ERRATA TO THE PAPER "MASS FORMULAS FOR LOCAL GALOIS REPRESENTATIONS AND QUOTIENT SINGULARITIES II: DUALITIES AND RESOLUTION OF SINGULARITIES"

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The paper "Mass formulas for local Galois representations and quotient singularities II: dualities and resolution of singularities" contains the following errors:

(1) The definition of the function $\mathbf{v}$ on page 829 should be multiplied with inertia degrees of relevant field extensions.
(2) In the example on page 833, at lines 14 and 15, the strong duality holds for $n = 2$ instead of $n = 1$ and the claimed weak duality does not hold.

The authors are grateful to Nigel Byott for letting them know the latter. We explain these errors in more detail below. The main results of the paper are not affected.

1. Functions $\mathbf{v}$ and $\mathbf{w}$

The definitions of functions $\mathbf{v}$ and $\mathbf{w}$ in [WY17, Section 4] as well as in [WY15] and [Yas17] are erroneous. With the notation of [WY17], for $\rho \in S_{K,\Gamma}$, let $\operatorname{Spec} M \to \operatorname{Spec} K$ be the corresponding étale Γ-torsor and $\operatorname{Spec} L$ a connected component of $\operatorname{Spec} M$. Let f_L be the inertia degree of L/K. Now the correct definition of $\mathbf{v}$ is

$$\mathbf{v}_\tau(\rho) := \frac{f_L}{\sharp\Gamma} \cdot \operatorname{length}\left(\frac{\mathcal{O}_M^{\oplus d}}{\mathcal{O}_M \cdot \Xi}\right).$$

Namely we modify the definition by multiplying with f_L. The correct definition of $\mathbf{w}_\tau$ is then $\mathbf{w}_\tau(\rho) := \dim u^{-1}(o) - \mathbf{v}_\tau(\rho)$ as it was in [WY17], but now with $\mathbf{v}$ of the new definition.

We need this change, mainly because we would like to have the equality $\mathbf{v}_\tau(\rho) = \mathbf{v}_{\tau_{\mathrm{nr}}}(\rho)$ for the induced representation $\tau_{\mathrm{nr}}\colon \Gamma \xrightarrow{\tau} \mathrm{GL}_d(\mathcal{O}_K) \hookrightarrow \mathrm{GL}_d(\mathcal{O}_{K_{\mathrm{nr}}})$, K_{nr} being the maximal unramified extension of K; this property was called "geometric" in [WY15]. Passing from K to K_{nr}, the length of $\mathcal{O}_M^{\oplus d}/\mathcal{O}_M \cdot \Xi$ increases by a factor of f_L. Indeed, if we express the transition to the maximal unramified extensionsince by the subscript nr, then we have $\mathcal{O}_{M_{\mathrm{nr}}}^{\oplus d}/\mathcal{O}_{M_{\mathrm{nr}}} \cdot \Xi_{\mathrm{nr}} = \left(\mathcal{O}_M^{\oplus d}/\mathcal{O}_M \cdot \Xi\right) \otimes_{\mathcal{O}_M} \mathcal{O}_{M_{\mathrm{nr}}}$. Tensoring with $\mathcal{O}_{M_{\mathrm{nr}}}$ raises the length of an $\mathcal{O}_M$-module by a factor of f_L. In [WY15, Lemma 3.4], we stated that $\mathbf{v}$ is geometric, which is however true only for the new definition. In its proof, we essentially proved the last equality of modules, however overlooked this raise of length. Also [WY15, Theorem 4.8 and Corollary 4.9], which relate $\mathbf{v}$ and $\mathbf{w}$ with the Artin and Swan conductors and was referenced in [WY17, Lemma 4.3], hold only with the corrected definitions of these functions. The last two equalities in the proof of [WY15, Theorem 4.8] should be corrected accordingly.

Fortunately, when Γ is a cylic group of prime order, the old and new definitions give the same functions $\mathbf{v}$ and $\mathbf{w}$. This is because in this case, either $f_L = 1$

or the length of the relevant module is zero. Thus this change does not affect computations of masses in [WY15, WY17] in this case, although ones in [WY17] contain an error concerning dualities, which will be explained in the next section. The other computations in these two papers as well as [Yas17] concern the case where Γ is the symmetric group S_n and the case where Γ is the wreath product $(\mathbb{Z}/2\mathbb{Z}) \wr S_n$. Masses computd there are valid for the new definitions of $\mathbf{v}$ and $\mathbf{w}$, since we use the mentioned relation with the Artin and Swan conductors.

2. Dualities

In what follows, a local field K will range over all unramified exntensions of a fixed non-archimedian local field and we regard computed masses as functions in the degree (denoted by r in [WY17]) of the extension.

Nigel Byott informed the authors that the claimed weak duality in the example "Quadratic extensions: characteristic zero" from [WY17, Section 5] does not hold. Before explaining it, we should note a typo there; $M(K,\Gamma,\mathbf{w}_{\sigma_n})$ at the 11th line, page 833 should be $M(K,\Gamma,-\mathbf{w}_{\sigma_n})$. Having said that, we computed there

$$\begin{aligned} M(K,\Gamma,\mathbf{v}_{\sigma_n}) &= 1+q^{-n+1}-q^{-n}+q^{-3n/2+1},\\ M(K,\Gamma,-\mathbf{w}_{\sigma_n}) &= q+q^{-n/2+1}. \end{aligned}$$

For $n=2$, since $M(K,\Gamma,\mathbf{v}_{\sigma_n}) = 1+q^{-1}$ and $M(K,\Gamma,-\mathbf{w}_{\sigma_n}) = q+1$, the strong duality holds. Recall that σ_n is the representation $\Gamma \to \mathrm{GL}_{2n}(\mathcal{O}_K)$ of $\Gamma = C_2$, the cyclic group of order 2, defined to be the direct sum of n copies of the regular representation. Since the degree of the representation σ_n is 2n, the weak duality [WY17, (4-2)] in this situation reads

$$\begin{aligned} &(1+q^{-n+1}-q^{-n}+q^{-3n/2+1})q^{2n}-(q+q^{-n/2+1})\\ &\qquad = (q^{-1}+q^{n/2-1})q^{2n}-(1+q^{n-1}-q^{n}+q^{3n/2-1}), \end{aligned}$$

which does not hold unless $n=2$. In particular, the answers to the two questions in [WY17, Question 5.2] are negative. Now there is no known example of representation $\tau\colon \Gamma \to \mathrm{GL}_d(\mathcal{O}_K)$ such that the strong duality fails but the weak duality holds.

The above error occured because the authors have overlooked the fact that whether the weak duality holds or not is not preserved by taking the direct sum with a trivial representation, unlike the strong duality. Actually we can show:

Proposition 2.1. *Consider representations $\tau_i\colon \Gamma \to \mathrm{GL}_{d_i}(\mathcal{O}_K)$, $i=0,1,2$, with τ_0 the trivial representation of degree $d_0>0$. Suppose that τ_2 is isomorphic to $\tau_1 \oplus \tau_0$ and that the strong duality fails for one (hence both) of τ_1 and τ_2. Then the weak duality fails for either τ_1 or τ_2 .*

Proof. On the contrary, suppose that the weak duality holds for both τ_1 and τ_2. Note that the masses $M(K,\Gamma,\mathbf{v})$ and $M(K,\Gamma,-\mathbf{w})$ do not change by adding a trivial representation. From the weak duality for the both cases, we get

$$M(K,\Gamma,\mathbf{v})(q^{d_1}-q^{d_2}) = \mathbb{D}(M(K,\Gamma,-\mathbf{w}))(q^{d_1}-q^{d_2}),$$

where d_i is the degree of τ_i. This implies the strong duality, a contradiction. □

Corollary 2.2. *If the strong duality fails for a representation* $\tau\colon \Gamma \to \mathrm{GL}_d(\mathcal{O}_K)$*, then the associated quotient scheme* $\mathbb{A}^d_{\mathcal{O}_K}/\Gamma$ *does not admit any ESWL resolution (equivariant simultaneous weak log resolution).*

Proof. If the weak duality fails, then $(\mathbb{A}^d_{\mathcal{O}_K}/\Gamma) \setminus Z$ with $Z \cong \operatorname{Spec} \mathcal{O}_K$ being the zero section has no ESWL resolution [WY17, p. 831]. Therefore $X := \mathbb{A}^d_{\mathcal{O}_K}/\Gamma$ has no ESWL resolution either. Let τ_i be a representation which is the direct sum of τ and a trivial representation of degree $i \geq 0$. From the above proposition, for every integer $i \geq 0$ with at most one exception, the weak duality fails for τ_i and the corresponding variety $X_i := \mathbb{A}^{d+i}_{\mathcal{O}_K}/\Gamma$ has no ESWL resolution. If X had an ESWL resolution $Y \to X$, then it would induce the ESWL resolution $Y \times_{\mathcal{O}_K} \mathbb{A}^i_{\mathcal{O}_K} \to X \times_{\mathcal{O}_K} \mathbb{A}^i_{\mathcal{O}_K} = X_i$, a contradiction. □

References

[WY15] Melanie Matchett Wood and Takehiko Yasuda. Mass formulas for local Galois representations and quotient singularities. I: a comparison of counting functions. *Int. Math. Res. Not. IMRN*, (23):12590–12619, 2015.

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