapsto \sharp^r X := \sharp X(k_r)$$

is admissible from the Grothendieck–Lefschetz trace formula.

Lemma 2.2. *The dual function $\mathbb{D} f$ does not depend on the choice of the expression of $f(r)$ as $1/(q^{cr}-1)\sum_{i=1}^{l} n_i \alpha_i^r$.*

Proof. Let

$$\frac{1}{q^{dr}-1} \sum_{j=1}^{l'} m_j \beta_j^r$$

be another such expression of $f(r)$. For every $r \in \mathbb{N}$, we have

$$(q^{dr}-1) \cdot \sum_{i=1}^{l} n_i \alpha_i^r = (q^{cr}-1) \cdot \sum_{j=1}^{l'} m_j \beta_j^r .$$

It suffices to show that for every $r \in \mathbb{N}$,

$$(q^{-dr}-1) \cdot \sum_{i=1}^{l} n_i \alpha_i^{-r} = (q^{-cr}-1) \cdot \sum_{j=1}^{l'} m_j \beta_j^{-r} .$$

This follows from the following claim:

Claim 2.3. *Suppose that for every r, we have*

$$\sum_{i=1}^{l} n_i \alpha_i^r = \sum_{j=1}^{l'} m_j \beta_j^r.$$

Suppose also that α_i, n_i, β_j and m_j are nonzero, that $\alpha_1, \dots, \alpha_l$ are distinct, and so are $\beta_1, \dots, \beta_{l'}$. Then we have $l = l'$, $\alpha_i = \beta_i$ and $n_i = m_i$, up to permutation.

In turn, this claim follows from:

Claim 2.4. *Suppose that $\alpha_1, \dots, \alpha_l$ are distinct nonzero complex numbers and that for every r,*

$$\sum_{i=1}^{l} n_i \alpha_i^r = 0.$$

Then $n_i = 0$ for every i.

The last claim follows from the regularity of the Vandermonde matrix

$$(\alpha_i^r)_{1 \le i \le l,\, 0 \le r \le l-1}. \qquad \square$$

For a smooth proper k-variety X of pure dimension d, the Poincaré duality for the l-adic cohomology shows that the function $r \mapsto \sharp^r X$ satisfies

$$\sharp^r X = q^{dr} \cdot \mathbb{D}(\sharp^r X).$$

3. Stringy point counts

Stringy point counts. Let K be a local field, that is, a finite extension of either $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$. We denote its residue field by k and its integer ring by $\mathcal{O}_K$. An *$\mathcal{O}_K$-variety* means an integral separated flat $\mathcal{O}_K$-scheme of finite type such that there exists an open dense subscheme $U \subset X$ which is smooth over $\mathcal{O}_K$. For an $\mathcal{O}_K$-variety X, we denote by X_k the closed fiber $X \otimes_{\mathcal{O}_K} k$ and by X_K the generic fiber $X \otimes_{\mathcal{O}_K} K$. For a constructible subset $C \subset X_k$, we let $X(\mathcal{O}_K)_C$ be the set of $\mathcal{O}_K$-points $\operatorname{Spec} \mathcal{O}_K \to X$ sending the closed point into C.

Suppose now that X is normal. From [Kollár 2013, Definition 1.5], X has a canonical sheaf $\omega_{X/\mathcal{O}_K}$. If d is the relative dimension of X over $\mathcal{O}_K$, then $\omega_{X/\mathcal{O}_K}$ coincides with $\Omega^d_{X/\mathcal{O}_K} = \bigwedge^d \Omega_{X/\mathcal{O}_K}$ on the $\mathcal{O}_K$-smooth locus of X. The canonical divisor $K_X = K_{X/\mathcal{O}_K}$ is defined as a divisor such that $\omega_{X/\mathcal{O}_K} = \mathcal{O}_X(K_X)$, which is determined up to linear equivalence. Let us suppose also that X is $\mathbb{Q}$-Gorenstein, that is, K_X is $\mathbb{Q}$-Cartier. Then, for a constructible subset $C \subset X_k$, we can define the *stringy point count along C*,

$$\sharp_{\mathrm{st}}(X)_C \in \mathbb{R}_{\ge 0} \cup \{\infty\},$$

as a certain p-adic volume of $X(\mathcal{O}_K)_C$. For details, see [Yasuda 2014b]. However, what is important for our purpose is not the definition but the explicit formula in the next subsection.

For $r \in \mathbb{N}$, let K_r be the unramified extension of K of degree r and let k_r be its residue field. We then define $\sharp^r_{\mathrm{st}}(X)_C$ to be the stringy point count $X \otimes_{\mathcal{O}_K} \mathcal{O}_{K_r}$ along the preimage of C in $(X \otimes_{\mathcal{O}_K} \mathcal{O}_{K_r}) \otimes_{\mathcal{O}_{K_r}} k_r = X \otimes_{\mathcal{O}_K} k_r$. We often regard $\sharp^r_{\mathrm{st}}(X)_C$ as a function in r. When X is smooth over $\mathcal{O}_K$, then $\sharp^r_{\mathrm{st}}(X)_C$ is equal to the number of k_r-points of X contained in C.

An explicit formula and the Poincaré duality. Let X be a normal $\mathbb{Q}$-Gorenstein variety over either $\mathcal{O}_K$ or k. For a proper birational morphism $f : Y \to X$ with Y normal, the relative canonical divisor

$$K_{Y/X} := K_Y - f^* K_X$$

is uniquely determined as a $\mathbb{Q}$-divisor on Y supported in the exceptional locus of f. We are especially interested in the case where Y is regular and $K_{Y/X}$ is a *simple normal crossing divisor*, a notion that we define on regular $\mathcal{O}_K$-varieties as follows.

Definition 3.1. Let Y be a regular variety over $\mathcal{O}_K$ and $E \subset Y$ a reduced closed subscheme of pure codimension one. We call E a *simple normal crossing divisor* if

- for every irreducible component E_0 of E and for every closed point $x \in E_0$ where Y is smooth over $\mathcal{O}_K$, the completion of E_0 at x is irreducible, and
- for every closed point $x \in Y$ where Y is smooth over $\mathcal{O}_K$, the support of E is, in a certain Zariski neighborhood, defined by a product $y_{i_1} \cdots y_{i_m}$ for a regular system of parameters $y_0 = \varpi, y_1, \dots, y_d \in \widehat{\mathcal{O}}_{Y,y}$ with ϖ a uniformizer of K and $0 \le i_1 < \cdots < i_m \le d$.

We call a $\mathbb{Q}$-divisor on Y a *simple normal crossing divisor* if its support is one.

The reason why we look only at closed points where Y is $\mathcal{O}_K$-smooth is that there is no $\mathcal{O}_K$-point passing through a closed point where Y is not $\mathcal{O}_K$-smooth, and hence such a point is irrelevant to stringy point counts.

Definition 3.2. Let $f : Y \to X$ be a proper birational morphism of varieties over either $\mathcal{O}_K$ or k such that X is normal and $\mathbb{Q}$-Gorenstein:

- We call f a *resolution* if Y is regular.
- We call f a *WL* (*weak log*) *resolution* if f is a resolution and the support of $K_{Y/X}$ is simple normal crossing.
- When f is defined over $\mathcal{O}_K$, we call f an *SWL* (*simultaneous weak log*) *resolution* if f is a WL resolution and Y is smooth over $\mathcal{O}_K$ or k.

Remark 3.3. The word *weak* indicates that we look only at the support of $K_{Y/X}$ rather than the whole exceptional locus. In particular, every resolution $Y \to X$ with $K_{Y/X} = 0$ is a WL resolution, whatever the exceptional locus is.

Remark 3.4. We cannot usually expect that an SWL resolution exists, unless X is thought of as an *equisingular* family over $\mathcal{O}_K$ in some sense. If $\mathcal{O}_K = k[\![t]\!]$ and if $X = X_0 \otimes_k k[\![t]\!]$ for some k-variety X_0, then X would be equisingular in any reasonable sense. In this situation, an SWL resolution of X exists if and only if a WL resolution of X_0 exists.

For a normal $\mathbb{Q}$-Gorenstein $\mathcal{O}_K$-variety X and a WL resolution $f : Y \to X$, we can uniquely decompose $K_{Y/X}$ as

$$K_{Y/X} = \sum_{h=1}^{l} a_h A_h + \sum_{i=1}^{m} b_i B_i + \sum_{j=1}^{n} c_j C_j, \tag{3-1}$$

where all the A_h, B_i, C_j are prime divisors on Y, the coefficients a_h, b_i, c_j are nonzero rational numbers, every A_h is contained in X_k and Y is $\mathcal{O}_K$-smooth at the generic point of A_h, every B_i is contained in X_k and Y is *not* $\mathcal{O}_K$-smooth at the generic point of B_i, and every C_j dominates $\operatorname{Spec} \mathcal{O}_K$. We denote by A_h° the locus in A_h where Y is $\mathcal{O}_K$-smooth. For a subset $J \subset \{1, \dots, n\}$, we define

$$C_J^\circ := \bigcap_{j \in J} C_j \Big\backslash \bigcup_{j \notin J} C_j.$$

For a constructible subset W of a k-variety Z, let $\sharp^r(W)$ denote the number of k_r-points of W.

Proposition 3.5 (An explicit formula, [Yasuda 2014b]). *Let $f : Y \to X$ be a WL resolution of a normal $\mathbb{Q}$-Gorenstein $\mathcal{O}_K$-variety X and write $K_{Y/X}$ as in (3-1). Let Y_{sm} be the $\mathcal{O}_K$-smooth locus of Y:*

(1) *We have $\sharp^r_{\mathrm{st}}(X)_C < \infty$ for all $r \in \mathbb{N}$ if and only if $c_j > -1$ for every j such that $C_j \cap f^{-1}(C) \cap Y_{\mathrm{sm}} \neq \varnothing$.*

(2) *If the two equivalent conditions of (1) hold, then*

$$\sharp^r_{\mathrm{st}}(X)_C = \sum_{h=1}^{l} q^{-r a_h} \sum_{J \subset \{1,\dots,n\}} \sharp^r(f^{-1}(C) \cap A_h^\circ \cap C_J^\circ) \prod_{j \in J} \frac{q^r - 1}{q^{r(1+c_j)} - 1}.$$

Proof. In [Yasuda 2014b] it was proved that $\sharp_{\mathrm{st}}(X)_C$ is equal to $\sharp_{\mathrm{st}}(Y, -K_{Y/X})_{f^{-1}(C)}$, the stringy point count of the pair $(Y, -K_{Y/X})$ along $f^{-1}(C)$. The proposition follows from the explicit formula for the stringy point count of a pair with a simple normal crossing divisor in the same paper. $\square$

In particular, if K has characteristic zero and if the generic fiber X_K has only log terminal singularities (for instance, quotient singularities), then $\sharp^r(X)_C < \infty$ for every r and C. The following is a direct consequence of the proposition.

Corollary 3.6. *Suppose that X is $\mathbb{Q}$-Gorenstein, that there exists a WL resolution $f : Y \to X$ and that $\sharp^r_{\mathrm{st}}(X)_C < \infty$ for every $r > 0$. Then the function $\sharp^r_{\mathrm{st}}(X)_C$ in the variable r is admissible.*

The following result was proved by Batyrev–Dais [1996] and Batyrev [1998] in a slightly different setting.

Corollary 3.7 (The Poincaré duality). *Let X be a d-dimensional proper normal $\mathbb{Q}$-Gorenstein $\mathcal{O}_K$-variety. Suppose that there exists an SWL resolution $f : Y \to X$ and that $\sharp^r_{\mathrm{st}}(X) < \infty$ for every $r \in \mathbb{N}$. We then have*

$$\sharp^r_{\mathrm{st}}(X) = \mathbb{D}(\sharp^r_{\mathrm{st}}(X)) \cdot q^{dr}.$$

Proof. We follow arguments in the proof of [Batyrev 1998, Theorem 3.7]. Since f is an SWL resolution, if we write $K_{Y/X}$ as in (3-1) then the terms $\sum a_h A_h$ and $\sum b_i B_i$ do not appear. Therefore Proposition 3.5 reads

$$\sharp^r_{\mathrm{st}}(X) = \sum_{J \subset \{1,\dots,n\}} \sharp^r(C_J^\circ) \prod_{j \in J} \frac{q^r - 1}{q^{r(1+c_j)} - 1}.$$

Putting

$$C_J := \bigcap_{j \in J} C_j,$$

we have $C_J := \bigsqcup_{J' \supset J} C^\circ_{J'}$ and $\sharp^r(C_J) = \sum_{J' \supset J} \sharp^r(C^\circ_{J'})$. From this and the inclusion-exclusion principle (or the Möbius inversion formula see, for instance, [Stanley 1997]), we deduce

$$\sharp^r(C_J^\circ) = \sum_{J' \supset J} (-1)^{\sharp J' - \sharp J} \sharp^r(C_{J'}).$$

Hence

$$\begin{aligned}
\sharp^r_{\mathrm{st}}(X) &= \sum_{J \subset \{1,\dots,n\}} \Bigl(\sum_{J' \supset J} (-1)^{\sharp J' - \sharp J} \sharp^r(C_{J'}) \Bigr) \prod_{j \in J} \frac{q^r - 1}{q^{r(1+c_j)} - 1} \\
&= \sum_{J \subset \{1,\dots,n\}} \sharp^r(C_J) \prod_{j \in J} \Bigl(\frac{q^r - 1}{q^{r(1+c_j)} - 1} - 1 \Bigr) \\
&= \sum_{J \subset \{1,\dots,n\}} \sharp^r(C_J) \prod_{j \in J} \Bigl(\frac{q^r - q^{r(1+c_j)}}{q^{r(1+c_j)} - 1} \Bigr).
\end{aligned}$$

Now we have

$$\mathbb{D}\Bigl(\frac{q^r - q^{r(1+c_j)}}{q^{r(1+c_j)} - 1} \Bigr) = \frac{q^{-r} - q^{r(-1-c_j)}}{q^{r(-1-c_j)} - 1} = \frac{q^r - q^{r(1+c_j)}}{q^{r(1+c_j)} - 1} \cdot q^{-r}.$$

Since $\sharp^r(C_J) = \sharp^r(C_J \otimes_K k)$ and $C_J \otimes_K k$ are proper smooth k-varieties of pure dimension $d - \sharp J$, the Poincaré duality for the l-adic cohomology gives

$$\mathbb{D}(\sharp^r(C_J)) = \sharp^r(C_J) \cdot q^{r(\sharp J - d)}.$$

Since $\mathbb{D} : \mathrm{AF} \to \mathrm{AF}$ is a ring homomorphism, the corollary follows. □

Set-theoretically free $\mathbb{G}_m$-actions.

Definition 3.8. An action of the group scheme $\mathbb{G}_m = \mathbb{G}_{m,k} = \operatorname{Spec} k[t, t^{-1}]$ on a k-variety Z is said to be *set-theoretically free* if for every field extension k'/k and every point $x \in Z(k')$, the stabilizer subgroup scheme $\mathrm{Stab}(x) \subset \mathbb{G}_{m,k'}$ consists of a single point.

A set-theoretically free $\mathbb{G}_m$-action naturally appears as the $\mathbb{G}_m$-action on the quotient variety $(\mathbb{A}^d_k/\Gamma) \setminus \{o\}$ with the origin o removed for a linear action of a finite group Γ on an affine space $\mathbb{A}^d_k$. In general, the $\mathbb{G}_m$-action may have nonreduced stabilizer subgroups and not be free. See [Yasuda 2014a] for such an example. If X is the quotient variety $Z/\mathbb{G}_m$ for a set-theoretically free $\mathbb{G}_m$-action on a smooth variety Z and if X is proper, then the function $\sharp^r X$ satisfies the same Poincaré duality as a smooth proper variety does. Note that X is not necessarily smooth because of nonreduced stabilizers. To prove the duality, we need to use Artin stacks.

Based on [Olsson 2015; Laszlo and Olsson 2008a; 2008b], Sun [2012] defined étale cohomology groups $H^i(\mathcal{X} \otimes_k \bar{k}, \overline{\mathbb{Q}}_l)$ and $H^i_c(\mathcal{X} \otimes_k \bar{k}, \overline{\mathbb{Q}}_l)$ for Artin stacks $\mathcal{X}$ of finite type over k and a prime number $l \neq p$. For $r \in \mathbb{N}$, we define

$$\sharp^r(\mathcal{X}) := \sum_i (-1)^i \mathrm{Tr}(F^r | H^i_c(\mathcal{X} \otimes_k \bar{k}, \overline{\mathbb{Q}}_l))$$

with F^r the r-iterated Frobenius action. This is a generalization of $\sharp^r X = \sharp X(k_r)$ for a k-variety X.

Proposition 3.9. *Let $\mathcal{X}$ be an Artin stack of finite type over k with finite diagonal, and $\overline{\mathcal{X}}$ its coarse moduli space. Then we have*

$$\sharp^r(\mathcal{X}) = \sharp^r(\overline{\mathcal{X}}).$$

Moreover, if $\mathcal{X}$ is smooth and proper of pure dimension d over k, then

$$\sharp^r(\mathcal{X}) = \mathbb{D}(\sharp^r(\mathcal{X})) \cdot q^{rd}.$$

Proof. In the proof of [Sun 2012, Proposition 7.3.2], Sun proved that

$$H^i_c(\mathcal{X} \otimes_k \bar{k}, \overline{\mathbb{Q}}_l) = H^i_c(\overline{\mathcal{X}} \otimes_k \bar{k}, \overline{\mathbb{Q}}_l)$$

and that if $\mathcal{X}$ is smooth and proper, then we have the Poincaré duality,

$$H^i(\mathcal{X} \otimes_k \bar{k}, \overline{\mathbb{Q}}_l)^\vee = H^{2d-i}(\mathcal{X} \otimes_k \bar{k}, \overline{\mathbb{Q}}_l) \otimes \overline{\mathbb{Q}}_l(d).$$

The proposition is a direct consequence of these results. □

Let Z be a k-variety endowed with a set-theoretically free $\mathbb{G}_m$-action. Then the quotient stack $\mathcal{W} := [Z/\mathbb{G}_m]$ is an Artin stack with finite diagonal and hence admits a coarse moduli space, which we write as $W = Z/\mathbb{G}_m$.

Proposition 3.10. *We have*

$$\sharp^r(\mathcal{W}) = \sharp^r(W) = \frac{\sharp^r(Z)}{q^r - 1}.$$

Proof. The left equality was proved in the last proposition. We show the right equality. For every field extension k'/k and every point $x \in Z(k')$, we have

$$\mathrm{Stab}(x) = \operatorname{Spec} k'[t, t^{-1}]/(t^{p^a} - 1)$$

for some nonnegative integer a. Let $U \subset Z$ be the locus where this number a takes the minimum value, and let $\overline{U} \subset W$ be its image. The algebraic spaces U and $\overline{U}$ are open dense subspaces of Z and W respectively. If we put

$$H := \operatorname{Spec} k[t, t^{-1}]/(t^{p^a} - 1),$$

then the given set-theoretically free $\mathbb{G}_m$-action on U induces a free action of the quotient group scheme $\mathbb{G}_m/H$, which is isomorphic to $\mathbb{G}_m$, on U. This action makes the projection $U \to \overline{U}$ a $\mathbb{G}_m$-torsor. From Hilbert's Theorem 90 [Milne 1980, p. 124], every $\mathbb{G}_m$-torsor is Zariski locally trivial, hence

$$\sharp^r(U) = (q^r - 1) \cdot \sharp^r(\overline{U}).$$

It is now easy to show $\sharp^r(Z) = (q^r - 1) \cdot \sharp^r(W)$ by induction. □

Definition 3.11. Let X be an $\mathcal{O}_K$-variety with a $\mathbb{G}_{m,\mathcal{O}_K}$-action. We suppose that the induced $\mathbb{G}_{m,k}$-action on X_k is set-theoretically free. For the quotient stack $[X/\mathbb{G}_{m,\mathcal{O}_K}]$, we put

$$\sharp^r_{\mathrm{st}}([X/\mathbb{G}_{m,\mathcal{O}_K}]) := \frac{\sharp^r_{\mathrm{st}}(X)}{q^r - 1}.$$

Definition 3.12. For a normal $\mathbb{Q}$-Gorenstein $\mathcal{O}_K$-variety X endowed with a $\mathbb{G}_{m,\mathcal{O}_K}$-action, we call a resolution $f : Y \to X$ an *ESWL* (*equivariant simultaneous weak log*) *resolution* if f is an SWL resolution and $\mathbb{G}_{m,\mathcal{O}_K}$-equivariant (that is, Y also has a $\mathbb{G}_{m,\mathcal{O}_K}$-action so that f is $\mathbb{G}_{m,\mathcal{O}_K}$-equivariant).

Proposition 3.13. *Let X be a normal $\mathbb{Q}$-Gorenstein $\mathcal{O}_K$-variety X endowed with a $\mathbb{G}_{m,\mathcal{O}_K}$-action. Suppose that the induced action of $\mathbb{G}_{m,k}$ on X_k is set-theoretically free and the quotient stack $[X_k/\mathbb{G}_{m,k}]$ is proper over k. Suppose also that there exists an ESWL resolution $f : Y \to X$ and that $\sharp^r_{\mathrm{st}}(X) < \infty$ for all r. Then*

$$\sharp^r_{\mathrm{st}}([X/\mathbb{G}_{m,\mathcal{O}_K}]) = \mathbb{D}(\sharp^r_{\mathrm{st}}([X/\mathbb{G}_{m,\mathcal{O}_K}])) \cdot q^{r(d-1)}$$

with d the dimension of X over $\mathcal{O}_K$.

Proof. From Proposition 3.5, if we write $K_{Y/X} = \sum_{j=1}^{n} c_j C_j$, then we obtain

$$\sharp^r_{\mathrm{st}}([X/\mathbb{G}_{m,\mathcal{O}_K}]) = \sum_{J\subset\{1,\dots,n\}} \frac{\sharp^r(C_J^\circ)}{q^r-1} \prod_{i\in J} \frac{q^r-1}{q^{r(1+c_j)}-1}$$

$$= \sum_{J\subset\{1,\dots,n\}} \frac{\sharp^r(C_J)}{q^r-1} \prod_{j\in J} \left(\frac{q^r - q^{r(1+c_j)}}{q^{r(1+c_j)}-1} \right)$$

as in the proof of Corollary 3.7. Each $C_J \cap Y_k$ is stable under the $\mathbb{G}_{m,k}$-action and the action on $C_J \cap Y_k$ is set-theoretically free. Hence $[(C_J \cap Y_k)/\mathbb{G}_{m,k}]$ is a smooth proper Artin k-stack of dimension $d-1-\sharp J$ with finite diagonal. From Proposition 3.9,

$$\frac{\sharp^r(C_J)}{q^r-1} = \mathbb{D}\left(\frac{\sharp^r(C_J)}{q^r-1}\right) \cdot q^{r(d-1-\sharp J)}.$$

Following the proof of Corollary 3.7 we obtain the result. □

4. Total masses of local Galois representations

Total masses. Let K be a local field and $G_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$ its absolute Galois group. For a finite group Γ, we define $S_{K,\Gamma}$ to be the set of continuous homomorphisms $G_K \to \Gamma$. To each representation $\tau : \Gamma \to \mathrm{GL}_d(\mathcal{O}_K)$, we can associate, among others, three functions $S_{K,\Gamma} \to \mathbb{R}$ denoted by $\boldsymbol{a}_\tau$, $\boldsymbol{v}_\tau$ and $\boldsymbol{w}_\tau$.

The first one, $\boldsymbol{a}_\tau$, is called the *Artin conductor*. For $\rho \in S_{K,\Gamma}$, let H be the image of ρ and L/K the associated Galois extension, having H as its Galois group. The Galois group H has the filtration by ramification subgroups with lower numbering

$$H \supset H_0 \supset H_1 \supset \cdots,$$

(see [Serre 1979]). We define

$$\boldsymbol{a}_\tau(\rho) := \sum_{i=0}^{\infty} \frac{1}{(H_0 : H_i)} \operatorname{codim}(K^d)^{H_i}.$$

The second function $\boldsymbol{v}_\tau$ is defined as follows. For $\rho \in S_{K,\Gamma}$ let $\operatorname{Spec} M \to \operatorname{Spec} K$ be the corresponding étale Γ-torsor and $\mathcal{O}_M$ the integer ring of M. Note that the spectrum of the extension L mentioned above is a connected component of $\operatorname{Spec} M$. Then the free $\mathcal{O}_M$-module $\mathcal{O}_M^{\oplus d}$ has two (left) Γ-actions: firstly the action induced from the map

$$\Gamma \xrightarrow{\tau} \mathrm{GL}_d(\mathcal{O}_K) \subset \mathrm{GL}_d(\mathcal{O}_M)$$

and secondly the diagonal action induced from the given Γ-action on $\mathcal{O}_M$. We define the associated *tuning submodule* $\Xi \subset \mathcal{O}_M^{\otimes d}$ to be the subset of those element on which the two actions coincide. It turns out that Ξ is a free $\mathcal{O}_K$-module of rank d. We define

$$\boldsymbol{v}_\tau(\rho) := \frac{1}{\sharp\Gamma} \cdot \operatorname{length}\left(\frac{\mathcal{O}_M^{\oplus d}}{\mathcal{O}_M \cdot \Xi}\right).$$

Note that we may also use $\mathcal{O}_L$ and the subgroup $H = \operatorname{Im}(\rho)$ in the definition of $\boldsymbol{v}_\tau$ instead of $\mathcal{O}_M$ and Γ.

The last function $\boldsymbol{w}_\tau$ is called the *weight function.* Let $\rho \in S_{K,\Gamma}$, M and $\Xi \subset \mathcal{O}_M^{\oplus d}$ be as above. We identify Ξ with the subgroup of Γ-equivariant maps of $\operatorname{Hom}_{\mathcal{O}_K}(\mathcal{O}_K^{\oplus d}, \mathcal{O}_M) = \mathcal{O}_M^{\oplus d}$. For an $\mathcal{O}_K$-basis $\phi_1, \dots, \phi_d$ of Ξ, we define an $\mathcal{O}_K$-algebra homomorphism $u^* : \mathcal{O}_K[x_1, \dots, x_d] \to \mathcal{O}_M[y_1, \dots, y_d]$ by

$$u^*(x_i) = \sum_{j=1}^{n} \phi_j(x_i) y_j.$$

Let $u : \mathbb{A}^d_{\mathcal{O}_M} \to \mathbb{A}^d_{\mathcal{O}_K}$ be the corresponding morphism of schemes and $o : \operatorname{Spec} k \hookrightarrow \mathbb{A}^d_{\mathcal{O}_K}$ the k-point at the origin. We define

$$\boldsymbol{w}_\tau(\rho) := \dim u^{-1}(o) - \boldsymbol{v}_\tau(\rho).$$

Remark 4.1. Our definition of $\boldsymbol{w}_\tau$ follows the one in [Yasuda 2014b] and is slightly different from the one in the earlier paper [Wood and Yasuda 2015]. Let us consider the left Γ-action on $\mathbb{A}^d_{\mathcal{O}_K}$ induced from the one on $\mathcal{O}_K[x_1, \dots, x_d]$. To be precise, an element $\gamma \in \Gamma$ gives the automorphism of $\mathbb{A}^d_{\mathcal{O}_K}$ corresponding to the automorphism on $\mathcal{O}_K[x_1, \dots, x_d]$ given by the inverse γ^{-1} of γ. In [Wood and Yasuda 2015] we defined

$$\boldsymbol{w}_\tau(\rho) := \operatorname{codim}((\mathbb{A}^d_k)^{H_0}, \mathbb{A}^d_k) - \boldsymbol{v}_\tau(\rho),$$

where H_0 is the zeroth ramification subgroup of H, that is, the inertia subgroup. However, as proved in [Yasuda 2014b], the two definitions coincide in the following three cases with which we are mainly concerned:

(1) The order of the given group Γ is invertible in k.

(2) The given representation τ is a permutation representation.

(3) The given local field K has positive characteristic, and hence $K = k((t))$, and the image of τ is contained in $\mathrm{GL}_d(k)$.

Definition 4.2. For a function $c : S_{K,\Gamma} \to \mathbb{R}$, we define the *total mass* of (K, Γ, c) as

$$M(K, \Gamma, c) := \frac{1}{\sharp\Gamma} \cdot \sum_{\rho :\in S_{K,\Gamma}} q^{-c(\rho)} \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

For $r \in \mathbb{N}$, let K_r/K be the unramified extension of degree r. We continue to denote the induced representations $\Gamma \xrightarrow{\tau} \mathrm{GL}_n(\mathcal{O}_K) \hookrightarrow \mathrm{GL}_n(\mathcal{O}_{K_r})$ by τ. Then we can consider total masses $M(K_r, \Gamma, \boldsymbol{a}_\tau)$, $M(K_r, \Gamma, \boldsymbol{v}_\tau)$ and $M(K_r, \Gamma, -\boldsymbol{w}_\tau)$, and regard them as functions in r.

Kedlaya studied total masses for $c = \boldsymbol{a}_\tau$ and used them to interpret and generalize the mass formula of Bhargava [2007]. Wood [2008] studied total masses for different choices of c. In the context of the wild McKay correspondence, explained below, functions $\boldsymbol{v}_\tau$ and $\boldsymbol{w}_\tau$ occur. For permutation representations, functions $\boldsymbol{a}_\tau$ and $\boldsymbol{v}_\tau$ are related as follows.

Lemma 4.3 [Wood and Yasuda 2015]. *If τ is a permutation representation, then*

$$2\boldsymbol{v}_\tau = \boldsymbol{v}_{\tau\oplus\tau} = \boldsymbol{a}_\tau.$$

The wild McKay correspondence. A representation $\tau : \Gamma \to \mathrm{GL}_d(\mathcal{O}_K)$ defines Γ-actions on the polynomial ring $\mathcal{O}_K[x_1, \dots, x_d]$ as above. Let X be the quotient variety $\mathbb{A}^d_{\mathcal{O}_K}/\Gamma = \operatorname{Spec} \mathcal{O}_K[x_1, \dots, x_d]^\Gamma$, which is a normal $\mathbb{Q}$-Gorenstein $\mathcal{O}_K$-variety. The following theorem, which we call the wild McKay correspondence, connects total masses and stringy point counts. This was proved in [Yasuda 2014a; Wood and Yasuda 2015] for special cases and in [Yasuda 2014b] in full generality.

Theorem 4.4 (The wild McKay correspondence). *Suppose that the representation τ is faithful and the quotient morphism $\mathbb{A}^d_{\mathcal{O}_K} \to X$ is étale in codimension one. We have*

$$\sharp^r_{\mathrm{st}}(X) = M(K_r, \Gamma, \boldsymbol{v}_\tau) \cdot q^{dr} \quad \textit{and} \quad \sharp^r_{\mathrm{st}}(X)_o = M(K_r, \Gamma, -\boldsymbol{w}_\tau).$$

The next corollary is a direct consequence of this theorem and Proposition 3.5.

Corollary 4.5. *If X admits a WL resolution and if $M(K_r, \Gamma, \boldsymbol{v}_\tau) < \infty$ (resp. $M(K_r, \Gamma, -\boldsymbol{w}_\tau) < \infty$) for every $r \in \mathbb{N}$, then the function $M(K_r, \Gamma, \boldsymbol{v}_\tau)$ (resp. $M(K_r, \Gamma, -\boldsymbol{w}_\tau)$) is admissible.*

Dualities of total masses. To discuss dualities, let us assume that both functions $M(K_r, \Gamma, \boldsymbol{v}_\tau)$ and $M(K_r, \Gamma, -\boldsymbol{w}_\tau)$ have finite values for all r and are admissible. The quotient variety X associated to a faithful representation $\tau : \Gamma \to \mathrm{GL}_d(\mathcal{O}_K)$ has a natural $\mathbb{G}_{m,\mathcal{O}_K}$-action. Let X^* be the complement of the zero section $\operatorname{Spec} \mathcal{O}_K \hookrightarrow X$. The $\mathbb{G}_{m,k}$-action on X^*_k is set-theoretically free. By definition, we have

$$\sharp^r_{\mathrm{st}}([X^*/\mathbb{G}_{m,\mathcal{O}_K}]) = \frac{\sharp^r_{\mathrm{st}}(X) - \sharp^r_{\mathrm{st}}(X)_o}{q^r - 1}.$$

If there exists an ESWL resolution $Y \to X^*$, then from Proposition 3.13, we have

$$\frac{\sharp^r_{\mathrm{st}}(X) - \sharp^r_{\mathrm{st}}(X)_o}{q^r - 1} = \mathbb{D}\left(\frac{\sharp^r_{\mathrm{st}}(X) - \sharp^r_{\mathrm{st}}(X)_o}{q^r - 1}\right) \cdot q^{r(d-1)}. \tag{4-1}$$

Remark 4.6. When $\Gamma = \mathbb{Z}/p\mathbb{Z} \subset \mathrm{GL}_d(k)$ with k a perfect field of characteristic $p > 0$, the motivic version of (4-1) was checked in [Yasuda 2014a, Proposition 6.36], not by using a resolution but by an explicit formula for motivic total masses.

If $\mathbb{A}^d_{\mathcal{O}_K} \to X$ is étale in codimension one, then the wild McKay correspondence, Theorem 4.4, shows that the last equality is equivalent to:

$$M(K_r, \Gamma, \boldsymbol{v}_\tau) \cdot q^{rd} - M(K_r, \Gamma, -\boldsymbol{w}_\tau) = \mathbb{D}(M(K_r, \Gamma, -\boldsymbol{w}_\tau)) \cdot q^{rd} - \mathbb{D}(M(K_r, \Gamma, \boldsymbol{v}_\tau)). \quad (4\text{-}2)$$

We call this the *weak duality*. The equality may be regarded as a constraint imposed by the existence of an ESWL resolution of X^*. If this duality fails for some representation τ, then the associated X^* would have no ESWL resolution. In particular, it is interesting to ask the following question.

Question 4.7. Suppose that K has positive characteristic so that $K = k((t))$, that $\tau : \Gamma \to \mathrm{GL}_d(\mathcal{O}_K)$ is a faithful representation factoring through $\mathrm{GL}_n(k)$ and that the quotient morphism $\mathbb{A}^d_{\mathcal{O}_K} \to \mathbb{A}^d_{\mathcal{O}_K}/\Gamma$ is étale in codimension one. Then, does the weak duality always hold?

If this has the negative answer and some representation τ does not satisfy (4-2), then the associated k-variety $X_k = \mathbb{A}^d_k/\Gamma$ does not admit any $\mathbb{G}_{m,k}$-equivariant WL resolution.

In some examples, even a stronger equality,

$$\mathbb{D}(M(K_r, \Gamma, -\boldsymbol{w}_\tau)) = M(K_r, \Gamma, \boldsymbol{v}_\tau), \quad (4\text{-}3)$$

holds: we call this the *strong duality*. A special case of this was the duality involving Bhargava's formula and the Hilbert scheme of points observed in [Wood and Yasuda 2015] (see Section 5). Actually, as far as the authors know, even the strong duality holds in all examples satisfying the assumptions of Question 4.7. It is thus natural to ask:

Question 4.8. With the same assumptions as in Question 4.7, does the strong duality always hold?

5. Examples

The tame case. We first consider the tame case. Suppose $p \nmid \sharp\Gamma$. Let $\tau : \Gamma \to \mathrm{GL}_d(\mathcal{O}_K)$ be a representation. Then $S_{K,\Gamma}$ is a finite set and every $\rho \in S_{K,\Gamma}$ factors through the Galois group G_K^{tame} of the maximal tamely ramified extension K^{tame}/K. It is topologically generated by two elements a and b with one relation $bab^{-1} = a^q$ (see [Neukirch et al. 2008, p. 410]). Here a is the topological generator of the inertia subgroup of G_K^{tame}. Therefore $S_{K,\Gamma}$ is in one-to-one correspondence with

$$\{(g, h) \in \Gamma^2 \mid hgh^{-1} = g^q\}.$$

The last set admits the involution $(g, h) \mapsto (g^{-1}, h)$. Let ι be the corresponding involution of $S_{K,\Gamma}$.

Lemma 5.1. *For $\rho \in S_{K,\Gamma}$, we have $\boldsymbol{v}_\tau(\iota(\rho)) = \boldsymbol{w}_\tau(\rho)$.*

Proof. Let K' be the completion of the maximal unramified extension of K. Both ρ and $\iota(\rho)$ define the same cyclic extension L/K' of order $l := \mathrm{ord}(g)$ with $(g, h) \in \Gamma^2$ corresponding to ρ. We can regard the element $a \in G_K^{\mathrm{tame}}$ as a generator of $\mathrm{Gal}(L/K')$. Let $\varpi \in L$ be a uniformizer and $\zeta \in K'$ an l-th root of unity such that $a(\varpi) = \zeta\varpi$.

If $\zeta^{a_1}, \dots, \zeta^{a_d}$ with $0 \le a_i < l$ are the eigenvalues of $g \in \Gamma \subset \mathrm{GL}_d(K')$, then, from [Wood and Yasuda 2015, Lemma 4.3], we have

$$\boldsymbol{v}_\tau(\rho) = \frac{1}{l}\sum_{i=1}^{l} a_i.$$

In the tame case, our definition of $\boldsymbol{w}_\tau$ coincides with the one in [Wood and Yasuda 2015] (see Remark 4.1). Therefore

$$\boldsymbol{w}_\tau(\rho) = \sharp\{i \mid a_i \neq 0\} - \boldsymbol{v}_\tau(\rho) = \frac{1}{l}\sum_{a_i \neq 0} l - a_i = \boldsymbol{v}_\tau(\iota(\rho)). \qquad \square$$

From the lemma,

$$\begin{aligned} M(K, \Gamma, -\boldsymbol{w}_\tau) &= \frac{1}{\sharp\Gamma}\sum_{\rho \in S_{K,\gamma}} q^{\boldsymbol{w}_\tau(\rho)} \\ &= \frac{1}{\sharp\Gamma}\sum_{\rho \in S_{K,\gamma}} q^{\boldsymbol{v}_\tau(\iota(\rho))} \\ &= \frac{1}{\sharp\Gamma}\sum_{\rho \in S_{K,\gamma}} q^{\boldsymbol{v}_\tau(\rho)} \\ &= \mathbb{D}(M(K, \Gamma, \boldsymbol{v}_\tau)). \end{aligned}$$

The strong duality thus holds in the tame case. Moreover it is a term-wise duality: $q^{\boldsymbol{w}_\tau(\rho)}$ and $q^{\boldsymbol{v}_\tau(\iota(\rho))}$ are dual to each other. This is no longer true in wild cases satisfying dualities.

Quadratic extensions: characteristic zero. We consider an unramified extension K of $\mathbb{Q}_2$ and compute some total masses counting quadratic extensions of K. Namely we consider the case $\Gamma = \mathbb{Z}/2\mathbb{Z}$. Let $\mathfrak{m}_K$ be the maximal ideal of $\mathcal{O}_K$, quadratic extensions are divided into 4 classes:

- The trivial extension $K \times K$.
- The unramified field extension of degree two.
- Ramified extensions with discriminant $\mathfrak{m}_K^2$. There are $2(q-1)$ of them.
- Ramified extensions with discriminant $\mathfrak{m}_K^3$. There are $2q$ of them.

For instance, this follows from Krasner's formula [1966] (see also [Serre 1978]).

They are in one-to-one correspondence with elements of $S_{K,\Gamma}$. Let σ be the two-dimensional representation of $\Gamma = \mathbb{Z}/2\mathbb{Z}$ by the transposition and for $n \in \mathbb{N}$, σ_n the direct sum of n copies of σ. From [Wood and Yasuda 2015, Lemma 2.6], the values $\boldsymbol{a}_\sigma(\rho)$, for $\rho \in S_{K,\Gamma}$, are respectively $0, 0, 2, 3$ depending on the corresponding class of quadratic extension. We have

$$\begin{aligned} M(K, \Gamma, \boldsymbol{v}_{\sigma_n}) &= M\big(K, \Gamma, \tfrac{1}{2}\boldsymbol{a}_{\sigma_n}\big) \\ &= \tfrac{1}{2}(1 + 1 + 2(q-1)q^{-2n/2} + 2q \cdot q^{-3n/2}) \\ &= 1 + q^{-n+1} - q^{-n} + q^{-3n/2+1}. \end{aligned}$$

As for the total mass with respect to $-\boldsymbol{w}_{\sigma_n}$, we have $\boldsymbol{w}_{\sigma_n}(\rho) = n - \boldsymbol{v}_{\sigma_n}(\rho)$ if ρ corresponds to a totally ramified extension, and $\boldsymbol{w}_{\sigma_n}(\rho) = \boldsymbol{v}_{\sigma_n}(\rho)$ otherwise. Therefore,

$$M(K, \Gamma, \boldsymbol{w}_{\sigma_n}) = \tfrac{1}{2}(1 + 1 + 2(q-1)q^{-n+n} + 2q \cdot q^{-3n/2+n}) = q + q^{-n/2+1}.$$

We easily see that the strong duality holds for $n = 1$. For $n > 1$, the strong duality does not hold, but the weak one holds.

This phenomenon seems to be rather general. For a few other examples of permutation representations σ of cyclic groups, the authors checked that the weak duality holds, but the strong duality does not always hold. From these computations as well as the example in Section 5, it is natural to ask:

Question 5.2. Suppose that the given representation τ is a permutation representation. Does the weak duality (4-2) always hold? Does the quotient scheme $\mathbb{A}^d_{\mathcal{O}_K}/\Gamma$ always admit an ESWL resolution?

Quadratic extensions: characteristic two. Next we consider the case $K = \mathbb{F}_q((t))$ with q a power of two and $\Gamma = \mathbb{Z}/2\mathbb{Z}$. As in the last example, there are exactly two unramified quadratic extensions of K: the trivial one $K \times K$ and the unramified field extension. As for the ramified extensions, for $i \in \mathbb{N}$, there are exactly $2(q-1)q^{i-1}$ extensions with discriminant $\mathfrak{m}_K^{2i}$. There is no extension with discriminant of odd exponent. This again follows from Krasner's formula.

Let σ be the two dimensional Γ-representation by transposition and σ_n the direct sum of n copies of it as above. We have

$$M(K, \Gamma, \boldsymbol{v}_{\sigma_n}) = 1 + \sum_{i=1}^{\infty}(q-1)q^{i-1}q^{-in} = 1 + \frac{(q-1)q^{-n}}{1-q^{-n+1}},$$

and

$$M(K, \Gamma, -\boldsymbol{w}_{\sigma_n}) = 1 + \sum_{i=1}^{\infty}(q-1)q^{i-1}q^{-in+n} = 1 + \frac{q-1}{1-q^{-n+1}},$$

We see that the strong duality and hence the weak duality hold for every n. Thus the answer to Question 4.8 in this case is positive.

For an integer $m \geq 0$, let us now consider the representation $\tau_m : \Gamma \to \mathrm{GL}_2(\mathcal{O}_K)$ given by the matrix

$$\begin{pmatrix} 1 & t^m \\ 0 & 1 \end{pmatrix}.$$

We have $\tau_0 \cong \sigma$. If $m > 0$, then τ_m is not defined over $k = \mathbb{F}_q$.

To compute the functions $\boldsymbol{v}_{\tau_m}$ and $\boldsymbol{w}_{\tau_m}$, let L be a ramified quadratic extension of K with discriminant $\mathfrak{m}_K^{2i}$, ι its unique K-involution and ϖ its uniformizer. Then $\delta(\varpi) := \iota(\varpi) - \varpi \in \mathcal{O}_K$ and $v_K(\delta(\varpi)) = i$ with v_K the normalized valuation of K. Therefore the tuning submodule Ξ is computed as follows:

$$\begin{aligned} \Xi &= \{(x, y) \in \mathcal{O}_L^{\oplus 2} \mid x + t^m y = \iota(x),\ y = \iota(y)\} \\ &= \{(x, t^{-m}\delta(x)) \mid x \in \mathcal{O}_L,\ v_K(\delta(x)) \geq m\} \\ &= \{(x, t^{-m}\delta(x)) \mid x \in \mathcal{O}_K \cdot 1 \oplus \mathcal{O}_K \cdot t^a \varpi\} \\ &= \langle (1, 0),\ (t^a\varpi, t^{a-m}\delta(\varpi))\rangle_{\mathcal{O}_K}, \end{aligned}$$

where $a = \sup\{0, m - i\}$. Therefore,

$$\frac{\mathcal{O}_L^{\oplus 2}}{\mathcal{O}_L \cdot \Xi} \cong \frac{\mathcal{O}_L}{\mathcal{O}_L \cdot t^{a-m}\delta(\varpi)} \cong \frac{\mathcal{O}_L}{\mathcal{O}_L \cdot t^{a-m+i}}.$$

For $\rho \in S_{K,\Gamma}$ corresponding to L,

$$\boldsymbol{v}_{\tau_m}(\rho) = a - m + i = \begin{cases} 0 & i \leq m, \\ i - m & i > m. \end{cases}$$

The map $u^* : \mathcal{O}_K[x, y] \to \mathcal{O}_L[X, Y]$ in the definition of $\boldsymbol{w}$ is explicitly given as

$$u^*(x) = X + t^a\varpi Y, \quad u^*(y) = t^{a-m}\delta(\varpi)Y.$$

The induced morphism $\operatorname{Spec} k[X, Y] \to \operatorname{Spec} k[x, y]$ is an isomorphism if $i \leq m$ and a linear map of rank one if $i > m$. Therefore

$$\boldsymbol{w}_{\tau_m}(\rho) = \begin{cases} 0 & i \leq m, \\ 1 - i + m & i > m. \end{cases}$$

To obtain quotient morphisms which are étale in codimension one, we consider the direct sums $\upsilon_{m,n} := \tau_m \oplus \sigma_n$. We have

$$\begin{aligned} M(K, \Gamma, \boldsymbol{v}_{\upsilon_{m,n}}) &= 1 + \sum_{i=1}^{m} (q-1)q^{i-1} \cdot q^{-in} + \sum_{i=m+1}^{\infty} (q-1)q^{i-1} \cdot q^{-i+m-in} \\ &= 1 + (q-1)q^{-n}\frac{1 - q^{(1-n)m}}{1 - q^{1-n}} + (q-1)\frac{q^{m-(m+1)n-1}}{1 - q^{-n}}, \end{aligned}$$

and

$$M(K,\Gamma,-\boldsymbol{w}_{\upsilon_{m,n}}) = 1+\sum_{i=1}^{m}(q-1)q^{i-1}\cdot q^{-in+n}+\sum_{i=m+1}^{\infty}(q-1)q^{i-1}\cdot q^{1-i+m-in+n}$$
$$= 1+(q-1)\frac{1-q^{(1-n)m}}{1-q^{1-n}}+(q-1)\frac{q^{m-mn}}{1-q^{-n}}.$$

One can easily see that the weak duality does not generally hold, for instance, by putting $(m,n)=(1,1)$. From our viewpoint, this is explained by noting that the entry t^m in the matrix defining τ_m causes the degeneration of singularities of $\mathbb{A}^d_{\mathcal{O}_K}/\Gamma$, namely it makes the family over $\mathcal{O}_K$ nonequisingular.

Bhargava's formula. We discuss an example from [Wood and Yasuda 2015] and the duality observed there. Let S_n be the n-th symmetric group and $\sigma : S_n \to \mathrm{GL}_d(\mathcal{O}_K)$ the standard permutation representation. According to Kedlaya [2007], the mass formula of Bhargava [2007] for étale extensions of a local field is formulated as

$$M(K,S_n,\boldsymbol{a}_\sigma)=\sum_{m=0}^{n-1}P(n,n-m)\cdot q^{-m}.$$

Here $P(n,n-m)$ denotes the number of partitions of n into exactly $n-m$ parts.

Let

$$\tau := \sigma\oplus\sigma : S_n \to \mathrm{GL}_{2n}(\mathcal{O}_K).$$

From Lemma 4.3, we have

$$M(K,S_n,\boldsymbol{v}_\tau)=M(K,S_n,\boldsymbol{a}_\sigma)=\sum_{m=0}^{n-1}P(n,n-m)\cdot q^{-m}.$$

In [Wood and Yasuda 2015], we verified that

$$M(K,S_n,-\boldsymbol{w}_\tau)=\sum_{m=0}^{n}P(n,n-m)\cdot q^{m}.$$

These show that the functions $M(K_r,S_n,-\boldsymbol{w}_\tau)$ and $M(K_r,S_n,\boldsymbol{v}_\tau)$ satisfy the strong duality (4-3). Thus the answer to Question 5.2 is positive in this situation.

The quotient scheme $X=\mathbb{A}^{2n}_{\mathcal{O}_K}/S_n$ associated to τ is nothing but the n-th symmetric product of the affine plane over $\mathcal{O}_K$. It admits a special ESWL resolution, namely the Hilbert–Chow morphism

$$f:\mathrm{Hilb}^n(\mathbb{A}^2_{\mathcal{O}_K})\to X,$$

from the Hilbert scheme of n points on $\mathbb{A}^2_{\mathcal{O}_K}$ defined over $\mathcal{O}_K$.

Using a stratification of $\mathrm{Hilb}^n(\mathbb{A}^2_k)$ into affine spaces [Ellingsrud and Strømme 1987; Conca and Valla 2008], we can directly show

$$\sharp\,\mathrm{Hilb}^n(\mathbb{A}^2_k) = \sum_{m=0}^{n} P(n, n-m)\cdot q^{2n-m}.$$

These together with Theorem 4.4 gives a new proof of Bhargava's formula without using the mass formula of Serre [1978] unlike [Bhargava 2007; Kedlaya 2007].

Kedlaya's formula. We suppose $p \neq 2$ and let G be the group of signed permutation matrices in $\mathrm{GL}_n(\mathcal{O}_K)$. A signed matrix is a matrix such that every row or column contains one and only one nonzero entry which is either 1 or -1. The group is isomorphic to the wreath product $(\mathbb{Z}/2\mathbb{Z}) \wr S_n$. Let

$$\iota : G \hookrightarrow \mathrm{GL}_n(\mathcal{O}_K)$$

be the inclusion and $\tau := \iota \oplus \iota$.

Lemma 5.3. *We have $\boldsymbol{a}_\iota = \boldsymbol{v}_\tau$.*

Proof. We construct a representation $\iota' : G \to \mathrm{GL}_{2n}(\mathcal{O}_K)$ by replacing entries of matrices in G as follows: replace 0 with $\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$, 1 with $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, and -1 with $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. The obtained ι' is a permutation representation of G and, from Lemma 4.3,

$$\boldsymbol{a}_{\iota'} = \boldsymbol{v}_{\iota'\oplus\iota'}.$$

If $\tau : G \to \mathrm{GL}_n(\mathcal{O}_K)$ is the trivial representation, then $\iota' \cong \iota \oplus \tau$. The facts that $\boldsymbol{a}_\tau \equiv \boldsymbol{v}_\tau \equiv 0$, $\boldsymbol{a}_{\iota\oplus\tau} = \boldsymbol{a}_\iota + \boldsymbol{a}_\tau$ and $\boldsymbol{v}_{\iota\oplus\tau} = \boldsymbol{v}_\iota + \boldsymbol{v}_\tau$, which were proved in [Wood and Yasuda 2015], prove the lemma. $\square$

We then have

$$M(K, G, \boldsymbol{v}_\tau) = M(K, G, \boldsymbol{a}_\iota) = \sum_{j=0}^{n}\sum_{i=0}^{j} P(j, i)P(n-j)q^{i-n}, \tag{5-1}$$

where the right equality is due to Kedlaya [2007, Remark 8.6] and $P(m)$ denotes the number of partitions of m.

We can construct a resolution of $X := \mathbb{A}^{2n}_{\mathcal{O}_K}/G$ as follows. First consider the involution of $\mathbb{A}^2_{\mathcal{O}_K} = \operatorname{Spec} \mathcal{O}_K[x, y]$ sending x to $-x$ and y to $-y$. Let

$$Z := \mathbb{A}^2_{\mathcal{O}_K}/(\mathbb{Z}/2\mathbb{Z}) = \operatorname{Spec} \mathcal{O}_K[x^2, xy, y^2]$$

be the associated quotient scheme. We have a natural isomorphism

$$X \cong S^n Z := Z^n/S_n.$$

Since Z is the trivial family of A_1-singularities over $\operatorname{Spec} \mathcal{O}_K$ (actually it is a toric variety), there exists the minimal resolution $W \to Z$ of Z such that W is $\mathcal{O}_K$-smooth, the exceptional locus is isomorphic to $\mathbb{P}^1_{\mathcal{O}_K}$ and $K_{W/Z}=0$. Let $\mathrm{Hilb}^n(W)$ be the Hilbert scheme of n points of W over $\mathcal{O}_K$, which is again smooth over $\mathcal{O}_K$. We have the proper birational morphism

$$f:\mathrm{Hilb}^n(W)\xrightarrow{\text{Hilbert–Chow}} S^nW\to S^nZ\xrightarrow{\sim} X$$

and $K_{\mathrm{Hilb}^n(W)/X}=0$.

We now compute $\sharp f^{-1}(o)(k)$. Let $E\subset W_k$ be the exceptional divisor of $W_k\to Z_k$, which is isomorphic to $\mathbb{P}^1_k$. Each k-point of $f^{-1}(o)$ corresponds to a zero-dimensional subscheme of W_k of length n supported in E. There exists a stratification

$$E=E_1\sqcup E_0$$

such that $E_1\cong\mathbb{A}^1_k$ is a coordinate line of an open subscheme $\mathbb{A}^2_k\cong U\subset W_k$ and $E_0\cong\operatorname{Spec}k$ is the origin of another open subscheme $\mathbb{A}^2_k\cong U'\subset W_k$. Let C_m be the set of zero-dimensional subschemes U of length m supported in E_1 and D_m the set of zero-dimensional subschemes of U' of length m supported in E_0. From [Conca and Valla 2008, Corollary 3.1],

$$\sharp C_m=P(m)q^m\quad\text{and}\quad\sharp D_m=\sum_{i=0}^{m}P(m,i)q^{m-i}.$$

Since $f^{-1}(o)(k)$ decomposes as

$$f^{-1}(o)(k)=\bigsqcup_{j=0}^{n}C_{n-j}\times D_j,$$

we have

$$\begin{aligned}\sharp f^{-1}(o)(k)&=\sum_{j=0}^{n}\biggl(P(n-j)q^{n-j}\sum_{i=0}^{j}P(j,i)q^{j-i}\biggr)\\&=\sum_{j=0}^{n}\sum_{i=0}^{j}P(j,i)P(n-j)q^{n-i}.\end{aligned}$$

From the wild McKay correspondence,

$$M(K,G,-\boldsymbol{w}_\tau)=\sum_{j=0}^{n}\sum_{i=0}^{j}P(j,i)P(n-j)q^{n-i}.$$

Comparing this with (5-1), we verify the strong duality for $M(K_r,G,-\boldsymbol{w}_\tau)$ and $M(K_r,G,\boldsymbol{v}_\tau)$.

In a similar way to computing $\sharp f^{-1}(o)(k)$, we can directly compute $\sharp\mathrm{Hilb}^n(W)(k)$ and obtain a new proof of Kedlaya's formula (5-1).

6. Concluding remarks and extra problems

As far as we computed, we observed the following phenomena. The weak duality and the strong duality may fail. Tame representations satisfy both the strong and weak dualities. Permutation representations satisfy the weak duality but not necessarily the strong duality. When $K = \mathbb{F}_q((t))$, representations defined over $\mathbb{F}_q$ satisfy both the strong and weak dualities.

We may interpret these as follows. We might be able to measure by the two dualities how equisingular the family $\mathbb{A}^d_{\mathcal{O}_K}/\Gamma$ is. If the strong duality holds, then the family would be very equisingular. If only the weak duality holds, then the family would be moderately so. If both dualities fail, then it is rather far from being equisingular. It may be interesting to look for a numerical invariant refining this measurement.

As we saw, the weak duality is derived from the Poincaré duality of stringy invariants if there exists an ESWL resolution. What about the strong duality? Is there any geometric interpretation of it?

Although it is still conjectural, there would be the motivic counterparts of total masses, stringy point counts and the McKay correspondence between them (see [Wood and Yasuda 2015; Yasuda 2013; 2014a; 2016]). We may discuss dualities in this motivic context as well. Indeed, some dualities for motivic invariants were verified in [Yasuda 2014a; 2016].

Acknowledgments

The authors would like to thank Kiran Kedlaya for helpful discussion. Yasuda thanks Max Planck Institute for Mathematics for its hospitality, where he stayed partly during this work. Wood was supported by NSF grants DMS-1147782 and DMS-1301690 and an American Institute of Mathematics Five Year Fellowship.

References

[Batyrev 1998] V. V. Batyrev, "Stringy Hodge numbers of varieties with Gorenstein canonical singularities", pp. 1–32 in *Integrable systems and algebraic geometry* (Kobe/Kyoto, 1997), edited by M.-H. Saito et al., World Sci. Publ., River Edge, NJ, 1998. MR Zbl

[Batyrev and Dais 1996] V. V. Batyrev and D. I. Dais, "Strong McKay correspondence, string-theoretic Hodge numbers and mirror symmetry", *Topology* **35**:4 (1996), 901–929. MR Zbl

[Bhargava 2007] M. Bhargava, "Mass formulae for extensions of local fields, and conjectures on the density of number field discriminants", *Int. Math. Res. Not.* **2007**:17 (2007), art. id. rnm052, 20 pp. MR Zbl

[Conca and Valla 2008] A. Conca and G. Valla, "Canonical Hilbert–Burch matrices for ideals of $k[x, y]$", *Michigan Math. J.* **57** (2008), 157–172. MR Zbl

[Ellingsrud and Strømme 1987] G. Ellingsrud and S. A. Strømme, "On the homology of the Hilbert scheme of points in the plane", *Invent. Math.* **87**:2 (1987), 343–352. MR Zbl

[Kedlaya 2007] K. S. Kedlaya, "Mass formulas for local Galois representations", *Int. Math. Res. Not.* **2007**:17 (2007), art. id. rnm021, 26 pp. MR Zbl

[Kollár 2013] J. Kollár, *Singularities of the minimal model program*, Cambridge Tracts in Mathematics **200**, Cambridge Univ. Press, 2013. MR Zbl

[Krasner 1966] M. Krasner, "Nombre des extensions d'un degré donné d'un corps $\mathfrak{p}$-adique", pp. 143–169 in *Les tendances géométriques en algèbre et théorie des nombres*, edited by M. Krasner, Colloques Intern. Centre National Recherche Sci. **143**, Editions Centre National Recherche Scientifique, Paris, 1966. MR Zbl

[Laszlo and Olsson 2008a] Y. Laszlo and M. Olsson, "The six operations for sheaves on Artin stacks, I: Finite coefficients", *Publ. Math. Inst. Hautes Études Sci.* 107 (2008), 109–168. MR Zbl

[Laszlo and Olsson 2008b] Y. Laszlo and M. Olsson, "The six operations for sheaves on Artin stacks, II: Adic coefficients", *Publ. Math. Inst. Hautes Études Sci.* 107 (2008), 169–210. MR Zbl

[Milne 1980] J. S. Milne, *Étale cohomology*, Princeton Mathematical Series **33**, Princeton Univ. Press, 1980. MR Zbl

[Neukirch et al. 2008] J. Neukirch, A. Schmidt, and K. Wingberg, *Cohomology of number fields*, 2nd ed., Grundlehren Math. Wissenschaften **323**, Springer, Berlin, 2008. MR Zbl

[Olsson 2015] M. Olsson, "Fujiwara's theorem for equivariant correspondences", *J. Algebraic Geom.* **24**:3 (2015), 401–497. MR Zbl

[Serre 1978] J.-P. Serre, "Une 'formule de masse' pour les extensions totalement ramifiées de degré donné d'un corps local", *C. R. Acad. Sci. Paris Sér. A-B* **286**:22 (1978), A1031–A1036. MR Zbl

[Serre 1979] J.-P. Serre, *Local fields*, Graduate Texts in Mathematics **67**, Springer, New York, 1979. MR Zbl

[Stanley 1997] R. P. Stanley, *Enumerative combinatorics, I*, Cambridge Studies in Advanced Mathematics **49**, Cambridge Univ. Press, 1997. Corrected reprint of the 1986 original. MR Zbl

[Sun 2012] S. Sun, "L-series of Artin stacks over finite fields", *Algebra Number Theory* **6**:1 (2012), 47–122. MR Zbl

[Wood 2008] M. M. Wood, "Mass formulas for local Galois representations to wreath products and cross products", *Algebra Number Theory* **2**:4 (2008), 391–405. MR Zbl

[Wood and Yasuda 2015] M. M. Wood and T. Yasuda, "Mass formulas for local Galois representations and quotient singularities, I: A comparison of counting functions", *Int. Math. Res. Not.* **2015**:23 (2015), 12590–12619. MR Zbl

[Yasuda 2013] T. Yasuda, "Toward motivic integration over wild Deligne–Mumford stacks", 2013. To appear in *Adv. Stud. Pure Math.* **74**. arXiv

[Yasuda 2014a] T. Yasuda, "The p-cyclic McKay correspondence via motivic integration", *Compos. Math.* **150**:7 (2014), 1125–1168. MR Zbl

[Yasuda 2014b] T. Yasuda, "The wild McKay correspondence and p-adic measures", 2014. To appear in *J. Eur. Math. Soc.* arXiv

[Yasuda 2016] T. Yasuda, "Wilder McKay correspondences", *Nagoya Math. J.* **221**:1 (2016), 111–164. MR Zbl

Communicated by Kiran S. Kedlaya
Received 2015-06-19 Accepted 2017-03-01

mmwood@math.wisc.edu
Department of Mathematics, University of Wisconsin, 480 Lincoln Drive, Madison, WI 53705, United States

American Institute of Mathematics, 360 Portage Avenue, Palo Alto, CA 94306, United States

takehikoyasuda@math.sci.osaka-u.ac.jp
Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka 560-0043, Japan

ALGEBRA AND NUMBER THEORY 11:4 (2017)
dx.doi.org/10.2140/ant.2017.11.841

A new equivariant in nonarchimedean dynamics

Robert Rumely

Let K be a complete, algebraically closed nonarchimedean valued field, and let $\varphi(z) \in K(z)$ have degree $d \geq 2$. We show there is a canonical way to assign nonnegative integer weights $w_\varphi(P)$ to points of the Berkovich projective line over K in such a way that $\sum_P w_\varphi(P) = d-1$. When φ has bad reduction, the set of points with nonzero weight forms a distributed analogue of the unique point which occurs when φ has potential good reduction. Using this, we characterize the minimal resultant locus of φ in analytic and moduli-theoretic terms: analytically, it is the barycenter of the weight-measure associated to φ; moduli-theoretically, it is the closure of the set of points where φ has semistable reduction, in the sense of geometric invariant theory.

Let K be a complete, algebraically closed nonarchimedean valued field with absolute value $|\cdot|$ and associated valuation $\operatorname{ord}(\cdot)$. Write $\mathcal{O}$ for the ring of integers of K, $\mathfrak{m}$ for its maximal ideal, and $\tilde{k}$ for its residue field. Let $\mathbf{P}^1_K$ be the Berkovich projective line over K: a compact, path-connected Hausdorff space which contains $\mathbb{P}^1(K)$ as a dense subset (a nonmetric tree in the sense of [Favre and Jonsson 2004]).

Our main result is:

Theorem A. *Let $\varphi(z) \in K(z)$ be a rational function with degree $d \geq 2$. There is a canonical way, determined by the variation in the resultant under conjugation of φ, to assign nonnegative integer weights $w_\varphi(P)$ to points of $\mathbf{P}^1_K$ in such a way that*

$$\sum_P w_\varphi(P) = d-1,$$

where $w_\varphi(P) = 0$ for each $P \in \mathbb{P}^1(K)$.

This work began during the ICERM Program “Complex and p-adic dynamics”, Spring 2012, and was carried out in part under the auspices of the University of Georgia VIGRE NSF grant DMS-0738586.
MSC2010: primary 37P50; secondary 11S82, 37P05.
Keywords: dynamics, minimal resultant locus, crucial set, repelling fixed points, nonarchimedean weight formula, geometric invariant theory.

We call the finite set of points which receive weight the *crucial set* of φ. Write δ_P for the Dirac measure at a point $P \in \mathbf{P}^1_K$. We call the discrete probability measure

$$\nu_\varphi := \frac{1}{d-1} \sum_P w_\varphi(P)\delta_P$$

the *crucial measure* of φ.

The crucial set and crucial measure are conjugation equivariants of φ which live in the interior of the Berkovich projective line. It is natural to ask about their meaning. The crucial set, decorated with the weights, adjacency data, and the reductions of φ at its points, appears to be a kind of "analytic special fiber" for φ, analogous to the algebraic special fiber for a model of a curve over a nonarchimedean valued field. When φ has potential good reduction in the sense of nonarchimedean dynamics, the crucial set consists of the unique point where φ has good reduction and that point has weight $d-1$. When φ has bad reduction, the single point of good reduction splits into a collection of points with total weight $d-1$.

Explicit formulas for the weights $w_\varphi(P)$ are given Section 6, and there are only finitely many ways a point can receive weight. The crucial set, decorated with adjacency data and information about how each point receives weight, appears to the classify possible reduction types for a functions φ of given degree: the author and students in a University of Georgia VIGRE group have shown that for quadratic rational functions and cubic polynomials [Doyle et al. 2016; Rumely et al. $\geq$ 2017], the finitely many possible configurations of the crucial set lead to a stratification of the dynamical moduli space which governs where φ lies in the moduli space, and is compatible with reduction $(\mathrm{mod}\, \mathfrak{m})$. This is analogous to the fact that for an elliptic curve E over a nonarchimedean local field, E has good reduction if and only if its j-invariant satisfies $|j_E| \leq 1$, while E has bad reduction (with semistable reduction a loop of curves), if and only if $|j_E| > 1$. It seems likely that such a stratification holds in general.

The crucial measure appears to be an intrinsic discrete approximation to the canonical invariant measure μ_φ supported on the Berkovich Julia set: Kenneth Jacobs [2014] has shown that the crucial measures of the iterates of φ converge weakly to μ_φ; for a given test function, the convergence is geometrically fast.

Our construction of the crucial set is an analytic one. It depends on the theory of the function $\mathrm{ordRes}_\varphi(\cdot)$ introduced in [Rumely 2015]. We associate to φ a canonical, finitely generated tree $\Gamma_{\mathrm{Fix,Repel}} \subset \mathbf{P}^1_K$. The weights $w_\varphi(P)$ and the crucial set are obtained by taking the Laplacian of $\mathrm{ordRes}_\varphi(\cdot)$ restricted to $\Gamma_{\mathrm{Fix,Repel}}$. It would be interesting to have a more algebraic construction.

Under conjugation of φ by an element $\gamma \in \mathrm{GL}_2(K)$ (given by $\varphi^\gamma = \gamma^{-1} \circ \varphi \circ \gamma$), the crucial set $\mathrm{Cr}(\varphi)$ and the crucial measure ν_φ have the same equivariance properties as fixed points: P is a fixed point of φ if and only if $\gamma^{-1}(P)$ is a fixed point of φ^γ.

Analogously, $\mathrm{Cr}(\varphi^\gamma) = \gamma^{-1}(\mathrm{Cr}(\varphi))$ and $\nu_{\varphi^\gamma} = \gamma^*(\nu_\varphi)$ (see Proposition 6.6). The construction of the crucial set applies in arbitrary characteristic. The crucial set is unchanged under enlargement of K (see Proposition 6.7), and is Galois-equivariant (see Proposition 6.8).

One application of the theory is as follows: by the formulas in Section 6, each repelling fixed point of φ in $\mathbf{H}^1_K := \mathbf{P}^1_K \setminus \mathbb{P}^1(K)$ receives weight. Hence φ can have at most $d-1$ nonclassical repelling fixed points (Corollary 6.2); previously it seems not to have been known that there were finitely many. In [Rumely 2014] it is shown that the bound $d-1$ is sharp.

A second application concerns the *minimal resultant locus* of φ, the set of points in $\mathbf{P}^1_K$ where, under change of coordinates, φ achieves its minimal resultant. We show that the minimal resultant locus is the *barycenter* of the crucial measure: the set of points P such that each component of $\mathbf{P}^1_K \setminus \{P\}$ has ν_φ-mass at most $\frac{1}{2}$. We also show that the minimal resultant locus characterizes the elements $\gamma \in \mathrm{GL}_2(K)$ for which φ^γ has semistable reduction in the sense of geometric invariant theory (see Theorems B and C below). Thus, the minimal resultant locus forms a bridge between analytic and moduli-theoretic properties of φ.

A third application is a new fixed point theorem, showing the existence of fixed points in balls which φ maps onto $\mathbf{P}^1_K$ (Theorem 4.6).

We will now give an overview of the paper. Suppose (F, G) is a normalized representation for φ: a pair of homogeneous functions $F(X, Y), G(X, Y) \in \mathcal{O}[X, Y]$ of degree d such that $\varphi(z) = F(z, 1)/G(z, 1)$ and some coefficient of F or G is a unit in $\mathcal{O}$. Such a pair (F, G) is unique up to scaling by a unit. Let $\mathrm{Res}(F, G)$ be the resultant of F and G; then $\mathrm{ordRes}(\varphi) := \mathrm{ord}(\mathrm{Res}(F, G))$ is well-defined and nonnegative.

Let $\mathbf{P}^1_K$ be the Berkovich projective line over K. By Berkovich's classification theorem (see [Berkovich 1990, p. 18]), points of $\mathbf{P}^1_K \setminus \{\infty\}$ correspond to discs $D(a, r) \subset K$ or to nested sequences of discs; points corresponding to discs with radius $r \in |K^\times|$ are said to be of type II. The point ζ_G corresponding to $D(0, 1)$ is called the *Gauss point*. The action of $\mathrm{GL}_2(K)$ on $\mathbb{P}^1(K)$ extends continuously to $\mathbf{P}^1_K$, and $\mathrm{GL}_2(K)$ acts transitively on type II points.

In [Rumely 2015] it is shown that the map $\gamma \mapsto \mathrm{ordRes}(\varphi^\gamma)$ factors through a function $\mathrm{ordRes}_\varphi(\,\cdot\,)$ on $\mathbf{P}^1_K$, given on type II points by

$$\mathrm{ordRes}_\varphi(\gamma(\zeta_G)) = \mathrm{ordRes}(\varphi^\gamma).$$

By [Rumely 2015, Theorem 0.1] the function $\mathrm{ordRes}_\varphi(\,\cdot\,)$ is piecewise affine and convex upwards on paths in $\mathbf{P}^1_K$ and it achieves a minimum. The set $\mathrm{MinResLoc}(\varphi)$ where the minimum occurs is called the *minimal resultant locus* of φ. For each $a \in \mathbb{P}^1(K)$, $\mathrm{MinResLoc}(\varphi)$ is contained in the tree $\Gamma_{\mathrm{Fix},\varphi^{-1}(a)}$ spanned by the classical fixed points of φ and the preimages of a. It is either a type II point or a segment joining two type II points.

In this paper, we show that the intersection of the trees $\Gamma_{\mathrm{Fix},\varphi^{-1}(a)}$ for all $a \in \mathbb{P}^1(K)$ is the tree $\Gamma_{\mathrm{Fix,Repel}}$ spanned by the classical fixed points of φ and the repelling fixed points in $\mathbf{H}^1_K$ (Theorem 4.2). Although $\mathrm{ordRes}_\varphi(\cdot)$ does not belong to the domain of the Laplacian on $\mathbf{P}^1_K$ (it is increasing too fast, in too many directions), one can compute its graph-theoretic Laplacian on the finitely generated tree $\Gamma_{\mathrm{Fix,Repel}}$. Comparing this Laplacian with the so-called "branching measure" μ_{Br} of $\Gamma_{\mathrm{Fix,Repel}}$ (which depends only on the topology of the tree), leads to the construction of the weights $w_\varphi(P)$ and the proof of Theorem A. An intrinsic formula for ν_φ is given in Corollary 6.5. Only points of type II can receive weight. The weights $w_\varphi(P)$, the crucial set, and the crucial measure are canonical because $\mathrm{ordRes}_\varphi(\cdot)$, $\Gamma_{\mathrm{Fix,Repel}}$, and μ_{Br} are canonical.

The following pair of theorems justify our claim that the minimal resultant locus forms a bridge between analytic and moduli-theoretic properties of φ:

Theorem B (analytic characterization of $\mathrm{MinResLoc}(\varphi)$). *Let $\varphi(z) \in K(z)$ have degree $d \geq 2$. Then* $\mathrm{MinResLoc}(\varphi)$ *is the barycenter of ν_φ in $\mathbf{P}^1_K$. If d is even,* $\mathrm{MinResLoc}(\varphi)$ *is a vertex of the tree Γ_φ spanned by the crucial set. If d is odd,* $\mathrm{MinResLoc}(\varphi)$ *is either a vertex or an edge of Γ_φ.*

Using geometric invariant theory (GIT), Silverman [1998] has constructed a moduli space $\mathcal{M}_d/\mathrm{Spec}(\mathbb{Z})$ for rational functions of degree d, modulo conjugacy. The construction involves the notions of semistable and stable reduction for φ, explained in Section 7. Building on work of Szpiro, Tepper, and Williams [2014], we show

Theorem C (moduli-theoretic characterization of $\mathrm{MinResLoc}(\varphi)$). *Let $\varphi(z) \in K(z)$ have degree $d \geq 2$. Suppose $\gamma \in \mathrm{GL}_2(K)$ and put $P = \gamma(\zeta_G)$, then*:

(A) *P belongs to* $\mathrm{MinResLoc}(\varphi)$ *if and only if φ^γ has semistable reduction in the sense of* GIT.

(B) $\mathrm{MinResLoc}(\varphi)$ *consists of the single point P if and only if φ^γ has stable reduction in the sense of* GIT.

The plan of the paper is as follows. In Section 1 we recall basic facts and definitions. In Sections 2 and 3 we establish some preliminaries concerning surplus multiplicities $s_\varphi(P, \vec{v})$ and repelling fixed points and in Section 4 we prove the tree intersection theorem (Theorem 4.2). In Section 5 we carry out slope computations needed to find the Laplacian of $\mathrm{ordRes}_\varphi(\cdot)$ on $\Gamma_{\mathrm{Fix,Repel}}$ and in Section 6 we prove Theorem A as Theorem 6.1. In Section 7, we prove Theorems B and C as Theorems 7.1 and 7.4, respectively.

1. Background

The Berkovich projective line is a path-connected Hausdorff space containing $\mathbb{P}^1(K)$. We denote the Berkovich projective line by $\mathbf{P}^1_K$ (in [Baker and Rumely 2010], it was written $\mathbb{P}^1_{\text{Berk}}$). By Berkovich's classification theorem (see e.g., [Baker and Rumely 2010, p. 5]), points of $\mathbf{P}^1_K \setminus \{\infty\}$ correspond to discs, or nested sequences of discs, in K. There are four kinds of points: Type I points are the points of $\mathbb{P}^1(K)$, which we regard as discs of radius 0. Type II points correspond to discs $D(a,r) = \{z \in K : |z-a| \le r\}$ such that r belongs to the value group $|K^\times|$. Type III points correspond to discs $D(a,r)$ with $r \notin |K^\times|$. We write $\zeta_{a,r}$ for the point corresponding to $D(a,r)$. The type II point $\zeta_{0,1}$ plays a special role; it is called the *Gauss point* and is denoted ζ_G. Type IV points serve to complete $\mathbf{P}^1_K$; they correspond to (cofinal equivalence classes of) sequences of nested discs with empty intersection. If $\{D(a_i, r_i)\}_{i\ge1}$ is such a sequence, by abuse of notation we continue to write $\zeta_{a,r}$ for the associated point in $\mathbf{P}^1_K$.

Paths in $\mathbf{P}^1_K$ correspond to ascending or descending chains of discs or unions of chains sharing an endpoint. For example the path from 0 to 1 in $\mathbf{P}^1_K$ corresponds to the chains $\{D(0,r) : 0 \le r \le 1\}$ and $\{D(1,r) : 1 \ge r \ge 0\}$; here $D(0,1) = D(1,1)$. Topologically, $\mathbf{P}^1_K$ is a tree: there is a unique path $[P, Q]$ between any two points $P, Q \in \mathbf{P}^1_K$. We write (P, Q) for the interior of that path, with similar notation for half-open segments.

If $P \in \mathbf{P}^1_K$, the *tangent space* T_P is the set of equivalence classes of paths $(P, x]$ as x varies over $\mathbf{P}^1_K \setminus \{P\}$; we call paths $(P, x]$ and $(P, y]$ *equivalent* if they share a common initial segment. We call elements of T_P *tangent vectors* or *directions* and denote them by vectors; given $\vec{v} \in T_P$, we write

$$B_P(\vec{v})^- = \{x \in \mathbf{P}^1_K : (P, x] \in \vec{v}\}$$

for the associated path-component of $\mathbf{P}^1_K \setminus \{P\}$. If $x \in B_P(\vec{v})^-$, we will say that x lies in the direction $\vec{v}$ at P. If P is of type I or type IV, then T_P has one element; if P is of type III, T_P has two elements; and if P is of type II then T_P is infinite. When $P = \zeta_G$, there is a natural one-to-one correspondence between elements of T_{ζ_G} and $\mathbb{P}^1(\tilde{k})$. More generally, for any type II point P, a map $\gamma \in \mathrm{GL}_2(K)$ with $\gamma(\zeta_G) = P$ induces a parametrization of T_P by $\mathbb{P}^1(\tilde{k})$; if $a \in \mathbb{P}^1(\tilde{k})$ we write $\vec{v}_a \in T_P$ for the corresponding direction.

The set $\mathbf{H}^1_K = \mathbf{P}^1_K \setminus \mathbb{P}^1(K)$ (written $\mathbb{H}_{\text{Berk}}$ in [Baker and Rumely 2010]) is called the *Berkovich upper half-space*; it carries a metric $\rho(x, y)$ called the *logarithmic path distance*, for which the length of the path corresponding to $\{D(a,r) : R_1 \le r \le R_2\}$ is $\log(R_2/R_1)$. (We normalize the function $\log t$ so that $\operatorname{ord}(x) = -\log|x|$.) There are two natural topologies on $\mathbf{P}^1_K$, called the *weak* and *strong* topologies. The weak topology on $\mathbf{P}^1_K$ is the coarsest one which makes the evaluation functionals $z \to |f(z)|$ continuous for all $f(z) \in K(z)$; under the weak topology, $\mathbf{P}^1_K$ is compact and $\mathbb{P}^1(K)$ is dense in it. The basic open sets for the weak topology are the path-components of $\mathbf{P}^1_K \setminus \{P_1, \ldots, P_n\}$ as $\{P_1, \ldots, P_n\}$ ranges over finite subsets of $\mathbf{H}^1_K$. The strong topology on $\mathbf{P}^1_K$ (which is finer than the weak topology) restricts to the topology on $\mathbf{H}^1_K$ induced by $\rho(x, y)$. The basic open sets for the strong topology are the $\rho(x, y)$-balls in $\mathbf{H}^1_K$ together with the basic open sets from the weak topology. Type II points are dense in $\mathbf{P}^1_K$ for both topologies.

The action of φ on $\mathbb{P}^1(K)$ extends functorially to $\mathbf{P}^1_K$. If φ is nonconstant, the induced map $\varphi : \mathbf{P}^1_K \to \mathbf{P}^1_K$ is surjective, open and continuous for both topologies and takes points of a given type to points of the same type; if $\varphi(P) = Q$, there is an induced surjection $\varphi_* : T_P \to T_Q$. The action of $\mathrm{GL}_2(K)$ on $\mathbb{P}^1(K)$ extends to an action on $\mathbf{P}^1_K$ which is continuous for both topologies and preserves the type of each point. $\mathrm{GL}_2(K)$ acts transitively on type II points. It preserves the logarithmic path distance:

$$\rho(\gamma(x), \gamma(y)) = \rho(x, y),$$

for all $x, y \in \mathbf{H}^1_K$ and $\gamma \in \mathrm{GL}_2(K)$.

At each $P \in \mathbf{P}^1_K$, the map φ has a *local degree* $\deg_\varphi(P)$ (called the multiplicity $m_\varphi(P)$ in [Baker and Rumely 2010]), which is a positive integer in the range $1 \le \deg_\varphi(P) \le d$. It has the property that for each $Q \in \mathbf{P}^1_K$,

$$\sum_{\varphi(P)=Q} \deg_\varphi(P) = d.$$

When $P \in \mathbb{P}^1(K)$, $\deg_\varphi(P)$ coincides with the classical algebraic multiplicity of φ at P.

For additional facts about $\mathbf{P}^1_K$, see [Berkovich 1990; Rivera-Letelier 2003b; Favre and Rivera-Letelier 2006; 2010; Baker and Rumely 2010; Faber 2013a; 2013b; Benedetto et al. 2014].

We will use two notions of "reduction" for φ. The first, which we simply call *the reduction of* φ, is defined as follows. If (F, G) is a normalized representation of φ, the reduction $\tilde{\varphi}$ is the rational map on $\mathbb{P}^1(\tilde{k})$ obtained by reducing F and $G \pmod{\mathfrak{m}}$ and eliminating common factors. Explicitly, let $\widetilde{F}, \widetilde{G} \in \tilde{k}[X, Y]$ be the reductions of $F, G \pmod{\mathfrak{m}}$ and put $\widetilde{A}(X, Y) = \mathrm{GCD}(\widetilde{F}(X, Y), \widetilde{G}(X, Y))$. Write $\widetilde{F}(X, Y) = \widetilde{A}(X, Y)\widetilde{F}_0(X, Y)$ and $\widetilde{G}(X, Y) = \widetilde{A}(X, Y)\widetilde{G}_0(X, Y)$. Then $\tilde{\varphi} : \mathbb{P}^1(\tilde{k}) \to \mathbb{P}^1(\tilde{k})$ is the map defined by $(X, Y) \mapsto (\widetilde{F}_0(X, Y) : \widetilde{G}_0(X, Y))$. If $\tilde{\varphi}$ has degree $d = \deg(\varphi)$, then φ is said to have *good reduction*.

The second, which we call the *reduction of* φ *at* P, is obtained by fixing a type II point $P \in \mathbb{H}_{\mathrm{Berk}}$, choosing a $\gamma \in \mathrm{GL}_2(K)$ with $\gamma(\zeta_G) = P$, and taking the reduction $\tilde{\varphi}_P$ of $\varphi^\gamma = \gamma^{-1} \circ \varphi \circ \gamma$ in the sense above. (If (F_P, G_P) is a normalized representation of φ^γ, we call (F_P, G_P) a *normalized representation of* φ *at* P.) When $P = \zeta_G$ and $\gamma = \mathrm{id}$, this notion of reduction coincides with previous one. The map $\tilde{\varphi}_P$ depends on the choice of γ, but if $\gamma' \in \mathrm{GL}_2(K)$ also satisfies $\gamma'(\zeta_G) = \zeta$ and $\tilde{\varphi}'_P$ is the reduction of φ at P corresponding to γ', there is an $\tilde{\eta} \in \mathrm{GL}_2(\tilde{k})$ such that $\tilde{\varphi}'_P = \tilde{\eta}^{-1} \circ \tilde{\varphi}_P \circ \tilde{\eta}$. Thus $\deg(\tilde{\varphi}_P)$, and the properties that $\tilde{\varphi}_P$ is constant, is nonconstant or is the identity map, are independent of the choice of γ with

$\gamma(\zeta_G) = P$. If φ does not have good reduction but after a change of coordinates by some $\gamma \in \mathrm{GL}_2(K)$ the map φ^γ has good reduction, then φ is said to have *potential good reduction*.

A theorem of Rivera-Letelier [2003a] (see [Baker and Rumely 2010, Corollary 9.27]) says that when P is of type II, then $\varphi(P) = P$ if and only if $\tilde{\varphi}_P$ is nonconstant. Rivera-Letelier shows that in this situation $\deg_\varphi(P) = \deg(\tilde{\varphi}_P)$; he calls P a *repelling fixed point* if $\deg(\tilde{\varphi}_P) \geq 2$ and an *indifferent fixed point* if $\deg(\tilde{\varphi}_P) = 1$. When $\varphi(P) = P$, the induced map $\varphi_* : T_P \to T_P$ on the tangent space comes from the action of $\tilde{\varphi}_P$ on $\mathbb{P}^1(\tilde{k})$. If $\varphi(P) \neq P$, the map $\tilde{\varphi}_P$ is constant, and in that case, if $\tilde{\varphi}_P(z) \equiv a \in \mathbb{P}^1(\tilde{k})$, then $\varphi(P)$ belongs to the ball $B_P(\vec{v}_a)^-$.

Another result of Rivera-Letelier [2003a, Lemma 2.1] (see [Faber 2013a, Proposition 3.10] for a definitive version) says that for each $P \in \mathbf{P}^1_K$ and each $\vec{v} \in T_P$, there are integers $m = m_\varphi(P, \vec{v}) \geq 1$ and $s = s_\varphi(P, \vec{v}) \geq 0$ such that

(a) for each $y \in B_{\varphi(P)}(\varphi_*(\vec{v}))^-$ there are exactly $m + s$ solutions to $\varphi(x) = y$ in $B_P(\vec{v})^-$ (counted with multiplicities), and

(b) for each $y \in \mathbf{P}^1_K \setminus \big(B_{\varphi(P)}(\varphi_*(\vec{v}))^- \cup \{\varphi(P)\}\big)$, there are exactly s solutions to $\varphi(x) = y$ in $B_P(\vec{v})^-$ (counted with multiplicities).

The number $m_\varphi(P, \vec{v})$ is called the *directional multiplicity* of φ at P in the direction $\vec{v}$, and $s_\varphi(P, \vec{v})$ is called the *surplus multiplicity* of φ at P in the direction $\vec{v}$. Several formulas relating $m_\varphi(P, \vec{v})$ to geometric quantities are given in [Baker and Rumely 2010, Theorem 9.26]. In particular, when P is of type II and $\varphi(P) = P$, then $m_\varphi(P, \vec{v}_a)$ is the algebraic multiplicity of $\tilde{\varphi}_P$ at a, for each for $a \in \mathbb{P}^1(\tilde{k})$. An important result due to Faber [2013a, Lemma 3.17] says that if P is of type II and $\varphi(P) = P$, if (F, G) is a normalized representation of φ at P and if $\widetilde{F}(X, Y) = \widetilde{A}(X, Y)\widetilde{F}_0(X, Y)$ and $\widetilde{G}(X, Y) = \widetilde{A}(X, Y)\widetilde{G}_0(X, Y)$, then $s_\varphi(P, \vec{v}_a)$ is the multiplicity of a as a root of $\widetilde{A}(X, Y)$.

Let $\varphi(z) \in K(z)$ have degree $d \geq 2$ and let (F, G) be a normalized representation of φ. Writing $F(X, Y) = f_d X^d + f_{d-1} X^{d-1} Y + \cdots + f_0 Y^d$ and $G(X, Y) = g_d X^d + g_{d-1} X^{d-1} Y + \cdots + g_0 Y^d$ the resultant of F and G is

$$\operatorname{Res}(F, G) = \det\left(\begin{bmatrix} f_d & f_{d-1} & \cdots & f_1 & f_0 & & & & \\ & f_d & f_{d-1} & \cdots & f_1 & f_0 & & & \\ & & & & \vdots & & & & \\ & & & & f_d & f_{d-1} & \cdots & f_1 & f_0 \\ g_d & g_{d-1} & \cdots & g_1 & g_0 & & & & \\ & g_d & g_{d-1} & \cdots & g_1 & g_0 & & & \\ & & & & \vdots & & & & \\ & & & & g_d & g_{d-1} & \cdots & g_1 & g_0 \end{bmatrix}\right). \tag{1}$$

The quantity $\mathrm{ordRes}(\varphi) := \mathrm{ord}(\mathrm{Res}(F, G))$ is independent of the choice of normalized representation; by construction, it is nonnegative. It is well-known (see e.g., [Silverman 2007, Theorem 2.15]) that φ has good reduction if and only if $\mathrm{ordRes}(\varphi) = 0$.

The starting point for the investigation in [Rumely 2015] was the following observation. By standard formulas for the resultant (see for example [Silverman 2007, p. 75]), for each $\gamma \in \mathrm{GL}_2(K)$ and each $\tau \in K^\times \cdot \mathrm{GL}_2(\mathcal{O})$, one has

$$\mathrm{ordRes}(\varphi^\gamma) = \mathrm{ordRes}(\varphi^{\gamma\tau}).$$

On the other hand, $K^\times \cdot \mathrm{GL}_2(\mathcal{O})$ is the stabilizer of the Gauss point. Since $\mathrm{GL}_2(K)$ acts transitively on the type II points in $\mathbf{P}^1_K$, the map $\gamma \mapsto \mathrm{ordRes}(\varphi^\gamma)$ factors through a well-defined function $\mathrm{ordRes}_\varphi(\,\cdot\,)$ on type II points given by

$$\mathrm{ordRes}_\varphi(\gamma(\zeta_G)) := \mathrm{ordRes}(\varphi^\gamma). \tag{2}$$

The main theorem in [Rumely 2015] (Theorem 0.1 of that paper) is:

Theorem 1.1. *Let $\varphi(z) \in K(z)$ and suppose $d = \deg(\varphi) \geq 2$. The function* $\mathrm{ordRes}_\varphi(\,\cdot\,)$ *on type II points extends to a function* $\mathrm{ordRes}_\varphi : \mathbf{P}^1_K \to [0, \infty]$ *which is continuous for the strong topology, is finite on $\mathbf{H}^1_K$, and takes the value ∞ on $\mathbb{P}^1(K)$. The extended function $f(\,\cdot\,) = \mathrm{ordRes}_\varphi(\,\cdot\,)$ has the following properties*:

(A) *$f(\,\cdot\,)$ is piecewise affine and convex upwards on each path in $\mathbf{P}^1_K$ relative to the logarithmic path distance*; *the slope of each affine piece is an integer m in the range $-(d^2+d) \leq m \leq d^2+d$ satisfying $m \equiv d^2+d \pmod{2d}$*; *the breaks between affine pieces occur at type II points.*

(B) *$f(\,\cdot\,)$ achieves a minimum value on $\mathbf{P}^1_K$. The set* $\mathrm{MinResLoc}(\varphi)$ *on which the minimum is taken is a single type II point if d is even and is either a single type II point or a segment with type II endpoints if d is odd.*

(C) *For each $a \in \mathbb{P}^1(K)$, the extended function $f(\,\cdot\,)$ is strictly increasing away from the tree $\Gamma_{\mathrm{Fix},\varphi^{-1}(a)} \subset \mathbf{P}^1_K$ spanned by the classical fixed points of φ and the preimages of a in $\mathbb{P}^1(K)$.*

In particular, $\mathrm{MinResLoc}(\varphi)$ *belongs to $\Gamma_{\mathrm{Fix},\varphi^{-1}(a)}$, for each $a \in \mathbb{P}^1(K)$.*

2. Identification lemmas

The surplus multiplicity $s_\varphi(P, \vec{v})$ plays an important role in the study of $\mathrm{ordRes}_\varphi(\,\cdot\,)$. It turns out that there is a relation between directions $\vec{v} \in T_P$ for which $s_\varphi(P, \vec{v}) > 0$ and directions containing type I fixed points. This is expressed in two lemmas which we call "identification lemmas", because they identify the directions which can have $s_\varphi(P, \vec{v}) > 0$.

If $P \in \mathbf{H}^1_K$ is a type II fixed point of φ and $\tilde{\varphi}_P$ is the reduction of φ at P, we say that P is id-indifferent for φ if $\tilde{\varphi}_P$ is the identity map. (This is part of a more general classification of fixed points given in Section 3.)

Definition 1 (directional and reduced fixed point multiplicities). Suppose $\varphi \in K(z)$ is nonconstant:

(A) For each $P \in \mathbf{H}^1_K$ and each $\vec{v} \in T_P$, we define the *directional fixed point multiplicity* $\#F_\varphi(P, \vec{v})$ to be the number of classical fixed points of φ in $B_P(\vec{v})^-$ (counting multiplicities).

(B) If $P \in \mathbf{H}^1_K$ is a type II fixed point of φ, which is not id-indifferent for φ, let $\tilde{\varphi}_P$ be the reduction of φ at P and parametrize T_P by $\mathbb{P}^1(\tilde{k})$ in such a way that $\varphi_*(\vec{v}_a) = \vec{v}_{\tilde{\varphi}_P(a)}$ for each $a \in \mathbb{P}^1(\tilde{k})$. If $\vec{v} = \vec{v}_a$, we define the *reduced fixed point multiplicity* $\#\widetilde{F}_\varphi(P, \vec{v})$ to be the multiplicity of a as a fixed point of $\tilde{\varphi}_P$.

Clearly $\sum_{\vec{v} \in T_P} F_\varphi(P, \vec{v}) = \deg(\varphi) + 1$. If P is a type II point with $\varphi(P) = P$ and P is not id-indifferent for φ, then $\sum_{\vec{v} \in T_P} \#\widetilde{F}_\varphi(P, \vec{v}) = \deg_\varphi(P) + 1$.

Lemma 2.1 (first identification lemma). *Suppose $P \in \mathbf{H}^1_K$ is of type II, and that $\varphi(P) = P$, but P is not* id-*indifferent. Let $\vec{v} \in T_P$; then*

$$\#F_\varphi(P, \vec{v}) = s_\varphi(P, \vec{v}) + \#\widetilde{F}_\varphi(P, \vec{v}). \tag{3}$$

In particular, $B_P(\vec{v})^-$ contains a type I fixed point of φ if and only if at least one of the following hold:

$$\varphi_*(\vec{v}) = \vec{v} \quad \textit{or} \quad s_\varphi(P, \vec{v}) > 0.$$

Proof. Choose $\gamma \in \mathrm{GL}_2(K)$ with $\gamma(P) = \zeta_G$ and put $\Phi = \varphi^\gamma$; then $\Phi(\zeta_G) = \zeta_G$. Note that $\xi \in \mathbb{P}^1(K)$ is fixed by φ if and only if $\gamma^{-1}(\xi)$ is fixed by Φ and that $\vec{v} \in T_P$ is fixed by φ_* if and only if $(\gamma^{-1})_*(\vec{v}) \in T_{\zeta_G}$ is fixed by Φ_*. Hence it suffices to prove the result under the assumption that $P = \zeta_G$.

Choose $F(X, Y), G(X, Y) \in \mathcal{O}[X, Y]$ so (F, G) is a normalized representation of φ. Let $\widetilde{F}, \widetilde{G} \in \tilde{k}[X, Y]$ be the reductions of F and G and put

$$\widetilde{A}(X, Y) = \mathrm{GCD}(\widetilde{F}(X, Y), \widetilde{G}(X, Y)).$$

Write $\widetilde{F}(X, Y) = \widetilde{A}(X, Y) \cdot \widetilde{F}_0(X, Y)$ and $\widetilde{G}(X, Y) = \widetilde{A}(X, Y) \cdot \widetilde{G}_0(X, Y)$. Then $\tilde{\varphi}$ is the map on $\mathbb{P}^1(\tilde{k})$ defined by $(X, Y) \mapsto (\widetilde{F}_0(X, Y) : \widetilde{G}_0(X, Y))$.

Put $H(X, Y) = XG(X, Y) - YF(X, Y)$ and $\widetilde{H}_0(X, Y) = X\widetilde{G}_0(X, Y) - Y\widetilde{F}_0(X, Y)$. Here $H(X, Y) \neq 0$ since $\deg(\varphi) > 1$ and $\widetilde{H}_0(X, Y) \neq 0$ since $\tilde{\varphi}(z) \not\equiv z$ by assumption. The fixed points of Φ are the zeros of $H(X, Y)$ in $\mathbb{P}^1(K)$ and the fixed points of $\tilde{\varphi}$ are the zeros of $\widetilde{H}_0(X, Y)$ in $\mathbb{P}^1(\tilde{k})$.

Reducing $H(X, Y)$ modulo $\mathfrak{m}$, we see that

$$\widetilde{H}(X, Y) = X\widetilde{G}(X, Y) - Y\widetilde{F}(X, Y) = \widetilde{A}(X, Y) \cdot \widetilde{H}_0(X, Y). \tag{4}$$

Since K is algebraically closed, $H(X, Y)$ factors over $K[X, Y]$ as a product of linear factors. After scaling the factors if necessary, we can assume that the factorization has the form

$$H(X, Y) = C \cdot \prod_{i=1}^{d+1} (b_i X - a_i Y), \tag{5}$$

where $\max(|a_i|, |b_i|) = 1$ for each $i = 1, \ldots, d+1$. We claim that $|C| = 1$ as well. To see this, note that if we choose u, v in $\mathcal{O}$ so that $(\tilde{u} : \tilde{v})$ is not a zero of either $\widetilde{A}(X, Y)$ or $\widetilde{H}_0(X, Y)$, then $\widetilde{H}(\tilde{u}, \tilde{v}) \neq 0$ by (4).

It follows that

$$\widetilde{H}(X, Y) = \widetilde{C} \cdot \prod_{i=1}^{d+1} (\tilde{b}_i X - \tilde{a}_i Y) \not\equiv 0. \tag{6}$$

Since $\tilde{k}[X, Y]$ is a unique factorization domain, comparing (4) and (6) we see that after reordering factors if necessary, there are constants $\widetilde{C}_1, \widetilde{C}_2 \in \tilde{k}^\times$ and an integer $0 \leq n \leq d$ such that

$$\widetilde{A}(X, Y) = \widetilde{C}_1 \cdot \prod_{i=1}^{n} (\tilde{b}_i X - \tilde{a}_i Y) \quad \text{and} \quad \widetilde{H}_0(X, Y) = \widetilde{C}_2 \cdot \prod_{i=n+1}^{d+1} (\tilde{b}_i X - \tilde{a}_i Y). \tag{7}$$

The fixed points of $\Phi(X, Y)$ are the points $(a_i : b_i) \in \mathbb{P}^1(K)$, $i = 1, \ldots, d+1$. By (4) and (7), each fixed point of $\Phi(X, Y)$ specializes to a zero of $\widetilde{H}_0(X, Y)$ or $\widetilde{A}(X, Y)$ and conversely each such zero is the specialization of a fixed point of $\Phi(X, Y)$. For a given $\vec{v} \in T_{\zeta_G}$, if $\tilde{a} \in \mathbb{P}^1(\tilde{k})$ is such that $\vec{v} = \vec{v}_{\tilde{a}}$, then $\#F_\varphi(P, \vec{v})$ is the multiplicity of $\tilde{a}$ as a zero of $\widetilde{H}(X, Y)$, $s_\varphi(P, \vec{v})$ is the multiplicity of $\tilde{a}$ as a zero of $\widetilde{A}(X, Y)$, and $\#\widetilde{F}_\varphi(P, \vec{v})$ is the multiplicity of $\tilde{a}$ as a root of $\widetilde{H}_0(X, Y)$. Thus $\#F_\varphi(P, \vec{v}) = s_\varphi(P, \vec{v}) + \#\widetilde{F}_\varphi(P, \vec{v})$. □

Lemma 2.2 (second identification lemma). *Suppose $P \in \mathbf{H}_K^1$ is of type II and that $\varphi(P) = Q \neq P$. Let $\vec{v} \in T_P$. Then $B_P(\vec{v})^-$ contains a type I fixed point of φ if and only if at least one of the following hold*

(A) $Q \in B_P(\vec{v})^-$,

(B) $P \in B_Q(\varphi_*(\vec{v}))^-$, *or*

(C) $s_\varphi(P, \vec{v}) > 0$.

Proof. Let $Q = \varphi(P)$. We claim that there is a $\gamma \in \mathrm{GL}_2(K)$ such that $\gamma(\zeta_G) = P$ and $\gamma^{-1}(Q) = \zeta_{0,r}$ for some $0 < r < 1$. To see this, take any $\tau_0 \in \mathrm{GL}_2(K)$ with $\tau_0(\zeta_G) = P$. Then $\tau_0^{-1}(Q) \neq \zeta_G$, since τ_0 is one-to-one. Let $\alpha \in \mathbb{P}^1(K)$ be such that the path $[\alpha, \zeta_G]$ contains $\tau_0^{-1}(Q)$ and choose $\tau_1 \in \mathrm{GL}_2(\mathcal{O})$ with $\tau_1(0) = \tau_0^{-1}(\alpha)$. (Such a τ_1 exists because $\mathrm{GL}_2(\mathcal{O})$ acts transitively on $\mathbb{P}^1(K)$). Setting $\gamma = \tau_0 \circ \tau_1$, we see that $\gamma(\zeta_G) = P$ and $\gamma(0) = \gamma_0(\tau_1(0)) = \alpha$. This means that

$\gamma^{-1}(Q)=\tau_1^{-1}(\tau_0^{-1}(Q))$ belongs to $\tau_1^{-1}([\alpha,\zeta_G])=[0,\zeta_G]$, so $\gamma^{-1}(Q)=\zeta_{0,r}$ for some $0<r<1$. Here $r\in|K^\times|$, since $\zeta_{0,r}=\gamma^{-1}(\varphi(P))$ is of type II. Since the fixed points of φ and conditions (A), (B), (C) are equivariant under conjugation, replacing φ with $\varphi^\gamma=\gamma^{-1}\circ\varphi\circ\gamma$ we are reduced to the case where $P=\zeta_G$ and $Q=\zeta_{0,r}$.

Take any $c\in K^\times$ with $|c|=r$. Put $\gamma_1(z)=\mathrm{id}$ and $\gamma_2(z)=cz$; then $\gamma_1(\zeta_G)=\zeta_G=P$ and $\gamma_2(\zeta_G)=\zeta_{0,r}=Q$. Put $\Phi=\gamma_2^{-1}\circ\varphi\circ\gamma_1$, noting that $\varphi(z)=c\cdot\Phi(z)$.

Choose $F(X,Y),G(X,Y)\in\mathcal{O}[X,Y]$ so (F,G) is a normalized representation of Φ. Let $\widetilde{F},\widetilde{G}\in\tilde{k}[X,Y]$ be their reductions and put

$$\widetilde{A}(X,Y)=\mathrm{GCD}(\widetilde{F}(X,Y),\widetilde{G}(X,Y)).$$

Write $\widetilde{F}(X,Y)=\widetilde{A}(X,Y)\cdot\widetilde{F}_0(X,Y)$ and $\widetilde{G}(X,Y)=\widetilde{A}(X,Y)\cdot\widetilde{G}_0(X,Y)$. The reduction $\widetilde{\Phi}$ of Φ is the map $(X,Y)\mapsto(\widetilde{F}_0(X,Y):\widetilde{G}_0(X,Y))$, so by Faber's theorem [2013a, Lemma 3.17], the directions $\vec{v}_a\in T_P$ with $s_\varphi(P,\vec{v}_a)>0$ (which are the same as the ones with $s_\Phi(P,\vec{v}_a)>0$) are precisely those corresponding to points $a\in\mathbb{P}^1(\tilde{k})$ with $\widetilde{A}(a)=0$.

Since $\varphi=c\cdot\Phi$, the pair (cF,G) is a normalized representation of φ. The fixed points of φ in $\mathbb{P}^1(K)$ are the zeros $(a_i:b_i)$ of $H(X,Y)=XG(X,Y)-Y\cdot cF(X,Y)$. As in the proof of Lemma 2.1, we can write

$$H(X,Y)=C\cdot\prod_{i=1}^{d+1}(b_iX-a_iY),\tag{8}$$

where $\max(|a_i|,|b_i|)=1$ for each $i=1,\dots,d+1$ and $|C|=1$. Absorbing C into the coefficients a_1,b_1, we can assume that $C=1$. Reducing $H(X,Y)$ modulo $\mathfrak{m}$ gives

$$\widetilde{H}(X,Y)=\prod_{i=1}^{d+1}(\tilde{b}_iX-\tilde{a}_iY)\not\equiv 0.\tag{9}$$

On the other hand, since $H(X,Y)=XG(X,Y)-Y\cdot cF(X,Y)$ with $|c|<1$, we also have

$$\widetilde{H}(X,Y)=X\widetilde{G}(X,Y)=X\cdot\widetilde{A}(X,Y)\cdot\widetilde{G}_0(X,Y).\tag{10}$$

Comparing (9) and (10), we see that after reordering the factors if necessary, there are constants $\widetilde{C}_1,\widetilde{C}_2,\widetilde{C}_3\in\tilde{k}^\times$ and an integer $1\le n\le d$ such that in $\tilde{k}[X,Y]$

$$X=\widetilde{C}_1\cdot(\tilde{b}_1X-\tilde{a}_1Y),\tag{11}$$

$$\widetilde{A}(X,Y)=\widetilde{C}_2\cdot\prod_{i=2}^{n}(\tilde{b}_iX-\tilde{a}_iY),\tag{12}$$

$$\widetilde{G}_0(X,Y)=\widetilde{C}_3\cdot\prod_{i=n+1}^{d+1}(\tilde{b}_iX-\tilde{a}_iY),\tag{13}$$

and each fixed point of φ corresponds to a factor of one of these terms.

From (11) it follows that $|a_1| < 1$ and $|b_1| = 1$. Thus the fixed point $(a_1 : b_1)$ belongs to $B_P(\vec{v}_0)^-$, where $\vec{v}_0 \in T_P$ is the tangent direction corresponding to $0 \in \mathbb{P}^1(\tilde{k})$. We can express this in a coordinate-free way by noting that $B_P(\vec{v}_0)^-$ is the ball containing $Q = \zeta_{0,r} = \varphi(P)$.

From (12) and the fact that the zeros of $\widetilde{A}(X, Y)$ correspond to the directions $\vec{v}_a$ for which $s_\varphi(P, \vec{v}_a) > 0$, we see that each direction $\vec{v} \in T_P$ with $s_\varphi(P, \vec{v}) > 0$ contains a fixed point of φ.

Finally, from (13), we see that each direction $\vec{v}_a \in T_P$ corresponding to a zero of $\widetilde{G}_0(X, Y)$ contains a fixed point of φ. However, the zeros of $\widetilde{G}_0(X, Y)$ are the poles of $\widetilde{\Phi}$ and if $\widetilde{\Phi}(a) = \infty$ then $\Phi_*(\vec{v}_a) = \vec{v}_\infty \in T_P$. Since $\varphi = c \cdot \Phi$ and $|c| < 1$, it follows that $\varphi_*(\vec{v}_a) \in T_Q = T_{\zeta_{0,r}}$ is the "upwards" direction $\vec{v}_{Q,\infty} \in T_Q$. This can be expressed a coordinate-free manner by noting that $B_Q(\varphi_*(\vec{v}_a))^- = B_Q(\vec{v}_{Q,\infty})^-$ is the unique ball containing $P = \zeta_G$. □

As a consequence of Lemmas 2.1 and 2.2 we obtain

Corollary 2.3. *Let P be of type II and suppose P is not* id*-indifferent for φ. Then for each $\vec{v} \in T_P$ such that $s_\varphi(P, \vec{v}) > 0$, the ball $B_P(\vec{v})^-$ contains a type I fixed point of φ.*

Remark. Later, when we have proved the tree intersection theorem (Theorem 4.2) we will establish a third identification lemma for id-indifferent points P (Lemma 4.5) and we will strengthen Corollary 2.3, obtaining a fixed point theorem for arbitrary balls $B_P(\vec{v})^-$ which φ maps onto $\mathbf{P}^1_K$ (Theorem 4.6).

3. Classification of fixed points in $\mathbf{H}^1_K$

Recall that a fixed point P of φ in $\mathbf{H}^1_K$ is called *indifferent* if $\deg_\varphi(P) = 1$ and *repelling* if $\deg_\varphi(P) > 1$. We begin by introducing two trees which will play an important role:

Definition 2 (the trees Γ_{Fix} and $\Gamma_{\text{Fix,Repel}}$). Suppose $\varphi(z) \in K(z)$ has degree $d \geq 2$. Let Γ_{Fix} be the tree in $\mathbf{P}^1_K$ spanned by the type I (classical) fixed points of φ. Let $\Gamma_{\text{Fix,Repel}}$ be the tree in $\mathbf{P}^1_K$ spanned by the type I fixed points of φ and the type II repelling fixed points of φ in $\mathbf{H}^1_K$.

Clearly $\Gamma_{\text{Fix}} \subset \Gamma_{\text{Fix,Repel}}$. Since φ has at most $d+1$ distinct type I fixed points, Γ_{Fix} is a finitely generated tree. It is possible that φ has a single type I fixed point of multiplicity $d+1$, in which case Γ_{Fix} is reduced to a point.

Below, we refine the notions of indifferent and repelling fixed points in $\mathbf{H}^1_K$.

Definition 3 (refined classification of indifferent fixed points in $\mathbf{H}^1_K$). Let $P \in \mathbf{H}^1_K$ be a type II indifferent fixed point of φ and let $\tilde{\varphi}_P$ be the reduction of φ at P. After a change of coordinates, $\tilde{\varphi}_P$ is of one of three types:

(A) $\tilde{\varphi}_P(z) = z$; in this case we will say P is id-*indifferent* for φ.

(B) $\tilde{\varphi}_P(z) = \tilde{a} \cdot z$ for some $\tilde{a} \in \tilde{k}^\times$ with $\tilde{a} \neq 1$; in this case we will say P is *multiplicatively indifferent* for φ and that $\tilde{\varphi}_P$ has reduced rotation number $\tilde{a}$. (We remark that the reduced rotation number is only well-defined as an element of $\{\tilde{a}, \tilde{a}^{-1}\}$: if coordinates on $\mathbb{P}^1(\tilde{k})$ are changed by conjugating with $1/z$, then $\tilde{a}$ is replaced by $\tilde{a}^{-1}$. If we need to be more precise, we will speak as follows. Note that the directions $\vec{v}_0, \vec{v}_\infty \in T_P$ corresponding to $0, \infty \in \mathbb{P}^1(\tilde{k})$ are the only directions $\vec{v} \in T_P$ fixed by φ_*. Let points $P_0 \in B_P(\vec{v}_0)^-$, $P_\infty \in B_P(\vec{v}_\infty)^-$ be given. We will say that φ has reduced rotation number $\tilde{a}$ for the axis (P_0, P_∞) and that it has reduced rotation number $\tilde{a}^{-1}$ for the axis (P_∞, P_0).)

(C) $\tilde{\varphi}_P(z) = z + \tilde{a}$ for some $\tilde{a} \in \tilde{k}$ with $\tilde{a} \neq 0$; in this case we will say that P is *additively indifferent* for φ, with reduced translation number $\tilde{a}$. The reduced translation number is well-defined: it is the same for any choice of coordinates on $\mathbb{P}^1(\tilde{k})$ with $\tilde{\varphi}_P(\infty) = \infty$.

Definition 4 (reduced multipliers for fixed directions). Suppose $P \in \mathbf{H}^1_K$ is a type II fixed point of φ and let $\vec{v} \in T_P$ be a direction with $\varphi_*(\vec{v}) = \vec{v}$ and $m_\varphi(P, \vec{v}) = 1$. Let $\tilde{\varphi}_P$ be the reduction of φ at P; without loss, assume coordinates have been chosen so that $\vec{v} = \vec{v}_0$. Then 0 is a fixed point of $\tilde{\varphi}_P$. Put $\tilde{a} = \tilde{\varphi}'_P(0) \in \tilde{k}$. We will call $\tilde{a}$ the *reduced multiplier* of φ in the direction $\vec{v}$.

Definition 5 (refined classification of repelling fixed points in $\mathbf{H}^1_K$). Let $\varphi(z) \in K(z)$ have degree $d \geq 2$ and let $P \in \mathbf{H}^1_K$ be a repelling fixed point of φ. The directions $\vec{v}_1, \ldots, \vec{v}_m \in T_P$ such that $B_P(\vec{v}_i)^-$ contains a classical fixed point of φ, will be called the *focal directions* of φ at P:

(A) P will be called *unifocused* (or just *focused*) if it has a unique focal direction $\vec{v}_1$.

(B) P will be called *bifocused* if it has exactly two distinct focal directions $\vec{v}_1, \vec{v}_2$.

(C) P will be called *multifocused* if it has $m \geq 3$ focal directions.

Below are some properties of unifocused repelling fixed points.

Proposition 3.1. *A repelling fixed point of φ in $\mathbf{H}^1_K$ is a focused repelling fixed point if and only if it does not belong to Γ_{Fix}. Each focused repelling fixed point is an endpoint of $\Gamma_{\mathrm{Fix,Repel}}$. If P is a focused repelling fixed point, with focus $\vec{v}_1$, then*:

(A) $\varphi_*(\vec{v}_1) = \vec{v}_1$, $m_\varphi(P, \vec{v}_1) = 1$, *and* $\#\widetilde{F}_\varphi(P, \vec{v}_1) = \deg_\varphi(P) + 1 \geq 3$.

(B) $s_\varphi(P, \vec{v}_1) = d - \deg_\varphi(P)$ *and* $\vec{v}_1$ *is the only direction* $\vec{v} \in T_P$ *for which one could have* $s_\varphi(P, \vec{v}) > 0$.

(C) *For each* $\vec{v} \in T_P$ *with* $\vec{v} \neq \vec{v}_1$ *we have* $\varphi_*(\vec{v}) \neq \vec{v}$ *and* $\varphi(B_P(\vec{v})^-)$ *is the ball* $B_P(\varphi_*(\vec{v}))^-$.

(D) *For each* $\vec{w} \in T_P$, *there is at least one* $\vec{v} \in T_P$ *with* $\vec{v} \neq \vec{v}_1$ *such that* $\varphi_*(\vec{v}) = \vec{w}$.

Proof. Let P be a repelling fixed point of φ. If $P \notin \Gamma_{\mathrm{Fix}}$ all the type I fixed points of φ lie in a single ball $B_P(\vec{v}_1)^-$, so P is a focused repelling fixed point. Conversely, suppose P is a focused repelling fixed point with focus $\vec{v}_1$. By assumption, all the type I fixed points of φ belong to $B_P(\vec{v}_1)^-$, so $\Gamma_{\mathrm{Fix}} \subset B_P(\vec{v}_1)^-$. Thus, $P \notin \Gamma_{\mathrm{Fix}}$.

We next show that if P is a focused repelling fixed point, it must be an endpoint of $\Gamma_{\mathrm{Fix,Repel}}$, and the only $\vec{v} \in T_P$ fixed by φ_* is $\vec{v}_1$. After a change of coordinates, we can assume that $P = \zeta_G$. Index directions $\vec{v} \in T_P$ by points $\tilde{\alpha} \in \mathbb{P}^1(\tilde{k})$ and choose coordinates so that $\vec{v}_1$ corresponds to $\tilde{\alpha} = 1$. Take any $\vec{v} \in T_P$ with $\vec{v} \neq \vec{v}_1$. Since $B_P(\vec{v})^-$ contains no type I fixed points, Lemma 2.1 shows that $s_\varphi(P, \vec{v}) = 0$ and $\varphi_*(\vec{v}) \neq \vec{v}$. Thus $\varphi(B_P(\vec{v})^-)$ is a ball, necessarily $B_P(\varphi_*(\vec{v})^-)$. If P were not an endpoint of $\Gamma_{\mathrm{Fix,Repel}}$, there would be a direction $\vec{v} \in T_P$ with $\vec{v} \neq \vec{v}_1$ such that $B_P(\vec{v})^-$ contained a repelling fixed point Q of φ. This is impossible, since $\varphi(B_P(\vec{v})^-) = B_P(\varphi_*(\vec{v}))^-$ where $\varphi_*(\vec{v}) \neq \vec{v}$, yet $Q \in B_P(\vec{v})^-$ and $\varphi(Q) = Q$. From the fact that $\varphi_*(\vec{v}) \neq \vec{v}$ for all $\vec{v} \neq \vec{v}_1$, it follows that $\varphi_*(\vec{v}_1) = \vec{v}_1$, since the action of φ_* on T_P corresponds to the action of the reduction $\tilde{\varphi}$ on $\mathbb{P}^1(\tilde{k})$ and $\tilde{\varphi}$ has at least one fixed point.

Since P is a repelling fixed point, necessarily $\deg_\varphi(P) \geq 2$. Since $\vec{v}_1$ is the only direction fixed by φ_*, $\tilde{\alpha} = 1$ is the only point of $\mathbb{P}^1(\tilde{k})$ fixed by $\tilde{\varphi}(z)$. Thus, $\tilde{\alpha} = 1$ is a fixed point of $\tilde{\varphi}$ of multiplicity $\deg(\tilde{\varphi}) + 1 = \deg_\varphi(P) + 1$. This means $\varphi_*(\vec{v}_1) = \vec{v}_1$ and $\#\widetilde{F}_\varphi(P, \vec{v}_1) = \deg_\varphi(P) + 1 \geq 3$. Since $\tilde{\alpha} = 1$ is a fixed point of $\tilde{\varphi}$ of multiplicity > 1, α is not a critical point of $\tilde{\varphi}$, so $m_\varphi(P, \vec{v}_1) = 1$.

Since all the classical fixed points belong to $B_P(\vec{v}_1)^-$, one has $\#F_\varphi(P, \vec{v}_1) = d+1$. By Lemma 2.1, $\#F_\varphi(P, \vec{v}_1) = s_\varphi(P, \vec{v}_1) + \#\widetilde{F}_\varphi(P, \vec{v}_1)$. We have seen above that $\#\widetilde{F}_\varphi(P, \vec{v}_1) = \deg_\varphi(P) + 1$, so $s_\varphi(P, \vec{v}_1) = d - \deg_\varphi(P)$. Since $\#F_\varphi(P, \vec{v}) = 0$ for all $\vec{v} \in T_P$ with $\vec{v} \neq \vec{v}_1$, Lemma 2.1 shows that $\vec{v}_1$ is the only direction in T_P for which one could have $s_\varphi(P, \vec{v}) > 0$.

Finally, we show that for each $\vec{w} \in T_P$, there is a $\vec{v} \in T_P$ with $\vec{v} \neq \vec{v}_1$ satisfying $\varphi_*(\vec{v}) = \vec{w}$. If $\vec{w} \neq \vec{v}_1$, this is trivial, since $\varphi_* : T_P \to T_P$ is surjective and $\varphi_*(\vec{v}_1) = \vec{v}_1$. If $\vec{w} = \vec{v}_1$, it follows from the fact that $m_\varphi(P, \vec{v}_1) = 1$, since

$$\sum_{\vec{v} \in T_P,\, \varphi_*(\vec{v}) = \vec{v}_1} m_\varphi(P, \vec{v}) = \deg_\varphi(P) \geq 2. \qquad \square$$

It may be worthwhile to note that focused repelling fixed points can exist. Here is a class of maps with a focused repelling fixed point at ζ_G, with focal direction $\vec{v}_0$:

Example (maps with a focused repelling fixed point at ζ_G). Fix $d \geq 2$ and let $\widetilde{L}(X, Y) \in \tilde{k}[X, Y]$ be a homogeneous form of degree $d-1$ with $\widetilde{L}(X, Y) \neq \widetilde{C} X^{d-1}$. Let $L_1(X, Y), L_2(X, Y) \in \mathcal{O}[X, Y]$ be homogeneous forms of degree $d-1$ whose reductions satisfy

$$\widetilde{L}_1(X, Y) \equiv \widetilde{L}_2(X, Y) \equiv \widetilde{L}(X, Y) \pmod{\mathfrak{m}}.$$

Put

$$F(X, Y) = XL_1(X, Y) \quad \text{and} \quad G(X, Y) = X^d + YL_2(X, Y).$$

For generic $L_1(X, Y)$ and $L_2(X, Y)$ we will have $\mathrm{GCD}(F, G) = 1$; if this fails, perturb $L_1(X, Y)$ or $L_2(X, Y)$ slightly. Henceforth we will assume $\mathrm{GCD}(F, G) = 1$.

Consider the map φ with representation (F, G). Write

$$\widetilde{F}(X, Y) = \widetilde{A}(X, Y)\widetilde{F}_0(X, Y) \quad \text{and} \quad \widetilde{G}(X, Y) = \widetilde{A}(X, Y)\widetilde{G}_0(X, Y).$$

The reduced map $\tilde{\varphi}$ is represented by $(\widetilde{F}_0, \widetilde{G}_0)$. Since $\widetilde{F}(X, Y) = X\widetilde{L}(X, Y)$ and $\widetilde{G}(X, Y) = X^d + Y\widetilde{L}(X, Y)$, we have

$$\widetilde{A}(X, Y) = \mathrm{GCD}(\widetilde{F}(X, Y), \widetilde{G}(X, Y)) = \mathrm{GCD}(X^{d-1}, \widetilde{L}(X, Y)).$$

Since $\widetilde{L}(X, Y) \neq \widetilde{C}X^{d-1}$, we must have $\widetilde{A}(X, Y) = X^s$ for some $0 \leq s \leq d-2$. Putting $n = d - s$, it follows that $\deg(\tilde{\varphi}) = n \geq 2$. Thus ζ_G is a repelling fixed point of φ and $\deg_\varphi(\zeta_G) = n$. Let $\vec{v}_0 \in T_{\zeta_G}$ be the direction corresponding to $0 \in \tilde{k} \subset \mathbb{P}^1(\tilde{k})$.

The fixed points of φ in $\mathbb{P}^1(K)$ are the zeros of $H(X, Y) = XG(X, Y) - YF(X, Y)$. The reduction of $H(X, Y)$ is

$$\begin{aligned}\widetilde{H}(X, Y) &= X\widetilde{G}(X, Y) - Y\widetilde{F}(X, Y)\\ &= X \cdot (X^d + Y\widetilde{L}_2(X, Y)) - Y \cdot (X\widetilde{L}_1(X, Y)) = X^{d+1},\end{aligned} \tag{14}$$

since $\widetilde{L}_1(X, Y) = \widetilde{L}_2(X, Y)$. By the theory of Newton polygons, the type I fixed points all belong to $B_{\zeta_G}(\vec{v}_0)^-$. We claim that ζ_G is a focused repelling fixed point for φ. To see this, note that Γ_{Fix} is contained in $B_{\zeta_G}(\vec{v}_0)^-$, so $\zeta_G \notin \Gamma_{\mathrm{Fix}}$. Since ζ_G is a repelling fixed point of φ, it belongs to $\Gamma_{\mathrm{Fix,Repel}}$. By Proposition 3.1, ζ_G must be a focused repelling fixed point.

By taking $\widetilde{L}(X, Y) = X^s Y^{d-1-s}$ for a given integer $0 \leq s \leq d-2$, we can arrange that $\widetilde{A}(X, Y) = X^s$. Thus for any pair (n, s) with $2 \leq n \leq d$ and $n + s = d$, there is a $\varphi \in K(z)$ of degree d which has a focused repelling fixed point at ζ_G with $\deg_\varphi(\zeta_G) = n$ and $s_\varphi(\zeta_G, \vec{v}_0) = s$. □

We next consider the properties of bifocused repelling fixed points. First, we will need a lemma.

Lemma 3.2. *Suppose $P \in \mathbf{H}^1_K$ is of type II and $\vec{v} \in T_P$. If there is a focused repelling fixed point $Q \in B_P(\vec{v})^-$ whose focus $\vec{v}_1$ points towards P, then $s_\varphi(P, \vec{v}) > 0$.*

Proof. We must show that $\varphi(B_P(\vec{v})^-) = \mathbf{P}^1_K$. Since $(\mathbf{P}^1_K \setminus B_Q(\vec{v}_1)^-) \subset B_P(\vec{v})^-$, it suffices to show that

$$\varphi\big(\mathbf{P}^1_K \setminus B_Q(\vec{v}_1)^-\big) = \mathbf{P}^1_K.$$

To see this, note that $\mathbf{P}^1_K \setminus B_Q(\vec{v}_1)^- = \{Q\} \cup \big(\bigcup_{\vec{v} \in T_Q, \vec{v} \neq \vec{v}_1} B_Q(\vec{v})^-\big)$. By assumption $\varphi(Q) = Q$. Since $\varphi_* : T_Q \to T_Q$ is surjective, Proposition 3.1 shows that for each

$\vec{w} \in T_Q$ there is at least one $\vec{v} \in T_Q$ with $\vec{v} \neq \vec{v}_1$ for which $\varphi_*(B_Q(\vec{v})^-) = B_Q(\vec{w})^-$. Since $\mathbf{P}^1_K = \{Q\} \cup \big(\bigcup_{\vec{w} \in T_Q} B_Q(\vec{w})^-\big)$, the claim follows. $\square$

Proposition 3.3. *A repelling fixed point of φ in $\mathbf{H}^1_K$ is a bifocused repelling fixed point if and only if it belongs Γ_{Fix} but is not a branch point of $\Gamma_{\mathrm{Fix,Repel}}$. Suppose P is a bifocused repelling fixed point and let $\vec{v}_1, \vec{v}_2 \in T_P$ be its focal directions, then*:

(A) $\varphi_*(\vec{v}) = \vec{v}$, $m_\varphi(P, \vec{v}) = 1$, *and* $\#\widetilde{F}_\varphi(P, \vec{v}) \geq 2$, *for at least one* $\vec{v} \in \{\vec{v}_1, \vec{v}_2\}$.

(B) *For each* $\vec{v} \in T_P$ *with* $\vec{v} \neq \vec{v}_1, \vec{v}_2$, *we have* $\varphi_*(\vec{v}) \neq \vec{v}$ *and* $s_\varphi(P, \vec{v}) = 0$.

Proof. Suppose P is a bifocused repelling fixed point. By definition, there are type I fixed points α_1, α_2 of φ belonging to distinct directions in T_P, so $P \in \Gamma_{\mathrm{Fix}}$. However, P cannot be a branch point of Γ_{Fix}, because if it were there would be at least three distinct directions in T_P containing type I fixed points. It also cannot be a branch point of $\Gamma_{\mathrm{Fix,Repel}}$ which is not a branch point of Γ_{Fix}, because if it were there would be at least one branch of $\Gamma_{\mathrm{Fix,Repel}} \setminus \Gamma_{\mathrm{Fix}}$ off P. If $\vec{v} \in T_P$ is the corresponding direction, then $B_P(\vec{v})^-$ would contain a type II repelling fixed point Q but no type I fixed points. Since Q is a focused repelling fixed point, whose focus $\vec{v}_1 \in T_Q$ points towards Γ_{Fix}, by Lemma 3.2 we would have $s_\varphi(P, \vec{v}) > 0$. However, this contradicts Lemma 2.1 since $B_P(\vec{v})^-$ contains no type I fixed points.

Conversely, suppose $P \in \Gamma_{\mathrm{Fix}}$ is a type II repelling fixed point but is not a vertex of $\Gamma_{\mathrm{Fix,Repel}}$. Then there are exactly two directions $\vec{v}_1, \vec{v}_2 \in T_P$ containing type I fixed points, so P is a bifocused repelling fixed point.

To prove assertions (A) and (B), let P be any bifocused repelling fixed point. After a change of coordinates, we can assume that $P = \zeta_G$ and that 0 and ∞ are fixed points of φ. Let (F, G) be a normalized representation of φ. Since $\varphi(z)$ fixes P, it has nonconstant reduction. Thus there are nonzero homogeneous polynomials

$$\widetilde{A}(X, Y), \widetilde{F}_0(X, Y), \widetilde{G}_0(X, Y) \in \tilde{k}[X, Y]$$

such that $(\widetilde{F}, \widetilde{G}) = (\widetilde{A} \cdot \widetilde{F}_0, \widetilde{A} \cdot \widetilde{G}_0)$, with $\mathrm{GCD}(\widetilde{F}_0, \widetilde{G}_0) = 1$. Since P is a repelling fixed point, we have $\delta := \deg_\varphi(P) \geq 2$ and $\deg(\widetilde{F}_0) = \deg(\widetilde{G}_0) = \delta$. Write $H(X, Y) = XG(X, Y) - YF(X, Y)$ and put $\widetilde{H}_0(X, Y) = X\widetilde{G}_0(X, Y) - Y\widetilde{F}_0(X, Y)$. Then $\widetilde{H}(X, Y) = \widetilde{A}(X, Y) \cdot \widetilde{H}_0(X, Y)$. Since $0, \infty \in \mathbb{P}^1(K)$ are fixed points of φ and P is a bifocused repelling fixed point, $\vec{v}_0, \vec{v}_\infty \in T_P$ are the only directions $\vec{v} \in T_P$ for which the balls $B_P(\vec{v})^-$ can contain type I fixed points. It follows from Lemma 2.1 that $\widetilde{H}(X, Y) = \tilde{c} \cdot X^\ell Y^{d+1-\ell}$ for some $\tilde{c} \in \tilde{k}^\times$ and some ℓ. Since $\widetilde{H}_0(X, Y) \mid \widetilde{H}(X, Y)$, we must have $\widetilde{H}_0(X, Y) = \tilde{h} \cdot X^{\ell_0} Y^{\delta+1-\ell_0}$ for some $\tilde{h} \in \tilde{k}^\times$ and some $0 \leq \ell_0 \leq \delta + 1$. Since $\delta + 1 \geq 3$, either $\#\widetilde{F}_\varphi(P, \vec{v}_0) \geq 2$ or $\#\widetilde{F}_\varphi(P, \vec{v}_\infty) \geq 2$. If $i \in \{1, 2\}$ is such that $\#\widetilde{F}_\varphi(P, \vec{v}_i) \geq 2$ then necessarily $\varphi_*(\vec{v}_i) = \vec{v}_i$ and $m_\varphi(P, \vec{v}_i) = 1$. This proves assertion (A).

For each $\vec{v} \in T_P$ with $\vec{v} \neq \vec{v}_0, \vec{v}_\infty$, there are no type I fixed point in $B_P(\vec{v})^-$, so Lemma 2.1 shows that $\varphi_*(\vec{v}) \neq \vec{v}$ and $s_\varphi(P, \vec{v}) = 0$. Thus assertion (B) holds. $\square$

Finally, we consider multifocused repelling fixed points. Recall that the valence of a point P in a graph Γ is the number of edges of Γ emanating from P.

Proposition 3.4. *A fixed point of φ in $\mathbf{H}_K^1$ is a multifocused repelling fixed point if and only if it is a repelling fixed point and is a branch point of Γ_{Fix}. If P is a multifocused repelling fixed point, then its valence in $\Gamma_{\mathrm{Fix,Repel}}$ is the same as its valence in Γ_{Fix}.*

Remark. Branch points of Γ_{Fix} need not be repelling fixed points: there can be branch points of Γ_{Fix} which are indifferent fixed points of φ or are moved by φ.

Proof of Proposition 3.4. If P is a multifocused repelling fixed point, then at least three directions in T_P contain type I fixed points of φ, so P is a branch point of Γ_{Fix}. Conversely, if a repelling fixed point P is a branch point of Γ_{Fix}, at least three directions in T_P contain type I fixed points of φ, so P is multifocused.

The second assertion can be reformulated as saying that if P is a multifocused repelling fixed point of φ, then there are no branches of $\Gamma_{\mathrm{Fix,Repel}} \setminus \Gamma_{\mathrm{Fix}}$ which fork off Γ_{Fix} at P. Suppose to the contrary that there were such a branch and let $\vec{v} \in T_P$ be the corresponding direction. By the same argument as in the proof of Proposition 3.3, $B_P(\vec{v})^-$ would contain a type II repelling fixed point Q but no type I fixed points. Since Q is a focused repelling fixed point, whose focus $\vec{v}_1$ points towards P, by Lemma 3.2 we would have $s_\varphi(P, \vec{v}) > 0$. However, this contradicts Lemma 2.1 since $B_P(\vec{v})^-$ contains no type I fixed points. $\square$

The fact that focused repelling fixed points are endpoints of $\Gamma_{\mathrm{Fix,Repel}}$, while bifocused repelling fixed points and multifocused repelling fixed points belong to Γ_{Fix}, leads one to ask about the nature of points of $\Gamma_{\mathrm{Fix,Repel}} \setminus \Gamma_{\mathrm{Fix}}$ which are not endpoints of $\Gamma_{\mathrm{Fix,Repel}}$.

Proposition 3.5. *Suppose P is a focused repelling fixed point of φ and let P_0 be the nearest point to P in Γ_{Fix}. Then each type II point $Q \in (P, P_0]$ is an* id*-indifferent fixed point of φ.*

Proof. Let Q be a type II point in $(P, P_0]$. Let $\vec{w} \in T_Q$ be the direction for which $P \in B_Q(\vec{w})^-$. The focus $\vec{v}_1$ of P points towards Γ_{Fix} and hence towards Q. By Lemma 3.2, we have $s_\varphi(Q, \vec{w}) > 0$. If $\varphi(Q) = Q$ but Q is not id-indifferent, this contradicts Lemma 2.1 since $B_Q(\vec{w})^-$ does not contain any type I fixed points of φ. If $\varphi(Q) \neq Q$, it contradicts Lemma 2.2 for the same reason. Thus, Q must be id-indifferent. $\square$

4. The tree intersection theorem

In this section we study the tree $\Gamma_{\mathrm{Fix,Repel}}$. We first note that it is never a single point.

Lemma 4.1. *Let $\varphi(z) \in K(z)$ have degree $d \geq 2$. Then either φ has at least two type I fixed points or it has one type I fixed point and one or more type II repelling fixed points. In either case, $\Gamma_{\mathrm{Fix,Repel}}$ has at least one edge.*

Proof. If φ has at least two type I fixed points, we are done. If it has only one, that point is necessarily a fixed point of multiplicity $d+1$, hence has multiplier 1 and is an indifferent fixed point. By a theorem of Rivera-Letelier (see [Rivera-Letelier 2003b, Theorem B] or [Baker and Rumely 2010, Theorem 10.82]), φ has at least one repelling fixed point in $\mathbf{P}^1_K$ (which may be in either $\mathbb{P}^1(K)$ or $\mathbf{H}^1_K$), so in this case φ must have a repelling fixed point in $\mathbf{H}^1_K$. □

Theorem 4.2 (the tree intersection theorem). *Let $\varphi \in K(z)$ have degree $d \geq 2$. Then $\Gamma_{\mathrm{Fix,Repel}}$ is the intersection of the trees $\Gamma_{\mathrm{Fix},\varphi^{-1}(a)}$, for all $a \in \mathbb{P}^1(K)$.*

Proof. Write Γ_0 for the intersection of the trees $\Gamma_{\mathrm{Fix},\varphi^{-1}(a)}$, for all $a \in \mathbb{P}^1(K)$.

We first show that $\Gamma_{\mathrm{Fix,Repel}} \subseteq \Gamma_0$. For this, it is enough to show that each repelling fixed point of φ in $\mathbf{H}^1_K$ belongs to Γ_0, since Γ_0 is connected and clearly contains the type I fixed points. Let P be a repelling fixed point of φ in $\mathbf{H}^1_K$ and fix $a \in \mathbb{P}^1(K)$. We claim that P belongs to $\Gamma_{\mathrm{Fix},\varphi^{-1}(a)}$.

By a theorem of Rivera-Letelier (see [Rivera-Letelier 2000, Proposition 5.1] or [Baker and Rumely 2010, Lemma 10.80]), P is of type II. By [Baker and Rumely 2010, Corollary 2.13(B)], there is a $\gamma \in \mathrm{GL}_2(K)$ for which $\gamma(\infty) = a$ and $\gamma(\zeta_G) = P$. After conjugating φ by γ, we can assume that $a = \infty$ and $P = \zeta_G$. Let (F, G) be a normalized representation of φ. The poles of φ are the zeros of $G(X, Y)$ and the fixed points of φ are the zeros of $H(X, Y) := YF(X, Y) - XG(X, Y)$. Let $\widetilde{F}, \widetilde{G} \in \tilde{k}[X, Y]$ be the reductions of F and G. Put

$$\widetilde{A} = \mathrm{GCD}(\widetilde{F}, \widetilde{G}).$$

Write $\widetilde{F} = \widetilde{A} \cdot \widetilde{F}_0$, and $\widetilde{G} = \widetilde{A} \cdot \widetilde{G}_0$. Then $\tilde{\varphi}$ is the map $(X, Y) \mapsto (\widetilde{F}_0(X, Y), \widetilde{G}_0(X, Y))$ on $\mathbb{P}^1(\tilde{k})$. Since P is a repelling fixed point of φ, we have $\tilde{d} := \deg(\tilde{\varphi}) \geq 2$.

Put $\widetilde{H}_0(X, Y) = Y\widetilde{F}_0(X, Y) - X\widetilde{G}_0(X, Y)$, so $\deg(\widetilde{H}_0) = \tilde{d} + 1$. The fixed points of $\tilde{\varphi}$ are the zeros of $\widetilde{H}_0(X, Y)$ in $\mathbb{P}^1(\tilde{k})$, listed with multiplicities. Since $\mathrm{GCD}(\widetilde{F}_0, \widetilde{G}_0) = 1$ and $\mathrm{GCD}(\widetilde{H}_0, \widetilde{G}_0) = \mathrm{GCD}(Y\widetilde{F}_0, \widetilde{G}_0)$, we must have either $\mathrm{GCD}(\widetilde{H}_0, \widetilde{G}_0) = 1$ or $\mathrm{GCD}(\widetilde{H}_0, \widetilde{G}_0) = Y$.

If $\mathrm{GCD}(\widetilde{H}_0, \widetilde{G}_0) = 1$, the fixed points and poles of $\tilde{\varphi}$ are disjoint. Since each fixed point of $\tilde{\varphi}$ is the reduction of at least one fixed point of φ (Lemma 2.1) and each pole of $\tilde{\varphi}$ is the reduction of at least one pole of φ, we conclude that φ has a fixed point z_0 and a pole z_1 lying in different directions in T_P. Thus P belongs to $[z_0, z_1]$ and $P \in \Gamma_{\mathrm{Fix},\varphi^{-1}(a)}$.

If $\mathrm{GCD}(\widetilde{H}_0, \widetilde{G}_0) = Y$, then Y divides $\widetilde{G}_0$, so $\tilde{\infty}$ is a pole of $\tilde{\varphi}$. This means φ has a pole z_1 in the ball $B_{\zeta_G}(\vec{v}_\infty)^- \subset \mathbb{P}^1(K)$. On the other hand, Y^2 cannot divide both $\widetilde{H}_0$ and $\widetilde{G}_0$. If Y^2 does not divide $\widetilde{H}_0$, then $\tilde{\varphi}$ has at least one fixed point in $\tilde{k}$, so φ has a fixed point z_0 in $\mathcal{O}$. Since z_0 and z_1 lie in different tangent directions at ζ_G, we conclude that $\zeta_G \in \Gamma_{\mathrm{Fix},\varphi^{-1}(a)}$ in this case. On the other hand, if Y^2 does not divide $\widetilde{G}_0$, then $\tilde{\varphi}$ has at least one pole in $\tilde{k}$, so φ has a pole $z_0 \in \mathcal{O}$, and again $P = \zeta_G \in \Gamma_{\mathrm{Fix},\varphi^{-1}(a)}$.

We next show that $\Gamma_{\mathrm{Fix,Repel}} = \Gamma_0$. Since $\Gamma_{\mathrm{Fix,Repel}} \subseteq \Gamma_0$, it will suffice to show that each endpoint of $\Gamma_{\mathrm{Fix,Repel}}$ which does not belong to Γ_{Fix} is an endpoint of the intersection of some set of trees $\{\Gamma_{\mathrm{Fix},\varphi^{-1}(a_i)} : i \in I\}$, for an appropriate index set I.

By Proposition 3.1, each endpoint of $\Gamma_{\mathrm{Fix,Repel}}$ not in Γ_{Fix} is a focused repelling fixed point of φ. Let $P \in \mathbf{H}^1_K$ be a focused repelling fixed point, with focus $\vec{v}_1$. Choose distinct directions $\vec{v}_2, \vec{v}_3 \in T_P \setminus \{\vec{v}_1\}$ and take $a_2 \in \mathbb{P}^1(K) \cap B_P(\vec{v}_2)^-$ and $a_3 \in \mathbb{P}^1(K) \cap B_P(\vec{v}_3)^-$. We claim that P is an endpoint of $\Gamma_{\mathrm{Fix},\varphi^{-1}(a_2)} \cap \Gamma_{\mathrm{Fix},\varphi^{-1}(a_3)}$. To see this, note that if $\xi_2 \in \mathbb{P}^1(K) \cap (\mathbf{P}^1_K \setminus B_P(\vec{v}_1)^-)$ is a solution to $\varphi(\xi_2) = a_2$ and $\xi_3 \in \mathbb{P}^1(K) \cap (\mathbf{P}^1_K \setminus B_P(\vec{v}_1)^-)$ is a solution to $\varphi(\xi_3) = a_3$, the directions $\vec{w}_2, \vec{w}_3 \in T_P$ such that $\xi_2 \in B_P(\vec{w}_2)^-$ and $\xi_3 \in B_P(\vec{w}_3)^-$ are necessarily distinct. This follows from Proposition 3.1, which asserts that for each $\vec{w} \in T_P$ with $\vec{w} \neq \vec{v}_1$, the image $\varphi(B_P(\vec{w})^-)$ is precisely $B_P(\varphi_*(\vec{w}))^-$. □

The tree $\Gamma_{\mathrm{Fix,Repel}}$ is spanned by finitely many points.

Corollary 4.3. *If $\varphi(z) \in K(z)$ has degree $d \geq 2$, then*

(A) *φ has at most d focused repelling fixed points, and*

(B) *$\Gamma_{\mathrm{Fix,Repel}}$ is a finitely generated tree with at most $2d+1$ endpoints. Each endpoint of $\Gamma_{\mathrm{Fix,Repel}}$ is either a type I fixed point or a type II focused repelling fixed point.*

Proof. If $\varphi(z)$ has no focused repelling fixed points, part (A) holds trivially. Otherwise, choose any focused repelling fixed point P. By Proposition 3.1, it is an endpoint of $\Gamma_{\mathrm{Fix,Repel}}$; let $\vec{v} \in T_P$ be a direction pointing away from $\Gamma_{\mathrm{Fix,Repel}}$. Fix a point $\alpha \in \mathbb{P}^1(K) \cap B_P(\vec{v})^-$ and consider the solutions $\xi_1, \ldots, \xi_d$ to $\varphi(z) = \alpha$. By Proposition 3.1, for each focused repelling fixed point Q there is a direction $\vec{w}_Q \in T_Q$ pointing away from $\Gamma_{\mathrm{Fix,Repel}}$ such that $\alpha \in \varphi(B_Q(\vec{w}_Q)^-)$; it follows that some ξ_i belongs to $B_Q(\vec{w}_Q)^-$. For distinct Q, the balls $B_Q(\vec{w}_Q)^-$ are pairwise disjoint. Thus φ has at most d focused repelling fixed points.

Since φ has at most $d+1$ type I fixed points, the tree $\Gamma_{\mathrm{Fix,Repel}}$ is spanned by at most $2d+1$ points. Its endpoints are clearly as claimed. □

We will sharpen Corollary 4.3 in Corollary 6.2. Below are some additional consequences of Theorem 4.2. A trivial but important one is:

Corollary 4.4. *Let $\varphi(z) \in K(z)$ have degree $d \geq 2$. Then* $\mathrm{MinResLoc}(\varphi)$ *is contained in* $\Gamma_{\mathrm{Fix,Repel}}$.

Proof. By Theorem 1.1, $\mathrm{MinResLoc}(\varphi)$ is contained in $\Gamma_{\mathrm{Fix},\varphi^{-1}(a)}$ for each $a \in \mathbb{P}^1(K)$. Hence it is contained in the intersection of those trees, which is $\Gamma_{\mathrm{Fix,Repel}}$. □

Another is an "identification lemma" for id-indifferent fixed points.

Lemma 4.5 (third identification lemma). *Suppose P is a type II* id-*indifferent fixed point of φ. Then for each $\vec{v} \in T_P$ such that $s_\varphi(P, \vec{v}) > 0$, the ball $B_P(\vec{v})^-$ contains either a type I fixed point or a type II focused repelling fixed point of φ.*

Proof. Suppose $s_\varphi(P, \vec{v}) > 0$. This means $\varphi(B_P(\vec{v})^-) = \mathbf{P}^1_K$. If $B_P(\vec{v})^-$ contains a type I fixed point, we are done.

If not, then there must be some $\vec{v}_0 \in T_P$ with $\vec{v}_0 \neq \vec{v}$ such that $B_P(\vec{v}_0)^-$ contains a type I fixed point. Since $\varphi(B_P(\vec{v})^-) = \mathbf{P}^1_K$, for each $a \in \mathbb{P}^1(K)$ there is a solution to $\varphi(x) = a$ in $B_P(\vec{v})^-$. The path from this solution to the type I fixed point passes through P, so $P \in \Gamma_{\mathrm{Fix},\varphi^{-1}(a)}$. Letting a vary, we see that

$$P \in \bigcap_{a \in \mathbb{P}^1(K)} \Gamma_{\mathrm{Fix},\varphi^{-1}(a)} = \Gamma_0 = \Gamma_{\mathrm{Fix,Repel}}.$$

Since P is id-indifferent, we must have $m_\varphi(P, \vec{v}) = 1$. Hence by a theorem of Rivera-Letelier (see [Rivera-Letelier 2003b, §4] or [Baker and Rumely 2010, Theorem 9.46]) there is a $Q \in B_P(\vec{v})^-$ such that φ maps the annulus $\mathrm{Ann}(P, Q)$ to itself and fixes each point in $[P, Q]$. Take any type II point $Z \in (P, Q)$ and let $\vec{w} \in T_Z$ be the direction towards Q. We claim that $s_\varphi(Z, \vec{w}) > 0$. If not, φ maps $B_Z(\vec{w})^-$ to a ball and that ball must be $B_Z(\vec{w})^-$ since $\varphi(Z) = Z$ and $\varphi_*(\vec{w}) = \vec{w}$. However, this means that

$$\begin{aligned}
\varphi(B_P(\vec{v})^-) &= \varphi(\mathrm{Ann}(P, Q) \cup B_Z(\vec{w})^-) \\
&= \varphi(\mathrm{Ann}(P, Q)) \cup \varphi(B_Z(\vec{w})^-) \\
&= \mathrm{Ann}(P, Q) \cup B_Z(\vec{w})^- \\
&= B_P(\vec{v})^-,
\end{aligned}$$

which contradicts that $s_\varphi(P, \vec{v}) > 0$. Hence it must be that $s_\varphi(Z, \vec{w}) > 0$ and, by the same argument as above, it follows that $Z \in \Gamma_{\mathrm{Fix,Repel}}$.

Thus $\Gamma_{\mathrm{Fix,Repel}}$ contains points in $B_P(\vec{v})^-$, so it has an endpoint in $B_P(\vec{v})^-$. Since $B_P(\vec{v})^-$ does not contain type I fixed points, the endpoint must be a focused repelling fixed point. □

As Lemma 4.5 shows, when P is a type II id-indifferent fixed point, the linkage between directions $\vec{v} \in T_P$ for which $s_\varphi(P, \vec{v}) > 0$ and directions containing a type I fixed point breaks down. It turns out that for id-indifferent fixed points it is the "primary terms" in a normalized representation at P which determine when $s_\varphi(P, \vec{v}) > 0$ and the "secondary terms" which determine the type I fixed points. This is made precise by the following class of examples:

Example (maps with an id-indifferent fixed point at ζ_G). Fix $d \geq 2$ and let $\widetilde{A}(X, Y) \in \tilde{k}[X, Y]$ be a nonzero homogeneous form of degree $d-1$. Lift $\widetilde{A}(X, Y)$

to $A(X,Y) \in \mathcal{O}[X,Y]$ and fix $\pi \in \mathcal{O}$ with $\mathrm{ord}(\pi) > 0$. Let $F_1(X,Y), G_1(X,Y) \in \mathcal{O}[X,Y]$ be arbitrary homogeneous forms of degree d. Put

$$F(X,Y) = A(X,Y) \cdot X + \pi F_1(X,Y),$$
$$G(X,Y) = A(X,Y) \cdot Y + \pi G_1(X,Y).$$

For generic $F_1(X,Y)$ and $G_1(X,Y)$ we will have $\mathrm{GCD}(F,G) = 1$; if that is the case, let φ be the map with normalized representation (F,G). Then

(1) $(\widetilde{F}(X,Y), \widetilde{G}(X,Y)) = \widetilde{A}(X,Y) \cdot (X,Y)$, while

(2) $H(X,Y) = XG(X,Y) - YF(X,Y) = \pi(XG_1(X,Y) - YF_1(X,Y))$.

Thus the directions $\vec{v} \in T_{\zeta_G}$ with $s_\varphi(\zeta_G, \vec{v}) > 0$ come from the roots of $\widetilde{A}(X,Y)$, while the type I fixed points of φ are the roots of $XG_1(X,Y) - YF_1(X,Y)$. □

By combining the three identification lemmas we obtain a new fixed point theorem which concerns balls which φ maps onto $\mathbf{P}^1_K$. Previously known fixed point theorems (see [Baker and Rumely 2010, Theorems 10.83, 10.85, and 10.86]) concern domains $D \subset \mathbf{P}^1_K$ whose image $\varphi_*(D)$ is another domain, with $D \subset \varphi_*(D)$ or $\varphi_*(D) \subset D$, or closed sets X with $\varphi(X) \subseteq X$.

Theorem 4.6 (full image fixed point theorem). *Let $\varphi(z) \in K(z)$, with $\deg(\varphi) \geq 2$. Suppose $P \in \mathbf{P}^1_K$ and $\vec{v} \in T_P$. If $\varphi(B_P(\vec{v})^-) = \mathbf{P}^1_K$, then $B_P(\vec{v})^-$ contains either a (classical) fixed point of φ in $\mathbb{P}^1(K)$ or a repelling fixed point of φ in $\mathbf{H}^1_K$.*

Proof. Given points $P, Q \in \mathbb{P}^1_{\mathrm{Berk}}$ with $Q \neq P$, write $\mathrm{Ann}(P,Q)$ for the component of $\mathbf{P}^1_K \setminus \{P,Q\}$ containing (P,Q).

Assume $\varphi(B_P(\vec{v})^-) = \mathbf{P}^1_K$. If P is of type II, then $s_\varphi(P,\vec{v}) > 0$ and the result follows by combining Lemmas 2.1, 2.2, and 4.5. If P is of type III or IV, one reduces to the case of type II as follows: There is a point $Q \in B_P(\vec{v})^-$ such that $\varphi(\mathrm{Ann}(P,Q)) = \mathrm{Ann}(\varphi(P),\varphi(Q))$. Take any type II point $Z \in (P,Q)$ and let $\vec{w} \in T_Z$ be the direction towards Q. We claim that $\varphi(B_Z(\vec{w})^-) = \mathbf{P}^1_K$. Otherwise, $\varphi_*(B_Z(\vec{w})^-)$ would be the ball $B_{\varphi(Z)}(\varphi_*(\vec{w}))^-$. By the mapping properties of annuli $\varphi_*(\vec{w})$ is the direction in $T_{\varphi(Z)}$ containing $\varphi(Q)$, so

$$\varphi(B_P(\vec{v})^-) = \varphi(\mathrm{Ann}(P,Q)) \cup \varphi(B_Z(\vec{w})^-) = \mathrm{Ann}(\varphi(P),\varphi(Q)) \cup B_{\varphi(Z)}(\varphi_*(\vec{w}))^-$$

omits the point $\varphi(P)$, contradicting that $\varphi(B_P(\vec{v})^-) = \mathbf{P}^1_K$.

If P is of type I and $B_P(\vec{v})^- = \mathbf{P}^1_K \setminus \{P\}$ contains no type I fixed points of φ, then P is the only type I fixed point of φ. Hence it is a fixed point of multiplicity $d+1 > 1$ and necessarily has multiplier 1. By a theorem of Rivera-Letelier (see [Rivera-Letelier 2003b, Theorem B] or [Baker and Rumely 2010, Theorem 10.82]) φ has at least one repelling fixed point in $\mathbf{P}^1_K$, so it has a repelling fixed point in $\mathbf{H}^1_K$, which clearly lies in $B_P(\vec{v})^-$. □

The author does not know whether Theorem 4.6 can be generalized to domains D with $\varphi(D) = \mathbf{P}^1_K$, when D has more than one boundary point.

5. Slope formulas

In this section we establish formulas for the slope of $\mathrm{ordRes}_\varphi(\,\cdot\,)$ at an arbitrary point $P \in \mathbf{H}^1_K$, in an arbitrary direction $\vec{v} \in T_P$. These formulas will be used in the proof of the weight formula in Section 6.

Let $f(z)$ be a function on $\mathbf{H}^1_K$ and write $\rho(P, Q)$ for the logarithmic path distance. Given a point $P \in \mathbf{H}^1_K$ and a direction $\vec{v} \in T_P$, the slope of f at P in the direction $\vec{v}$ is defined to be

$$\partial_{\vec{v}} f(P) = \lim_{\substack{Q \to P \\ Q \in B_P(\vec{v})^-}} \frac{F(Q) - F(P)}{\rho(Q, P)}, \tag{15}$$

provided the limit exists. For any two points $P_1, P_2 \in B_P(\vec{v})^-$, the paths $[P, P_1]$ and $[P, P_2]$ share a common initial segment, so the limit in (15) exists if and only if it exists for Q restricted to $[P, P_1]$. Since $\rho(P, Q)$ is invariant under the action of $\mathrm{GL}_2(K)$ on $\mathbf{H}^1_K$, $\partial_{\vec{v}} F(P)$ is independent of the choice of coordinates.

For the remainder of this section we will take $f(\,\cdot\,) = \mathrm{ordRes}_\varphi(\,\cdot\,)$.

Definition 6 (tangent space and valence in $\Gamma_{\mathrm{Fix,Repel}}$). For a point $P \in \Gamma_{\mathrm{Fix,Repel}}$, we write $T_{P,\mathrm{FR}}$ for the tangent space to P in $\Gamma_{\mathrm{Fix,Repel}}$, the set of directions $\vec{v} \in T_P$ such that there is an edge of $\Gamma_{\mathrm{Fix,Repel}}$ emanating from P in the direction $\vec{v}$. We write $v_{\mathrm{FR}}(P)$ for the valence of P in $\Gamma_{\mathrm{Fix,Repel}}$, the number of edges of $\Gamma_{\mathrm{Fix,Repel}}$ emanating from P.

Definition 7 (shearing directions). If P is a fixed point of φ in $\mathbf{P}^1_K$ (of any type I, II, III, or IV), a direction $\vec{v} \in T_P$ will be called a *shearing direction* if there is a type I fixed point in $B_P(\vec{v})^-$ but $\varphi_*(\vec{v}) \neq \vec{v}$. Let $N_{\mathrm{Shearing}}(P)$ be the number of shearing directions in T_P.

We will now give formulas for the slope of $\mathrm{ordRes}_\varphi(\,\cdot\,)$ at each type II point P, in each direction $\vec{v} \in T_P$. The formulas break into several cases, depending on the nature of P. For the convenience of the reader, we first state all the formulas, then give the proofs.

Proposition 5.1. *Let $P \in \mathbf{H}^1_K$ be a type II fixed point of φ which is not* id*-indifferent. Then for each $\vec{v} \in T_P$, the slope of $f(\,\cdot\,) = \mathrm{ordRes}_\varphi(\,\cdot\,)$ at P in the direction $\vec{v}$ is*

$$\begin{aligned} \partial_{\vec{v}} f(P) &= (d^2 - d) - 2d \cdot \#F_\varphi(P, \vec{v}) + 2d \cdot \max(1, \#\widetilde{F}_\varphi(P, \vec{v})) \\ &= (d^2 - d) - 2d \cdot s_\varphi(P, \vec{v}) + 2d \cdot \begin{cases} 0 & \text{if } \varphi_*(\vec{v}) = \vec{v}, \\ 1 & \text{if } \varphi_*(\vec{v}) \neq \vec{v}. \end{cases} \end{aligned} \tag{16}$$

Furthermore, if $P \in \Gamma_{\mathrm{Fix,Repel}}$, *then*

$$\sum_{\vec{v} \in T_{P,\mathrm{FR}}} \partial_{\vec{v}} f(P) = (d^2 - d) \cdot (v_{\mathrm{FR}}(P) - 2) + 2d \cdot (\deg_{\varphi}(P) - 1 + N_{\mathrm{Shearing}}(P)). \quad (17)$$

Proposition 5.2. *Let* $P \in \mathbf{H}^1_K$ *be a type II* id-*indifferent fixed point of* φ. *Then for each* $\vec{v} \in T_P$, *the slope of* $f(\cdot) = \mathrm{ordRes}_{\varphi}(\cdot)$ *at* P *in the direction* $\vec{v}$ *is*

$$\partial_{\vec{v}} f(P) = (d^2 - d) - 2d \cdot s_{\varphi}(P, \vec{v}). \quad (18)$$

Furthermore, if $P \in \Gamma_{\mathrm{Fix,Repel}}$, *then*

$$\sum_{\vec{v} \in T_{P,\mathrm{FR}}} \partial_{\vec{v}} f(P) = (d^2 - d) \cdot (v_{\mathrm{FR}}(P) - 2). \quad (19)$$

Proposition 5.3. *Let* $P \in \mathbf{H}^1_K$ *be a type II point with* $\varphi(P) \neq P$. *Then for each* $\vec{v} \in T_P$, *the slope of* $f(\cdot) = \mathrm{ordRes}_{\varphi}(\cdot)$ *at* P *in the direction* $\vec{v}$ *is*

$$\partial_{\vec{v}} f(P) = d^2 + d - 2d \cdot \#F_{\varphi}(P, \vec{v}). \quad (20)$$

Furthermore, if $P \in \Gamma_{\mathrm{Fix,Repel}}$, *then*

$$\sum_{\vec{v} \in T_{P,\mathrm{FR}}} \partial_{\vec{v}} f(P) = (d^2 - d) \cdot (v_{\mathrm{FR}}(P) - 2) + 2d \cdot (v_{\mathrm{FR}}(P) - 2). \quad (21)$$

Finally, we consider the "slope of $\mathrm{ordRes}_{\varphi}(\cdot)$ in the direction away from Q", at a type I point Q. Since $\mathrm{ordRes}_{\varphi}(Q) = \infty$, the definition of the slope given in (15) does not make sense. Instead, for a suitable Q_1 near Q, we show that $\mathrm{ordRes}_{\varphi}(\cdot)$ has constant slope in the direction of Q_1, for each $P \in (Q, Q_1)$.

Proposition 5.4. *Let* Q *be a type I point. Then there is a* $Q_1 \in \mathbf{H}^1_K$ *such that for each* $P \in (Q, Q_1)$, *the slope of* $f(\cdot) = \mathrm{ordRes}_{\varphi}(\cdot)$ *at* P, *in the direction* $\vec{v}_1 \in T_P$ *which points towards* Q_1, *is*

$$\partial_{\vec{v}_1} f(P) = \begin{cases} -(d^2 - d) & \text{if } Q \text{ is fixed by } \varphi, \\ -(d^2 + d) & \text{if } Q \text{ is not fixed by } \varphi. \end{cases}$$

We will now prove Propositions 5.1–5.4.

Proof of Proposition 5.1. After a change of coordinates, we can assume that $P = \zeta_G$ and $\vec{v} = \vec{v}_0$. Let (F, G) be a normalized representation of φ; write $F(X, Y) = a_d X^d + \cdots + a_0 Y^d$ and $G(X, Y) = b_d X^d + \cdots + b_0 Y^d$. For each $A \in K^{\times}$ we have, by [Rumely 2015, formula 13],

$$\begin{aligned} &\mathrm{ordRes}_{\varphi}(\zeta_{0,|A|}) - \mathrm{ordRes}_{\varphi}(\zeta_G) \\ &\qquad = (d^2 + d)\mathrm{ord}(A) - 2d \cdot \min\Bigl(\min_{0 \le i \le d} \mathrm{ord}(A^i a_i), \min_{0 \le j \le d} \mathrm{ord}(A^{j+1} b_j)\Bigr). \end{aligned}$$

By hypothesis, the reductions $\widetilde{F}$ and $\widetilde{G}$ are nonzero; thus there are indices i and j such that $\tilde{a}_i \neq 0$ and $\tilde{b}_j \neq 0$ or equivalently that $\mathrm{ord}(a_i) = 0$ and $\mathrm{ord}(b_j) = 0$. Let ℓ_1 be the least index i for which $\mathrm{ord}(a_i) = 0$ and let ℓ_2 be the least index j for which $\mathrm{ord}(b_j) = 0$. If $\mathrm{ord}(A) > 0$ and $\mathrm{ord}(A)$ is sufficiently small, then

$$\begin{aligned}\min\Big(\min_{0\le i\le d}\mathrm{ord}(A^i a_i),\ \min_{0\le j\le d}\mathrm{ord}(A^{j+1}b_j)\Big) &= \min\big(\mathrm{ord}(A^{\ell_1}a_{\ell_1}),\ \mathrm{ord}(A^{\ell_2+1}b_{\ell_2})\big)\\ &= \min(\ell_1, \ell_2+1)\cdot \mathrm{ord}(A).\end{aligned}$$

It follows that the slope of $\mathrm{ordRes}_\varphi(\,\cdot\,)$ at P in the direction $\vec{v} = \vec{v}_0$ is

$$d^2 + d - 2d\cdot\min(\ell_1, \ell_2+1). \tag{22}$$

We will now reformulate (22) using dynamical invariants. Letting $\widetilde{F}, \widetilde{G} \in \tilde{k}[X, Y]$ be the reductions of F and G, we can factor $\widetilde{F} = \widetilde{A}\cdot\widetilde{F}_0$ and $\widetilde{G} = \widetilde{A}\cdot\widetilde{G}_0$ where $\widetilde{A} = \mathrm{GCD}(\widetilde{F}, \widetilde{G})$. Write $\mathrm{ord}_X(\widetilde{F})$ and $\mathrm{ord}_X(\widetilde{G})$ for the power with which X occurs as a factor of $\widetilde{F}$ and $\widetilde{G}$. Using Faber's theorem [2013a, Lemma 3.17], it follows that

$$s_\varphi(P, \vec{v}_0) = \mathrm{ord}_X(\widetilde{A}) = \min\big(\mathrm{ord}_X(\widetilde{F}), \mathrm{ord}_X(\widetilde{G})\big) = \min(\ell_1, \ell_2).$$

On the other hand, since P is not id-indifferent for φ, by Lemma 2.1

$$s_\varphi(P, \vec{v}_0) = \#F_\varphi(P, \vec{v}_0) - \#\widetilde{F}_\varphi(P, \vec{v}_0).$$

Note that $\#\widetilde{F}_\varphi(P, \vec{v}_0) = 0$ holds if and only if $\mathrm{ord}_X(\widetilde{F}_0) = 0$, which in turn holds if and only if $\ell_1 \le \ell_2$. Thus, if $\#\widetilde{F}_\varphi(P, \vec{v}_0) = 0$, then $\ell_1 \le \ell_2$ and

$$\min(\ell_1, \ell_2+1) = \ell_1 = s_\varphi(P, \vec{v}_0) = \#F_\varphi(P, \vec{v}_0).$$

On the other hand, if $\#\widetilde{F}_\varphi(P, \vec{v}_0) > 0$ then $\ell_2 < \ell_1$ and $s_\varphi(P, \vec{v}_0) = \ell_2$, so

$$\min(\ell_1, \ell_2+1) = \ell_2 + 1 = s_\varphi(P, \vec{v}_0) + 1 = \#F_\varphi(P, \vec{v}_0) - \#\widetilde{F}_\varphi(P, \vec{v}_0) + 1.$$

It follows that

$$\begin{aligned}\min(\ell_1, \ell_2+1) &= \begin{cases}\#F_\varphi(P, \vec{v}_0) & \text{if } \#\widetilde{F}_\varphi(P, \vec{v}_0) = 0,\\ \#F_\varphi(P, \vec{v}_0) - \#\widetilde{F}_\varphi(P, \vec{v}_0) + 1 & \text{if } \#\widetilde{F}_\varphi(P, \vec{v}_0) > 0\end{cases}\\ &= \#F_\varphi(P, \vec{v}_0) - \max(1, \#\widetilde{F}_\varphi(P, \vec{v}_0)) + 1.\end{aligned} \tag{23}$$

Inserting (23) in (22) yields the first formula in (16). The second formula in (16) follows from $s_\varphi(P, \vec{v}) = \#F_\varphi(P, \vec{v}) - \#\widetilde{F}_\varphi(P, \vec{v})$, since $\varphi_*(\vec{v}) = \vec{v}$ if and only if $\#\widetilde{F}_\varphi(P, \vec{v}) > 0$.

If $P \in \Gamma_{\mathrm{Fix,Repel}}$, note that by Lemma 2.1, for a direction $\vec{v} \in T_{P,\mathrm{FR}}$, then $\varphi_*(\vec{v}) \neq \vec{v}$ if and only if $\vec{v}$ is a shearing direction. To obtain (17), sum the second formula in (16) over all $\vec{v} \in T_{P,\mathrm{FR}}$, getting

$$\sum_{\vec{v}\in T_{P,\mathrm{FR}}} \partial_{\vec{v}} f(P) = (d^2 - d)\cdot v_{\mathrm{FR}}(P) - 2d\cdot \sum_{\vec{v}\in T_{P,\mathrm{FR}}} s_\varphi(P, \vec{v}) + 2d\cdot N_{\mathrm{Shearing}}(P). \tag{24}$$

By Lemma 2.1, $T_{P,\mathrm{FR}}$ contains all $\vec{v} \in T_P$ such that $s_\varphi(P, \vec{v}) > 0$. It follows that

$$\sum_{\vec{v} \in T_{P,\mathrm{FR}}} s_\varphi(P, \vec{v}) = d - \deg_\varphi(P).$$

Inserting this in (24) yields (17). □

Proof of Proposition 5.2. After a change of coordinates, we can assume that $P = \zeta_G$ and $\vec{v} = \vec{v}_0$. Let (F, G) be a normalized representation of φ; write $F(X, Y) = a_d X^d + \cdots + a_0 Y^d$ and $G(X, Y) = b_d X^d + \cdots + b_0 Y^d$. Letting ℓ_1 and ℓ_2 be the least index such that $\mathrm{ord}(a_i) = 0$ and $\mathrm{ord}(b_j) = 0$, respectively, just as in Proposition 5.1 one sees that the slope of $\mathrm{ordRes}_\varphi(\,\cdot\,)$ at P in the direction $\vec{v} = \vec{v}_0$ is

$$d^2 + d - 2d \cdot \min(\ell_1, \ell_2 + 1). \tag{25}$$

Since P is id-indifferent for φ, the reductions $\widetilde{F}$ and $\widetilde{G}$ are nonzero and if $\widetilde{A} = \mathrm{GCD}(\widetilde{F}, \widetilde{G})$ then $\widetilde{F}(X, Y) = X \cdot \widetilde{A}(X, Y)$ and $\widetilde{G}(X, Y) = Y \cdot \widetilde{A}(X, Y)$. Thus $\ell_1 = \mathrm{ord}_X(\widetilde{A}(X, Y)) + 1$ and $\ell_2 = \mathrm{ord}_X(\widetilde{A}(X, Y))$. By Faber's theorem [2013a, Lemma 3.17], we have $s_\varphi(P, \vec{v}_0) = \mathrm{ord}_X(\widetilde{A})$. Hence

$$\min(\ell_1, \ell_2 + 1) = \mathrm{ord}_X(\widetilde{A}) + 1 = s_\varphi(P, \vec{v}_0) + 1. \tag{26}$$

Inserting (26) in (25) yields (18).

When $P \in \Gamma_{\mathrm{Fix,Repel}}$, to obtain (19) sum (18) over all $\vec{v} \in T_{P,\mathrm{FR}}$, getting

$$\sum_{\vec{v} \in T_{P,\mathrm{FR}}} \partial_{\vec{v}} f(P) = (d^2 - d) \cdot v_{\mathrm{FR}}(P) - 2d \cdot \sum_{\vec{v} \in T_{P,\mathrm{FR}}} s_\varphi(P, \vec{v}). \tag{27}$$

By Lemma 4.5, $T_{P,\mathrm{FR}}$ contains all $\vec{v} \in T_P$ such that $s_\varphi(P, \vec{v}) > 0$. Since $\deg_\varphi(P) = 1$, it follows that

$$\sum_{\vec{v} \in T_{P,\mathrm{FR}}} s_\varphi(P, \vec{v}) = d - \deg_\varphi(P) = d - 1.$$

Inserting this in (27), and simplifying, yields (19). □

Proof of Proposition 5.3. To prove (20) we use the machinery from Lemma 2.2. As in that lemma, we choose $\gamma \in \mathrm{GL}_2(K)$ such that $\gamma(\zeta_G) = P$ and $\gamma^{-1}(\varphi(P)) = \zeta_{0,r}$, for some $r \in |K^\times|$ with $0 < r < 1$. By replacing φ with φ^γ we can assume that $P = \zeta_G$ and $\varphi(P) = \zeta_{0,r}$.

Let $c \in K^\times$ be such that $|c| = r$; put $\Phi(z) = (1/c)\varphi(z)$. Then $\Phi(\zeta_G) = \zeta_G$, so Φ has nonconstant reduction and $\varphi(z) = c \cdot \Phi(z)$. Let (F, G) be a normalized representation of Φ; then (cF, G) is a normalized representation of φ. Using this representation, put $H(X, Y) = XG(X, Y) - Y \cdot cF(X, Y)$; this yields $\widetilde{H}(X, Y) = X \cdot \widetilde{G}(X, Y)$. On the other hand, if the type I fixed points of φ (listed with multiplicities) are $(a_i : b_i)$ for $i = 1, \ldots, d+1$, normalized so that $\max(|a_i|, |b_i|) = 1$ for each i, there is a constant C with $|C| = 1$ such that $H(X, Y) = C \cdot \prod_{i=1}^{d+1}(b_i X - a_i Y)$. Reducing this $\pmod{\mathfrak{m}}$ gives

$$X\widetilde{G}(X, Y) = \widetilde{H}(X, Y) = \widetilde{C} \cdot \prod_{i=1}^{d+1}(\tilde{b}_i X - \tilde{a}_i Y). \tag{28}$$

We will now consider what this means for the slope of $\mathrm{ordRes}_\varphi(\,\cdot\,)$ at P in a direction $\vec{v} \in T_P$. We parametrize the directions $\vec{v} \in T_P$ by points $a \in \mathbb{P}_1(\tilde{k})$. We will consider three cases, corresponding to the directions $\vec{v}_0$, $\vec{v}_a$ for $0 \neq a \in \tilde{k}$, and $\vec{v}_\infty$.

First consider the direction $\vec{v}_0 \in T_P$. We use the normalized representation (cF, G) for φ, expanding $cF(X, Y) = a_d X^d + \cdots + a_0 Y^d$ and $G(X, Y) = b_d X^d + \cdots + b_0 Y^d$. As usual, for each $A \in K^\times$,

$$\begin{aligned}
&\mathrm{ordRes}_\varphi(\zeta_{0,|A|}) - \mathrm{ordRes}_\varphi(\zeta_G) \\
&\qquad = (d^2 + d)\mathrm{ord}(A) - 2d \cdot \min\Big(\min_{0\le i\le d} \mathrm{ord}(A^i a_i),\ \min_{0\le j\le d} \mathrm{ord}(A^{j+1} b_j)\Big).
\end{aligned} \tag{29}$$

Let $N = N_0 = \#F_\varphi(P, \vec{v}_0)$ be the number of fixed points of φ in $B_P(\vec{v}_0)^-$. By (28) we have $X^{N-1} \parallel \widetilde{G}(X, Y)$, so $\mathrm{ord}(b_\ell) > 0$ for $\ell = 0, \ldots, N-2$ and $\mathrm{ord}(b_{N-1}) = 0$. Since $|c| < 1$ we see that $\mathrm{ord}(a_\ell) > 0$ for all ℓ. It follows that if $\mathrm{ord}(A) > 0$ and $\mathrm{ord}(A)$ is sufficiently small, then

$$\min\Big(\min_{0\le i\le d} \mathrm{ord}(A^i a_i),\ \min_{0\le j\le d} \mathrm{ord}(A^{j+1} b_j)\Big) = \mathrm{ord}(A^{(N-1)+1} b_{N-1}) = N \cdot \mathrm{ord}(A).$$

Inserting this in (29) shows that the slope of $\mathrm{ordRes}_\varphi(\,\cdot\,)$ at P in the direction $\vec{v}_0$ is

$$\partial_{\vec{v}_0} f(P) = d^2 + d - 2d \cdot N_0 = d^2 + d - 2d \cdot \#F_\varphi(P, \vec{v}_0). \tag{30}$$

Next consider a direction $\vec{v}_a \in T_P$, where $0 \neq a \in \tilde{k}$. Choose an $\alpha \in \mathcal{O}$ with $\tilde{\alpha} = a$ and conjugate φ by

$$\gamma_\alpha = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}.$$

A normalized representation for $\varphi_\alpha := (\varphi)^{\gamma_\alpha}$ is given by

$$\begin{pmatrix} F_\alpha(X, Y) \\ G_\alpha(X, Y) \end{pmatrix} = \begin{bmatrix} 1 & -\alpha \\ 0 & 1 \end{bmatrix} \cdot \begin{pmatrix} cF \\ G \end{pmatrix} \cdot \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} = \begin{pmatrix} cF(X + \alpha Y, Y) - \alpha G(X + \alpha Y, Y) \\ G(X + \alpha Y, Y) \end{pmatrix}.$$

Reducing this gives $\widetilde{F}_\alpha(X, Y) = -a\widetilde{G}(X + aY, Y)$ and $\widetilde{G}_\alpha(X, Y) = \widetilde{G}(X + aY, Y)$.

Let $N = N_a = \#F_\varphi(P, \vec{v}_a)$ be the number of fixed points of φ in $B_P(\vec{v}_a)^-$; from (28) it follows that $(X - aY)^N \parallel \widetilde{G}(X, Y)$. The direction $\vec{v}_\alpha$ for φ pulls back to $\vec{v}_0$ for φ_α, so $N = \#F_{\varphi_\alpha}(P, \vec{v}_0)$. Thus $X^N \parallel \widetilde{F}_\alpha(X, Y)$ and $X^N \parallel \widetilde{G}_\alpha(X, Y)$. Expanding $F_\alpha(X, Y) = a_d X^d + \cdots + a_0 Y^d$ and $G_\alpha(X, Y) = b_d X^d + \cdots + b_0 Y^d$, we see that $\mathrm{ord}(a_\ell), \mathrm{ord}(b_\ell) > 0$ for $\ell = 0, \ldots, N-1$, while $\mathrm{ord}(a_N) = \mathrm{ord}(b_N) = 0$.

If $\mathrm{ord}(A) > 0$ is sufficiently small, it follows that

$$\min\bigl(\min_{0\le i\le d}\mathrm{ord}(A^i a_i),\ \min_{0\le j\le d}\mathrm{ord}(A^{j+1}b_j)\bigr) = \mathrm{ord}(A^N a_N) = N\cdot\mathrm{ord}(A).$$

Inserting this in (29) shows that the slope of $\mathrm{ordRes}_\varphi(\,\cdot\,)$ at P in the direction $\vec{v}_a$ is

$$\partial_{\vec{v}_a} f(P) = d^2 + d - 2d\cdot N_a = d^2 + d - 2d\cdot \#F_\varphi(P,\vec{v}_a). \tag{31}$$

Finally, consider the direction $\vec{v}_\infty \in T_P$. Conjugate φ by

$$\gamma_\infty = \begin{bmatrix} 0 & 1\\ 1 & 0\end{bmatrix}.$$

A normalized representation for $\varphi_\infty := (\varphi)^{\gamma_\infty}$ is given by

$$\begin{pmatrix} F_\infty(X,Y)\\ G_\infty(X,Y)\end{pmatrix} = \begin{bmatrix} 0 & 1\\ 1 & 0\end{bmatrix}\cdot\begin{pmatrix} cF\\ G\end{pmatrix}\cdot\begin{bmatrix} 0 & 1\\ 1 & 0\end{bmatrix} = \begin{pmatrix} G(Y,X)\\ cF(Y,X)\end{pmatrix}.$$

Reducing this gives $\widetilde{F}_\infty(X,Y) = \widetilde{G}(Y,X)$ and $\widetilde{G}_\infty(X,Y) = 0$.

Let $N = N_\infty = \#F_\varphi(P,\vec{v}_\infty)$ be the number of fixed points of φ in $B_P(\vec{v}_\infty)^-$; by formula (28) we have $Y^N \parallel \widetilde{G}(X,Y)$. The direction $\vec{v}_\infty$ for φ pulls back to $\vec{v}_0$ for φ_∞, so $N = \#F_{\varphi_\infty}(P,\vec{v}_0)$. Thus $X^N \parallel \widetilde{F}_\infty(X,Y)$. Writing $F_\infty(X,Y) = a_d X^d + \cdots + a_0 Y^d$ and $G_\infty(X,Y) = b_d X^d + \cdots + b_0 Y^d$, we see that $\mathrm{ord}(a_\ell) > 0$ for $\ell = 0,\ldots,N-1$, while $\mathrm{ord}(a_N) = 0$; we have $\mathrm{ord}(b_\ell) > 0$ for all ℓ. Thus if $\mathrm{ord}(A) > 0$ is sufficiently small,

$$\min\bigl(\min_{0\le i\le d}\mathrm{ord}(A^i a_i),\ \min_{0\le j\le d}\mathrm{ord}(A^{j+1}b_j)\bigr) = \mathrm{ord}(A^N a_N) = N\cdot\mathrm{ord}(A).$$

Inserting this in (29) shows that the slope of $\mathrm{ordRes}_\varphi(\,\cdot\,)$ at P in the direction $\vec{v}_\infty$ is

$$\partial_{\vec{v}_\infty} f(P) = d^2 + d - 2d\cdot N_\infty = d^2 + d - 2d\cdot \#F_\varphi(P,\vec{v}_\infty). \tag{32}$$

Combining (30), (31) and (32) yields (20).

If $P \in \Gamma_{\mathrm{Fix,Repel}}$, to obtain (19) sum (20) over all $\vec{v} \in T_{P,\mathrm{FR}}$, getting

$$\sum_{\vec{v}\in T_{P,\mathrm{FR}}} \partial_{\vec{v}} f(P) = (d^2+d)\cdot v_{\mathrm{FR}}(P) - 2d\cdot \sum_{\vec{v}\in T_{P,\mathrm{FR}}} \#F_\varphi(P,\vec{v}). \tag{33}$$

By Lemma 2.2, $T_{P,\mathrm{FR}}$ contains all $\vec{v}\in T_P$ such that $\#F_\varphi(P,\vec{v}) > 0$. Since φ has $d+1$ type I fixed points (counting multiplicities), it follows that

$$\sum_{\vec{v}\in T_{P,\mathrm{FR}}} \#F_\varphi(P,\vec{v}) = d+1.$$

Inserting this in (33) yields (21). $\square$

Proof of Proposition 5.4. After a change of coordinates, we can assume that $Q = 0$. Since $\mathrm{ordRes}_\varphi(\,\cdot\,)$ is piecewise affine on paths in $\mathbf{H}^1_K$ (relative to the logarithmic path distance), it suffices to prove the result when $P = \zeta_{0,r}$ is a type II point with r sufficiently small and $\vec{v}_1 \in T_P$ is the direction $\vec{v}_\infty$.

Write $F(X, Y) = a_d X^d + \cdots + a_0 Y^d$, $G(X, Y) = b_d X^d + \cdots + b_0 Y^d$ and let (F, G) be a normalized representation of φ. Take $A \in K^\times$ and put $r = |A|$; by [Rumely 2015, formula (19)]

$$\begin{aligned}&\mathrm{ordRes}_\varphi(\zeta_{0,r})\\ &= \mathrm{ordRes}(F, G) + (d^2 + d)\mathrm{ord}(A) - 2d \min\Bigl(\min_{0\le\ell\le d} \mathrm{ord}(A^\ell a_\ell), \min_{0\le\ell\le d} \mathrm{ord}(A^{\ell+1} b_\ell)\Bigr).\end{aligned}$$

First suppose that Q is fixed by φ, so $\varphi(0) = 0$. In this situation $a_0 = 0$ and $b_0 \neq 0$, so if $r = |A|$ is sufficiently small, then

$$\min\Bigl(\min_{0\le\ell\le d} \mathrm{ord}(A^\ell a_\ell), \min_{0\le\ell\le d} \mathrm{ord}(A^{\ell+1} b_\ell)\Bigr)$$

coincides with

$$\min\bigl(\mathrm{ord}(Aa_1), \mathrm{ord}(Ab_0)\bigr) = \mathrm{ord}(A) + \min\bigl(\mathrm{ord}(a_1), \mathrm{ord}(b_0)\bigr).$$

For such r we have

$$\mathrm{ordRes}_\varphi(\zeta_{0,r}) = \mathrm{ordRes}(F, G) - 2d \min\bigl(\mathrm{ord}(a_1), \mathrm{ord}(b_0)\bigr) + (d^2 - d)\mathrm{ord}(A),$$

and since $\mathrm{ord}(A)$ measures the logarithmic path distance and increases as $r \to 0$, the slope of $\mathrm{ordRes}_\varphi(\,\cdot\,)$ at $\zeta_{0,r}$ in the direction $\vec{v}_\infty$ is $-(d^2 - d)$.

Next suppose Q is not fixed by φ, so $\varphi(0) \neq 0$. In this case $a_0 \neq 0$. If $r = |A|$ is sufficiently small, then $\min\bigl(\min_{0\le\ell\le d} \mathrm{ord}(A^\ell a_\ell), \min_{0\le\ell\le d} \mathrm{ord}(A^{\ell+1} b_\ell)\bigr) = \mathrm{ord}(a_0)$. For such r we have

$$\mathrm{ordRes}_\varphi(\zeta_{0,r}) = \mathrm{ordRes}(F, G) - 2d\,\mathrm{ord}(a_0) + (d^2 + d)\mathrm{ord}(A),$$

so the slope of $\mathrm{ordRes}_\varphi(\,\cdot\,)$ at $\zeta_{0,r}$ in the direction $\vec{v}_\infty$ is $-(d^2 + d)$. $\square$

6. The weight formula and the crucial set

In this section, we prove the weight formula (Theorem A) as Theorem 6.1. We then give an intrinsic formula for the crucial measure (Corollary 6.5) and note some of the functoriality properties of the crucial set and crucial measure.

We begin by defining weights $w_\varphi(P)$, motivated by Propositions 5.1, 5.2, and 5.3.

Definition 8 (weights). For each $P \in \mathbf{P}^1_K$, let $w_\varphi(P)$ be the following nonnegative integer:

(1) If $P \in \mathbf{H}^1_K$ and P is fixed by φ, put

$$w_\varphi(P) = \deg_\varphi(P) - 1 + N_{\mathrm{Shearing}}(P).$$

(2) If $P \in \mathbf{H}^1_K$ and P is not fixed by φ, let $v(P)$ be the number of directions $\vec{v} \in T_P$ such that $B_P(\vec{v})^-$ contains a type I fixed point of φ, and put

$$w_\varphi(P) = \max(0, v(P) - 2).$$

(3) If $P \in \mathbb{P}^1(K)$, put $w_\varphi(P) = 0$.

Our main theorem (Theorem A) is:

Theorem 6.1 (weight formula). *Let $\varphi(z) \in K(z)$ have degree $d \geq 2$. Then*

$$\sum_{P \in \mathbf{P}^1_K} w_\varphi(P) = d - 1. \tag{34}$$

Corollary 6.2. *Let $\varphi(z) \in K(z)$ have degree $d \geq 2$. Then*

$$\sum_{\substack{\text{repelling fixed points} \\ P \in \mathbf{H}^1_K}} (\deg_\varphi(P) - 1) \leq d - 1. \tag{35}$$

In particular φ can have most $d-1$ repelling fixed points in $\mathbf{H}^1_K$.

In [Rumely 2014] it is shown that for each $d \geq 2$, there exist functions $\varphi \in K(z)$ of degree d which have $d-1$ repelling fixed points in $\mathbf{H}^1_K$, so Corollary 6.2 is sharp.

Although the weight formulas in Definition 8 are mysterious, they arise naturally, as shown by the proof of Theorem 6.1 and by Corollary 6.5 which gives an intrinsic formula for the crucial measure ν_φ.

The author's efforts to understand the weights lead to the following observations. It is not surprising that for a fixed point in $\mathbb{H}_{\text{Berk}}$, the degree should enter into the weight. Given that (35) holds, if there are infinitely many fixed points, most must have degree 1 and weight 0, so the contribution $\deg_\varphi(P) - 1$ to the weight is understandable. The contribution from the shearing directions is more unexpected. The presence of a shearing direction $\vec{v} \in T_P$ indicates a difference between the local and global behavior of φ: " near P" points in the direction $\vec{v}$ are moved to another direction $\vec{w}$ but "far away from P" there is a classical fixed point in the direction $\vec{v}$. There is a hidden similarity between the weight formulas for fixed points and points that are moved. If P is fixed by φ, there need be no shearing directions, and all the shearing directions count in the weight. For a point $P \in \mathbf{H}^1_K$ which is moved by φ, there are automatically some shearing directions: If P is outside $\Gamma_{\text{Fix,Repel}}$ all the classical fixed points lie in one direction and that direction is sheared. If P is in $\Gamma_{\text{Fix,Repel}}$ but is not a branch point, there are two directions containing classical fixed points and those directions are sheared. Such points must have weight 0. However, if P is a branch point of $\Gamma_{\text{Fix,Repel}}$, there are more than two shearing directions, and the weight counts the excess.

The following proposition makes the weights more explicit and describes which points have positive weight and which have weight zero:

Proposition 6.3 (properties of weights). *Let $P \in \mathbf{P}^1_K$. If P is a focused repelling fixed point, then $w_\varphi(P) = \deg_\varphi(P) - 1$. If P is an additively indifferent fixed point in Γ_{Fix} or is a multiplicatively indifferent fixed point which is a branch point of Γ_{Fix}, then $w_\varphi(P) = N_{\mathrm{Shearing}}(P)$. If P is an* id-*indifferent fixed point, then $w_\varphi(P) = 0$. If P is a nonfixed branch point of Γ_{Fix}, then $w_\varphi(P) = v(P) - 2$. For all P, $w_\varphi(P)$ is an integer with $0 \le w_\varphi(P) \le d - 1$ and:*

(A) *$w_\varphi(P) > 0$ if*
   (1) *P is a type II repelling fixed point of φ, or*
   (2) *P is an additively indifferent fixed point of φ which belongs to Γ_{Fix}, or*
   (3) *P is a multiplicatively indifferent fixed point of φ which is a branch point of Γ_{Fix}, or*
   (4) *P is a branch point of Γ_{Fix} which is moved by φ.*

(B) *$w_\varphi(P) = 0$ if*
   (1) *$P \notin \Gamma_{\mathrm{Fix,Repel}}$, or*
   (2) *P is not of type II, or*
   (3) *P is an additively indifferent fixed point which is not in Γ_{Fix}, or*
   (4) *P is a multiplicatively indifferent fixed point which is not a branch point of Γ_{Fix}, or*
   (5) *P is an* id-*indifferent fixed point, or*
   (6) *P is moved by φ and is not a branch point of Γ_{Fix}.*

Proof. If P is a focused repelling fixed point, the unique direction $\vec{v} \in T_P$ such that $B_P(\vec{v})^-$ contains type I fixed points is fixed by φ_*, so $N_{\mathrm{Shearing}}(P) = 0$ and $w_\varphi(P) = \deg_\varphi(P) - 1$. If P is an additively indifferent or multiplicatively indifferent fixed point, then $\deg_\varphi(P) = 1$ so $w_\varphi(P) = N_{\mathrm{Shearing}}(P)$. If P is id-indifferent, then $\deg_\varphi(P) = 1$ and each $\vec{v} \in T_P$ is fixed by φ_*, so $N_{\mathrm{Shearing}}(P) = 0$; thus $w_\varphi(P) = 0$. If P is a nonfixed branch point of Γ_{Fix}, then $v(P) > 2$ so $w_\varphi(P) = v(P) - 2$.

Clearly each type II repelling fixed point has $w_\varphi(P) > 0$. By results of Rivera-Letelier [2003b, Lemmas 5.3 and 5.4] (or [Baker and Rumely 2010, Lemma 10.80]), each fixed point of type III or IV has degree 1 and each of its tangent directions is fixed by φ_*, hence $w_\varphi(P) = 0$. By definition, each point of type I has $w_\varphi(P) = 0$.

Each additively indifferent fixed point $P \notin \Gamma_{\mathrm{Fix}}$ has type I fixed points in exactly one tangent direction and Lemma 2.1 shows that direction is fixed by φ_*; hence $w_\varphi(P) = 0$. On the other hand, each additively indifferent fixed point $P \in \Gamma_{\mathrm{Fix}}$ has type I fixed points in at least two tangent directions, one of which must be a shearing direction since an additively indifferent fixed point has exactly one fixed direction; hence $w_\varphi(P) > 0$.

Likewise, a multiplicatively indifferent fixed point P has two fixed directions and Lemma 2.1 shows each of them must contain type I fixed points; thus P belongs to Γ_{Fix}. If P is not a branch point of Γ_{Fix}, its two fixed directions are the only ones containing type I fixed points, so it has no shearing directions, and $w_\varphi(P) = 0$. If P is a branch point of Γ_{Fix}, there is at least one $\vec{v} \in T_P$ containing a type I fixed point besides the two fixed directions and that direction is a shearing direction, so $w_\varphi(P) > 0$.

If $P \in \mathbf{H}_K^1$ is not fixed by φ, there are three possibilities. Recall that $v(P)$ is the number of directions in T_P containing a type I fixed point. If $P \notin \Gamma_{\mathrm{Fix}}$, there is only one direction $\vec{v} \in T_P$ containing type I fixed points, so $v(P) = 1$, giving $w_\varphi(P) = 0$. If $P \in \Gamma_{\mathrm{Fix}}$ but P is not a branch point of Γ_{Fix}, then $v(P) = 2$, giving $w_\varphi(P) = 0$. If P is a branch point of Γ_{Fix}, then $v(P) \geq 3$, so $w_\varphi(P) > 0$.

The fact that $0 \leq w_\varphi(P) \leq d-1$ follows from Theorem 6.1, whose proof only uses the formulas in Definition 8. □

Using Proposition 6.3, the weight formula (34) in Theorem 6.1 can be made more explicit:

$$\sum_{\substack{\text{focused}\\ \text{repelling f.p.s}}} (\deg_\varphi(P) - 1) + \sum_{\substack{\text{bifocused and}\\ \text{multifocused f.p.s}}} (\deg_\varphi(P) - 1 + N_{\text{Shearing}}(P))$$
$$+ \sum_{\substack{\text{additively}\\ \text{indifferent f.p.s in } \Gamma_{\text{Fix}}}} N_{\text{Shearing}}(P) + \sum_{\substack{\text{multiplicatively}\\ \text{indifferent f.b.p.s of } \Gamma_{\text{Fix}}}} N_{\text{Shearing}}(P) + \sum_{\substack{\text{nonfixed}\\ \text{b.p.s of } \Gamma_{\text{Fix}}}} (v(P) - 2)$$
$$= d - 1, \tag{36}$$

where f.p.s, b.p.s, and f.b.p.s stand for fixed points, branch points, and fixed branch points, respectively.

The idea for the proof of Theorem 6.1 is to take the Laplacian of $\mathrm{ordRes}_\varphi(\cdot)$ on finite metrized subgraphs of $\Gamma_{\mathrm{Fix,Repel}}$; by simplifying the Laplacians and taking a limit over a cofinal sequence of subgraphs, we obtain the result. By a finite metrized graph we mean a connected subgraph with a finite number of vertices and edges, such that each edge is equipped with a metric which makes it isometric to an interval of finite length in $\mathbb{R}$. For subgraphs of $\mathbf{H}_K^1$ we will always take the metric to be the one induced by the logarithmic path distance $\rho(x, y)$.

If $\Gamma \subset \mathbf{H}_K^1$ is a finite metrized subgraph, let $\mathrm{CPA}(\Gamma)$ be the space of functions on Γ which are continuous and piecewise affine (with a finite number of pieces) with respect to the logarithmic path distance. For each $P \in \Gamma$, we write $T_{P,\Gamma}$ for the tangent space to P in Γ, namely the set of $\vec{v} \in T_P$ such that there is an edge of Γ emanating from P in the direction $\vec{v}$. If the valence of P in Γ is $v(P)$, then $T_{P,\Gamma}$ has $v(P)$ elements. If $f \in \mathrm{CPA}(\Gamma)$, the Laplacian of F (as defined in [Baker and Rumely 2007]) is the measure

$$\Delta_\Gamma(f) = \sum_{P \in \Gamma} -\Big(\sum_{\vec{v} \in T_{P,\Gamma}} \partial_{\vec{v}}(f)(P)\Big)\delta_P(z), \tag{37}$$

where $\delta_P(z)$ is the Dirac measure at P. Here $\Delta_\Gamma(F)$ is a discrete measure, since the inner sum in (37) is 0 at any P which is not a branch point of Γ and where F does not change slope. It is well known (see for example [Baker and Rumely 2007]) that $\Delta_\Gamma(F)$ has total mass 0. The proof is as follows. Note that for any partition of Γ into finitely many segments $[P_i, Q_i]$ (disjoint except for their endpoints), such that the restriction of F to each $[P_i, Q_i]$ is affine, the slopes of F at P_i and Q_i, in the directions pointing into $[P_i, Q_i]$, are negatives of each other.

Before proving Theorem 6.1, we will need an identity relating the number of endpoints of a tree to the valences of its internal branch points. If Γ is a finite graph and $P \in \Gamma$, the valence $v_\Gamma(P)$ is the number of edges of Γ incident at P.

Lemma 6.4. *Let Γ be a finite tree with D endpoints. Then*

$$D - \sum_{\text{b.p.s } P \in \Gamma} (v_\Gamma(P) - 2) = 2, \tag{38}$$

where b.p.s stands for "branch points".

Proof. This is a reformulation of Euler's formula $V - E + F = 2$ for planar graphs.

If B is the number of branch points of Γ, then $V = D + B$. Each edge has two endpoints, so $E = \frac{1}{2}\big(\sum_{\text{vertices}} v(P)\big)$. Since Γ is a tree, if it is embedded as a planar graph then $F = 1$. Inserting these in Euler's formula yields

$$2D + 2B - \sum_{\text{e.p.s}} v(P) - \sum_{\text{b.p.s}} v(P) = 2, \tag{39}$$

where e.p.s stands for end points. At each endpoint we have $v(P) = 1$, so $\sum_{\text{e.p.s}} v(P) = D$. There are B branch points, so $2B = \sum_{\text{b.p.s}} 2$. Combining terms in (39) gives (38). □

Proof of Theorem 6.1. The idea is to restrict $\operatorname{ordRes}_\varphi(\cdot)$ to finite subgraphs of $\Gamma_{\text{Fix,Repel}}$, take its Laplacian and simplify. Since $\Gamma_{\text{Fix,Repel}}$ has branches of infinite length, to apply the theory of graph Laplacians for metrized graphs from [Baker and Rumely 2007], we cut off segments at the type I endpoints of $\Gamma_{\text{Fix,Repel}}$, obtaining a finite metrized tree Γ_{FR}. We then exhaust $\Gamma_{\text{Fix,Repel}}$ by a sequence of such trees.

For each type I fixed point α_i of φ, choose a type II point $Q_i \in \Gamma_{\text{Fix,Repel}}$ close enough to α_i that

(1) there are no branch points of $\Gamma_{\text{Fix,Repel}}$ in $[Q_i, \alpha_i]$, and

(2) at each $P \in [Q_i, \alpha_i)$, the slope of $\operatorname{ordRes}_\varphi(\cdot)$ at P in the (unique) direction $\vec{v} \in T_P$ pointing away from α_i in $\Gamma_{\text{Fix,Repel}}$ is $-(d^2 - d)$.

Since any two paths emanating from α_i share a common initial segment, such Q_i exist by Proposition 5.4.

Let Γ_{FR} be the subtree of $\Gamma_{\mathrm{Fix,Repel}}$ spanned by the focused repelling fixed points and the points Q_i. Suppose there are D_1 distinct type I fixed points (ignoring multiplicities) and D_2 focused repelling fixed points. Then the number of endpoints of Γ_{FR} is

$$D = D_1 + D_2.$$

Let $f(\cdot)$ be the restriction of $\mathrm{ordRes}_\varphi(\cdot)$ to Γ_{FR}. Then f belongs to $\mathrm{CPA}(\Gamma_{\mathrm{FR}})$.

For each $P \in \Gamma_{\mathrm{FR}}$, write $T_{P,\mathrm{FR}}$ for its tangent space in Γ_{FR} (the set of $\vec{v} \in T_P$ such that there is an edge of Γ_{FR} emanating from P in the direction $\vec{v}$). If $P \in \Gamma_{\mathrm{FR}} \setminus \{Q_1, \dots, Q_{D_1}\}$, then $T_{P,\mathrm{FR}} = T_{P,\mathrm{FR}}$ and the valence of P in Γ_{FR} coincides with $v_{\mathrm{FR}}(P)$. If $P \in \{Q_1, \dots, Q_{D_1}\}$ then $T_{P,\mathrm{FR}}$ consists of the single direction $\vec{v}_{P,1} \in T_P$ pointing into Γ_{FR}.

For the Laplacian $\Delta_{\mathrm{FR}}(f)$ we have

$$-\Delta_{\mathrm{FR}}(f) = \sum_{P \in \{Q_1,\dots,Q_{D_1}\}} \partial_{\vec{v}_{P,1}} f(P)\delta_P(z) + \sum_{P \in \Gamma_{\mathrm{FR}} \setminus \{Q_1,\dots,Q_{D_1}\}} \left(\sum_{\vec{v} \in T_{P,\mathrm{FR}}} \partial_{\vec{v}} f(P) \right) \delta_P(z). \quad (40)$$

By Proposition 5.4, if $P \in \{Q_1, \dots, Q_{D_1}\}$ then $\partial_{\vec{v}_{P,1}} f(P) = -(d^2 - d)$. We claim that if $P \in \Gamma_{\mathrm{FR}} \setminus \{Q_1, \dots, Q_{D_1}\}$, then

$$\sum_{\vec{v} \in T_{P,\mathrm{FR}}} \partial_{\vec{v}} f(P) = (d^2 - d) \cdot (v_{\mathrm{FR}}(P) - 2) + 2d \cdot w_\varphi(P). \quad (41)$$

If $P \in \Gamma_{\mathrm{FR}} \setminus \{Q_1, \dots, Q_{D_1}\}$ is of type II and is a repelling fixed point, or is a multiplicatively or additively indifferent fixed point, then (41) follows from Proposition 5.1 and the definition of $w_\varphi(P)$. If P is an id-indifferent fixed point, then (41) follows from Proposition 5.2 since $w_\varphi(P) = 0$ by Proposition 6.3.

If $P \in \Gamma_{\mathrm{FR}} \setminus \{Q_1, \dots, Q_{D_1}\}$ is a type II point with $\varphi(P) \neq P$, then by Propositions 3.1 and 3.5, P belongs to Γ_{Fix} and is not a point where a branch of $\Gamma_{\mathrm{Fix,Repel}} \setminus \Gamma_{\mathrm{Fix}}$ attaches to Γ_{Fix}. It follows that $v_{\mathrm{FR}}(P)$ (which is the valence of P in Γ_{FR} and $\Gamma_{\mathrm{Fix,Repel}}$) coincides with $v(P)$ (its valence in Γ_{Fix}). Hence (41) follows from Proposition 5.3 and the definition of $w_\varphi(P)$.

If $P \in \Gamma_{\mathrm{FR}} \setminus \{Q_1, \dots, Q_{D_1}\}$ is a type III point, then $v_{\mathrm{FR}}(P) = 2$ and $w_\varphi(P) = 0$, so the right side of (41) is 0. The left side of (41) is also 0, since $\mathrm{ordRes}_\varphi(\cdot)$ can change slope only at points of type II (see Theorem 1.1). Each $P \in \Gamma_{\mathrm{FR}} \setminus \{Q_1, \dots, Q_{D_1}\}$ is either of type II or type III, so this establishes (41) in all cases.

Since $\Delta_{\mathrm{FR}}(f)$ has total mass 0, it follows from (40) and (41) that

$$D_1 \cdot (-(d^2 - d)) + \sum_{P \in \Gamma_{\mathrm{FR}} \setminus \{Q_1,\dots,Q_{D_1}\}} \Big((d^2 - d) \cdot (v_{\mathrm{FR}}(P) - 2) + 2d \cdot w_\varphi(P) \Big) = 0. \quad (42)$$

If P is a focused repelling fixed point, then $v_{\mathrm{FR}}(P)=1$, so $v_{\mathrm{FR}}(P)-2=-1$. If P is not an endpoint or a branch point of Γ_{FR} then $v_{\mathrm{FR}}(P)-2=0$. Moving the terms in (42) involving (d^2-d) to the right side, and noting that there are D_2 focused repelling fixed points, it follows that

$$\sum_{P\in\Gamma_{\mathrm{FR}}\setminus\{Q_1,\dots,Q_{D_1}\}} 2d\cdot w_\varphi(P)=(d^2-d)\cdot\Bigl(D_1+D_2-\sum_{\text{b.p.s of }\Gamma_{\mathrm{FR}}}(v_{\mathrm{FR}}(P)-2)\Bigr), \tag{43}$$

where b.p.s stands for branch points. By Lemma 6.4 the sum on the right side of (43) equals 2. Dividing through by $2d$ gives

$$\sum_{P\in\Gamma_{\mathrm{FR}}\setminus\{Q_1,\dots,Q_{D_1}\}} w_\varphi(P)=d-1. \tag{44}$$

If we let the endpoints points Q_i approach the type I fixed points α_i, the corresponding graphs Γ_{FR} exhaust $\Gamma_{\mathrm{Fix,Repel}}\cap\mathbf{H}^1_K$. By Proposition 6.3, we have $w_\varphi(P)=0$ for each type I fixed point and for each $P\in\mathbf{P}^1_K\setminus\Gamma_{\mathrm{Fix,Repel}}$. Thus

$$\sum_{P\in\mathbf{P}^1_K} w_\varphi(P)=d-1. \qquad \square$$

Definition 9 (the crucial set, crucial weights, and crucial measure). The set of points $P\in\mathbf{P}^1_K$ with weight $w_\varphi(P)>0$ will be called the *crucial set* and their weights $w_\varphi(P)$ will be called the *crucial weights*. The probability measure

$$\nu_\varphi=\frac{1}{d-1}\sum_{P\in\mathbf{P}^1_K} w_\varphi(P)\delta_P(z) \tag{45}$$

will be called the *crucial measure* of φ.

The crucial set consists of the repelling fixed points in $\mathbf{H}^1_K$, the indifferent fixed points with a shearing direction, and the branch points of Γ_{Fix} which are moved by φ. It has at most $d-1$ elements. If φ has potential good reduction, it consists of the unique point where $\deg_\varphi(P)=d$. However, it can also consist of a single point in other ways: for each $k=1,\dots,d-1$, it could be a fixed point P with $\deg_\varphi(P)=k$ and $d-k$ shearing directions or it could be a branch point of Γ_{Fix} with valence $d+1$, which is moved. All of these possibilities actually occur; see [Rumely 2014].

If Γ is a finite metrized graph, there is a natural measure of total mass 1 attached to Γ, its "Branching Measure" $\mu_{\Gamma,\mathrm{Br}}$. This measure was introduced in [Chinburg and Rumely 1993], where it was called the canonical measure and used to define a pairing on the Arakelov divisor class group of a curve. It was studied in detail in [Baker and Rumely 2007] and was generalized by Shouwu Zhang [1993], who used it to construct the dualizing metric for the Serre duality theorem in nonarchimedean Arakelov theory.

When Γ is a tree, $\mu_{\Gamma,\mathrm{Br}}$ is supported on the branch points and endpoints of Γ and is given by the formula

$$\mu_{\Gamma,\mathrm{Br}} = \sum_{P\in\Gamma} \bigl(1 - \tfrac{1}{2} v_\Gamma(P)\bigr)\delta_P(z).$$

This formula makes sense even when Γ has edges of infinite length. Note that $\mu_{\Gamma,\mathrm{Br}}$ gives mass $\frac{1}{2}$ to each endpoint of Γ and negative mass $1-\frac{1}{2}v_\Gamma(P)$ to each branch point, so it is not in general a positive measure. The fact that $\mu_{\Gamma,\mathrm{Br}}$ has total mass 1 follows from Lemma 6.4. When $\Gamma = \Gamma_{\mathrm{Fix,Repel}}$, we will write μ_{Br} for its branching measure.

We can obtain an intrinsic formula for ν_φ by taking the Laplacian of $\mathrm{ordRes}_\varphi(\,\cdot\,)$ on the tree $\Gamma_{\mathrm{Fix,Repel}}$: suitably normalized, the Laplacian decomposes as the difference of μ_{Br} and ν_φ. Note that μ_{Br} depends only on the topology of $\Gamma_{\mathrm{Fix,Repel}}$, while ν_φ encodes analytic properties of φ. Thus, ν_φ is the analytic part of a decomposition of the normalized Laplacian into topological and analytic parts.

Corollary 6.5. *Let $\varphi(z) \in K(z)$ have degree $d \geq 2$. Let $f(\,\cdot\,)$ be the restriction of* $\mathrm{ordRes}_\varphi(\,\cdot\,)$ *to $\Gamma_{\mathrm{Fix,Repel}}$ and let $\Delta_{\mathrm{Fix,Repel}}(f)$ be its Laplacian. Then*

$$\frac{1}{2d(d-1)}\Delta_{\mathrm{Fix,Repel}}(f) = \mu_{\mathrm{Br}} - \nu_\varphi.$$

Proof. Since $\Gamma_{\mathrm{Fix,Repel}}$ has edges of infinite length and $\mathrm{ordRes}_\varphi(\,\cdot\,)$ is unbounded, the Laplacian $\Delta_{\mathrm{Fix,Repel}}(f)$ must be defined in a more sophisticated way than formula (37). (See [Baker and Rumely 2010, §5] and [Jonsson 2015, §2] for general discussions of Laplacians on mildly infinite graphs.) Here is a distributional definition for it. Let $\mathcal{C}(\Gamma_{\mathrm{Fix,Repel}})$ be the space of continuous functions $\eta : \Gamma_{\mathrm{Fix,Repel}} \to \mathbb{R}$. Since $\Gamma_{\mathrm{Fix,Repel}}$ is compact, such functions are necessarily bounded. Let $\Lambda_f : \mathcal{C}(\Gamma_{\mathrm{Fix,Repel}}) \to \mathbb{R}$ be the continuous linear functional defined by

$$\Lambda_f(\eta) = \varinjlim_{\Gamma\subset\Gamma_{\mathrm{Fix,Repel}}} \int_\Gamma \eta\,\Delta_\Gamma(f),$$

where the limit is taken over the set of finite subgraphs $\Gamma \subset \Gamma_{\mathrm{Fix,Repel}}$, partially ordered by inclusion. Then $\Delta_{\mathrm{Fix,Repel}}(f)$ is the Radon measure representing Λ_f.

Let $\Gamma = \Gamma_{\mathrm{FR}}$ be a finite subgraph of $\Gamma_{\mathrm{Fix,Repel}}$ of the type considered in the proof of Theorem 6.1. Reformulating (40) using (41) and Proposition 5.4, one has

$$\Delta_\Gamma(f) = \sum_{P\in\{Q_1,\dots,Q_{D_1}\}} (d^2-d)\delta_P - \sum_{P\in\Gamma\setminus\{Q_1,\dots,Q_{D_1}\}} \bigl((d^2-d)(v_\Gamma(P)-2) + 2d\cdot w_\varphi(P)\bigr)\delta_P.$$

Using the definitions of $\mu_{\Gamma,\mathrm{Br}}$ and ν_φ, this can be rewritten as

$$\Delta_\Gamma(f) = 2d(d-1)\cdot(\mu_{\Gamma,\mathrm{Br}} - \nu_\varphi).$$

Letting $Q_1, \ldots, Q_{D_1}$ approach the type I endpoints of $\Gamma_{\mathrm{Fix,Repel}}$, one sees that for any test function $\eta \in \mathcal{C}(\Gamma_{\mathrm{Fix,Repel}})$,

$$\int_\Gamma \eta\,\mu_{\Gamma,\mathrm{Br}} \to \int_{\Gamma_{\mathrm{Fix,Repel}}} \eta\,\mu_{\mathrm{Br}}.$$

Hence $\Delta_{\mathrm{Fix,Repel}}(f) = 2d(d-1)\cdot(\mu_{\mathrm{Br}} - \nu_\varphi)$. □

We close this section by noting some functoriality properties of the crucial set. First, the weights, crucial set, and crucial measure have the same equivariance properties under conjugation as fixed points:

Proposition 6.6. *Let $\varphi(z) \in K(z)$ have degree $d \geq 2$ and let $\gamma \in \mathrm{GL}_2(K)$. Viewing γ as a linear fractional transformation, put $\varphi^\gamma = \gamma^{-1}\circ\varphi\circ\gamma$. Then the weights, crucial set, and crucial measure satisfy*

$$w_{\varphi^\gamma}(\gamma^{-1}(P)) = w_\varphi(P), \quad \mathrm{Cr}(\varphi^\gamma) = \gamma^{-1}(\mathrm{Cr}(\varphi)), \quad \textit{and} \quad \nu_{\varphi^\gamma} = \gamma^*(\nu_\varphi).$$

Proof. The group $\mathrm{GL}_2(K)$ acts transitively on type II points and, by the definition of $\mathrm{ordRes}_\varphi(\cdot)$ on type II points, for each $\tau \in \mathrm{GL}_2(K)$, if $P = \tau(\zeta_G)$ then $\mathrm{ordRes}_\varphi(\tau(\zeta_G)) = \mathrm{ord}(\mathrm{Res}(\varphi^\tau))$. It follows that

$$\mathrm{ordRes}_{\varphi^\gamma}(\tau(\zeta_G)) = \mathrm{ord}(\mathrm{Res}((\varphi^\gamma)^\tau)) = \mathrm{ordRes}_\varphi(\gamma(\tau(\zeta_G))).$$

Since $\mathrm{ordRes}_\varphi(\cdot)$ is determined by continuity from its values on type II points, we have $\mathrm{ordRes}_{\varphi^\gamma}(P) = \mathrm{ordRes}_\varphi(\gamma(P))$ for all $P \in \mathbf{P}^1_K$. Replacing P by $\gamma^{-1}(P)$ shows that

$$\mathrm{ordRes}_{\varphi^\gamma}(\gamma^{-1}(P)) = \mathrm{ordRes}_\varphi(P).$$

Furthermore, P is a fixed point of φ if and only if $\gamma^{-1}(P)$ is a fixed point of φ^γ and in that case the points have the same local degrees. This implies that $\Gamma_{\mathrm{Fix,Repel}}(\varphi^\gamma) = \gamma^{-1}(\Gamma_{\mathrm{Fix,Repel}}(\varphi))$. The assertions in the Proposition follow by taking Laplacians. □

Next, we show the crucial set and crucial measure are preserved under extension of the ground field. If $L \supset K$ is another complete, algebraically closed field, there is a canonical embedding $\mathbf{P}^1_K \hookrightarrow \mathbf{P}^1_L$. This is a special case of a general result about what Berkovich [1990] called *peaked points* and Poineau [2013] renamed *universal points*. Poineau[1] shows that whenever K is algebraically closed and L/K is any field extension, if X_K is any Berkovich space over K, then there is a canonical embedding $X_K \hookrightarrow X_L$.

[1]The author thanks one of the anonymous referees for bringing Poineau's result to his attention.

Proposition 6.7. *Let $\varphi(z) \in K(z)$ have degree $d \geq 2$ and let $L \supseteq K$ be another complete algebraically closed valued field, whose valuation extends the one on K. Using subscripts L and K to denote objects computed over the corresponding ground fields, we have*

$$\mathrm{Cr}_L(\varphi) = \mathrm{Cr}_K(\varphi) \quad \textit{and} \quad \nu_{\varphi,L} = \nu_{\varphi,K}.$$

Proof. Faber [2013a, §4] constructs the embedding $\mathbf{P}^1_K \hookrightarrow \mathbf{P}^1_L$ in an elementary way and shows that it respects the action of φ on points and tangent spaces and preserves path distances and degrees of points. By the formulas in Definition 8, for each $P \in \mathbf{P}^1_K$ we have $w_{\varphi,L}(P) \geq w_{\varphi,K}(P)$. Applying Theorem 6.1 over L and over K we see that

$$\sum_{P \in \mathbf{P}^1_L} w_{\varphi,L}(P) = d - 1 = \sum_{P \in \mathbf{P}^1_K} w_{\varphi,K}(P).$$

Hence $w_{\varphi,L}(P) = w_{\varphi,K}(P)$ for each $P \in \mathbf{P}^1_K$ and $w_{\varphi,L}(P) = 0$ for each $P \in \mathbf{P}^1_L \backslash \mathbf{P}^1_K$. The assertions in the proposition follow from this. □

Finally, the weights, crucial set, and crucial measures are Galois-equivariant. Recall that if σ is a continuous automorphism of K, then the action of σ on K extends to an action of σ on $\mathbf{P}^1_K$ which is continuous for both the weak and strong topologies, respects the logarithmic path distance, and is given on points of type II and type III by $\sigma(\zeta_{a,r}) = \zeta_{\sigma(a),r}$.

Proposition 6.8. *Let $\varphi(z) \in K(z)$ have degree $d \geq 2$ and let σ be a continuous automorphism of K. Then the weights, crucial set, and crucial measure satisfy*

$$w_{\sigma(\varphi)}(\sigma(P)) = w_\varphi(P), \quad \mathrm{Cr}(\sigma(\varphi)) = \sigma(\mathrm{Cr}(\varphi)), \quad \textit{and} \quad \nu_{\sigma(\varphi)} = \sigma_*(\nu_\varphi).$$

Proof. Since σ respects absolute value on K and since the resultant is defined by a determinant in the coefficients of φ, it follows that for each $\tau \in \mathrm{GL}_2(K)$,

$$\mathrm{ord}(\mathrm{Res}(\varphi^\tau)) = \mathrm{ord}(\mathrm{Res}(\sigma(\varphi^\tau))) = \mathrm{ord}(\mathrm{Res}(\sigma(\varphi)^{\sigma(\tau)})).$$

Since $\zeta_G = \zeta_{0,1}$, the description of the action of σ on type II points shows that $\sigma(\zeta_G) = \zeta_G$. This means that for the type II point $P = \tau(\zeta_G)$, one has $\sigma(P) = \sigma(\tau)(\zeta_G)$ and hence

$$\begin{aligned}\mathrm{ordRes}_{\sigma(\varphi)}(\sigma(P)) = \mathrm{ordRes}_{\sigma(\varphi)}(\sigma(\tau)(\zeta_G)) &= \mathrm{ord}(\mathrm{Res}(\sigma(\varphi)^{\sigma(\tau)})) \\ &= \mathrm{ord}(\mathrm{Res}(\varphi^\tau)) = \mathrm{ordRes}_\varphi(\tau(\zeta_G)) = \mathrm{ordRes}_\varphi(P).\end{aligned}$$

By continuity, this holds for all $P \in \mathbf{P}^1_K$.

Furthermore, P is a fixed point of φ if and only if $\sigma(P)$ is a fixed point of $\sigma(\varphi)$ and in that case the points have the same local degrees. This implies that

$\Gamma_{\text{Fix,Repel}}(\sigma(\varphi)) = \sigma(\Gamma_{\text{Fix,Repel}}(\varphi))$. The assertions in the proposition follow by taking Laplacians. $\square$

7. Characterizations of the minimal resultant locus

In this section, we characterize $\text{MinResLoc}(\varphi)$ from analytic and moduli-theoretic standpoints. We first give an analytic characterization. Before stating it we need two definitions.

Definition 10 (barycenter). The barycenter of a finite positive measure ν on $\mathbf{P}^1_K$ is the set of points $Q \in \mathbf{P}^1_K$ such that for each direction $\vec{v} \in T_Q$, at most half the mass of ν lies in $B_Q(\vec{v})^-$.

The notion of the barycenter of a measure on $\mathbf{P}^1_K$ is due to Benedetto and Rivera-Letelier, who used it in unpublished work on the "$s \log s$" bound for the number of k-rational preperiodic points of $\varphi(z) \in k(z)$, when k is a number field.

Definition 11 (the crucial tree). The crucial tree Γ_φ is the subtree of $\Gamma_{\text{Fix,Repel}}$ spanned by the crucial set of φ. We define the *vertices* of Γ_φ to be the points of the crucial set (whether they are endpoints or interior points of Γ_φ) and the branch points of Γ_φ. The *edges* of Γ_φ are the closed segments between adjacent vertices.

Theorem 7.1 (dynamical characterization of $\text{MinResLoc}(\varphi)$). *Let $\varphi(z) \in K(z)$ have degree $d \geq 2$. Then* $\text{MinResLoc}(\varphi)$ *is the barycenter of the crucial measure ν_φ. Equivalently, a point $Q \in \mathbf{P}^1_K$ belongs to* $\text{MinResLoc}(\varphi)$ *if and only if for each $\vec{w} \in T_Q$*

$$\sum_{P \in B_Q(\vec{w})^-} w_\varphi(P) \leq \frac{d-1}{2}. \tag{46}$$

If d is even, then $\text{MinResLoc}(\varphi)$ *is a vertex of the crucial tree Γ_φ. If d is odd, then* $\text{MinResLoc}(\varphi)$ *is either a vertex or an edge of Γ_φ.*

Proof. To show that $\text{MinResLoc}(\varphi)$ is the barycenter of ν_φ, we must show for each $Q \in \mathbf{P}^1_K$ that $Q \in \text{MinResLoc}(\varphi)$ if and only if for each $\vec{w} \in T_Q$,

$$\nu_\varphi(B_Q(\vec{w})^-) \leq \tfrac{1}{2}.$$

If $Q \notin \Gamma_{\text{Fix,Repel}}$ this is trivial: $Q \notin \text{MinResLoc}(\varphi)$ since

$$\text{MinResLoc}(\varphi) \subset \Gamma_{\text{Fix,Repel}} \cap \mathbf{H}^1_K,$$

while if $\vec{w} \in T_Q$ is the direction towards $\Gamma_{\text{Fix,Repel}}$ then $\nu_\varphi(B_Q(\vec{v})^-) = 1$. Similar reasoning applies when $Q \in \Gamma_{\text{Fix,Repel}} \cap \mathbb{P}^1(K)$.

Let $Q \in \Gamma_{\text{Fix,Repel}} \cap \mathbf{H}^1_K$ and write $f(\cdot) = \text{ordRes}_\varphi(\cdot)$. Then $Q \in \text{MinResLoc}(\varphi)$ if and only if $\partial_{\vec{w}} f(Q) \geq 0$ for each $\vec{w} \in T_Q$. If $\vec{w} \in T_Q$ points away from $\Gamma_{\text{Fix,Repel}}$ then $\partial_{\vec{w}} f(Q) > 0$ and $\nu_\varphi(B_Q(\vec{w})^-) = 0$. Hence it is enough to show that for each

$\vec{w} \in T_{Q,\mathrm{FR}}$, we have $\partial_{\vec{w}} f(Q) \ge 0$ if and only if $\nu_\varphi(B_Q(\vec{w})^-) \le \frac{1}{2}$, or equivalently that (46) holds. For this, we use an argument like the one in the proof of Theorem 6.1.

Let Γ be the graph consisting of the part of $\Gamma_{\mathrm{Fix,Repel}}$ in $B_Q(\vec{w})^-$, together with Q. The endpoints of Γ are Q and the type I fixed points and focused repelling fixed points of φ in $B_Q(\vec{w})^-$. Let D_1' be the number of type I endpoints and let D_2' be the number of focused repelling endpoints, so Γ has $D' = 1 + D_1' + D_2'$ endpoints in all.

Suppose the type I fixed points of φ in $B_Q(\vec{w})^-$ are $\alpha_1, \dots, \alpha_{D_1'}$. For each α_i, choose a type II point Q_i in $\Gamma_{\mathrm{Fix,Repel}}$ close enough to α_i that

(1) there are no branch points of Γ in $[Q_i, \alpha_i]$,

(2) at each $P \in [Q_i, \alpha_i)$, the slope of $\mathrm{ordRes}_\varphi(\,\cdot\,)$ at P in the (unique) direction $\vec{v}_i \in T_P$ pointing away from α_i in Γ is $-(d^2-d)$, and

(3) each $P \in [Q_i, \alpha_i]$ has weight $w_\varphi(P) = 0$.

Since only finitely many points have positive weight, Proposition 5.4 shows that such Q_i exist. Let $\widehat{\Gamma}$ be the finite metrized graph gotten by cutting the terminal segments $(Q_i, \alpha_i]$ off of Γ.

To simplify notation, write Q_0 for Q and let $\vec{v}_0 = \vec{w} \in T_Q$. Then the Laplacian $\Delta_{\widehat{\Gamma}}(f)$ satisfies

$$-\Delta_{\widehat{\Gamma}}(f) = \sum_{P \in \{Q_0, Q_1, \dots, Q_{D_1'}\}} \partial_{\vec{v}_i} f(P) \delta_P(z) + \sum_{P \in \widehat{\Gamma} \setminus \{Q_0, Q_1, \dots, Q_{D_1'}\}} \Bigg(\sum_{\vec{v} \in T_{P,\mathrm{FR}}} \partial_{\vec{v}} f(P) \Bigg) \delta_P(z). \tag{47}$$

Put $L = \partial_{\vec{w}} f(Q) = \partial_{\vec{v}_0} f(Q_0)$. By Proposition 5.4, if $P \in \{Q_1, \dots, Q_{D_1'}\}$ then $\partial_{\vec{v}_i} f(P) = -(d^2-d)$. By formula (41), if $P \in \widehat{\Gamma} \setminus \{Q_0, Q_1, \dots, Q_{D_1'}\}$, then

$$\sum_{\vec{v} \in T_{P,\mathrm{FR}}} \partial_{\vec{v}} f(P) = (d^2-d) \cdot (v_{\widehat{\Gamma}}(P) - 2) + 2d \cdot w_\varphi(P).$$

Inserting these values in (47) and using that $\Delta_{\widehat{\Gamma}}(f)$ has total mass 0, we see that

$$0 = L + D_1' \cdot (-(d^2-d)) + \sum_{P \in \widehat{\Gamma} \setminus \{Q_0, Q_1, \dots, Q_{D_1'}\}} \big((d^2-d) \cdot (v_{\widehat{\Gamma}}(P) - 2) + 2d \cdot w_\varphi(P) \big).$$

If $P \in \widehat{\Gamma}$ is not an endpoint or a branch point, then $(d^2-d) \cdot (v_{\widehat{\Gamma}}(P) - 2) = 0$. There are D_2' focused repelling endpoints of $\widehat{\Gamma}$ in $B_Q(\vec{w})^-$ and for each of them we have $(d^2-d) \cdot (v_{\widehat{\Gamma}}(P) - 2) = -(d^2-d)$. Hence

$$L = \Bigg(D_1' + D_2' - \sum_{\text{b.p.s of } \widehat{\Gamma}} (v_{\widehat{\Gamma}}(P) - 2) \Bigg) \cdot (d^2-d) - 2d \cdot \sum_{P \in \widehat{\Gamma} \cap B_P(\vec{w})^-} w_\varphi(P). \tag{48}$$

Since $\widehat{\Gamma}$ has $D' = 1 + D'_1 + D'_2$ endpoints, Lemma 6.4 gives

$$D'_1 + D'_2 - \sum_{\text{b.p.s of } \widehat{\Gamma}} (v_{\widehat{\Gamma}}(P) - 2) = 1. \tag{49}$$

It follows from (48) and (49) that $L \geq 0$ if and only if (46) holds. Thus Q belongs to $\mathrm{MinResLoc}(\varphi)$ if and only if Q is in the barycenter of ν_φ.

Clearly $\mathrm{MinResLoc}(\varphi) \subseteq \Gamma_\varphi$. To see that $\mathrm{MinResLoc}(\varphi)$ is either a vertex or an edge of Γ_φ, note that as a point P moves along Γ_φ, the distribution of ν_φ-mass in the various directions $\vec{v} \in T_P$ can change only when P passes through a vertex.

If $\mathrm{MinResLoc}(\varphi)$ consists of a vertex of Γ_φ, then we are done. Otherwise, $\mathrm{MinResLoc}(\varphi)$ contains a point Q in the interior of an edge e of Γ_φ. Since there are precisely two directions $\vec{v} \in T_Q$ for which the balls $B_Q(\vec{v})^-$ can contain ν_φ-mass and since each has mass at most $\frac{1}{2}$, each must have mass exactly $\frac{1}{2}$. This continues to hold for all P in the interior of e but it changes when P reaches an endpoint of e.

At an endpoint P_0 of e, for the direction $\vec{v}_0 \in T_{P_0}$ pointing into e we still have $\nu_\varphi(B_{P_0}(\vec{v}_0)^-) = \frac{1}{2}$, so for all $\vec{v} \in T_{P_0}$ with $\vec{v} \neq \vec{v}_0$ we necessarily have $\nu_\varphi(B_P(\vec{w})^-) \leq \frac{1}{2}$ and P_0 belongs to $\mathrm{MinResLoc}(\varphi)$. If P_0 is an endpoint of Γ_φ, this is all that needs to be said. If P_0 is a vertex of Γ_φ which is not a branch point, then P_0 belongs to the crucial set, so $\nu_\varphi(\{P_0\}) > 0$. Hence when P moves outside e, for the direction $\vec{v} \in T_P$ pointing towards e we will have $\nu_\varphi(B_P(\vec{v})^-) > \frac{1}{2}$, so $P \notin \mathrm{MinResLoc}(\varphi)$. Finally, if P_0 is a branch point of Γ_φ, there are at least two directions $\vec{v}_1, \vec{v}_2 \in T_{P_0}$ with $\vec{v}_1, \vec{v}_2 \neq \vec{v}_0$ such that $\nu_\varphi(B_{P_0}(\vec{v}_i)^-) > 0$. Hence when P moves outside e, for the direction $\vec{v} \in T_P$ pointing towards e, we will again have $\nu_\varphi(B_P(\vec{v})^-) > \frac{1}{2}$ and $P \notin \mathrm{MinResLoc}(\varphi)$. □

We next give a moduli-theoretic characterization of $\mathrm{MinResLoc}(\varphi)$.

The basic theorem in geometric invariant theory (GIT) concerning moduli spaces of rational functions is due to Silverman. Let R be a commutative ring with 1 and let $\varphi : \mathbb{P}^1_R \to \mathbb{P}^1_R$ be a morphism of degree $d \geq 2$. Fixing homogeneous coordinates on $\mathbb{P}^1_R$, the map φ corresponds to a pair of homogeneous functions $F(X, Y) = f_d X^d + \cdots + f_0 Y^d$ and $G(X, Y) = g_d X^d + \cdots + g_0 Y^d$ in $R[X, Y]$ such that $\mathrm{Res}(F, G)$ is a unit in R. Writing $f = (f_d, \ldots, f_0)$ and $g = (g_d, \ldots, g_0)$, let

$$Z_\varphi = Z_{f,g} = (f_d : \cdots : f_0 : g_d : \cdots : g_0) \in \mathbb{P}^{2d+1}(R)$$

be the point corresponding to φ. If $\mu = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{GL}_2(R)$, the usual conjugation action of μ on φ defined by

$$\begin{aligned} \varphi^\mu &= \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} \circ \begin{pmatrix} F \\ G \end{pmatrix} \circ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \\ &= \begin{pmatrix} \delta F(\alpha X + \beta Y, \gamma X + \delta Y) - \beta G(\alpha X + \beta Y, \gamma X + \delta Y) \\ -\gamma F(\alpha X + \beta Y, \gamma X + \delta Y) + \alpha G(\alpha X + \beta Y, \gamma X + \delta Y) \end{pmatrix} \end{aligned}$$

induces an algebraic action of GL_2 on $\mathbb{P}^{2d+1}$. If $R = \Omega$ is an algebraically closed field, the groups $\mathrm{GL}_2(\Omega)$ and $\mathrm{SL}_2(\Omega)$ have the same orbits in $\mathbb{P}^{2d+1}(\Omega)$. For technical reasons, in GIT it is better to work with the action of SL_2; Silverman [1998, Theorems 1.1 and 1.3] proves:

Theorem 7.2 (Silverman). *There are open subschemes of* $\mathbb{P}^{2d+1}/\mathrm{Spec}(\mathbb{Z})$

$$\mathrm{Rat}_d \subseteq (\mathbb{P}^{2d+1})^s \subseteq (\mathbb{P}^{2d+1})^{ss}$$

(*the subschemes of rational morphisms of degree d, stable points, and semistable points*), *which are invariant under the conjugation action of* SL_2, *such that the quotients*

$$M_d = \mathrm{Rat}_d / \mathrm{SL}_2, \quad M_d^s = (\mathbb{P}^{2d+1})^s / \mathrm{SL}_2, \quad \textit{and} \quad M_d^{ss} = (\mathbb{P}^{2d+1})^{ss} / \mathrm{SL}_2$$

exist. M_d is a dense open subset of M_d^s and M_d^{ss}, M_d and M_d^s are geometric quotients, and M_d^{ss} is a categorical quotient which is proper and of finite type over $\mathbb{Z}$.

Geometrical and categorical quotients are quotients with certain desirable properties (see [Mumford et al. 1994] for the definitions). The fact that M_d is a geometric quotient includes the fact over any algebraically closed field Ω, the $\mathrm{SL}_2(\Omega)$ orbits of rational functions of degree d are in one-to-one correspondence with the points of $M_d(\Omega)$. The spaces M_d^s and M_d^{ss} are called the spaces of stable and semistable conjugacy classes of rational maps, respectively. Loosely, the stable locus $(\mathbb{P}^{2d+1})^s$ is the largest subscheme such that for any algebraically closed field Ω, the $\mathrm{SL}_2(\Omega)$-orbits of points in $(\mathbb{P}^{2d+1})^s(\Omega)$ are closed and are in one-to-one correspondence with the points of $M_d^s(\Omega)$. Loosely, the semistable locus $(\mathbb{P}^{2d+1})^{ss}$ is the largest subscheme for which a quotient makes sense: $\mathrm{SL}_2(\Omega)$-orbits of points in $(\mathbb{P}^{2d+1})^{ss}(\Omega)$ need not be closed but if one defines two orbits to be equivalent if their closures meet, points of $M_d^{ss}(\Omega)$ correspond to equivalence classes of $\mathrm{SL}_2(\Omega)$-orbits. Each equivalence class of orbits in $(\mathbb{P}^{2d+1})^{ss}(\Omega)$ contains a unique minimal closed orbit.

Recall that K is a complete, algebraically closed nonarchimedean valued field with ring of integers $\mathcal{O}$ and residue field $\tilde{k}$:

Definition 12 (semistable and stable reduction). Let $\varphi(z) \in K(z)$ have degree $d \geq 2$ and let (F, G) be a normalized representation of φ. Writing $F(X, Y) = f_d X^d + \cdots + f_0 Y^d$ and $G(X, Y) = g_d X^d + \cdots g_0 Y^d$, let

$$Z_\varphi = (f_d : \cdots : f_0 : g_d : \cdots : g_0) \in \mathbb{P}^{2d+1}(K)$$

be the point corresponding to φ. We will say that φ has *semistable reduction* if

$$\widetilde{Z}_\varphi = (\tilde{f}_d : \cdots : \tilde{f}_0 : \tilde{g}_d : \cdots : \tilde{g}_0) \in \mathbb{P}^{2d+1}(\tilde{k})$$

belongs to $(\mathbb{P}^{2d+1})^{ss}(\tilde{k})$ and that φ has *stable reduction* if $\widetilde{Z}_\varphi$ belongs to $(\mathbb{P}^{2d+1})^s(\tilde{k})$.

Making explicit the Hilbert–Mumford numerical criteria, Silverman [1998, Proposition 2.2] gives necessary and sufficient conditions for a point to be semistable or stable.

Proposition 7.3. *Let* SL_2 *act on* $\mathbb{P}^{2d+1}$ *as above and suppose* $\widetilde{Z} \in \mathbb{P}^{2d+1}(\tilde{k})$. *Then*:

(A) $\widetilde{Z}$ *belongs to* $(\mathbb{P}^{2d+1})^{ss}(\tilde{k})$ *if and only if for each* $\tilde{\tau} \in \mathrm{SL}_2(\tilde{k})$, *when* $\widetilde{Z}^{\tilde{\tau}}$ *is written as* $(\tilde{a}_d : \cdots : \tilde{a}_0 : \tilde{b}_d : \cdots : \tilde{b}_0)$, *either there is some* i *with* $(d+1)/2 \le i \le d$ *such that* $\tilde{a}_i \neq 0$ *or there is some* i *with* $(d-1)/2 \le i \le d$ *such that* $\tilde{b}_i \neq 0$.

(B) $\widetilde{Z}$ *belongs to* $(\mathbb{P}^{2d+1})^{s}(\tilde{k})$ *if and only if for each* $\tilde{\tau} \in \mathrm{SL}_2(\tilde{k})$, *when* $\widetilde{Z}^{\tilde{\tau}}$ *is written as* $(\tilde{a}_d : \cdots : \tilde{a}_0 : \tilde{b}_d : \cdots : \tilde{b}_0)$, *either there is some* i *with* $(d+1)/2 < i \le d$ *such that* $\tilde{a}_i \neq 0$ *or there is some* i *with* $(d-1)/2 < i \le d$ *such that* $\tilde{b}_i \neq 0$.

Proof. This is [Silverman 1998, Proposition 2.2] in the special case $\Omega = \tilde{k}$, with two modifications. Silverman formulates conditions (A) and (B) as characterizing the "unstable" and "not stable" points of $\mathbb{P}^{2d+1}(\tilde{k})$. Our assertions are the contrapositives of his: by definition, "semistable" is "not unstable" and "stable" is "not *not stable*". Second, Silverman writes $\varphi(z) = (a_0 z^d + \cdots + a_d)/(b_0 z^d + \cdots + b_d)$, indexing coefficients of rational function in the opposite order. We have adjusted his coefficient ranges to account for this. $\square$

The connection between semistability and having minimal resultant is due to Szpiro, Tepper, and Williams, who proved the implication "semistable reduction implies minimal resultant" in part A of the following theorem using a moduli-theoretic argument (see [Szpiro et al. 2014, Theorem 3.3]). Here we establish the converse as well; however, the theorem of Szpiro, Tepper, and Williams holds in arbitrary dimension n, whereas ours applies only when $n = 1$.

Theorem 7.4 (moduli-theoretic characterization of $\mathrm{MinResLoc}(\varphi)$). *Let* $\varphi(z) \in K(z)$ *have degree* $d \ge 2$. *Suppose* $\gamma \in \mathrm{GL}_2(K)$ *and put* $P = \gamma(\zeta_G)$, *then*:

(A) P *belongs to* $\mathrm{MinResLoc}(\varphi)$ *if and only if* φ^γ *has semistable reduction in the sense of* GIT.

(B) $\mathrm{MinResLoc}(\varphi)$ *consists of the single point* P *if and only if* φ^γ *has stable reduction in the sense of* GIT.

Proof. We first show (A). Fixing γ, put $P = \gamma(\zeta_G)$. Note that $P \in \mathrm{MinResLoc}(\varphi)$ if and only if $\partial_{\vec{v}} \mathrm{ordRes}_\varphi(P) \ge 0$ for each $\vec{v} \in T_P$. After replacing φ with φ^γ, we can assume that $P = \zeta_G$. Let $Z_\varphi \in \mathbb{P}^{2d+1}(K)$ be the point corresponding to φ and let $\widetilde{Z}_\varphi \in \mathbb{P}^{2d+1}(\tilde{k})$ be its reduction. We will index directions $\vec{v} \in T_{\zeta_G}$ by points $a \in \mathbb{P}^1(\tilde{k})$.

Fix a direction $\vec{v}_a \in T_{\zeta_G}$ and choose $\tilde{\tau} \in \mathrm{SL}_2(\tilde{k})$ so that $\tilde{\tau}(\infty) = a$. We will show that $\partial_{\vec{v}_a} \mathrm{ordRes}_\varphi(\zeta_G) \ge 0$ if and only if the condition of Proposition 7.3(A) holds for $(\widetilde{Z}_\varphi)^{\tilde{\tau}}$.

Lift $\tilde{\tau}$ to $\tau \in \mathrm{SL}_2(\mathcal{O})$. Let (F, G) be a normalized representation of φ^τ and write $F(X, Y) = a_d X^d + \cdots + a_0 Y^d$ and $G(X, Y) = b_d X^d + \cdots + b_0 Y^d$. Since the action of SL_2 commutes with reduction, it follows that

$$(\widetilde{Z}_\varphi)^{\tilde{\tau}} = (\widetilde{Z_{\varphi^\tau}}) = (\tilde{a}_d : \cdots \tilde{a}_0 : \tilde{b}_d : \cdots : \tilde{b}_0).$$

Since $\tau_*(\vec{v}_\infty) = \vec{v}_a$, the function $\mathrm{ordRes}_\varphi(\,\cdot\,)$ is nondecreasing in the direction $\vec{v}_a$ at ζ_G if and only if $\mathrm{ordRes}_{\varphi^\tau}(\,\cdot\,)$ is nondecreasing in the direction $\vec{v}_\infty$. Take $A \in K^\times$ with $|A| \geq 1$. By [Rumely 2015, formula (13)], we have

$$\begin{aligned}
\mathrm{ordRes}&_{\varphi^\tau}(\zeta_{0,|A|}) - \mathrm{ordRes}_{\varphi^\tau}(\zeta_G) \\
&= (d^2 + d)\mathrm{ord}(A) - 2d \cdot \min_{0 \leq i \leq d} \bigl(\mathrm{ord}(A^i a_i), \mathrm{ord}(A^{i+1} b_i)\bigr) \\
&= \max_{0 \leq i \leq d} \bigl((d^2 + d - 2di)\mathrm{ord}(A) - 2d\mathrm{ord}(a_i), \\
&\qquad\qquad (d^2 + d - 2d(i+1))\mathrm{ord}(A) - 2d\mathrm{ord}(b_i)\bigr). \qquad (50)
\end{aligned}$$

The terms in (50) which are nondecreasing as $|A|$ increases are the ones involving $\mathrm{ord}(a_i)$ for $(d+1)/2 \leq i \leq d$ and the ones involving $\mathrm{ord}(b_i)$ for $(d-1)/2 \leq i \leq d$.

Since (F, G) is normalized, we have $\mathrm{ord}(a_i) \geq 0$ and $\mathrm{ord}(b_i) \geq 0$ for all i and there is at least one i for which $\mathrm{ord}(a_i) = 0$ or $\mathrm{ord}(b_i) = 0$. Hence the terms with $\mathrm{ord}(a_i) > 0$ or $\mathrm{ord}(b_i) > 0$ cannot be the maximal ones in (50) when $\mathrm{ord}(A)$ is near 0. It follows that $\partial_{\vec{v}_\infty} \mathrm{ordRes}_{\varphi^\tau}(\zeta_G) \geq 0$ if and only if

$$\begin{aligned}
&\mathrm{ord}(a_i) = 0 \quad \text{for some } i \text{ with } (d+1)/2 \leq i \leq d, \text{ or} \\
&\mathrm{ord}(b_i) = 0 \quad \text{for some } i \text{ with } (d-1)/2 \leq i \leq d.
\end{aligned}$$

Since $\mathrm{ord}(a_i) = 0$ if and only if $\tilde{a}_i \neq 0$ and $\mathrm{ord}(b_i) = 0$ if and only if $\tilde{b}_i \neq 0$, these are precisely the conditions on $(\widetilde{Z}_\varphi)^{\tilde{\tau}}$ from Proposition 7.3(A).

Since $\tau_*(\vec{v}_\infty)$ runs over all $\vec{v}_a \in T_P$ as $\tilde{\tau}$ runs over $\mathrm{SL}_2(\tilde{k})$, it follows that P belongs to $\mathrm{MinResLoc}(\varphi)$ if and only if φ^γ has semistable reduction.

Part (B) is proved similarly, using that $P = \gamma(\zeta_G)$ is the unique point in $\mathrm{MinResLoc}(\varphi)$ if and only if $\partial_{\vec{v}} \mathrm{ordRes}_\varphi(P) > 0$ for each $\vec{v} \in T_P$ and that the terms in (50) which are increasing as $|A|$ increases are the ones involving $\mathrm{ord}(a_i)$ for $(d+1)/2 < i \leq d$ and the ones involving $\mathrm{ord}(b_i)$ for $(d-1)/2 < i \leq d$. □

References

[Baker and Rumely 2007] M. Baker and R. Rumely, "Harmonic analysis on metrized graphs", *Canad. J. Math.* **59**:2 (2007), 225–275. MR Zbl

[Baker and Rumely 2010] M. Baker and R. Rumely, *Potential theory and dynamics on the Berkovich projective line*, Mathematical Surveys and Monographs **159**, Amer. Math. Soc., 2010. MR Zbl

[Benedetto et al. 2014] R. Benedetto, P. Ingram, R. Jones, and A. Levy, "Attracting cycles in p-adic dynamics and height bounds for postcritically finite maps", *Duke Math. J.* **163**:13 (2014), 2325–2356. MR Zbl

[Berkovich 1990] V. G. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*, Mathematical Surveys and Monographs **33**, Amer. Math. Soc., 1990. MR Zbl

[Chinburg and Rumely 1993] T. Chinburg and R. Rumely, "The capacity pairing", *J. Reine Angew. Math.* **434** (1993), 1–44. MR Zbl

[Doyle et al. 2016] J. R. Doyle, K. Jacobs, and R. Rumely, "Configuration of the crucial set for a quadratic rational map", *Res. Number Theory* **2** (2016), 11–16. MR Zbl

[Faber 2013a] X. Faber, "Topology and geometry of the Berkovich ramification locus for rational functions, I", *Manuscripta Math.* **142**:3-4 (2013), 439–474. MR Zbl

[Faber 2013b] X. Faber, "Topology and geometry of the Berkovich ramification locus for rational functions, II", *Math. Ann.* **356**:3 (2013), 819–844. MR Zbl

[Favre and Jonsson 2004] C. Favre and M. Jonsson, *The valuative tree*, Lecture Notes in Mathematics **1853**, Springer, 2004. MR Zbl

[Favre and Rivera-Letelier 2006] C. Favre and J. Rivera-Letelier, "Équidistribution quantitative des points de petite hauteur sur la droite projective", *Math. Ann.* **335**:2 (2006), 311–361. MR Zbl

[Favre and Rivera-Letelier 2010] C. Favre and J. Rivera-Letelier, "Théorie ergodique des fractions rationnelles sur un corps ultramétrique", *Proc. Lond. Math. Soc.* **100**:1 (2010), 116–154. MR Zbl

[Jacobs 2014] K. Jacobs, "An equidistribution result for dynamical systems on the Berkovich projective line", preprint, 2014. arXiv

[Jonsson 2015] M. Jonsson, "Dynamics of Berkovich spaces in low dimensions", pp. 205–366 in *Berkovich spaces and applications*, edited by A. Ducros et al., Lecture Notes in Math. **2119**, Springer, 2015. MR Zbl

[Mumford et al. 1994] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, 2nd ed., Ergeb. Math. Grenzgeb. **34**, Springer, 1994. MR Zbl

[Poineau 2013] J. Poineau, "Les espaces de Berkovich sont angéliques", *Bull. Soc. Math. France* **141**:2 (2013), 267–297. MR Zbl

[Rivera-Letelier 2000] J. Rivera-Letelier, "Sur la structure des ensembles de Fatou p-adiques", preprint, 2000. arXiv

[Rivera-Letelier 2003a] J. Rivera-Letelier, "Dynamique des fonctions rationnelles sur des corps locaux", pp. 147–230 in *Journées de Géométrie Algébrique d'Orsay* (Orsay, 1992), Astérisque **287**, Société Mathématique de France, 2003. MR Zbl

[Rivera-Letelier 2003b] J. Rivera-Letelier, "Espace hyperbolique p-adique et dynamique des fonctions rationnelles", *Compositio Math.* **138**:2 (2003), 199–231. MR Zbl

[Rumely 2014] R. Rumely, "The geometry of the minimal resultant locus", preprint, 2014. arXiv

[Rumely 2015] R. Rumely, "The minimal resultant locus", *Acta Arith.* **169**:3 (2015), 251–290. MR Zbl

[Rumely et al. $\geq$ 2017] R. Rumely, E. Harvey, A. Varghese, and L. Watson, "Configurations of the crucial set for a cubic polynomial", in preparation.

[Silverman 1998] J. H. Silverman, "The space of rational maps on $\mathbb{P}^1$", *Duke Math. J.* **94**:1 (1998), 41–77. MR Zbl

[Silverman 2007] J. H. Silverman, *The arithmetic of dynamical systems*, Graduate Texts in Mathematics **241**, Springer, 2007. MR Zbl

[Szpiro et al. 2014] L. Szpiro, M. Tepper, and P. Williams, "Semi-stable reduction implies minimality of the resultant", *J. Algebra* **397** (2014), 489–498. MR Zbl

[Zhang 1993] S. Zhang, "Admissible pairing on a curve", *Invent. Math.* **112**:1 (1993), 171–193. MR Zbl

Communicated by Joseph H. Silverman
Received 2015-08-27 Revised 2017-01-13 Accepted 2017-02-11

rr@math.uga.edu Department of Mathematics, University of Georgia, Athens, GA 30602, United States

ALGEBRA AND NUMBER THEORY 11:4 (2017)
dx.doi.org/10.2140/ant.2017.11.885

On pairs of p-adic L-functions for weight-two modular forms

Florian Sprung

Dedicated to Barry, Joe, and Rob

The point of this paper is to give an explicit p-adic analytic construction of two Iwasawa functions, $L_p^\sharp(f,T)$ and $L_p^\flat(f,T)$, for a weight-two modular form $\sum a_n q^n$ and a good prime p. This generalizes work of Pollack who worked in the supersingular case and also assumed $a_p=0$. These Iwasawa functions work in tandem to shed some light on the Birch and Swinnerton-Dyer conjectures in the cyclotomic direction: we bound the rank and estimate the growth of the Šafarevič–Tate group in the cyclotomic direction analytically, encountering a new phenomenon for small slopes.

1. Introduction

Let $f=\sum a_n q^n$ be a weight-two modular form. The idea of attaching a p-adic L-function to f goes back to at least Mazur and Swinnerton-Dyer in the case where the associated abelian variety A_f is an elliptic curve. They analytically constructed a power series $L_\alpha(f,T)$, whose behavior at special values $\zeta_{p^n}-1$, corresponding to finite layers $\mathbb{Q}_n$ of the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}_\infty$ of $\mathbb{Q}$, should mirror that of the rational points $A_f(\mathbb{Q}_n)$ and the Šafarevič–Tate group $Ш(A_f/\mathbb{Q}_n)$ in view of the Birch and Swinnerton-Dyer conjectures. Identifying algebraic numbers with p-adic numbers via a fixed embedding $\overline{\mathbb{Q}}\hookrightarrow\mathbb{C}_p$, we may give a_p a valuation v. Their crucial assumption was that p be good and $v=0$ (p is *ordinary*), so that $L_\alpha(f,T)$ is an Iwasawa function, i.e., analytic on the closed unit disc. Since $L_\alpha(f,T)$ is nonzero by work of Rohrlich [1984], we can extract Iwasawa invariants, responsible for that behavior of $L_\alpha(f,T)$ which under the main conjecture corresponds to a

This material is based upon work supported by the National Science Foundation under agreement No. DMS-1128155. Any opinions, findings and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

MSC2010: primary 11G40; secondary 11F67, 11R23.

Keywords: Birch and Swinnerton-Dyer, p-adic L-function, elliptic curve, modular form, Šafarevič–Tate group, Iwasawa Theory.

bound for $\mathrm{rank}(A_f(\mathbb{Q}_\infty))$ and a description of the size of $\text{Ш}(A_f/\mathbb{Q}_n)[p^\infty]$. Skinner and Urban [2014] settled the main conjecture in many cases.

The construction of $L_\alpha(f, T)$ has been generalized to the supersingular (i.e., $v > 0$) case as well [Amice and Vélu 1975; Višik 1976], in which there are two power series $L_\alpha(f, T)$ and $L_\beta(f, T)$. They are not Iwasawa functions, and thus not amenable for estimates for $\mathrm{rank}(A_f(\mathbb{Q}_\infty))$ or $\text{Ш}(A_f/\mathbb{Q}_n)$ directly. Nevertheless, the results of Rohrlich show that $L_\alpha(f, T)$ and $L_\beta(f, T)$ vanish at finitely many special values $\zeta_{p^n} - 1$, so that the analytic rank of $A_f(\mathbb{Q}_\infty)$ is bounded. His results are effective.[1]

The main theorem (Theorem 2.14) of Part I in this paper obtains a *pair* of appropriate Iwasawa functions in the general good reduction case so that p can be ordinary or supersingular. This pair is unique when p is supersingular, and generalize the results of Pollack in the $a_p = 0$ case. Note that the hypothesis $a_p = 0$ is very restrictive since the vast majority of supersingular modular abelian varieties have modular forms failing this condition. The philosophy of using *pairs* of objects has its origins in the work of Perrin-Riou [1990; 1993] in the supersingular case in which the pair consisting of $L_\alpha(f, T)$ and $L_\beta(f, T)$ is considered as one object as a power series with coefficients in the Dieudonné module. Our main theorem generalizes the construction of a pair of functions in the case of elliptic curves and supersingular primes [Sprung 2012] via Kato's zeta element. As pointed out in [Lei et al. 2010, Remark 5.26], the methods in [Sprung 2012] extend to the case $v \geqslant \frac{1}{2}$ as well. In this paper, we have completely isolated the analytic aspects of the theory and are thus able to treat the much harder case $v < \frac{1}{2}$. In the supersingular case, we prove a functional equation for this pair, which corrects a corresponding statement in [Pollack 2003] when reduced to the $a_p = 0$ case. Also, we give a quick proof that $L_\alpha(f, T)$ and $L_\beta(f, T)$ have finitely many common zeros, as conjectured by Greenberg.

Part 2 is dedicated to the estimates connected to the Birch and Swinnerton-Dyer conjectures. We bound the p-adic analytic rank of $A_f(\mathbb{Q}_\infty)$, which is the number of zeros of $L_\alpha(f^\sigma, T)$ at cyclotomic points $T = \zeta_{p^n} - 1$ summed over all Galois conjugates f^σ of f. This analytic rank does not depend on the choice of α in the supersingular case, as shown in Proposition 7.1. When p is ordinary, α is chosen to be a p-adic unit. This is hard even when f is ordinary, since f^σ may not be ordinary, i.e., we may have $v = 0$ but $v^\sigma > 0$, where v^σ is the valuation for a_p^σ for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We overcome this difficulty by giving an upper bound in terms of the Iwasawa invariants of our pair of Iwasawa functions in the supersingular case. Apart from our Iwasawa-theoretic arguments, the key ingredient for this upper bound is to find any nonjump in the analytic rank in the cyclotomic tower, i.e.,

[1] In the elliptic curve case, $L_\alpha(\zeta_{p^n} - 1) \neq 0$ for $p^n > \max(10^{1000}N^{170}, 10^{120p}p^{7p}, 10^{6000}p^{420})$, where N is the conductor of $E = A_f$ [Rohrlich 1984, p. 416].

any n such that $L_\alpha(f, \zeta_{p^n} - 1) \neq 0$. Note that this is a much weaker corollary to Rohrlich's theorem (stating that almost all n give rise to nonjumps). This manifests itself in the fact that the first such n is typically very small. We also give another upper bound that assumes $a_p = 0$ (so that all f^σ are supersingular), and is due to Pollack under the further assumption $p \equiv 3 \pmod 4$, which our new proof removes. This upper bound is in most cases not as sharp as the more general one. Note that the upper bounds are also upper bounds for the corresponding algebraic objects (i.e., that rank of $A_f(\mathbb{Q}_\infty)$) in view of classical work of Perrin-Riou [1990, Lemme 6.10].

We then give growth formulas for the analytic size of $\text{Ш}(A_f/\mathbb{Q}_n)[p^\infty]$, unifying results of Mazur (who assumed $v^\sigma = 0$ for all σ) and Pollack (who assumed $v^\sigma = \infty$ for all σ, i.e., $a_p = 0$), and finishing this problem in most of the good reduction cases.[2] For example, we finish this problem when A_f is an elliptic curve, where there are infinitely many remaining cases all for which $p = 2$ or $p = 3$ in view of the Hasse bound. In Mazur's case, this formula was governed by $L_\alpha(f^\sigma, T)$, while in the $a_p = 0$ case, Pollack's Iwasawa functions alternated responsibility for the growth at even n and odd n. The reason we can cover the remaining cases is that our estimates result from both of our Iwasawa functions working *in tandem*, giving rise to several growth formula scenarios in these remaining cases, illustrating their difficulty even when A_f is an elliptic curve. In the ordinary case (and some special subcases of the supersingular case), one of the Iwasawa functions dominates, and only the invariants of that function are visible in the estimates. When $0 < v^\sigma < \frac{1}{2}$, we encounter a mysterious phenomenon: the estimates depend further on which one of (up to infinitely many) progressively smaller intervals v^σ lies in, and the roles of the Iwasawa functions generally alternate in adjacent intervals. We suspect the answer to the following question is very deep: *Where does this phenomenon come from and why does it occur?*

We now state our results more precisely. We work in the context of weight-two modular forms and a good (coprime to the level) prime p. The functions $L_\alpha(f, T)$ and $L_\beta(f, T)$ are named after the roots α and β of the Hecke polynomial $X^2 - a_p X + \epsilon(p)p$, ordered such that $\text{ord}_p(\alpha) \leqslant \text{ord}_p(\beta)$. In the supersingular case (i.e., when $v := \text{ord}_p(a_p) > 0$), we can now trace the p-adic L-functions back to a pair of Iwasawa functions when $v = \infty$ (i.e., $a_p = 0$) thanks to the methods of Pollack [2003].

In Part I, we prove the following theorem.

Theorem 1.1. *Let $f = \sum a_n q^n$ be a modular form of weight two and p be a good prime. We let $\Lambda = \mathcal{O}[\![T]\!]$, where $\mathcal{O}$ is the ring of integers of the completion at a prime above p of $\mathbb{Q}((a_n)_{n\in\mathbb{N}}, \epsilon(\mathbb{Z}))$.*

[2] The remaining cases we term the *sporadic cases*, which shouldn't occur in nature: For $v^\sigma > 0$, we are in the sporadic case when the μ-invariants differ in a specific way and $v^\sigma = \frac{1}{2}p^{-k}$ for $k \in \mathbb{N}$ and the valuation of $\sigma(a_p)^2 - \epsilon(p)\Phi_p(\zeta_{p^{k+2}})$ is exactly $2v^\sigma(1 + p^{-1} - p^{-2})$ for n with a fixed parity with respect to k; see Definition 8.3.

(1) *When p is a supersingular prime, we have*

$$\big(L_\alpha(f,T),L_\beta(f,T)\big)=\big(L_p^\sharp(f,T),L_p^\flat(f,T)\big)\mathcal{L}og_{\alpha,\beta}(1+T),$$

for two power series $L_p^\sharp(f,T)$ and $L_p^\flat(f,T)$ which are elements of Λ, and $\mathcal{L}og_{\alpha,\beta}(1+T)$ is an explicit 2×2 matrix of functions converging on the open unit disc.

(2) *When p is ordinary, we can write*

$$L_\alpha(f,T)=L_p^\sharp(f,T)\log_\alpha^\sharp(1+T)+L_p^\flat(f,T)\log_\alpha^\flat(1+T),$$

for some nonunique Iwasawa functions $L_p^\sharp(f,T)$ and $L_p^\flat(f,T)$, where $\log_\alpha^\sharp(T)$ and $\log_\alpha^\flat(T)$ are the entries in the first column of $\mathcal{L}og_{\alpha,\beta}(1+T)$. They are functions converging on the closed unit disc.

In the supersingular case, our vector $(L_p^\sharp(f,T),L_p^\flat(f,T))$ is related to the vector $(L_\alpha(f,T),L_\beta(f,T))$ much like the completed Riemann zeta function is related to the original zeta function: since $L_\alpha(f,T)$ and $L_\beta(f,T)$ are not Iwasawa functions, they have infinitely many zeros in the open unit disk. The analogue of the Gamma factor is the matrix $\mathcal{L}og_{\alpha,\beta}(1+T)$. It removes zeros of linear combinations of $L_\alpha(f,T)$ and $L_\beta(f,T)$, producing the vector of Iwasawa functions with finitely many zeros. Its definition for odd p is

$$\mathcal{L}og_{\alpha,\beta}(1+T):=\lim_{n\to\infty}\mathcal{C}_1\cdots\mathcal{C}_nC^{-(n+2)}\begin{pmatrix}-1&-1\\ \beta&\alpha\end{pmatrix},$$

where

$$\mathcal{C}_i:=\begin{bmatrix}a_p&1\\ -\epsilon(p)\Phi_{p^i}(1+T)&0\end{bmatrix},\quad C:=\begin{bmatrix}a_p&1\\ -\epsilon(p)p&0\end{bmatrix},$$

and $\Phi_n(X)$ is the n-th cyclotomic polynomial.

As one immediate corollary of Theorem 1.1, we obtain that $L_\alpha(f,T)$ and $L_\beta(f,T)$ have finitely many common zeros, as conjectured by Greenberg [2001].

When $p=2$ and $a_2=0$, our construction of $\mathcal{L}og_{\alpha,\beta}(1+T)$ explains a seemingly artificial extra factor of $\frac{1}{2}$ in Pollack's [2003] corresponding half-logarithm. The theorem also shows that for $p=2$, the functions $L_2^\sharp(T)$ and $L_2^\flat(T)$ of [Sprung 2012] in $\Lambda\otimes\mathbb{Q}$ are in fact elements of Λ in the strong Weil case.

Our proof of Theorem 1.1 is completely p-adic analytic, generalizing the arguments of Pollack [2003] when $v=\infty$ (i.e., $a_p=0$). Recall that the methods in [Sprung 2012] extend to the case $v\geqslant\frac{1}{2}$ ([Lei et al. 2010, Remark 5.26]). However, the situation for the *remaining* (and more difficult) valuations when $v<\frac{1}{2}$ is more involved. This part forms the technical heart of the first half of the paper, in which one major new tool is Lemma 4.17, which gives an explicit expansion of the terms of $\mathcal{L}og_{\alpha,\beta}(1+T)$. We use *valuation matrices*, an idea introduced in [Sprung 2013], to scrutinize the growth properties of the functions in its columns: They grow like $L_\alpha(f,T)$ and $L_\beta(f,T)$ for $v \geqslant \frac{1}{2}$, and at most as fast as these functions when $v < \frac{1}{2}$, proving, e.g., that the entries in the first column are Iwasawa functions when $v = 0$.

We also construct a completed version $\widehat{\mathcal{L}og}_{\alpha,\beta}$ of $\mathcal{L}og_{\alpha,\beta}$, and similarly $\widehat{L}_p^\sharp$ and $\widehat{L}_p^\flat$, and then prove functional equations for these completed objects.

Theorem 1.2. *Let p be supersingular. Then under the change of variables*

$$(1+T) \mapsto (1+T)^{-1},$$

$\widehat{\mathcal{L}og}_{\alpha,\beta}(1+T)$ is invariant, and the vector $(\widehat{L}_p^\sharp(T), \widehat{L}_p^\flat(T))$ is invariant up to a root number of the form $-\epsilon(-1)(1+T)^{-\log_\gamma(N)}$. A similar statement holds for $p = 2$.

For a precise definition of the root number, we refer to Section 2.

We derive functional equations for $L_p^\sharp$ and $L_p^\flat$ in some cases as well, which correct a corresponding statement in [Pollack 2003] (where $a_p = 0$), which is off by a unit factor. The algebraic version of the functional equation by Kim [2008] when $a_p = 0$ is still correct, since it is given up to units.

Part II is concerned with applications involving the invariants of the Birch and Swinnerton-Dyer (BSD) conjectures in the cyclotomic direction:

- Choose a subset $\mathcal{G}_f$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\{f^\sigma\}_{\sigma\in\mathcal{G}_f}$ contains each Galois conjugate of f once. Each zero of $L_\alpha(f^\sigma,T)$ or $L_\beta(f^\sigma,T)$ at $T = \zeta_{p^n} - 1$ (counted with multiplicity) with $\sigma \in \mathcal{G}_f$ should, in view of BSD for number fields, contribute toward the jump in the ranks, $\mathrm{rank}(A_f(\mathbb{Q}_n)) - \mathrm{rank}(A_f(\mathbb{Q}_{n-1}))$, where $\mathbb{Q}_n$ is the n-th layer in the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q} = \mathbb{Q}_0$. More specifically, the number of zeroes r_∞^{an} at all $T = \zeta_{p^n} - 1$ of all $L_\alpha(f^\sigma,T)$ is an analytic upper bound for $r_\infty = \lim_{n\to\infty} \mathrm{rank}\, A_f(\mathbb{Q}_n)$. Denote by $r_\infty^{\mathrm{an}}(f^\sigma)$ the σ-part of r_∞^{an}, i.e., the number of such zeros of $L_\alpha(f^\sigma,T)$, so that $r_\infty^{\mathrm{an}} = \sum_{\sigma\in\mathcal{G}_f} r_\infty^{\mathrm{an}}(f^\sigma)$. When f^σ is ordinary, $r_\infty^{\mathrm{an}}(f^\sigma)$ is bounded by the λ-invariant λ^σ of $L_\alpha(f^\sigma,T)$.

For the case in which f^σ is supersingular, $L_\alpha(f^\sigma,T)$ and $L_\beta(f^\sigma,T)$ are known to have finitely many zeroes of the form $\zeta_{p^n} - 1$, by a theorem of Rohrlich. By only assuming the much weaker corollary that $L_\alpha(f^\sigma,T)$ does not vanish at some $\zeta_{p^n} - 1$, we give an explicit upper bound:

Theorem 1.3. *Let $\lambda_\sharp$ and $\lambda_\flat$ be the λ-invariants of $L_p^\sharp(f^\sigma,T)$ and $L_p^\flat(f^\sigma,T)$. Put*

$$q_n^\sharp := \left\lfloor \frac{p^n}{p+1} \right\rfloor \text{ if } n \text{ is odd}, \quad \text{and} \quad q_n^\sharp := q_{n+1}^\sharp \text{ for even } n,$$

$$q_n^\flat := \left\lfloor \frac{p^n}{p+1} \right\rfloor \text{ if } n \text{ is even}, \quad \text{and} \quad q_n^\flat := q_{n+1}^\flat \text{ for odd } n,$$

$$\nu_\sharp := \text{largest odd integer } n \geqslant 1 \text{ such that } \lambda_\sharp \geqslant p^n - p^{n-1} - q_n^\sharp,$$

$$\nu_\flat := \text{largest even integer } n \geqslant 2 \text{ such that } \lambda_\flat \geqslant p^n - p^{n-1} - q_n^\flat,$$

$$\nu := \max(\nu_\sharp, \nu_\flat).$$

(1) *Assume* $\mu_\sharp = \mu_\flat$. *Then the* σ*-part* $r_\infty^{\mathrm{an}}(f^\sigma)$ *of the cyclotomic analytic rank* r_∞ *for* $\lim_{n\to\infty} \operatorname{rank} A_f(\mathbb{Q}_n)$ *is bounded above by*

$$\min(q_\nu^\sharp + \lambda_\sharp, q_\nu^\flat + \lambda_\flat).$$

(2) *For the case* $\mu_\sharp \neq \mu_\flat$, *there is a similar bound of the form* $q_\nu^* + \lambda_*$, *where* $* \in \{\sharp, \flat\}$. *We refer the reader to Theorem 7.8 for a precise formulation.*

(3) *When* $a_p = 0$, *another analytic upper bound is given by* $\lambda_\sharp + \lambda_\flat$.

Let $\mathcal{G}_f^{\mathrm{ord}} = \{\sigma \in \mathcal{G}_f : f^\sigma \text{ is ordinary}\}$ and $\mathcal{G}_f^{\mathrm{ss}} = \{\sigma \in \mathcal{G}_f : f^\sigma \text{ is supersingular}\}$. When $\sigma \in \mathcal{G}_f^{\mathrm{ss}}$, denote by $\lambda_\natural^\sigma$ the minimum of the bounds from Theorem 1.3.

Corollary 1.4. *For a prime of good reduction*, r_∞^{an} *is bounded above as follows*:

$$r_\infty^{\mathrm{an}} \leqslant \sum_{\sigma \in \mathcal{G}_f^{\mathrm{ord}}} \lambda^\sigma + \sum_{\sigma \in \mathcal{G}_f^{\mathrm{ss}}} \lambda_\natural^\sigma.$$

The *Kurihara terms* $q_n^{\sharp/\flat}$ in Theorem 1.3 are p-power sums, e.g., when $\nu > 1$ and ν is odd, we have

$$q_\nu^\sharp = p^{\nu-1} - p^{\nu-2} + p^{\nu-3} - p^{\nu-4} + \cdots + p^2 - p.$$

The $\nu \in \mathbb{N}$ is chosen according to an explicit algorithm that measures the contribution of $\mathcal{L}og_{\alpha,\beta}(1+T)$ to the cyclotomic zeroes. For example, when $\mu_\sharp = \mu_\flat$ and $\lambda_\sharp < p-1$ and $\lambda_\flat < (p-1)^2$, we have $\nu = 0$, in which case $q_0^\sharp = q_0^\flat = 0$ and the bound is simply $\min(\lambda_\sharp, \lambda_\flat)$, which is very much in the spirit of the bound in the ordinary case.

The bound for the case $a_p = 0$, $\lambda_\sharp + \lambda_\flat$, is a generalization of work of Pollack [2003]. When p is odd, this bound is in most (computationally known) cases weaker than the above one, but there are cases in which it is stronger. It is interesting to ask for an *optimal bound*.

• For the leading term part, we know from the discussion above that $L_\alpha(f, T)$ and $L_\beta(f, T)$ don't vanish at $T = \zeta_{p^n} - 1$ for $n \gg 0$, so that these values should encode $\#(\text{Ш}(A_f/\mathbb{Q}_n)[p^\infty])/\#(\text{Ш}(A_f/\mathbb{Q}_{n-1})[p^\infty])$, i.e., the jumps in the p-primary parts of Ш at the n-th layer of the cyclotomic $\mathbb{Z}_p$-extension.

In the ordinary case, a classical result of Mazur gives an estimate for

$$\#\text{Ш}^{\mathrm{an}}(A_f/\mathbb{Q}_n) := \frac{L^{(r_n^{\mathrm{an}'})}(A_f/\mathbb{Q}_n, 1)\#A_f^{\mathrm{tor}}(\mathbb{Q}_n)\#\widehat{A}_f^{\mathrm{tor}}(\mathbb{Q}_n)\sqrt{D(\mathbb{Q}_n)}}{(r_n^{\mathrm{an}'})!\,\Omega_{A_f/\mathbb{Q}_n} R(A_f/\mathbb{Q}_n)\,\mathrm{Tam}(A_f/\mathbb{Q}_n)}.$$

His analytic estimate for $e_n := \mathrm{ord}_p(\#\text{Ш}^{\mathrm{an}}(A_f/\mathbb{Q}_n))$ when f^σ are all ordinary says that for $n \gg 0$,

$$e_n - e_{n-1} = \sum_{\sigma \in \mathcal{G}_f} \mu^\sigma(p^n - p^{n-1}) + \lambda^\sigma - r_\infty^{\mathrm{an}}(f^\sigma),$$

much in the spirit of Iwasawa's famous class number formula. Here, μ^σ and λ^σ are the Iwasawa invariants of $L_\alpha(f^\sigma, T)$. We prove a theorem that estimates e_n in the general good reduction case in terms of the Iwasawa invariants of $L_p^\sharp(f^\sigma, T)$ and $L_p^\flat(f^\sigma, T)$ and $v^\sigma = \mathrm{ord}_p(a_p^\sigma)$:

Given an integer n, we now define two generalized Kurihara terms $q_n^*(v^\sigma)$ for $* \in \{\sharp, \flat\}$ which are *continuous* in $v^\sigma \in [0, \infty]$. They are each a sum of a truncated Kurihara term and a multiple of $p^n - p^{n-1}$. For fixed n, they are piecewise linear in v.

Definition 1.5. For a real number $v > 0$, let $k \in \mathbb{Z}^{\geqslant 1}$ be the smallest positive integer such that $v \geqslant p^{-k}/2$.

$$q_n^\sharp(v) := \begin{cases} (p^n - p^{n-1})kv + \left\lfloor \dfrac{p^{n-k}}{p+1} \right\rfloor & \text{when } n \not\equiv k \bmod (2), \\ (p^n - p^{n-1})((k-1)v) + \left\lfloor \dfrac{p^{n+1-k}}{p+1} \right\rfloor & \text{when } n \equiv k \bmod (2), \end{cases}$$

$$q_n^\flat(v) := \begin{cases} (p^n - p^{n-1})((k-1)v) + p\left\lfloor \dfrac{p^{n-k}}{p+1} \right\rfloor + p - 1 & \text{when } n \not\equiv k \bmod (2), \\ (p^n - p^{n-1})kv + p\left\lfloor \dfrac{p^{n-1-k}}{p+1} \right\rfloor + p - 1 & \text{when } n \equiv k \bmod (2). \end{cases}$$

We also put

$$q_n^*(\infty) := \lim_{v \to \infty} q_n^*(v), \quad \text{and} \quad q_n^*(0) := \lim_{v \to 0} q_n^*(v) = \begin{cases} 0 & \text{when } * = \sharp, \\ p-1 & \text{when } * = \flat. \end{cases}$$

Definition 1.6 (modesty algorithm). Given $v \in [0, \infty]$, an integer n, integers $\lambda_\sharp$ and $\lambda_\flat$, and rational numbers $\mu_\sharp$ and $\mu_\flat$, choose $* \in \{\sharp, \flat\}$ via

$$* = \begin{cases} \sharp & \text{if } (p^n - p^{n-1})\mu_\sharp + \lambda_\sharp + q_n^\sharp(v) < (p^n - p^{n-1})\mu_\flat + \lambda_\flat + q_n^\flat(v), \\ \flat & \text{if } (p^n - p^{n-1})\mu_\flat + \lambda_\flat + q_n^\flat(v) < (p^n - p^{n-1})\mu_\sharp + \lambda_\sharp + q_n^\sharp(v). \end{cases}$$

Theorem 1.7. *Let $e_n := \mathrm{ord}_p(\#\text{Ш}^{\mathrm{an}}(A_f/\mathbb{Q}_n))$. Then for $n \gg 0$, we have*

$$e_n - e_{n-1} = \sum_{\sigma \in \mathcal{G}_f} \left(\mu_*^\sigma (p^n - p^{n-1}) + \lambda_*^\sigma + q_n^*(v^\sigma) \right) - r_\infty^{\mathrm{an}},$$

where:

(1) $* \in \{\sharp, \flat\}$ *is chosen according to the modesty algorithm* (Definition 1.6) *with the choice $v = v^\sigma$ when $v^\sigma \neq p^{-k}/2$. Note that one input of the algorithm is the parity of the integer k such that $v^\sigma \in \left[\frac{1}{2}p^{-k}, \frac{1}{2}p^{-k+1}\right)$.*

(2) *The term $q_n^*(v^\sigma)$ is replaced by a modified Kurihara term $q_n^*(v^\sigma, v_2^\sigma)$, when $v^\sigma = p^{-k}/2$, that depends further on the valuation v_2^σ of $(a_p^\sigma)^2 - \epsilon(p)\Phi_p(\zeta_{p^{k+2}})$ when $\mu_\sharp^\sigma \neq \mu_\flat^\sigma$. Further, $* \in \{\sharp, \flat\}$ is chosen according to a generalized modesty algorithm which also depends on v_2^σ. We refer the reader to Theorem 8.5 for a precise formulation.*

	$v=0$			$0<v<\infty$		$v=\infty$	
	$\lambda_\sharp<\lambda'_\flat$	$\lambda_\sharp>\lambda'_\flat$	$\lambda_\sharp=\lambda'_\flat$	n odd	n even	n odd	n even
$\mu_\sharp=\mu_\flat$	$\sharp$	$\flat$	excluded	$\sharp$	$\flat$	$\sharp$	$\flat$
$\mu_\sharp<\mu_\flat$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\flat$
$\mu_\flat<\mu_\sharp$	$\flat$	$\flat$	$\flat$	$\flat$	$\flat$	$\sharp$	$\flat$

Table 1. Table for Corollary 1.9, with $\lambda'_\flat$ denoting $\lambda_\flat+p-1$.

Remark 1.8. The (generalized) modesty algorithm doesn't work for some excluded ("sporadic") cases. When $v^\sigma=0$, the excluded case is $\mu_\sharp^\sigma=\mu_\flat^\sigma$, and $\lambda_\sharp^\sigma=\lambda_\flat^\sigma+p-1$, which can be remedied by adhering to the ordinary theory; see Theorem 8.5. The other excluded cases occur when $v^\sigma=p^{-k}/2$ and $v_2^\sigma=p^{-k}(1+p^{-1}-p^{-2})$ and an inequality of μ-invariants (or λ-invariants) is satisfied. This case should conjecturally not occur. See Definition 8.3 for details.

In less precise terms, the theorem above states that *our formulas in the supersingular case approach Mazur's formula in the ordinary case as $k\to\infty$*, and that *during this approach, the roles of $\sharp$ and $\flat$ may switch as the parity of k does*. The simplest scenario is when the μ-invariants are equal. Here, a switch in the parity of k *always* causes a switch in the role of $\sharp$ and $\flat$. It is a mystery why these formulas appear in this way, but we invite the reader to ponder this phenomenon by looking at Figure 1.

In the supersingular case, Greenberg, Iovita, and Pollack (in unpublished work around 2005) generalized the approach of Perrin-Riou of extracting invariants $\mu_\pm$ and $\lambda_\pm$ from the classical p-adic L-functions for a modular form f, $L_p(f,\alpha,T)$ and $L_p(f,\beta,T)$, which they used for their estimates (under the assumption $\mu_+=\mu_-$). Our formulas match theirs exactly in those cases, although the techniques are different.

We write out explicitly the elliptic curve case of the above theorem for the convenience of the readers, and since it hints at a unification of the ordinary and supersingular theories:

Corollary 1.9. *Let E be an elliptic curve over $\mathbb{Q}$, p a prime of good reduction, $v=\operatorname{ord}_p(a_p)$ and $e_n:=\operatorname{ord}_p(\#\text{Ш}^{\mathrm{an}}(E/\mathbb{Q}_n))$, and $\mu_{\sharp/\flat}$ and $\lambda_{\sharp/\flat}$ the Iwasawa invariants of $L_p^{\sharp/\flat}(E,T)$. Then for $n\gg 0$,*

$$e_n-e_{n-1}=\mu_*(p^n-p^{n-1})+\lambda_*-r_\infty^{\mathrm{an}}+\min(1,v)q_n^*,$$

where the q_n^* are the Kurihara terms from Theorem 1.3 and $*\in\{\sharp,\flat\}$ is chosen according to Table 1.

In particular, there are *three* different possible formulas for the growth of the Šafarevič–Tate group when $a_p\neq 0$ and p is supersingular, one for each scenario of

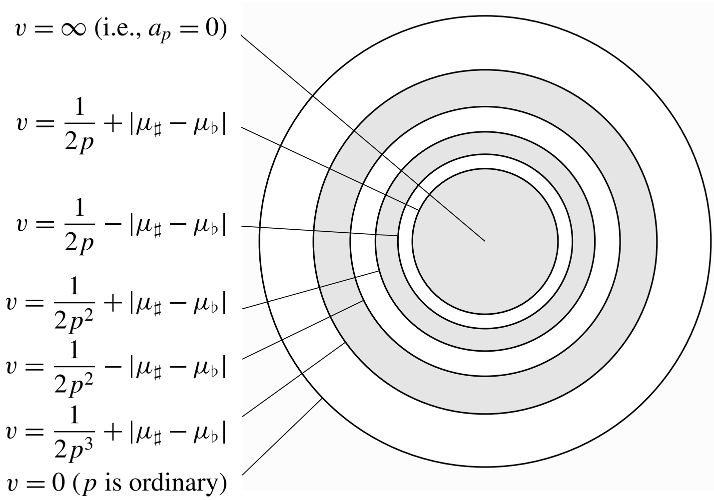

Figure 1. The locus inside the p-adic unit disc in which the modesty algorithm chooses $\sharp$ or $\flat$ when $(p-1)/(4p^4)<|\mu_\sharp-\mu_\flat|<(p-1)/(4p^3)$. Here, $v=\operatorname{ord}_p(a_p)$ indicates the possible valuations of a_p inside the unit disc. At the center, we have $a_p=0$, i.e., $v=\infty$. On the edge, we have $v=0$, so that p is ordinary. In the central shaded region, the formula involves $\mu_\sharp$, $\lambda_\sharp$ for odd n and $\mu_\flat$, $\lambda_\flat$ for even n, while in the second shaded region, $\mu_\sharp$ and $\lambda_\sharp$ are part of the formula for even n, and $\mu_\flat$ and $\lambda_\flat$ for odd n. In the outermost shaded region, the roles are flipped yet again and the $\sharp$-invariants come into play for odd n, and the $\flat$-invariants for even n. When $\mu_\sharp<\mu_\flat$, the formulas are only controlled by the $\mu_\sharp$ and $\lambda_\sharp$ in the nonshaded regions.

comparing μ-invariants. As visible when $a_p\neq 0$, the Šafarevič–Tate group Ш tries to stay as small as possible (it is “modest”) during its ascent along the cyclotomic $\mathbb{Z}_p$-extension by choosing smaller Iwasawa-invariants. The analytic estimates in the case $v=\infty$ (i.e., $a_p=0$) were given by Pollack [2003], see also the many computations in [Stein and Wuthrich 2013].

Thanks to the work of Kurihara [2002], Perrin-Riou [2003], Kobayashi [2003], and the work in [Sprung 2013], we now understand the algebraic side of the corollary (i.e., the elliptic curve case) quite well in the supersingular case when p is odd.[3] The formulas also match the algebraic ones of Kurihara and Otsuki [2006] when $p=2$. For the unknown cases (in which $p=2$), the formulas thus serve as a prediction of how $Ш(E/\mathbb{Q}_n)[p^\infty]$ grows.

[3]These works answer a comment by Coates and Sujatha who wrote in their textbook [2000, p. 56] only 15 years ago that when looking at the p-primary part of $Ш(E/\mathbb{Q}_n)$ as $n\to\infty$, “…nothing seems to be known about the asymptotic behavior of the order …”

Organization of Paper. This paper consists of two parts. Part I is mainly concerned with the construction of our pair of Iwasawa functions: In Section 2, we introduce *Mazur–Tate symbols*, which inherit special values of L-functions to construct the classical p-adic L-functions of Amice, Vélu, and Višik, and state the main theorem. In Section 3, we give a quick application, answering a question by Greenberg. In Section 4, we scrutinize the logarithm matrix $\mathcal{L}og_{\alpha,\beta}$ and prove its basic properties. In Section 5, we then put this information together to rewrite the p-adic L-functions from Section 2 in terms of the new p-adic L-functions $L_p^\sharp$ and $L_p^\flat$, proving the main theorem. Part II is devoted to the BSD-theoretic aspects as one climbs up the cyclotomic tower: in Section 6, we give the necessary preparation, Section 7 is concerned with the two upper bounds on the Mordell–Weil rank, and Section 8 scrutinizes the size of Ш.

Outlook. Pottharst [2012] constructs an algebraic counterpart (*Selmer modules*) to the pair $L_\alpha(f, T)$, $L_\beta(f, T)$ in the supersingular case, which hints at an algebraic counterpart to $\mathcal{L}og_{\alpha,\beta}(1+T)$ as well, along with algebraic versions of each of our analytic applications; these are equivalent under an Iwasawa main conjecture. A proof of the main conjecture in terms of $L_\alpha(f, T)$ in the ordinary case is due to Skinner and Urban [2014], building on [Kato 2004]. See also [Rubin 1991] for the CM case. The work of Lei, Loeffler, and Zerbes [Lei et al. 2010] constructs pairs of Iwasawa functions (in $\mathbb{Q}\otimes\Lambda$) out of Berger–Li–Zhu's basis [Berger et al. 2004] of *Wach modules*, see also [Loeffler and Zerbes 2013]. They match the Iwasawa functions in this paper when $a_p = 0$ (which shows that the functions in [Lei et al. 2010] actually live in Λ), which hints at an explicit relationship between our methods and theirs. For the higher weight case, there are generalizations of $L_p^\sharp(f, T)$ and $L_p^\flat(f, T)$, which is forthcoming work. Their invariants are already sometimes visible (see, e.g., [Pollack and Weston 2011]). It would be nice to generalize the pairs of p-adic BSD conjectures formulated in [Sprung 2015] to modular abelian varieties as well. For the ordinary case, the generalization of [Mazur et al. 1986] is [Balakrishnan et al. 2016]. Another challenge is formulating p-adic BSD for a bad prime. See [Colmez 2004] for an overview of p-adic BSD, mainly in the good reduction case. Apart from [Mazur et al. 1986], a hint for what to do can be found in Delbourgo's [1998] formulation of Iwasawa theory.

Part I. The pair of Iwasawa functions $L_p^\sharp$ and $L_p^\flat$

2. The p-adic L-function of a modular form

In this section, we recall the classical p-adic L-functions given in [Amice and Vélu 1975; Mazur et al. 1986; Višik 1976; Mazur and Swinnerton-Dyer 1974], in the case of weight-two modular forms. We give a construction via *queue sequences* which we scrutinize carefully enough to arrive at the decomposition of the classical p-adic L-functions as linear combinations of Iwasawa functions $L_p^\sharp$ and $L_p^\flat$. This is the main theorem of this paper, which will be proved in Sections 4 and 5, and upon which the applications (Sections 3, 6, and 7) depend.

Let f be a weight-two modular form with character ϵ which is an eigenform for the Hecke operators T_n with eigenvalue a_n. We also fix forever a good (i.e., coprime to the level) prime p. Given integers a, m, the *period* of f is

$$\varphi\Big(f, \frac{a}{m}\Big) := 2\pi i \int_{i\infty}^{a/m} f(z)\, dz.$$

The following theorem puts these transcendental periods into the algebraic realm (see also [Greenberg and Stevens 1994, Theorem 3.5.4]).

Theorem 2.1 [Manin 1973, Proposition 9.2c]. *There is a finite extension $K(f)$ of $\mathbb{Q}$ with integer ring $\mathcal{O}(f)$ and nonzero complex numbers $\Omega_f^\pm$ such that the following are in $\mathcal{O}(f)$ for all $a, m \in \mathbb{Z}$:*

$$\Big[\frac{a}{m}\Big]_f^+ := \frac{\varphi(f, a/m) + \varphi(f, -a/m)}{2\Omega_f^+} \quad \textit{and} \quad \Big[\frac{a}{m}\Big]_f^- := \frac{\varphi(f, a/m) - \varphi(f, -a/m)}{2\Omega_f^-}.$$

Convention 2.2 (for Part II). We choose the convention that $\prod_{f^\sigma} \Omega_{f^\sigma}^\pm = \Omega_{A_f}^\pm$, where the f^σ run over all Galois conjugates of f (see, e.g., [Balakrishnan et al. 2016, (2.4)]), and $\Omega_{A_f}^\pm$ are the Neron periods as in [Manin 1971, §8.10]. We also use the convention that any period Ω with an omitted sign denotes Ω^+.

The $[a/m]_f^\pm$ are called *modular symbols*. For p-adic considerations, we fix an embedding $\overline{\mathbb{Q}} \to \mathbb{C}_p$ of an algebraic closure of $\mathbb{Q}$ inside the completion of an algebraic closure of $\mathbb{Q}_p$. This all allows us to construct p-adic L-functions as follows: Denote by $\mathbb{C}^0(\mathbb{Z}_p^\times)$ the $\mathbb{C}_p$-valued step functions on $\mathbb{Z}_p^\times$. Let a be an integer prime to p, and denote by $\mathbf{1}_U$ the characteristic function of an open set U. We let ord_p be the valuation associated to p so that $\mathrm{ord}_p(p) = 1$. Let α be a root of the Hecke polynomial $X^2 - a_p X + \epsilon(p)p$ of f so that $\mathrm{ord}_p(\alpha) < 1$, and denote the conjugate root by β. Without loss of generality, we assume throughout this paper that $\mathrm{ord}_p(\alpha) \leqslant \mathrm{ord}_p(\beta)$. We define a linear map $\mu_{f,\alpha}^\pm$ from $\mathbb{C}^0(\mathbb{Z}_p^\times)$ to $\mathbb{C}_p$ by setting

$$\mu_{f,\alpha}^\pm(\mathbf{1}_{a+p^n\mathbb{Z}_p}) = \frac{1}{\alpha^{n+1}} \left(\Big[\frac{a}{p^n}\Big]_f^\pm, \Big[\frac{a}{p^{n-1}}\Big]_f^\pm\right) \begin{pmatrix} \alpha \\ -\epsilon(p) \end{pmatrix}.$$

Remark 2.3. The maps $\mu_{f,\alpha}^\pm$ are not measures, but $\mathrm{ord}_p(\alpha)$-admissible measures. See, e.g., [Pollack 2003]. For background on measures, see [Washington 1982, Section 12.2].

Theorem 2.4. *We can extend the maps $\mu_{f,\alpha}^\pm$ to all analytic functions on $\mathbb{Z}_p^\times$.*

This can be done by locally approximating analytic functions by *step functions*, since $\mu^{\pm}_{f,\alpha}$ are $\operatorname{ord}_p(\alpha)$-admissible measures. That is, we look at their Taylor series expansions and ignore the higher order terms. For an explicit construction, see [Amice and Vélu 1975] or [Višik 1976]. Since characters χ of $\mathbb{Z}_p^\times$ are locally analytic functions, we thus obtain an element

$$L_p(f, \alpha, \chi) := \mu^{\operatorname{sign}(\chi)}_{f,\alpha}(\chi).$$

Now since $\mathbb{Z}_p^\times \cong (\mathbb{Z}/2p\mathbb{Z})^\times \times (1+2p\mathbb{Z}_p)$, we can write a character χ on $\mathbb{Z}_p^\times$ as a product

$$\chi = \omega^i \chi_u,$$

with $0 \leqslant i < |\Delta|$ for some $u \in \mathbb{C}_p$ with $|u-1|_p < 1$, where χ_u sends the topological generator $\gamma = 1+2p$ of $1+2p\mathbb{Z}_p$ to u, and where $\omega : \Delta \to \mathbb{Z}_p^\times \in \mathbb{C}_p$ is the usual embedding of the $|\Delta|$-th roots of unity in $\mathbb{Z}_p$ so that ω^i is a tame character of $\Delta = (\mathbb{Z}/2p\mathbb{Z})^\times$. Using this product, we can identify the open unit disc of $\mathbb{C}_p$ with characters χ on $\mathbb{Z}_p^\times$ having the same tame character ω^i. Thus if we fix i, we can regard $L_p(f, \alpha, \omega^i \chi_u)$ as a function on the open unit disc. We can go even further:

Theorem 2.5 [Višik 1976; Mazur et al. 1986; Amice and Vélu 1975; Pollack 2003]. *Fix a tame character $\omega^i : \Delta = (\mathbb{Z}/2p\mathbb{Z})^\times \to \mathbb{C}_p$. Then the function $L_p(f, \alpha, \omega^i \chi_u)$ is an analytic function converging on the open unit disc.*

We can thus form its power series expansion about $u = 1$. For convenience, we change variables by setting $T = u-1$ and denote $L_p(f, \alpha, \omega^i \chi_u)$ by $L_p(f, \alpha, \omega^i, T)$.

Denote by $\zeta = \zeta_{p^n}$ a primitive p^n-th root of unity. We can then regard $\omega^i \chi_\zeta$ as a character of $(\mathbb{Z}/p^N\mathbb{Z})^\times$, where $N = n+1$ if p is odd and $N = n+2$ if $p = 2$. Given any character ψ of $(\mathbb{Z}/p^N\mathbb{Z})^\times$, let $\tau(\psi)$ be the Gauß sum $\sum_{a\in(\mathbb{Z}/p^N\mathbb{Z})^\times} \psi(a)\zeta^a_{p^N}$.

Theorem 2.6 [Amice and Vélu 1975; Višik 1976]. *The above $L_p(f, \alpha, \omega^i, T)$ interpolate as follows*:

$$\begin{aligned} L_p(f, \alpha, \omega^i, \zeta - 1) &= \frac{p^N}{\alpha^N \tau(\omega^{-i}\chi_{\zeta^{-1}})} \frac{L(f_{\omega^{-i}\chi_{\zeta^{-1}}}, 1)}{\Omega_f^{\omega^i(-1)}} \quad \textit{if } i \neq 0 \textit{ and } \zeta - 1 \neq 0, \\ L_p(f, \alpha, \omega^0, 0) &= \left(1 - \frac{1}{\alpha}\right)^2 \frac{L(f, 1)}{\Omega_f^+}. \end{aligned} \tag{1}$$

Queue sequences and Mazur–Tate elements. Denote by μ_{p^n} the group of p^n-th roots of unity, and put $\mathcal{G}_N := \operatorname{Gal}(\mathbb{Q}(\mu_{p^N}))$. We let $\mathbb{Q}_n$ be the unique subextension of $\mathbb{Q}(\mu_{p^N})$ with Galois group isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$ and put $\Gamma_n := \operatorname{Gal}(\mathbb{Q}_n/\mathbb{Q})$. We also let $\Gamma := \operatorname{Gal}\big(\bigcup_n \mathbb{Q}_n/\mathbb{Q}\big)$. We then have an isomorphism

$$\mathcal{G}_N \cong \Delta \times \Gamma_n.$$

We let $K := K(f)_v$ be the completion of $K(f)$ from Theorem 2.1 by the prime v of $K(f)$ over p determined by $\mathrm{ord}_p(\cdot)$ and denote by $\mathcal{O}$ the ring of integers of K. Let $\Lambda_n = \mathcal{O}[\Gamma_n]$ be the finite version of the Iwasawa algebra at level n. We need two maps $\nu = \nu_{n-1/n}$ and $\pi = \pi_{n/n-1}$ to construct queue sequences: π is the natural projection from Λ_n to Λ_{n-1}, and the map $\Lambda_{n-1} \xrightarrow{\nu_{n-1/n}} \Lambda_n$ we define by

$$\nu_{n-1/n}(\sigma) = \sum_{\tau \mapsto \sigma, \tau \in \Gamma_n} \tau.$$

We let $\Lambda = \mathcal{O}[\![\Gamma]\!] = \varprojlim_{\pi_{n/n-1}} \mathcal{O}[\Gamma_n]$ be the Iwasawa algebra. We identify Λ with $\mathcal{O}[\![T]\!]$ by sending our topological generator $\gamma = 1 + 2p$ of $\Gamma \cong \mathbb{Z}_p$ to $1 + T$. This induces an isomorphism between Λ_n and $\mathcal{O}[\![T]\!]/((1+T)^{p^n} - 1)$.

Definition 2.7. A *queue sequence* is a sequence of elements $(\Theta_n)_n \in (\Lambda_n)_n$ such that

$$\pi\Theta_n = a_p\Theta_{n-1} - \epsilon(p)\nu\Theta_{n-2} \quad \text{when } n \geqslant 2.$$

Definition 2.8. For $a \in \mathcal{G}_N$, denote its projection onto Δ by $\bar{a}$, and let $i : \Delta \hookrightarrow \mathcal{G}_N$ be the standard inclusion, so that $a/i(\bar{a}) \in \Gamma_n$. Define $\log_\gamma(a)$ to be the smallest positive integer such that the image of $\gamma^{\log_\gamma(a)}$ under the projection from Γ to Γ_n equals $a/i(\bar{a})$. We then have a natural map $i : \Delta \hookrightarrow \mathcal{G}_\infty$ which allows us to extend this definition to any $a \in \mathcal{G}_\infty = \varprojlim_N \mathcal{G}_N$: let $\log_\gamma(a)$ be the unique element of $\mathbb{Z}_p^\times$ such that $\gamma^{\log_\gamma(a)} = a/i(\bar{a})$.

Example 2.9. We make the identification $\mathcal{G}_N \cong (\mathbb{Z}/p^N\mathbb{Z})^\times$ by identifying σ_a with a, where $\sigma_a(\zeta) = \zeta^a$ for $\zeta \in \mu_{p^N}$. This allows us to construct the *Mazur–Tate element*, which is

$$\vartheta_N^\pm := \sum_{a \in (\mathbb{Z}/p^N)^\times} \left[\frac{a}{p^N}\right]_f^\pm \sigma_a \in \mathcal{O}[\mathcal{G}_N].$$

For each character $\omega^i : \Delta \to \mathbb{C}_p^\times$, put

$$\varepsilon_{\omega^i} = \frac{1}{\#\Delta} \sum_{\tau \in \Delta} \omega^i(\tau)\tau^{-1}.$$

We can take isotypical components $\varepsilon_{\omega^i}\vartheta_N$ of the Mazur–Tate elements, which can be regarded as elements of $\Lambda_n \cong \mathcal{O}[\![T]\!]/((1+T)^{p^n} - 1)$. Denote these *Mazur–Tate elements associated to the tame character* ω^i by

$$\theta_n(\omega^i, T) := \varepsilon_{\omega^i}\vartheta_N^{\mathrm{sign}(\omega^i)}.$$

We extend ω^i to all of $(\mathbb{Z}/p^N\mathbb{Z})^\times$ by precomposing with the natural projection onto Δ, and can thus write these elements explicitly as elements of Λ_n:

$$\theta_n(\omega^i, T) = \sum_{a \in (\mathbb{Z}/p^N\mathbb{Z})^\times} \left[\frac{a}{p^N}\right]_f^{\mathrm{sign}(\omega^i)} \omega^i(a)(1+T)^{\log_\gamma(a)}.$$

When $\omega^i = \mathbf{1}$ is the trivial character, we simply write $\theta_n(T)$ instead of $\theta_n(\mathbf{1}, T)$. For a fixed tame character ω^i, the associated Mazur–Tate elements $\theta_n(\omega^i, T)$ form a queue sequence. For a proof, see [Mazur et al. 1986, (4.2)].

We can now explicitly approximate $L_p(f, \alpha, \omega^i, T)$ by Riemann sums:

Definition 2.10. Put

$$L^{\pm}_{N,\alpha} := \sum_{a \in (\mathbb{Z}/p^N\mathbb{Z})^\times} \mu^{\pm}_{f,\alpha}(\mathbf{1}_{a+p^N\mathbb{Z}_p})\sigma_a \in \mathbb{C}_p[\mathcal{G}_N],$$

so we get the representation

$$\varepsilon_{\omega^i} L^{\operatorname{sign}(\omega^i)}_{N,\alpha}(T) = \sum_{a \in (\mathbb{Z}/p^N\mathbb{Z})^\times} \mu^{\operatorname{sign}(\omega^i)}_{f,\alpha}(\mathbf{1}_{a+p^n\mathbb{Z}_p})\omega^i(a)(1+T)^{\log_\gamma(a)}.$$

Note that the homomorphism $\nu : \Gamma_{n-1} \to \Gamma_n$ extends naturally to a homomorphism from $\mathcal{G}_{N-1}$ to $\mathcal{G}_N$, also denoted by ν.

Lemma 2.11. *Let $n \geqslant 0$, i.e., $N \geqslant 1$ for odd p, and $N \geqslant 2$ for $p = 2$. Then*

$$(L^{\pm}_{N,\alpha}, L^{\pm}_{N,\beta}) = (\vartheta^{\pm}_N, \nu\vartheta^{\pm}_{N-1}) \begin{pmatrix} \alpha^{-N} & \beta^{-N} \\ -\epsilon(p)\alpha^{-(N+1)} & -\epsilon(p)\beta^{-(N+1)} \end{pmatrix}.$$

Proof. This follows from the definitions. □

Proposition 2.12. *As functions converging on the open unit disc, we have*

$$L_p(f, \alpha, \omega^i, T) = \lim_{n\to\infty} \varepsilon_{\omega^i} L^{\operatorname{sign}(\omega^i)}_{N,\alpha}(T).$$

Proof. Approximation by Riemann sums, and decomposition into tame characters. □

Corollary 2.13. *Let p be supersingular. Then both α and β have valuation strictly less than one, so we can reconstruct the p-adic L-functions by the Mazur–Tate elements*:

$$\begin{aligned}\big(L_p(f, \alpha, \omega^i, T), L_p(f, \beta, \omega^i, T)\big) \\ = \lim_{n\to\infty} \big(\theta_n(\omega^i, T), \nu\theta_{n-1}(\omega^i, T)\big) \begin{pmatrix} \alpha^{-N} & \beta^{-N} \\ -\epsilon(p)\alpha^{-(N+1)} & -\epsilon(p)\beta^{-(N+1)} \end{pmatrix}.\end{aligned}$$

In the ordinary case, we have $\operatorname{ord}_p(\alpha) = 0 < 1$, *so*

$$L_p(f, \alpha, \omega^i, T) = \lim_{n\to\infty} \big(\theta_n(\omega^i, T), \nu\theta_{n-1}(\omega^i, T)\big) \begin{pmatrix} \alpha^{-N} \\ -\epsilon(p)\alpha^{-(N+1)} \end{pmatrix}.$$

Proof. This follows from Lemma 2.11. □

In Section 4, we define an explicit 2×2 matrix

$$\widehat{\mathcal{L}og}_{\alpha,\beta}(1+T)=\begin{pmatrix}\widehat{\log}^{\sharp}_{\alpha}(1+T) & \widehat{\log}^{\sharp}_{\beta}(1+T)\\ \widehat{\log}^{\flat}_{\alpha}(1+T) & \widehat{\log}^{\flat}_{\beta}(1+T)\end{pmatrix}$$

that encodes convenient behavior of the Mazur–Tate elements. We prove that the entries are functions convergent on the open unit disc when p is supersingular, and that $\widehat{\log}^{\sharp}_{\alpha}(1+T)$ and $\widehat{\log}^{\flat}_{\alpha}(1+T)$ converge on the closed unit disc in the ordinary case. This assertion is the main lemma (Lemma 4.4). A corollary of the construction, Remark 4.5, says that the determinant of $\widehat{\mathcal{L}og}_{\alpha,\beta}(1+T)$ converges and vanishes precisely at $\zeta_{p^n}=1$ for $n\geqslant 0$. Lemma 4.4 is the key ingredient to proving our main theorem:

Theorem 2.14. *Fix a tame character ω^i.*

(a) *When p is supersingular, there is a unique vector of two Iwasawa functions*

$$\widehat{\overrightarrow{L}}_p(f,\omega^i,T)=\big(\widehat{L}^{\sharp}_p(f,\omega^i,T),\widehat{L}^{\flat}_p(f,\omega^i,T)\big)\in\Lambda^{\oplus 2}$$

such that

$$\big(L_p(f,\alpha,\omega^i,T),L_p(f,\beta,\omega^i,T)\big)=\big(\widehat{L}^{\sharp}_p(f,\omega^i,T),\widehat{L}^{\flat}_p(f,\omega^i,T)\big)\widehat{\mathcal{L}og}_{\alpha,\beta}(1+T).$$

(b) *When p is ordinary, there is vector*

$$\widehat{\overrightarrow{L}}_p(f,\omega^i,T)=\big(\widehat{L}^{\sharp}_p(f,\omega^i,T),\widehat{L}^{\flat}_p(f,\omega^i,T)\big)\in\Lambda^{\oplus 2}$$

such that

$$\widehat{\overrightarrow{L}}_p(f,\omega^i,T)\begin{pmatrix}\widehat{\log}^{\sharp}_{\alpha}(1+T)\\ \widehat{\log}^{\flat}_{\alpha}(1+T)\end{pmatrix}=L_p(f,\alpha,\omega^i,T)$$

and

$$\widehat{\overrightarrow{L}}_p(f,\omega^i,0)\begin{pmatrix}\widehat{\log}^{\sharp}_{\beta}(1)\\ \widehat{\log}^{\flat}_{\beta}(1)\end{pmatrix}\ \textit{is given by (1) with } \zeta=1 \textit{ and } \alpha \textit{ replaced by } \beta.$$

Once $\widehat{\overrightarrow{L}}_p(f,\omega^i,T)$ is fixed, all (other) such vectors are given by

$$\widehat{\overrightarrow{L}}_p(f,\omega^i,T)+g(T)T\big(-\widehat{\log}^{\flat}_{\alpha}(1+T),\widehat{\log}^{\sharp}_{\alpha}(1+T)\big)$$

for $g(T)\in\Lambda$. In particular, the value of $\widehat{\overrightarrow{L}}_p(f,\omega^i,0)$ is uniquely determined.

Statements analogous to parts (a) *and* (b) *also hold for objects without the hats.*

3. A question by Greenberg

To motivate our theorem, we give a quick application. Greenberg [2001] conjectured that $L_p(f,\alpha,\omega^i,T)$ and $L_p(f,\beta,\omega^i,T)$ have finitely many common zeros (in the elliptic curve case) when p is supersingular and $i=0$. In this section, we work in the general supersingular case.

Theorem 3.1 [Rohrlich 1984]. *$L_p(f,\alpha,\omega^i,T)$ and $L_p(f,\beta,\omega^i,T)$ vanish at only finitely many $T=\zeta_{p^n}-1$.*

Proof. Since these functions interpolate well (Theorem 2.6), the result follows from the original theorem of Rohrlich [1984], which guarantees that $L(f,\chi,1)=0$ at finitely many characters of p-power order. □

Theorem 3.2. *$L_p(f,\alpha,\omega^i,T)$ and $L_p(f,\beta,\omega^i,T)$ have finitely many common zeros. In particular, Greenberg's conjecture is true.*

Proof. When a zero is not a p-power root of unity minus 1, it is one of the finitely many zeros of $L_p^\sharp(f,\omega^i,T)$ and $L_p^\flat(f,\omega^i,T)$, since $\det \mathcal{L}og_{\alpha,\beta}(1+T)$ doesn't vanish there. For the other zeros, use Rohrlich's theorem. □

Remark 3.3. Pollack already found a different proof in the case $a_p=0$ ([Pollack 2003, Corollary 5.12]).

4. The logarithm matrix $\widehat{\mathcal{L}og}_{\alpha,\beta}(1+T)$

Definition of the matrix $\mathcal{L}og_{\alpha,\beta}(1+T)$ and convergence of entries. In this section, we construct a matrix $\mathcal{L}og_{\alpha,\beta}(1+T)$ whose entries are functions converging on the open unit disc in the supersingular case. In the ordinary case, its first column converges on the closed unit disc. They directly generalize the four functions $\log_{\alpha/\beta}^{\sharp/\flat}$ from [Sprung 2012] and the three functions $\log_p^+$, $\log_p^-\cdot\alpha$, $\log_p^-\cdot\beta$ from [Pollack 2003], all of which concern the supersingular case. We also construct a completed version $\widehat{\mathcal{L}og}_{\alpha,\beta}(1+T)$.

Definition 4.1. Let $i\geqslant 1$. We *complete* the p^i-th cyclotomic polynomial by putting

$$\widehat{\Phi}_{p^i}(1+T):=\Phi_{p^i}(1+T)/(1+T)^{\frac{1}{2}p^{i-1}(p-1)},$$

except when $p=2$ and $i=1$: to avoid branch cuts (square roots), we set

$$\widehat{\Phi}_2(1+T):=\Phi_2(1+T).$$

Definition 4.2. Define the following matrices:

$$\mathcal{C}_i:=\mathcal{C}_i(1+T):=\begin{pmatrix} a_p & 1\\ -\epsilon(p)\Phi_{p^i}(1+T) & 0\end{pmatrix},\quad \text{and}\quad C:=\mathcal{C}_i(1)=\begin{pmatrix} a_p & 1\\ -\epsilon(p)p & 0\end{pmatrix}.$$

Definition 4.3. Recall that ord_p is the valuation on $\mathbb{C}_p$ normalized by $\mathrm{ord}_p(p) = 1$. Put

$$v = \mathrm{ord}_p(a_p) \quad \text{and} \quad w = \mathrm{ord}_p(\alpha).$$

Lemma 4.4 (main lemma). *Recall that $N = n+1$ if p is odd, and $N = n+2$ if $p = 2$. We put*

$$\mathcal{L}og_{\alpha,\beta}(1+T) := \lim_{n\to\infty} \mathcal{C}_1 \cdots \mathcal{C}_n C^{-(N+1)} \begin{pmatrix} -1 & -1 \\ \beta & \alpha \end{pmatrix}.$$

Then the entries in the left column of $\mathcal{L}og_{\alpha,\beta}(1+T)$ and $\widehat{\mathcal{L}og}_{\alpha,\beta}(1+T)$ are well-defined as power series and converge on the open unit disc. When $v > 0$ (the supersingular case) or $T = \zeta_{p^n} - 1$ with $n \geqslant 0$, we can say the same about all entries.

Remark 4.5. We call $\mathcal{L}og_{\alpha,\beta}(1+T)$ the logarithm matrix. The reason for this name is that for odd p, we have

$$\det \mathcal{L}og_{\alpha,\beta}(1+T) = \frac{\log_p(1+T)}{T} \times \frac{\beta-\alpha}{(\epsilon(p)p)^2}.$$

For $p = 2$, the above exponent of 2 has to be replaced by a 3.

Remark 4.6. After this paper was written, Antonio Lei found a more streamlined proof of Lemma 4.4. This can be found in the proof of [Lei 2014, Theorem 1.5], which relies on techniques of Perrin-Riou [1994, §1.2.1]. The methods below uses the technique of valuation matrices developed below instead of those in [Perrin-Riou 1994, §1.2.1].

Convention 4.7. Whenever we encounter an expression $\mathcal{E}$ involving $\Phi_{p^i}(1+T)$, we let $\widehat{\mathcal{E}}$ be the corresponding expression involving $\widehat{\Phi}_{p^i}(1+T)$. For example, we let $\widehat{\mathcal{C}}_i$ be $\mathcal{C}_i$ with $-\epsilon(p)\widehat{\Phi}_{p^i}(1+T)$ in the lower left entry instead of $-\epsilon(p)\Phi_{p^i}(1+T)$, and

$$\widehat{\mathcal{L}og}_{\alpha,\beta}(1+T) = \lim_{n\to\infty} \widehat{\mathcal{C}}_1 \cdots \widehat{\mathcal{C}}_n C^{-(N+1)} \begin{pmatrix} -1 & -1 \\ \beta & \alpha \end{pmatrix}.$$

Observation 4.8. For $i > n \geqslant 0$, we have

$$\widehat{\mathcal{C}}_i(T+1)|_{T=\zeta_{p^n}-1} = \widehat{\mathcal{C}}_i(\zeta_{p^n}) = \mathcal{C}_i(\zeta_{p^n}) = C.$$

Definition 4.9. For a matrix $M = (m_{i,j})_{i,j}$ with entries $m_{i,j}$ in the domain of a valuation val, let the *valuation matrix* $[M]$ *of* M be the matrix consisting of the valuations of the entries:

$$[M] := [\mathrm{val}(m_{i,j})]_{i,j}.$$

Let $N = (n_{j,k})_{j,k}$ be another matrix so that we can form the product MN. Valuation matrices have the following *valuative multiplication* operation:

$$[M][N] := \left[\min_j (m_{i,j} + n_{j,k})\right]_{i,k}.$$

We also define the *valuation* $\mathrm{val}(M)$ *of* M to be the minimum of the entries in the valuation matrix:

$$\mathrm{val}(M) := \min\{\mathrm{val}(m_{i,j})\}.$$

Definition 4.10. Let $0 < r < 1$. Denote by $|\cdot|_p = p^{-\mathrm{ord}_p(\cdot)}$ the p-adic absolute value. For $f(T) \in \mathbb{C}_p[\![T]\!]$ convergent on the open unit disc, we define its *valuation at* r to be

$$v_r(f(T)) := \inf_{|z|_p < r} \mathrm{ord}_p(f(z)).$$

We define the *valuation at* 0 to be

$$v_0(f(T)) := \mathrm{ord}_p(f(0)).$$

Lemma 4.11. *Let* val *be a valuation, and* M *and* N *be matrices as above allowing a matrix product* MN. *Then* $\mathrm{val}(MN) \geqslant \mathrm{val}(M) + \mathrm{val}(N)$.

Proof. Term by term, the valuations of the entries of $[MN]$ are at least as big as those of $[M][N]$. □

Notation 4.12. Let M be a matrix whose coefficients are in $\mathbb{C}_p[\![T]\!]$. With respect to v_r we may then define the *valuation matrix of* M *at* r and denote it by $[M]_r$. We similarly define the *valuation of* M *at* r and denote it by $v_r(M)$. When these terms don't depend on r (e.g., when the entries of M are constants), we drop the subscript r.

Example 4.13.[4] Denote the logarithm with base p by $\log_{(p)}$ to distinguish it from the p-adic logarithm $\log_p$ of Iwasawa.

$$v_r(\Phi_{p^n}(1+T)) = \begin{cases} 1 & \text{when } r \leqslant p^{-(p^{n-1}(p-1))^{-1}}, \\ -\log_{(p)}(r)\,p^{n-1}(p-1) & \text{when } r \geqslant p^{-(p^{n-1}(p-1))^{-1}}. \end{cases}$$

Example 4.14. $v_r\big((1+T)^{\frac{1}{2}p^{n-1}(p-1)}\big) = \frac{1}{2}p^{n-1}(p-1)v_r(1+T) = 0.$

In what follows, we give the arguments needed for our main lemma (Lemma 4.4) for $\widehat{\mathcal{L}og}_{\alpha,\beta}(1+T)$. From Example 4.14, the proof for $\mathcal{L}og_{\alpha,\beta}(1+T)$ follows by taking the hat off the relevant expressions.

[4] This essentially appears in [Pollack 2003, lemma 4.5]. It seems that he meant to write $p^{-v_r(\Phi_n(1+T))} \sim r^{p^{n-1}(p-1)}$ in the proof.

Definition 4.15. Assume $\beta \neq \alpha$. We put $\rho := \alpha/\beta$ and let

$$\widehat{\Upsilon}_n := \frac{1}{\beta-\alpha}\Big(\beta - \frac{\widehat{\Phi}_{p^n}(1+T)}{\alpha}\Big), \quad H_n := \begin{pmatrix} -1 & -\rho^{n+1} \\ \rho^{-n} & \rho \end{pmatrix},$$

$$\begin{aligned}\widehat{M}_n &:= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + H_n \widehat{\Upsilon}_n \\ &= \begin{pmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{pmatrix} \begin{pmatrix} -1 & -1 \\ \beta & \alpha \end{pmatrix}^{-1} \widehat{\mathcal{C}}_n \begin{pmatrix} -1 & -1 \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} \alpha^{-n-1} & 0 \\ 0 & \beta^{-n-1} \end{pmatrix},\\ H_{a,n_1,n_2,\dots,n_l} &:= H_a H_{a+n_1+1} H_{a+n_1+n_2+2} \cdots H_{a+n_1+n_2+\dots+n_l+l},\\ \widehat{\Upsilon}_{a,n_1,n_2,\dots,n_l} &:= \widehat{\Upsilon}_a \widehat{\Upsilon}_{a+n_1+1} \widehat{\Upsilon}_{a+n_1+n_2+2} \cdots \widehat{\Upsilon}_{a+n_1+n_2+\dots+n_l+l}.\end{aligned}$$

Note that $\mathcal{L}og_{\alpha,\beta}$ differs from $\lim_{n\to\infty} \widehat{M}_1 \widehat{M}_2 \cdots \widehat{M}_n$ by multiplication by $\left(\begin{smallmatrix} -1 & -1 \\ \beta & \alpha \end{smallmatrix}\right)$ on the left and a diagonal matrix on the right.

Lemma 4.16. $H_{a,n_1,n_2,\dots,n_l} =$

$$H_a \begin{pmatrix} (-1)^l (1-\rho^{-n_1})(1-\rho^{-n_2}) \cdots (1-\rho^{-n_l}) & 0 \\ 0 & \rho^l (1-\rho^{n_1})(1-\rho^{n_2}) \cdots (1-\rho^{n_l}) \end{pmatrix}.$$

Proof. This follows from the fact that

$$H_a H_{a+b+1} = H_a \begin{pmatrix} -(1-\rho^{-b}) & 0 \\ 0 & \rho(1-\rho^b) \end{pmatrix}. \qquad \square$$

Lemma 4.17 (expansion lemma).

$$\widehat{M}_1 \cdots \widehat{M}_n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{a\geqslant 1}^{n} H_a \widehat{\Upsilon}_a + \sum_{l \geqslant 1} \sum_{\substack{a \geqslant 1,\, n_i \geqslant 1 \\ l+a+\sum_{i\geqslant 1}^{l} n_i \leqslant n}} H_{a,n_1,n_2,\dots,n_l} \widehat{\Upsilon}_{a,n_1,n_2,\dots,n_l}.$$

Proof. We prove this by induction. For $n = 1$, this is just the definition. Now assume the lemma holds for n. We want to show that it holds for $n+1$. The entries in the very last sum are products of $(l+1)$ matrices $H_m \widehat{\Upsilon}_m$ whose subindices are bounded above by n. In fact, the entries are all such matrix products with the two following conditions on the subindices m:

- The m's are off by at least 2.
- The m's are in ascending order.

Note that the first sum corresponds to the $l = 0$ case.

We want to show that

$$(\widehat{M}_1 \cdots \widehat{M}_n)\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + H_{n+1} \widehat{\Upsilon}_{n+1}\right) = \widehat{M}_1 \cdots \widehat{M}_n + \widehat{M}_1 \cdots \widehat{M}_n H_{n+1} \widehat{\Upsilon}_{n+1}$$

satisfies Lemma 4.17. Each time when multiplying $H_{n+1}\widehat{\Upsilon}_{n+1}$ with the product of $(l+1)$ matrices that satisfy the conditions in the bullet points above, we obtain a product of $(l+2)$ matrices whose subindices are now bounded by $n+1$. The subindices are still in ascending order, and we may assume they are also off by at least 2, since

$$H_a H_{a+1} = \begin{pmatrix} -1 & -\rho^{a+1} \\ \rho^{-a} & \rho \end{pmatrix} \begin{pmatrix} -1 & -\rho^{a+2} \\ \rho^{-a-1} & \rho \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \qquad \square$$

Proof of the main lemma (Lemma 4.4). We want to prove that the sums involved in Lemma 4.17 converge as $n \to \infty$.

When $\beta = \alpha$, $w = \operatorname{ord}_p(\alpha) = \frac{1}{2}$, so that the arguments of [Sprung 2012, Lemma 4.4] work; see [Lei et al. 2010, Remark 5.26]. Thus, suppose $\beta \neq \alpha$. Fix $r < 1$.

For convergence of the first sum, we see the terms in the valuation matrix $[H_a\widehat{\Upsilon}_a]_r$ are bounded below by the terms in

$$\begin{bmatrix} a & 2wa \\ (2-2w)a & a \end{bmatrix}$$

up to a constant independent from a or n. This follows from Example 4.13 and the fact that $\prod_{i \geqslant 1}^{a-1} \Phi_{p^i}(1+T)$ divides $\widehat{\Upsilon}_a$: Indeed, $\widehat{\Upsilon}_a$ vanishes at $T = \zeta_{p^i} - 1$ for $1 \leqslant i \leqslant a-1$, since $\Phi_{p^a}(\zeta_{p^i}) = p$. All terms in this valuation matrix go to ∞ as a does, except for the upper-right term when $w = 0$, or the lower left term when $w = \frac{1}{2}$.

Now we handle the second (double) sum. We want to show that

$$H_{a,n_1,n_2,\dots,n_l}\widehat{\Upsilon}_{a,n_1,n_2,\dots,n_l}$$

is bounded. Note that we have $\operatorname{ord}_p(\rho(1-\rho^m)) \geqslant (2w-1)(1+m)$ so that

$$[\widehat{\Upsilon}_{a+m}\rho(1-\rho^m)]_r \geqslant 2wm + a + C$$

for some constant C independent of a or m. Assume for the moment that we are in the supersingular case. Lemma 4.16 and the easier fact that $[\widehat{\Upsilon}_{a+m}\rho(1-\rho^{-m})]_r \geqslant a+m$ then shows that all entries in the valuation matrix of $H_{a,n_1,n_2,\dots,n_l}\widehat{\Upsilon}_{a,n_1,n_2,\dots,n_l}$, except possibly for the lower-left term, have terms bounded below by the corresponding entries of $H_a\widehat{\Upsilon}_a$.

Thus, three of the terms of $\widehat{\mathcal{L}og}_{\alpha,\beta}(1+T)$ converge in the supersingular case. Since $\det \widehat{\mathcal{L}og}_{\alpha,\beta}(1+T)$ converges as well, all terms of $\widehat{\mathcal{L}og}_{\alpha,\beta}(1+T)$ converge. For $\mathcal{L}og_{\alpha,\beta}(1+T)$, we take our hats off.

For the ordinary case, analogous arguments hold for the terms in the left column.

$\square$

The rate of growth.

Definition 4.18. For $f(T), g(T) \in \mathbb{C}_p[\![T]\!]$ converging on the open unit disc, we say $f(T)$ is $O(g(T))$ if

$$p^{-v_r(f(T))} \text{ is } O(p^{-v_r(g(T))}) \quad \text{as } r \to 1^-,$$

i.e., there is an $r_0 < 1$ and a constant C such that

$$v_r(g(T)) < v_r(f(T)) + C \quad \text{when } 1 > r > r_0.$$

If $f(T)$ is $O(g(T))$ and $g(T)$ is $O(f(T))$, we say "$f(T)$ grows like $g(T)$," and write $f(T) \sim g(T)$.

Example 4.19. $1 \sim T \sim \Phi_p(1+T)$. Also, $\det \mathcal{L}og_{\alpha,\beta} \sim \log_p(1+T)$ by Remark 4.5.

Proposition 4.20 (growth lemma). *When $v > \frac{1}{2}$, the entries of $\widehat{\mathcal{L}og}_{\alpha,\beta}(1+T)$ and $\mathcal{L}og_{\alpha,\beta}(1+T)$ grow like $\log_p(1+T)^{\frac{1}{2}}$. When $v \leqslant \frac{1}{2}$, the entries in the left column are $O(\log_p(1+T)^v)$, and those in the right $O(\log_p(1+T)^{1-v})$.*

We give the proof for $\mathcal{L}og_{\alpha,\beta}(1+T)$, since it is similar for the case $\widehat{\mathcal{L}og}_{\alpha,\beta}(1+T)$. Before beginning with the proof, let us name the quantities from Example 4.13:

Definition 4.21. $\quad e_{n,r} := v_r(\Phi_{p^n}(1+T)) = \min\bigl(1, -\log_{(p)}(r)(p^n - p^{n-1})\bigr).$

Lemma 4.22. *The entries of $\mathcal{L}og_{\alpha,\beta}(1+T)$ are $O(\log_p(1+T)^{1-w})$.*

Proof. It suffices to prove this for $\lim_{n\to\infty} M_1 \cdots M_n$, where the M_i are as in Definition 4.15. Note that

$$M_n = \Phi_{p^n}(1+T)\begin{pmatrix} 1/\alpha & 1/\beta \\ -1/\alpha & -1/\beta \end{pmatrix} + \begin{pmatrix} -\alpha & -\alpha \\ -\beta & -\beta \end{pmatrix},$$

so that for $r < 1$,

$$[M_n]_r \geqslant e_{n,r} + w - 1 \geqslant (1-w)(e_{n,r} - 1).$$

Hence,

$$[M_1 \cdots M_n]_r \geqslant (1-w)\sum_{i\geqslant 1}^{n}(e_{i,r} - 1) = (1-w)\prod_{i\geqslant 1}^{n}\left[\frac{\Phi_{p^i}(1+T)}{p}\right]_r.$$

Taking limits, the result follows. □

We implicitly used diagonalization in an earlier proof, but write it out for convenience:

Observation 4.23. Let m be an integer. Then

$$\begin{pmatrix} -1 & -1 \\ \beta & \alpha \end{pmatrix}\begin{pmatrix} \alpha^m & 0 \\ 0 & \beta^m \end{pmatrix} = C^m \begin{pmatrix} -1 & -1 \\ \beta & \alpha \end{pmatrix}.$$

Proof of the growth lemma (Proposition 4.20). We first treat the case $v = 0 = \mathrm{ord}_p(\alpha)$. When $n \geqslant 1$,

$$[\mathcal{C}_1 \cdots \mathcal{C}_n]_r = [\mathcal{C}_1]_r \cdots [\mathcal{C}_n]_r = \begin{bmatrix} 0 & 0 \\ e_{1,r} & e_{1,r} \end{bmatrix}.$$

By Observation 4.23, the valuation matrix of the left column of $\mathcal{C}_1 \cdots \mathcal{C}_n \left(\begin{smallmatrix} -1 & -1 \\ \beta & \alpha \end{smallmatrix}\right)$ and of $\mathcal{L}og_{\alpha,\beta}$ is $\left[\begin{smallmatrix} 0 \\ e_{1,r} \end{smallmatrix}\right]$. Thus, these entries are $O(\Phi_p(1+T))$. Since we have $\Phi_p(1+T) \sim 1$ by Example 4.19, they are indeed $O(1)$.

Next, we assume $0 < v \leqslant \frac{1}{2}$. Given r, let i be the largest integer such that $e_{i,r} < 2v$. Without loss of generality, assume i is even. We then compute

$$\begin{aligned}[\mathcal{C}_1 \cdots \mathcal{C}_i]_r &= [\mathcal{C}_1]_r \cdots [\mathcal{C}_i]_r \\ &= \begin{bmatrix} e_{2,r} + e_{4,r} + \cdots + e_{i,r} & v + e_{2,r} + \cdots + e_{i-2,r} \\ v + e_{1,r} + e_{3,r} + \cdots + e_{i-1,r} & e_{1,r} + \cdots + e_{i-1,r} \end{bmatrix}.\end{aligned}$$

Remembering that $e_{i+1,r} \geqslant 2v$, we see that for $n > i$,

$$\begin{aligned}&[\mathcal{C}_1 \cdots \mathcal{C}_n]_r \\ &\quad = \begin{bmatrix} (n-i)v + e_{2,r} + e_{4,r} + \cdots + e_{i,r} & (n-i-1)v + e_{2,r} + \cdots + e_{i,r} \\ \geqslant (n-i-1)v + e_{1,r} + \cdots + e_{i-1,r} & \geqslant (n-i)v + e_{1,r} + \cdots + e_{i-1,r} \end{bmatrix},\end{aligned}$$

where by $\geqslant x$ we have denoted an unspecified entry that is greater than or equal to x. By Observation 4.23, we have that $\left[\mathcal{C}_1 \cdots \mathcal{C}_n C^{-(N+1)} \left(\begin{smallmatrix} -1 & -1 \\ \beta & \alpha \end{smallmatrix}\right)\right]_r$ is

$$\begin{bmatrix} \geqslant (n-N-i)v + e_{2,r} + \cdots + e_{i,r} & \geqslant (n+N-i)v - N + e_{2,r} + \cdots + e_{i,r} \\ \geqslant (n-N-i+1)v + e_{1,r} + \cdots + e_{i-1,r} & \geqslant (n+N-i+1)v - N + e_{1,r} + \cdots + e_{i-1,r} \end{bmatrix}.$$

Now let $m := i - \lfloor \mathrm{ord}_p(2v) \rfloor$. We then have $e_{m-h,r} \cdot 2v \leqslant e_{i-h,r}$ for any $h < i$, and $e_{M,r} = 1$ for any $M > m$. Thus, the top-left entry of $\left[\mathcal{C}_1 \cdots \mathcal{C}_n C^{-(N+1)} \left(\begin{smallmatrix} -1 & -1 \\ \beta & \alpha \end{smallmatrix}\right)\right]_r$ is, up to a constant independent from r, greater than or equal to

$$2v(e_{m-i+2,r} + e_{m-i+4,r} + \cdots + e_{m,r}) - iv = 2v \cdot v_r \Bigg(\prod_{\substack{k \geqslant m-i+2 \\ k \text{ even}}}^{m} \frac{\Phi_{p^k}(1+T)}{p} \Bigg).$$

Now note that

$$\prod_{\text{even } k \geqslant 2}^{\infty} \frac{\Phi_{p^k}(1+T)}{p} \sim \log_p(1+T)^{\frac{1}{2}}.$$

Using similar arguments, one obtains the appropriate bound for the lower left entry.[5] The claim for the case $0 \leqslant v \leqslant \frac{1}{2}$ thus follows from Lemma 4.22.

[5]The same arguments show it for the right entries when $v = \frac{1}{2}$, although this already follows from Lemma 4.22.

Lastly, we treat the case $v > \frac{1}{2}$. Without loss of generality, let $n > 1$ be even. Then

$$[\mathcal{C}_1 \cdots \mathcal{C}_n]_r = [\mathcal{C}_1]_r \cdots [\mathcal{C}_n]_r = \begin{bmatrix} e_{2,r} + e_{4,r} + \cdots + e_{n,r} & v + e_{1,r} + \cdots + e_{n-2,r} \\ v + e_{1,r} + \cdots + e_{n-1,r} & e_{1,r} + \cdots + e_{n-1,r} \end{bmatrix}.$$

From Observation 4.23, $\operatorname{ord}_p(\alpha) = \operatorname{ord}_p(\beta) = \frac{1}{2}$, and $e_{n,r} \leqslant 1$, we compute

$$\left[\mathcal{C}_1 \cdots \mathcal{C}_n C^{-(N+1)} \begin{pmatrix} -1 & -1 \\ \beta & \alpha \end{pmatrix}\right]_r = \begin{bmatrix} -N/2 + e_{2,r} + \cdots + e_{n,r} & -N/2 + e_{2,r} + \cdots + e_{n,r} \\ (1-N)/2 + e_{1,r} + \cdots + e_{n-1,r} & (1-N)/2 + e_{1,r} + \cdots + e_{n-1,r} \end{bmatrix}.$$

Up to a constant independent from r, these entries are

$$v_r\Bigg(\prod_{k\geqslant 2,\ k \text{ even}}^{n} \frac{\Phi_{p^k}(1+T)}{p}\Bigg) = e_{2,r} + \cdots + e_{n,r} - \frac{n}{2},$$
$$v_r\Bigg(\prod_{k\geqslant 1,\ k \text{ odd}}^{n} \frac{\Phi_{p^k}(1+T)}{p}\Bigg) = e_{1,r} + \cdots + e_{n-1,r} - \frac{n}{2}.$$

But from [Pollack 2003, Lemma 4.5], we have

$$\prod_{\text{even } k\geqslant 2}^{\infty} \frac{\Phi_{p^k}(1+T)}{p} \sim \prod_{\text{odd } k\geqslant 1}^{\infty} \frac{\Phi_{p^k}(1+T)}{p} \sim \log_p(1+T)^{\frac{1}{2}},$$

from which the assertion follows for the case $v > \frac{1}{2}$. □

Lemma 4.24. *When $v = 0$, the functions $\widehat{\log}_\alpha^\sharp(1+T)$ and $\widehat{\log}_\alpha^\flat(1+T)$ are in Λ.*

Proof. Observation 4.23 and Remark 4.5. □

The functional equation.

Proposition 4.25. *Under the change of variable $(1+T) \mapsto (1+T)^{-1}$, the first column of $\widehat{\mathcal{L}og}_{\alpha,\beta}(1+T)$ is invariant. When all four entries of $\widehat{\mathcal{L}og}_{\alpha,\beta}(1+T)$ converge, then*

$$\widehat{\mathcal{L}og}_{\alpha,\beta}(1+T) = \widehat{\mathcal{L}og}_{\alpha,\beta}((1+T)^{-1}) \quad \text{if } p \text{ is odd},$$
$$\begin{pmatrix} 1 & 0 \\ 0 & (1+T)^{-1} \end{pmatrix} \widehat{\mathcal{L}og}_{\alpha,\beta}(1+T) = \widehat{\mathcal{L}og}_{\alpha,\beta}((1+T)^{-1}) \quad \text{if } p = 2.$$

Proof. All $\widehat{\mathcal{C}}_i(1+T)$ are invariant under the change of variables $1+T \mapsto 1/(1+T)$, except $\widehat{\mathcal{C}}_1(1+T)$ if $p = 2$, where we have

$$\widehat{\mathcal{C}}_1\Big(\frac{1}{1+T}\Big) = \begin{pmatrix} 1 & 0 \\ 0 & (1+T)^{-1} \end{pmatrix} \widehat{\mathcal{C}}_1(1+T).$$ □

The functional equation in the case $a_p = 0$. The entries of $\widehat{\mathcal{L}og}_{\alpha,\beta}(1+T)$, when $a_p = 0$, are off by units from the corresponding ones in $\mathcal{L}og_{\alpha,\beta}(1+T)$. More precisely, denote by $\log_p^{\pm}(1+T)$ Pollack's [2003] half-logarithms:

$$\log_p^+(T) := \frac{1}{p}\prod_{j\geqslant 1}\frac{\Phi_{p^{2j}}(1+T)}{p}, \quad \text{and} \quad \log_p^-(T) := \frac{1}{p}\prod_{j\geqslant 1}\frac{\Phi_{p^{2j-1}}(1+T)}{p}.$$

We then have

$$\mathcal{L}og_{\alpha,\beta}(1+T) = \begin{cases} \dfrac{1}{\epsilon(p)}\begin{pmatrix} \log_p^+(T) & \log_p^+(T) \\ \log_p^-(T)\alpha & \log_p^-(T)\beta \end{pmatrix} & \text{when } p \text{ is odd,} \\ \dfrac{1}{\epsilon(2)}\begin{pmatrix} \dfrac{-1}{\epsilon(2)2}\log_2^+(T)\alpha & \dfrac{-1}{\epsilon(2)2}\log_2^+(T)\beta \\ \log_2^-(T) & \log_2^-(T) \end{pmatrix} & \text{when } p = 2. \end{cases}$$

Setting $U^{\pm}(1+T) := \widehat{\log_p^{\pm}(T)}/\log_p^{\pm}(T)$, we obtain

$$\widehat{\mathcal{L}og}_{\alpha,\beta}(1+T) = \begin{pmatrix} U^+(1+T) & 0 \\ 0 & U^-(1+T) \end{pmatrix}\mathcal{L}og_{\alpha,\beta}(1+T).$$

Now put

$$W^+(1+T) = \frac{U^+(1+T)}{U^+((1+T)^{-1})} = \prod_{j\geqslant 1}(1+T)^{-p^{2j-1}(p-1)},$$

and

$$W^-(1+T) = \begin{cases} \dfrac{U^-(1+T)}{U^-((1+T)^{-1})} = \prod_{j\geqslant 1}(1+T)^{-p^{2j-2}(p-1)} & \text{for odd } p, \\ \dfrac{U^-(1+T)}{(1+T)U^-((1+T)^{-1})} = (1+T)^{-1}\prod_{j\geqslant 2}(1+T)^{-p^{2j-2}(p-1)} & \text{when } p = 2. \end{cases}$$

We can finally arrive at the corrected statement of [Pollack 2003, Lemma 4.6]:

Lemma 4.26. *We have*

$$\log_p^+(T)W^+(1+T) = \log_p^+\Big(\frac{1}{1+T} - 1\Big),$$
$$\log_p^-(T)W^-(1+T) = \log_p^-\Big(\frac{1}{1+T} - 1\Big).$$

Proof. This follows from what has been said above, or by going through the proof of [Pollack 2003, Lemma 4.6] on noting that the units $U^{\pm}(1+T) \neq 1$. □

5. The two p-adic L-functions $\widehat{L}_p^{\sharp}(f, T)$ and $\widehat{L}_p^{\flat}(f, T)$

In this section, we construct Iwasawa functions $\widehat{L}_p^{\sharp}(f, T)$ and $\widehat{L}_p^{\flat}(f, T)$. We present the arguments with the completions. The corresponding noncompleted arguments can be recovered by taking off the hat above any expression $\widehat{xyz}$ and replacing it by xyz. Instead of working with the matrices $\widehat{\mathcal{C}}_i$ and C, we make our calculations easier via the following definitions:

Definition 5.1. We put

$$\widehat{\mathcal{A}}_i := \begin{pmatrix} a_p & \widehat{\Phi}_{p^i}(1+T) \\ -\epsilon(p) & 0 \end{pmatrix}, \quad A := \begin{pmatrix} a_p & p \\ -\epsilon(p) & 0 \end{pmatrix}, \quad \tilde{A} := \begin{pmatrix} a_p & 1 \\ -\epsilon(p) & 0 \end{pmatrix}.$$

Definition 5.2. For any integer i, put $Y_{2i} := p^{-i} A^{2i}$, and $Y_{2i+1} = Y_{2i} \tilde{A}$.

Proposition 5.3 (tandem lemma). *Fix $n \in \mathbb{N}$. Assume that for any $i \in \mathbb{N}$, we are given functions $Q_i = Q_i(T)$ such that $Q_i \in \Phi_{p^i}(1+T)\mathcal{O}[T]$ whenever $i \leqslant n$, and $(Q_{n+1}, Q_n)Y_{n'-n} = (Q_{n'+1}, Q_{n'})$ for any $n' \in \mathbb{N}$. Then*

$$(Q_{n+1}, Q_n) = (\widetilde{q}_1, q_0)\widehat{\mathcal{A}}_1 \cdots \widehat{\mathcal{A}}_n \quad \text{with } \widetilde{q}_1, q_0 \in \mathcal{O}[T].$$

Proof. We inductively show that $(Q_{n+1}, Q_n) = (\widetilde{q}_{i+1}, q_i)\widehat{\mathcal{A}}_{i+1} \cdots \widehat{\mathcal{A}}_n$ for $\widetilde{q}_{i+1}, q_i \in \mathcal{O}[T]$ with $0 \leqslant i \leqslant n$: Note that at the base step $i = n$, the product of the $\widehat{\mathcal{A}}$'s is empty so that we indeed have $(\widetilde{q}_{n+1}, q_n) = (Q_{n+1}, Q_n)$. For the inductive step, let $i \geqslant 1$. Then we have

$$\begin{aligned} (Q_{n+1}, Q_n) &= (\widetilde{q}_{i+1}, q_i)A^{n-i} && \text{by evaluation at } \zeta_{p^i} - 1, \\ (Q_{n+1}, Q_n)Y_{i-n} &= (Q_{i+1}, Q_i) && \text{by assumption.} \end{aligned}$$

We thus have

$$(\widetilde{q}_{i+1}, q_i)A^{n-i}Y_{i-n} = (Q_{i+1}, 0) \quad \text{at } \zeta_{p^i} - 1,$$

whence q_i vanishes at $\zeta_{p^i} - 1$. We hence write $q_i = \widehat{\Phi}_{p^i}(1+T) \cdot \widetilde{q}_i$ for some $\widetilde{q}_i \in \mathcal{O}[T]$. Now put $(\widetilde{q}_i, q_{i-1}) := (\widetilde{q}_{i+1}, \widetilde{q}_i)\tilde{A}^{-1}$. Then $(\widetilde{q}_{i+1}, q_i) = (\widetilde{q}_i, q_{i-1})\widehat{\mathcal{A}}_i$. □

Observation 5.4. Let $(\Theta_n)_n$ be a queue sequence and $\pi : \Lambda_n \to \Lambda_{n-1}$ be the projection. Then for $n \geqslant 2$, we have $\pi(\Theta_n, \nu\Theta_{n-1}) = (\Theta_{n-1}, \nu\Theta_{n-2})A$.

Proof. Definition 2.7. □

Proposition 5.5. *Let $(\Theta_n)_n$ be a queue sequence and $0 \leqslant n' \leqslant n$. When lifting elements of Λ_n to $\mathcal{O}[T]$, the second entry of $(\Theta_n, \nu\Theta_{n-1})Y_{n'-n}$ vanishes at $\zeta_{p^{n'}} - 1$.*

Proof. Denote by $\pi_{n/n'}$ the projection from Λ_n to $\Lambda_{n'}$. By the above observation, the second entry of

$$\pi_{n/n'}(\Theta_n, \nu\Theta_{n-1})Y_{n'-n} = (\Theta_{n'}, \nu\Theta_{n'-1})A^{n-n'}Y_{n'-n}$$

is contained in the ideal $(\Phi_{n'}) \subset \Lambda_{n'}$. Thus, its preimage under $\pi_{n/n'}$ is in the ideal $(\Phi_{n'}) \subset \Lambda_n$. □

Corollary 5.6. *Let $(\Theta_n)_n$ be a queue sequence. Then*

$$(\Theta_n, \nu\Theta_{n-1}) = \widehat{\Upsilon}_n \widehat{\mathcal{C}}_1 \cdots \widehat{\mathcal{C}}_n \tilde{A}^{-1} \quad \textit{for some } \widehat{\Upsilon}_n \in \Lambda_n^{\oplus 2}.$$

Proof. We identify elements of Λ_n by their corresponding representative in $\mathcal{O}[T]$ and use Proposition 5.5. Then, we can apply the tandem lemma (Proposition 5.3), and project back to $\Lambda_n^{\oplus 2}$. □

Corollary 5.7. (We rewrite the Riemann sum approximations of Definition 2.10) *For some $\widehat{\overrightarrow{L}}{}_{p,n}^{\omega^i} \in \mathcal{O}[T]^{\oplus 2}$,*

$$\begin{aligned}\left(\varepsilon_{\omega^i} L_{N,\alpha}^{\operatorname{sign}(\omega^i)}, \varepsilon_{\omega^i} L_{N,\beta}^{\operatorname{sign}(\omega^i)}\right) &= \widehat{\overrightarrow{L}}{}_{p,n}^{\omega^i} \widehat{\mathcal{C}}_1 \cdots \widehat{\mathcal{C}}_n \tilde{A}^{-1} \begin{pmatrix} \alpha^{-N} & \beta^{-N} \\ -\alpha^{-(N+1)} & -\beta^{-(N+1)} \end{pmatrix} \\ &= \widehat{\overrightarrow{L}}{}_{p,n}^{\omega^i} \widehat{\mathcal{C}}_1 \cdots \widehat{\mathcal{C}}_n C^{-(N+1)} \begin{pmatrix} -1 & -1 \\ \beta & \alpha. \end{pmatrix}.\end{aligned}$$

Proof. We know that $(\alpha^{N+1} L_{N,\alpha}, \beta^{N+1} L_{N,\beta}) = (\vartheta_N, \nu\vartheta_{N-1}) \left(\begin{smallmatrix} \alpha & \beta \\ -\epsilon & -\epsilon \end{smallmatrix}\right)$. The isotypical components of ϑ_N form queue sequences. Now apply Corollary 5.6 and lift back to $\mathcal{O}[T]^{\oplus 2}$. □

The above $\widehat{\overrightarrow{L}}{}_{p,n}^{\omega^i}$ are not unique, so we take limits by regarding the polynomials as elements of $\Lambda_n^{\oplus 2}$:

Definition 5.8. We define

$$\widehat{\overrightarrow{L}}{}_p^{\omega^i} := \lim_{n\to\infty} \widehat{\overrightarrow{L}}{}_{p,n}^{\omega^i} \in \Lambda^{\oplus 2}/\mathfrak{M},$$

where $\mathfrak{M}$ is given by the next definition.

Definition 5.9. We put $\mathfrak{M} := \varprojlim_n \mathfrak{M}_n$, where

$$\mathfrak{M}_n := \ker\left(\times \widehat{\mathcal{C}}_1 \cdots \widehat{\mathcal{C}}_n C^{-(N+1)} \begin{pmatrix} -1 & -1 \\ \beta & \alpha \end{pmatrix}\right) \subset \Lambda_n \oplus \Lambda_n.$$

Proposition 5.10. *For supersingular p, $\mathfrak{M}$ is trivial. For ordinary p,*

$$\mathfrak{M} \cong T\Lambda(-\log_\alpha^\flat \oplus \log_\alpha^\sharp) \subset \Lambda \oplus \Lambda.$$

Proof. Since

$$C^{-(N+1)} \begin{pmatrix} -1 & -1 \\ \beta & \alpha \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} -\alpha^{-(N+1)} & 0 \\ 0 & \beta^{-(N+1)} \end{pmatrix},$$

we have

$$\mathfrak{M}_n = p^{\operatorname{ord}_p(\alpha)(N+1)} (\alpha^{N+1}\Lambda_n \oplus \beta^{N+1}\Lambda_n) \begin{pmatrix} \alpha & 1 \\ -\beta & -1 \end{pmatrix} \widehat{\mathcal{C}}_n^* \cdots \widehat{\mathcal{C}}_1^* (\beta - \alpha) T,$$

where $\widehat{\mathcal{C}}_i^*$ is the adjugate of $\widehat{\mathcal{C}}_i$ (see also [Sprung 2012, Lemma 5.8]).

Since the matrix product to the right of $(\alpha^{N+1}\Lambda_n \oplus \beta^{N+1}\Lambda_n)$ has Λ_n-integral coefficients, we see that $\mathfrak{M}_n \subset p^{\mathrm{ord}_p(\alpha)(N+1)}\Lambda_n^{\oplus 2}$ so that $\varprojlim_n \mathfrak{M}_n = 0$ when $\mathrm{ord}_p(\alpha) > 0$. In the ordinary case, only the terms involving a power of β go to zero in the limit, whence the result. □

Proof of Theorem 2.14. We give the proof with the hats, since the proof for the expressions without the hats is the same. Part a follows from taking limits of $\widehat{\overrightarrow{L}}{}^{\omega^i}_{p,n}$ together with the main lemma (Lemma 4.4) and the above Proposition 5.10 (triviality of $\mathfrak{M}$). For part b, the proof is the same up to the description of $\mathfrak{M}$ and Proposition 5.10, which gives rise to the term $g(T)T(-\widehat{\log}_\alpha^\flat \oplus \widehat{\log}_\alpha^\sharp)$. Now use Lemma 4.24. □

Now that we have finally proved Theorem 2.14, we can give the following corollary:

Corollary 5.11. *Pick T such that $\mathcal{L}og_{\alpha,\beta}(1+T)$ and $\widehat{\mathcal{L}og}_{\alpha,\beta}(1+T)$ converge in all entries and are invertible. Then*

$$\begin{aligned}&\big(\widehat{L}_p^\sharp(f,\omega^i,T),\widehat{L}_p^\flat(f,\omega^i,T)\big)\\&\qquad=\big(L_p^\sharp(f,\omega^i,T),L_p^\flat(f,\omega^i,T)\big)\mathcal{L}og_{\alpha,\beta}(1+T)\widehat{\mathcal{L}og}_{\alpha,\beta}(1+T)^{-1}.\end{aligned}$$

Remark 5.12. In our setup so far, we have worked with the periods $\Omega_f^\pm$. In the case of an elliptic curve E over $\mathbb{Q}$, one can alternatively use the real and imaginary Néron periods $\Omega_E^\pm$. These real and imaginary Néron periods are defined as follows.

Definition 5.13. Decompose $H_1(E,\mathbb{R}) = H_1(E,\mathbb{R})^+ \oplus H_1(E,\mathbb{R})^-$, where complex conjugation acts as $+1$ on the first summand and as -1 on the second. Put $H_1^\pm(E,\mathbb{Z}) := H_1(E,\mathbb{Z})^\pm \cap H_1(E,\mathbb{R})$. Choose generators $\delta^\pm$ of $H_1(E,\mathbb{Z})^\pm$ such that the following integrals are positive:

$$\Omega_E^\pm := \begin{cases}\int_{\delta^\pm}\omega_E & \text{if } E(\mathbb{R}) \text{ is connected,}\\ 2\cdot\int_{\delta^\pm}\omega_E & \text{if not.}\end{cases}$$

Convention 5.14. When working with these periods, we may define modular symbols and p-adic L-functions analogously, and write E wherever we have written f before.

In view of [Breuil et al. 2001] and [Wiles 1995], we have a modular parametrization $\pi : X_0(N) \to E$, such that $\pi^*(\omega_E) = c \cdot f_E \cdot (dq)/q$ for some normalized weight-two newform f_E of level N. The constant c is called the *Manin constant* for π. It is known to be an integer (see [Edixhoven 1991, Proposition 2]) and conjectured to be 1. See [Manin 1972, § 5].

We note that the analogue of Theorem 2.1 is not necessarily satisfied when one replaces $\Omega_f^\pm$ by $\Omega_E^\pm$, but the following is known (see [Pollack 2003, Remarks 5.4, 5.5]):

	$L_p^\sharp(f,\omega^i,0)$	$L_p^\flat(f,\omega^i,0)$
p odd, $i=0$	$\bigl(-a_p^2+2a_p+p-1\bigr)\dfrac{L(f,1)}{\Omega_f^+}$	$(2-a_p)\dfrac{L(f,1)}{\Omega_f^+}$
p odd, $i\neq 0$	$-pa_p\dfrac{L(f,\omega^{-i},1)}{\tau(\omega^{-i})\Omega_f^{\omega^i(-1)}}$	$-p\dfrac{L(f,\omega^{-i},1)}{\tau(\omega^{-i})\Omega_f^{\omega^i(-1)}}$
$p=2$, $i=0$	$\bigl(-a_p^3+2a_p^2+2pa_p-a_p-2p\bigr)\dfrac{L(f,1)}{\Omega_f^+}$	$\bigl(-a_p^2+2a_p+p-1\bigr)\dfrac{L(f,1)}{\Omega_f^+}$
$p=2$, $i\neq 0$	$-p^2a_p\dfrac{L(f,\omega^{-i},1)}{\tau(\omega^{-i})\Omega_f^{\omega^i(-1)}}$	$-p^2\dfrac{L(f,\omega^{-i},1)}{\tau(\omega^{-i})\Omega_f^{\omega^i(-1)}}$

Table 2. Table of the special values for a good prime p.

Theorem 5.15 (imitation of Theorem 2.1). *Let E be a strong Weil curve over $\mathbb{Q}$, and p be a prime of good reduction. Then*:

(1) [Abbes and Ullmo 1996, théorème A] *p does not divide c.*

(2) [Manin 1972, Theorem 3.3] *If $a_p \not\equiv 1 \bmod p$, we have*

$$2\left[\frac{a}{p^n}\right]_E^{\pm} \in c^{-1}\mathbb{Z}, \quad \textit{so} \quad 2\left[\frac{a}{p^n}\right]_E^{\pm} \in \mathbb{Z}_p.$$

Corollary 5.16. *When $a_p \not\equiv 1 \bmod p$, $L_p^\sharp(E,\omega^i,T)$ and $L_p^\flat(E,\omega^i,T)$ and their completions are in Λ. In particular, the 2-adic L-functions $L_2^\sharp(E,\omega^i,T)$ and $L_2^\flat(E,\omega^i,T)$ from* [Sprung 2012, Definition 6.1] *agree with those of this paper and are consequently elements of Λ, rather than $\Lambda\otimes\mathbb{Q}$.*

Proof. This follows from Theorem 2.14 and what has just been said. For $p=2$, we exploit the following symmetry in the isotypical components of the Riemann sums $L_{N,\alpha}^{\pm}$ and $L_{N,\beta}^{\pm}$: From $\eta^{\pm}(a/m)=\pm\eta^{\pm}(-a/m)$, we can conclude that $\omega^i(a)\eta^{\pm}(a/m)=\pm\omega^i(-a)\eta^{\pm}(-a/m)$. □

Corollary 5.17 (analogue of Theorem 2.14). *When $a_p \not\equiv 1 \bmod p$, the statement of Theorem 2.14 with f formally replaced by E is still valid. When $a_p \equiv 1 \bmod p$ or E is not a strong Weil curve, we can say the same with the added caveat that $L_p^\sharp(E,\omega^i,T)$, $L_p^\flat(E,\omega^i,T)$, and their completions are elements of $\mathbb{Q}\otimes\Lambda$.*

From Theorem 2.6, we can give a table of the special values for a good prime p, as shown in Table 2. In view of these special values, it seems reasonable to make the following conjecture:

Conjecture 5.18. *Let f be a modular form as above, and let p be a good supersingular prime. When p is odd, $\widehat{L}_p^\flat(f,\omega^i,T)$ and $L_p^\flat(f,\omega^i,T)$ are not identically zero, and $\widehat{L}_p^\sharp(f,\omega^i,T)$ and $L_p^\sharp(f,\omega^i,T)$ are not identically zero when $a_p\neq 2$. When $p=2$, the power series $\widehat{L}_2^\sharp(f,\omega^i,T)$ and $L_2^\sharp(f,\omega^i,T)$ are not identically zero, and $\widehat{L}_2^\flat(f,\omega^i,T)$ and $L_2^\flat(f,\omega^i,T)$ are not identically zero when $a_2\neq 1$.*

The functional equation in the supersingular case. Let f be a weight-two modular form of level N and nebentype ϵ which is an eigenform for all T_n. Recall also $\log_\gamma(\cdot)$ introduced in Definition 2.8. We denote by $f^*(z) = w_N(f(z)) = \epsilon(-1) f(-1/(Nz))$ the involuted form of f under the Atkin-Lehner / Fricke operator, as in [Mazur et al. 1986, (5.1)], and let $\alpha^* = \alpha/\epsilon(p)$ and $\beta^* = \beta/\epsilon(p)$.

Theorem 5.19. *Let p be a supersingular prime so that $(p, N) = 1$, i.e., $N \in \mathbb{Z}_p^\times \cong \mathcal{G}_\infty$. Then*

$$\begin{aligned}&\big(\widehat{L}_p^\sharp(f, \omega^i, T),\ \widehat{L}_p^\flat(f, \omega^i, T)\big)\widehat{\mathcal{L}og}_{\alpha,\beta}(1+T)\\&= -\epsilon(-1)\omega^{-i}(-N)(1+T)^{-\log_\gamma(N)}\\&\qquad\times\Big(\widehat{L}_p^\sharp\Big(f^*, \omega^{-i}, \frac{1}{1+T}-1\Big),\ \widehat{L}_p^\flat\Big(f^*, \omega^{-i}, \frac{1}{1+T}-1\Big)\Big)\widehat{\mathcal{L}og}_{\alpha^*,\beta^*}(1+T).\end{aligned}$$

Corollary 5.20. *For an elliptic curve E over $\mathbb{Q}$ and a good supersingular prime p, let c_N be the sign of f, i.e., $f^* := -c_N f$* (see [Mazur et al. 1986, §18]). *We then have*

$$\widehat{L}_p^\sharp(E, \omega^i, T) = -(1+T)^{-\log_\gamma(N)}\omega^i(-N)c_N\widehat{L}_p^\sharp\Big(E, \omega^i, \frac{1}{1+T}-1\Big),$$
$$\widehat{L}_p^\flat(E, \omega^i, T) = -(1+T)^{-\log_\gamma(N)}\omega^i(-N)c_N\widehat{L}_p^\flat\Big(E, \omega^i, \frac{1}{1+T}-1\Big).$$

When $a_p = 0$, we can give an explicit functional equation for the noncompleted p-adic L-functions, which corrects [Pollack 2003, Theorem 5.13] *in the case $i = 0$:*

$$L_p^\sharp(E, \omega^i, T) = -(1+T)^{-\log_\gamma(N)}\omega^i(-N)c_N W^+(1+T)L_p^\sharp\Big(E, \omega^i, \frac{1}{1+T}-1\Big),$$
$$L_p^\flat(E, \omega^i, T) = -(1+T)^{-\log_\gamma(N)}\omega^i(-N)c_N W^-(1+T)L_p^\flat\Big(E, \omega^i, \frac{1}{1+T}-1\Big).$$

Proof of Theorem 5.19. This follows from the functional equations for $\widehat{L}_p(f,\alpha,\omega^i,T)$ and $\widehat{L}_p(f, \beta, \omega^i, T)$, which formally display exactly the same invariance under the substitution $T \mapsto (1+T)^{-1} - 1$, see [Mazur et al. 1986, §17, (17.3)]. The rest is invariance of $\widehat{\mathcal{L}og}_{\alpha,\beta}(1+T)$ under $T \mapsto (1+T)^{-1} - 1$, see Proposition 4.25. □

Part II. Invariants coming from the conjectures of Birch and Swinnerton-Dyer in the cyclotomic direction

6. The conjectures about the rank and leading coefficient

We scrutinize what happens when $T = \zeta_{p^n} - 1$ for $n \geqslant 1$: we estimate BSD-theoretic quantities in the cyclotomic direction, using the pairs of Iwasawa invariants of $L_p^\sharp$ and $L_p^\flat$ (which match those of $\widehat{L}_p^\sharp$ and $\widehat{L}_p^\flat$ when used, see Corollary 8.9).

Choose $\mathcal{G}_f \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\left\{f^\sigma = \sum \sigma(a_n)q^n\right\}_{\sigma\in\mathcal{G}_f}$ contains each Galois conjugate of f once.

Definition 6.1. For $\sigma \in \mathcal{G}_f$, the σ-parts of the (p-adic) analytic ranks of $A_f(\mathbb{Q}_n)$ and of $A_f(\mathbb{Q}_\infty)$ are

$$r_n^{\mathrm{an}}(f^\sigma) = \sum_{\zeta:\ p^n\text{-th roots of unity}} \mathrm{ord}_{\zeta-1}(L_p(f^\sigma, \alpha, T)),$$

$$r_\infty^{\mathrm{an}}(f^\sigma) := \lim_{n\to\infty} r_n^{\mathrm{an}}(f^\sigma) = \sum_{\zeta:\ \textit{all}\ p\text{-power roots of unity}} \mathrm{ord}_{\zeta-1}(L_p(f^\sigma, \alpha, T)).$$

Note that by a theorem of Rohrlich [1984], r_∞^{an} is a finite integer.

We can then estimate the p-adic analytic rank of $A_f(\mathbb{Q}_n)$ and of $A_f(\mathbb{Q}_\infty)$ by setting

$$r_n^{\mathrm{an}} := \sum_{\sigma\in\mathcal{G}_f} r_n^{\mathrm{an}}(f^\sigma) \quad \text{and} \quad r_\infty^{\mathrm{an}} := \sum_{\sigma\in\mathcal{G}_f} r_\infty^{\mathrm{an}}(f^\sigma). \tag{2}$$

Conjecturally, r_∞^{an} should agree with the complex analytic rank of $A_f(\mathbb{Q}_\infty)$ defined by the order of vanishing of the Hasse–Weil series $L(A_f/\mathbb{Q}_\infty, s)$ at $s = 1$.

Definition 6.2. We let d_n be the normalized jump in the ranks of A_f at level $\mathbb{Q}_n$:

$$d_n := \frac{\mathrm{rank}(A_f(\mathbb{Q}_n)) - \mathrm{rank}(A_f(\mathbb{Q}_{n-1}))}{p^n - p^{n-1}}.$$

Denote by $D(\mathbb{Q}_n)$ the discriminant, by $R(A_f/\mathbb{Q}_n)$ the regulator, by $\mathrm{Tam}(A_f/\mathbb{Q}_n)$ the product of the Tamagawa numbers, and let $\Omega_{A_f/\mathbb{Q}_n} = (\Omega_{A_f/\mathbb{Q}})^{p^n}$, where $\Omega_{A_f/\mathbb{Q}}$ is the real period of A_f. We also denote by $\widehat{A}_f$ the dual of A_f.

Conjecture 6.3 (cyclotomic BSD). *Let ζ_{p^n} be a primitive p^n-th root of unity, $d_n^{\mathrm{an}}(f)$ the order of vanishing of $L_p(f, \alpha, T)$ at $T = \zeta_{p^n} - 1$, and $r_n^{\mathrm{an}'}(f)$ the order of vanishing of the complex L-series $L(f/\mathbb{Q}_n, s) := \prod_{\chi\in\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})} L(f, \chi, s)$ at $s = 1$. Then*

$$d_n^{\mathrm{an}}(f) = \frac{r_n^{\mathrm{an}'}(f) - r_{n-1}^{\mathrm{an}'}(f)}{p^n - p^{n-1}} \quad \textit{and} \quad \sum_\sigma d_n^{\mathrm{an}}(f^\sigma) = d_n.$$

In particular, the order of vanishing $r_n^{\mathrm{an}'}$ of $L(A_f/\mathbb{Q}_n, s) := \prod_\sigma L(f^\sigma/\mathbb{Q}_n, s)$ at $s = 1$ is d_n.

In view of this conjecture, we put (see [Manin 1971, Remark 8.5]):

$$\#\text{Ш}^{\mathrm{an}}(A_f/\mathbb{Q}_n) := \frac{L^{(r_n^{\mathrm{an}'})}(A_f/\mathbb{Q}_n, 1)\#A_f^{\mathrm{tor}}(\mathbb{Q}_n)\#\widehat{A}_f^{\mathrm{tor}}(\mathbb{Q}_n)\sqrt{D(\mathbb{Q}_n)}}{(r_n^{\mathrm{an}'})!\,\Omega_{A_f/\mathbb{Q}_n} R(A_f/\mathbb{Q}_n)\,\mathrm{Tam}(A_f/\mathbb{Q}_n)}.$$

Our notation of $d_n^{\mathrm{an}}(f)$, which is independent of the choice ζ_{p^n}, is justified as follows:

Lemma 6.4.
$$d_n^{\mathrm{an}}(f) = \frac{r_n^{\mathrm{an}}(f) - r_{n-1}^{\mathrm{an}}(f)}{p^n - p^{n-1}}.$$

We postpone the proof until after Lemma 7.4.

Remark 6.5. It is not clear (at least not to the author) how to relate the leading Taylor coefficient of $L_p(f, \alpha, T)$ at $T = \zeta_{p^n} - 1$ to the size of the Šafarevič–Tate groups, even when A_f is an elliptic curve. (For a relative version, see [Mazur and Swinnerton-Dyer 1974, §9.5, Conjecture 4].)

7. The Mordell–Weil rank in the cyclotomic direction

We now give an upper bound for $r_\infty^{\mathrm{an}}(f)$. When f is ordinary at p, we have the estimate $\lambda \geqslant r_\infty^{\mathrm{an}}(f)$, where λ is the λ-invariant of $L_p(f, \alpha, T)$. This section is devoted to the more complicated supersingular scenario. We give two different upper bounds. To obtain an upper bound on r_∞^{an}, one then simply sums the bounds on $r_\infty^{\mathrm{an}}(f^\sigma)$. (*Note that f^σ may be ordinary or supersingular at p independently of whether f was!*) Recall that the weight of f is two in this paper, so that trivial zeros of p-adic L-functions are not an issue.

Proposition 7.1. *Let f be a weight-two modular form and p be a good supersingular prime. If ζ is a p^n-th root of unity, then we have*

$$\operatorname{ord}_{\zeta-1} L_p(f, \alpha, T) = \operatorname{ord}_{\zeta-1} L_p(f, \beta, T).$$

Proof. For $n = 0$, this is [Pollack 2003, Lemma 6.6]. Thus, let $n > 0$. Let us first prove that $L_p(f, \alpha, \zeta - 1) = 0$ if and only if $L_p(f, \beta, \zeta - 1) = 0$: Observation 4.8 allows us to conclude that

$$\begin{aligned}
&\big(L_p(f, \alpha, \zeta - 1), L_p(f, \beta, \zeta - 1)\big) \\
&\qquad = \overrightarrow{L_p}(f, \zeta - 1)\mathcal{L}og_{\alpha,\beta}(\zeta - 1) \\
&\qquad = \overrightarrow{L_p}(f, \zeta - 1)\begin{pmatrix} * & * \\ * & * \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \Phi_{p^n}(\zeta) \end{pmatrix}\begin{pmatrix} -1 & -1 \\ \beta & \alpha \end{pmatrix}\begin{pmatrix} \alpha^{-N} & 0 \\ 0 & \beta^{-N} \end{pmatrix} \\
&\qquad = \overrightarrow{L_p}(f, \zeta - 1)\begin{pmatrix} * & * \\ * & * \end{pmatrix}\begin{pmatrix} -1 & -1 \\ \beta\Phi_{p^n}(\zeta) & \alpha\Phi_{p^n}(\zeta) \end{pmatrix}\begin{pmatrix} \alpha^{-N} & 0 \\ 0 & \beta^{-N} \end{pmatrix}
\end{aligned}$$

for some 2×2-matrix $\left(\begin{smallmatrix} * & * \\ * & * \end{smallmatrix}\right)$ with entries in $\overline{\mathbb{Q}}$. But $\Phi_{p^n}(\zeta) = 0$, so

$$L_p(f, \alpha, \zeta - 1) = 0 \quad \text{implies} \quad \overrightarrow{L_p}(\zeta - 1)\begin{pmatrix} * & * \\ * & * \end{pmatrix} = (0, *).$$

Thus, we can conclude that $L_p(f, \beta, \zeta - 1) = 0$. A symmetric argument shows $L_p(f, \beta, \zeta - 1) = 0$ implies $L_p(f, \alpha, \zeta - 1) = 0$.

The rest is induction: Fixing $k \in \mathbb{N}$ and assuming

$$L_p^{(i)}(f, \alpha, \zeta - 1) = L_p^{(i)}(f, \beta, \zeta - 1) = 0 \quad \text{for } 0 \leqslant i < k,$$

we have

$$\big(L_p^{(k)}(f,\alpha,\zeta-1),\, L_p^{(k)}(f,\beta,\zeta-1)\big) = \overrightarrow{L}_p^{(k)}(f,\zeta-1)\mathcal{L}og_{\alpha,\beta}(\zeta-1)$$

by the product rule. By the above argument, $L_p^{(k)}(f,\alpha,\zeta-1)=0$ if and only if $L_p^{(k)}(f,\beta,\zeta-1)=0$. □

Corollary 7.2. *Let $a_p=0$. Then we have $r_\infty^{\mathrm{an}}(f)\leqslant\lambda_\sharp+\lambda_\flat$.*

This has already been proved by Pollack in the elliptic curve case when $p\equiv 3 \bmod 4$ and $a_p=0$. He derived Proposition 7.1 in this case by a very clever argument involving Gauß sums (for which $p\equiv 3 \bmod 4$ is needed) and the functional equation (which is simple enough for elliptic curves).

Proof of Corollary 7.2. The proof of [Pollack 2003, Corollary 6.8] now works in the desired generality, since the only hard ingredient was Proposition 7.1. □

Definition 7.3. For any integer n, let Ξ_n be the matrix such that $\mathcal{L}og_{\alpha,\beta}=\mathcal{C}_1\cdots\mathcal{C}_n\,\Xi_n$.

Lemma 7.4. *Fix an integer n and let $m\leqslant n$. Then*

$$\operatorname{ord}_{\zeta_{p^m}-1}L_p(f,\alpha,T)=j \quad \textit{if and only if} \quad \operatorname{ord}_{\zeta_{p^m}-1}\overrightarrow{L_p}(f,T)\mathcal{C}_1\cdots\mathcal{C}_n=j.$$

Proof. Since $\det\Xi_n(\zeta_{p^m}-1)\neq 0$, we have by induction on i and Proposition 7.1 that

$$L_p^{(i)}(f,\alpha,\zeta_{p^m}-1)=0 \quad \text{for } i\leqslant j-1 \text{ but not for } i=j$$

is equivalent to

$$\overrightarrow{L}_p^{(i)}(f,T)\mathcal{C}_1\cdots\mathcal{C}_n(\zeta_{p^m}-1)=(0,0) \quad \text{for } i\leqslant j-1 \text{ but not for } i=j,$$

which is equivalent to $\operatorname{ord}_{\zeta_{p^m}-1}\overrightarrow{L_p}(f,T)\mathcal{C}_1\cdots\mathcal{C}_n=j$. □

Proof of Lemma 6.4. The entries of the vector $\overrightarrow{L_p}(f,T)\mathcal{C}_1\cdots\mathcal{C}_n$ are up to units polynomials, so for $m\leqslant n$, we have

$$\operatorname{ord}_{\zeta_{p^m}-1}\overrightarrow{L_p}(f,T)\mathcal{C}_1\cdots\mathcal{C}_n=\operatorname{ord}_{\zeta'_{p^m}-1}\overrightarrow{L_p}(f,T)\mathcal{C}_1\cdots\mathcal{C}_n$$

for any two primitive p^m-th roots of unity ζ_{p^m} and ζ'_{p^m}. From Lemma 7.4,

$$\operatorname{ord}_{\zeta_{p^m}-1}L_p(f,\alpha,T)=\operatorname{ord}_{\zeta'_{p^m}-1}L_p(f,\alpha,T).$$ □

Notation 7.5. Given $x\in\mathbb{Q}$, we let $\lfloor x\rfloor$ be the largest integer $\leqslant x$.

Definition 7.6. We define the n-th $\sharp/\flat$-*Kurihara terms* $q_n^{\sharp/\flat}$ and some auxiliary integers $\nu_{\sharp/\flat}$, $\widetilde{\nu}_{\sharp/\flat}$.

$$q_n^\sharp := \left\lfloor \frac{p^n}{p+1} \right\rfloor \text{ if } n \text{ is odd,} \quad \text{and} \quad q_n^\sharp := q_{n+1}^\sharp \text{ for even } n,$$

$$q_n^\flat := \left\lfloor \frac{p^n}{p+1} \right\rfloor \text{ if } n \text{ is even,} \quad \text{and} \quad q_n^\flat := q_{n+1}^\flat \text{ for odd } n,$$

$$\nu_\sharp := \text{largest odd integer } n \geqslant 1 \text{ such that } \lambda_\sharp \geqslant p^n - p^{n-1} - q_n^\sharp,$$

$$\nu_\flat := \text{largest even integer } n \geqslant 2 \text{ such that } \lambda_\flat \geqslant p^n - p^{n-1} - q_n^\flat,$$

$$\widetilde{\nu}_\flat := \text{largest odd integer } n \geqslant 3 \text{ such that } \lambda_\flat \geqslant p^n - p^{n-1} - pq_{n-1}^\flat - (p-1)^2,$$

$$\widetilde{\nu}_\sharp := \text{largest even integer } n \geqslant 2 \text{ such that } \lambda_\sharp \geqslant p^n - p^{n-1} - pq_{n-1}^\sharp.$$

In case no such integer exists, we put respectively $\nu_\sharp := 0$, $\nu_\flat := 0$, $\widetilde{\nu}_\sharp := 0$, but $\widetilde{\nu}_\flat := 1$.

Note that explicitly, we have

$$q_n^\sharp = p^{n-1} - p^{n-2} + p^{n-3} - p^{n-4} + \cdots + p^2 - p \quad \text{for odd } n > 1,$$
$$q_n^\flat = p^{n-1} - p^{n-2} + p^{n-3} - p^{n-4} + \cdots + p - 1 \quad \text{for even } n > 0.$$

Convention 7.7. We define the μ-invariant of the 0-function to be ∞.

Theorem 7.8.
- *When* $|\mu_\sharp - \mu_\flat| \leqslant v = \operatorname{ord}_p(a_p)$ (*e.g., when* $a_p = 0$), *put* $\nu = \max(\nu_\sharp, \nu_\flat)$. *We then have* $r_\infty^{\mathrm{an}}(f) \leqslant \min(q_\nu^\sharp + \lambda_\sharp, q_\nu^\flat + \lambda_\flat)$.
- *When* $\mu_\sharp > \mu_\flat + v$, *put* $\nu = \max(\nu_\flat, \widetilde{\nu}_\flat)$. *We then have*
$$r_\infty^{\mathrm{an}}(f) \leqslant \begin{cases} \min(q_\nu^\flat + \lambda_\flat, q_{\nu-1}^\flat - (p-1)^2 + \lambda_\flat) & \textit{when } \nu \neq 1, \\ \min(q_1^\flat + \lambda_\flat, q_1^\sharp + \lambda_\sharp) & \textit{when } \nu = 1. \end{cases}$$
- *When* $\mu_\flat > \mu_\sharp + v$, *put* $\nu = \max(\nu_\sharp, \widetilde{\nu}_\sharp)$. *We then have*
$$r_\infty^{\mathrm{an}}(f) \leqslant \min(pq_{\nu-1}^\sharp + \lambda_\sharp, q_\nu^\sharp + \lambda_\sharp).$$

Definition 7.9. For a vector $\overrightarrow{a} = (a_\sharp, a_\flat) \in \Lambda^{\oplus 2}$, we define its λ-invariant as $\lambda(\overrightarrow{a}) := \min(\lambda(a_\sharp), \lambda(a_\flat))$.

Proof of Theorem 7.8. We handle the first case first. Denote $L_p(f, \alpha, T)$ by L_α. In the proof, we justify the two equality signs in the following equation:

$$\sum_{\substack{\text{all } p\text{-power} \\ \text{roots of unity } \zeta}} \operatorname{ord}_{\zeta-1} L_\alpha = \sum_{\substack{\zeta \text{ s.t. } \zeta^{p^n}=1 \\ \text{and } n \leqslant \nu}} \operatorname{ord}_{\zeta-1} L_\alpha = \sum_{\substack{\zeta \text{ s.t. } \zeta^{p^n}=1 \\ \text{and } n \leqslant \nu}} \operatorname{ord}_{\zeta-1} \overrightarrow{L}_p \mathcal{C}_1 \cdots \mathcal{C}_\nu$$

The result then follows on noting that the last term is bounded by $\lambda(\overrightarrow{L}_p \mathcal{C}_1 \cdots \mathcal{C}_\nu)$.

We justify the first equality sign. By Proposition 7.1, $\operatorname{ord}_{\zeta_{p^n}-1} L_\alpha = \operatorname{ord}_{\zeta_{p^n}-1} L_\beta$. Since we have $\det \Xi_n|_{T=\zeta_{p^n}-1} \neq 0$, we can say that $\operatorname{ord}_{\zeta_{p^n}-1} L_\alpha = 0$ if and only if

$\overrightarrow{L}_p\mathcal{C}_1\cdots\mathcal{C}_n|_{T=\zeta_{p^n}-1}\neq(0,0)$. Since $\lambda(\overrightarrow{L}_p\mathcal{C}_1\cdots\mathcal{C}_n)$ is bounded above by $\lambda_\sharp+q_n^\sharp$ and $\lambda_\flat+q_n^\flat$, we have that

$$p^n-p^{n-1}>\min(\lambda_\sharp+q_n^\sharp,\lambda_\flat+q_n^\flat)\quad\text{implies}\quad\overrightarrow{L}_p\mathcal{C}_1\cdots\mathcal{C}_n|_{\zeta_{p^n}-1}\neq(0,0).$$

Now $\lambda_\flat+q_n^\flat<p^n-p^{n-1}$ for some even n implies $\lambda_\flat+q_m^\flat<p^m-p^{m-1}$ for any even $m\geqslant n$. Similarly, $\lambda_\sharp+q_n^\sharp<p^n-p^{n-1}$ for some odd n implies $\lambda_\sharp+q_m^\sharp<p^m-p^{m-1}$ for any odd $m\geqslant n$. Thus,

$$m>\nu\quad\text{implies}\quad\operatorname{ord}_{\zeta_{p^m}-1}L_\alpha=0.$$

The second equality sign follows from Lemma 7.4 applied to $n=\nu$.

In the other cases, similar arguments hold, with the following caveats: in the second case,

$$\lambda(\overrightarrow{L}_p\mathcal{C}_1\cdots\mathcal{C}_n)=\begin{cases}\lambda_\flat+q_n^\flat & \text{when } n \text{ is even},\\ \lambda_\flat+pq_{n-1}^\flat-(p-1)^2 & \text{when } n \text{ is odd and } n\neq1,\\ \lambda_\flat+q_1^\flat & \text{when } n=1,\end{cases}$$

while in the third case,

$$\lambda(\overrightarrow{L}_p\mathcal{C}_1\cdots\mathcal{C}_n)=\begin{cases}\lambda_\sharp+pq_{n-1}^\sharp & \text{when } n \text{ is even},\\ \lambda_\sharp+q_n^\sharp & \text{when } n \text{ is odd}.\end{cases}$$

The asymmetry in the second case comes from

$$(L_p^\sharp,L_p^\flat)\mathcal{C}_1\equiv(-\Phi_pL_p^\flat,L_p^\sharp)\mod a_p. \qquad\square$$

Comparing this bound with the sum of λ-invariants bound of Corollary 7.2, we find that it is in most cases sharper. (The exception is when $p=2$, in which case it is *never sharper*. Here, the cases when the bounds match is when there is an odd ν so that $\lambda_\sharp=q_\nu^\sharp$ and $\lambda_\flat\leqslant q_\nu^\flat$ or there is an even ν so that $\lambda_\flat=q_\nu^\flat$ and $\lambda_\sharp\leqslant q_\nu^\sharp$.) When p is odd and f is elliptic modular, this bound is strictly sharper in all known cases (see the tables of Perrin-Riou [2003] and Pollack [2002]), except when:

(1) $\lambda_\flat=0$ and $\lambda_\sharp<p-1$,

(2) $p=3$, and

$(\lambda_\sharp,\lambda_\flat)\in\{(0,6),(1,5),(1,6),(2,4),(2,5),(2,6),(12,2),(13,x)\text{ with }x\leqslant5\}$.

The following corollary gives a bound that is in the spirit of the bound in the ordinary case:

Corollary 7.10. *Assume $\lambda_\sharp<p-1$ and $\lambda_\flat<p^2-p-p+1=(p-1)^2$. Then $r_\infty^{\text{an}}(f)\leqslant\min(\lambda_\sharp,\lambda_\flat)$ for $\mu_\flat\leqslant\mu_\sharp+v$, while $r_\infty^{\text{an}}(f)\leqslant\lambda_\sharp$ when $\mu_\flat>\mu_\sharp+v$.*

Proof. Indeed, we have $\nu_\sharp=\nu_\flat=\widetilde{\nu}_\sharp=\widetilde{\nu}_\flat-1=0$ in this case. $\square$

Example 7.11. When p is odd and $\lambda_\sharp = \lambda_\flat = 1$, we have $r_\infty^{\mathrm{an}} = r_\infty^{\mathrm{an}}(f) \leqslant 1$; see [Perrin-Riou 2003, Proposition 7.17] for the elliptic curve case. This case is very common numerically.

We thank Robert Pollack for pointing out the following example in which the sum of the λ-invariants is not a bound for $r_\infty^{\mathrm{an}}(f)$ as in Corollary 7.2. Our proposition explains the bound:

Example 7.12. Consider E37A. For the prime 3, we have $a_3 = -3$, and at this prime 3, we have $\lambda_\sharp = 1$, while $\lambda_\flat = 5$, and $r_\infty^{\mathrm{an}} = r_\infty^{\mathrm{an}}(f) = 7$. In this case $\nu_\sharp = 0$ and $\nu_\flat = 2$. Thus, the bound for r_∞^{an} is $\min(q_2^\flat + 5, q_2^\sharp + 1) = \min(3-1+5, 3^2-3+1) = 7$. Note that $r_\infty^{\mathrm{an}} = 7 > \lambda_\sharp + \lambda_\flat = 6$.

8. The special value of the L-function of f in the cyclotomic direction

The purpose of this section is to prove a special value formula for modular forms of weight two in the cyclotomic direction that estimates the size of $\text{Ш}^{\mathrm{an}}(A_f/\mathbb{Q}_n)[p^\infty]$. We encounter an unexpected phenomenon when $v = \mathrm{ord}_p(a_p) < \frac{1}{2}$.

Definition 8.1. Put

$$\mathcal{C}_i(a, 1+T) := \begin{pmatrix} a & 1 \\ -\epsilon(p)\Phi_{p^i}(1+T) & 0 \end{pmatrix}.$$

We now put $\mathcal{H}_a^i(1+T) := \mathcal{C}_1(a, 1+T) \cdots \mathcal{C}_i(a, 1+T)$.

Definition 8.2. For an element a in the closed unit disc of $\mathbb{C}_p$, let $v := \mathrm{ord}_p(a) \geqslant 0$. When $v > 0$, let $k \in \mathbb{Z}^{\geqslant 1}$ be the smallest positive integer such that $v \geqslant p^{-k}/2$. We now let $v_m = v_m(a)$ be the upper left entry in the valuation matrix of $\mathcal{H}_a^m(\zeta_{p^{k+2}} - 1)$.

Given further an integer n, we now define two functions $q_n^*(v, v_2)$ for $* \in \{\sharp, \flat\}$.

When $\infty > v \geqslant p^{-k}/2$, we put $\delta := \min(v_2 - 2v, (p-1)p^{-k-2})$. Note that $\delta = 0$ when $v > p^{-k}/2$, since $v_2 = \mathrm{ord}_p(a^2 - \Phi_{p^2}(\zeta_{p^{k+2}}))$. We define

$$q_n^\sharp(v, v_2) := \begin{cases} (p^n - p^{n-1})kv + \left\lfloor \dfrac{p^{n-k}}{p+1} \right\rfloor & \text{if } n \not\equiv k \bmod 2 \\ (p^n - p^{n-1})((k-1)v + \delta) + \left\lfloor \dfrac{p^{n+1-k}}{p+1} \right\rfloor & \text{if } n \equiv k \bmod 2, \end{cases}$$

$$q_n^\flat(v, v_2) := \begin{cases} (p^n - p^{n-1})((k-1)v + \delta) + p\left\lfloor \dfrac{p^{n-k}}{p+1} \right\rfloor + p - 1 & \text{if } n \not\equiv k \bmod 2 \\ (p^n - p^{n-1})kv + p\left\lfloor \dfrac{p^{n-1-k}}{p+1} \right\rfloor + p - 1 & \text{if } n \equiv k \bmod 2. \end{cases}$$

Note that the tail terms

$$\left\lfloor \frac{p^{n-k}}{p+1} \right\rfloor \quad \text{and} \quad \left\lfloor \frac{p^{n+1-k}}{p+1} \right\rfloor$$

appearing in $q_n^\sharp(v, v_2)$ are equal to $q_{n-k}^\sharp$. For $n > k$, those for $q_n^\flat(v, v_2)$, i.e.,

$$p\left\lfloor \frac{p^{n-k}}{p+1} \right\rfloor + p - 1 \quad \text{and} \quad p\left\lfloor \frac{p^{n-1-k}}{p+1} \right\rfloor + p - 1,$$

are both $q_{n-k}^\flat$. For $v = \infty$, we define

$$q_n^*(\infty, v_2) := \lim_{v\to\infty} q_n^*(v, v_2).$$

Finally, for $v = 0$, we similarly put

$$q_n^*(0, v_2) := \lim_{v\to 0} q_n^*(v, v_2) = \begin{cases} 0 & \text{when } * = \sharp, \\ p-1 & \text{when } * = \flat. \end{cases}$$

(We use this seemingly strange adherence to the symbol v_2 simply for uniform notation.)

Definition 8.3. The *sporadic case* (for v and v_2) occurs if $v = 0$ and $\mu_\sharp = \mu_\flat$ and $\lambda_\sharp = \lambda_\flat + p - 1$, or if $v = p^{-k}/2$ and $v_2 = 2v(1 + p^{-1} - p^{-2})$ and

$$n \not\equiv k \bmod 2 \quad \text{and} \quad \begin{cases} \mu_\sharp - \mu_\flat > v - 2v/(p^3+p^2) & \text{or,} \\ \mu_\sharp - \mu_\flat = v - 2v/(p^3+p^2) & \text{and} \quad \lambda_\sharp > \lambda_\flat, \end{cases}$$

or

$$n \equiv k \bmod 2 \quad \text{and} \quad \begin{cases} \mu_\sharp - \mu_\flat < 2v/(p^3+p^2) - v & \text{or,} \\ \mu_\sharp - \mu_\flat = 2v/(p^3+p^2) - v & \text{and} \quad \lambda_\sharp \leqslant \lambda_\flat. \end{cases}$$

Definition 8.4 (modesty algorithm). Given a in the closed unit disc, an integer n, integers $\lambda_\sharp$ and $\lambda_\flat$, and rational numbers $\mu_\sharp$ and $\mu_\flat$, choose $* \in \{\sharp, \flat\}$ via

$$* = \begin{cases} \sharp & \text{if } (p^n - p^{n-1})\mu_\sharp + \lambda_\sharp + q_n^\sharp(v, v_2) < (p^n - p^{n-1})\mu_\flat + \lambda_\flat + q_n^\flat(v, v_2), \\ \flat & \text{if } (p^n - p^{n-1})\mu_\flat + \lambda_\flat + q_n^\flat(v, v_2) < (p^n - p^{n-1})\mu_\sharp + \lambda_\sharp + q_n^\sharp(v, v_2). \end{cases}$$

Theorem 8.5. *Let f be a modular form of weight two which is a normalized eigenform for all T_n with eigenvalue a_n and p a good prime. Let $v = v(a_p)$ and $v_2 = v_2(a_p)$ (via Definition 8.2). For a character χ of $\mathbb{Z}_p^\times$ with order p^n, denote by $\tau(\chi)$ the Gauß sum. Let n be large enough that*

$$\operatorname{ord}_p\big(L_p^{\sharp/\flat}(f, \zeta_{p^n} - 1)\big) = \mu_{\sharp/\flat} + \frac{\lambda_{\sharp/\flat}}{p^n - p^{n-1}},$$

and suppose that $n > k$ when $v > 0$, and that we are not in the sporadic case. Then

$$\operatorname{ord}_p\left(\tau(\chi)\frac{L(f, \chi^{-1}, 1)}{\Omega_f}\right) = \mu_* + \frac{1}{p^n - p^{n-1}}\big(\lambda_* + q_n^*(v, v_2)\big),$$

and $* \in \{\sharp, \flat\}$ is chosen according to the modesty algorithm (Definition 8.4).

See Figure 1 in the introduction for an illustration of this theorem.

Proof. Let p be odd, since the other case is similar. Letting $\chi(\gamma)=\zeta_{p^n}$, the interpolation property implies

$$L_p(f,\alpha,\zeta_{p^n}-1)=\frac{1}{\alpha^{n+1}}\frac{p^{n+1}}{\tau(\chi^{-1})}\frac{L(f,\chi^{-1},1)}{\Omega_f}.$$

Now $\alpha^{n+1}L_p(f,\alpha,\zeta_{p^n}-1)$ has the desired p-adic valuation by Proposition 8.12 below and Theorem 2.14. □

For $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, let $\mu^\sigma_{\sharp/\flat}$ and $\lambda^\sigma_{\sharp/\flat}$ be the μ- and λ-invariants of $L_p^{\sharp/\flat}(f^\sigma,T)$, and let $v^\sigma=v(a_p^\sigma)$ and $v_2^\sigma=v_2(a_p^\sigma)$. For $v^\sigma=0$, put $q_n^\natural(v^\sigma,v_2^\sigma)=0$ and let $\mu^\sigma_\natural$ and $\lambda^\sigma_\natural$ be the μ- and λ-invariants of $L_p(f^\sigma,\alpha,T)$.

Corollary 8.6. *Let $p^{e_n}:=\text{Ш}^{\mathrm{an}}(A_f/\mathbb{Q}_n)[p^\infty]$. Suppose we are not in the sporadic case for any pair v^σ, v_2^σ with $v^\sigma>0$. Then for $n\gg 0$,*

$$e_n-e_{n-1}=\sum_{\sigma\in\mathcal{G}_f}\mu^\sigma_*(p^n-p^{n-1})+\lambda^\sigma_*+q_n^*(v^\sigma,v_2^\sigma)-r_\infty^{\mathrm{an}}(f^\sigma),$$

where $*\in\{\sharp,\flat\}$ is chosen according to the modesty algorithm (Definition 8.4), *except when $v^\sigma=0$* (and we are in the sporadic case), *in which case $*:=\natural$.*

Proof. This follows from Theorem 8.5 in the same way that [Pollack 2003, Proposition 6.10] follows from [Pollack 2003, Proposition 6.9(3)]: The idea is to pick n large enough that $\mathrm{ord}_p(\#A_f(\mathbb{Q}_n))=\mathrm{ord}_p(\#A_f(\mathbb{Q}_{n-1}))$, $L(A_f,\chi,1)\neq 0$ for χ of order p^n, and $\mathrm{ord}_p(\mathrm{Tam}(A_f/\mathbb{Q}_n))=\mathrm{ord}_p(\mathrm{Tam}(A_f/\mathbb{Q}_{n-1}))$. Noting that $R(A_f/\mathbb{Q}_n)=p^{r_n}R(A_f/\mathbb{Q}_{n-1})$ and by computing $D(\mathbb{Q}_n)$,

$$\begin{aligned}e_n-e_{n-1}&=\mathrm{ord}_p\Bigg(\prod_{\chi\text{ of order }p^n}\frac{L(A_f/\mathbb{Q},\chi^{-1},1)}{\Omega_{A_f/\mathbb{Q}}}\Bigg)+p^{n-1}(p-1)\cdot\frac{n+1}{2}-r_\infty^{\mathrm{an}}\\&=\mathrm{ord}_p\Bigg(\prod_{\chi\text{ of order }p^n}\tau(\chi)\frac{L(A_f/\mathbb{Q},\chi^{-1},1)}{\Omega_{A_f/\mathbb{Q}}}\Bigg)-r_\infty^{\mathrm{an}}\\&=\mathrm{ord}_p\Bigg(\prod_{\chi\text{ of order }p^n}\tau(\chi)\prod_{\sigma\in\mathcal{G}_f}\frac{L(f^\sigma,\chi^{-1},1)}{\Omega_{f^\sigma}}\Bigg)-\sum_{\sigma\in\mathcal{G}_f}r_\infty^{\mathrm{an}}(f^\sigma).\end{aligned}$$ □

Corollary 8.7. *If A_f is an elliptic curve*, $\mathrm{ord}_p(L(A_f,1)/\Omega_{A_f})=0$, $a_p\not\equiv 1 \bmod p$, *p is odd, and $p\nmid\mathrm{Tam}(A_f/\mathbb{Q}_n)$, then $e_0=e_1=0=\mu_{\sharp/\flat}=\lambda_{\sharp/\flat}=r_\infty^{\mathrm{an}}$ and the above formulas are valid for $n\geqslant 2$.*

Proof. We can pick $n=0$ by [Kurihara 2002, Proposition 1.2] and the arguments of its proof, invoking [Greenberg 1999, Proposition 3.8] and Theorem 5.15. □

Definition 8.8 (the invariants $\mu_\pm$ and $\lambda_\pm$). Perrin-Riou, (resp. Greenberg, Iovita, and Pollack) defined invariants $\mu_\pm$ (resp. $\lambda_\pm$) as follows. Let p be a supersingular odd prime.[6] Let $(Q_n)_n \in \Lambda_n$ be a queue sequence. Let π be a generator of the maximal ideal of $\mathcal{O}$ such that $\pi^m = p$. When $Q_n \neq 0$, we define $\mu'(Q_n)$ to be the unique integer such that

$$Q_n \in (\pi)^{\mu'(Q_n)}\Lambda_n - (\pi)^{\mu'(Q_n)+1}\Lambda_n.$$

Further, we let $\lambda(Q_n)$ be the unique integer such that

$$\pi^{-\mu'(Q_n)} Q_n \bmod \pi \in \tilde{I}_n^{\lambda(Q_n)} - \tilde{I}_n^{\lambda(Q_n)+1},$$

where $\tilde{I}_n$ is the augmentation ideal of $\mathcal{O}/\pi\mathcal{O}[\Gamma_n]$. Finally, we put $\mu(Q_n) := m\mu'(Q_n)$. Then for even (resp. odd) n, $\mu(Q_n)$ stabilizes to a minimum constant value μ_+ (resp. μ_-). When $\mu_+ = \mu_-$, put

$$\lambda_+ := \lim_{n\to\infty} \lambda(Q_{2n}) - q_{2n}^{\flat} \quad \text{and} \quad \lambda_- := \lim_{n\to\infty} \lambda(Q_{2n+1}) - q_{2n+1}^{\sharp}.$$

Corollary 8.9. *When $\mu_\sharp$ and $\lambda_\sharp$ (resp. $\mu_\flat$ and $\lambda_\flat$) appear in the estimates of Theorem 8.5, they are the Iwasawa invariants of $\widehat{L}_p^\sharp$ (resp. $\widehat{L}_p^\flat$). When $\mu_\sharp = \mu_\flat$, we define $\mu_\pm$ and $\lambda_\pm$ via the queue sequences that gave rise to $L_p^\sharp$ and $L_p^\flat$, and have*

$$\mu_\sharp = \mu_+, \quad \lambda_\sharp = \lambda_+, \quad \mu_\flat = \mu_-, \quad \textit{and} \quad \lambda_\flat = \lambda_-.$$

Proof. The Kurihara terms $q_n^*(v, v_2)$ come from appropriate valuation matrices of $\mathcal{L}og_{\alpha,\beta}$ and $\widehat{\mathcal{L}og}_{\alpha,\beta}$, which are the same. Thus, the Iwasawa invariants of $\widehat{L}_p^{\sharp/\flat}$ and of $L_p^{\sharp/\flat}$ match. We can calculate the p-primary part of the special value in Theorem 8.5 using the appropriate queue sequences.[7] Since $\mu_+ = \mu_-$, we are a posteriori not in the sporadic case, so that our formulas match. □

Remarks in the elliptic curve case. For the remainder of this subsection, assume $A_f = E$ is an elliptic curve. Then in the supersingular case, Corollary 8.6 generalizes [Pollack 2003, Proposition 6.10], which works under the assumption $a_p = 0$. For an algebraic version of this Corollary 8.6 for supersingular primes, see [Kobayashi 2003, Theorem 10.9] in the case $a_p = 0$ and odd p, and [Sprung 2013, Theorem 3.13] for any odd supersingular prime.

Remark 8.10. These formulas are compatible with Perrin-Riou's [2003] formulas. Note that she assumes that p is odd, and that $\mu_+ = \mu_-$ or $a_p = 0$ in [Perrin-Riou 2003, Theorem 6.1(4)]; see also [Sprung 2013, Theorem 5.1] . Her invariants match ours, by Corollary 8.9. For $p = 2$, our results are compatible with [Kurihara and Otsuki 2006, Theorem 0.1(2)] (which determines the structure of the 2-primary

[6] Perrin-Riou makes the assumption that p is odd. For $p = 2$, one could define the $\mu_\pm$ and $\lambda_\pm$ in the same way but switch the signs so that they agree with the Iwasawa invariants of $L_p^\pm$ in the case $a_p = 0$.

[7] This has been explicitly done in an unpublished preprint of Greenberg, Iovita, and Pollack.

component of $\text{Ш}(A_f/\mathbb{Q}_n)$ under the assumption $a_2 = \pm 2$ in the elliptic curve case and other conditions, which force the Iwasawa invariants to vanish).

In the ordinary case, the estimate for $n \gg 0$ is

$$e_n - e_{n-1} = (p^n - p^{n-1})\mu + \lambda - r_\infty^{\mathrm{an}},$$

where μ and λ are the Iwasawa invariants of $L_p(E, \alpha, T)$. Thus, we obtain:

Corollary 8.11. *In the ordinary case, let λ be the λ-invariant of $L_p(E, \alpha, T)$. Then*

$$\lambda = \begin{cases} \lambda_\sharp & \text{when } \mu_\sharp < \mu_\flat \quad \text{or} \quad \mu_\sharp = \mu_\flat \text{ and } \lambda_\sharp < \lambda_\flat + p - 1, \\ \lambda_\flat & \text{when } \mu_\flat < \mu_\sharp \quad \text{or} \quad \mu_\flat = \mu_\sharp \text{ and } \lambda_\flat < \lambda_\sharp + 1 - p. \end{cases}$$

Tools for the proof of Theorem 8.5.

Proposition 8.12. *Suppose we have $(L^\sharp(T), L^\flat(T)) \in \mathcal{O}[\![T]\!]^{\oplus 2}$, where $\mathcal{O}$ is the ring of integers of some finite extension of $\mathbb{Q}_p$. Rewrite $L^\sharp(T) := p^{\mu_\sharp} \times P^\sharp(T) \times U^\sharp(T)$ for a distinguished polynomial $P^\sharp(T)$ with λ-invariant $\lambda_\sharp$ and a unit $U^\sharp(T)$. Note that $\mu_\sharp \in \mathbb{Q}$. Rewrite $L^\flat(T)$ similarly to extract $\mu_\flat$ and $\lambda_\flat$. Suppose we are not in the sporadic case. Let a and k be as in Definition 8.2, and e_n the left entry of the* 1×2 *valuation matrix of*

$$\big(L^\sharp(\zeta_{p^n} - 1), L^\flat(\zeta_{p^n} - 1)\big)\mathcal{H}_a^{n-1}(\zeta_{p^n} - 1).$$

Then for n large enough so that $n > k$ and $\mathrm{ord}_p(L^{\sharp/\flat}(\zeta_{p^n} - 1)) = \mu_{\sharp/\flat} + \dfrac{\lambda_{\sharp/\flat}}{p^n - p^{n-1}}$, *we have*

$$e_n = \mu_* + \frac{\lambda_*}{p^n - p^{n-1}} + \frac{q_n^*(v, v_2)}{p^n - p^{n-1}},$$

where $* \in \{\sharp, \flat\}$ is chosen according to the modesty algorithm.

Proof. From Lemma 8.14 and Lemma 8.15 below, it follows that the valuation matrix of the above expression is a product (of valuation matrices) of the form

$$\left[\mu_\sharp + \frac{\lambda_\sharp}{p^n - p^{n-1}}, \mu_\flat + \frac{\lambda_\flat}{p^n - p^{n-1}}\right] \begin{bmatrix} \dfrac{q_n^\sharp(v, v_2)}{p^n - p^{n-1}} & * \\ \dfrac{q_n^\flat(v, v_2)}{p^n - p^{n-1}} & * \end{bmatrix},$$

except when $v = p^{-k}/2$ and $v_2 = p^{-k}(1 + p^{-1} - p^{-2})$, in which case one of the two entries shown in the right valuation matrix is the actual entry, while the other is a lower estimate, see Lemma 8.15. The leading term of $P^{\sharp/\flat}(T)$ dominates by assumption, so the modesty algorithm (Definition 8.4) chooses the correct subindex. □

Lemma 8.13. *When $v>0$ and $n>k+3$, the valuation matrix $[\mathcal{H}_a^{n-k-2}(\zeta_{p^n}-1)]$ is*

$$\begin{cases}\begin{bmatrix} p^{2-n}+p^{4-n}+p^{6-n}+\cdots+p^{-k-2} & v+p^{2-n}+\cdots+p^{-k-4} \\ v+p^{1-n}+\cdots+p^{-k-3} & p^{1-n}+\cdots+p^{-k-3}\end{bmatrix} & \text{if } n\equiv k \bmod 2,\\ \begin{bmatrix} v+p^{2-n}+\cdots+p^{-k-3} & p^{2-n}+\cdots+p^{-k-3} \\ p^{1-n}+\cdots+p^{-k-2} & v+p^{1-n}+\cdots+p^{-k-4}\end{bmatrix} & \text{if } n\not\equiv k \bmod 2.\end{cases}$$

Proof. Multiplication of valuation matrices and induction. □

Lemma 8.14. *With notation as above, assume $v=0$. Then*

$$\left[\mathcal{H}_a^{n-1}(\zeta_{p^n}-1)\right]=\begin{bmatrix} 0 & 0 \\ p^{1-n} & p^{1-n}\end{bmatrix}.$$

Proof. Multiplication of valuation matrices. □

Given a real number x, recall that "$\geqslant x$" denotes an unknown quantity greater than or equal to x.

Lemma 8.15. *When $v>0$ and $n>k$, we have*

$$(p^n-p^{n-1})[\mathcal{H}_a^{n-1}(\zeta_{p^n}-1)]=\begin{bmatrix} q_n^\sharp(v,v_2) & q_n^\sharp(v,v_2)-v \\ q_n^\flat(v,v_2) & q_n^\flat(v,v_2)-v\end{bmatrix},$$

unless $v=\frac{1}{2}p^{-k}$ *and* $v_2=2v(1+p^{-1}-p^{-2})$.

When $v=\frac{1}{2}p^{-k}$ and $v_2=2v(1+p^{-1}-p^{-2})$, we have

$$(p^n-p^{n-1})[\mathcal{H}_a^{n-1}(\zeta_{p^n}-1)] = \begin{cases}\begin{bmatrix} \geqslant q_n^\sharp(v,v_2) & \geqslant q_n^\sharp(v,v_2)-v \\ q_n^\flat(v,v_2) & q_n^\flat(v,v_2)-v\end{bmatrix} & \text{when } n\equiv k \bmod 2,\\ \begin{bmatrix} q_n^\sharp(v,v_2) & q_n^\sharp(v,v_2)-v \\ \geqslant q_n^\flat(v,v_2) & \geqslant q_n^\flat(v,v_2)-v\end{bmatrix} & \text{when } n\not\equiv k \bmod 2.\end{cases}$$

Proof. We give the proof for the case $n\equiv k \bmod 2$ and $n\geqslant k+4$. (The case where $n\not\equiv k \bmod 2$ and $n\geqslant 5$ is similar, and the excluded cases are easier variants of these calculations.[8]) We have

$$\mathcal{H}_a^{n-1}(\zeta_{p^n}-1)=\mathcal{H}_a^{n-k-2}(\zeta_{p^n}-1)\mathcal{H}_a^{k+1}(\zeta_{p^n}^{p^{n-k-2}}-1),$$

[8]For $n=k+1$, we directly verify $[\mathcal{H}_a^k(\zeta_{p^{k+1}}-1)]=\left[\begin{smallmatrix} kv & (k-1)v \\ (k-1)v+p^{-k} & (k-2)+p^{-k}\end{smallmatrix}\right]$.

whose valuation matrix is the product of valuation matrices:

$$\left(\begin{bmatrix} \dfrac{p^{-k}}{p^2-1} & v+\dfrac{p^{-k-2}}{p^2-1} \\ v+\dfrac{p^{-k-1}}{p^2-1} & \dfrac{p^{-k-1}}{p^2-1} \end{bmatrix} - \begin{bmatrix} \dfrac{p^{2-n}}{p^2-1} & \dfrac{p^{2-n}}{p^2-1} \\ \dfrac{p^{1-n}}{p^2-1} & \dfrac{p^{1-n}}{p^2-1} \end{bmatrix}\right) \begin{bmatrix} v_{k+1} & v_k \\ kv+p^{-1-k} & (k-1)v+p^{-1-k} \end{bmatrix},$$

by Lemma 8.13, where the lower entries in the last valuation matrix are calculated by induction just as in Lemma 8.13 above. The first column of $[\mathcal{H}_a^{n-1}(\zeta_{p^n}-1)]$ is

$$\begin{bmatrix} \min(v_{k+1},\,(k+1)v+p^{-1-k}-p^{-k-2})+\dfrac{p^{-k}-p^{2-n}}{p^2-1} \\ \min(v_{k+1},\,(k-1)v+p^{-1-k})+v+\dfrac{p^{-k-1}-p^{1-n}}{p^2-1}, \end{bmatrix}.$$

as long as the two terms involved in $\min(\cdot\,,\cdot)$ are different.

If $2v>p^{-k}$, we have $v_{k+1}=(k-1)v+p^{-k}$, so the first column of $[\mathcal{H}_a^{n-1}(\zeta_{p^n}-1)]$ is

$$\begin{bmatrix} (k-1)v+p^{-k}+\dfrac{p^{-k}-p^{2-n}}{p^2-1} \\ kv+p^{-1-k}+\dfrac{p^{-k-1}-p^{1-n}}{p^2-1} \end{bmatrix}.$$

The difficult part is the case $2v=p^{-k}$. From the lemma below, we find that the expression for the lower term is the same as when $2v>p^{-k}$.

We claim that the upper term is the same as well (i.e., the minimum is v_{k+1}) when $v_2<p^{-k}(1+1/p-1/p^2)$, while the minimum is the other term when $v_2>p^{-k}(1+1/p-1/p^2)$. For $v_2\geqslant p^{1-k}$, this follows at once from the below lemma, since $v_{k+1}\geqslant (k-1)v+p^{1-k}$; so the real difficulty is when $p^{1-k}>v_2\geqslant p^{-k}$: Here, the lemma below tells us that $v_{k+1}=v_2+(k-1)p^{-k}/2$, from which we obtain our claim. Note that when $v_2=p^{-k}(1+1/p-1/p^2)$, we obtain our desired inequality. □

Lemma 8.16. *In the above situation, let $m\geqslant 2$. We then have $v_m=(m-2)v+v_2$ when $v_2<p^{1-k}$ and $v_m\geqslant (m-2)v+p^{1-k}$ if not.*

Proof. Explicit decomposition of the valuation matrix of $\mathcal{H}_a^k(\zeta_{p^n}^{p^{n-k-1}}-1)$. □

Acknowledgments

We are grateful to our advisors, Barry Mazur, Robert Pollack, and Joseph Silverman, for many inspiring conversations and many pieces of advice in the writing of this paper, and for making the author a (better) mathematician. We also thank Christian Wuthrich for a helpful conversation about regulators at the Iwasawa 2012 conference, and Tadashi Ochiai and Daniel Disegni for pointing out inaccuracies in an earlier version of this paper. We thank J. Pottharst for enlightening comments, and Diana

Davis and Maxime Bourque for help with typesetting the picture. We are also grateful to the anonymous referees and the typesetter for improving our paper.

References

[Abbes and Ullmo 1996] A. Abbes and E. Ullmo, "À propos de la conjecture de Manin pour les courbes elliptiques modulaires", *Compositio Math.* **103**:3 (1996), 269–286. MR Zbl

[Amice and Vélu 1975] Y. Amice and J. Vélu, "Distributions p-adiques associées aux séries de Hecke", pp. 119–131 in *Journées arithmétiques de Bordeaux* (Bordeaux, 1974), Astérisque **24-25**, Soc. Math. France, Paris, 1975. MR Zbl

[Balakrishnan et al. 2016] J. S. Balakrishnan, J. S. Müller, and W. A. Stein, "A p-adic analogue of the conjecture of Birch and Swinnerton-Dyer for modular abelian varieties", *Math. Comp.* **85**:298 (2016), 983–1016. MR Zbl

[Berger et al. 2004] L. Berger, H. Li, and H. J. Zhu, "Construction of some families of 2-dimensional crystalline representations", *Math. Ann.* **329**:2 (2004), 365–377. MR Zbl

[Breuil et al. 2001] C. Breuil, B. Conrad, F. Diamond, and R. Taylor, "On the modularity of elliptic curves over $\mathbb{Q}$: wild 3-adic exercises", *J. Amer. Math. Soc.* **14**:4 (2001), 843–939. MR Zbl

[Coates and Sujatha 2000] J. Coates and R. Sujatha, *Galois cohomology of elliptic curves*, Tata Institute of Fundamental Research Lectures on Mathematics **88**, Narosa, New Delhi, 2000. MR Zbl

[Colmez 2004] P. Colmez, "La conjecture de Birch et Swinnerton-Dyer p-adique", exposé 919, pp. 251–319 in *Séminaire Bourbaki 2002/2003*, Astérisque **294**, Soc. Math. France, Paris, 2004. MR Zbl

[Delbourgo 1998] D. Delbourgo, "Iwasawa theory for elliptic curves at unstable primes", *Compositio Math.* **113**:2 (1998), 123–153. MR Zbl

[Edixhoven 1991] B. Edixhoven, "On the Manin constants of modular elliptic curves", pp. 25–39 in *Arithmetic algebraic geometry* (Texel, 1989), edited by G. van der Geer et al., Progr. Math. **89**, Birkhäuser, Boston, 1991. MR Zbl

[Greenberg 1999] R. Greenberg, "Iwasawa theory for elliptic curves", pp. 51–144 in *Arithmetic theory of elliptic curves* (Cetraro, Italy, 1997), edited by C. Viola, Lecture Notes in Math. **1716**, Springer, Berlin, 1999. MR Zbl

[Greenberg 2001] R. Greenberg, "Iwasawa theory: past and present", pp. 335–385 in *Class field theory: its centenary and prospect* (Tokyo, 1988), edited by K. Miyake, Adv. Stud. Pure Math. **30**, Math. Soc. Japan, Tokyo, 2001. MR Zbl

[Greenberg and Stevens 1994] R. Greenberg and G. Stevens, "On the conjecture of Mazur, Tate, and Teitelbaum", pp. 183–211 in *p-adic monodromy and the Birch and Swinnerton-Dyer conjecture* (Boston, 1991), edited by B. Mazur and G. Stevens, Contemp. Math. **165**, American Mathematical Society, Providence, RI, 1994. MR Zbl

[Kato 2004] K. Kato, "p-adic Hodge theory and values of zeta functions of modular forms", pp. 117–290 in *Cohomologies p-adiques et applications arithmétiques, III*, edited by P. Berthelot et al., Astérisque **295**, Soc. Math. France, Paris, 2004. MR Zbl

[Kim 2008] B. D. Kim, "The algebraic functional equation of an elliptic curve at supersingular primes", *Math. Res. Lett.* **15**:1 (2008), 83–94. MR Zbl

[Kobayashi 2003] S.-i. Kobayashi, "Iwasawa theory for elliptic curves at supersingular primes", *Invent. Math.* **152**:1 (2003), 1–36. MR Zbl

[Kurihara 2002] M. Kurihara, "On the Tate–Shafarevich groups over cyclotomic fields of an elliptic curve with supersingular reduction, I", *Invent. Math.* **149**:1 (2002), 195–224. MR Zbl

[Kurihara and Otsuki 2006] M. Kurihara and R. Otsuki, "On the growth of Selmer groups of an elliptic curve with supersingular reduction in the $\mathbb{Z}_2$-extension of $\mathbb{Q}$", *Pure Appl. Math. Q.* **2**:2, part 2 (2006), 557–568. MR Zbl

[Lei 2014] A. Lei, "Factorisation of two-variable p-adic L-functions", *Canad. Math. Bull.* **57**:4 (2014), 845–852. MR Zbl

[Lei et al. 2010] A. Lei, D. Loeffler, and S. L. Zerbes, "Wach modules and Iwasawa theory for modular forms", *Asian J. Math.* **14**:4 (2010), 475–528. MR Zbl

[Loeffler and Zerbes 2013] D. Loeffler and S. L. Zerbes, "Wach modules and critical slope p-adic L-functions", *J. Reine Angew. Math.* **679** (2013), 181–206. MR Zbl

[Manin 1971] Ju. I. Manin, "Cyclotomic fields and modular curves", *Uspehi Mat. Nauk* **26**:6 (1971), 7–71. In Russian; translated in *Russian Math. Surveys* **26**:6 (1971), 7–78. MR Zbl

[Manin 1972] Ju. I. Manin, "Parabolic points and zeta functions of modular curves", *Izv. Akad. Nauk SSSR Ser. Mat.* **36**:1 (1972), 19–66. In Russian; translated in *Math. USSR-Izv.* **6**:1 (1972), 19–64. MR Zbl

[Manin 1973] Ju. I. Manin, "Periods of parabolic forms and p-adic Hecke series", *Mat. Sb. (N.S.)* **92 (134)**:3 (1973), 378–401. In Russian; translation in *Math. USSR-Sb.* **21**:3 (1973) 371–393. MR Zbl

[Mazur and Swinnerton-Dyer 1974] B. Mazur and P. Swinnerton-Dyer, "Arithmetic of Weil curves", *Invent. Math.* **25** (1974), 1–61. MR Zbl

[Mazur et al. 1986] B. Mazur, J. Tate, and J. Teitelbaum, "On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer", *Invent. Math.* **84**:1 (1986), 1–48. MR Zbl

[Perrin-Riou 1990] B. Perrin-Riou, "Théorie d'Iwasawa p-adique locale et globale", *Invent. Math.* **99**:2 (1990), 247–292. MR Zbl

[Perrin-Riou 1993] B. Perrin-Riou, "Fonctions L p-adiques d'une courbe elliptique et points rationnels", *Ann. Inst. Fourier* (*Grenoble*) **43**:4 (1993), 945–995. MR Zbl

[Perrin-Riou 1994] B. Perrin-Riou, "Théorie d'Iwasawa des représentations p-adiques sur un corps local", *Invent. Math.* **115**:1 (1994), 81–161. MR Zbl

[Perrin-Riou 2003] B. Perrin-Riou, "Arithmétique des courbes elliptiques à réduction supersingulière en p", *Experiment. Math.* **12**:2 (2003), 155–186. MR Zbl

[Pollack 2002] R. Pollack, "Tables of Iwasawa invariants of elliptic curves", 2002, Available at http://math.bu.edu/people/rpollack/Data/data.html.

[Pollack 2003] R. Pollack, "On the p-adic L-function of a modular form at a supersingular prime", *Duke Math. J.* **118**:3 (2003), 523–558. MR Zbl

[Pollack and Weston 2011] R. Pollack and T. Weston, "Mazur–Tate elements of nonordinary modular forms", *Duke Math. J.* **156**:3 (2011), 349–385. MR Zbl

[Pottharst 2012] J. Pottharst, "Cyclotomic Iwasawa theory of motives", submitted, 2012, Available at http://tinyurl.com/cyclotomic.

[Rohrlich 1984] D. E. Rohrlich, "On L-functions of elliptic curves and cyclotomic towers", *Invent. Math.* **75**:3 (1984), 409–423. MR Zbl

[Rubin 1991] K. Rubin, "The "main conjectures" of Iwasawa theory for imaginary quadratic fields", *Invent. Math.* **103**:1 (1991), 25–68. MR

[Skinner and Urban 2014] C. Skinner and E. Urban, "The Iwasawa main conjectures for GL_2", *Invent. Math.* **195**:1 (2014), 1–277. MR Zbl

[Sprung 2012] F. E. I. Sprung, "Iwasawa theory for elliptic curves at supersingular primes: a pair of main conjectures", *J. Number Theory* **132**:7 (2012), 1483–1506. MR Zbl

[Sprung 2013] F. Sprung, "The Šafarevič–Tate group in cyclotomic $\mathbb{Z}_p$-extensions at supersingular primes", *J. Reine Angew. Math.* **681** (2013), 199–218. MR Zbl

[Sprung 2015] F. Sprung, "A formulation of p-adic versions of the Birch and Swinnerton-Dyer conjectures in the supersingular case", *Res. Number Theory* **1** (2015), Art. 17, 13. MR

[Stein and Wuthrich 2013] W. Stein and C. Wuthrich, "Algorithms for the arithmetic of elliptic curves using Iwasawa theory", *Math. Comp.* **82**:283 (2013), 1757–1792. MR Zbl

[Višik 1976] M. M. Višik, "Nonarchimedean measures associated with Dirichlet series", *Mat. Sb. (N.S.)* **99(141)**:2 (1976), 248–260. In Russian; translated in *Math. USSR-Sb.* **28**:2 (1976), 216–228. MR Zbl

[Washington 1982] L. C. Washington, *Introduction to cyclotomic fields*, Graduate Texts in Mathematics **83**, Springer, New York, 1982. MR Zbl

[Wiles 1995] A. Wiles, "Modular elliptic curves and Fermat's last theorem", *Ann. of Math.* (2) **141**:3 (1995), 443–551. MR Zbl

Communicated by Christopher Skinner
Received 2016-04-10 Revised 2016-12-16 Accepted 2017-01-13

fsprung@princeton.edu *School of Mathematics, Institute for Advanced Study & Princeton University, 1 Einstein Dr, Princeton, NJ 08540, United States*

ALGEBRA AND NUMBER THEORY 11:4 (2017)
dx.doi.org/10.2140/ant.2017.11.929

On Hilbert's 17th problem in low degree

Olivier Benoist

Artin solved Hilbert's 17th problem, proving that a real polynomial in n variables that is positive semidefinite is a sum of squares of rational functions, and Pfister showed that only 2^n squares are needed.

In this paper, we investigate situations where Pfister's theorem may be improved. We show that a real polynomial of degree d in n variables that is positive semidefinite is a sum of $2^n - 1$ squares of rational functions if $d \leq 2n - 2$. If n is even or equal to 3 or 5, this result also holds for $d = 2n$.

Introduction

Hilbert's 17th problem. Let $\boldsymbol{R}$ be a real closed field, for instance the field $\mathbb{R}$ of real numbers, and let $n \geq 1$. A polynomial $f \in \boldsymbol{R}[X_1, \dots, X_n]$ is said to be positive semidefinite if $f(x_1, \dots, x_n) \geq 0$ for all $x_1, \dots, x_n \in \boldsymbol{R}$. As an odd degree polynomial changes sign, such a polynomial has even degree.

Artin [1927] answered Hilbert's 17th problem by proving that a positive semidefinite polynomial $f \in \boldsymbol{R}[X_1, \dots, X_n]$ is a sum of squares of rational functions.[1] This theorem was later improved by Pfister [1967, Theorem 1], who showed that it is actually the sum of 2^n squares of rational functions. We refer to [Pfister 1995, Chapter 6] for a nice account of these classical results.

In two variables, the situation is very well understood. Hilbert [1888] showed that a positive semidefinite polynomial $f \in \boldsymbol{R}[X_1, X_2]$ of degree ≤ 4 is a sum of 3 squares of rational functions,[2] and Cassels, Ellison and Pfister [Cassels et al. 1971] gave an example of a positive semidefinite polynomial $f \in \boldsymbol{R}[X_1, X_2]$ of degree 6 that is not a sum of 3 squares of rational functions.

Our goal is to prove an analogue of Hilbert's result — that in low degree, less squares are needed — in more than two variables:

MSC2010: primary 11E25; secondary 14F20, 14P99.

Keywords: Hilbert's 17th problem, sums of squares, real algebraic geometry, Bloch–Ogus theory.

[1] Hilbert himself [1888] had given examples of positive semidefinite polynomials that are not sums of squares of polynomials.

[2] In fact, in this exceptional case, Hilbert actually showed that squares of polynomials suffice. We will not consider this question in what follows, and refer the interested reader to [Pfister and Scheiderer 2012].

Theorem 0.1. *Let $f \in \boldsymbol{R}[X_1, \dots, X_n]$ be a positive semidefinite polynomial of degree d. Suppose that one of the following holds*:

(i) $d \leq 2n - 2$.

(ii) $d = 2n$, *and either* n *is even*, $n = 3$, *or* $n = 5$.

Then f *is a sum of* $2^n - 1$ *squares in* $\boldsymbol{R}(X_1, \dots, X_n)$.

Of course, when $d = 2$, the classification of quadratic forms over $\boldsymbol{R}$ shows the much stronger result that $n + 1$ squares are enough. However, to the best of our knowledge, our theorem is already new for $d = 4$ and $n \geq 3$.

Dependence on the degree. The question whether the bound 2^n in Pfister's aforementioned theorem is optimal is natural and well known [Pfister 1971, §4, Problem 1]. It is often formulated in the following equivalent way, where the Pythagoras number $p(K)$ of a field K is the smallest number p such that every sum of squares in K is a sum of p squares:

Question 0.2. Do we have $p(\boldsymbol{R}(X_1, \dots, X_n)) = 2^n$?

When $n \geq 2$, the best known result is that $n + 2 \leq p(\boldsymbol{R}(X_1, \dots, X_n)) \leq 2^n$ [Pfister 1995, p. 97], where the upper bound is Pfister's theorem and the lower bound is an easy consequence of the Cassels–Ellison–Pfister theorem.

Our main theorem does not address this question directly; it explores the opposite direction, that is, the values of the degree for which Pfister's bound may be improved. However, Theorem 0.1 gives insights into Question 0.2. The bound $d \leq 2n$ has a natural geometric origin (it reflects the rational connectedness of an associated algebraic variety), and it would be natural to expect that Theorem 0.1 cannot be extended to degrees $d \geq 2n + 2$.

In view of Theorem 0.1, it is natural to ask whether the bound $d \leq 2n - 2$ may be improved to $d \leq 2n$ for every odd value of n. When $n = 1$, this is not the case because $X_1^2 + 1$ is not a square. On the other hand, when $n \geq 3$ is odd, we reduce this question to a geometric coniveau estimate (Proposition 6.3). When $n = 3$, it is very easy to check. We also verify it when $n = 5$, following an argument of Voisin. This explains the hypotheses on the degree in Theorem 0.1.

Strategy of the proof. In two variables, the theorems of Hilbert and Cassels–Ellison–Pfister quoted above have received geometric proofs by Colliot-Thélène [1992, Remark 2; 1993]. His idea is to consider the homogenization F of f and to introduce the algebraic surface $Y := \{Z^2 + F = 0\}$. Then, whether or not f may be written as a sum of three squares in $\boldsymbol{R}(X_1, X_2)$ depends on the injectivity of the map $\mathrm{Br}(\boldsymbol{R}) \to \mathrm{Br}(\boldsymbol{R}(Y))$, which may be studied by geometrical methods.

We follow the same strategy in more variables. Proposition 3.2 and Proposition 3.3 translate the property that f is a sum of $2^n - 1$ squares in $\boldsymbol{R}(X_1, \dots, X_n)$ into a cohomological property of (a resolution of singularities of) the variety Y. The group that plays a role analogous to that of the Brauer group in two variables is a degree n unramified cohomology group.

It remains to show that when the degree of f is small, some class in a degree n unramified cohomology group vanishes. This is more difficult than the corresponding result in two variables, as these groups are harder to control than Brauer groups. Our main tool to achieve this is Bloch–Ogus theory.

Structure of the paper. The first two sections gather general cohomological results for varieties over $\boldsymbol{R}$, which are used throughout the text. It will be very important for us to use cohomology with integral coefficients (as opposed to 2-torsion coefficients). For this reason, Section 1 is devoted to general properties of the 2-adic cohomology[3] of varieties over $\boldsymbol{R}$.

In Section 2, we recall the basics of Bloch–Ogus theory, then focus on the specific properties of it over real closed fields. In particular, we adapt to our needs a strategy of Colliot-Thélène and Scheiderer [1996] to compare the Bloch–Ogus theory of a variety over $\boldsymbol{R}$ and over the algebraic closure $\boldsymbol{C}$ of $\boldsymbol{R}$, and explain in our context consequences of the Bloch–Kato conjectures discovered by Bloch and Srinivas [1983] and extended by Colliot-Thélène and Voisin [2012].

We study when a positive semidefinite polynomial $f \in \boldsymbol{R}[X_1, \dots, X_n]$ is a sum of $2^n - 1$ squares of rational functions in Section 3. We successively relate this property to the level of the function field $\boldsymbol{R}(Y)$ of the variety $Y := \{Z^2 + F = 0\}$ in Proposition 3.2 (this is due to Pfister), to degree n unramified cohomology of Y in Proposition 3.3 (an important tool is Voevodsky's solution [2003] to the Milnor conjecture) and to degree $n+1$ cohomology of Y in Proposition 3.5 (this is the crucial step, which uses Bloch–Ogus theory, and where the rational connectedness of Y plays a role).

Section 4 contains the cohomological computations on the variety Y that are relevant to apply the results of Section 3. Section 4D will only be useful when n is odd and $d = 2n$, and is complemented by a geometric coniveau estimate in Section 5. The reader who is not interested in our partial and conditional results when $n \geq 3$ is odd and $d = 2n$ may skip them.

Section 6 completes the proof of Theorem 0.1. For a generic choice of f (that is, when the degree of f is maximal among the values allowed in the statement of Theorem 0.1, and Y is a smooth variety), this is an immediate consequence of the results obtained so far. In general, we do not know how to apply this argument directly, because we do not have a good control on the geometry of (a resolution of singularities of) Y. Instead, we rely on a specialization argument. This argument

[3]It would also have been possible to work with equivariant Betti cohomology over the field $\mathbb{R}$ of real numbers [Krasnov 1994], and with its semialgebraic counterpart over a general real closed field.

reduces Theorem 0.1 to the generic case, but over a bigger real closed field. In particular, even if one is only interested in proving Theorem 0.1 over $\mathbb{R}$, one has to work over real closed fields that are not necessarily archimedean.

1. Cohomology of real varieties

Let $\boldsymbol{R}$ be a real closed field and $\boldsymbol{C}$ be an algebraic closure of $\boldsymbol{R}$. We denote the Galois group by $G := \mathrm{Gal}(\boldsymbol{C}/\boldsymbol{R}) \simeq \mathbb{Z}/2\mathbb{Z}$. A variety over $\boldsymbol{R}$ is a separated scheme of finite type over $\boldsymbol{R}$.

1A. *2-adic cohomology.* If X is a variety over $\boldsymbol{R}$, we denote by $H^k(X, \mathbb{Z}/2^r\mathbb{Z}(j))$ its étale cohomology groups. These cohomology groups are finite; this follows from the Hochschild–Serre spectral sequence

$$E_2^{p,q} = H^p(G, H^q(X_{\boldsymbol{C}}, \mathbb{Z}/2^r\mathbb{Z}(j))) \Rightarrow H^{p+q}(X, \mathbb{Z}/2^r\mathbb{Z}(j))$$

using the facts that $X_{\boldsymbol{C}}$ has finite cohomological dimension [SGA 4_3 1973, X Corollaire 4.3], that the groups $H^q(X_{\boldsymbol{C}}, \mathbb{Z}/2^r\mathbb{Z}(j))$ are finite [SGA 4_3 1973, XVI Théorème 5.1] and that a finite G-module has finite cohomology.

Let us define $H^k(X, \mathbb{Z}_2(j)) := \varprojlim_r H^k(X, \mathbb{Z}/2^r\mathbb{Z}(j))$. Since the Galois cohomology of finite G-modules is finite, [Jannsen 1988, Remark 3.5(c)] shows that these groups coincide with the continuous étale cohomology groups defined by Jannsen. In particular, we have a Hochschild–Serre spectral sequence [Jannsen 1988, Remark 3.5(b)]:

$$E_2^{p,q} = H^p(G, H^q(X_{\boldsymbol{C}}, \mathbb{Z}_2(j))) \Rightarrow H^{p+q}(X, \mathbb{Z}_2(j)). \tag{1-1}$$

We also freely use the cup-products, cohomology groups with support, cycle class maps and Gysin morphisms defined by Jannsen [1988].

Note that since $G = \mathbb{Z}/2\mathbb{Z}$, the sheaves $\mathbb{Z}/2^r\mathbb{Z}(j)$ only depend on the parity of j, hence so do all the cohomology groups considered above.

Let ω be the generator of $H^1(\boldsymbol{R}, \mathbb{Z}_2(1)) \simeq \mathbb{Z}/2\mathbb{Z}$. We denote also by ω its reduction modulo 2: the generator of $H^1(\boldsymbol{R}, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$. If $k \geq 1$, their powers ω^k generate $H^k(\boldsymbol{R}, \mathbb{Z}_2(k)) \simeq \mathbb{Z}/2\mathbb{Z}$ and $H^k(\boldsymbol{R}, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$, and we still denote by ω^k their pull-backs to any variety X over $\boldsymbol{R}$.

1B. *Comparison with geometric cohomology.* Let $\pi : \mathrm{Spec}(\boldsymbol{C}) \to \mathrm{Spec}(\boldsymbol{R})$ be the base-change morphism, and fix $j \in \mathbb{Z}$. There is a natural short exact sequence of étale sheaves on $\mathrm{Spec}(\boldsymbol{R})$: $0 \to \mathbb{Z}/2^r\mathbb{Z}(j) \to \pi_*\mathbb{Z}/2^r\mathbb{Z} \to \mathbb{Z}/2^r\mathbb{Z}(j+1) \to 0$, as one checks at the level of G-modules. They fit together to form a short exact sequence of 2-adic sheaves on $\mathrm{Spec}(\boldsymbol{R})$: $0 \to \mathbb{Z}_2(j) \to \pi_*\mathbb{Z}_2 \to \mathbb{Z}_2(j+1) \to 0$.

Let X be a variety over $\boldsymbol{R}$, and let us still denote by $\pi : X_{\boldsymbol{C}} \to X$ the base-change morphism. Notice that by the Leray spectral sequence, we have $H^k(X, \pi_*\mathbb{Z}/2^r\mathbb{Z}) =$

$H^k(X_{\boldsymbol{C}}, \mathbb{Z}/2^r\mathbb{Z})$. Now pull-back the exact sequence of 2-adic sheaves $0 \to \mathbb{Z}_2(j) \to \pi_*\mathbb{Z}_2 \to \mathbb{Z}_2(j+1) \to 0$ on X and take continuous étale cohomology. We obtain a long exact sequence

$$\cdots \to H^k(X, \mathbb{Z}_2(j)) \xrightarrow{\pi^*} H^k(X_{\boldsymbol{C}}, \mathbb{Z}_2) \xrightarrow{\pi_*} H^k(X, \mathbb{Z}_2(j+1)) \xrightarrow{\omega} H^{k+1}(X, \mathbb{Z}_2(j)) \to \cdots \quad (1\text{-}2)$$

in which the boundary map $H^k(X, \mathbb{Z}_2(j+1)) \to H^{k+1}(X, \mathbb{Z}_2(j))$ is the cup-product by the class of the extension $0 \to \mathbb{Z}_2(j) \to \pi_*\mathbb{Z}_2 \to \mathbb{Z}_2(j+1) \to 0$, which is the nonzero class $\omega \in H^1(G, \mathbb{Z}_2(1)) \simeq \mathbb{Z}/2\mathbb{Z}$.

1C. ***Cohomological dimension.*** Recall first the following well-known statement, which goes back to Artin.

Proposition 1.1. *Let X be an integral variety over $\boldsymbol{R}$. The following are equivalent*:

(i) *$\boldsymbol{R}(X)$ is formally real, that is, -1 is not a sum of squares in $\boldsymbol{R}(X)$.*

(ii) *X has a smooth $\boldsymbol{R}$-point.*

(iii) *$X(\boldsymbol{R})$ is Zariski-dense in X.*

Proof. By the Artin–Lang homomorphism theorem [Bochnak et al. 1998, Theorem 4.1.2], if (i) holds, every open affine subset of X contains an $\boldsymbol{R}$-point, proving (iii). Conversely, if $X(\boldsymbol{R})$ were Zariski-dense in X, -1 could not be a sum of squares in $\boldsymbol{R}(X)$, because we would get a contradiction by evaluating this identity at an $\boldsymbol{R}$-point outside of the poles of the rational functions that appear. That (ii) implies (iii) is a consequence of the implicit function theorem [Bochnak et al. 1998, Corollary 2.9.8], and the converse is trivial. □

From this proposition, it is possible to deduce estimates on the cohomological dimension of varieties X over $\boldsymbol{R}$ without $\boldsymbol{R}$-points. For the cohomological dimension of $\boldsymbol{R}(X)$, this follows from a theorem of Serre [1965] and Artin–Schreier theory. The cohomological dimension of an arbitrary variety X may then be controlled using [SGA 4_3 1973, X Corollaire 4.2].

Here, we point out places in the literature where the statements we need are explicitly formulated.

Proposition 1.2. *Let X be an integral variety of dimension n over $\boldsymbol{R}$ such that $X(\boldsymbol{R}) = \varnothing$.*

(i) *$\boldsymbol{R}(X)$ has cohomological dimension n.*

(ii) *X has étale cohomological dimension $\leq 2n$.*

(iii) *If X is affine, X has étale cohomological dimension $\leq n$.*

Proof. The first statement is [Colliot-Thélène and Parimala 1990, Proposition 1.2.1], where it is attributed to Ax.

The second (resp. third) statement follows from [Scheiderer 1994, Corollary 7.21], noticing that the real spectrum of X is empty by Proposition 1.1 and using that X_C has étale cohomological dimension $\leq 2d$ (resp. $\leq d$) by [SGA 4_3 1973, X Corollaire 4.3] (resp. [SGA 4_3 1973, XIV Corollaire 3.2]). □

2. Bloch–Ogus theory

2A. ***Gersten's conjecture.*** In this subsection, let X be a smooth variety over $\boldsymbol{R}$.

We want to apply Bloch–Ogus theory to the cohomology groups $H^k(X, \mathbb{Z}_2(j))$. For this purpose, one needs to check the validity of Gersten's conjecture for this cohomology theory. There are two ways to do so.

First, the formal properties of continuous étale cohomology proven by Jannsen [1988] allow one to prove that associating to a variety X over $\boldsymbol{R}$ its continuous étale cohomology groups $H^k(X, \mathbb{Z}_2(j))$ is part of a Poincaré duality theory with supports in the sense of Bloch and Ogus [1974, Definition 1.3], in the same way as it is proven for étale cohomology with finite coefficients [Bloch and Ogus 1974, §2]. Then it is possible to apply [Bloch and Ogus 1974, Theorem 4.2].

Another possibility is to use the axioms of [Colliot-Thélène et al. 1997], which are easier to check. That these axioms hold for continuous étale cohomology is explained for instance in [Kahn 2012, §3C], allowing us to apply [Colliot-Thélène et al. 1997, Corollary 5.1.11].

Let us now explain the meaning of Gersten's conjecture in our context. We define $\mathcal{H}^k_X(j)$ to be the Zariski sheaf on X that is the sheafification of $U \mapsto H^k(U, \mathbb{Z}_2(j))$. Moreover, if $z \in X$ is a point with closure $Z \subset X$, we define[4]

$$H^k_{\rightarrow}(z, \mathbb{Z}_2(j)) := \varinjlim_{U \subset Z} H^k(U, \mathbb{Z}_2(j)), \tag{2-1}$$

where U runs over all nonempty open subsets of Z. We define $\iota_z : z \to X$ to be the inclusion, and we consider the skyscraper sheaves $\iota_{z_*} H^k_{\rightarrow}(z, \mathbb{Z}_2(j))$ on X. Finally, we set $X^{(c)}$ to be the set of codimension c points in X. Then the sheaves $\mathcal{H}^k_X(j)$ admit Cousin resolutions (see either [Bloch and Ogus 1974, (4.2.2)] or [Colliot-Thélène et al. 1997, Corollary 5.1.11] taking into account purity [Jannsen 1988, (3.21)] to obtain the precise form below):

$$\begin{aligned} 0 \to \mathcal{H}^k_X(j) \to \bigoplus_{z \in X^{(0)}} \iota_{z_*} H^k_{\rightarrow}(z, \mathbb{Z}_2(j)) \to \bigoplus_{z \in X^{(1)}} \iota_{z_*} H^{k-1}_{\rightarrow}(z, \mathbb{Z}_2(j-1)) \\ \to \dots \to \bigoplus_{z \in X^{(k)}} \iota_{z_*} H^0_{\rightarrow}(z, \mathbb{Z}_2(j-k)) \to 0. \end{aligned} \tag{2-2}$$

[4]Beware that since continuous étale cohomology does not commute with inverse limit of schemes, this group does not coincide in general with the continuous Galois cohomology of the residue field of z.

The way this Cousin resolution is constructed, from a coniveau spectral sequence, shows that the arrows in (2-2) are given by maps in long exact sequences of cohomology with support, also called residue maps.

Since the sheaves in this resolution are flasque, the Cousin complex obtained by taking its global sections computes the Zariski cohomology of $\mathcal{H}^k_X(j)$. For instance, this implies that $H^0(X, \mathcal{H}^k_X(j))$ coincides with the unramified cohomology group $H^k_{\mathrm{nr}}(X, \mathbb{Z}_2(j))$, that is, the subgroup of $H^k_{\to}(\eta, \mathbb{Z}_2(j))$ on which all residues at codimension 1 points of X vanish.

The exactness of (2-2) allows us to compute the second page of the coniveau spectral sequence for X mentioned above. As shown in [Bloch and Ogus 1974, Corollary 6.3] or [Colliot-Thélène et al. 1997, Corollary 5.1.11], it reads

$$E_2^{p,q} = H^p(X, \mathcal{H}^q_X(j)) \Rightarrow H^{p+q}(X, \mathbb{Z}_2(j)). \tag{2-3}$$

Recall that the filtration induced by this spectral sequence on $H^k(X, \mathbb{Z}_2(j))$ is the coniveau filtration, where a class $\alpha \in H^k(X, \mathbb{Z}_2(j))$ has coniveau $\geq c$ if it vanishes in the complement of a closed subset of codimension c of X.

2B. ***Bloch–Ogus theory over $\boldsymbol{R}$.*** If X is a variety over $\boldsymbol{R}$, we still denote by $\pi : X_{\boldsymbol{C}} \to X$ the natural morphism, and we naturally view $X_{\boldsymbol{C}}$ as a variety over $\boldsymbol{R}$. The following proposition was proved in [Colliot-Thélène and Scheiderer 1996, Lemma 2.2.1] over $\mathbb{R}$ and with 2-torsion coefficients, but the proof goes through, and we include it for completeness.

Proposition 2.1. *Let X be a smooth variety over $\boldsymbol{R}$ and fix $j \in \mathbb{Z}$. Then there exists a long exact sequence of Zariski sheaves on X:*

$$\cdots \to \mathcal{H}^k_X(j) \to \pi_*\mathcal{H}^k_{X_{\boldsymbol{C}}} \to \mathcal{H}^k_X(j+1) \to \mathcal{H}^{k+1}_X(j) \to \cdots. \tag{2-4}$$

Moreover, the sheaf $\pi_*\mathcal{H}^k_{X_{\boldsymbol{C}}}$ coincides with the sheafification of $U \mapsto H^k(U_{\boldsymbol{C}}, \mathbb{Z}_2)$ and its cohomology groups are $H^q(X, \pi_*\mathcal{H}^k_{X_{\boldsymbol{C}}}) = H^q(X_{\boldsymbol{C}}, \mathcal{H}^k_{X_{\boldsymbol{C}}})$ for any $k, q \geq 0$.

Proof. Let $x \in X$. If V is a neighborhood of $\pi^{-1}(x)$ in $X_{\boldsymbol{C}}$, the sheaf $\mathcal{H}^k_V$ has a flasque Cousin resolution (2-2). Taking global sections and taking the limit over all such neighborhoods V gives a complex that is exact in positive degree (the argument for étale cohomology with finite coefficients is [Colliot-Thélène et al. 1997, Proposition 2.1.2], and the corresponding effaceability condition for continuous étale cohomology follows from [Colliot-Thélène et al. 1997, Theorem 5.1.10]). As a consequence,

$$\varinjlim_V H^p(V, \mathcal{H}^k_V) = 0 \quad \text{for } p > 0. \tag{2-5}$$

Considering the coniveau spectral sequences (2-3) for every V and taking (2-5) into account shows that

$$\varinjlim_V H^k(V, \mathbb{Z}_2) = \varinjlim_V H^0(V, \mathcal{H}^k_V). \tag{2-6}$$

Note that in both (2-5) and (2-6), it is possible to restrict to neighborhoods of the form $U_{\boldsymbol{C}}$ for $U \subset X$ because they form a cofinal family.

Now, the exact sequences obtained by applying (1-2) to all open subsets of X fit together to induce a long exact sequence of Zariski presheaves on X. By exactness of sheafification, one obtains a long exact sequence of Zariski sheaves on X:

$$\cdots \to \mathcal{H}_X^k(j) \to \mathcal{F}^k \to \mathcal{H}_X^k(j+1) \to \mathcal{H}_X^{k+1}(j) \to \cdots,$$

where $\mathcal{F}^k$ is the sheafification of $U \mapsto H^k(U_{\boldsymbol{C}}, \mathbb{Z}_2)$. The universal property of sheafification gives a morphism $\mathcal{F}^k \to \pi_*\mathcal{H}^k_{X_{\boldsymbol{C}}}$. The map induced on stalks at $x \in X$ is precisely (2-6), hence an isomorphism. It follows that $\mathcal{F}^k \simeq \pi_*\mathcal{H}^k_{X_{\boldsymbol{C}}}$, completing the construction of (2-4).

If $k \geq 0$ and $p > 0$, the stalk of $R^p\pi_*\mathcal{H}^k_{X_{\boldsymbol{C}}} = 0$ at $x \in X$ is given by (2-5), hence trivial. It follows that $R^p\pi_*\mathcal{H}^k_{X_{\boldsymbol{C}}}$ vanishes, and the Leray spectral sequence for π implies the last statement of the proposition. □

2C. ***Consequences of the Bloch–Kato conjecture.*** The following proposition is due to Bloch and Srinivas [1983, proof of Theorem 1] for $k \leq 2$ and to Colliot-Thélène and Voisin [2012, Théorème 3.1] in general. Since both references work over an algebraically closed field, and since the latter uses Betti cohomology, we repeat the proof to emphasize that it works in our setting.

Proposition 2.2. *Let X be a smooth variety over $\boldsymbol{R}$. Then for every $k \geq 0$, the sheaf $\mathcal{H}_X^{k+1}(k)$ is torsion free.*

Proof. Since it is a sheaf of $\mathbb{Z}_2$-modules, it suffices to prove that it has no 2-torsion. Consider the exact sequence of 2-adic sheaves on X: $0 \to \mathbb{Z}_2(k) \xrightarrow{2} \mathbb{Z}_2(k) \to \mu_2^{\otimes k} \to 0$. Taking long exact sequences of continuous cohomology over every open subset $U \subset X$ to get a long exact sequence of presheaves on X and sheafifying it gives a long exact sequence of sheaves on X, part of which is

$$\mathcal{H}_X^k(k) \to \mathcal{H}_X^k(\mu_2^{\otimes k}) \to \mathcal{H}_X^{k+1}(k) \xrightarrow{2} \mathcal{H}_X^{k+1}(k),$$

where $\mathcal{H}_X^k(\mu_2^{\otimes k})$ is the sheafification of $U \mapsto H^k(U, \mu_2^{\otimes k})$. Consequently, it suffices to prove the surjectivity of $\mathcal{H}_X^k(k) \to \mathcal{H}_X^k(\mu_2^{\otimes k})$.

On an open set $U \subset X$, the Kummer exact sequence $0 \to \mu_2 \to \mathbb{G}_m \xrightarrow{2} \mathbb{G}_m \to 0$ induces a boundary map $H^0(U, \mathcal{O}_U^*) \to H^1(U, \mu_2)$. These maps sheafify to $\mathcal{O}_X^* \to \mathcal{H}_X^1(\mu_2)$, inducing via cup-products a morphism of sheaves $(\mathcal{O}_X^*)^{\otimes k} \to \mathcal{H}_X^k(\mu_2^{\otimes k})$. It is explained in [Colliot-Thélène and Voisin 2012, end of section 2.2] how Gersten's conjecture for Milnor K-theory proven by Kerz [2009] and the Bloch–Kato conjecture proven by Rost and Voevodsky (since we only need this conjecture at the prime 2, Voevodsky's work [2003, Corollary 7.4] on Milnor's conjecture is sufficient here) imply the surjectivity of this morphism.

Over an open set $U \subset X$, the boundary maps $H^0(U, \mathcal{O}_U^*) \to H^1(U, \mu_{2^r})$ for the Kummer exact sequences

$$0 \to \mu_{2^r} \to \mathbb{G}_m \xrightarrow{2^r} \mathbb{G}_m \to 0$$

fit together to induce a map $H^0(U, \mathcal{O}_U^*) \to \varprojlim_r H^1(U, \mu_{2^r}) = H^1(U, \mathbb{Z}_2(1))$. Again, this sheafifies to a morphism $\mathcal{O}_X^* \to \mathcal{H}_X^1(1)$, inducing via cup-products a morphism of sheaves $(\mathcal{O}_X^*)^{\otimes k} \to \mathcal{H}_X^k(k)$ lifting $(\mathcal{O}_X^*)^{\otimes k} \to \mathcal{H}_X^k(\mu_2^{\otimes k})$. The surjectivity of $\mathcal{H}_X^k(k) \to \mathcal{H}_X^k(\mu_2^{\otimes k})$ now follows from surjectivity of $(\mathcal{O}_X^*)^{\otimes k} \to \mathcal{H}_X^k(\mu_2^{\otimes k})$. □

In [Bloch and Srinivas 1983; Colliot-Thélène and Voisin 2012], the authors worked over an algebraically closed field, and the Tate twist was not essential for the result to hold. Here, it is very important; it is not true in general that the sheaf $\mathcal{H}_X^k(k)$ has no torsion.

As in these references, the following are straightforward corollaries.

Corollary 2.3. *Let X be a smooth variety over $\boldsymbol{R}$ and $k \geq 0$. Then*

$$H_{\mathrm{nr}}^{k+1}(X, \mathbb{Z}_2(k)) = H^0(X, \mathcal{H}_X^{k+1}(k))$$

is torsion free.

Corollary 2.4. *Let X be an integral variety over $\boldsymbol{R}$ with generic point η and $k \geq 0$. Then $H_{\rightarrow}^{k+1}(\eta, \mathbb{Z}_2(k))$ is torsion free.*

Proof. If $\alpha \in H^{k+1}(U, \mathbb{Z}_2(k))$ is a torsion class on a smooth open subset $U \subset X$, it vanishes in $H_{\mathrm{nr}}^{k+1}(U, \mathbb{Z}_2(k))$ by Corollary 2.3, hence on an open subset $V \subset U$. □

Another application of Proposition 2.2 is as follows.

Proposition 2.5. *Let X be a smooth variety over $\boldsymbol{R}$. Then for every $k \geq 0$, there is an exact sequence*

$$0 \to \mathcal{H}_X^{k-1}(k) \to \pi_* \mathcal{H}_{X_C}^{k-1} \to \mathcal{H}_X^{k-1}(k+1) \to \mathcal{H}_X^k(k) \to \pi_* \mathcal{H}_{X_C}^k \to \mathcal{H}_X^k(k+1) \to 0.$$

Proof. Let us prove that the long exact sequence (2-4) splits into these shorter exact sequences. It suffices to prove that, for $k \geq 0$, the morphism $\mathcal{H}_X^{k-1}(k) \to \pi_* \mathcal{H}_{X_C}^{k-1}$ is injective. The composition

$$\mathcal{H}_X^{k-1}(k) \to \pi_* \mathcal{H}_{X_C}^{k-1} \to \mathcal{H}_X^{k-1}(k)$$

is multiplication by 2. Consequently, the kernel of $\mathcal{H}_X^{k-1}(k) \to \pi_* \mathcal{H}_{X_C}^{k-1}$ is of 2-torsion. Since $\mathcal{H}_X^{k-1}(k)$ is torsion free by Proposition 2.2, this kernel is trivial, as required. □

Proposition 2.6. *Let X be an integral variety over $\boldsymbol{R}$ with generic point η. Then for every $k \geq 0$, there is an exact sequence*

$$\begin{aligned} 0 \to H_{\rightarrow}^{k-1}(\eta, \mathbb{Z}_2(k)) \to H_{\rightarrow}^{k-1}(\eta, \pi_* \mathbb{Z}_2) \to H_{\rightarrow}^{k-1}(\eta, \mathbb{Z}_2(k+1)) \\ \to H_{\rightarrow}^{k}(\eta, \mathbb{Z}_2(k)) \to H_{\rightarrow}^{k}(\eta, \pi_* \mathbb{Z}_2) \to H_{\rightarrow}^{k}(\eta, \mathbb{Z}_2(k+1)) \to 0. \end{aligned}$$

Proof. Take the direct limit of the long exact sequence (1-2) applied to all open subsets of X; it splits into exact sequences of length six by the same argument as in the proof of Proposition 2.5, using Corollary 2.4 instead of Corollary 2.3. $\square$

3. Sums of squares and unramified cohomology

3A. ***Sums of squares and level.*** Let $n \geq 1$, consider a nonzero positive semidefinite polynomial $f \in \boldsymbol{R}[X_1, \dots, X_n]$ and its homogenization $F \in \boldsymbol{R}[X_0, \dots, X_n]$. Notice that since an odd degree polynomial over $\boldsymbol{R}$ changes sign, f and F must have even degree. This allows us to consider the double cover Y of $\mathbb{P}^n_{\boldsymbol{R}}$ ramified over $\{F = 0\}$ defined by the equation $Y := \{Z^2 + F = 0\}$ in the weighted projective space $\mathbb{P}(1, \dots, 1, \deg(F)/2)$.

Lemma 3.1. *The variety Y is integral, $\boldsymbol{R}(Y)$ is not formally real, and if $\widetilde{Y} \to Y$ is a resolution of singularities, then $\widetilde{Y}(\boldsymbol{R}) = \varnothing$.*

Proof. To prove that Y is integral, one has to check that $-f$ is not a square in $\boldsymbol{R}(X_1, \dots, X_n)$, or equivalently, that it is not a square in $\boldsymbol{R}[X_1, \dots, X_n]$. But if it were, f would be negative on $\boldsymbol{R}^n$, hence zero on $\boldsymbol{R}^n$ by positivity, hence zero by Zariski-density of $\boldsymbol{R}^n$ in $\boldsymbol{C}^n$. This is a contradiction.

The $\boldsymbol{R}$-points of $\widetilde{Y}$ necessarily lie above $\boldsymbol{R}$-points of Y, hence, by positivity of F, above zeroes of F. Consequently, $\widetilde{Y}(\boldsymbol{R})$ is not Zariski-dense in $\widetilde{Y}$. Applying Proposition 1.1 using the smoothness of $\widetilde{Y}$ shows that $\widetilde{Y}(\boldsymbol{R}) = \varnothing$, and that $\boldsymbol{R}(Y)$ is not formally real. $\square$

Recall that the level $s(K) \in \mathbb{N}^* \cup \{\infty\}$ of a field K is ∞ if -1 is not a sum of squares in K, and the smallest s such that -1 is a sum of s squares otherwise. In the latter case, it has been shown by Pfister [1965, Satz 4] to be a power of 2.

Proposition 3.2. *The polynomial f is a sum of $2^n - 1$ squares in $\boldsymbol{R}(X_1, \dots, X_n)$ if and only if $\boldsymbol{R}(Y)$ has level $< 2^n$. Conversely, the polynomial f is not a sum of $2^n - 1$ squares in $\boldsymbol{R}(X_1, \dots, X_n)$ if and only if $\boldsymbol{R}(Y)$ has level 2^n.*

Proof. Proposition 1.1 shows that $\boldsymbol{R}(X_1, \dots, X_n)$ is formally real and Artin's solution [1927] to Hilbert's 17th problem shows that f is a sum of squares in $\boldsymbol{R}(X_1, \dots, X_n)$.

Then [Lam 1980, Chapter 11, Theorem 2.7] applies and shows that f is a sum of $2^n - 1$ squares in $\boldsymbol{R}(X_1, \dots, X_n)$ if and only if $\boldsymbol{R}(Y)$ has level $< 2^n$ (this is essentially due to Pfister; the statement we have used is very close and its proof is identical to [Pfister 1965, Satz 5]).

Since $\boldsymbol{R}(Y)$ is not formally real by Lemma 3.1, Pfister [1967, Theorem 2] has shown that its level is $\leq 2^n$. This concludes the proof. $\square$

3B. ***Level and unramified cohomology.*** To apply Proposition 3.2, we need to control the level of the function field of a variety over $\boldsymbol{R}$. The following proposition relates it to one of its unramified cohomology groups. The equivalence (i)$\Leftrightarrow$(ii) is hinted at in [Colliot-Thélène 1993, bottom of p. 236], at least for $n=3$. I am grateful to Olivier Wittenberg for explaining to me that the implication (ii)$\Rightarrow$(iii) holds.

Proposition 3.3. *Let X be a smooth integral variety over $\boldsymbol{R}$, and fix $n \geq 1$. The following assertions are equivalent*:

(i) *The function field $\boldsymbol{R}(X)$ has level $< 2^n$.*

(ii) *The map $H^n(\boldsymbol{R}, \mathbb{Z}/2\mathbb{Z}) \to H^n(\boldsymbol{R}(X), \mathbb{Z}/2\mathbb{Z})$ vanishes.*

(iii) *The map $H^n(\boldsymbol{R}, \mathbb{Z}_2(n)) \to H^n_{\mathrm{nr}}(X, \mathbb{Z}_2(n))$ vanishes.*

Proof. Consider the property that the level of $\boldsymbol{R}(X)$ is $< 2^n$. It is equivalent to the fact that -1 is a sum of 2^n-1 squares in $\boldsymbol{R}(X)$, hence to the fact that the Pfister quadratic form $q := \langle 1, 1\rangle^{\otimes n}$ is isotropic over $\boldsymbol{R}(X)$. By a theorem of Elman and Lam [1972, Corollary 3.3], this is equivalent to the vanishing of the symbol $\{-1\}^n$ in the Milnor K-theory group $K_n^M(\boldsymbol{R}(X))/2$. By Voevodsky's proof [2003, Corollary 7.4] of the Milnor conjecture, the natural map

$$K_n^M(\boldsymbol{R}(X))/2 \to H^n(\boldsymbol{R}(X), \mathbb{Z}/2\mathbb{Z})$$

is an isomorphism, so our property is equivalent to the vanishing of $H^n(\boldsymbol{R}, \mathbb{Z}/2\mathbb{Z}) \to H^n(\boldsymbol{R}(X), \mathbb{Z}/2\mathbb{Z})$. We have proven that (i) and (ii) are equivalent.

Suppose that (iii) holds and let η be the generic point of X. The definition of $H^n_{\mathrm{nr}}(X, \mathbb{Z}_2(n))$ as a subgroup of $H^n_{\to}(\eta, \mathbb{Z}_2(n))$ shows that $H^n(\boldsymbol{R}, \mathbb{Z}_2(n)) \to H^n_{\to}(\eta, \mathbb{Z}_2(n))$ vanishes. Then we have a commutative diagram

$$\begin{tikzcd} H^n(\boldsymbol{R}, \mathbb{Z}_2(n)) \arrow[r] \arrow[d, "\cong"] & H^n_{\to}(\eta, \mathbb{Z}_2(n)) \arrow[d] \\ H^n(\boldsymbol{R}, \mathbb{Z}/2\mathbb{Z}) \arrow[r] & H^n_{\to}(\eta, \mathbb{Z}/2\mathbb{Z}) \end{tikzcd}$$

where the groups on the right are defined as inductive limits on the open subsets of X as in (2-1), showing that $H^n(\boldsymbol{R}, \mathbb{Z}/2\mathbb{Z}) \to H^n_{\to}(\eta, \mathbb{Z}/2\mathbb{Z})$ vanishes. Since étale cohomology commutes with such limits [SGA 4_2 1972, VII Corollaire 5.8], $H^n_{\to}(\eta, \mathbb{Z}/2\mathbb{Z})$ is nothing but the Galois cohomology group $H^n(\boldsymbol{R}(X), \mathbb{Z}/2\mathbb{Z})$, proving (ii).

Suppose conversely that (ii) holds, and let $U \subset X$ be an open subset such that ω^n vanishes in $H^n(U, \mathbb{Z}/2\mathbb{Z})$. Consider the following commutative exact diagram, where the lines are (1-2):

$$\begin{CD}
H^{n-1}(U,\mathbb{Z}_2(n-1)) @>{\omega}>> H^n(U,\mathbb{Z}_2(n)) @>>> H^n(U_{\boldsymbol{C}},\mathbb{Z}_2)\\
@VV{2}V @VV{2}V @VV{2}V\\
H^{n-1}(U,\mathbb{Z}_2(n-1)) @>{\omega}>> H^n(U,\mathbb{Z}_2(n)) @>>> H^n(U_{\boldsymbol{C}},\mathbb{Z}_2)\\
@. @VVV @.\\
@. H^n(U,\mathbb{Z}/2\mathbb{Z}) @.
\end{CD}$$

Look at $\omega^n \in H^n(U,\mathbb{Z}_2(n))$. By hypothesis, it vanishes in $H^n(U,\mathbb{Z}/2\mathbb{Z})$, hence may be written 2α for some $\alpha\in H^n(U,\mathbb{Z}_2(n))$. Since $\omega^n\in H^n(U,\mathbb{Z}_2(n))$ is the image of $\omega^{n-1}\in H^{n-1}(U,\mathbb{Z}_2(n-1))$, $\alpha_{\boldsymbol{C}}\in H^n(U_{\boldsymbol{C}},\mathbb{Z}_2)$ is a 2-torsion class. By Corollary 2.4, any torsion class in $H^n(U_{\boldsymbol{C}},\mathbb{Z}_2)$ vanishes on an open subset: up to shrinking U, we may assume that $\alpha_{\boldsymbol{C}}=0$, hence that there is $\beta\in H^{n-1}(U,\mathbb{Z}_2(n-1))$ such that $\beta\cdot\omega=\alpha$. Then $\omega^n=\beta\cdot 2\omega=0\in H^n(U,\mathbb{Z}_2(n))$, proving (iii). □

3C. ***From degree n to degree n+1 cohomology.*** Condition (iii) in Proposition 3.3 means that ω^n has coniveau ≥ 1. Proposition 3.5 uses Bloch–Ogus theory to relate this property to the coniveau of ω^{n+1}.

Fix an integer $n\geq 1$ and let X be a smooth variety over $\boldsymbol{R}$. The coniveau spectral sequence (2-3) induces two maps $H^n(X,\mathbb{Z}_2(n))\xrightarrow{\phi}H^n_{\mathrm{nr}}(X,\mathbb{Z}_2(n))$ and $K:=\operatorname{Ker}\big[H^{n+1}(X,\mathbb{Z}_2(n+1))\to H^{n+1}_{\mathrm{nr}}(X,\mathbb{Z}_2(n+1))\big]\xrightarrow{\psi}H^1(X,\mathcal{H}^n_X(n+1))$.

Cup-product with ω gives morphisms $H^n(X,\mathbb{Z}_2(n))\xrightarrow{\omega}H^{n+1}(X,\mathbb{Z}_2(n+1))$ and $H^n_{\mathrm{nr}}(X,\mathbb{Z}_2(n))\xrightarrow{\omega}H^{n+1}_{\mathrm{nr}}(X,\mathbb{Z}_2(n+1))$. Let $I:=\{\alpha\in H^n(X,\mathbb{Z}_2(n))\mid \alpha\cdot\omega\in K\}$ and $I_{\mathrm{nr}}:=\{\alpha\in H^n_{\mathrm{nr}}(X,\mathbb{Z}_2(n))\mid\alpha\cdot\omega=0\}$.

Finally, Proposition 2.5 gives an exact sequence of sheaves on X:

$$0\to\mathcal{H}^n_X(n+1)\to\pi_*\mathcal{H}^n_{X_{\boldsymbol{C}}}\to\mathcal{H}^n_X(n)\xrightarrow{\omega}\mathcal{H}^{n+1}_X(n+1)\to\cdots. \tag{3-1}$$

Taking cohomology, we obtain an exact sequence:

$$0\to H^n_{\mathrm{nr}}(X,\mathbb{Z}_2(n+1))\to H^n_{\mathrm{nr}}(X_{\boldsymbol{C}},\mathbb{Z}_2)\to I_{\mathrm{nr}}\xrightarrow{\delta}H^1(X,\mathcal{H}^n_X(n+1)). \tag{3-2}$$

Lemma 3.4. *Let X be a smooth variety over $\boldsymbol{R}$. The diagram*

$$\begin{CD}
I @>{\omega}>> K\\
@V{\phi}VV @VV{\psi}V\\
I_{\mathrm{nr}} @>{\delta}>> H^1(X,\mathcal{H}^n_X(n+1))
\end{CD}$$

constructed above commutes.

Proof. Let $\alpha\in I$. By hypothesis, the class $\alpha\cdot\omega\in H^{n+1}(X,\mathbb{Z}_2(n+1))$ vanishes on an open subset $U\subset X$. Let $D:=X\setminus U$ be endowed with its reduced structure.

The description of $H^1(X,\mathcal{H}^n_X(n+1))$ as a cohomology group of the Cousin complex (2-2) shows that if $X^\circ\subset X$ is an open subset whose complement has codimension ≥ 2, then the restriction $H^1(X, \mathcal{H}^n_X(n+1)) \to H^1(X^\circ, \mathcal{H}^n_{X^\circ}(n+1))$ is injective. Consequently, to prove that $\psi(\alpha \cdot \omega) = \delta \circ \phi(\alpha)$, it is possible to remove from X a closed subset of codimension ≥ 2. This allows us to suppose that D is smooth of pure codimension 1.

Our next task is to concretely identify $\delta \circ \phi(\alpha)$. The cohomology theory with supports in the sense of [Colliot-Thélène et al. 1997, Definition 5.1.1], which to a variety X over $\boldsymbol{R}$ and a closed subset $Z \subset X$ associates the groups $H^k_Z(X, \pi_*\mathbb{Z}_2) = H^k_{Z_{\boldsymbol{C}}}(X_{\boldsymbol{C}}, \mathbb{Z}_2)$, satisfies axioms COH1 and COH3 by [Colliot-Thélène et al. 1997, 5.5(1)], hence COH2 by [Colliot-Thélène et al. 1997, Proposition 5.3.2]. It follows from [Colliot-Thélène et al. 1997, Corollary 5.1.11] that the sheafification of $U \mapsto H^n(U_{\boldsymbol{C}}, \mathbb{Z}_2)$ (that is, $\pi_*\mathcal{H}^n_{X_{\boldsymbol{C}}}$ by Proposition 2.1) admits a Cousin resolution by flasque sheaves, and the same goes for $\pi_*\mathcal{H}^{n+1}_{X_{\boldsymbol{C}}}$. These resolutions fit together with the Cousin resolutions (2-2) of $\mathcal{H}^n_X(n+1)$, $\mathcal{H}^n_X(n)$, $\mathcal{H}^{n+1}_X(n+1)$ and $\mathcal{H}^{n+1}_X(n)$, giving rise to a diagram that is an exact sequence of flasque resolutions for the exact sequence of sheaves (3-1) by Proposition 2.6. Let us only draw the relevant part of the diagram containing the Cousin resolutions for $\mathcal{H}^n_X(n+1)$, $\pi_*\mathcal{H}^{n+1}_{X_{\boldsymbol{C}}}$ and $\mathcal{H}^n_X(n)$:

$$\begin{tikzcd}
\bigoplus_{z\in X^{(0)}} \iota_{z*} H^n_{\to}(z, \mathbb{Z}_2(n+1)) \arrow[r] \arrow[d] & \bigoplus_{z\in X^{(1)}} \iota_{z*} H^{n-1}_{\to}(z, \mathbb{Z}_2(n)) \arrow[r] \arrow[d] & \cdots \\
\bigoplus_{z\in X^{(0)}} \iota_{z*} H^n_{\to}(z, \pi_*\mathbb{Z}_2) \arrow[r] \arrow[d] & \bigoplus_{z\in X^{(1)}} \iota_{z*} H^{n-1}_{\to}(z, \pi_*\mathbb{Z}_2) \arrow[r] \arrow[d] & \cdots \\
\bigoplus_{z\in X^{(0)}} \iota_{z*} H^n_{\to}(z, \mathbb{Z}_2(n)) \arrow[r] & \bigoplus_{z\in X^{(1)}} \iota_{z*} H^{n-1}_{\to}(z, \mathbb{Z}_2(n-1)) \arrow[r] & \cdots
\end{tikzcd} \tag{3-3}$$

It is now possible to give a description of $\delta \circ \phi(\alpha)$ by a diagram chase in the diagram obtained by taking the global sections of (3-3). More precisely, α induces a class

$$\phi(\alpha) \in \operatorname{Ker}\Big[\bigoplus_{z\in X^{(0)}} H^n_{\to}(z, \mathbb{Z}_2(n)) \to \bigoplus_{z\in X^{(1)}} H^{n-1}_{\to}(z, \mathbb{Z}_2(n-1))\Big].$$

Lifting it in $\bigoplus_{z\in X^{(0)}} H^n_{\to}(z, \pi_*\mathbb{Z}_2)$ by the hypothesis that $\alpha \in I$, pushing it to $\bigoplus_{z\in X^{(1)}} H^{n-1}_{\to}(z, \pi_*\mathbb{Z}_2)$ and lifting it again to $\bigoplus_{z\in X^{(1)}} H^{n-1}_{\to}(z, \mathbb{Z}_2(n))$ gives a cohomology class of degree one of the complex of global sections of the Cousin resolution of $\mathcal{H}^n_X(n+1)$ representing $\delta \circ \phi(\alpha) \in H^1(X, \mathcal{H}^n_X(n+1))$.

At this point, consider the following commutative diagram, whose rows are exact sequences of cohomology with support, whose columns are instances of (1-2), and where the coefficient ring $\mathbb{Z}_2$ has been omitted:

$$\begin{tikzcd}
 & & H^n(U, n+1) \arrow[r] \arrow[d] & H^{n-1}(D, n) \arrow[d, "\pi^*"] \\
 & & H^n(U_{\boldsymbol{C}}) \arrow[r, "\partial"] \arrow[d, "\pi_*"] & H^{n-1}(D_{\boldsymbol{C}}) \arrow[d] \\
H^{n-2}(D, n-1) \arrow[r] \arrow[d] & H^n(X, n) \arrow[r, "j^*"] \arrow[d, "\omega"] & H^n(U, n) \arrow[r] \arrow[d] & H^{n-1}(D, n-1) \\
H^{n-1}(D, n) \arrow[r, "i_*"] \arrow[d] & H^{n+1}(X, n+1) \arrow[r] & H^{n+1}(U, n+1) & \\
H^{n-1}(D_{\boldsymbol{C}}) & & &
\end{tikzcd}$$

Here we have denoted by $i : D \to X$ and $j : U \to X$ the inclusions, and by ∂ the residue map. By our choice of U, $\alpha \in H^n(X, \mathbb{Z}_2(n))$ vanishes in $H^{n+1}(U, \mathbb{Z}_2(n+1))$. Chasing the diagram, there are two ways to construct a (not well-defined) class in $H^{n-1}(D, \mathbb{Z}_2(n))$. First, we may consider a class $\beta \in H^{n-1}(D, \mathbb{Z}_2(n))$ such that $i_*\beta = \alpha \cdot \omega$. Second, we may lift $j^*\alpha$ along π_*, apply the residue map ∂, and lift the resulting class along π^* to obtain $\gamma \in H^{n-1}(D, \mathbb{Z}_2(n))$.

Our diagram has been constructed from the diagram of distinguished triangles in the derived category of 2-adic sheaves on X:

$$\begin{tikzcd}
i_* Ri^! \mathbb{Z}_2(n+1) \arrow[r] \arrow[d] & \mathbb{Z}_2(n+1) \arrow[r] \arrow[d] & Rj_* j^* \mathbb{Z}_2(n+1) \arrow[r] \arrow[d] & {} \\
i_* Ri^! \pi_* \mathbb{Z}_2 \arrow[r] \arrow[d] & \pi_* \mathbb{Z}_2 \arrow[r] \arrow[d] & Rj_* j^* \pi_* \mathbb{Z}_2 \arrow[r] \arrow[d] & {} \\
i_* Ri^! \mathbb{Z}_2(n) \arrow[r] \arrow[d] & \mathbb{Z}_2(n) \arrow[r] \arrow[d] & Rj_* j^* \mathbb{Z}_2(n) \arrow[r] \arrow[d] & {} \\
{} & {} & {} &
\end{tikzcd}$$

A homological algebra lemma due to Jannsen [2000, lemma, p. 268], applied exactly as in [Jannsen 2000, proof of Theorem 2], shows that the images of β and γ in $H^{n-1}(D_{\boldsymbol{C}}, \mathbb{Z}_2)$ that are well-defined up to the image of $H^n(U, \mathbb{Z}_2(n+1))$ coincide up to a sign. It follows that β and γ, well-defined up to the images of $H^n(U, \mathbb{Z}_2(n+1))$ and $H^{n-2}(D, \mathbb{Z}_2(n-1))$ in $H^{n-1}(D, \mathbb{Z}_2(n))$, coincide up to a sign.

Now notice that β and γ induce classes in $\bigoplus_{z\in X^{(1)}} H^{n-1}_{\to}(z, \mathbb{Z}_2(n))$. Our explicit description of $\delta\circ\phi(\alpha)$ shows that β is a representative of it as a cohomology class of degree one of the Cousin complex. On the other hand, γ has been constructed by lifting $\alpha\cdot\omega$ along the Gysin morphism $H^{n-1}(D, \mathbb{Z}_2(n)) \to H^{n+1}(X, \mathbb{Z}_2(n+1))$. By construction of the coniveau spectral sequence [Bloch and Ogus 1974, §3; Colliot-Thélène et al. 1997, §1], γ is a representative of $\psi(\alpha\cdot\omega)$ as a cohomology class of degree one of the Cousin complex.

At this point, we have proven that $\psi(\alpha\cdot\omega) = [\gamma] = -[\beta] = -\delta\circ\phi(\alpha)$. Since this element is 2-torsion because ω is, one has in fact $\psi(\alpha\cdot\omega) = \delta\circ\phi(\alpha)$, as wanted. □

Proposition 3.5. *Let X be a smooth projective variety over $\boldsymbol{R}$, and fix $n \geq 1$. Consider the following assertions*:

(i) *The class $\omega^n \in H^n(X, \mathbb{Z}_2(n))$ has coniveau ≥ 1.*

(ii) *The class $\omega^{n+1} \in H^{n+1}(X, \mathbb{Z}_2(n+1))$ has coniveau ≥ 2.*

Then (i) *implies* (ii). *Moreover, if $\mathrm{CH}_0(X_{\boldsymbol{C}})$ is supported on a closed subvariety of $X_{\boldsymbol{C}}$ of dimension $n-1$, then the converse holds.*

Proof. Either (i) or (ii) implies that ω^{n+1} has coniveau ≥ 1, or equivalently that it vanishes in $H^{n+1}_{\mathrm{nr}}(X, \mathbb{Z}_2(n+1))$. Let us suppose this is the case; in particular, $\omega^n \in I$.

By the coniveau spectral sequence (2-3), ω^n has coniveau ≥ 1 in X if and only if its class in $H^n_{\mathrm{nr}}(X, \mathbb{Z}_2(n))$ vanishes, and ω^{n+1} has coniveau ≥ 2 if and only if its class in $H^1(X, \mathcal{H}^n_X(n+1))$ vanishes. Then consider the diagram

$$\begin{tikzcd} H^n(\boldsymbol{R}, \mathbb{Z}_2(n)) \arrow[r, "\omega", "\cong"'] \arrow[d] & H^{n+1}(\boldsymbol{R}, \mathbb{Z}_2(n+1)) \arrow[d] \\ I_{\mathrm{nr}} \arrow[r, "\delta"] & H^1(X, \mathcal{H}^n_X(n+1)) \end{tikzcd}$$

which is commutative by Lemma 3.4. Contemplating it shows that (i) implies (ii).

Conversely, if $\mathrm{CH}_0(X_{\boldsymbol{C}})$ is supported on a closed subvariety of $X_{\boldsymbol{C}}$ of dimension $n-1$, we have $H^n_{\mathrm{nr}}(X_{\boldsymbol{C}}, \mathbb{Z}_2) = 0$ by [Colliot-Thélène and Voisin 2012, Proposition 3.3(ii)]. Indeed, the argument given there for Betti cohomology over $\mathbb{C}$, which relies on decomposition of the diagonal, works as well for 2-adic cohomology over $\boldsymbol{C}$. It then follows from the exact sequence (3-2) that δ is injective, proving that (ii) implies (i). □

4. Cohomology of smooth double covers

Recall the notation of Section 3A. The polynomial $F \in \boldsymbol{R}[X_0, \dots, X_n]$ is the homogenization of a nonzero positive semidefinite polynomial $f \in \boldsymbol{R}[X_1, \dots, X_n]$.

Its degree d is even. We introduced the double cover Y of $\mathbb{P}^n_{\boldsymbol{R}}$ ramified over $\{F=0\}$ defined by the equation $Y := \{Z^2+F=0\}$.

Throughout this section, we make the additional hypothesis that $\{F=0\}$ is smooth so that Y is smooth. By Lemma 3.1, $Y(\boldsymbol{R})=\varnothing$. The main goal of this section is to prove Propositions 4.8 and 4.12.

4A. ***Geometric cohomology.*** We first collect some needed results on the cohomology of $Y_{\boldsymbol{C}}$. They follow from general theorems on the cohomology of weighted complete intersections due to Dimca [1985]. When $n=3$, we could also have applied [Clemens 1983, Corollary 1.19 and Lemma 1.23].

Proposition 4.1. *Let $H_{\boldsymbol{C}} \in H^2(Y_{\boldsymbol{C}}, \mathbb{Z}_2(1))$ be the class of $\mathcal{O}_{\mathbb{P}^n_{\boldsymbol{C}}}(1)$.*

(i) *The cohomology groups $H^k(Y_{\boldsymbol{C}}, \mathbb{Z}_2)$ have no torsion.*

(ii) *If $k \neq n$ is odd, $H^k(Y_{\boldsymbol{C}}, \mathbb{Z}_2)=0$.*

(iii) *If $0 \leq l < \frac{n}{2}$, $H^{2l}(Y_{\boldsymbol{C}}, \mathbb{Z}_2) \simeq \mathbb{Z}_2(l)$ as a G-module, and is generated by $H^l_{\boldsymbol{C}}$.*

(iv) *If $\frac{n}{2} < l \leq n$, $H^{2l}(Y_{\boldsymbol{C}}, \mathbb{Z}_2) \simeq \mathbb{Z}_2(l)$ as a G-module, and has a generator α_l such that $2\alpha_l = H^l_{\boldsymbol{C}}$.*

Proof. It suffices to prove the equalities as $\mathbb{Z}_2$-modules (this means that it is possible to forget the twist indicating the action of G), because one recovers the correct twist by noticing that the relevant cohomology groups are rationally generated by algebraic cycles.

Using the fact that $Y_{\boldsymbol{C}}$ is defined over an algebraically closed subfield that may be embedded in $\mathbb{C}$ together with the invariance of étale cohomology under an extension of algebraically closed fields, it suffices to prove the lemma when $\boldsymbol{C}=\mathbb{C}$. Moreover, by comparison with Betti cohomology, it suffices to prove it for Betti cohomology.

Since $Y_{\boldsymbol{C}}$ is a strongly smooth weighted complete intersection in the sense of [Dimca 1985], its cohomology groups have no torsion by Proposition 6(ii) of that paper. Moreover, its Betti numbers in degree $k \neq n$ are computed in Dimca's Proposition 6(i).

If $l < \frac{n}{2}$, the class $H^l_{\boldsymbol{C}} \in H^{2l}(Y_{\boldsymbol{C}}, \mathbb{Z}_2)$ cannot be divisible by 2 because the intersection product $\frac{1}{2}H^l_{\boldsymbol{C}} \cdot \frac{1}{2}H^l_{\boldsymbol{C}} \cdot H^{n-2l}_{\boldsymbol{C}} = \frac{1}{2}$ would not be an integer. It follows that $H^l_{\boldsymbol{C}}$ generates $H^{2l}(Y_{\boldsymbol{C}}, \mathbb{Z}_2)$.

If $l > \frac{n}{2}$, since $H^l_{\boldsymbol{C}} \cdot H^{n-l}_{\boldsymbol{C}} = 2$, it follows by Poincaré duality that $H^{2l}(Y_{\boldsymbol{C}}, \mathbb{Z}_2)$ is generated by a class α_l such that $2\alpha_l = H^l_{\boldsymbol{C}}$. □

4B. ***Preparation for a deformation argument.*** In the next subsections, we perform some computations on the cohomology of Y. One of the arguments, in the proofs of Lemma 4.7 and Proposition 4.12, is a reduction to the Fermat double cover $Y^\dagger := \{Z^2+F^\dagger=0\}$, where $F^\dagger := X_0^d+\cdots+X_n^d$ is the Fermat equation. This deformation argument relies on a little bit of semialgebraic geometry, so it is convenient to collect the relevant lemmas here.

Let $V := \boldsymbol{R}[X_0, \dots, X_n]_d$ be the space of degree d homogeneous polynomials viewed as an algebraic variety over $\boldsymbol{R}$. The discriminant $\Delta \subset V$ is the closed algebraic subvariety parametrizing equations that do not define smooth hypersurfaces in $\mathbb{P}^n_{\boldsymbol{R}}$. It is irreducible, and a general point of Δ defines a hypersurface with only one ordinary double point as singularities. Let $\Delta' \subset \Delta$ be the closed algebraic subvariety parametrizing singular hypersurfaces that do not have only one ordinary double point as singularities; it has codimension ≥ 2 in V. We view the sets of $\boldsymbol{R}$-points $V(\boldsymbol{R})$, $\Delta(\boldsymbol{R})$ and $\Delta'(\boldsymbol{R})$ as semialgebraic sets. Define

$$\Pi := \{H \in V(\boldsymbol{R}) \mid H(x_0, \dots, x_n) > 0 \text{ for every } (x_0, \dots, x_n) \in \boldsymbol{R}^{n+1} \setminus (0, \dots, 0)\}.$$

Lemma 4.2. *The set $\Pi \subset V(\boldsymbol{R})$ is convex, open and semialgebraic. Moreover, the polynomials F and $F^\dagger$ belong to Π.*

Proof. It is immediate that Π is convex. We prove that the complement of Π is a closed semialgebraic set. By homogeneity of H, it coincides with the projection to $V(\boldsymbol{R})$ of

$$Q := \{(H, x_0, \dots, x_n) \in V(\boldsymbol{R}) \times \mathbb{S}^n \mid H(x_0, \dots, x_n) \leq 0\},$$

where $\mathbb{S}^n := \{(x_0, \dots, x_n) \in \boldsymbol{R}^{n+1} \mid x_0^2 + \cdots + x_n^2 = 1\}$ is the unit sphere.

That it is semialgebraic follows from the Tarski–Seidenberg theorem [Bochnak et al. 1998, Theorem 2.2.1]. To check that it is closed, it suffices to check that its intersection with every closed hypercube in $V(\boldsymbol{R})$ is closed, which follows from [Bochnak et al. 1998, Theorem 2.5.8].

That $F^\dagger \in \Pi$ is clear. We know that $F \geq 0$ because it is positive semidefinite. Moreover, it cannot vanish on $\boldsymbol{R}^{n+1} \setminus (0, \dots, 0)$ because $Y(\boldsymbol{R}) \neq \varnothing$ by Lemma 3.1. This shows that $F \in \Pi$. □

Now choose a general affine subspace $W \subset V$ of dimension 2 that contains F and $F^\dagger$.

Lemma 4.3. *The set $\Pi \cap W(\boldsymbol{R}) \cap \Delta(\boldsymbol{R})$ is finite.*

Proof. Let $H \in \Pi \cap W(\boldsymbol{R}) \cap \Delta(\boldsymbol{R})$. Since $H \in \Delta(\boldsymbol{R})$, $\{H = 0\} \subset \mathbb{P}^n_{\boldsymbol{R}}$ is a singular hypersurface. Since $H \in \Pi$, $\{H = 0\}$ has no real point. Consequently, $\{H = 0\}$ has (geometrically) at least two singular points: any singular point and its distinct complex conjugate. This shows that $\Pi \cap W(\boldsymbol{R}) \cap \Delta(\boldsymbol{R}) \subset W(\boldsymbol{R}) \cap \Delta'(\boldsymbol{R})$. But if W has been chosen to properly intersect Δ', the variety $W \cap \Delta'$ is already finite. □

Lemma 4.4. *There exists a variety S over $\boldsymbol{R}$, two points $s, s^\dagger \in S(\boldsymbol{R})$ and a morphism $\rho : S \to W \setminus \Delta$ such that $S(\boldsymbol{R})$ is semialgebraically connected, $\rho(s) = F$, $\rho(s^\dagger) = F^\dagger$ and $\rho(S(\boldsymbol{R})) \subset \Pi$.*

Proof. Choose a coordinate system on W for which F has coordinate $(-1,0)$ and $F^\dagger$ has coordinate $(1,0)$. By Lemma 4.2, the segment $[F, F^\dagger]$ is included in Π and $W(\boldsymbol{R})\setminus\Pi$ is closed and semialgebraic. Consequently, combining Proposition 2.2.8(ii) and Theorem 2.5.8 of [Bochnak et al. 1998], we see that the distance between $[F, F^\dagger]$ and $W(\boldsymbol{R})\setminus\Pi$ is positive. It follows that if $\varepsilon\in\boldsymbol{R}$ is small enough, the ellipse $\{x^2+y^2/\varepsilon\leq 1\}\subset W(\boldsymbol{R})$, which contains F and $F^\dagger$, is included in Π.

Now, consider the double cover $\rho: W' := \{x^2+y^2/\varepsilon+z^2=1\}\to W$ and define $S:=\rho^{-1}(W\setminus\Delta)\subset W'$. That $\rho(S(\boldsymbol{R}))$ is included in Π and contains F and $F^\dagger$ follows from our choice of the ellipse. The semialgebraic set $W'(\boldsymbol{R})$ is a sphere $\mathbb{S}^2$, and $S(\boldsymbol{R})$ is the complement of a finite number of points in it by Lemma 4.3. This allows us to show by hand that it is semialgebraically path-connected, hence semialgebraically connected by [Bochnak et al. 1998, Proposition 2.5.13]. □

Over the base S, there is a smooth projective family $\mathcal{Y}\xrightarrow{p} S$ obtained by pulling back by ρ the universal family of smooth double covers over $W\setminus\Delta$. In particular, $\mathcal{Y}_s\simeq Y$ and $\mathcal{Y}_{s^\dagger}\simeq Y^\dagger$. Since $\rho(S(\boldsymbol{R}))\subset\Pi$, we see that $\mathcal{Y}(\boldsymbol{R})=\varnothing$.

4C. ***Cohomology over R when d ≡ 0[4].*** We start with a general lemma.

Lemma 4.5. *Let X be a smooth projective geometrically integral variety of dimension n over $\boldsymbol{R}$ such that $X(\boldsymbol{R})=\varnothing$. Then*:

(i) $H^{2n}(X,\mathbb{Z}_2(n))\simeq\mathbb{Z}_2$.

(ii) $H^{2n}(X,\mathbb{Z}_2(n+1))\simeq\mathbb{Z}/2\mathbb{Z}$.

Proof. We use the exact sequence (1-2), as well as Proposition 1.2(ii).

Consider $H^{2n}(X,\mathbb{Z}_2(n))\to H^{2n}(X_{\boldsymbol{C}},\mathbb{Z}_2)\xrightarrow{\pi_*} H^{2n}(X,\mathbb{Z}_2(n+1))\to 0$. The cohomology class of a closed point in $H^{2n}(X,\mathbb{Z}_2(n))$ pulls back to twice the cohomology class of a closed point in $H^{2n}(X_{\boldsymbol{C}},\mathbb{Z}_2)$. This shows that $H^{2n}(X,\mathbb{Z}_2(n+1))$ is torsion. From $H^{2n}(X,\mathbb{Z}_2(n+1))\to H^{2n}(X_{\boldsymbol{C}},\mathbb{Z}_2)\to H^{2n}(X,\mathbb{Z}_2(n))\to 0$, we deduce that $\mathbb{Z}_2\simeq H^{2n}(X_{\boldsymbol{C}},\mathbb{Z}_2)\to H^{2n}(X,\mathbb{Z}_2(n))$ is an isomorphism. The composition $H^{2n}(X,\mathbb{Z}_2(n))\xrightarrow{\pi^*}H^{2n}(X_{\boldsymbol{C}},\mathbb{Z}_2)\xrightarrow{\pi_*}H^{2n}(X,\mathbb{Z}_2(n))$ being multiplication by 2, we see that the image of $H^{2n}(X,\mathbb{Z}_2(n))\xrightarrow{\pi^*}H^{2n}(X_{\boldsymbol{C}},\mathbb{Z}_2)$ has index 2, so that $H^{2n}(X,\mathbb{Z}_2(n+1))=\mathbb{Z}/2\mathbb{Z}$. □

We need information about $\omega^{2n}\in H^{2n}(Y,\mathbb{Z}_2(2n))$ provided by Lemma 4.7 below. As a first step towards this result, we deal with the Fermat double cover $Y^\dagger := \{Z^2+F^\dagger=0\}$, where $F^\dagger := X_0^d+\cdots+X_n^d$.

Lemma 4.6. *Suppose n is odd and $d\equiv 0[4]$. Then $\omega^{2n}\in H^{2n}(Y^\dagger,\mathbb{Z}_2(2n))$ is zero.*

Proof. The morphism $\mu: Y^\dagger\to Q^n$ to $Q^n := \{Z^2+T_0^2+\cdots+T_n^2=0\}\subset\mathbb{P}^{n+1}_{\boldsymbol{R}}$ defined by $T_i = X_i^{d/2}$ has even degree because $d\equiv 0[4]$. By Lemma 4.5 applied to

$Y^\dagger$ and Q^n, there is a commutative diagram with surjective vertical arrows:

$$\begin{tikzcd}
\mathbb{Z}_2 = H^{2n}(Q^n_{\boldsymbol{C}}, \mathbb{Z}_2) \arrow[r, "\mu^*_{\boldsymbol{C}}"] \arrow[d] & H^{2n}(Y^\dagger_{\boldsymbol{C}}, \mathbb{Z}_2) = \mathbb{Z}_2 \arrow[d] \\
\mathbb{Z}/2\mathbb{Z} = H^{2n}(Q^n, \mathbb{Z}_2(2n)) \arrow[r, "\mu^*"] & H^{2n}(Y^\dagger, \mathbb{Z}_2(2n)) = \mathbb{Z}/2\mathbb{Z}
\end{tikzcd}$$

Since $\mu^*_{\boldsymbol{C}}$ is the multiplication by the even number $\deg(\mu)$, μ^* vanishes. Hence so does the composite $H^{2n}(\boldsymbol{R}, \mathbb{Z}_2(2n)) \to H^{2n}(Q^n, \mathbb{Z}_2(2n)) \xrightarrow{\mu^*} H^{2n}(Y^\dagger, \mathbb{Z}_2(2n))$. □

We deduce the same result for Y using a deformation argument:

Lemma 4.7. *Suppose n is odd and $d \equiv 0[4]$. Then $\omega^{2n} \in H^{2n}(Y, \mathbb{Z}_2(2n))$ is zero.*

Proof. Lemma 4.6 and the diagram

$$\begin{tikzcd}
H^{2n}(\boldsymbol{R}, \mathbb{Z}_2(2n)) \arrow[r, "\cong"] \arrow[d] & H^{2n}(\boldsymbol{R}, \mathbb{Z}/2\mathbb{Z}) \arrow[d] \\
H^{2n}(Y^\dagger, \mathbb{Z}_2(2n)) \arrow[r] & H^{2n}(Y^\dagger, \mathbb{Z}/2\mathbb{Z})
\end{tikzcd}$$

show that $H^{2n}(\boldsymbol{R}, \mathbb{Z}/2\mathbb{Z}) \to H^{2n}(Y^\dagger, \mathbb{Z}/2\mathbb{Z})$ vanishes.

Now consider the family $\mathcal{Y} \xrightarrow{p} S$ constructed at the end of Section 4B. The varieties Y and $Y^\dagger$ are members of this family and $S(\boldsymbol{R})$ is semialgebraically connected. If we were working over the field $\mathbb{R}$ of real numbers, we would use topological arguments (namely a G-equivariant version of Ehresmann's theorem applied to the fibration $p_{\mathbb{C}}^{-1}(S(\mathbb{R})) \to S(\mathbb{R})$) to show that $H^{2n}(\mathbb{R}, \mathbb{Z}/2\mathbb{Z}) \to H^{2n}(Y, \mathbb{Z}/2\mathbb{Z})$ vanishes as well. Over an arbitrary real closed field $\boldsymbol{R}$, the corresponding tools have been developed by Scheiderer [1994] and the topological arguments may be replaced by [Scheiderer 1994, Corollary 17.21].

Let us explain more precisely how to apply this result. In doing so, we freely use the notations of [Scheiderer 1994]. Consider the composition

$$\begin{aligned}
H^{2n}(\boldsymbol{R}, \mathbb{Z}/2\mathbb{Z}) \to H^{2n}(\mathcal{Y}, \mathbb{Z}/2\mathbb{Z}) \xleftarrow{\sim} H^{2n}(\mathcal{Y}_b, \mathbb{Z}/2\mathbb{Z}) \\
\to H^0(S_b, R^{2n} p_{b*}\mathbb{Z}/2\mathbb{Z}) \to H^0(S_r, i^* R^{2n} p_{b*}\mathbb{Z}/2\mathbb{Z}),
\end{aligned}$$

where the isomorphism $H^{2n}(\mathcal{Y}_b, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\sim} H^{2n}(\mathcal{Y}, \mathbb{Z}/2\mathbb{Z})$ follows from [Scheiderer 1994, Example 2.14], taking into account Proposition 1.1 and the fact that $\mathcal{Y}(\boldsymbol{R}) = \varnothing$. By proper base change [Scheiderer 1994, Theorem 16.2(b)] and comparing étale and b-cohomology using [Scheiderer 1994, Example 2.14] once again, we see that the stalk of $i^* R^{2n} p_{b*}\mathbb{Z}/2\mathbb{Z}$ is $H^{2n}(Y, \mathbb{Z}/2\mathbb{Z})$ at s and $H^{2n}(Y^\dagger, \mathbb{Z}/2\mathbb{Z})$ at $s^\dagger$. We have proven above that $H^{2n}(\boldsymbol{R}, \mathbb{Z}/2\mathbb{Z})$ vanishes in the stalk at $s^\dagger$. But we know that the sheaf $i^*R^{2n}p_{b*}\mathbb{Z}/2\mathbb{Z}$ is locally constant on S_r by [Scheiderer 1994, Corollary 17.20(b)], and that S_r is connected by [Bochnak et al. 1998, Proposition 7.5.1(i)] and because $S(\boldsymbol{R})$ is semialgebraically connected. Consequently, $H^{2n}(\boldsymbol{R},\mathbb{Z}/2\mathbb{Z})$ also vanishes in the stalk at s, so that $H^{2n}(\boldsymbol{R},\mathbb{Z}/2\mathbb{Z})\to H^{2n}(Y,\mathbb{Z}/2\mathbb{Z})$ is zero.

To conclude that $\omega^{2n}\in H^{2n}(Y,\mathbb{Z}_2(2n))$ vanishes, consider the exact diagram

$$\begin{tikzcd}
 & H^{2n}(\boldsymbol{R},\mathbb{Z}_2(2n)) \arrow[r, "\cong"] \arrow[d] & H^{2n}(\boldsymbol{R},\mathbb{Z}/2\mathbb{Z}) \arrow[d] \\
H^{2n}(Y,\mathbb{Z}_2(2n)) \arrow[r, "2"] & H^{2n}(Y,\mathbb{Z}_2(2n)) \arrow[r] & H^{2n}(Y,\mathbb{Z}/2\mathbb{Z})
\end{tikzcd}$$

and notice that the multiplication by 2 map is zero by Lemma 4.5. □

Proposition 4.8. *Suppose that $d\equiv 0[4]$. Then $\omega^{n+1}\in H^{n+1}(Y,\mathbb{Z}_2(n+1))$ is zero.*

Proof. Suppose not, and let $k\geq n+1$ be such that $\omega^k\in H^k(Y,\mathbb{Z}_2(k))$ is nonzero and $\omega^{k+1}\in H^{k+1}(Y,\mathbb{Z}_2(k+1))$ vanishes. By Proposition 1.2(ii), k exists and $k\leq 2n$.

Consider the short exact sequence (1-2) applied to Y:

$$H^k(Y,\mathbb{Z}_2(k+1))\xrightarrow{\pi^*}H^k(Y_{\boldsymbol{C}},\mathbb{Z}_2)\xrightarrow{\pi_*}H^k(Y,\mathbb{Z}_2(k))\xrightarrow{\omega}H^{k+1}(Y,\mathbb{Z}_2(k+1)).$$

By hypothesis, $\omega^k\in\operatorname{Im}(\pi_*)$. By Proposition 4.1(ii), since $\omega^k\in H^k(Y,\mathbb{Z}_2(k))$ is nonzero, k has to be even, say $k=2l$.

If l were even, we would have $H^k(Y_{\boldsymbol{C}},\mathbb{Z}_2(k+1))^G=0$ by Proposition 4.1(iv), and the Hochschild–Serre spectral sequence (1-1) would show that

$$\pi^*:H^k(Y,\mathbb{Z}_2(k+1))\to H^k(Y_{\boldsymbol{C}},\mathbb{Z}_2)$$

is zero. Consequently, $\operatorname{Im}(\pi_*)$ has no torsion by Proposition 4.1(iv). This is a contradiction and shows that l is odd.

Let $H\in H^2(Y,\mathbb{Z}_2(1))$ be the class of $\mathcal{O}_{\mathbb{P}^n_{\boldsymbol{R}}}(1)$. Since $\pi^*H^l=H^l_{\boldsymbol{C}}$, and taking into account Proposition 4.1(iv), the only class in $\operatorname{Im}(\pi_*)$ that may be nonzero is $\pi_*\alpha_l$. Consequently, we have $\omega^k=\pi_*\alpha_l$.

Choose by the Bertini theorem an l-dimensional linear subspace $\mathbb{P}^l_{\boldsymbol{R}}\subset\mathbb{P}^n_{\boldsymbol{R}}$ that is transverse to the smooth hypersurface $\{F=0\}$, and define $i:Z\hookrightarrow Y$ to be the inverse image of $\mathbb{P}^l_{\boldsymbol{R}}$ in Y; it is a smooth double cover of $\mathbb{P}^l_{\boldsymbol{R}}$ ramified over $\{F=0\}\cap\mathbb{P}^l_{\boldsymbol{R}}$. Consider the following commutative diagram, where the left horizontal arrows are restrictions, the right horizontal arrows are Gysin morphisms and the vertical ones are those appearing in the exact sequence (1-2):

$$\begin{tikzcd}
H^k(Y_{\boldsymbol{C}},\mathbb{Z}_2) \arrow[r, "i^*_{\boldsymbol{C}}"] \arrow[d, "\pi_*"] & H^k(Z_{\boldsymbol{C}},\mathbb{Z}_2) \arrow[r, "i_{\boldsymbol{C}*}"] \arrow[d, "\pi_*"] & H^{2n}(Y_{\boldsymbol{C}},\mathbb{Z}_2) \arrow[d, "\pi_*"] \\
H^k(Y,\mathbb{Z}_2(k)) \arrow[r, "i^*"] & H^k(Z,\mathbb{Z}_2(k)) \arrow[r, "i_*"] & H^{2n}(Y,\mathbb{Z}_2(n+l))
\end{tikzcd}$$

Look at $\alpha_l \in H^k(Y_{\boldsymbol{C}}, \mathbb{Z}_2)$. We have $i_* i^* \pi_* \alpha_l = i_* i^* \omega^k = i_* \omega^k = 0$ by Lemma 4.7 applied to Z. On the other hand, $\pi_* i_{\boldsymbol{C}*} i^*_{\boldsymbol{C}} \alpha_l = \pi_*(\alpha_l \cdot [Z_{\boldsymbol{C}}]) = \pi_*(\alpha_l \cdot H_{\boldsymbol{C}}^{n-l}) = \pi_* \alpha_n \neq 0$ because it is the generator of $H^{2n}(Y, \mathbb{Z}_2(n+l)) \simeq \mathbb{Z}/2\mathbb{Z}$ as seen in Lemma 4.5. This is a contradiction. □

4D. ***Cohomology over R when n is odd.*** As in the previous paragraph, we reduce the computations we need to the case of a quadric. To do this, we collect a few results about the cohomology of quadrics. Analogues with 2-torsion coefficients of some of these computations appear in [Kahn and Sujatha 2000, §4].

We denote by $Q^n := \{Z^2 + T_0^2 + \cdots + T_n^2 = 0\} \subset \mathbb{P}^{n+1}_{\boldsymbol{R}}$ the n-dimensional projective anisotropic quadric, by $U^n := Q^n \setminus Q^{n-1}$ its affine counterpart and by $H \in H^2(Q^n, \mathbb{Z}_2(1))$ the class of $\mathcal{O}_{\mathbb{P}^{n+1}_{\boldsymbol{R}}}(1)$.

Lemma 4.9. *The cohomology groups of U^n are as follows*:

$$H^k(U^n, \mathbb{Z}_2(j)) = \begin{cases} \mathbb{Z}_2 & \textit{if } k = 0 \textit{ and } j \equiv 0[2], \\ \mathbb{Z}/2\mathbb{Z} \cdot \omega^k & \textit{if } 1 \leq k \leq n \textit{ and } j \equiv k[2], \\ \mathbb{Z}_2 & \textit{if } k = n \textit{ and } j \equiv n+1[2], \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The geometric cohomology groups of U^n are easily computed as G-modules from the description of the geometric cohomology groups of Q^n and Q^{n-1} as G-modules given in [SGA 7_{II} 1973, XII Théorème 3.3], and from the long exact sequence of cohomology with support associated with $Q^{n-1}_{\boldsymbol{C}} \subset Q^n_{\boldsymbol{C}}$:

$$\cdots \to H^{k-2}(Q^{n-1}_{\boldsymbol{C}}, \mathbb{Z}_2(-1)) \to H^k(Q^n_{\boldsymbol{C}}, \mathbb{Z}_2) \to H^k(U^n_{\boldsymbol{C}}, \mathbb{Z}_2) \to \cdots .$$

One gets

$$H^k(U^n_{\boldsymbol{C}}, \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{if } k = 0, \\ \mathbb{Z}_2(n+1) & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the Hochschild–Serre spectral sequence (1-1) for U^n. The only possibly nonzero arrows in this spectral sequence are the $d_n : E^{p,n}_{n+1} \to E^{p+n+1,0}_{n+1}$. These are necessarily surjective as $H^k(U^n, \mathbb{Z}_2(j)) = 0$ for $k > n$ by Proposition 1.2(iii). This allows us to compute the spectral sequence entirely, and to deduce the lemma. □

Lemma 4.10. *Suppose that n is odd. Then $H^n(Q^n, \mathbb{Z}_2(n)) \simeq (\mathbb{Z}/2\mathbb{Z})^{\lceil n/4 \rceil}$ generated by $\omega^n, \omega^{n-4}H^2, \ldots, \omega^{n-4\lfloor n/4 \rfloor}H^{2\lfloor n/4 \rfloor}$. Moreover, the 2-torsion subgroup of $H^{n+1}(Q^n, \mathbb{Z}_2(n+1))$ is $H^{n+1}(Q^n, \mathbb{Z}_2(n+1))[2] \simeq (\mathbb{Z}/2\mathbb{Z})^{\lceil n/4 \rceil}$, generated by $\omega^{n+1}, \omega^{n-3}H^2, \ldots, \omega^{n+1-4\lfloor n/4 \rfloor}H^{2\lfloor n/4 \rfloor}$.*

Proof. Fix $r \geq 0$. The Gysin morphism $H^{n-2r-2}(Q^{n-r-1}, \mathbb{Z}_2(n-r-1)) \to H^{n-2r}(Q^{n-r}, \mathbb{Z}_2(n-r))$ is part of a long exact sequence of cohomology with supports. In this exact sequence, the morphisms

$$H^{n-2r-1}(Q^{n-r}, \mathbb{Z}_2(n-r)) \to H^{n-2r-1}(U^{n-r}, \mathbb{Z}_2(n-r)),$$
$$H^{n-2r}(Q^{n-r}, \mathbb{Z}_2(n-r)) \to H^{n-2r}(U^{n-r}, \mathbb{Z}_2(n-r))$$

are surjective. Indeed, in the degrees that come up, all the cohomology of U^{n-r} comes from the base field by Lemma 4.9, hence a fortiori from Q^{n-r}.

It follows that this Gysin morphism is injective, with a cokernel naturally isomorphic to $H^{n-2r}(U^{n-r}, \mathbb{Z}_2(n-r)) = H^{n-2r}(\boldsymbol{R}, \mathbb{Z}_2(n-r))$. This allows us to compute $H^{n-2r}(Q^{n-r}, \mathbb{Z}_2(n-r))$ by decreasing induction on r and shows, since n is odd, that $H^n(Q^n, \mathbb{Z}_2(n)) \simeq (\mathbb{Z}/2\mathbb{Z})^{\lceil n/4 \rceil}$ generated by $\omega^n, \omega^{n-4}H^2, \dots, \omega^{n-4\lfloor n/4 \rfloor} H^{2\lfloor n/4 \rfloor}$.

Considering the long exact sequence (1-2), and using our knowledge of the geometric cohomology of Q^n to get

$$0 \to H^n(Q^n, \mathbb{Z}_2(n)) \xrightarrow{\omega} H^{n+1}(Q^n, \mathbb{Z}_2(n+1)) \to H^{n+1}(Q^n_{\boldsymbol{C}}, \mathbb{Z}_2) \simeq \mathbb{Z}_2, \quad (4\text{-}1)$$

the lemma follows. □

In the remainder of this section, we continue to suppose that n is odd. We consider the generator γ of $H^{n+1}(Q^n_{\boldsymbol{C}}, \mathbb{Z}_2) \simeq \mathbb{Z}_2$ such that $2\gamma = H_{\boldsymbol{C}}^{(n+1)/2}$ [SGA 7_{II} 1973, XII Théorème 3.3].

If $n \equiv 1[4]$, define $\delta := \pi_*\gamma \in H^{n+1}(Q^n, \mathbb{Z}_2(n+1))$. If $n \equiv 3[4]$, define $\delta := \pi_*\gamma - H^{(n+1)/2} \in H^{n+1}(Q^n, \mathbb{Z}_2(n+1))$.

Lemma 4.11. *Suppose that n is odd. Then $\delta \in H^{n+1}(Q^n, \mathbb{Z}_2(n+1))[2]$. Moreover, δ does not belong to the subgroup of $H^{n+1}(Q^n, \mathbb{Z}_2(n+1))[2]$ generated by $\omega^{n-3}H^2, \dots, \omega^{n+1-4\lfloor n/4 \rfloor} H^{2\lfloor n/4 \rfloor}$.*

Proof. We suppose that $n \equiv 1[4]$. The arguments when $n \equiv 3[4]$ are analogous.

Since G acts on $H^{n+1}(Q^n_{\boldsymbol{C}}, \mathbb{Z}_2) \simeq \mathbb{Z}_2$ by multiplication by $(-1)^{(n+1)/2}$ and since $n+1 \not\equiv \frac{1}{2}(n+1)[2]$, the natural morphism $H^{n+1}(Q^n, \mathbb{Z}_2(n+1)) \to H^{n+1}(Q^n_{\boldsymbol{C}}, \mathbb{Z}_2)$ is zero. From the exact sequence (4-1), this implies that $H^{n+1}(Q^n, \mathbb{Z}_2(n+1))$ is a 2-torsion group, so that δ is 2-torsion.

Let us check that δ is nonzero. From the exact sequence

$$H^{n+1}(Q^n, \mathbb{Z}_2(n)) \xrightarrow{\pi^*} H^{n+1}(Q^n_{\boldsymbol{C}}, \mathbb{Z}_2) \xrightarrow{\pi_*} H^{n+1}(Q^n, \mathbb{Z}_2(n+1)),$$

we see that it suffices to prove that there does not exist a cohomology class $\zeta \in H^{n+1}(Q^n, \mathbb{Z}_2(n))$ such that $\pi^*\zeta = \gamma$. If such a class existed, $\pi^*(\zeta \cdot H^{(n-1)/2})$ would be a generator of $H^{2n}(Q^n_{\boldsymbol{C}}, \mathbb{Z}_2)$. This would contradict the fact, proven in Lemma 4.5, that the cokernel of

$$H^{2n}(Q^n, \mathbb{Z}_2(n)) \to H^{2n}(Q^n_{\boldsymbol{C}}, \mathbb{Z}_2)$$

is $H^{2n}(Q^n, \mathbb{Z}_2(n+1)) \simeq \mathbb{Z}/2\mathbb{Z}$.

Consider now the commutative diagram below, whose horizontal arrows are Gysin morphisms:

$$\begin{tikzcd} H^{n-3}(Q^{n-2},\mathbb{Z}_2(n-1)) \arrow[r] \arrow[d, "\omega"] & H^{n-1}(Q^{n-1},\mathbb{Z}_2(n)) \arrow[r] \arrow[d, "\omega"] & H^{n+1}(Q^{n},\mathbb{Z}_2(n+1)) \arrow[d, "\omega"] \\ H^{n-2}(Q^{n-2},\mathbb{Z}_2(n)) \arrow[r] & H^{n}(Q^{n-1},\mathbb{Z}_2(n+1)) \arrow[r] & H^{n+2}(Q^{n},\mathbb{Z}_2(n+2)) \end{tikzcd} \tag{4-2}$$

Let us show that the lower horizontal arrows of (4-2) are injective. Considering $\omega^{n-1} \in H^{n-1}(Q^{n-1},\mathbb{Z}_2(n+1))$ and using Lemma 4.9 shows that

$$H^{n-1}(Q^{n-1},\mathbb{Z}_2(n+1)) \to H^{n-1}(U^{n-1},\mathbb{Z}_2(n+1))$$

is surjective, so that the map

$$H^{n-2}(Q^{n-2},\mathbb{Z}_2(n)) \to H^{n}(Q^{n-1},\mathbb{Z}_2(n+1))$$

is injective. Since $H^{n+1}(U^n,\mathbb{Z}_2(n+2)) = 0$ by Lemma 4.9,

$$H^{n}(Q^{n-1},\mathbb{Z}_2(n+1)) \to H^{n+2}(Q^{n},\mathbb{Z}_2(n+2))$$

is also injective.

Suppose for contradiction that δ may be written as a linear combination of the classes $\omega^{n-3}H^2,\dots,\omega^{n+1-4\lfloor n/4\rfloor}H^{2\lfloor n/4\rfloor}$. Then it is the image by the Gysin morphism $H^{n-3}(Q^{n-2},\mathbb{Z}_2(n-1)) \to H^{n+1}(Q^{n},\mathbb{Z}_2(n+1))$ of a class ϵ that is a linear combination of $\omega^{n-3},\dots,\omega^{n+1-4\lfloor n/4\rfloor}H^{2\lfloor n/4\rfloor-2}$. The image of ϵ in $H^{n+2}(Q^n,\mathbb{Z}_2(n+2))$ is $\delta\cdot\omega = \pi_*\gamma\cdot\omega = 0$. By the injectivity result just proved, $\epsilon\cdot\omega = 0$. It follows that $\omega^{n-2},\dots,\omega^{n+2-4\lfloor n/4\rfloor}H^{2\lfloor n/4\rfloor-2}$ are not independent in $H^{n-2}(Q^{n-2},\mathbb{Z}_2(n))$, contradicting Lemma 4.10. □

We return to our double cover $Y \to \mathbb{P}^n_{\boldsymbol{R}}$. We recall from Proposition 4.1 that $H^{n+1}(Y_{\boldsymbol{C}},\mathbb{Z}_2) = \mathbb{Z}_2(n+1)$ with a generator $\alpha := \alpha_{(n+1)/2}$ such that $2\alpha = H_{\boldsymbol{C}}^{(n+1)/2}$.

If $n \equiv 1[4]$, define $\beta := \pi_*\alpha \in H^{n+1}(Y,\mathbb{Z}_2(n+1))$. If $n \equiv 3[4]$, define instead $\beta := \pi_*\alpha - H^{(n+1)/2} \in H^{n+1}(Y,\mathbb{Z}_2(n+1))$, where $H \in H^2(Y,\mathbb{Z}_2(1))$ is the class of $\mathcal{O}_{\mathbb{P}^n_{\boldsymbol{R}}}(1)$.

Proposition 4.12. *Suppose that n is odd. Then ω^{n+1} is a linear combination of $\omega^{n-3}H^2,\dots,\omega^{n+1-4\lfloor n/4\rfloor}H^{2\lfloor n/4\rfloor}$ and β in $H^{n+1}(Y,\mathbb{Z}_2(n+1))$.*

Proof. Using a deformation argument as in the proof of Lemma 4.7, one reduces to the case of the Fermat double cover $Y^\dagger := \{Z^2 + F^\dagger = 0\}$, where $F^\dagger := X_0^d + \cdots + X_n^d$.

As in Lemma 4.6, we consider the morphism $\mu : Y^\dagger \to Q^n$ to the quadric $Q^n := \{Z^2 + T_0^2 + \cdots + T_n^2 = 0\} \subset \mathbb{P}^{n+1}_{\boldsymbol{R}}$, defined by $T_i = X_i^{d/2}$. Note that $\mu^*H = \left(\frac{d}{2}\right)H$, so that $\mu^*_{\boldsymbol{C}}\gamma = \left(\frac{d}{2}\right)^{(n+1)/2}\alpha$ and $\mu^*\delta = \left(\frac{d}{2}\right)^{(n+1)/2}\beta$.

By Lemma 4.10, the group $H^{n+1}(Q^n,\mathbb{Z}_2(n+1))[2]$ is freely generated by $\omega^{n+1},\omega^{n-3}H^2,\dots,\omega^{n+1-4\lfloor n/4\rfloor}H^{2\lfloor n/4\rfloor}$. By Lemma 4.11, δ does not belong to the subgroup generated by $\omega^{n-3}H^2, \dots, \omega^{n+1-4\lfloor n/4\rfloor}H^{2\lfloor n/4\rfloor}$. Thus, ω^{n+1} is a linear combination of $\delta, \omega^{n-3}H^2, \dots, \omega^{n+1-4\lfloor n/4\rfloor}H^{2\lfloor n/4\rfloor}$ in $H^{n+1}(Q^n, \mathbb{Z}_2(n+1))[2]$.

Pulling back this relation by μ, it follows that ω^{n+1} is a linear combination of $\beta, \omega^{n-3}H^2, \dots, \omega^{n+1-4\lfloor n/4\rfloor}H^{2\lfloor n/4\rfloor}$ in $H^{n+1}(Y^\dagger, \mathbb{Z}_2(n+1))$, as wanted. □

Corollary 4.13. *Suppose that n is odd. If $\alpha \in H^{n+1}(Y_{\boldsymbol{C}}, \mathbb{Z}_2)$ has coniveau ≥ 2, then $\omega^{n+1} \in H^{n+1}(Y, \mathbb{Z}_2(n+1))$ has coniveau ≥ 2.*

Proof. The statement follows from Proposition 4.12, because the cohomology classes $\omega^{n-3}H^2, \dots, \omega^{n+1-4\lfloor n/4\rfloor}H^{2\lfloor n/4\rfloor}$, as well as $H^{(n+1)/2}$ when $n \neq 1$, obviously have coniveau ≥ 2 as multiples of H^2. □

5. A geometric coniveau computation

In this section, we fix an odd integer $n \geq 3$ and take $d := 2n$.

We work over an algebraically closed field $\boldsymbol{C}$ of characteristic 0. We consider $F \in \boldsymbol{C}[X_0, \dots, X_n]_d$ a homogeneous degree d polynomial such that $\{F = 0\}$ is smooth. Let Y be the double cover of $\mathbb{P}^n_{\boldsymbol{C}}$ ramified over $\{F = 0\}$ defined by the equation $Y := \{Z^2 + F = 0\}$, and $H := \mathcal{O}_{\mathbb{P}^n_{\boldsymbol{C}}}(1)$.

By Proposition 4.1, $H^{n+1}(Y, \mathbb{Z}_2)$ has a generator α such that $2\alpha = H^{(n+1)/2}$. In order to apply Corollary 4.13, we need to answer positively the following:

Question 5.1. Does α have coniveau ≥ 2?

When $n = 3$, Y is a sextic double solid, and this is very easy:

Lemma 5.2. *If $n = 3$, α has coniveau ≥ 2.*

Proof. A dimension count (see for instance Lemma 5.6 below) shows that Y contains a line, that is, a curve of degree 1 against H. The cohomology class of such a curve is α, so that α is algebraic, hence of coniveau 2. □

In what follows, we answer Question 5.1 positively when $n = 5$, following an argument of Voisin. We comment on the $n \geq 7$ case in Section 5D.

Proposition 5.3. *If $n = 5$, α has coniveau ≥ 2.*

5A. *Reductions.* The following reductions are standard. We include them because we do not know a convenient reference.

Lemma 5.4. *Let $\boldsymbol{C} \subset \boldsymbol{C}'$ be an extension of algebraically closed fields. Then the answer to Question 5.1 is positive for Y over $\boldsymbol{C}$ if and only if it is for $Y_{\boldsymbol{C}'}$ over $\boldsymbol{C}'$.*

Proof. It is clear that if α has coniveau ≥ 2 over $\boldsymbol{C}$, it has coniveau ≥ 2 over $\boldsymbol{C}'$.

Suppose conversely that there is a closed subset $Z \subset Y_{\boldsymbol{C}'}$ of codimension ≥ 2 such that $\alpha_{\boldsymbol{C}'}$ vanishes in $Y_{\boldsymbol{C}'} \setminus Z$. Taking an extension of finite type of $\boldsymbol{C}$ over which Z is defined, spreading out and shrinking the base gives a smooth integral variety B over $\boldsymbol{C}$, and a subvariety $Z_B \subset Y \times B$ of codimension ≥ 2 in the fibers of $p_2 : Y \times B \to B$ such that $p_1^*\alpha$ vanishes in the generic geometric fiber of $p_2 : (Y \times B \setminus Z_B) \to B$. The existence of cospecialization maps for smooth morphisms [SGA 4½ 1977, Arcata V (1.6)] implies that α vanishes in every geometric fiber of $p_2 : (Y \times B \setminus Z_B) \to B$. Taking the fiber over a $\boldsymbol{C}$-point of B shows that α has coniveau ≥ 2. □

Lemma 5.5. *In order to answer Question 5.1 positively in general, it suffices to answer it over the field $\mathbb{C}$ of complex numbers, for a general choice of F.*

Proof. Let $U \subset \mathbb{C}[X_0, \dots, X_n]_d$ be a Zariski-open subset of degree d polynomials F as in the hypothesis: for $F \in U(\mathbb{C})$, $\{F = 0\}$ is smooth and α is of coniveau ≥ 2.

Let K be an algebraic closure of the function field $\mathbb{C}(U)$, let F_K be the generic polynomial and let Y_K be the associated universal double cover. Choosing an isomorphism $K \simeq \mathbb{C}$ such that the induced polynomial $F_{\mathbb{C}}$ belongs to U shows that the answer to Question 5.1 is positive for Y_K.

Let us now deal with the case $\boldsymbol{C} = \mathbb{C}$. It is possible to find the spectrum T of a strictly henselian discrete valuation ring and a morphism $T \to \mathbb{C}[X_0, \dots, X_n]_d$ sending the closed point of T to the polynomial associated to Y and its generic point to the generic point of U. Let Y_T be the induced family of double covers. Up to replacing T by a finite extension, there exists a codimension 2 subset $Z \subset Y_T$ flat over T such that α vanishes in the complement of Z in the generic geometric fiber. Using cospecialization maps again shows that α vanishes in the complement of Z in the special fiber, so that the answer to Question 5.1 is positive for Y.

In general, choose an algebraically closed subfield of finite transcendence degree of $\boldsymbol{C}$ over which Y is defined, embed it in $\mathbb{C}$, and apply Lemma 5.4. □

5B. ***The variety of lines.*** We define $F(Y)$ to be the Fano variety of lines of Y, that is, the Hilbert scheme of Y parametrizing degree 1 curves in Y. We also introduce the universal family $I \subset Y \times F(Y)$ and denote by $q : I \to Y$ and $p : I \to F(Y)$ the natural projections.

The following lemma is well known for hypersurfaces [Barth and Van de Ven 1978/79, §3] or complete intersections [Debarre and Manivel 1998, Théorème 2.1], and the proof in our situation is similar.

Lemma 5.6. *If Y is general, $F(Y)$ is smooth, nonempty and of dimension $n - 2$.*

Proof. First, an easy dimension count shows that if F is general, $\{F = 0\}$ contains no line [Barth and Van de Ven 1978/79, §3]. It follows that for such a general Y, $F(Y)$ is a double étale cover of the variety $G(Y)$ of lines in $\mathbb{P}^n_{\boldsymbol{C}}$ on which the restriction of F is a square.

One introduces the space $V \subset \boldsymbol{C}[X_0, \dots, X_n]_d$ of degree d polynomials F such that $\{F = 0\}$ is smooth. We consider the universal double cover $Y_V \to V$, and the universal Fano variety of lines $F(Y)_V \to V$, viewed as a double cover of $G(Y)_V$

that is étale generically over V. Looking at the natural projection from $G(Y)_V$ to the grassmannian of lines in $\mathbb{P}^n_{\boldsymbol{C}}$, one sees that $G(Y)_V$ is smooth of dimension $\dim(V)+n-2$. Since we are in characteristic 0, the general fiber of $G(Y)_V \to V$ is smooth, and it remains to show that this morphism is dominant, or that $F(Y)_V \to V$ is dominant. We do it by finding one point at which it is smooth.

To do so, we fix a line L with equations $X_2 = \cdots = X_n = 0$, and a degree d polynomial F such that the restriction of F to L is the square of $H \in \boldsymbol{C}[X_0, X_1]_n$. The double cover Y may be viewed naturally as the zero-locus of a section of $\mathcal{O}(d)$ in the total space $E \to \mathbb{P}^n_{\boldsymbol{C}}$ of the line bundle $\mathcal{O}_{\mathbb{P}^n_{\boldsymbol{C}}}(n)$. The inverse image of L in Y splits into the union of two lines. Let Λ be one of them. The normal exact sequence $0 \to N_{\Lambda/Y} \to N_{\Lambda/E} \to N_{Y/E}|_{\Lambda} \to 0$ reads

$$0 \to N_{\Lambda/Y} \to \mathcal{O}(1)^{\oplus n-1} \oplus \mathcal{O}(n) \to \mathcal{O}(2n) \to 0.$$

The same computation as the one carried out in [Barth and Van de Ven 1978/79, §2] for hypersurfaces shows that the last arrow is given by $(\partial F/\partial X_2, \ldots, \partial F/\partial X_n, H)$. Consequently, if $H^0(\Lambda, \mathcal{O}(2n))$ is generated by multiples of $\partial F/\partial X_2, \ldots, \partial F/\partial X_n$ and H, then $H^1(\Lambda, N_{\Lambda/Y}) = 0$ and Λ corresponds to a smooth point of the relative Hilbert scheme $F(Y)_V \to V$, as wanted. It is easy to find a polynomial F satisfying this condition. □

Lemma 5.7. *If there exists a smooth Y over $\mathbb{C}$ such that $F(Y)$ is smooth of dimension $n-2$, and a cohomology class $\zeta \in H^{n-1}(F(Y), \mathbb{Z}_2)$ such that $q_* p^* \zeta$ is an odd multiple of α, then Question 5.1 has a positive answer.*

Proof. By Lemma 5.5 and Lemma 5.6, it suffices to consider a double cover over the complex numbers whose variety of lines is smooth of dimension $n-2$. By Ehresmann's theorem, the existence of a cohomology class ζ as in our hypothesis does not depend on Y (as long as Y and $F(Y)$ are smooth). Consequently, it suffices to answer Question 5.1 for a double cover Y for which such a ζ exists.

On the one hand $2\alpha = H^{(n+1)/2}$ is algebraic, hence of coniveau ≥ 2. On the other hand ζ has coniveau ≥ 1 because it vanishes on any affine open subset of $F(Y)$. It follows that $q_* p^* \zeta$ has coniveau ≥ 2 because $q : I \to Y$ is not dominant by dimension. Combining these two assertions, we see that α has coniveau ≥ 2. □

5C. ***A degeneration argument.*** In this subsection, we set $n = 5$ and $d = 10$, and we prove Proposition 5.3 by checking the hypothesis of Lemma 5.7.

To do so, we choose four homogeneous polynomials $P \in \mathbb{C}[X_0, \ldots, X_5]_5$, $G \in \mathbb{C}[X_1, X_2, X_3]_5$, and $Q_1, Q_2 \in \mathbb{C}[X_0, \ldots, X_5]_9$, and we set

$$F_0 := X_0^{10} + PG + Q_1 X_4 + Q_2 X_5 \in \mathbb{C}[X_0, \ldots, X_5]_{10}.$$

The reason for this choice is that F_0 restricts to a square on the cone $\Gamma := \{G = X_4 = X_5 = 0\}$, so that the inverse image of Γ in $Y_0 := \{Z^2 + F_0 = 0\}$ has two irreducible components, giving rise to two 1-dimensional families of lines in Y_0. We denote by $\Phi_0 \subset F(Y_0)$ the curve corresponding to one of these families; it is naturally isomorphic to $\{G = 0\} \subset \mathbb{P}^2_{\mathbb{C}}$.

Lemma 5.8. *If P, G, Q_1, Q_2 have been chosen general, then Y_0 is smooth, Φ_0 is smooth and $F(Y_0)$ is smooth of dimension 3 along Φ_0.*

Proof. To check that the general zero-locus of such an F_0 is smooth, it suffices to deal with equations of the form $\lambda X_0^{10} + \mu X_1^{10} + Q_1 X_4 + Q_2 X_5 \in \mathbb{C}[X_0, \dots, X_5]_{10}$. These form a linear system, so a general one among these is smooth outside of the base locus $\{X_0 = X_1 = X_4 = X_5 = 0\}$ by the Bertini theorem. But there exists a particular one that is smooth on the base locus: take $Q_1 = X_2^9$ and $Q_2 = X_3^9$. It follows that the general one is smooth everywhere.

To conclude, fix G such that $\Phi_0 \simeq \{G = 0\} \subset \mathbb{P}^2_{\mathbb{C}}$ is smooth. It suffices to prove that for every $\Lambda \in \Phi_0$, $F(Y_0)$ is smooth of dimension 3 at Λ, with possible exceptions on a codimension 2 subset of the parameter space for P, Q_1 and Q_2. By the computations in the proof of Lemma 5.6, we need to show that, outside such a subset, $H^0(\Lambda, \mathcal{O}(10))$ is generated by multiples of $\partial F_0/\partial X_2, \dots, \partial F_0/\partial X_5$ and X_0^5.

This amounts to showing that, outside of a codimension 2 subset of the parameter space for $P \in \mathbb{C}[X_0, X_1]_5$ and $Q_1, Q_2 \in \mathbb{C}[X_0, X_1]_9$, $H^0(\mathbb{P}^1, \mathcal{O}(10))$ is generated by multiples of X_0^5, PX_1^4, Q_1 and Q_2. This is easy to see, by exhibiting a complete curve in the projectivized parameter space avoiding the bad locus. $\square$

Now, let Δ be a small enough disk in $\mathbb{C}[X_0, \dots, X_5]_{10}$ centered around the polynomial F_0 given by Lemma 5.8. Let Y_Δ be the family of double covers over it, and $F(Y)_\Delta$ the corresponding family of varieties of lines.

Recall that Φ_0 is a smooth proper subvariety of the smooth locus of the special fiber $F(Y_0)$. Using the flow of a vector field as in the proof of Ehresmann's theorem, one sees that Φ_0 deforms (as a differentiable submanifold) to nearby fibers for which $F(Y_t)$ is smooth, giving rise to a cohomology class $\zeta_t = [\Phi_t] \in H^{n-1}(F(Y_t), \mathbb{Z}_2)$. We compute $q_* p^* \zeta_t = q_*[p^{-1}(\Phi_t)] = q_*[p^{-1}(\Phi_0)]$; it is the cycle class of the cone Γ that is equal to 5α.

5D. *Remarks.* When $n \geq 7$, an argument analogous to that of Section 5C fails, because one gets a double cover Y_0 whose variety of lines is singular along a codimension 2 subset of the subvariety Φ_0 that we would like to deform to nearby fibers $F(Y_t)$.

It might still be possible to show, by another argument, the existence of a cohomology class ζ allowing application of Lemma 5.7. To do so, one would need to compute part of the integral cohomology of $F(Y)$. The rational cohomology of $F(Y)$ in the required degree is well understood thanks to [Debarre and Manivel 1998, Théorème 3.4] (where the computations are carried out in the analogous setting of hypersurfaces or complete intersections). However, Debarre and Manivel's approach, relying on the Hodge decomposition, does not allow control of the integral cohomology groups of $F(Y)$.

6. Proof of the main theorem

We now come back to our main goal: the proof of Theorem 0.1.

6A. ***The generic case.*** Let us first put together what we have obtained so far. Fix $n \geq 2$. Define $d(n)$ by setting $d(n) := 2n$ if n is even or equal to 3 or 5 and $d(n) := 2n-2$ if $n \geq 7$ is odd.

Proposition 6.1. *Let $f \in \boldsymbol{R}[X_1, \dots, X_n]$ be a positive semidefinite polynomial of degree $d(n)$ whose homogenization F defines a smooth hypersurface in $\mathbb{P}^n_{\boldsymbol{R}}$. Then f is a sum of $2^n - 1$ squares in $\boldsymbol{R}(X_1, \dots, X_n)$.*

Proof. We consider the double cover Y of $\mathbb{P}^n_{\boldsymbol{R}}$ ramified over $\{F = 0\}$ defined by the equation $Y := \{Z^2 + F = 0\}$. The variety Y is smooth, $Y(\boldsymbol{R}) = \varnothing$ by Lemma 3.1, and computing that the anticanonical bundle $-K_Y = \mathcal{O}_{\mathbb{P}^n_{\boldsymbol{R}}}(n+1-d(n)/2)$ of Y is ample, one sees that $Y_{\boldsymbol{C}}$ is Fano, hence rationally connected.

By Proposition 3.2, we need to show that the level of $\boldsymbol{R}(Y)$ is $< 2^n$. Applying Proposition 3.3 (iii)$\Rightarrow$(i), we have to prove that $\omega^n \in H^n(Y, \mathbb{Z}_2(n))$ has coniveau ≥ 1. Finally, since $Y_{\boldsymbol{C}}$ is rationally connected, the converse Proposition 3.5 (ii)$\Rightarrow$(i) holds: we only have to check that $\omega^{n+1} \in H^{n+1}(Y, \mathbb{Z}_2(n+1))$ has coniveau ≥ 2.

When $n \neq 3$ or 5, $d(n) \equiv 0[4]$ so that $\omega^{n+1} \in H^{n+1}(Y, \mathbb{Z}_2(n+1))$ vanishes by Proposition 4.8.

When $n = 3$ or 5, $\omega^{n+1} \in H^{n+1}(Y, \mathbb{Z}_2(n+1))$ is seen to be of coniveau ≥ 2 by combining Corollary 4.13 and either Lemma 5.2 when $n = 3$ or Proposition 5.3 when $n = 5$. $\square$

6B. ***A specialization argument.*** We do not know how to deal with singular equations using the same arguments because one has too little control on the geometry of (a resolution of singularities of) the variety Y. Instead, we rely on a specialization argument, that will also take care of the lower values of the degree.

Theorem 6.2 (Theorem 0.1). *Let $f \in \boldsymbol{R}[X_1, \dots, X_n]$ be a positive semidefinite polynomial of degree $\leq d(n)$. Then f is a sum of $2^n - 1$ squares in $\boldsymbol{R}(X_1, \dots, X_n)$.*

Proof. Consider $g := f + t\big(1 + \sum_{i=1}^n X_i^{d(n)}\big) \in \boldsymbol{R}(t)[X_1, \dots, X_n]$. It is a degree $d(n)$ polynomial whose homogenization defines a smooth hypersurface in $\mathbb{P}^n_{\boldsymbol{R}}$, because so does its specialization $1 + \sum_{i=1}^n X_i^{d(n)}$. Let $\mathbf{S} := \bigcup_r \boldsymbol{R}((t^{1/r}))$ be a real closed extension of $\boldsymbol{R}(t)$. By Artin's solution [1927] to Hilbert's 17th problem, f is a sum of squares in $\boldsymbol{R}(X_1, \dots, X_n)$, hence still a positive semidefinite polynomial viewed in $\mathbf{S}[X_1, \dots, X_n]$. Consequently, since $t = (t^{1/2})^2$ is a square in $\mathbf{S}$,

$g \in \mathbf{S}[X_1, \ldots, X_n]$ is a positive semidefinite polynomial. Applying Proposition 6.1 over the real closed field $\mathbf{S}$, we see that g is a sum of $2^n - 1$ squares in $\mathbf{S}(X_1, \ldots, X_n)$: one has $g = \sum_{i=1}^{2^n-1} h_i^2$.

Consider the t-adic valuation on $\mathbf{S}$. Applying n times successively [Bourbaki 1985, Chapitre VI §10, Proposition 2], we can extend it to a valuation v on $\mathbf{S}(X_1, \ldots, X_n)$ that is trivial on $\boldsymbol{R}(X_1, \ldots, X_n)$, and whose residue field is isomorphic to $\boldsymbol{R}(X_1, \ldots, X_n)$. Note that these choices imply that $v(g) = 0$ and that the reduction of g modulo v is $f \in \boldsymbol{R}(X_1, \ldots, X_n)$.

Define $m := \inf_i v(h_i)$ and notice that $m \leq 0$ because $v(g) = 0$. Suppose for contradiction that $m < 0$ and let j be such that $v(h_j) = m$. Then it is possible to reduce the equality $gh_j^{-2} = \sum_{i=1}^{2^n-1} (h_i h_j^{-1})^2$ modulo v. This is absurd because we get a nontrivial sum of squares that is zero in $\boldsymbol{R}(X_1, \ldots, X_n)$. This shows that $m = 0$. Consequently, it is possible to reduce the equality $g = \sum_{i=1}^{2^n-1} h_i^2$ modulo v, showing that f is a sum of $2^n - 1$ squares in $\boldsymbol{R}(X_1, \ldots, X_n)$ as wanted. □

We conclude by stating explicitly the following consequence of our proof.

Proposition 6.3. *If $n \geq 3$ is odd and if Question 5.1 has a positive answer, then Theorem 0.1 also holds in n variables and degree $d = 2n$.*

Acknowledgements

I have benefited from numerous discussions with Olivier Wittenberg, which have shaped my understanding of the cohomology of real algebraic varieties and have been very important for the completion of this work.

I am grateful to Claire Voisin for explaining to me the coniveau computation contained in Section 5 that was used to deal with the $n = 5$ and $d = 10$ case of Theorem 0.1.

References

[Artin 1927] E. Artin, "Über die Zerlegung definiter Funktionen in Quadrate", *Abh. Math. Sem. Univ. Hamburg* **5**:1 (1927), 100–115. MR Zbl

[Barth and Van de Ven 1978/79] W. Barth and A. Van de Ven, "Fano varieties of lines on hypersurfaces", *Arch. Math.* (*Basel*) **31**:1 (1978/79), 96–104. MR Zbl

[Bloch and Ogus 1974] S. Bloch and A. Ogus, "Gersten's conjecture and the homology of schemes", *Ann. Sci. École Norm. Sup.* (4) **7** (1974), 181–201. MR Zbl

[Bloch and Srinivas 1983] S. Bloch and V. Srinivas, "Remarks on correspondences and algebraic cycles", *Amer. J. Math.* **105**:5 (1983), 1235–1253. MR Zbl

[Bochnak et al. 1998] J. Bochnak, M. Coste, and M.-F. Roy, *Real algebraic geometry*, Ergebnisse der Mathematik (3) **36**, Springer, Berlin, 1998. MR Zbl

[Bourbaki 1985] N. Bourbaki, *Éléments de mathématique: Algébre commutative, Chapitres 5 á 7*, Masson, Paris, 1985. MR Zbl

[Cassels et al. 1971] J. W. S. Cassels, W. J. Ellison, and A. Pfister, "On sums of squares and on elliptic curves over function fields", *J. Number Theory* **3** (1971), 125–149. MR Zbl

[Clemens 1983] C. H. Clemens, "Double solids", *Adv. in Math.* **47**:2 (1983), 107–230. MR Zbl

[Colliot-Thélène 1992] J.-L. Colliot-Thélène, "Real rational surfaces without a real point", *Arch. Math.* (*Basel*) **58**:4 (1992), 392–396. MR Zbl

[Colliot-Thélène 1993] J.-L. Colliot-Thélène, "The Noether–Lefschetz theorem and sums of 4-squares in the rational function field $\mathbf{R}(x, y)$", *Compositio Math.* **86**:2 (1993), 235–243. MR Zbl

[Colliot-Thélène and Parimala 1990] J.-L. Colliot-Thélène and R. Parimala, "Real components of algebraic varieties and étale cohomology", *Invent. Math.* **101**:1 (1990), 81–99. MR Zbl

[Colliot-Thélène and Scheiderer 1996] J.-L. Colliot-Thélène and C. Scheiderer, "Zero-cycles and cohomology on real algebraic varieties", *Topology* **35**:2 (1996), 533–559. MR Zbl

[Colliot-Thélène and Voisin 2012] J.-L. Colliot-Thélène and C. Voisin, "Cohomologie non ramifiée et conjecture de Hodge entière", *Duke Math. J.* **161**:5 (2012), 735–801. MR Zbl

[Colliot-Thélène et al. 1997] J.-L. Colliot-Thélène, R. T. Hoobler, and B. Kahn, "The Bloch–Ogus–Gabber theorem", pp. 31–94 in *Algebraic K-theory* (Toronto, 1996), edited by V. P. Snaith, Fields Institute Communications **16**, American Mathematical Society, Providence, RI, 1997. MR Zbl

[Debarre and Manivel 1998] O. Debarre and L. Manivel, "Sur la variété des espaces linéaires contenus dans une intersection complète", *Math. Ann.* **312**:3 (1998), 549–574. MR Zbl

[Dimca 1985] A. Dimca, "Monodromy and Betti numbers of weighted complete intersections", *Topology* **24**:3 (1985), 369–374. MR Zbl

[Elman and Lam 1972] R. Elman and T. Y. Lam, "Pfister forms and K-theory of fields", *J. Algebra* **23** (1972), 181–213. MR Zbl

[Hilbert 1888] D. Hilbert, "Ueber die Darstellung definiter Formen als Summe von Formenquadraten", *Math. Ann.* **32**:3 (1888), 342–350. MR Zbl

[Jannsen 1988] U. Jannsen, "Continuous étale cohomology", *Math. Ann.* **280**:2 (1988), 207–245. MR Zbl

[Jannsen 2000] U. Jannsen, "Letter from Jannsen to Gross on higher Abel–Jacobi maps", pp. 261–275 in *The arithmetic and geometry of algebraic cycles* (Banff, AB, 1998), edited by B. B. Gordon et al., NATO Sci. Ser. C Math. Phys. Sci. **548**, Kluwer Academic Publishers, Dordrecht, Netherlands, 2000. MR Zbl

[Kahn 2012] B. Kahn, "Classes de cycles motiviques étales", *Algebra Number Theory* **6**:7 (2012), 1369–1407. MR Zbl

[Kahn and Sujatha 2000] B. Kahn and R. Sujatha, "Motivic cohomology and unramified cohomology of quadrics", *J. Eur. Math. Soc.* (*JEMS*) **2**:2 (2000), 145–177. MR Zbl

[Kerz 2009] M. Kerz, "The Gersten conjecture for Milnor K-theory", *Invent. Math.* **175**:1 (2009), 1–33. MR Zbl

[Krasnov 1994] V. A. Krasnov, "On the equivariant Grothendieck cohomology of a real algebraic variety and its application", *Izv. Ross. Akad. Nauk Ser. Mat.* **58**:3 (1994), 36–52. In Russian; translated in *Russian Acad. Sci. Izv. Math.* **44**:3 (1995), 461–477. MR Zbl

[Lam 1980] T. Y. Lam, *The algebraic theory of quadratic forms*, Benjamin/Cummings Publishing Co., Reading, MA, 1980. MR Zbl

[Pfister 1965] A. Pfister, "Zur Darstellung von -1 als Summe von Quadraten in einem Körper", *J. London Math. Soc.* **40** (1965), 159–165. MR Zbl

[Pfister 1967] A. Pfister, "Zur Darstellung definiter Funktionen als Summe von Quadraten", *Invent. Math.* **4** (1967), 229–237. MR Zbl

[Pfister 1971] A. Pfister, "Sums of squares in real function fields", pp. 297–300 in *Actes du Congrès International des Mathématiciens, Tome 1* (Nice, 1970), Gauthier-Villars, Paris, 1971. MR Zbl

[Pfister 1995] A. Pfister, *Quadratic forms with applications to algebraic geometry and topology*, London Mathematical Society Lecture Note Series **217**, Cambridge University Press, 1995. MR Zbl

[Pfister and Scheiderer 2012] A. Pfister and C. Scheiderer, "An elementary proof of Hilbert's theorem on ternary quartics", *J. Algebra* **371** (2012), 1–25. MR Zbl

[Scheiderer 1994] C. Scheiderer, *Real and étale cohomology*, Lecture Notes in Mathematics **1588**, Springer, Berlin, 1994. MR Zbl

[Serre 1965] J.-P. Serre, "Sur la dimension cohomologique des groupes profinis", *Topology* **3** (1965), 413–420. MR Zbl

[SGA $4\frac{1}{2}$ 1977] P. Deligne, *Cohomologie étale* (Séminaire de Géométrie Algébrique du Bois Marie), Lecture Notes in Mathematics **569**, Springer, Berlin, 1977. MR Zbl

[SGA 4_2 1972] M. Artin, A. Grothendieck, and J. L. Verdier, *Théorie des topos et cohomologie étale des schémas, Tome 2: Exposés V–VIII* (Séminaire de Géométrie Algébrique du Bois Marie 1963–1964), Lecture Notes in Mathematics **270**, Springer, Berlin, 1972. MR Zbl

[SGA 4_3 1973] M. Artin, A. Grothendieck, and J. L. Verdier, *Théorie des topos et cohomologie étale des schémas, Tome 3: Exposés IX–XIX* (Séminaire de Géométrie Algébrique du Bois Marie 1963–1964), Lecture Notes in Mathematics **305**, Springer, Berlin, 1973. MR Zbl

[SGA 7_{II} 1973] P. Deligne and N. Katz, *Groupes de monodromie en géométrie algébrique, II: Exposés X–XXII* (Séminaire de Géométrie Algébrique du Bois Marie 1967–1969), Lecture Notes in Mathematics **340**, Springer, Berlin, 1973. MR Zbl

[Voevodsky 2003] V. Voevodsky, "Motivic cohomology with $\mathbf{Z}/2$-coefficients", *Publ. Math. Inst. Hautes Études Sci.* **98** (2003), 59–104. MR Zbl

Communicated by Jean-Louis Colliot-Thélène
Received 2016-07-11 Revised 2017-01-05 Accepted 2017-02-03

olivier.benoist@unistra.fr *Institut de Recherche Mathématique Avancée, UMR 7501, Université de Strasbourg et CNRS, 7 rue René Descartes, 67000 Strasbourg, France*

ALGEBRA AND NUMBER THEORY 11:4 (2017)
dx.doi.org/10.2140/ant.2017.11.961

Gowers norms of multiplicative functions in progressions on average

Xuancheng Shao

Let μ be the Möbius function and let $k \geq 1$. We prove that the Gowers U^k-norm of μ restricted to progressions $\{n \leq X : n \equiv a_q \pmod q\}$ is $o(1)$ on average over $q \leq X^{1/2-\sigma}$ for any $\sigma > 0$, where $a_q \pmod q$ is an arbitrary residue class with $(a_q, q) = 1$. This generalizes the Bombieri–Vinogradov inequality for μ, which corresponds to the special case $k = 1$.

1. Introduction

A basic problem in analytic number theory is to understand the distribution of primes, or other related arithmetic functions such as the Möbius function μ and the Liouville function λ, in arithmetic progressions when the modulus is relatively large. In this direction, the Bombieri–Vinogradov inequality leads us almost half way to the ultimate goal, if we average over the moduli.

Theorem (Bombieri–Vinogradov). *Let $X, Q \geq 2$, and let $A \geq 2$. Assume that $Q \leq X^{1/2}(\log X)^{-B}$ for some sufficiently large $B = B(A)$. Then for all but at most $Q(\log X)^{-A}$ moduli $q \leq Q$, we have*

$$\sup_{(a,q)=1} \left| \sum_{\substack{n \leq X \\ n \equiv a \pmod q}} \Lambda(n) - \frac{1}{\varphi(q)} \sum_{n \leq X} \Lambda(n) \right| \ll_A \frac{X}{Q(\log X)^A}.$$

The same statement holds for the Möbius function μ and the Liouville function λ.

See [Iwaniec and Kowalski 2004, Chapter 17] for its proof and applications. In this paper, we investigate a higher order generalization of the Bombieri–Vinogradov inequality, which measures more refined distributional properties. This higher order version involves Gowers norms, a central tool in additive combinatorics. We refer the readers to [Tao and Vu 2006, Chapter 11] for the basic definitions and applications. In particular, $\|f\|_{U^k(Y)}$ stands for the U^k-norm of the function f on the interval $[0, Y] \cap \mathbb{Z}$.

Shao was supported by a Glasstone Research Fellowship.
MSC2010: primary 11P32; secondary 11B30, 11N13.
Keywords: multiplicative functions, Bombieri–Vinogradov theorem, Gowers norms.

For any arithmetic function $f : \mathbb{Z} \to \mathbb{C}$ and any residue class $a \pmod{q}$, denote by $f(q \cdot + a)$ the function $m \mapsto f(qm+a)$. Precisely we study the Gowers U^k-norm of f restricted to progressions $\{n \le X : n \equiv a \pmod{q}\}$, i.e., the U^k-norm of the functions $f(q \cdot + a)$ on $[0, X/q] \cap \mathbb{Z}$.

Corollary 1.1. *Let $X, Q \ge 2$, let k be a positive integer, let $A \ge 2$, and let $\varepsilon > 0$. Assume that $Q \le X^{1/2}(\log X)^{-B}$ for some sufficiently large $B = B(k, A, \varepsilon)$. Then for all but at most $Q(\log X)^{-A}$ moduli $q \le Q$, we have*

$$\sup_{\substack{0 \le a < q \\ (a,q)=1}} \|\mu(q \cdot + a)\|_{U^k(X/q)} \le \varepsilon.$$

The same statement holds for the Liouville function λ.

The Bombieri–Vinogradov inequality is the $k = 1$ case of Corollary 1.1 (qualitatively), since the U^1-norm of a function is the same as the absolute value of its average. By the inverse theorem for Gowers norms [Green et al. 2012], Corollary 1.1 is a straightforward consequence of the following result.

Theorem 1.2. *Let $X, Q \ge 2$ be parameters with $10Q^2 \le X$. Associated to each $Q \le q < 2Q$ we have*

1. *a residue class $a_q \pmod{q}$ with $0 \le a_q < q$, $(a_q, q) = 1$;*
2. *a nilmanifold G_q/Γ_q of dimension at most some $d \ge 1$, equipped with a filtration $(G_q)_\bullet$ of degree at most some $s \ge 1$ and a $(\log X)$-rational Malcev basis $\mathcal{X}_q$;*
3. *a polynomial sequence $g_q : \mathbb{Z} \to G_q$ adapted to $(G_q)_\bullet$;*
4. *a Lipschitz function $\varphi_q : G_q/\Gamma_q \to \mathbb{C}$ with $\|\varphi_q\|_{\mathrm{Lip}(\mathcal{X}_q)} \le 1$.*

Let $\psi_q : \mathbb{Z} \to \mathbb{C}$ be the function defined by $\psi_q(n) = \varphi_q(g_q(n)\Gamma_q)$. Then for any $A \ge 2$, the bound

$$\left| \sum_{\substack{n \le X \\ n \equiv a_q \pmod{q}}} \mu(n)\psi_q\left(\frac{n - a_q}{q}\right) \right| \ll_{A,d,s} \frac{X}{Q} \cdot \frac{\log\log X}{\log(X/Q^2)} \tag{1-1}$$

holds for all but at most $Q(\log X)^{-A}$ moduli $Q \le q < 2Q$. The same statement holds for the Liouville function λ.

See [Green and Tao 2012a] for the precise definitions of nilmanifolds and the associated data appearing in the statement. To avoid confusions later on, we point out that the Lipschitz norm is defined by

$$\|\varphi_q\|_{\mathrm{Lip}(\mathcal{X}_q)} = \|\varphi_q\|_\infty + \sup_{x \ne y} \frac{|\varphi_q(x) - \varphi_q(y)|}{d(x, y)},$$

where $d(\cdot\,,\cdot)$ is the metric induced by $\mathcal{X}_q$. In particular $\|\varphi_q\|_\infty \le \|\varphi_q\|_{\mathrm{Lip}(\mathcal{X}_q)}$.

To understand this paper, however, it is not essential to know these definitions, as long as one is willing to accept certain results about nilsequences as black boxes, many of which can be found in [Green and Tao 2012a]. The readers are thus encouraged to consider the following special case when the nilmanifolds are the torus $\mathbb{R}/\mathbb{Z}$, the polynomial sequences are genuine polynomials of degree at most s, and the Lipschitz functions are $\varphi(x) = \mathrm{e}(x) = e^{2\pi i x}$.

Theorem (main theorem, special case). *Let $X, Q \geq 2$ be parameters with $10Q^2 \leq X$, and let $s \geq 1$. Then for any $A \geq 2$, the bound*

$$\sup_{\substack{0\leq a<q \\ (a,q)=1}} \sup_{\alpha_1,\dots,\alpha_s\in\mathbb{R}} \Bigg| \sum_{\substack{n\leq X \\ n\equiv a \pmod q}} \mu(n)\,\mathrm{e}(\alpha_s n^s + \cdots + \alpha_1 n) \Bigg| \ll_{A,s} \frac{X}{Q}\cdot\frac{\log\log X}{\log(X/Q^2)}$$

holds for all but at most $Q(\log X)^{-A}$ moduli $Q \leq q < 2Q$. The same statement holds for the Liouville function λ.

Without restricting to arithmetic progressions (i.e., when $Q = O(1)$), the discorrelation between the Möbius function and nilsequences was studied by Green and Tao [2012b], as part of their program to count the number of solutions to linear equations in prime variables.

The rest of the paper is organized as follows. In Section 2 we reduce Theorem 1.2 to the minor arc case (Proposition 2.1). This reduction process is summarized in Lemma 2.4, using a factorization theorem for nilsequences [Green and Tao 2012a, Theorem 1.19]. In fact, one can obtain analogues of Theorem 1.2 for all 1-bounded multiplicative functions satisfying the Bombieri–Vinogradov estimate, such as indicator functions of smooth numbers (see [Fouvry and Tenenbaum 1996; Harper 2012] and the references therein). See [Frantzikinakis and Host 2017] for a previous work on Gowers norms of multiplicative functions, and also [Matthiesen 2016] for a generalization to some not necessarily bounded multiplicative functions. However, we will not seek for such generality here since any such result can be easily deduced from Lemma 2.4 and Proposition 2.1 as needed.

The rest of the argument applies to all bounded multiplicative functions. In Section 3 we consider the minor arc case using an orthogonality criterion. The idea, going back to Montgomery and Vaughan [1977] and Kátai [1986] (see also [Bourgain et al. 2013; Harper 2011]), is that one can make do with type-II estimates (or bilinear estimates) in a very restricted range when dealing with bounded multiplicative functions. This is the reason that we are unable to prove Theorem 1.2 for the primes, which would require type-II estimates in an inaccessible range, and also the reason that one saves no more than $\log X$ in the bound (1-1). In fact, to get this saving we use a quantitatively superior argument of Ramaré [2009], which received a lot of attention recently [Matomäki et al. 2015; Green 2016] following its use in

Matomäki and Radziwiłł's recent breakthrough [2016]. Finally the required type-II estimates will be proved in Section 4.

2. Technical reductions

In this section, we reduce Theorem 1.2 to the following minor arc, or equidistributed, case. See [Green and Tao 2012a, Definition 1.2] for the precise definition about equidistribution of nilsequences.

Proposition 2.1. *Let $X, Q \geq 2$ be parameters with $10Q^2 \leq X$. Let $\eta \in \left(0, \frac{1}{2}\right)$. Let $\mathcal{Q} \subset [Q, 2Q)$ be an arbitrary subset. Associated to each $q \in \mathcal{Q}$ we have*

1. *a residue class $a_q \pmod q$ with $0 \leq a_q < q$, $(a_q, q) = 1$, and an arbitrary interval $I_q \subset [0, X]$;*
2. *a nilmanifold G_q/Γ_q of dimension at most some $d \geq 1$, equipped with a filtration $(G_q)_\bullet$ of degree at most some $s \geq 1$ and an η^{-c}-rational Malcev basis $\mathcal{X}_q$ for some sufficiently small $c = c(d, s) > 0$;*
3. *a polynomial sequence $g_q : \mathbb{Z} \to G_q$ adapted to $(G_q)_\bullet$ such that $\{g_q(m)\}_{1 \leq m \leq X/q}$ is totally η-equidistributed;*
4. *a Lipschitz function $\varphi_q : G_q/\Gamma_q \to \mathbb{C}$ with $\|\varphi_q\|_\infty \leq 1$ and $\int \varphi_q = 0$.*

Let $\psi_q : \mathbb{Z} \to \mathbb{C}$ be the function defined by $\psi_q(n) = \varphi_q(g_q(n)\Gamma_q)$. Let $f : \mathbb{Z} \to \mathbb{C}$ be a multiplicative function with $|f(n)| \leq 1$. Then

$$\sum_{q \in \mathcal{Q}} \Bigg| \sum_{\substack{n \in I_q \\ n \equiv a_q \ (\mathrm{mod}\ q)}} f(n) \psi_q\left(\frac{n - a_q}{q}\right) \Bigg| \ll \frac{\log \eta^{-1}}{\log(X/Q^2)} \sum_{q \in \mathcal{Q}} \left(\frac{|I_q|}{q} + 1\right) + \eta^c X \log X \max_{q \in \mathcal{Q}} \|\varphi_q\|_{\mathrm{Lip}(\mathcal{X}_q)},$$

for some constant $c = c(d, s) > 0$.

Thus one obtains a saving of (at most) $\log \eta^{-1} / \log(X/Q^2)$ compared to the trivial bound. The attentive reader may notice an extra factor $\log X$ in the second term of the bound, which prevents one from taking any $\eta = o(1)$ and still getting a nontrivial estimate. This extra factor mainly comes from the type-II estimate (Lemma 3.3); see the comments after its statement. It won't be a concern for us since we will take η to be a large negative power of $\log X$.

To deduce Theorem 1.2 from Proposition 2.1, we may assume that A is sufficiently large depending on d and s, that X is sufficiently large depending on A, d and s, and that $Q \leq X^{1/2}(\log X)^{-B}$ for some sufficiently large $B = B(A)$, since otherwise the bound (1-1) is trivial. In particular, it suffices to establish the bound $O_A(Q(\log X)^{-2A})$ for the number of exceptional moduli.

Reducing to completely multiplicative functions. The first technical step of the reduction is to pass from the Möbius function μ to its completely multiplicative cousin λ. In this subsection we deduce Theorem 1.2 for μ, assuming that it has already been proved for the Liouville function λ. This step is summarized in the following lemma.

Lemma 2.2. *Let $X \geq 2$ be large, and let $a \pmod q$ be a residue class with $(a,q)=1$. Let $\varepsilon \in (0,1)$, and assume that $q \leq \varepsilon X^{1/2}(\log X)^{-3}$. Let $f : \mathbb{Z} \to \mathbb{C}$ be a multiplicative function with $|f(n)| \leq 1$, and let $f' : \mathbb{Z} \to \mathbb{C}$ be the completely multiplicative function defined by $f'(p) = f(p)$ for each prime p. Let $c : \mathbb{Z} \to \mathbb{C}$ be an arbitrary function with $|c(n)| \leq 1$. If*

$$\Bigg| \sum_{\substack{n \leq X \\ n \equiv a \pmod q}} f(n)c(n) \Bigg| \geq \varepsilon \frac{X}{q},$$

then there is a positive integer $\ell \ll \varepsilon^{-3}$ with $(\ell, q) = 1$, such that

$$\Bigg| \sum_{\substack{n \leq X/\ell \\ n \equiv a\ell^{-1} \pmod q}} f'(n)c(\ell n) \Bigg| \gg \varepsilon \frac{X}{\ell q}.$$

To deduce Theorem 1.2 for μ, apply Lemma 2.2 with $f = \mu$ (so that $f' = \lambda$) and $\varepsilon = C \log\log X / \log(X/Q^2)$ for some large constant C depending on A. For each q satisfying

$$\Bigg| \sum_{\substack{n \leq X \\ n \equiv a_q \pmod q}} \mu(n)\psi_q\Big(\frac{n-a_q}{q}\Big) \Bigg| \geq \varepsilon \frac{X}{q}, \tag{2-1}$$

Lemma 2.2 produces a positive integer $\ell = \ell_q \ll \varepsilon^{-3}$ with $(\ell_q, q) = 1$, such that

$$\Bigg| \sum_{\substack{n \leq X/\ell_q \\ n \equiv a_q\ell_q^{-1} \pmod q}} \lambda(n)\psi_q\Big(\frac{\ell_q n-a_q}{q}\Big) \Bigg| \gg \varepsilon \frac{X}{\ell_q q}.$$

For each $\ell \ll \varepsilon^{-3}$, apply Theorem 1.2 for λ, with X replaced by X/ℓ, to conclude that there are at most $Q(\log X)^{-3A}$ moduli q satisfying (2-1) with $\ell_q = \ell$. It follows that the total number of moduli q satisfying (2-1) is $O(\varepsilon^{-3}Q(\log X)^{-3A}) = O(Q(\log X)^{-2A})$, as desired.

In the remainder of this subsection, we give a standard proof of Lemma 2.2, starting with a basic lemma.

Lemma 2.3. *Let $f : \mathbb{Z} \to \mathbb{C}$ be a multiplicative function with $|f(n)| \leq 1$, and let $f' : \mathbb{Z} \to \mathbb{C}$ be the completely multiplicative function defined by $f'(p) = f(p)$ for* *each prime p. Let g be the multiplicative function with $f = f' * g$. Then for any $N \geq 2$ we have*

$$\sum_{n \geq N} \frac{|g(n)|}{n} \ll N^{-1/2}(\log N)^2 \quad \textit{and} \quad \sum_{n \leq N} |g(n)| \ll N^{1/2}(\log N)^2.$$

Proof. It is easy to see that $g(p) = 0$ and $|g(p^k)| \leq 2$ for every prime p. Set $\sigma = \frac{1}{2} + 1/(10 \log N)$ so that $\sigma \in \left(\frac{1}{2}, 1\right)$ and $N^\sigma \asymp N^{1/2}$. By Rankin's trick we have

$$\sum_{n \geq N} \frac{|g(n)|}{n} \leq \sum_n \frac{|g(n)|}{n} \left(\frac{n}{N}\right)^{1-\sigma} \ll N^{-1/2} \sum_n |g(n)| n^{-\sigma},$$

and similarly

$$\sum_{n \leq N} |g(n)| \leq \sum_n |g(n)| \left(\frac{N}{n}\right)^\sigma \ll N^{1/2} \sum_n |g(n)| n^{-\sigma}.$$

Thus it suffices to establish the bound

$$\sum_n |g(n)| n^{-\sigma} \ll (\log N)^2.$$

We may write the Dirichlet series associated to $|g|$ in terms of its Euler product:

$$\sum_n |g(n)| n^{-\sigma} = \prod_p \left(1 + |g(p^2)| p^{-2\sigma} + |g(p^3)| p^{-3\sigma} + \cdots\right).$$

Since $|g(p^k)| \leq 2$, we may bound it by

$$\prod_p (1 + p^{-2\sigma} + p^{-4\sigma} + \cdots)^2 (1 + p^{-3\sigma} + p^{-6\sigma} + \cdots)^2 = \zeta(2\sigma)^2 \zeta(3\sigma)^2.$$

Since $\zeta(3\sigma) \ll 1$ and $\zeta(2\sigma) \ll (2\sigma - 1)^{-1}$, the desired bound follows immediately. □

Proof of Lemma 2.2. Write $f = f' * g$ for some multiplicative function g. We have

$$\sum_{\substack{n \leq X \\ n \equiv a \pmod q}} f(n)c(n) = \sum_{\substack{\ell n \leq X \\ \ell n \equiv a \pmod q}} f'(n) g(\ell) c(\ell n) = \sum_{\substack{\ell \leq X \\ (\ell, q) = 1}} g(\ell) \Bigg(\sum_{\substack{n \leq X/\ell \\ n \equiv a\ell^{-1} \pmod q}} f'(n) c(\ell n) \Bigg).$$

Let $L \geq 2$ be a parameter. Using the trivial bound $O(X/\ell q + 1)$ for the inner sum, we may apply Lemma 2.3 to bound the total contributions from those terms with $\ell \geq L$ by

$$\frac{X}{q} \sum_{\ell \geq L} \frac{|g(\ell)|}{\ell} + \sum_{\ell \leq X} |g(\ell)| \ll \frac{X}{q L^{1/2}} (\log L)^2 + X^{1/2} (\log X)^2.$$

We may choose $L \ll \varepsilon^{-3}$ such that the first term above is negligible compared to the lower bound $\varepsilon X/q$, and the second term is already negligible compared to $\varepsilon X/q$ by the assumption on q. It follows that

$$\sum_{\substack{\ell \le L \\ (\ell,q)=1}} |g(\ell)| \Bigg| \sum_{\substack{n \le X/\ell \\ n \equiv a\ell^{-1} \ (\mathrm{mod}\ q)}} f'(n)c(\ell n) \Bigg| \gg \varepsilon \frac{X}{q}.$$

Since $\sum |g(\ell)|\ell^{-1} \ll 1$, there is some $\ell \le L$ with $(\ell, q) = 1$ such that

$$\Bigg| \sum_{\substack{n \le X/\ell \\ n \equiv a\ell^{-1} \ (\mathrm{mod}\ q)}} f'(n)c(\ell n) \Bigg| \gg \varepsilon \frac{X}{\ell q}.$$

This completes the proof of the lemma. □

Reducing to equidistributed nilsequences. We now use the factorization theorem [Green and Tao 2012a, Theorem 1.19] to reduce arbitrary nilsequences to equidistributed ones. This step is summarized in the following lemma, the proof of which is similar to arguments in [Green and Tao 2012b, Section 2].

Lemma 2.4. *Let $X \ge 2$ be large, and let $a \pmod q$ be a residue class with $0 \le a < q$, $(a, q) = 1$. Let $\varepsilon \in \left(0, \frac{1}{2}\right)$. Let $f : \mathbb{Z} \to \mathbb{C}$ be a completely multiplicative function with $|f(n)| \le 1$. Given*

- *a nilmanifold G/Γ of dimension at most some $d \ge 1$, equipped with a filtration $G_\bullet$ of degree at most some $s \ge 1$ and a M_0-rational Malcev basis $\mathcal{X}$ for some $M_0 \ge \varepsilon^{-1}$;*
- *a polynomial sequence $g : \mathbb{Z} \to G$ adapted to $G_\bullet$; and*
- *a Lipschitz function $\varphi : G/\Gamma \to \mathbb{C}$ with $\|\varphi\|_{\mathrm{Lip}(\mathcal{X})} \le 1$,*

let $\psi : \mathbb{Z} \to \mathbb{C}$ be the function defined by $\psi(n) = \varphi(g(n)\Gamma)$. Assume that

$$\Bigg| \sum_{\substack{n \le X \\ n \equiv a \ (\mathrm{mod}\ q)}} f(n)\psi\left(\frac{n-a}{q}\right) \Bigg| \ge \varepsilon \frac{X}{q}.$$

For any $A \ge 2$ large enough depending on d and s, we may find $M_0 \le M \le M_0^{O_A(1)}$, an interval $I \subset [0, X]$ with $|I| \gg X/M^3$, a positive integer q' with $q \mid q'$ and $q' \le qM$, a residue class $a' \pmod{q'}$ with $0 \le a' < q'$, $(a', q') = 1$, and moreover

- *a nilmanifold G'/Γ' of dimension at most d, equipped with a filtration $G'_\bullet$ of degree at most s and a $M^{O_{d,s}(1)}$-rational Malcev basis $\mathcal{X}'$;*
- *a polynomial sequence $g' : \mathbb{Z} \to G'$ adapted to $G'_\bullet$ such that $\{g'(m)\}_{1 \le m \le X/q}$ is totally M^{-A}-equidistributed; and*

- *a Lipschitz function* $\varphi' : G'/\Gamma' \to \mathbb{C}$ *with* $\|\varphi'\|_\infty \le 1$ *and* $\|\varphi'\|_{\mathrm{Lip}(\mathcal{X}')} \le M^{O_{d,s}(1)}$,

such that

$$\Bigg| \sum_{\substack{n\in I \\ n\equiv a' \pmod{q'}}} f(n)\psi'\Big(\frac{n-a'}{q'}\Big)\Bigg| \gg \varepsilon\frac{|I|}{q'},$$

where $\psi' : \mathbb{Z} \to \mathbb{C}$ *is the function defined by* $\psi'(n) = \varphi'(g'(n)\Gamma')$.

To deduce Theorem 1.2 for λ from Proposition 2.1, apply Lemma 2.4 with $f = \lambda$, $\varepsilon = C \log\log X/\log(X/Q^2)$ for some large constant C depending on A, and $M_0 = (\log X)^C$ for some large constant C depending on d and s. For each q satisfying

$$\Bigg| \sum_{\substack{n\le X \\ n\equiv a_q \pmod{q}}} \lambda(n)\psi_q\Big(\frac{n-a_q}{q}\Big)\Bigg| \ge \varepsilon\frac{X}{q}, \tag{2-2}$$

Lemma 2.4 produces M_q, I_q, $a' \pmod{q'}$, G'/Γ', and $\psi' = \varphi' \circ g'$, all of which depend on q (some of these dependencies are suppressed for notational convenience), such that

$$\Bigg| \sum_{\substack{n\in I_q \\ n\equiv a' \pmod{q'}}} \lambda(n)\psi'\Big(\frac{n-a'}{q'}\Big)\Bigg| \gg \varepsilon\frac{|I_q|}{q'}.$$

Divide the possible values of M_q into $O_A(1)$ subintervals of the form $[M^{1/2}, M]$ with $M_0 \le M \le M_0^{O_A(1)}$. Given M, let $\mathcal{Q} = \mathcal{Q}_M$ be the set of moduli $q \in \mathcal{Q}$ such that $M^{1/2} \le M_q \le M$, and let $\mathcal{Q}' = \mathcal{Q}'_M$ be the set of q' arising from $q \in \mathcal{Q}$. It suffices to show that

$$|\mathcal{Q}| \ll Q(\log X)^{-2A}.$$

Since q'/q is a positive integer at most M, each q' occurs with multiplicity at most M. Thus $|\mathcal{Q}| \le M|\mathcal{Q}'|$. Before applying Proposition 2.1 we need to ensure that each φ' has average 0. For $q' \in \mathcal{Q}'$ either

$$\Bigg| \sum_{\substack{n\in I_q \\ n\equiv a' \pmod{q'}}} \lambda(n)\Bigg| \gg \varepsilon\frac{|I_q|}{q'} \gg \frac{X}{q'(\log X)^{O_A(1)}} \tag{2-3}$$

or

$$\Bigg| \sum_{\substack{n\in I_q \\ n\equiv a' \pmod{q'}}} \lambda(n)\Big(\psi'\Big(\frac{n-a'}{q'}\Big) - \int\varphi'\Big)\Bigg| \gg \varepsilon\frac{|I_q|}{q'}. \tag{2-4}$$

To bound the number of q' satisfying (2-3), note that the Bombieri–Vinogradov inequality (for λ) is applicable since $q' \le 2QM \le X^{1/2}(\log X)^{-B/2}$ (recall the assumption that $Q \le X^{1/2}(\log X)^{-B}$ for some large B). By choosing b large enough, we may ensure that the number of q' satisfying (2-3) is at most QM^{-A}.

Now let $\mathcal{Q}_2' \subset \mathcal{Q}'$ be the set of $q' \in \mathcal{Q}'$ satisfying (2-4). To bound the size of $\mathcal{Q}_2'$, we apply Proposition 2.1 after replacing each φ' by $\varphi' - \int \varphi'$ and dyadically dividing the possible values of q'. This leads to

$$\varepsilon \sum_{q' \in \mathcal{Q}_2'} \frac{|I_q|}{q'} \ll \frac{\log M^A}{\log(X/Q^2M^2)} \sum_{q' \in \mathcal{Q}_2'} \left(\frac{|I_q|}{q'} + 1 \right) + M^{-cA} X (\log X)^2 M^{O_{d,s}(1)},$$

for some $c = c(d, s) > 0$. The first term on the right can be made negligible compared to the left-hand side, if the constant C in the choice of ε is taken large enough in terms of A. Hence

$$\varepsilon \frac{X}{M^3} \cdot \frac{|\mathcal{Q}_2'|}{QM} \ll \varepsilon \sum_{q' \in \mathcal{Q}'} \frac{|I_q|}{q'} \ll M^{-cA+O_{d,s}(1)} X.$$

It follows that $|\mathcal{Q}_2'| \ll QM^{-cA+O_{d,s}(1)}$. Combining the estimates for the two types of q' together, we obtain

$$|\mathcal{Q}| \le M|\mathcal{Q}'| \ll QM^{-cA+O_{d,s}(1)} \ll Q(\log X)^{-2A},$$

if the constant C in the choice of M is large enough depending on d and s. This completes the deduction of Theorem 1.2.

Proof of Lemma 2.4. Let C be a large constant (depending on d and s), and apply the factorization theorem [Green and Tao 2012a, Theorem 1.19] to find $CM_0 \le M \le M_0^{O_A(1)}$, a rational subgroup $\widetilde{G} \subset G$, a Malcev basis $\widetilde{\mathcal{X}}$ for $\widetilde{G}/\widetilde{\Gamma}$ (where $\widetilde{\Gamma} = \Gamma \cap \widetilde{G}$) in which each element is an M-rational combination of the elements of $\mathcal{X}$, and a decomposition $g = s\tilde{g}\gamma$ into polynomial sequences $s, \tilde{g}, \gamma : \mathbb{Z} \to G$ with the following properties:

(1) s is $(M, X/q)$-smooth in the sense that $d(s(n), \mathrm{id}) \le M$ and $d(s(n), s(n-1)) \le qM/X$ for each $1 \le n \le X/q$.

(2) $\tilde{g}$ takes values in $\widetilde{G}$; moreover $\{\tilde{g}(n)\}_{1 \le n \le X/q}$ is totally M^{-CA}-equidistributed in $\widetilde{G}/\widetilde{\Gamma}$ (using the metric induced by the Malcev basis $\widetilde{\mathcal{X}}$).

(3) γ is M-rational in the sense that for each $n \in \mathbb{Z}$, $\gamma(n)^r \in \Gamma$ for some $1 \le r \le M$. Moreover, γ is periodic with period $t \le M$.

We may assume that $X \ge qM^3$, since otherwise the conclusion holds trivially. After a change of variables $n = qm + a$, we may rewrite the assumption as

$$\left| \sum_{m \le X/q} f(qm + a)\psi(m) \right| \gg \varepsilon \frac{X}{q}.$$

Dividing $[0, X/q]$ into $O(M^2)$ intervals of equal length and then further dividing them into residue classes modulo t, we may find an interval $J \subset [0, X]$ with

$|J| \asymp X/M^2$ and some residue class $b \pmod t$, such that

$$\Bigg|\sum_{\substack{m\equiv b \pmod t\\ qm+a\in J}} f(qm+a)\varphi\big(s(m)\tilde{g}(m)\gamma(m)\big)\Bigg| \gg \varepsilon\frac{|J|}{qt}. \tag{2-5}$$

Pick any m_0 counted in the sum (i.e., $m_0 \equiv b \pmod t$ and $qm_0+a \in J$), and note that we may replace $s(m)$ in (2-5) by $s(m_0)$ with a negligible error, since φ has Lipschitz norm at most 1 and

$$d\big(s(m)\tilde{g}(m)\gamma(m), s(m_0)\tilde{g}(m)\gamma(m)\big) = d\big(s(m), s(m_0)\big) \ll M^{-1},$$

for all m with $qm+a \in J$ by the right invariance of d and the smoothness property of s. Moreover, by the periodicity of γ, we may replace $\gamma(m)$ in (2-5) by $\gamma(m_0)$. Now let g' be the polynomial sequence defined by

$$g'(m) = \gamma(m_0)^{-1}\tilde{g}(tm+b)\gamma(m_0),$$

taking values in $G' = \gamma(m_0)^{-1}\widetilde{G}\gamma(m_0)$, and let φ' be the automorphic function on G' defined by

$$\varphi'(x) = \varphi\big(s(m_0)\gamma(m_0)x\big).$$

The desired properties about G'/Γ', g', and φ' can be established via standard "quantitative nillinear algebra" (see the claim at the end of [Green and Tao 2012b, Section 2]). After a change of variables replacing m by $tm+b$, the inequality (2-5) can be rewritten as

$$\Bigg|\sum_{m:\, qtm+c\in J} f(qtm+c)\varphi'(g'(m))\Bigg| \gg \varepsilon\frac{|J|}{qt},$$

where $c = qb+a$. This is almost what we need, but there is the slight issue that c may not be coprime with t. Let $d = (c,t)$ so that $d \le M$. Let $q' = qt/d$ and $a' = c/d$ so that $(a',q') = 1$. Let $I = d^{-1}J$ so that $|I| \gg X/M^3$. Since f is completely multiplicative, we have

$$\Bigg|\sum_{m:\, q'm+a'\in I} f(q'm+a')\varphi'(g'(m))\Bigg| \gg \varepsilon\frac{|I|}{q'}.$$

This completes the proof of the lemma. □

3. The minor arc case: Proof of Proposition 2.1

In this section we prove Proposition 2.1, which is the minor arc case of our main theorem and applies to all 1-bounded multiplicative functions. For convenience write

$$T = \sum_{q\in\mathcal{Q}}\bigg(\frac{|I_q|}{q}+1\bigg).$$

We may assume that

$$(X/Q^2)^{-1/20} < \eta < (\log X)^{-C},$$

for some sufficiently large $C = C(d, s) > 0$, since otherwise the bound is trivial. We may further assume that $|I_q| \geq X^{0.9}$ for each $q \in \mathcal{Q}$, since the contributions from those q with $|I_q| \leq X^{0.9}$ are trivially acceptable. After multiplying each φ_q by an appropriate scalar, it suffices to prove the desired inequality with the absolute value sign removed. Set

$$Y = \eta^{-1} \quad \text{and} \quad Z = \left(\frac{X}{Q^2}\right)^{1/20},$$

so that $2 \leq Y < Z \leq X^{1/20}$. Let F be the function defined by

$$F(n) = \sum_{\substack{q \in \mathcal{Q},\ n \in I_q \\ n \equiv a_q \ (\mathrm{mod}\ q)}} \psi_q\left(\frac{n - a_q}{q}\right).$$

Clearly F is supported on $[0, X]$. The desired bound can be rewritten as

$$\sum_{n \leq X} f(n) F(n) \ll \frac{\log Y}{\log Z} \cdot T + \eta^c X \log X \max_{q \in \mathcal{Q}} \|\varphi_q\|_{\mathrm{Lip}(\mathcal{X}_q)}. \tag{3-1}$$

As alluded to in the introduction, this will be proved using an orthogonality criterion for multiplicative functions. A general principle of this type is given in [Green 2016, Proposition 2.2]. In the notations there, the terms giving rise to E_{triv} and E_{sieve} will be dealt with by Lemmas 3.1 and 3.2, respectively. In particular, the E_{sieve} term leads to the first bound in (3-1). The bilinear (type-II) sum E_{bilinear} will be dealt with in Lemma 3.3, leading to the second bound in (3-1).

Unfortunately we cannot directly apply [Green 2016, Proposition 2.2], since for example our function F is not necessarily bounded. In the remainder of this section we reproduce the argument from [Green 2016] with suitable modifications to prove (3-1). Recall the definition of Ramaré's weight function:

$$w(n) = \frac{1}{\#\{Y \leq p < Z : p \mid n\} + 1}.$$

Introduce also the function $\mu^2_{[Y,Z)}$, the indicator function of the set of integers n that are not divisible by the square of any prime $p \in [Y, Z)$. To prove (3-1), we first dispose of those terms with $\mu^2_{[Y,Z)}(n) = 0$:

$$\sum_{\substack{n \leq X \\ \mu^2_{[Y,Z)}(n) = 0}} |f(n) F(n)| \leq \sum_{Y \leq p < Z} \sum_{\substack{n \leq X \\ p^2 \mid n}} |F(n)|.$$

The following lemma will be used repeatedly.

Lemma 3.1. *For any positive integer $D \le X^{0.4}$ we have*

$$\sum_{\substack{n \le X \\ D \mid n}} |F(n)| \ll \frac{T}{D}.$$

Proof. Using the trivial bound

$$|F(n)| \le \sum_{\substack{q \in \mathcal{Q},\ n \in I_q \\ n \equiv a_q \pmod q}} 1, \tag{3-2}$$

we obtain

$$\sum_{\substack{n \le X \\ D \mid n}} |F(n)| \le \sum_{q \in \mathcal{Q}} \sum_{\substack{n \in I_q,\ D \mid n \\ n \equiv a_q \pmod q}} 1.$$

Since $(a_q, q) = 1$, the inner sum over n is nonempty unless $(q, D) = 1$, in which case it is $O(|I_q|/qD)$. The conclusion follows immediately. □

Since $Z^2 \le X^{0.4}$, Lemma 3.1 implies that

$$\sum_{\substack{n \le X \\ \mu^2_{[Y,Z)}(n) = 0}} |f(n)F(n)| \ll T \sum_{Y \le p < Z} \frac{1}{p^2} \ll \frac{T}{Y}.$$

Hence the contributions from those $n \le X$ with $\mu^2_{[Y,Z)}(n) = 0$ are acceptable. If $\mu^2_{[Y,Z)}(n) = 1$, then we have the Ramaré identity

$$\sum_{\substack{Y \le p < Z \\ p \mid n}} w\left(\frac{n}{p}\right) = \begin{cases} 1 & \text{if } p \mid n \text{ for some } Y \le p < Z, \\ 0 & \text{otherwise.} \end{cases}$$

The following lemma disposes of those n not divisible by any $p \in [Y, Z)$:

Lemma 3.2. *We have*

$$\sum_{n \le X} |F(n)| \cdot \mathbf{1}_{\left(n, \prod_{Y \le p < Z} p\right) = 1} \ll \frac{\log Y}{\log Z} \cdot T.$$

Proof. Using (3-2), we can bound the left-hand side by

$$\sum_{q \in \mathcal{Q}} \sum_{\substack{n \in I_q \\ n \equiv a_q \pmod q}} \mathbf{1}_{\left(n, \prod_{Y \le p < Z} p\right) = 1}.$$

Consider the inner sum for a fixed q. Writing $d = \left(q, \prod_{Y \le p < Z} p\right)$, we may bound the inner sum using a standard upper bound sieve (since $Z^2 \le |I_q|/q$) to obtain

$$\sum_{\substack{n\in I_q\\ n\equiv a_q \pmod q}} \mathbf{1}_{(n,\prod_{Y\le p<Z} p)=1} \ll \frac{|I_q|}{q} \prod_{\substack{Y\le p<Z\\ p\nmid q}} \left(1-\frac{1}{p}\right) \ll \frac{|I_q|}{q} \cdot \frac{\log Y}{\log Z} \cdot \frac{d}{\varphi(d)}.$$

On the other hand, since $d \mid \prod_{Y\le p<Z} p$ and $d \le q$ we have

$$\frac{d}{\varphi(d)} \le \prod_{Y\le p<W} \left(1-\frac{1}{p}\right)^{-1},$$

where $W \sim Y + \log q$. Thus $d/\varphi(d) = O(1)$ since $Y \ge \log q$, and the conclusion of the lemma follows. □

Thus we can restrict to those n with $\mu^2_{[Y,Z)}(n) = 1$ and having at least one prime divisor $p \in [Y, Z)$. By the Ramaré identity, we need to estimate

$$\Sigma := \sum_{\substack{n\le X\\ \mu^2_{[Y,Z)}(n)=1}} f(n)F(n) \sum_{\substack{Y\le p<Z\\ p\mid n}} w\left(\frac{n}{p}\right).$$

Writing $m = n/p$ and using the multiplicativity of f, we obtain

$$\Sigma = \sum_{\substack{m\le X/Y\\ \mu^2_{[Y,Z)}(m)=1}} w(m)f(m) \sum_{\substack{Y\le p<Z\\ p\le X/m\\ (m,p)=1}} f(p)F(pm). \tag{3-3}$$

The condition $\mu^2_{[Y,Z)}(m) = 1$ can be dropped since the contribution from those m divisible by $\tilde{p}^2$ for some $\tilde{p} \in [Y, Z)$ is at most

$$\sum_{Y\le \tilde{p}<Z} \sum_{Y\le p<Z} \sum_{\substack{m\le X/p\\ \tilde{p}^2\mid m}} |F(pm)| \ll T \sum_{Y\le \tilde{p}<Z} \sum_{Y\le p<Z} \frac{1}{p\tilde{p}^2} \ll \frac{T}{Y} \log\log X,$$

by an application of Lemma 3.1. Similarly, the condition $(m, p) = 1$ in (3-3) can also be dropped since the contribution from the terms with $p \mid m$ is at most

$$\sum_{Y\le p<Z} \sum_{\substack{m\le X/p\\ p\mid m}} |F(pm)| \ll T \sum_{Y\le p<Z} \frac{1}{p^2} \ll \frac{T}{Y},$$

by Lemma 3.1. Both these bounds are acceptable. Thus it remains to bound

$$\Sigma' := \sum_{m\le X/Y} w(m)f(m) \sum_{\substack{Y\le p<Z\\ p\le X/m}} f(p)F(pm).$$

Dyadically dividing the range $[Y, Z)$ for p, we consider

$$\Sigma'(P) := \sum_{m \le X/P} w(m) f(m) \sum_{\substack{P \le p < 2P \\ p \le X/m}} f(p) F(pm)$$

for $P \in [Y, Z)$. Use the trivial bound $|w(m) f(m)| \le 1$ and apply the Cauchy–Schwarz inequality to obtain

$$|\Sigma'(P)|^2 \ll \frac{X}{P} \sum_{m \le X/P} \left| \sum_{\substack{P \le p < 2P \\ p \le X/m}} f(p) F(pm) \right|^2.$$

After expanding the square and changing the order of summation, we obtain

$$\begin{aligned} |\Sigma'(P)|^2 &\ll \frac{X}{P} \sum_{P \le p, p' < 2P} f(p) \overline{f(p')} \sum_{m \le \min(X/p, X/p')} F(pm) \overline{F(p'm)} \\ &\ll \frac{X}{P} \sum_{P \le p, p' < 2P} \left| \sum_{m \le \min(X/p, X/p')} F(pm) \overline{F(p'm)} \right|. \end{aligned}$$

Set $K = P$, $L = X/P$, and $\delta = \eta^c$ for some $c > 0$ small enough depending on d and s. The following lemma, whose proof will be given in Section 4, gives the necessary estimates for the type-II (bilinear) sums appearing above.

Lemma 3.3. *Let $K, L, Q \ge 2$ be parameters with $10Q^2 \le L$. Let $\delta \in \left(0, \frac{1}{2}\right)$. Associated to each $Q \le q < 2Q$ we have*

1. *a residue class $a_q \pmod q$ with $0 \le a_q < q$, $(a_q, q) = 1$, and an arbitrary interval I_q;*
2. *a nilmanifold G_q/Γ_q of dimension at most some $d \ge 1$, equipped with a filtration $(G_q)_\bullet$ of degree at most some $s \ge 1$ and a δ^{-1}-rational Malcev basis $\mathcal{X}_q$;*
3. *a polynomial sequence $g_q : \mathbb{Z} \to G_q$ adapted to $(G_q)_\bullet$;*
4. *a Lipschitz function $\varphi_q : G_q/\Gamma_q \to \mathbb{C}$ with $\|\varphi_q\|_{\mathrm{Lip}(\mathcal{X}_q)} \le 1$ and $\int \varphi_q = 0$.*

Let $\psi_q : \mathbb{Z} \to \mathbb{C}$ be the function defined by $\psi_q(n) = \varphi_q(g_q(n)\Gamma_q)$, and let F be the function defined by

$$F(n) = \sum_{\substack{Q \le q < 2Q \\ n \in I_q \\ n \equiv a_q \ (\mathrm{mod}\ q)}} \psi_q\left(\frac{n - a_q}{q}\right).$$

For each $k, k' \in [K, 2K)$, let $I(k, k') \subset [0, L]$ be an arbitrary interval. Suppose that

$$\sum_{K \le k, k' < 2K} \left| \sum_{\ell \in I(k,k')} F(k\ell) \overline{F(k'\ell)} \right| \ge \delta K^2 L, \tag{3-4}$$

and that $K^{-c} < \delta < (\log Q)^{-1}$ for some sufficiently small $c = c(d, s) > 0$. Then the polynomial sequence $\{g_q(m)\}_{1\le m\le KL/Q}$ fails to be totally $\delta^{O_{d,s}(1)}$-equidistributed for some $Q \le q < 2Q$.

To complete the proof of Proposition 2.1, note that the hypotheses $10Q^2 \le L$ and $K^{-c} < \delta < (\log Q)^{-1}$ in Lemma 3.3 are satisfied by our choices of Y and Z. By setting $\varphi_q = 0$ for $q \notin \mathcal{Q}$ and renormalizing (replacing φ_q by $\varphi_q/\|\varphi_q\|_{\mathrm{Lip}(\mathcal{X}_q)}$), we may apply Lemma 3.3 to conclude that

$$|\Sigma'(P)|^2 \ll \frac{X}{P} \cdot \eta^c P X \max_{q\in\mathcal{Q}} \|\varphi_q\|^2_{\mathrm{Lip}(\mathcal{X}_q)}.$$

The desired bound (3-1) follows after summing over P dyadically.

Remark 3.4. Instead of using simply the trivial bound $|w(n)| \le 1$, one may appeal to [Green 2016, Lemma 2.1] to dispose of the extra $\log X$ factor that appeared when summing over P dyadically. We will, however, not bother with this since the type-II estimates we use already have an extra logarithmic factor anyways.

4. Type-II estimates

In this section we prove Lemma 3.3. We start with the following lemma, needed to treat composite moduli.

Lemma 4.1. *Let $Q \ge 2$ and $Q \le R \le 4Q^2$. Let E be the set of pairs (q, q') with $Q \le q, q' < 2Q$ and $R \le [q, q'] < 2R$. For each $Q \le q < 2Q$ and $R \le r < 2R$, let*

$$m_q(r) = \#\big\{Q \le q' < 2Q : (q, q') \in E, [q, q'] = r\big\}.$$

Then for any $m_0 \ge 1$ we have

$$\#\big\{(q, q') \in E : m_q([q, q']) \ge m_0\big\} \ll m_0^{-1} R \log Q.$$

Proof. By a dyadic division, it suffices to show that

$$\#\big\{(q, q') \in E : m_0 \le m_q([q, q']) < 2m_0\big\} \ll m_0^{-1} R \log Q,$$

for any $m_0 \ge 1$. Call the left-hand side above $N(m_0)$. For any $D \ge 1$, let $\sigma_D(q)$ be the number of divisors of q in the range $[D, 8D]$. A moment's thought reveals that $m_q(r) \le \sigma_D(q)$, where $D = Q^2/2R$. Indeed, each q' with $[q, q'] = r$ gives rise to a divisor (q, q') of q in the range $[D, 8D]$, and moreover (q, q') is uniquely determined by q' via $(q, q') = qq'/r$. It follows that

$$N(m_0) = \sum_{\substack{Q\le q<2Q \\ \sigma_D(q)\ge m_0}} \sum_{\substack{R\le r<2R \\ m_0\le m_q(r)<2m_0}} m_q(r).$$

Since $m_q(r) = 0$ unless $q \mid r$, the inner sum over r is $O(m_0 R/Q)$. It thus suffices to show that

$$\#\{Q \le q < 2Q : \sigma_D(q) \ge m_0\} \ll m_0^{-2} Q \log Q,$$

for any $D \ge 1$. We may assume that $D \le Q^{1/2}$ since otherwise we may replace D by $Q/8D$. By the second moment method, we have

$$\#\{Q \le q < 2Q : \sigma_D(q) \ge m_0\} \le \frac{1}{m_0^2} \sum_{Q \le q < 2Q} \sigma_D(q)^2.$$

After expanding the square and changing the order of summation, the right-hand side above is

$$\frac{1}{m_0^2} \sum_{D \le d_1, d_2 \le 8D} \sum_{\substack{Q \le q < 2Q \\ [d_1, d_2] \mid q}} 1 \ll \frac{Q}{m_0^2} \sum_{D \le d_1, d_2 \le 8D} \frac{1}{[d_1, d_2]} \ll \frac{Q}{m_0^2} \log D.$$

This completes the proof of the lemma. □

There are two places in the proof of Lemma 3.3 where we lose a factor of $\log Q$ (and hence the assumption that $\delta < (\log Q)^{-1}$). One place is from dyadically decomposing the possible values of $[q, q']$, and the other from the conclusion of Lemma 4.1. If one is only interested in prime moduli, then this extra loss can certainly be saved.

Proof of Lemma 3.3. In this proof, all implied constants are allowed to depend on d and s. For $k, k' \in [K, 2K)$, we may write

$$\sum_{\ell \in I(k,k')} F(k\ell) \overline{F(k'\ell)} = \sum_{Q \le q, q' < 2Q} \sum_{\substack{\ell \in I(k,k',q,q') \\ k\ell \equiv a_q \pmod{q} \\ k'\ell \equiv a_{q'} \pmod{q'}}} \psi_q\left(\frac{k\ell - a_q}{q}\right) \overline{\psi_{q'}\left(\frac{k'\ell - a_{q'}}{q'}\right)}, \tag{4-1}$$

for some interval $I(k, k', q, q') \subset I(k, k')$. The solution to the simultaneous congruence conditions

$$k\ell \equiv a_q \pmod{q} \quad \text{and} \quad k'\ell \equiv a_{q'} \pmod{q'}$$

takes the form

$$\ell \equiv a(k, k', q, q') \pmod{[q, q']},$$

for some $0 \le a(k, k', q, q') < [q, q']$. It is possible that no solutions exist, in which case we may simply set $I(k, k', q, q')$ to be empty and assign an arbitrary value to $a(k, k', q, q')$. After a change of variables $\ell = [q, q']m + a(k, k', q, q')$, the inner sum over ℓ in (4-1) can be rewritten as

$$\sum_{m\in J(k,k',q,q')} \psi_q\left(\frac{k[q,q']}{q}m+b\right)\overline{\psi_{q'}\left(\frac{k'[q,q']}{q'}m+b'\right)},$$

for some interval $J(k,k',q,q') \subset \big[0, L/[q,q']\big]$, where

$$b=\frac{1}{q}\big(ka(k,k',q,q')-a_q\big) \quad\text{and}\quad b'=\frac{1}{q'}\big(k'a(k,k',q,q')-a_{q'}\big).$$

In principle b and b' depend on k, k', q and q', but to simplify notations we drop this dependence, as the precise nature of b and b' is unimportant, apart from the obvious facts that $0\le b\le k[q,q']/q$ and $0\le b'\le k'[q,q']/q'$. Consider the polynomial sequence $g_{k,k',q,q'}:\mathbb{Z}\to G_q\times G_{q'}$ defined by

$$g_{k,k',q,q'}(m)=\left(g_q\left(\frac{k[q,q']}{q}m+b\right), g_{q'}\left(\frac{k'[q,q']}{q'}m+b'\right)\right),$$

and the Lipschitz function $\varphi_{q,q'}: G_q/\Gamma_q\times G_{q'}/\Gamma_{q'}\to\mathbb{C}$ defined by

$$\varphi_{q,q'}(x,x')=\varphi_q(x)\overline{\varphi_{q'}(x')}.$$

Then the type-II sum from (4-1) can be written as

$$\sum_{\ell\in I(k,k')} F(k\ell)\overline{F(k'\ell)}=\sum_{Q\le q,q'<2Q}\ \sum_{m\in J(k,k',q,q')}\varphi_{q,q'}(g_{k,k',q,q'}(m)).$$

After dyadically dividing the possible values of $[q,q']$, we deduce from the hypothesis (3-4) that

$$\sum_{K\le k,k'<2K}\ \sum_{\substack{Q\le q,q'<2Q\\ R\le[q,q']<2R}}\left|\sum_{m\in J(k,k',q,q')}\varphi_{q,q'}(g_{k,k',q,q'}(m))\right|\gg \delta^2K^2L, \tag{4-2}$$

for some $Q\le R\le 4Q^2$, where we used the assumption that $\delta<(\log Q)^{-1}$. For the rest of the proof fix such an R. Hence there is a subset T consisting of quadruples (k,k',q,q') with $R\le[q,q']<2R$, such that

$$|T|\gg \delta^{O(1)}K^2R, \tag{4-3}$$

and for $(k,k',q,q')\in T$ we have

$$\left|\sum_{m\in J(k,k',q,q')}\varphi_{q,q'}(g_{k,k',q,q'}(m))\right|\gg\frac{\delta^2L}{R}. \tag{4-4}$$

Since $\int\varphi_{q,q'}=0$, (4-4) implies that the sequence $\{g_{k,k',q,q'}(m)\}_{0\le m\le L/R}$ fails to be $\delta^{O(1)}$-equidistributed. Hence by [Green and Tao 2012a, Theorem 2.9], there is a nontrivial horizontal character $\chi_{q,q'} = \chi_{k,k',q,q'} : G_q \times G_{q'} \to \mathbb{C}$ with $\|\chi_{q,q'}\| \ll \delta^{-O(1)}$, such that

$$\|\chi_{q,q'} \circ g_{k,k',q,q'}\|_{C^\infty(L/R)} \ll \delta^{-O(1)}. \tag{4-5}$$

We have tacitly assumed that $\chi_{q,q'}$ is independent of k and k', since this can be achieved after pigeonholing in the $\delta^{-O(1)}$ possible choices of $\chi_{q,q'}$ and enlarging the constant $O(1)$ in (4-3) appropriately. More explicitly, if we write

$$\chi_{q,q'} \circ g_{k,k',q,q'}(m) = \sum_{i=0}^{s} \beta_i(k,k',q,q')m^i, \tag{4-6}$$

for some coefficients $\beta_i(k,k',q,q') \in \mathbb{R}$, then (4-5) combined with [Green and Tao 2012b, Lemma 3.2] implies that there is a positive integer $r = O(1)$ such that

$$\|r\beta_i(k,k',q,q')\| \ll \delta^{-O(1)}\left(\frac{L}{R}\right)^{-i}, \tag{4-7}$$

for each $1 \le i \le s$ and $(k,k',q,q') \in T$. Write

$$\chi_{q,q'} = \left(\chi^{(1)}_{q,q'}, \chi^{(2)}_{q,q'}\right),$$

where $\chi^{(1)}_{q,q'}$ and $\chi^{(2)}_{q,q'}$ are horizontal characters on G_q and $G_{q'}$, respectively, with

$$\|\chi^{(1)}_{q,q'}\| \ll \delta^{-O(1)} \quad \text{and} \quad \|\chi^{(2)}_{q,q'}\| \ll \delta^{-O(1)}.$$

Write also

$$\chi^{(1)}_{q,q'} \circ g_q(n) = \sum_{i=0}^{s} \alpha_i(q,q')n^i \quad \text{and} \quad \chi^{(2)}_{q,q'} \circ g_{q'}(n) = \sum_{i=0}^{s} \alpha'_i(q,q')n^i,$$

for some coefficients $\alpha_i(q,q'), \alpha'_i(q,q') \in \mathbb{R}$.

Claim. *There exists a sequence of subsets $E_1 \subset \cdots \subset E_s$ of pairs (q,q') with $R \le [q,q'] < 2R$ and a sequence of positive integers $r_1 \ge \cdots \ge r_s$ with $r_{i+1} \mid r_i$ for each i, such that $|E_1| \gg \delta^{O(1)}R$, $r_1 \ll \delta^{-O(1)}$, and moreover*

$$\|r_i\alpha_i(q,q')\| \ll \delta^{-O(1)}\left(\frac{KL}{Q}\right)^{-i} \quad \textit{and} \quad \|r_i\alpha'_i(q,q')\| \ll \delta^{-O(1)}\left(\frac{KL}{Q}\right)^{-i},$$

for each $1 \le i \le s$ and $(q,q') \in E_i$.

Note that the claim actually implies the bounds

$$\|r_i\alpha_j(q,q')\| \ll \delta^{-O(1)}\left(\frac{KL}{Q}\right)^{-j} \quad \text{and} \quad \|r_i\alpha'_j(q,q')\| \ll \delta^{-O(1)}\left(\frac{KL}{Q}\right)^{-j},$$

for each $1 \le i \le j \le s$ and $(q,q') \in E_j$.

Assuming the claim, we may conclude the proof of the lemma as follows: Pick an arbitrary pair $(q,q') \in E_1$. Since $\chi_{q,q'}$ is nontrivial, either $\chi^{(1)}_{q,q'}$ or $\chi^{(2)}_{q,q'}$

is nontrivial. Without loss of generality, assume that $\chi^{(1)}_{q,q'}$ is nontrivial. The diophantine information about $\alpha_i(q,q')$ from the claim implies that

$$\|r_1\chi^{(1)}_{q,q'}\circ g_q\|_{C^\infty(KL/Q)} \ll \delta^{-O(1)}.$$

Thus by [Frantzikinakis and Host 2017, Lemma 5.3], the polynomial sequence g_q fails to be totally $\delta^{O(1)}$-equidistributed.

It remains to establish the claim. Start by finding a subset E of pairs (q,q') with $R \le [q,q'] < 2R$, such that the following properties hold:

(1) $|E| \gg \delta^{O(1)}R$.

(2) For each pair $(q,q')\in E$, there are at least $\delta^{O(1)}K^2$ pairs (k,k') such that $(k,k',q,q')\in T$.

(3) For each pair $(q,q')\in E$, there are at most $\delta^{-O(1)}$ pairs $(q,\tilde q')\in E$ with $[q,q']=[q,\tilde q']$, and similarly there are at most $\delta^{-O(1)}$ pairs $(\tilde q,q')\in E$ with $[q,q']=[\tilde q,q']$.

Indeed, from the bound (4-3) we may first find E satisfying (1) and (2), and then apply Lemma 4.1 with $m_0=\delta^{-C}$ for some sufficiently large C to remove a small number of pairs from E, so that property (3) is satisfied.

Construct $\{E_i\}$ and $\{r_i\}$ in the claim by downward induction on i as follows. Take E_{s+1} equal to the E just constructed and $r_{s+1}=r$ from (4-7). Now let $1\le i\le s$, and suppose that E_j and r_j have already been constructed for $j>i$ satisfying the desired properties. First we show that for each pair $(q,q')\in E_{i+1}$, there is a positive integer $\tilde r(q,q')\ll\delta^{-O(1)}$ such that

$$\left\|\tilde r(q,q')r_{i+1}\alpha_i(q,q')\left(\frac{[q,q']}{q}\right)^i\right\| \ll \delta^{-O(1)}\left(\frac{KL}{R}\right)^{-i}, \tag{4-8}$$

and similarly with $\alpha_i(q,q')$ replaced by $\alpha'_i(q,q')$. To prove this, fix $(q,q')\in E_{i+1}$, and for the purpose of simplifying notations we drop the dependence on q and q' so that

$$\alpha_i=\alpha_i(q,q'),\quad \alpha'_i=\alpha'_i(q,q'),\quad\text{and}\quad \beta_i(k,k')=\beta_i(k,k',q,q').$$

From the definition of $g_{k,k',q,q'}$ we see the following relationship between the coefficients α_i, α'_i, and $\beta_i(k,k')$:

$$\beta_i(k,k')=\left(\frac{k[q,q']}{q}\right)^i\sum_{i\le j\le s}\alpha_j b^{j-i}+\left(\frac{k'[q,q']}{q'}\right)^i\sum_{i\le j\le s}\alpha'_j b'^{j-i}. \tag{4-9}$$

Write $\tilde\beta_i(k,k')$ for the contribution from the term with $j=i$:

$$\tilde\beta_i(k,k')=\left(\frac{k[q,q']}{q}\right)^i\alpha_i+\left(\frac{k'[q,q']}{q'}\right)^i\alpha'_i.$$

By the induction hypothesis, $\|r_{i+1}\alpha_j\|$ and $\|r_{i+1}\alpha'_j\|$ are small for $j > i$. Combined with the bound $0 \le b, b' \ll KR/Q$, this implies that the terms with $j > i$ are negligible:

$$\left\|r_{i+1}(\beta_i(k,k') - \tilde{\beta}_i(k,k'))\right\| \ll \delta^{-O(1)}\Big(\frac{L}{R}\Big)^{-i}.$$

It follows from (4-7) that

$$\left\|r_{i+1}\tilde{\beta}_i(k,k')\right\| \ll \delta^{-O(1)}\Big(\frac{L}{R}\Big)^{-i}, \tag{4-10}$$

whenever $(k,k',q,q') \in T$. Since $(q,q') \in E$, this holds for at least $\delta^{O(1)}K^2$ pairs (k,k'). Choose k' such that (4-10) holds whenever $k \in \mathcal{K}$, for some subset $\mathcal{K}$ with $|\mathcal{K}| \gg \delta^{O(1)}K$. Since

$$\tilde{\beta}_i(k,k') - \tilde{\beta}_i(\tilde{k},k') = \alpha_i\left(\frac{[q,q']}{q}\right)^i (k^i - \tilde{k}^i),$$

it follows that for $k, \tilde{k} \in \mathcal{K}$ we have

$$\left\|r_{i+1}\alpha_i\left(\frac{[q,q']}{q}\right)^i (k^i - \tilde{k}^i)\right\| \ll \delta^{-O(1)}\Big(\frac{L}{R}\Big)^{-i}.$$

Since $\delta > K^{-c}$ for some sufficiently small $c > 0$, the desired inequality (4-8) follows from a standard recurrence result such as [Green and Tao 2012a, Lemma 4.5]. The analogous bound for α'_i can be proved in a similar way.

Now that we have established (4-8), define $\widetilde{E}_i \subset E_{i+1}$ to be a subset with $|\widetilde{E}_i| \gg \delta^{O(1)}|E_{i+1}| \gg \delta^{O(1)}R$, such that $\tilde{r}(q,q')$ take a common value $\tilde{r}$ for $(q,q') \in \widetilde{E}_i$. We say that a pair $(q,q') \in \widetilde{E}_i$ is *typical* if there are at least $\delta^{O(1)}R/Q$ pairs $(q,\tilde{q}') \in \widetilde{E}_i$ with $\chi^{(1)}_{q,q'} = \chi^{(1)}_{q,\tilde{q}'}$, and similarly there are at least $\delta^{O(1)}R/Q$ pairs $(\tilde{q},q') \in \widetilde{E}_i$ with $\chi^{(2)}_{q,q'} = \chi^{(2)}_{\tilde{q},q'}$. Define $E_i \subset \widetilde{E}_i$ to be the set of typical pairs in $\widetilde{E}_i$. By choosing the constant $O(1)$ in the definition of typical pairs sufficiently large, we may ensure that $|E_i| \gg \delta^{O(1)}R$.

Now let $(q,q') \in E_i$. Since (q,q') is typical, there exists a subset $\mathcal{Q}(q,q')$ with $|\mathcal{Q}(q,q')| \gg \delta^{O(1)}R/Q$, such that $(q,\tilde{q}') \in \widetilde{E}_i$ and $\chi^{(1)}_{q,q'} = \chi^{(1)}_{q,\tilde{q}'}$ for all $\tilde{q}' \in \mathcal{Q}(q,q')$. Thus $\alpha(q,q') = \alpha(q,\tilde{q}')$ for all $\tilde{q}' \in \mathcal{Q}(q,q')$, and by applying (4-8) to $(q,\tilde{q}')$ we obtain

$$\left\|\tilde{r}r_{i+1}\alpha_i(q,q')\left(\frac{[q,\tilde{q}']}{q}\right)^i\right\| \ll \delta^{-O(1)}\Big(\frac{KL}{R}\Big)^{-i},$$

for each $\tilde{q}' \in \mathcal{Q}(q,q')$. Since $\widetilde{E}_i \subset E$, property (3) of the set E implies that

$$\#\left\{\frac{[q,\tilde{q}']}{q} : \tilde{q}' \in \mathcal{Q}(q,q')\right\} \gg \delta^{O(1)}|\mathcal{Q}(q,q')| \gg \delta^{O(1)}\frac{R}{Q}.$$

By a standard recurrence result such as [Green and Tao 2012a, Lemma 4.5], there exists $r_i \ll \delta^{-O(1)}\tilde{r}r_{i+1} \ll \delta^{-O(1)}$ such that

$$\|r_i\alpha_i(q,q')\| \ll \delta^{-O(1)}\left(\frac{KL}{Q}\right)^{-i},$$

as desired. The analogous bound for $\alpha_i'(q,q')$ can be proved in a similar way. This finishes the proof of the claim, and also the proof of Lemma 3.3. □

References

[Bourgain et al. 2013] J. Bourgain, P. Sarnak, and T. Ziegler, "Disjointness of Moebius from horocycle flows", pp. 67–83 in *From Fourier analysis and number theory to Radon transforms and geometry*, edited by H. M. Farkas et al., Dev. Math. **28**, Springer, New York, 2013. MR Zbl

[Fouvry and Tenenbaum 1996] E. Fouvry and G. Tenenbaum, "Répartition statistique des entiers sans grand facteur premier dans les progressions arithmétiques", *Proc. London Math. Soc.* (3) **72**:3 (1996), 481–514. MR Zbl

[Frantzikinakis and Host 2017] N. Frantzikinakis and B. Host, "Higher order Fourier analysis of multiplicative functions and applications", *J. Amer. Math. Soc.* **30**:1 (2017), 67–157. MR Zbl

[Green 2016] B. Green, "A note on multiplicative functions on progressions to large moduli", 2016. To appear in *Proc. Royal Soc. Edinburgh, Ser. A*. arXiv

[Green and Tao 2012a] B. Green and T. Tao, "The quantitative behaviour of polynomial orbits on nilmanifolds", *Ann. of Math.* (2) **175**:2 (2012), 465–540. MR Zbl

[Green and Tao 2012b] B. Green and T. Tao, "The Möbius function is strongly orthogonal to nilsequences", *Ann. of Math.* (2) **175**:2 (2012), 541–566. MR Zbl

[Green et al. 2012] B. Green, T. Tao, and T. Ziegler, "An inverse theorem for the Gowers $U^{s+1}[N]$-norm", *Ann. of Math.* (2) **176**:2 (2012), 1231–1372. MR Zbl

[Harper 2011] A. J. Harper, "A different proof of a finite version of Vinogradov's bilinear sum inequality", unpublished note, 2011, Available at http://tinyurl.com/adamharper.

[Harper 2012] A. J. Harper, "Bombieri–Vinogradov and Barban–Davenport–Halberstam type theorems for smooth numbers", preprint, 2012. arXiv

[Iwaniec and Kowalski 2004] H. Iwaniec and E. Kowalski, *Analytic number theory*, American Mathematical Society Colloquium Publications **53**, American Mathematical Society, Providence, RI, 2004. MR Zbl

[Kátai 1986] I. Kátai, "A remark on a theorem of H. Daboussi", *Acta Math. Hungar.* **47**:1-2 (1986), 223–225. MR Zbl

[Matomäki and Radziwiłł 2016] K. Matomäki and M. Radziwiłł, "Multiplicative functions in short intervals", *Ann. of Math.* (2) **183**:3 (2016), 1015–1056. MR Zbl

[Matomäki et al. 2015] K. Matomäki, M. Radziwiłł, and T. Tao, "An averaged form of Chowla's conjecture", *Algebra Number Theory* **9**:9 (2015), 2167–2196. MR Zbl

[Matthiesen 2016] L. Matthiesen, "Generalized Fourier coefficients of multiplicative functions", preprint, 2016. arXiv

[Montgomery and Vaughan 1977] H. L. Montgomery and R. C. Vaughan, "Exponential sums with multiplicative coefficients", *Invent. Math.* **43**:1 (1977), 69–82. MR Zbl

[Ramaré 2009] O. Ramaré, *Arithmetical aspects of the large sieve inequality*, Harish-Chandra Research Institute Lecture Notes **1**, Hindustan Book Agency, New Delhi, 2009. MR Zbl

[Tao and Vu 2006] T. Tao and V. Vu, *Additive combinatorics*, Cambridge Studies in Advanced Mathematics **105**, Cambridge Univ. Press, 2006. MR Zbl

Communicated by Peter Sarnak
Received 2016-07-15 Revised 2016-12-02 Accepted 2017-01-29

xuancheng.shao@maths.ox.ac.uk *Mathematical Institute, University of Oxford, Radcliffe Observatory Quarter, Woodstock Road, Oxford OX2 6GG, United Kingdom*

ALGEBRA AND NUMBER THEORY 11:4 (2017)
dx.doi.org/10.2140/ant.2017.11.983

The degree of the Gauss map of the theta divisor

Giulio Codogni, Samuel Grushevsky and Edoardo Sernesi

We study the degree of the Gauss map of the theta divisor of principally polarised complex abelian varieties. Thanks to this analysis, we obtain a bound on the multiplicity of the theta divisor along irreducible components of its singular locus. We spell out this bound in several examples, and we use it to understand the local structure of isolated singular points. We further define a stratification of the moduli space of ppavs by the degree of the Gauss map. In dimension four, we show that this stratification gives a weak solution of the Schottky problem, and we conjecture that this is true in any dimension.

1. Introduction

Let (A, Θ) be a complex g-dimensional principally polarised abelian variety (ppav), where, by abuse of notation, we write Θ both for an actual symmetric divisor of A and for its first Chern class. The Gauss map

$$\mathcal{G} : \Theta \dashrightarrow \mathbb{P}^{g-1}$$

is the rational map given by the complete linear system $L := \mathcal{O}(\Theta)|_\Theta$. If (A, Θ) is indecomposable (not a product of lower-dimensional ppavs), then $\mathcal{G}$ is a rational dominant generically finite map, and we are interested in its degree. Since a basis of $H^0(\Theta, L)$ is given by the partial derivatives of the theta function restricted to the theta divisor, the scheme-theoretic base locus of the Gauss map is equal to the singular locus $\mathrm{Sing}(\Theta)$, endowed with the scheme structure defined by the partial derivatives of the theta function.

Given an irreducible component V of $\mathrm{Sing}(\Theta)$, we denote by V_{red} the reduced scheme structure on V, and then denote $\mathrm{mult}_{V_{\mathrm{red}}} \Theta$ the vanishing multiplicity of the

Codogni is funded by the FIRB 2012 “Moduli spaces and their applications” and the “ERC StG 307119 - Stab AGDG”. Research of Grushevsky is supported in part by National Science Foundation under the grant DMS-15-01265, and by a Simons Fellowship in Mathematics (Simons Foundation grant #341858 to Samuel Grushevsky).
MSC2010: primary 14K10; secondary 14C17, 14H42.
Keywords: Gauss map, principally polarised abelian varieties, Schottky problem, V-cycles, excess intersection formula.

theta function along V_{red}. We denote by $\deg_V \Theta$ the intersection number $\Theta^d \cdot V$ on A, where $d = \dim V$; note that $\deg_V \Theta \geq 1$ for any V, since the theta divisor is ample. Our main result relates the multiplicity of the singularities of the theta divisor with the degree of the Gauss map.

Theorem 1.1. *Let (A, Θ) be an indecomposable principally polarised abelian variety of dimension $g \geq 3$. Let $\{V_i\}_{i\in I}$ be the set of irreducible components of* $\mathrm{Sing}(\Theta)$; *denote $d_i := \dim V_i$ and $m_i := \mathrm{mult}_{V_{i,\mathrm{red}}} \Theta$. Then the following inequalities hold*:

$$\sum_{i\in I} m_i (m_i - 1)^{g-d_i-1} \deg_{V_{i,\mathrm{red}}} \Theta \leq g! - \deg \mathcal{G} \leq g! - 4,$$

where $\deg \mathcal{G}$ *is the degree of the Gauss map.*

In Section 3 we discuss the improvements of this result in various cases, by using the known lower bounds on the degree of the theta divisor on subschemes of A. In particular, as an easy corollary we obtain a special case of a well-known result of Ein and Lazarsfeld [1997].

Corollary 1.2. *For any indecomposable ppav (A, Θ) such that the algebraic cohomology group $HA^{2,2}(A) = \mathbb{Z}$, the divisor Θ is smooth in codimension* 1.

Remark 1.3. For the case of an isolated singular point $z \in \mathrm{Sing}(\Theta)$ of multiplicity m, the bound of Theorem 1.1 gives $m(m-1)^{g-1} \leq g! - 4$, which asymptotically for $g \to \infty$ behaves like $m \leq g/e$, where e is the Euler number.

Recall that the inequality $\mathrm{mult}_z \Theta \leq g$ at any point of any ppav is a famous result of Kollár [1995], while Smith and Varley [1996] prove that there exists a point z such that $\mathrm{mult}_z \Theta = g$ if and only if the ppav is the product of g elliptic curves. Conjecturally (see the discussion in [Grushevsky and Hulek 2013]) for indecomposable ppavs the multiplicity of the theta divisor at any point is at most $\lfloor (g+1)/2 \rfloor$ — this is the maximum possible multiplicity for Jacobians of smooth curves. The bound we get for multiplicity of isolated singular points is thus asymptotically better than this conjecture.

Since the multiplicity $m_i \geq 2$, we also get the following:

Corollary 1.4. *The number of isolated singular points of the theta divisor is at most $(g! - 4)/2$.*

Remark 1.5. Recall that by definition a *vanishing theta-null* for a ppav is an even two-torsion point contained in the theta divisor. Conjecturally for an indecomposable ppav the number of vanishing theta-nulls is less than $2^{g-1}(2^g + 1) - 3^g$, and this conjecture was recently investigated in detail by Auffarth, Pirola, Salvati Manni [Auffarth et al. 2017]. The corollary above provides a better bound than this conjecture for all ppavs for genus up to 8 (this is the range in which $(g!-4)/2 < 2^{g-1}(2^g+1)-3^g$) such that every vanishing theta-null is an isolated point of $\mathrm{Sing}(\Theta)$.

We use the machinery of Vogel's v-cycles to obtain the proof of our result. As we only need to apply this machinery in a specific case, we give a direct elementary construction of v-cycles, and a self-contained argument for our results, not relying on the general v-cycle literature.

Related results and approaches. Using the results of [Ein et al. 1996; Nakamaye 2000] one can obtain a slightly weaker version of the bound on the multiplicity of singularities claimed in Theorem 1.1. More recently, Mustaţă and Popa applied their newly developed theory of Hodge ideals to also get a slightly weaker multiplicity bound; see [Mustata and Popa 2016, Section 29]. Neither of these works is directly related to the degree of the Gauss map. R. Varley explained to us that Corollary 1.4 can be obtained by extending the techniques from [Smith and Varley 1996] and utilizing the machinery developed in [Fulton 1984, Section 4.4 and Chapter 11].

In the spirit of the Andreotti–Mayer loci defined as

$$N_k^{(g)} := \{(A, \Theta) \in \mathcal{A}_g \mid \dim \operatorname{Sing}(\Theta) \geq k\},$$

we define the *Gauss loci*

$$G_d^{(g)} := \{(A, \Theta) \in \mathcal{A}_g \mid \deg(\mathcal{G} : \Theta \dashrightarrow \mathbb{P}^{g-1}) \leq d\}.$$

In this spirit, we conjecture that the degree of the Gauss map gives a weak solution to the Schottky problem:

Conjecture 1.6. The Gauss loci $G_d^{(g)}$ are closed for any d and g. The closure of the locus of Jacobians of smooth curves $\overline{\mathcal{J}}_g$ is an irreducible component of $G_{d'}^{(g)}$, where $d' = \binom{2g-2}{g-1}$. The closure of the locus of Jacobians of smooth hyperelliptic curves $\overline{\mathcal{H}}_g$ is an irreducible component of $G_{2^{g-1}}^{(g)}$.

In making this conjecture, we recall that the degree of the Gauss map is known for all Jacobians of smooth curves: it is equal to 2^{g-1} if the curve is hyperelliptic, and to $\binom{2g-2}{g-1}$ if the curve is not hyperelliptic; see [Andreotti 1958, Proof of Proposition 10].

In Proposition 4.2, we prove Conjecture 1.6 for $g = 4$ (after easily verifying it for any $g < 4$). We also show that all relevant Gauss loci are distinct in genus 4, i.e., that for any $2 \leq d \leq 12$ there exists an abelian fourfold such that the degree of its Gauss map is equal to $2d$.

Already when $g = 5$, we expect not all the even degree Gauss loci $G_{2d}^{(5)}$ for $2 \leq d \leq 60$ to be distinct, and in general the following natural question remains completely open:

Question 1.7. For a given $g \geq 5$, what is the set of possible degrees of the Gauss map for g-dimensional ppavs? Equivalently, for which d is $G_{2d}^{(g)} \setminus G_{2d-2}^{(g)}$ nonempty?

2. Generalities about the Gauss map and multiplicities of singularities

Our paper is focussed on studying the degree and the geometry of the Gauss map for abelian varieties. We first recall the general setup for working with generically finite rational maps to projective spaces.

Let X be an n-dimensional complex projective variety (we implicitly assume all varieties in this paper to be irreducible, unless stated otherwise), L a line bundle on X, and $W \subseteq H^0(X, L)$ a vector subspace such that the rational map

$$f = |W| : X \dashrightarrow \mathbb{P}W^\vee$$

is generically finite and dominant. Let B be the scheme-theoretic base locus of f. Our focus will be the discrepancy of f:

Definition 2.1 (discrepancy). In the above setup, the *discrepancy* δ of f is

$$\delta := \deg_X L - \deg f.$$

If f is regular, the discrepancy δ is equal to zero. In general, naively, δ tells us how much of the degree of L is absorbed by the base locus B. Note that the discrepancy is not always positive: the presence of the base locus could even increase the degree, as shown in Example 2.4. We will show that when L is ample and the base locus is not empty, the discrepancy is strictly positive.

The following formula for the discrepancy in terms of Segre classes is given in [Fulton 1984, Proposition 4.4]:

$$\delta = \int_B (1 + c_1(L))^n \cap s(B, X).$$

For our purposes in this paper we prefer to avoid Segre classes and use the language of Vogel's cycles, called v-cycles for short. The theory of v-cycles is fully developed and described in detail in [Flenner et al. 1999]; other references are [Vogel 1984; van Gastel 1991]. However, as the applications we need in this paper are limited, for our purposes a limited version of the theory suffices. We thus prefer to give an entirely self-contained exposition of the machinery we use — avoiding most technicalities, and not having to rely on the literature on the subject.

In the setup above, we will define effective cycles V^j with support contained in B and the so-called residual cycles R^j such that the support of none of their irreducible components is contained in B, with $j = 0, \ldots, n$. Both V^j and R^j are equidimensional of dimension $n - j$, unless they are empty.

These cycles are defined inductively, according to Vogel's intersection algorithm. We let $V^0 := \varnothing$, and let $R^0 := X$. For any $j > 0$ assume we have already defined effective cycles V^{j-1} and R^{j-1} of pure dimension $n - j + 1$. Since none of the irreducible components of R^{j-1} have support contained in B, the zero locus $D_j \subset X$

of a generic section $s_j \in W$ does not contain any irreducible component of R^{j-1}. We thus write $D_j \cap R^{j-1} = \sum_k a_k E_k$, where the E_k are reduced and irreducible Cartier divisors on R^{j-1}. We order the divisors E_k in such a way that E_k is supported within B for $k = 1, \dots, t$ and is not supported within B for $k > t$. Then we let

$$V^j := \sum_{k=1}^{t} a_k E_k \quad \text{and} \quad R^j := \sum_{k>t} a_k E_k.$$

The cycles R^j and V^j are effective of pure dimension $n - j$. The cycle V^j can possibly be empty; on the other hand, for D_j general, the cycle R^j is nonempty. Since

$$D_j \cap R^{j-1} = V^j + R^j,$$

and since D_j is the zero locus of a section of L, it follows that

$$\deg_{R^j} L = \deg_{R^{j-1}} L - \deg_{V^j} L. \tag{1}$$

Theorem 2.2 (formula for the discrepancy). *In the setup above*, *the discrepancy is given by*

$$\delta = \sum_{j=1}^{n} \deg_{V^j} L.$$

Proof. For $0 \le j \le n$, consider the restricted morphisms

$$f_j := f|_{R^j} : R^j \dashrightarrow \mathbb{P}H_j,$$

where H_j is the linear subspace of $W^\vee$ that is the common zero locus of the sections $s_1, \dots, s_j \in W$. Since R^j is the closure of $f^{-1}(\mathbb{P}H_j)$ in X, we have

$$\deg f = \deg f_j$$

for any j. We now take $j = n$. Since R^n is zero-dimensional and it is supported outside B, the map

$$f_n \colon R^n \to \mathbb{P}H_n$$

is a regular morphism, and thus

$$\deg f = \deg f_n = \deg_{R^n} L.$$

To compute $\deg_{R^n} L$, we apply formula (1) n times to obtain

$$\deg_{R^n} L = \deg_{R^0} L - \sum_{j=1}^{n} \deg_{V^j} L.$$

The theorem follows from observing that $\deg_{R^0} L = \deg_X L$. $\square$

Example 2.3. Let ℓ be a line in $\mathbb{P}^3$ and let $W \subset H^0(\mathbb{P}^3, \mathcal{O}(3))$ define a generic 3-dimensional linear system of cubics containing ℓ. This linear system gives a rational map with finite fibres

$$f = |W| : \mathbb{P}^3 \dashrightarrow \mathbb{P}W^\vee.$$

Let us run Vogel's algorithm to describe the v-cycles and compute the discrepancy. We thus pick 4 generic cubics $D_1, \ldots, D_4 \in W$, and have $R^1 = D_1$ and $V^1 = \varnothing$. The intersection $D_2 \cap R^1$ is a reducible curve $C \cup \ell$; thus $R^2 = C$ and $V^2 = \ell$. The intersection $C \cap D_3$, is equal to the union of $C \cap \ell$ and r reduced points not contained in ℓ, and thus $V^3 = C \cap \ell$ and R^3 consists of the other r reduced points. To describe $C \cap \ell$, notice that the arithmetic genus of $C \cup \ell$ is 10. The curve C is a component of a complete intersection; since we know the degree of all components of this complete intersection, we can show that the genus of C is 7 using a standard formula; see for example [Sernesi 1986, Proposition 11.6]. We conclude that $C \cap \ell$ consists of 4 reduced points.

The discrepancy of the morphism associated to W is

$$\delta = \deg_\ell \mathcal{O}(3) + \deg_{V^3} \mathcal{O}(3) = 3 + 4 = 7.$$

Hence the rational map given by W has degree $\deg_{\mathbb{P}^3} \mathcal{O}(3) - \delta = 27 - 7 = 20$.

We can also compute this discrepancy using Segre classes. Notice that ℓ is regularly embedded in $\mathbb{P}^3$, so its Segre class is just the inverse of the Chern class of the normal bundle. Let p be the class of a point in ℓ. We have

$$\delta = \int_\ell (1+3p)^3(1-2p) = \int_\ell (1+9p-2p) = 9 - 2 = 7.$$

It is worth remarking that the Segre classes formula expresses the discrepancy as a difference, whereas the v-cycles formula expresses the discrepancy as a sum of positive contributions.

We now modify slightly the previous example to obtain a map with negative discrepancy (which is of course impossible for an ample line bundle L). The highlight of the following example is that the discrepancy is negative because the degree of L on the v-cycle V^2 is negative; in particular, L is not nef.

Example 2.4 (negative discrepancy). Keeping the notation of Example 2.3, we consider the blow-up $\pi : X \to \mathbb{P}^3$ of $\mathbb{P}^3$ at generic points $p_1, \ldots, p_k$ on the line ℓ; in particular, we assume that these points are not contained in the curve C described in Example 2.3. Let $E_1, \ldots, E_k$ be the exceptional divisors and let $E = \sum E_i$. On X, we take as line bundle $L = \pi^*\mathcal{O}(3) - E$, and as linear system we take the proper transform $\widetilde{W}$ of W; denote by $\tilde{\ell}$ the proper transform of ℓ.

The v-cycles are the proper transform of the v-cycles of the previous example, so $V^2 = \tilde{\ell}$, and V^3 consists again of 4 reduced points on $\tilde{\ell}$. The discrepancy is now

$$\delta = \deg_{\tilde{\ell}} L + \deg_{V^3} L = (3-k) + 4 = 7-k,$$

which is negative for $k > 7$. The degree of the map is

$$\deg_X L - \delta = (27-k) - (7-k) = 20.$$

As expected, this degree agrees with the degree of the map described in Example 2.3; the reason is that the two morphisms coincide outside the exceptional locus of the blow-up.

Before giving an application of Theorem 2.2, we need to recall the notion of multiplicity. Let Z be an irreducible subscheme of an irreducible scheme X; the *multiplicity* $\mathrm{mult}_Z X$ *of X along Z* is defined using the notion of Samuel multiplicity as follows. Let $(R, \mathfrak{m})$ be the local ring of X at (the generic point of) Z_{red}. In this ring, Z is defined by an $\mathfrak{m}$-primary ideal $\mathfrak{q}$. For t sufficiently large, the Hilbert function $h(t) := \mathrm{len}(R/\mathfrak{q}^t)$ is a polynomial in t. We normalise this polynomial by multiplying it by $\deg(h(t))! = \mathrm{codim}(Z, X)!$, and the *Samuel multiplicity* $\mathfrak{e}(\mathfrak{q}, R)$ is then defined to be the leading term of the normalised Hilbert polynomial.

Then one defines

$$\mathrm{mult}_Z X := \mathfrak{e}(\mathfrak{q}, R).$$

The following commutative algebra result, roughly speaking, reduces the computation of the Samuel multiplicity to the case of local complete intersections.

Proposition 2.5. *Let $(R, \mathfrak{m})$ be a d-dimensional Cohen–Macaulay local ring with infinite residue field K. Let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal of R. Let $t_1, \dots, t_m$ be a set of generators of $\mathfrak{q}$, and let $s_1, \dots, s_d$ be generic linear combinations of the t_i. Then the ideal $I := (s_1, \dots, s_d) \subseteq \mathfrak{q}$ computes the Samuel multiplicity of $\mathfrak{q}$; that is*

$$\mathfrak{e}(\mathfrak{q}, R) = \mathfrak{e}(I, R) = \ell(R/I).$$

Proof. From [Matsumura 1986, Theorems 14.13 and 14.14 on p. 112] it follows that $\mathfrak{e}(\mathfrak{q}, R) = \mathfrak{e}(I, R)$. Since R is Cohen–Macaulay, $\underline{s} := s_1, \dots, s_d$ is a regular R-sequence; thus

$$\ell(R/I) = \ell(H_0(\underline{s}, R)) = \sum_i (-1)^i \ell(H_i(\underline{s}, R)) = \mathfrak{e}(I, R)$$

by [op. cit., p. 109]. □

Before applying this result, let us recall that by definition the class of a closed irreducible subscheme $Z \subset X$ in the Chow group of X is

$$[Z] = \ell(\mathcal{O}_{X, Z_{\mathrm{red}}}/\mathcal{I}_Z) \cdot [Z_{\mathrm{red}}],$$

as explained for example in [Voisin 2002, Section 21.1.1].

Corollary 2.6 (bound on the discrepancy). *In the setup above, assume that X is Cohen–Macaulay, let $B = \bigcup_{i=1}^r B_i$ be the decomposition of the scheme-theoretic base locus of f into its scheme-theoretic irreducible components, and let $\mathfrak{e}_i := \mathrm{mult}_{B_i} X$. If L is nef, then*

$$\delta \geq \sum_i \mathfrak{e}_i \deg_{B_{i,\mathrm{red}}} L.$$

In particular, if B is nonempty and L is ample, then the discrepancy δ is strictly positive.

Proof. Fix an irreducible component B_i of B and denote by $k := \dim X - \dim B_i$. We will single out an irreducible component Z_i of V^k such that $Z_{i,\mathrm{red}} = B_{i,\mathrm{red}}$ and

$$\mathfrak{e}_i = \mathrm{mult}_{B_i} X = \mathrm{mult}_{Z_i} X = \ell(\mathcal{O}_{X, Z_{i,\mathrm{red}}} / \mathcal{I}_{Z_i}).$$

Since B_i is contained in each divisor D_j, it follows that B_i is contained in R^{k-1}, and, for dimensional reasons, $B_{i,\mathrm{red}}$ is the support of an irreducible component Z_i of $R^{k-1} \cap D_k$. By construction of the v-cycles, this means that Z_i is an irreducible component of V^k. In particular, we have $Z_{i,\mathrm{red}} = B_{i,\mathrm{red}}$. Let $(R, \mathfrak{m})$ be the local ring of X at the generic point of $Z_{i,\mathrm{red}}$. The ideal $\mathfrak{q}$ defining B_i in the local ring R is generated by the image in R of the linear system W. Recall that s_j is the section defining D_j. The subscheme Z_i is defined in R by the image of the sections $s_1, \dots, s_k$. These sections vanish on B_i, so they lie in $\mathfrak{q}$. The sections are generic elements of W, so we can apply Proposition 2.5 to conclude that $\ell(\mathcal{O}_{X, Z_{i,\mathrm{red}}} / \mathcal{I}_{Z_i}) = \mathrm{mult}_{Z_i} X = \mathrm{mult}_{B_i} X$.

We now use the nefness of L, which implies that its degree is nonnegative on any subscheme $Y \subset X$. Thus Theorem 2.2 implies

$$\delta = \sum_{j=1}^n \deg_{V^j} L \geq \sum_{i=1}^r \deg_{Z_i} L.$$

where $Z_1, \dots, Z_r$ is the list of irreducible components of v-cycles associated to the irreducible components $B_1, \dots, B_r$ of the base locus by the procedure described above. (Note that if B is zero-dimensional we do not need to get rid of higher codimension v-cycles, so the inequality becomes an equality.)

The degree $\deg_{Z_i} L$ is an intersection number which can be computed in the Chow group of X, so $\deg_{Z_i} L = \mathfrak{e}_i \deg_{B_{i,\mathrm{red}}} L$. □

Let us compute the cycles Z_i of the previous proof in a particular example.

Example 2.7. Let X be the singular quadric surface defined by $x^2 + y^2 + w^2 = 0$ in $\mathbb{P}^3$, with coordinates $[x : y : w : z]$; let $p = [0 : 0 : 0 : 1]$ be the singular point. Let L be $\mathcal{O}_{\mathbb{P}^3}(1)|_X$, and consider the linear system W on X generated by the restrictions of x, y, and w. The base locus is the point p endowed with the reduced scheme structure. On the other hand, if we run Vogel's algorithm using as sections $s_1 = x$, $s_2 = y$, and $s_3 = w$, the v-cycle V^2 is defined in $\mathbb{P}^3$ by the ideal $(x^2+y^2+w^2, x, y)$; thus V^2 is supported at p but is not reduced. This is a case where Z and B have the same support and the same multiplicity in X, but they are different as schemes.

Remark 2.8 (zero-dimensional base locus). Suppose $B = \bigcup_i B_i$, where each B_i is a possibly nonreduced scheme supported on a closed point p_i. In this case, as explained in the proof of Corollary 2.6,

$$\delta = \deg_{V^n} L = \sum \operatorname{mult}_{B_i} X.$$

This formula has a down to earth proof. Take n general effective Cartier divisors D_i in $|W|$: they intersect in a point $q \in \mathbb{P}W^\vee$; the intersection of the D_i in X consists of the points p_i counted with multiplicity $\operatorname{mult}_{B_i} X$ and other $\deg_X L - \sum \operatorname{mult}_{B_i} X$ distinct reduced points q_i. These points q_i are exactly the fibre of f over q, proving the formula for δ in this case.

The case we are interested in is when $X = \Theta$ is the theta divisor of a ppav (A, Θ) of dimension g, and $f = \mathcal{G}$ is the Gauss map. We thus obtain

$$\deg \mathcal{G} = g! - \delta.$$

3. The multiplicity bound for theta divisors

In this section we prove Theorem 1.1 bounding the multiplicity of the theta divisor, by an analysis of the discrepancy associated to the Gauss map. It is well-known [Birkenhake and Lange 2004, Section 4.4] that for an indecomposable ppav the Gauss map is dominant and has positive degree. We start with the following observation.

Lemma 3.1. *For an indecomposable ppav the degree of the Gauss map* $\deg \mathcal{G}$ *is even. If* $g \geq 3$, *then*

$$4 \leq \deg \mathcal{G} \leq g!.$$

Moreover, $\deg \mathcal{G} = g!$ *if and only if the theta divisor is smooth.*

Proof. Let $\iota : z \mapsto -z$ be the involution of the ppav; since the Gauss map is invariant under ι, its degree is even.

If the degree of the Gauss map for a ppav of dimension $g \geq 3$ were equal to 2, then Θ/ι would be rational. However, the quotient Θ/ι cannot be rational because there exist nonzero holomorphic ι-invariant 2-forms on Θ.

The inequality $\deg \mathcal{G} \leq g!$ follows from Corollary 2.6 and the fact that the Gauss map is given by an ample line bundle. The last claim holds because the base locus of the Gauss map is exactly $\operatorname{Sing}(\Theta)$. ∎

Since the base locus of the Gauss map is equal to $\mathrm{Sing}(\Theta)$, the results of the previous section combined with this lemma yield

$$\delta = g! - \deg \mathcal{G} \leq g! - 4,$$

where δ is the discrepancy associated to $\mathcal{G}$. To translate this into geometric information about the singularities of the theta divisor, we need another commutative algebra statement.

Proposition 3.2. *Let $D \subset X$ be an irreducible divisor in an n-dimensional smooth variety X, and let Z be a d-dimensional irreducible component of* $\mathrm{Sing}(D)$. *Let $m := \mathrm{mult}_{Z_{\mathrm{red}}} D$ and $\mathfrak{e} := \mathrm{mult}_Z D$. Then*

$$\mathfrak{e} \geq m(m-1)^{n-d-1}.$$

Proof. Let $(S, \mathfrak{n})$ be the local ring of $Z_{\mathrm{red}} \subset X$. It is an $(n-d)$-dimensional regular local ring. A local equation of D is an element $f \in \mathfrak{n}^m$. Let $(R, \mathfrak{m}) = (S/(f), \mathfrak{n}/(f))$ be the local ring of $Z_{\mathrm{red}} \subset D$. It is a Cohen–Macaulay ring of dimension $n-d-1$. Since $m := \mathrm{mult}_{Z_{\mathrm{red}}} D$,

$$\mathfrak{e}(\mathfrak{m}, R) = m.$$

Let $\mathfrak{q} := (f_1, \ldots, f_n)$ be the ideal of $Z \subset D$, where the f_i are the classes of the partial derivatives of f. Note that $\mathfrak{q} \subseteq \mathfrak{m}^{m-1}$. Let $s_1, \ldots, s_{n-d-1} \in \mathfrak{q}$ be a system of parameters. Then

$$\ell(R/(s_1, \ldots, s_{n-d-1})) \geq (m-1)^{n-d-1} \mathfrak{e}(\mathfrak{m}, R) = (m-1)^{n-d-1} m$$

by [Matsumura 1986, Theorem 14.9 on p. 109]. Choosing $s_1, \ldots, s_{n-d-1}$ to be a general linear combination of $f_1, \ldots, f_n$ we can apply Proposition 2.5. Then

$$\mathfrak{e}(\mathfrak{q}, R) = \ell(R/(s_1, \ldots, s_{n-d-1})) \geq (m-1)^{n-d-1} m,$$

proving the proposition. □

We can now prove our main result.

Proof of Theorem 1.1. The theorem follows by combining the bound $\delta \leq g! - 4$, Corollary 2.6, and Proposition 3.2. □

The estimates on the degree of the Gauss map and on the multiplicity of the singularities of Theorem 1.1 can be improved using the vast literature on lower bounds of $\deg_{V_{i,\mathrm{red}}} \Theta$ for d-dimensional subvarieties $V \subseteq A$. For any p we denote by $HA^{p,p}(A)$ the algebraic cohomology, that is the subgroup of $H^{p,p}(A, \mathbb{C}) \cap H^{2p}(A, \mathbb{Z})$ generated by the dual classes to the fundamental cycles of $(g-p)$-dimensional subvarieties of A (so in particular $HA^{1,1}$ is the Néron–Severi group).

When $d = 1$, a refinement of Matsusaka's criterion due to Ran states that if V generates the abelian variety, then $\deg_V \Theta \geq g$, with equality if and only if A is a

Jacobian and V is the Abel–Jacobi curve [Ran 1981, Corollary 2.6 and Theorem 3]. Combining this with Theorem 1.1 immediately yields the following:

Corollary 3.3. *For a one-dimensional irreducible component V of* $\mathrm{Sing}(\Theta)$, *denote by $m := \mathrm{mult}_{V_{\mathrm{red}}}\Theta$. If A is a simple abelian variety that is not a Jacobian*, *then*

$$m(m-1)^{g-2}(g+1) \leq g! - \deg \mathcal{G} \leq g! - 4.$$

In general for $d = \dim V > 1$, Ran [1981, Corollary II.6] shows that

$$\deg_V \Theta \geq \binom{g}{d}$$

if V is nondegenerate in the sense of that paper.

If the class of V in $HA^{g-d,g-d}(A)$ is an integer multiple of the so-called minimal class

$$\theta_{g-d} := \frac{1}{(g-d)!}\Theta^{g-d}$$

(which must be the case if $HA^{g-d,g-d}(A) = \mathbb{Z}$), then we have the bound

$$\deg_V \Theta \geq \frac{g!}{(g-d)!},$$

by noticing that V must then be a positive integral multiple of the minimal class, since it has positive intersection number with Θ^d, we get in particular the following:

Corollary 3.4. *For a d-dimensional irreducible component V of* $\mathrm{Sing}(\Theta)$, *denote by $m := \mathrm{mult}_{V_{\mathrm{red}}}\Theta$. If $HA^{g-d,g-d}(A) = \mathbb{Z}$*, *then*

$$m(m-1)^{g-d-1}\frac{g!}{(g-d)!} \leq g! - \deg \mathcal{G} \leq g! - 4.$$

Note that the case of $d = g-2$ of the above corollary gives precisely Corollary 1.2.

By applying Theorem 1.1, we can also obtain further results for the case of isolated singular points, extensively studied in the literature. For an isolated point z of $\mathrm{Sing}(\Theta)$, let B_z be the irreducible component of $\mathrm{Sing}(\Theta)$ supported at z, and let $\mathfrak{e}(z) := \mathrm{mult}_{B_z}\Theta$. We first need the following proposition:

Proposition 3.5. *If $z \in$* $\mathrm{Sing}(\Theta)$ *is an isolated double point such that the Hessian matrix of θ* (*second partial derivatives*) *at z has rank at least $g-1$*, *then there exists a local coordinate system in which locally the theta function can be written as*

$$\theta = x_1^2 + \cdots + x_{g-1}^2 + x_g^\ell$$

for some $\ell \geq 2$. In this case $\mathfrak{e}(z) = \ell$.

Proof. The existence of a local coordinate system where θ can be written as above follows immediately from the holomorphic Morse lemma [Żołądek 2006, Theorem 2.26].

Let s_i be as in Proposition 2.5; since we are interested just in the linear span of the s_i, applying Gauss elimination algorithm we can assume that $s_i = x_i$, for $i = 1, \dots, g-2$ and $s_{g-1} = x_{g-1} + x_g^{\ell-1}$. Since

$$\mathfrak{e} = \dim_{\mathbb{C}} S/(f, s_1, \dots, s_{n-d-1}),$$

where S is the local ring of the abelian variety at z, the statement follows. □

In genus 4 the proposition immediately gives the following corollary, which was obtained by completely different methods in [Smith and Varley 2012].

Corollary 3.6. *Let $(A, \Theta) \in \mathcal{A}_4$ be a Jacobian of a smooth nonhyperelliptic curve with a semicanonical g_3^1 (i.e., with a theta-null). Then locally around the unique singular point z of the theta divisor, the theta function can be written in an appropriate local coordinate system as $\theta = x_1^2 + x_2^2 + x_3^2 + x_4^4$.*

Proof. Recall that the degree of the Gauss map for any nonhyperelliptic genus 4 Jacobian is $\binom{2\cdot 4-2}{4-1} = 20 = 4! - 4$, so the discrepancy δ is equal to 4.

By [Arbarello et al. 1985, p. 232], in this case $\operatorname{Sing}(\Theta)_{\mathrm{red}} = \{p\}$ and the rank of the Hessian of the theta function at p is equal to 3. The result now follows from Proposition 3.5 and Corollary 2.6. □

Remark 3.7. Smith and Varley [2012] prove that coordinates x_i in the corollary above can furthermore be chosen to be equivariant, i.e., such that they all change sign under the involution $z \mapsto -z$ of the ppav. This also follows from the above proof: indeed, in Proposition 3.5 note that if the point z is two-torsion, i.e., fixed by the involution, then since the theta divisor is also invariant under the involution, as a set, one can use the equivariant version of the holomorphic Morse lemma to ensure that the coordinates x_i are equivariant under the involution as well.

The proposition also immediately yields the following easy general bound, which seems not to have been previously known.

Corollary 3.8. *If the point z is an isolated singular point of Θ, and the rank of the Hessian of the theta function at z is at least $g-1$, then, with the notation of Proposition 3.5, we have $\ell \leq g! - 4$.*

In a similar spirit, we can also prove the following bound

Corollary 3.9. *A theta divisor contains at most $\frac{1}{2}(g! - 4)$ isolated singular points.*

Proof. For any isolated point $z \in \operatorname{Sing}(\Theta)$, we can apply the multiplicity one criterion, which states that a primary ideal in a local ring has Samuel multiplicity one if and only if the ring is regular and the ideal is the maximal ideal, to see that $\mathfrak{e}(z) \geq 2$; references for the criterion are [Fulton 1984, Proposition 7.2] or [Nowak 1997]. Then the corollary follows from Theorem 1.1. □

Perhaps the next interesting case of ppavs with singular theta divisors beyond Jacobians of curves are intermediate Jacobians of smooth cubic threefolds. In this case the degree of the Gauss map can be computed from the geometric description given by Clemens and Griffiths [1972]; see [Krämer 2016, Remark 7(b)]. Let us sketch the computation. In [Clemens and Griffiths 1972], the authors consider the Fano surface S of lines of a cubic 3-fold F and a natural rational map:

$$\Psi : S \times S \longrightarrow \Theta,$$

where $\Theta \subset J(F)$ is the theta divisor. They prove that Ψ is generically 6 to 1, [op. cit., p. 348]. The composition $\mathcal{G} \cdot \Psi$ associates to an ordered pair of lines $(\ell_1, \ell_2) \in S \times S$ the hyperplane $H := \langle \ell_1, \ell_2 \rangle \subset \mathbb{P}^4$. The degree of $\mathcal{G} \cdot \Psi$ equals the number of ordered pairs of nonintersecting lines contained in the cubic surface $H \cap F$ for a general $H \in \mathbb{P}^{4\vee}$. There are 27×16 ordered pairs of nonintersecting lines on a nonsingular cubic surface; therefore the degree of $\mathcal{G}$ is

$$\deg(\mathcal{G}) = \frac{\deg(\mathcal{G} \circ \Psi)}{\deg(\Psi)} = \frac{27 \times 16}{6} = 72.$$

We remark that since the degree of the Gauss map for Jacobians of genus 5 curves is either 16 (for hyperelliptic curves) or $\binom{8}{4} = 70$ (for nonhyperelliptic curves), this computation already shows that the intermediate Jacobian of a smooth cubic threefold is not a Jacobian of a curve, allowing one to bypass a longer argument for this in [Clemens and Griffiths 1972].

Our machinery gives a quick alternative computation for this degree.

Proposition 3.10. *The degree of the Gauss map of the intermediate Jacobian of a smooth cubic threefold is equal to* 72.

Proof. It is well-known [Beauville 1982] that $B = \operatorname{Sing}(\Theta)$ is supported on a single point z, which has multiplicity 3; the tangent cone of that singularity is the cubic threefold. In this case, as explained in Remark 2.8, the discrepancy δ equals the multiplicity $\mathfrak{e}$ of Θ along B.

We use Proposition 2.5 to compute $\mathfrak{e}$. Locally near its singularity, the theta function is a cubic polynomial defining the cubic, plus higher order terms. Since the cubic is smooth, its partial derivatives are independent, and thus in the setup of Proposition 2.5 we can choose s_i to be the theta function (which vanishes at z to order 3), and four of its partial derivatives (each of which vanishes to second order). Thus we compute the Samuel multiplicity of z, or equivalently $\deg_B \Theta$, to be equal to $3 \cdot 2^4 = 48$. It follows that the discrepancy is equal to 48. Thus the degree of the Gauss map in this case is $\deg \mathcal{G} = 5! - 48 = 72$. $\square$

4. The stratification by the degree of the Gauss map

The moduli space of ppavs $\mathcal{A}_g$ is an orbifold of complex dimension $g(g+1)/2$. The classical Andreotti–Mayer loci $N_k^{(g)}$ are defined to be the loci of ppav such that $\dim \operatorname{Sing}(\Theta) \geq k$. These loci are closed. In analogy with the Andreotti–Mayer loci, we define the Gauss loci.

Definition 4.1 (Gauss loci). For any $d \in \mathbb{Z}_{\geq 0}$ we let $G_d^{(g)} \subseteq \mathcal{A}_g$ be the set of all ppavs for which the degree of the Gauss map is less than or equal to d.

For a decomposable ppav, the Gauss map is not dominant, so the degree of the Gauss map is 0; in other words $\mathcal{A}_g^{\mathrm{dec}} \subset G_0^{(g)}$, where we denoted by $\mathcal{A}_g^{\mathrm{dec}}$ the locus of decomposable ppavs. By definition we have $G_0^{(g)} \subseteq G_1^{(g)} \subseteq \cdots \subseteq G_{g!}^{(g)} = \mathcal{A}_g$. Since $\deg \mathcal{G}$ is always even, $G_{2d}^{(g)} = G_{2d+1}^{(g)}$ for any d. For $g \geq 3$, since the degree of the Gauss map on an indecomposable ppav is at least 4, $G_0^{(g)} = G_1^{(g)} = G_2^{(g)} = G_3^{(g)} = \mathcal{A}_g^{\mathrm{dec}}$.

We now discuss in detail the Gauss loci in low genus. In genus $g = 2$ the theta divisor of any indecomposable principally polarised abelian surface is smooth, and the degree of the Gauss map is equal to two. To summarise,

$$G_0^{(2)} = \mathcal{A}_2^{\mathrm{dec}} \quad \text{and} \quad G_2^{(2)} = \mathcal{A}_2.$$

In genus 3 the theta divisor of any nonhyperelliptic Jacobian is smooth, while the theta divisor of a hyperelliptic Jacobian has a unique singular point, which is an ordinary double point. Thus,

$$G_0^{(3)} = G_2^{(3)} = \mathcal{A}_3^{\mathrm{dec}}, \quad G_4^{(3)} = \overline{\mathcal{H}}_3, \quad \text{and} \quad G_6^{(3)} = \mathcal{A}_3,$$

where $\mathcal{H}_3$ denotes the locus of hyperelliptic Jacobians, and we note that the boundary of $\mathcal{H}_3$ in $\mathcal{A}_3$ is in fact equal to the decomposable locus. We note that in this case the Gauss loci are equal to the Andreotti–Mayer loci, but already in genus 4 this is not the case.

We now turn to genus 4. Recall that $N_1^{(4)} = \overline{\mathcal{H}}_4$, and the degree of the Gauss map for a Jacobian of a smooth hyperelliptic genus 4 curve is equal to $2^{4-1} = 8$; this is to say $\mathcal{H}_4 \subset G_8^{(4)} \setminus G_6^{(4)}$. We also recall from [Beauville 1977] that $N_0^{(4)} = \mathcal{J}_4 \cup \theta_{\mathrm{null}}$, where $\mathcal{J}_4$ denotes the locus of Jacobians, and θ_{null} denotes the theta-null divisor — the locus of ppavs with a singular 2-torsion point on the theta divisor. Recall also that the degree of the Gauss map for any nonhyperelliptic genus 4 Jacobian is equal to 20. Furthermore, as is well-known, and as also immediately follows from our bound for the discrepancy, an indecomposable 4-dimensional ppav cannot have an isolated singular point z on the theta divisor with $m = \operatorname{mult}_z \Theta > 2$ (since $3 \cdot 2^3 = 24 > 20 = 4! - 4$). Finally, Grushevsky and Salvati Manni [2008] showed that if the theta divisor of a 4-dimensional indecomposable ppav has a double point that is not ordinary, it is the Jacobian of a curve. To summarise (and further using

[Beauville 1977]), if a 4-dimensional indecomposable ppav that is not a Jacobian has a singular theta divisor, the only singularities of it are at two-torsion points (vanishing theta-nulls), and each of those is an ordinary double point.

We can now describe the Gauss loci in genus 4.

Proposition 4.2. (1) *All the Gauss loci $G_{2d}^{(4)}$ are closed.*

(2) *All the Gauss loci $G_{2d}^{(4)}$, for $d = 2, \ldots, 12$ are distinct.*

(3) *The locus $G_4^{(4)}$ is equal to the union of the locus of decomposable ppavs $\mathcal{A}_4^{\mathrm{dec}} = G_0^{(4)}$ (which has two irreducible components $\mathcal{A}_1 \times \mathcal{A}_3$, of dimension 7, and $\mathcal{A}_2 \times \mathcal{A}_2$ of dimension 6), and one point $\{A_V\}$, the fourfold discovered by Varley [1986].*

(4) *The 9-dimensional locus $\mathcal{J}_4$ is an irreducible component of $G_{20}^{(4)}$, which has another 8-dimensional irreducible component.*

(5) *The 7-dimensional locus $\mathcal{H}_4$ is an irreducible component of $G_8^{(4)}$, which has another 2-dimensional irreducible component.*

Proof. Indeed, the above results imply that for a 4-dimensional indecomposable ppav that is not a Jacobian, the only possible singularities of the theta divisor are theta-nulls that are ordinary double points, each of which contributes precisely two to the discrepancy by Proposition 3.5. The degree of the Gauss map is then $24-2k$, where k is the number of vanishing theta constants. Since the vanishing of an even theta constant defines a divisor on a level cover $\mathcal{A}_4(4, 8)$ of $\mathcal{A}_4$, the locus where a given collection of theta constants vanishes is always closed in $\mathcal{A}_4(4, 8)$, and thus its image is closed in $\mathcal{A}_4$. Thus all the Gauss loci are closed within $\mathcal{A}_4 \setminus N_1^{(4)}$. Since $N_1^{(4)} = \overline{\mathcal{H}}_4 \cup \mathcal{A}_4^{\mathrm{dec}} \subset G_8^{(4)}$, it follows that $G_{2d}^{(4)}$ are closed for $d \geq 4$, and to prove (1) it remains to show that $G_4^{(4)}$ and $G_6^{(4)}$ are closed. The statement about $G_4^{(4)}$ follows from (3), which we will prove shortly. To prove that $G_6^{(4)}$ is closed, all we need to show is that a family in $G_6^{(4)}$ can not degenerate to a smooth hyperelliptic Jacobian, which lies in $G_8^{(4)} \setminus G_6^{(4)}$. We will do this once we are done with the rest of the proof.

To show that all Gauss loci are distinct, recall that Varley [1986] constructed an indecomposable principally polarised abelian fourfold A_V, which is not a Jacobian, but has 10 vanishing even theta-nulls (and note that the above discussion reproves that a 4-dimensional indecomposable ppav can have at most $k = 10$ vanishing theta constants). Thus the degree of the Gauss map for A_V is equal to 4, so that $A_V \in G_4^{(4)}$. Debarre [1987] showed that A_V is the unique indecomposable ppav in $\mathcal{A}_4$ such that the singular locus of the theta divisor consists of 10 isolated points; in our language, this implies (3), which also shows that the locus $G_4^{(4)}$ is closed. Thus locally on $\mathcal{A}_4(4, 8)$ the point A_V is defined by the vanishing of the corresponding 10 theta constants. The vanishing locus of each theta constant is a hypersurface in

$\mathcal{A}_4(4,8)$, and since all 10 of them locally intersect in expected codimension 10, any subset also locally intersects in expected codimension. Thus for any $k \le 10$, there is a $(10-k)$-dimensional locus of ppavs near A_V with exactly k vanishing theta constants (and which do not lie on $\mathcal{J}_4$, since A_V is not a Jacobian). Thus locally near A_V there exist ppavs with the degree of the Gauss map equal to $24-2k$, for any $k \le 10$.

Furthermore, by the above discussion we see that there exists an irreducible $(10-k)$-dimensional component of $G^{(4)}_{24-2k}$ containing A_V. Taking $k=2$ and $k=8$ shows the existence of irreducible components of $G^{(4)}_{20} \setminus \overline{\mathcal{J}}_4$ and of $G^{(4)}_{8} \setminus \overline{\mathcal{H}}_4$ of the claimed dimensions. Furthermore, we note that the boundary of $\mathcal{H}_4$ within $\mathcal{A}_4$ is

$$(\overline{\mathcal{H}}_2 \times \overline{\mathcal{H}}_2) \cup (\overline{\mathcal{H}}_1 \times \overline{\mathcal{H}}_3) = (\mathcal{A}_2 \times \mathcal{A}_2) \cup (\mathcal{A}_1 \times \mathcal{H}_3) \subsetneq \mathcal{A}_4^{\mathrm{dec}} = (\mathcal{A}_2 \times \mathcal{A}_2) \cup (\mathcal{A}_1 \times \mathcal{A}_3).$$

Recalling that $\overline{\mathcal{J}}_4$ and $\overline{\mathcal{H}}_4$ are irreducible, and that the Gauss map on these loci generically has degree 20 and 8, respectively, proves parts 4 and 5.

Finally, we return to statement (1). Suppose we have a family A_t of ppavs contained in $G^{(4)}_6$, degenerating to some $A_0 \in \overline{\mathcal{H}}_4$. We want to prove that A_0 cannot be a smooth hyperelliptic Jacobian, which is equivalent to showing that $A_0 \in \mathcal{A}_4^{\mathrm{dec}}$. Indeed, a generic A_t must be a ppav that has at least 9 vanishing theta constants, and thus the same configuration of 9 theta constants must also vanish on A_0. It is known that for any smooth hyperelliptic Jacobian there are exactly 10 vanishing even theta constants, every three of them forming an azygetic triple (this results seems to go back to Riemann originally, we refer to [Mumford 1984] for a detailed discussion). Thus if among the 9 vanishing theta constants on A_t there exists a syzygetic triple, then a syzygetic triple of theta constants vanishes at A_0, and so A_0 cannot be a smooth hyperelliptic Jacobian. On the other hand, if an azygetic 9-tuple of theta constants vanishes on A_t, then A_t is a hyperelliptic Jacobian or a product, by [Igusa 1981, Lemma 6]. But then since $A_t \in G^{(4)}_6$, it follows that $A_t \in \mathcal{A}_4^{\mathrm{dec}}$. $\square$

In genus 5, if the theta divisor is smooth, the degree of the Gauss map is 120. For intermediate Jacobians of cubic threefolds the degree of the Gauss map is 72, for the Jacobian of any nonhyperelliptic curve it is 70, while for the Jacobian of any hyperelliptic curve it is 16. In this case our results show that if all singular points of theta divisor are isolated (i.e., on $N^{(5)}_0 \setminus N^{(5)}_1$), there are at most $(5!-4)/2 = 58$ such singular points — which in particular is much better than the conjectured bound of $2^4(2^5+1) - 3^5 = 285$ for the number of vanishing even theta constants in this case; see [Auffarth et al. 2017] for the recent results on this conjecture.

All irreducible components of the Andreotti–Mayer locus $N^{(5)}_1$ were described explicitly by Debarre [1988, Proposition 8.2] via the Prym construction, using the fact that the Prym map is dominant onto $\mathcal{A}_5$. While it appears possible to compute the degree of the Gauss map for a generic point of every irreducible component of $N_1^{(5)}$, determining the degree of the Gauss map for every point of $N_1^{(5)}$ would require computing the degree of the Gauss map for Pryms of all admissible covers that arise as degenerations, which involves a large number of cases, and which we thus do not undertake. Furthermore, we have $G_{16}^{(5)} \supset \mathcal{H}_5$, and Debarre showed that $N_2^{(5)} \setminus \mathcal{A}_5^{\mathrm{dec}} = \mathcal{H}_5$ [Ciliberto and van der Geer 2000, Section 1], while Ein and Lazarsfeld [1997] showed that $N_3^{(5)} = \mathcal{A}_5^{\mathrm{dec}}$, which is thus equal to $G_4^{(5)}$. We do not know whether various Gauss loci are distinct.

In higher genus, other than the classical case of Jacobians, the degree of the Gauss map is known for a generic Prym: Verra [2001] showed that it is equal to $D(g)+2^{g-3}$, where $D(g)$ is the degree of the variety of all quadrics of rank at most 3 in $\mathbb{P}^{g-1}$. We note that for $g=5$ a generic ppav is a Prym, with a smooth theta divisor, and thus the result of Verra recovers the degree $5! = 120$ of the Gauss map in this case.

Acknowledgments

We thank P. Aluffi, R. Lazarsfeld, R. Salvati Manni, A. Verra and F. Viviani for many stimulating conversations, and R. Auffarth, T. Krämer, S. Schreider and R. Varley for valuable comments on a preliminary version of this paper.

References

[Andreotti 1958] A. Andreotti, "On a theorem of Torelli", *Amer. J. Math.* **80** (1958), 801–828. MR Zbl

[Arbarello et al. 1985] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris, *Geometry of algebraic curves, I*, Grundlehren Math. Wissenschaften **267**, Springer, New York, 1985. MR Zbl

[Auffarth et al. 2017] R. Auffarth, G. P. Pirola, and R. Salvati Manni, "Torsion points on theta divisors", *Proc. Amer. Math. Soc.* **145**:1 (2017), 89–99. MR Zbl

[Beauville 1977] A. Beauville, "Prym varieties and the Schottky problem", *Invent. Math.* **41**:2 (1977), 149–196. MR Zbl

[Beauville 1982] A. Beauville, "Les singularités du diviseur Θ de la Jacobienne intermédiaire de l'hypersurface cubique dans $\mathbf{P}^4$", pp. 190–208 in *Algebraic threefolds* (Varenna, Italy, 1981), edited by A. Conte, Lecture Notes in Math. **947**, Springer, Berlin, 1982. MR Zbl

[Birkenhake and Lange 2004] C. Birkenhake and H. Lange, *Complex abelian varieties*, 2nd ed., Grundlehren Math. Wissenschaften **302**, Springer, Berlin, 2004. MR Zbl

[Ciliberto and van der Geer 2000] C. Ciliberto and G. van der Geer, "The moduli space of abelian varieties and the singularities of the theta divisor", pp. 61–81 in *Surveys in differential geometry*, vol. 7, edited by S.-T. Yau, Int. Press, Somerville, MA, 2000. MR Zbl

[Clemens and Griffiths 1972] C. H. Clemens and P. A. Griffiths, "The intermediate Jacobian of the cubic threefold", *Ann. of Math.* (2) **95** (1972), 281–356. MR

[Debarre 1987] O. Debarre, "Annulation de thêtaconstantes sur les variétés abéliennes de dimension quatre", *C. R. Acad. Sci. Paris Sér. I Math.* **305**:20 (1987), 885–888. MR Zbl

[Debarre 1988] O. Debarre, "Sur les variétés abéliennes dont le diviseur theta est singulier en codimension 3", *Duke Math. J.* **57**:1 (1988), 221–273. MR Zbl

[Ein and Lazarsfeld 1997] L. Ein and R. Lazarsfeld, "Singularities of theta divisors and the birational geometry of irregular varieties", *J. Amer. Math. Soc.* **10**:1 (1997), 243–258. MR Zbl

[Ein et al. 1996] L. Ein, R. Lazarsfeld, and M. Nakamaye, "Zero-estimates, intersection theory, and a theorem of Demailly", pp. 183–207 in *Higher-dimensional complex varieties* (Trento, 1994), edited by M. Andreatta and T. Peternell, de Gruyter, Berlin, 1996. MR Zbl

[Flenner et al. 1999] H. Flenner, L. O'Carroll, and W. Vogel, *Joins and intersections*, Springer, Berlin, 1999. MR Zbl

[Fulton 1984] W. Fulton, *Intersection theory*, Ergebnisse der Mathematik (3) **2**, Springer, Berlin, 1984. MR Zbl

[van Gastel 1991] L. J. van Gastel, "Excess intersections and a correspondence principle", *Invent. Math.* **103**:1 (1991), 197–222. MR Zbl

[Grushevsky and Hulek 2013] S. Grushevsky and K. Hulek, "Geometry of theta divisors: a survey", pp. 361–390 in *A celebration of algebraic geometry* (Cambridge, 2011), edited by B. Hassett et al., Clay Math. Proc. **18**, American Mathematical Society, Providence, RI, 2013. MR Zbl

[Grushevsky and Salvati Manni 2008] S. Grushevsky and R. Salvati Manni, "Jacobians with a vanishing theta-null in genus 4", *Israel J. Math.* **164** (2008), 303–315. MR Zbl

[Igusa 1981] J.-i. Igusa, "On the irreducibility of Schottky's divisor", *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **28**:3 (1981), 531–545. MR Zbl

[Kollár 1995] J. Kollár, *Shafarevich maps and automorphic forms*, Princeton Univ. Press, 1995. MR Zbl

[Krämer 2016] T. Krämer, "Cubic threefolds, Fano surfaces and the monodromy of the Gauss map", *Manuscripta Math.* **149**:3 (2016), 303–314. MR Zbl

[Matsumura 1986] H. Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics **8**, Cambridge Univ. Press, 1986. MR Zbl

[Mumford 1984] D. Mumford, *Tata lectures on theta, II*, Progress in Mathematics **43**, Birkhäuser, Boston, 1984. MR Zbl

[Mustata and Popa 2016] M. Mustata and M. Popa, "Hodge ideals", preprint, 2016. arXiv

[Nakamaye 2000] M. Nakamaye, "Zero estimates on abelian varieties", pp. 285–298 in *Recent progress in intersection theory* (Bologna, 1997), edited by G. Ellingsrud et al., Birkhäuser, Boston, 2000. MR Zbl

[Nowak 1997] K. J. Nowak, "A proof of the criterion for multiplicity one", *Univ. Iagel. Acta Math.* 35 (1997), 247–249. MR Zbl

[Ran 1981] Z. Ran, "On subvarieties of abelian varieties", *Invent. Math.* **62**:3 (1981), 459–479. MR Zbl

[Sernesi 1986] E. Sernesi, *Topics on families of projective schemes*, Queen's Papers in Pure and Applied Mathematics **73**, Queen's University, Kingston, ON, 1986. MR

[Smith and Varley 1996] R. Smith and R. Varley, "Multiplicity g points on theta divisors", *Duke Math. J.* **82**:2 (1996), 319–326. MR Zbl

[Smith and Varley 2012] R. Smith and R. Varley, "Deformations of isolated even double points of corank one", *Proc. Amer. Math. Soc.* **140**:12 (2012), 4085–4096. MR Zbl

[Varley 1986] R. Varley, "Weddle's surfaces, Humbert's curves, and a certain 4-dimensional abelian variety", *Amer. J. Math.* **108**:4 (1986), 931–951. MR Zbl

[Verra 2001] A. Verra, "The degree of the Gauss map for a general Prym theta-divisor", *J. Algebraic Geom.* **10**:2 (2001), 219–246. MR Zbl arXiv

[Vogel 1984] W. Vogel, *Lectures on results on Bezout's theorem*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics **74**, Springer, Berlin, 1984. MR Zbl

[Voisin 2002] C. Voisin, *Théorie de Hodge et géométrie algébrique complexe*, Cours Spécialisés **10**, Société Mathématique de France, Paris, 2002. MR Zbl

[Żołądek 2006] H. Żołądek, *The monodromy group*, Instytut Mat. Polskiej Akad. Nauk. Mono. Mat. (New Series) **67**, Birkhäuser, Basel, 2006. MR Zbl

Communicated by János Kollár
Received 2016-08-17 Revised 2017-01-10 Accepted 2017-02-11

codogni@mat.uniroma3.it *Dipartimento di Matematica e Fisica, Università Roma Tre, Largo San Leonardo Murialdo, I-00146 Rome, Italy*

sam@math.stonybrook.edu *Mathematics Department, Stony Brook University, Stony Brook, NY 11794-3651, United States*

sernesi@mat.uniroma3.it *Dipartimento di Matematica e Fisica, Università Roma Tre, Largo San Leonardo Murialdo, I-00146 Roma, Italy*

ALGEBRA AND NUMBER THEORY 11:4 (2017)
dx.doi.org/10.2140/ant.2017.11.1003

Correction to the article Frobenius and valuation rings

Rankeya Datta and Karen E. Smith

Volume **10**:5 (2016), 1057–1090

Theorem 5.1 is corrected in the paper *Frobenius and valuation rings*.

The characterization of F-finite valuation rings of function fields (Theorem 5.1) is incorrect. The corrected characterization is as follows:

Theorem 0.1. *Let K be a finitely generated extension of an F-finite ground field k. Let V be any nontrivial valuation ring of K/k. Then V is F-finite if and only if V is divisorial.*

A proof of this corrected theorem follows a complete accounting of affected statements in the paper. Notation and theorem and page numbers are as in [Datta and Smith 2016].

1. List of affected statements

The source of the error is Proposition 2.2.1 (page 1061), which is misquoted from the original source [Bourbaki 1998]. Specifically, the last sentence in the statement of Proposition 2.2.1 should read: *Furthermore, equality holds **if** the integral closure of R_ν in L is a finitely generated R_ν-module,* not ***if and only if***. The statement *is* correct, however, under the additional assumption that the valuation ring R_ν is *Noetherian* [Bourbaki 1998, VI, §8.5, Remark (1)].

This error has consequences in the following statements from the paper.

THEOREM 4.3.1 should read: *Let V be a valuation ring of an F-finite field K of prime characteristic p. If V is F-finite, then*

$$[\Gamma : p\Gamma][\kappa : \kappa^p] = [K : K^p],$$

where Γ is the value group and κ is the residue field of V.

Datta was partially supported by the Juha Heinonen Memorial Graduate Fellowship at the University of Michigan. Smith was partially supported by NSF grant DMS-1501625.
MSC2010: primary 13A35; secondary 13F30, 14B05.
Keywords: F-finite valuation rings, divisorial valuations in prime characteristic.

The paper had incorrectly stated the converse, which *does* hold if the valuation ring is Noetherian [Bourbaki 1998, VI, §8.5, Remark (1)].

COROLLARY 4.3.2 should read: *With hypotheses as in Theorem 4.3.1, if* $[K : K^p] = [\kappa : \kappa^p]$, *then* V *is F-finite.* See Section 2 below for the proof. The original statement holds as stated if V is Noetherian.

EXAMPLES 4.4.1 and 4.4.2 (page 1069): The computations here are correct but the conclusions are not. Rather, Theorem 0.1 ensures these valuation rings are *not* F-finite.

EXAMPLE 4.5.1 (page 1070) is correct as stated, as it follows from the correct implication of Theorem 4.3.1.

PROPOSITION 4.6.1 (page 1071): This statement is correct. However, the proof of (iii) is not, because it cited the incorrect implication of Proposition 2.2.1. The corrected proof is in Section 3 below.

THEOREM 5.1 (page 1072): The equivalence of (ii) and (iii) is correct, and both imply (i). Under the additional assumption that the valuation is discrete, also (i) implies (ii) and (iii). The proof of Theorem 5.1 in the paper proves:

Theorem 1.1. *Let* K *be a finitely generated field extension of an F-finite ground field* k *of characteristic* p. *The following are equivalent for a valuation* v *on* K/k:

(i) *The valuation* v *is Abhyankar.*

(ii) $[\Gamma : p\Gamma][\kappa : \kappa^p] = [K : K^p]$, *where* Γ *is the value group, and* κ *is the residue field of* v.

COROLLARY 5.2 (page 1072) is correct as stated; it is a consequence of Theorem 0.1.

COROLLARY 6.6.3 (page 1086) is correct as stated; the proof should invoke Theorem 0.1 instead of Theorem 5.1.

The remaining statements in Section 5 (Remark 5.3, Proposition 5.4, Lemma 5.5, Proposition 5.6, Lemma 5.7, Lemma 5.8) and Section 6 are correct, because they do not rely on Proposition 2.2.1 or its consequences.

2. Proof of Theorem 0.1

Lemma 2.1. *Let* $(V, \mathfrak{m})$ *be a valuation ring of prime characteristic* p. *If* $\mathfrak{m}$ *is not principal, then*

$$\mathfrak{m} = \mathfrak{m}^{[p]}.$$

Proof. Take any $x \in \mathfrak{m}$. It suffices to show that there exists $y \in \mathfrak{m}$ such that $y^p \mid x$. Note that since $\mathfrak{m}$ is not principal, the value group Γ of the corresponding valuation v *does not have a smallest element* > 0. This means we can choose $\beta_0 \in \Gamma$ such that $0 < \beta_0 < v(x)$, and then also choose $\beta_1 \in \Gamma$ such that

$$0 < \beta_1 < \min\{\beta_0, v(x) - \beta_0\}.$$

Let $y_1 \in \mathfrak{m}$ be such that $v(y_1) = \beta_1$. Then

$$v(y_1^2) = 2\beta_1 < \beta_0 + (v(x) - \beta_0) = v(x).$$

Thus, $y_1^2 \mid x$. Repeating this argument inductively, we can find $y_n \in \mathfrak{m}$ such that $y_n^2 \mid y_{n-1}$ for all $n \in \mathbb{N}$. For $n > \log_2 p$, we have that $y_n^{2^n}$, and hence y_n^p, divides x. □

Lemma 2.2. *Let $(V, \mathfrak{m}, \kappa)$ be a valuation ring of characteristic p. Then the dimension of $V/\mathfrak{m}^{[p]}$ over κ^p is*

(a) $[\kappa : \kappa^p]$ *if $\mathfrak{m}$ is not finitely generated.*

(b) $p[\kappa : \kappa^p]$ *if $\mathfrak{m}$ is finitely generated.*

Proof. Consider the short exact sequence of κ^p-vector spaces

$$0 \to \mathfrak{m}/\mathfrak{m}^{[p]} \to V/\mathfrak{m}^{[p]} \to \kappa \to 0. \tag{1}$$

If $\mathfrak{m}$ is not finitely generated, then Lemma 2.1 implies that $\mathfrak{m}/\mathfrak{m}^{[p]} = 0$, and (a) follows. Otherwise, $\mathfrak{m}$ is principal, so $\mathfrak{m}^{[p]} = \mathfrak{m}^p$ and we have a filtration

$$\mathfrak{m} \supsetneq \mathfrak{m}^2 \supsetneq \cdots \supsetneq \mathfrak{m}^{p-1} \supsetneq \mathfrak{m}^{[p]} = \mathfrak{m}^p.$$

Since $\mathfrak{m}^i/\mathfrak{m}^{i+1} \cong \kappa$, we see that $\dim_{\kappa^p}(\mathfrak{m}/\mathfrak{m}^{[p]}) = (p-1)[\kappa : \kappa^p]$. From the short exact sequence (1), $\dim_{\kappa^p}(V/\mathfrak{m}^{[p]}) = p[\kappa : \kappa^p]$, proving (b). □

Proof of Theorem 0.1. If V is divisorial, it is a localization of a finitely generated algebra over the F-finite ground field k, hence it is F-finite.

For the converse, let $\mathfrak{m}$ be the maximal ideal and κ the residue field of V. Since V is F-finite, by (the corrected) Theorem 4.3.1,

$$[\Gamma : p\Gamma][\kappa : \kappa^p] = [K : K^p], \tag{2}$$

where Γ is the value group of V. Thus, the valuation is Abhyankar by Theorem 1.1 above. To show it is divisorial, we need to show it is discrete.

Because the value group of an Abhyankar valuation is finitely generated, $\Gamma \cong \mathbb{Z}^{\oplus s}$ for some integer $s \geq 1$. Hence it suffices to show that $s = 1$.

Since V is F-finite, we know V is free over V^p of rank $[K : K^p]$. Tensoring with the residue field κ^p of V^p, we have that $V/\mathfrak{m}^{[p]}$ is also a free κ^p-module of rank $[K : K^p]$. Thus, from (2) we have

$$\begin{aligned} \dim_{\kappa^p} V/\mathfrak{m}^{[p]} &= [K : K^p] = [\Gamma : p\Gamma][\kappa : \kappa^p] \\ &= \left|\mathbb{Z}^{\oplus s}/p\mathbb{Z}^{\oplus s}\right|[\kappa : \kappa^p] = p^s[\kappa : \kappa^p]. \end{aligned} \tag{3}$$

But now, since $p^s \neq 1$, Lemma 2.2 forces $s = 1$, and the proof is complete. □

Remark 2.3. Theorem 0.1 does not hold without *some* assumption on the field K. For instance, if K is perfect, Frobenius is an isomorphism (hence a finite map) for *any* valuation ring of K. The proof of Theorem 0.1 does show that an F-finite valuation ring with finitely generated value group must be Noetherian (hence discrete), without any restriction on its fraction field. Furthermore, the proof also shows that a valuation ring cannot be F-finite if its value group Γ satisfies $[\Gamma : p\Gamma] > p$.

Proof of revised COROLLARY 4.3.2. In general, $[\Gamma : p\Gamma][\kappa : \kappa^p] \leq [K : K^p]$ by [Bourbaki 1998, VI, §8.1, Lemma 2]. Hence our hypothesis forces $[\Gamma : p\Gamma] = 1$. Thus $\Gamma = p\Gamma$, so that Γ cannot have a smallest positive element, which means that the maximal ideal $\mathfrak{m}$ of V is not finitely generated. Lemma 2.2(a) now ensures $\dim_{\kappa^p}(V/\mathfrak{m}^{[p]}) = [\kappa : \kappa^p] = [K : K^p]$. Then V is F-finite by [Bourbaki 1998, VI, §8.5, Theorem 2(c)]. □

3. Proof of Proposition 4.6.1(iii)

We recall PROPOSITION 4.6.1(iii): *Let $K \hookrightarrow L$ be a finite extension of F-finite fields of characteristic p. Let w be a valuation on L and v its restriction to K. Then the valuation ring of v is F-finite if and only if the valuation ring of w is F-finite.*

Lemma 3.1. *With notation as in Proposition 4.6.1(iii), the maximal ideal of the valuation ring of v is finitely generated if and only if the maximal ideal of the valuation ring of w is finitely generated.*

Proof. For ideals in a valuation ring, finite generation is the same as being principal. Principality of the maximal ideal is equivalent to the value group having a smallest element > 0. Thus, it suffices to show that the value group Γ_v of v has this property if and only if Γ_w does.

Assume Γ_w has a smallest element $g > 0$. We claim that for each $t \in \mathbb{N}$, the only positive elements of Γ_w less than tg are $g, 2g, \ldots, (t-1)g$. Indeed, suppose $0 < h < tg$. Since g is smallest, $g \leq h < tg$, whence $0 \leq h - g < (t-1)g$. So by induction, $h - g = ig$ for some $i \in \{0, 1, \ldots, t-2\}$, and hence h is among $g, 2g, \ldots, (t-1)g$.

Now, because $[\Gamma_w : \Gamma_v] \leq [L : K] < \infty$ by [Bourbaki 1998, VI, §8.1, Lemma 2], every element of Γ_w/Γ_v is torsion. Let n be the smallest positive integer such that $ng \in \Gamma_v$. We claim that ng is the smallest positive element of Γ_v. Indeed, the only positive elements smaller than ng in Γ_w are $g, 2g, \ldots, (n-1)g$, and none of these are in Γ_v by our choice of n.

Conversely, if Γ_v has a smallest element $h > 0$, then the set

$$S := \{g \in \Gamma_w : 0 < g < h\}$$

is finite because for distinct g_1, g_2 in this set, their classes in Γ_w/Γ_v are also distinct, while Γ_w/Γ_v is a finite group. Then the smallest positive element of Γ_w is the smallest element of S, or h if S is empty. □

Proof of Proposition 4.6.1(iii). A *necessary and sufficient* condition for the F-finiteness of a valuation ring $(V, \mathfrak{m}, \kappa)$ with F-finite fraction field K is, by [Bourbaki 1998, VI, §8.5, Theorem 2(c)], that

$$\dim_{\kappa^p}(V/\mathfrak{m}^{[p]}) = [K : K^p]. \tag{4}$$

Lemma 2.2 gives a formula for $\dim_{\kappa^p}(V/\mathfrak{m}^{[p]})$ in terms of $[\kappa : \kappa^p]$ that depends on whether the maximal ideal is finitely generated, which is the same for v and w by Lemma 3.1. Proposition 4.6.1(i) and (ii) tell us that both $[K : K^p] = [L : L^p]$ and $[\kappa_v : \kappa_v^p] = [\kappa_w : \kappa_w^p]$. Thus Lemma 2.2 and (4) guarantee that the valuation ring of v is F-finite if and only if the valuation ring of w is F-finite. □

References

[Bourbaki 1998] N. Bourbaki, *Commutative algebra, Chapters 1–7*, Springer, Berlin, 1998. MR Zbl

[Datta and Smith 2016] R. Datta and K. E. Smith, "Frobenius and valuation rings", *Algebra Number Theory* **10**:5 (2016), 1057–1090. MR Zbl

Communicated by János Kollár
Received 2016-12-03 Accepted 2017-02-03

rankeya@umich.edu *Department of Mathematics, University of Michigan, 2074 East Hall, 530 Church Street, Ann Arbor, MI 48109-1043, United States*

kesmith@umich.edu *Department of Mathematics, University of Michigan, 2074 East Hall, 530 Church Street, Ann Arbor, MI 48109-1043, United States*

Guidelines for Authors

Authors may submit manuscripts in PDF format on-line at the Submission page at the ANT website.

Originality. Submission of a manuscript acknowledges that the manuscript is original and and is not, in whole or in part, published or under consideration for publication elsewhere. It is understood also that the manuscript will not be submitted elsewhere while under consideration for publication in this journal.

Language. Articles in *ANT* are usually in English, but articles written in other languages are welcome.

Length There is no a priori limit on the length of an *ANT* article, but *ANT* considers long articles only if the significance-to-length ratio is appropriate. Very long manuscripts might be more suitable elsewhere as a memoir instead of a journal article.

Required items. A brief abstract of about 150 words or less must be included. It should be self-contained and not make any reference to the bibliography. If the article is not in English, two versions of the abstract must be included, one in the language of the article and one in English. Also required are keywords and subject classifications for the article, and, for each author, postal address, affiliation (if appropriate), and email address.

Format. Authors are encouraged to use LaTeX but submissions in other varieties of TeX, and exceptionally in other formats, are acceptable. Initial uploads should be in PDF format; after the refereeing process we will ask you to submit all source material.

References. Bibliographical references should be complete, including article titles and page ranges. All references in the bibliography should be cited in the text. The use of BibTeX is preferred but not required. Tags will be converted to the house format, however, for submission you may use the format of your choice. Links will be provided to all literature with known web locations and authors are encouraged to provide their own links in addition to those supplied in the editorial process.

Figures. Figures must be of publication quality. After acceptance, you will need to submit the original source files in vector graphics format for all diagrams in your manuscript: vector EPS or vector PDF files are the most useful.

Most drawing and graphing packages (Mathematica, Adobe Illustrator, Corel Draw, MATLAB, etc.) allow the user to save files in one of these formats. Make sure that what you are saving is vector graphics and not a bitmap. If you need help, please write to graphics@msp.org with details about how your graphics were generated.

White space. Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal's preferred fonts and layout.

Proofs. Page proofs will be made available to authors (or to the designated corresponding author) at a Web site in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.

Algebra & Number Theory

Volume 11 No. 4 2017