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The motivic Donaldson–Thomas invariants of (-2)-curves

Ben Davison and Sven Meinhardt

We calculate the motivic Donaldson–Thomas invariants for (-2)-curves arising from 3-fold flopping contractions in the minimal model program. We translate this geometric situation into the machinery developed by Kontsevich and Soibelman, and using the results and framework developed earlier by the authors we describe the monodromy on these invariants. In particular, in contrast to all existing known Donaldson–Thomas invariants for small resolutions of Gorenstein singularities these monodromy actions are nontrivial.

1. Introduction

Motivic Donaldson–Thomas invariants were introduced in [Kontsevich and Soibelman 2008], as a generalisation of the classical theory of Donaldson–Thomas invariants initiated in [Thomas 2000]. At the same time Joyce [2006a; 2006b; 2007a; 2007b; 2007c; 2007d; 2008] and Joyce and Song [2012] rigorously extended classical Donaldson–Thomas theory to take care of the technicalities involved in dealing with strictly semistable coherent sheaves on Calabi–Yau 3-folds, and in this framework formulated a deep integrality conjecture regarding the resulting Donaldson–Thomas invariants. Assuming the more ambitious framework of [Kontsevich and Soibelman 2008], integrality properties of generalised Donaldson–Thomas invariants are conjecturally obtained by taking the Euler characteristic of motivic Donaldson–Thomas invariants, after multiplication by the motive $\mathbb{C}^*$; such statements are supposed to be a shadow of the fact that these invariants, which are a

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priori only stack valued, are in fact variety valued, so that taking Euler characteristic is legitimate, and produces integers.

If $Y \to X$ is a small resolution of a toric Gorenstein singularity, the calculation of the motivic Donaldson–Thomas invariants of Y has by now received a fairly comprehensive treatment; see [Behrend et al. 2013; Morrison et al. 2012; Morrison and Nagao 2015]. Let $\mathrm{K}^{\hat{\mu}}(\mathrm{Var} / \mathrm{Spec}(\mathbb{C}))$ be the ring of $\hat{\mu}$-equivariant varieties, then there is a ring homomorphism

$$\mathrm{K}^{\hat{\mu}}(\mathrm{Var} / \mathrm{Spec}(\mathbb{C}))[\mathbb{L}^{1/2}] \to \mathbb{Z}[q^{1/2}],$$

obtained by first taking the Hodge spectrum, a homomorphism to the ring of polynomials in fractional powers of two variables u and v, and then specialising $u = v = q^{1/2}$. Furthermore, this is a retraction of rings, since there is a right inverse taking $q^{1/2}$ to $-\mathbb{L}^{1/2}$. The Donaldson–Thomas invariants that arise in the study of the above toric resolutions $Y \to X$ all lie in the obviously very well-understood subring that is the image of this retract.

By contrast, the ring $\mathrm{K}^{\hat{\mu}}(\mathrm{Var} / \mathrm{Spec}(\mathbb{C}))[\mathbb{L}^{1/2}]$, as a whole, has a rich ring structure, with the product given by Looijenga's "exotic" convolution product (see [Looijenga 2002; Guibert et al. 2006; Kontsevich and Soibelman 2008]), and a pre-λ-ring structure utilised in [Davison and Meinhardt 2015] to express the motivic DT invariants of the one loop quiver with potential — this was the first case to really make use of this extra structure.

The present paper represents perhaps the first case where "natural" Donaldson–Thomas invariants living in the interesting part of the ring $\mathrm{K}^{\hat{\mu}}(\mathrm{Var} / \mathrm{Spec}(\mathbb{C}))[\mathbb{L}^{1/2}]$ are discussed. Of course, the question of naturalness here is subjective — we are appealing to the sensibilities of algebraic geometers, in that we consider an example that is manifestly a part of 3-dimensional geometry, as opposed to the case of the one loop quiver with potential, which in the homogeneous case gives rise to the algebra $\mathbb{C}[x] / (x^d)$, which looks rather more like zero-dimensional geometry. More specifically, we consider the motivic Donaldson–Thomas invariants of (-2)-curves, which are, for us, resolutions $Y_d \to X_d$ of singularities as defined in (2). In birational geometry and physics, these curves have a very long and rich history; see [Reid 1983; Laufer 1981; Kollár 1989; Katz and Morrison 1992] for example.

Our paper also seems to represent the first serious attempt to calculate Donaldson–Thomas invariants while keeping as true as possible to the framework of [Kontsevich and Soibelman 2008]. A side effect of this approach is that some discussion of orientation data is necessitated. It is hoped that seeing this aspect of the story in action will help to demystify it a little. For the sake of those who would like to swap the (ever decreasingly) conjectural framework of [loc. cit.] for the single very reasonable-looking conjecture of [Davison and Meinhardt 2015], we prove a slight variant of our main result at the end of the paper, avoiding all mention of orientation data, cyclic 3-Calabi–Yau categories, and minimal potentials.

In both cases we work with an algebraic model of the derived category of compactly supported coherent sheaves on Y_d, provided by considering modules over an algebra A_{Q_{-2},W_d}, which is the free path algebra of the quiver in Figure 1, quotiented by some relations determined by the noncommutative derivatives of a potential W_d. Our main result is Theorem 5.4, which states that

$$\begin{aligned}&\Phi_{Q_{-2},W_d}\big([\mathcal{X}^{\mathrm{nilp}}_{Q_{-2},W_d}]\big)\\ &=\mathrm{Sym}\bigg(\sum_{n\geq 0}\frac{\mathbb{L}^{-1/2}(1-[\mu_{d+1}])}{\mathbb{L}^{1/2}-\mathbb{L}^{-1/2}}\big(\hat{e}_{(n,n+1)}+\hat{e}_{(n+1,n)}\big)+\sum_{n\geq 1}\frac{\mathbb{L}^{-1/2}+\mathbb{L}^{-3/2}}{\mathbb{L}^{1/2}-\mathbb{L}^{-1/2}}\hat{e}_{(n,n)}\bigg), \quad (1)\end{aligned}$$

where the quantity on the left hand side is by definition the motivic generating series for nilpotent modules over A_{Q_{-2},W_d}, which on the geometric side of Van den Bergh's equivalence corresponds to counting coherent sheaves on the exceptional locus of $Y_d \to X_d$. In more detail, the variables $\hat{e}_{(n,m)}$ keep track of the Chern classes of the sheaves we are counting, under the transformation

$$(n,m)\mapsto (n-m)[C_d]+m[\mathrm{pt}],$$

where C_d is the exceptional curve of the resolution $Y_d \to X_d$. By the definition of motivic Donaldson–Thomas invariants Ω^{nilp}, (1) implies that they are given by

$$\Omega^{\mathrm{nilp}}(\boldsymbol{n})=\begin{cases}\mathbb{L}^{-1/2}(1-[\mu_{d+1}]) & \text{if } \boldsymbol{n}=(n,n+1) \text{ or } \boldsymbol{n}=(n+1,n),\\ \mathbb{P}^1\cdot\mathbb{L}^{-3/2} & \text{if } \boldsymbol{n}=(n,n).\end{cases}$$

Here μ_{d+1} is considered as a μ_{d+1}-equivariant variety in the natural way, and so we have indeed produced motivic Donaldson–Thomas invariants with nontrivial monodromy, arising "in nature," e.g., string theory, and confirmed integrality, all the way up to the motivic level, for the Donaldson–Thomas invariants of (−2)-curves.

The structure of the paper is as follows. In Section 2 we collect facts regarding the algebraic geometry and noncommutative algebraic geometry of (−2)-curves, in particular introducing the explicit noncommutative algebra A_{Q_{-2},W_d} whose noncommutative Donaldson–Thomas theory we subsequently study. This version of Donaldson–Thomas theory is motivic; in Section 3 we explain what the word "motivic" means, by introducing all the relevant technicalities on motivic vanishing cycles, motivic Hall algebras, and pre-λ-ring structures on "naive" Grothendieck rings of motives. These are the rings in which motivic DT invariants live. In Section 4 we explain how these invariants are defined; we introduce requisite definitions and facts regarding 3-Calabi–Yau categories and orientation data. Orientation data is a concept introduced in [Kontsevich and Soibelman 2008], and is an extra structure that one must put on a 3-Calabi–Yau category in order to be able to define motivic DT invariants for that category. Furthermore, the motivic DT invariants will in general depend on the choice that we make. We recall how to control this choice with Proposition 4.8, which states that for the natural choice of orientation data provided by the presentation of a Jacobi algebra as the algebra arising from a quiver with potential, the integration map agrees with an analogue of the integration map considered by Behrend, Bryan and Szendrői [Behrend et al. 2013]. Finally, in Section 5 we present our results. To start with, we work within the framework of motivic Donaldson–Thomas theory established by Kontsevich and Soibelman [2008] to prove Theorem 5.4, which is (1), and concerns the Donaldson–Thomas theory of sheaves on Y_d supported on the exceptional locus — in particular we calculate the contribution of the exceptional curve itself. Secondly, we present a calculation of the motivic DT invariants of the category of compactly supported sheaves on the whole of Y_d, working with the somewhat more down-to-earth integration map of Behrend, Bryan and Szendrői, and a conjectural identity regarding motivic vanishing cycles.

2. The geometry of (-2)-curves

We study the motivic Donaldson–Thomas invariants of local (-2)-curves, which are defined in the following way, following [Reid 1983, Section 5]. We assume that $f : Y \to X$ is a resolution of a Gorenstein complex 3-fold singularity with exceptional curve $C \cong \mathbb{P}^1$, satisfying the conditions that $f^*\omega_X \cong \omega_Y,\ \omega_Y \cdot C = 0$, and

$$N_{C|Y} \cong \mathcal{O}_C \oplus \mathcal{O}_C(-2) \quad \text{or} \quad N_{C|Y} \cong \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1).$$

Then (see [Reid 1983, (5.13)] and the surrounding discussion) we may assume that X is one of the singularities

$$X_d = \operatorname{Spec}\big(\mathbb{C}[x, y, z, w] \,/\, (x^2 + y^2 + (z + w^d)(z - w^d))\big) \tag{2}$$

for $d \geq 1$, and Y is given by one of the two resolutions provided by blowing up along $0 = x = z \pm w^d$. We denote by Y_d the blowup along $0 = x = z + w^d$, and by Y_d^+ the blowup along $0 = x = z - w^d$. We will refer to the exceptional rational curve in Y_d always as C_d, to make it clear which resolution of singularities it belongs to. The birational morphism $Y_d \dashrightarrow Y_d^+$ is the flop of the curve C_d, and there is an equivalence of categories

$$\mathrm{D^b}(\mathrm{Coh}(Y_d)) \to \mathrm{D^b}(\mathrm{Coh}(Y_d^+)) \tag{3}$$

with Fourier–Mukai kernel $\mathcal{O}_{Y_d \times_{X_d} Y_d^+}$. This is an example of a generalised spherical twist; see [Toda 2007]. This equivalence is not given by an equivalence of the hearts of these two categories (even though they are in fact equivalent, as there is an obvious isomorphism of schemes $Y_d \to Y_d^+$). As in [Reid 1983, (5.3)] one defines the width[1] of C_d to be the length of the component of the moduli space of coherent sheaves on Y_d containing $\mathcal{O}_{C_d}$. One can show from the explicit description of X_d and Y_d that the width of $C_d \subset Y_d$ is d.

For the purposes of this paper we will be interested in a derived equivalence that is different to that of (3). That is, we will be interested in a derived equivalence between the category of coherent sheaves on Y_d and the category of finitely generated right modules $\texttt{Mod-}A_{Q_{-2},W_d}$ for a noncommutative algebra A_{Q_{-2},W_d}. The approach to defining and studying Donaldson–Thomas invariants of categories of coherent sheaves is as initiated by Szendrői [2008], where the case of the "noncommutative conifold" is considered, and indeed we will recover (motivic) Donaldson–Thomas invariants for the noncommutative conifold, as it is a (−2)-curve of width 1.[2]

The existence of the algebra A_{Q_{-2},W_d} satisfying

$$\mathrm{D^b}(\texttt{Mod-}A_{Q_{-2},W_d}) \xrightarrow{\sim} \mathrm{D^b}(\mathrm{Coh}(Y_d)) \tag{4}$$

is provided by the results of Van den Bergh [2004, Theorem 5.1]. It will help to have an explicit description of Y_d. It is covered by two coordinate patches

$$U_1 = \mathrm{Spec}(\mathbb{C}[x, y_1, y_2]) \quad \text{and} \quad U_2 = \mathrm{Spec}(\mathbb{C}[w, z_1, z_2]),$$

which are glued along

$$\begin{cases} x = w^{-1} \\ z_1 = x^2 y_1 + x y_2^d \\ z_2 = y_2. \end{cases}$$

In the case of the conifold, after the change of coordinates

$$\begin{cases} z_1' = w z_1 - z_2, \\ z_2' = -(1+w) z_1 + z_2, \\ y_1' = y_1, \\ y_2' = (x+1) y_1 + y_2, \end{cases}$$

we recover the usual presentation of the resolved conifold as the total space of the bundle $\mathcal{O}_{C_1}(-1) \oplus \mathcal{O}_{C_1}(-1)$ over $C_1 \cong \mathbb{P}^1$. Define $\mathcal{O}_{Y_d}(-n) := \mathcal{O}_{Y_d}(nD)$, where D is the divisor cut out by the equation $x = 0$ in the above coordinate patches. Then by Van den Bergh's theorem, we have a derived equivalence as in (4) if we set

$$A_{Q_{-2},W_d} := \mathrm{End}_{Y_d}(E_d), \tag{5}$$

[1] Not to be confused with the *length* of C_d, which is an entirely different invariant introduced by Kollár [Clemens et al. 1988] and used in the classification by S. Katz and D. Morrison [1992] of irreducible small resolutions of Gorenstein 3-fold singularities.

[2] Note that the motivic Donaldson–Thomas invariants we obtain for the conifold differ from those of [Szendrői 2008; Morrison et al. 2012]; this is a result of a different choice of orientation data, in the terminology of [Kontsevich and Soibelman 2008]. We revisit this subtle point in Remark 5.8.

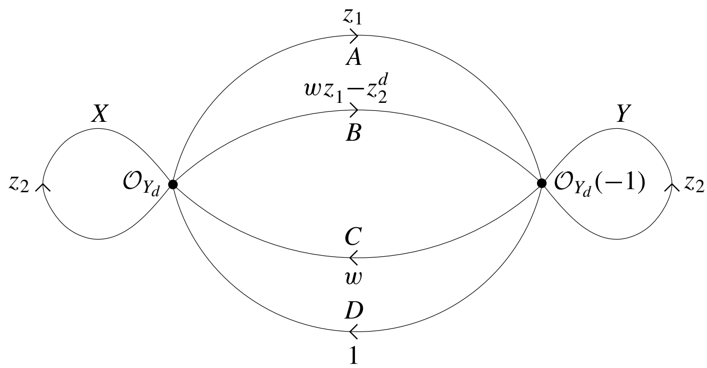

Figure 1. The quiver Q_{-2}. The vertices are marked with the summands of the bundle E_d, and the arrows are marked with morphisms between these summands.

where we define

$$E_d := \mathcal{O}_{Y_d} \oplus \mathcal{O}_{Y_d}(-1).$$

We follow the convention of [Aspinwall and Katz 2006], representing morphisms between the two line bundles $\mathcal{O}_{Y_d}$ and $\mathcal{O}_{Y_d}(-1)$ by elements of $\mathbb{C}[w, z_1, z_2]$ under the identifications $\Gamma(U_2, \mathcal{O}_{Y_d}) \cong \mathbb{C}[w, z_1, z_2] \cong \Gamma(U_2, \mathcal{O}_{Y_d}(-1))$. The endomorphism algebra can then be represented by the quiver algebra depicted in Figure 1. We have the relations

$$\begin{cases} AX = YA, \\ BX = YB, \\ XC = CY, \\ XD = DY, \end{cases} \quad \text{and} \quad \begin{cases} X^d = CA - DB, \\ Y^d = AC - BD. \end{cases} \tag{6}$$

It follows that A_{Q_{-2}, W_d} admits a superpotential description in the sense of [Ginzburg 2006], with quiver Q_{-2} depicted in Figure 1 and superpotential given by

$$W_d = \frac{1}{d+1} X^{d+1} - \frac{1}{d+1} Y^{d+1} - XCA + XDB + YAC - YBD. \tag{7}$$

That is, we have an isomorphism

$$A_{Q_{-2}, W_d} \cong \mathbb{C}Q_{-2} \,/\, \langle \partial W_d \,/\, \partial E \mid E \in \mathrm{E}(Q_{-2}) \rangle, \tag{8}$$

where for a general quiver Q and $W \in \mathbb{C}Q \,/\, [\mathbb{C}Q, \mathbb{C}Q]$ given by a single cycle, and $E \in \mathrm{E}(Q)$ an arrow,

$$\partial W \,/\, \partial E := \sum_{\substack{aEb = W \\ a \text{ and } b \text{ paths in } Q}} ba, \tag{9}$$

and for general W, $\partial W \,/\, \partial E$ is defined by extending linearly.

Definition 2.1. For a general quiver Q with potential W, we define $A_{Q,W}$ in the same way as in (8). This is called the Jacobi algebra associated to the pair (Q, W).

Remark 2.2. In the case of the conifold (i.e., if $d = 1$) there is a simpler Jacobi algebra presentation of the noncommutative resolution $\mathrm{End}_{Y_d}(E_d)$ [Szendrői 2008]; see also Remark 5.8. The quiver is given by Q_{con}, which is Q_{-2} with the two loops X and Y removed; see Figure 3. One sets $W_{\mathrm{con}} = ACBD - ADBC$, and one can show directly that $A_{Q_{\mathrm{con}},W_{\mathrm{con}}} \cong A_{Q_{-2},W_1}$. Note that the relations (6) imply the relations given by the noncommutative derivatives of W_{con}, considered as a superpotential for Q_{-2}. As a result one may consider the morphisms assigned to X and Y for a A_{Q_{-2},W_d}-module M as being together an endomorphism of a module M_{con} for $A_{Q_{\mathrm{con}},W_{\mathrm{con}}}$, where M_{con} in turn is determined by the morphisms assigned to A, B, C, and D by M, via the forgetful map.

3. Naive Grothendieck rings of motives

3A. ***A pre-λ-ring of motives.*** In this section we recall the construction of "naive" Grothendieck pre-λ-rings of $\hat{\mu}$-equivariant motives, or motives carrying a monodromy action. The reason for introducing such rings is that they are the natural home of motivic vanishing cycles, which carry monodromy actions in analogy with their sheaf-theoretic cousins. The reason for taking special care of the monodromy is that while in general the map induced on naive Grothendieck rings of motives by forgetting the monodromy will be a homomorphism of underlying groups, it will fail to respect the multiplication or pre-λ-ring operations. In particular, both the "integration map" (18) of Kontsevich and Soibelman and the map (19) generalising the map exploited by Behrend, Bryan and Szendrői [Behrend et al. 2013] will fail to be algebra homomorphisms for general quivers with potential if we forget monodromy.

For $\mathfrak{M}$ an Artin stack locally of finite type over $\mathbb{C}$ we define $\mathrm{K}_0(\mathrm{St}^{\mathrm{aff}}/\mathfrak{M})$ to be the Abelian group which is generated by isomorphism classes of morphisms

$$X \xrightarrow{f} \mathfrak{M}$$

of finite type, with X a separated reduced stack over $\mathbb{C}$ satisfying the condition that each of its $\mathbb{C}$-points has affine stabiliser, subject to the relations

$$[X \xrightarrow{f} \mathfrak{M}] \sim [Z \xrightarrow{f|_Z} \mathfrak{M}] + [X \setminus Z \xrightarrow{f|_{X\setminus Z}} \mathfrak{M}],$$

for $Z \subset X$ a closed substack of X. If $(\mathfrak{M}, \epsilon, 0)$, with

$$\epsilon : \mathfrak{M} \times \mathfrak{M} \to \mathfrak{M} \quad \text{and} \quad 0 : \mathrm{Spec}(\mathbb{C}) \to \mathfrak{M},$$

is a (commutative) monoid in the category of Artin stacks over $\mathbb{C}$ with ϵ of finite type, then $\mathrm{K}_0(\mathrm{St}^{\mathrm{aff}}/\mathfrak{M})$ acquires via convolution the structure of a (commutative)

$\mathrm{K}_0(\mathrm{St}^{\mathrm{aff}}/\operatorname{Spec}(\mathbb{C}))$-algebra, and the inclusion

$$[X \xrightarrow{f} \operatorname{Spec}(\mathbb{C})] \mapsto [X \xrightarrow{0\circ f} \mathfrak{M}].$$

There are obvious G-equivariant versions $\mathrm{K}_0^G(\mathrm{St}^{\mathrm{aff}}/\mathfrak{M})$ of the above groups and rings for G-equivariant stacks or monoids $\mathfrak{M}$, where we work with G-equivariant morphisms and assume that every point in X lies in a G-equivariant affine neighbourhood. Again we consider $\mathrm{K}_0^G(\mathrm{St}^{\mathrm{aff}}/\mathfrak{M})$ as a $\mathrm{K}_0(\mathrm{St}^{\mathrm{aff}}/\operatorname{Spec}(\mathbb{C}))$-algebra if $\mathfrak{M}$ is a monoid in the category of locally finite type G-equivariant Artin stacks with finite type monoid map. For technical reasons it is better to make the following modifications to $\mathrm{K}_0^G(\mathrm{St}^{\mathrm{aff}}/\mathfrak{M})$, forming the modified ring $\mathrm{K}^G(\mathrm{St}^{\mathrm{aff}}/\mathfrak{M})$:

(1) If $X' \xrightarrow{\pi} X$ is a G-equivariant vector bundle of rank r then we impose the relation
$$[X' \xrightarrow{f\circ\pi} \mathfrak{M}] \sim \mathbb{L}^r \cdot [X \xrightarrow{f} \mathfrak{M}]$$
in $\mathrm{K}^G(\mathrm{St}^{\mathrm{aff}}/\mathfrak{M})$, where $\mathbb{L}$ is the class of the affine line $\mathbb{A}^1_{\mathbb{C}}$ in $\mathrm{K}(\mathrm{St}^{\mathrm{aff}}/\operatorname{Spec}(\mathbb{C}))$.

(2) In addition, we complete with respect to the topology having as closed neighbourhoods of zero the subgroups
$$\mathrm{K}_{\mathfrak{U}} := \{L \in \mathrm{K}_0^G(\mathrm{St}^{\mathrm{aff}}/\mathfrak{M}) \text{ such that } L|_{\mathfrak{U}} = 0\}$$
for $\mathfrak{U} \subset \mathfrak{M}$ an open substack. In the sequel we always complete with respect to the analogous system of neighbourhoods, so for example the statement of Proposition 3.1 concerns expressions with infinitely many denominators $[\mathrm{GL}_{\mathbb{C}}(n)]^{-1}$ if the stack $\mathfrak{M}$ is not of finite type. If the base stack $\mathfrak{M}$ is of finite type this second modification makes no difference.

We define in the natural way the subgroup (or subring, if $\mathfrak{M}$ is a monoid) $\mathrm{K}^G(\mathrm{Var}/\mathfrak{M})$, spanned by classes $[X \to \mathfrak{M}]$ for X a G-equivariant variety over $\mathbb{C}$.

By [Ekedahl 2009, Proposition 1.1],
$$[\mathrm{GL}_{\mathbb{C}}(n)] = \prod_{0\le i\le n-1} (\mathbb{L}^n - \mathbb{L}^i)$$
in $\mathrm{K}(\mathrm{Var}/\operatorname{Spec}(\mathbb{C}))$.

Proposition 3.1 [Ekedahl 2009, Theorem 1.2]. *The natural map*
$$\Psi : \mathrm{K}^G(\mathrm{Var}/\mathfrak{M})[[\mathrm{GL}_{\mathbb{C}}(n)]^{-1}, n \in \mathbb{N}] \to \mathrm{K}^G(\mathrm{St}^{\mathrm{aff}}/\mathfrak{M})$$
is an isomorphism.

For a morphism $h : \mathfrak{M} \to \mathfrak{T}$ of locally finite type Artin stacks we define
$$h^* : \mathrm{K}^G(\mathrm{St}^{\mathrm{aff}}/\mathfrak{T}) \to \mathrm{K}^G(\mathrm{St}^{\mathrm{aff}}/\mathfrak{M})$$

via the pullback. If h is representable there is an equality

$$h^* = \Psi \circ (h^*)|_{\mathrm{Var}} \circ \Psi^{-1}$$

where $(h^*)|_{\mathrm{Var}}$ is the $\mathrm{K}(\mathrm{Var}\,/\,\mathrm{Spec}(\mathbb{C}))[[\mathrm{GL}_{\mathbb{C}}(n)]^{-1}, n \in \mathbb{N}]$-linear extension of the restriction of h^* to a map

$$\mathrm{K}^G(\mathrm{Var}\,/\,\mathfrak{T}) \to \mathrm{K}^G(\mathrm{Var}\,/\,\mathfrak{M}).$$

We define

$$\int_h : \mathrm{K}^G(\mathrm{Var}\,/\,\mathfrak{M}) \to \mathrm{K}^G(\mathrm{Var}\,/\,\mathfrak{T})$$

via composition with h, if h is of finite type. For $j : \mathfrak{M}' \hookrightarrow \mathfrak{M}$ an inclusion of a finite type substack we write $\int_{\mathfrak{M}'} := \int_h \circ j^*$, where $h : \mathfrak{M}' \to \mathrm{Spec}(\mathbb{C})$ is the structure morphism.

We will briefly recall the framework of [Davison and Meinhardt 2015]. Let $(\mathfrak{M}, \epsilon, 0)$ be a monoid in the category of Artin stacks, locally of finite type, with ϵ of finite type. We wish to define a "naive" Grothendieck pre-λ-ring of motives over $\mathfrak{M}$, where such motives are to carry a monodromy action. When it comes to defining the pre-λ-ring operations, it turns out to be most instructive to consider such motives via their associated mapping tori. For this reason, we will be interested in the group $\mathrm{K}^{\mathbb{G}_m,n}(\mathrm{St}^{\mathrm{aff}}/\mathbb{A}^1_{\mathfrak{M}})$, the naive Grothendieck group of $\mathbb{G}_m$-equivariant stacks over

$$\mathbb{A}^1_{\mathfrak{M}} := \mathbb{A}^1_{\mathbb{C}} \times \mathfrak{M}.$$

The stack $\mathbb{A}^1_{\mathfrak{M}}$ is given the $\mathbb{G}_m$-action that is trivial on $\mathfrak{M}$ and acts with weight n on $\mathbb{A}^1_{\mathbb{C}}$. The $\mathbb{G}_m$-equivariant projection map

$$p : \mathbb{A}^1_{\mathbb{C}} \times \mathfrak{M} \to \mathfrak{M}$$

induces a map

$$p^* : \mathrm{K}^{\mathbb{G}_m}(\mathrm{St}^{\mathrm{aff}}/\mathfrak{M}) \to \mathrm{K}^{\mathbb{G}_m,n}(\mathrm{St}^{\mathrm{aff}}/\mathbb{A}^1_{\mathfrak{M}})$$

and we denote by $\mathfrak{I}_n$ the image of this map. We give $\mathfrak{M}$ the trivial μ_n-action, where μ_n denotes the group of n-th roots of unity in $\mathbb{C}^*$. The map $\mathrm{K}^{\mu_n}(\mathrm{St}^{\mathrm{aff}}/\,\mathfrak{M}) \to \mathrm{K}^{\mathbb{G}_m,n}(\mathrm{St}^{\mathrm{aff}}/\mathbb{A}^1_{\mathfrak{M}})\,/\,\mathfrak{I}_n$ given by

$$[Y \xrightarrow{f} \mathfrak{M}] \mapsto [Y \times_{\mu_n} \mathbb{G}_m \xrightarrow{(y,z)\mapsto(z^n, f(y))} \mathbb{A}^1_{\mathbb{C}} \times \mathfrak{M}]$$

is an isomorphism. For each $a \geq 1$ there is a natural morphism $\mu_{an} \to \mu_n,\ z \mapsto z^a$, and this induces an inclusion $\mathrm{K}^{\mu_n}(\mathrm{St}^{\mathrm{aff}}/\,\mathfrak{M}) \to \mathrm{K}^{\mu_{an}}(\mathrm{St}^{\mathrm{aff}}/\,\mathfrak{M})$, and we define $\mathrm{K}^{\hat{\mu}}(\mathrm{St}^{\mathrm{aff}}/\,\mathfrak{M})[\mathbb{L}^{-1/2}]$ to be the group obtained by taking the direct limit of these inclusions, and then adding a formal square root to the inverse of $\mathbb{L}$.

Definition 3.2. Given a ring R, always assumed to be commutative, a pre-λ-ring structure on R is given by a map $\sigma : R \to R[\![T]\!]$ satisfying

- $\sigma(0) = 1$,
- $\sigma(a) = 1 + aT$ modulo $T^2 \cdot R[\![T]\!]$,
- $\sigma(a+b) = \sigma(a)\sigma(b)$.

We define the operations $\sigma^n(r)$ via

$$\sigma(r) = \sum_{i\geq 0} \sigma^i(r) T^i,$$

and we define $\mathrm{Sym}(r) = \sum_{i\geq 0} \sigma^i(r)$ when this infinite sum exists.[3] Finally, if R is a pre-λ-ring we define a pre-λ-ring structure on $R[\![X]\!]$ by setting

$$\sigma^n(r \cdot X^i) := \sigma^n(r) \cdot X^{i \cdot n},$$

extending to polynomials in X by the equation $\sigma(a+b) = \sigma(a)\sigma(b)$, and completing with respect to the ideal generated by X.

We always assume that $\sigma(1) = (1-T)^{-1}$; in other words we always pick 1 to be a line element. Using the above notation,

$$\mathrm{Sym}\biggl(\sum_{i\geq 1} a_i \cdot X^i\biggr) = \sum_{\pi} \prod_i (\sigma^{\pi(i)}(a_i))\, X^{i\pi(i)},$$

where the sum is over all partitions π, and we denote by $\pi(i)$ the number of parts of π of size i.

Proposition 3.3 [Davison and Meinhardt 2015, Lemma 4.1]. *Let $(\mathfrak{M}, \epsilon, 0)$ be a commutative monoid in the category of locally finite type schemes over $\mathbb{C}$ with ϵ of finite type. Consider the map*

$$+ : \mathbb{A}^1_{\mathfrak{M}} \times \mathbb{A}^1_{\mathfrak{M}} \to \mathbb{A}^1_{\mathfrak{M}}, \qquad ((z_1, x_1), (z_2, x_2)) \mapsto (z_1 + z_2, \epsilon(x_1, x_2))$$

making $\mathbb{A}^1_{\mathfrak{M}}$ into a commutative monoid. The Abelian group $\mathrm{K}^{\mathbb{G}_m,n}(\mathrm{Var}/\mathbb{A}^1_{\mathfrak{M}})$ has the structure of a pre-λ-ring if we set

$$[X_1 \xrightarrow{f_1} \mathbb{A}^1_{\mathfrak{M}}] \cdot [X_2 \xrightarrow{f_2} \mathbb{A}^1_{\mathfrak{M}}] = [X_1 \times X_2 \xrightarrow{f_1 \times f_2} \mathbb{A}^1_{\mathfrak{M}} \times \mathbb{A}^1_{\mathfrak{M}} \xrightarrow{+} \mathbb{A}^1_{\mathfrak{M}}]$$

and

$$\sigma^n([X \xrightarrow{f} \mathbb{A}^1_{\mathfrak{M}}]) = [\mathrm{Sym}^n X \xrightarrow{\mathrm{Sym}^n f} \mathrm{Sym}^n \mathbb{A}^1_{\mathfrak{M}} \xrightarrow{+} \mathbb{A}^1_{\mathfrak{M}}],$$

for varieties X, X_1, X_2. Furthermore, this induces a pre-λ-ring structure on the quotient $\mathrm{K}^{\mu_n}(\mathrm{Var}/\mathfrak{M})$, which is preserved by the embeddings $\mathrm{K}^{\mu_n}(\mathrm{Var}/\mathfrak{M}) \to \mathrm{K}^{\mu_{an}}(\mathrm{Var}/\mathfrak{M})$, giving rise to a pre-λ-ring structure on $\mathrm{K}^{\hat{\mu}}(\mathrm{Var}/\mathfrak{M})$.

[3] This will be the case for $r \in F^1 R$ if R is a complete filtered ring with filtration $F^* R$ such that $\sigma^i(F^j R) \subset F^{i \cdot j} R$ for all $i, j \in \mathbb{N}$.

Remark 3.4. Using Proposition 3.1, there is a unique extension of this pre-λ-ring structure to $\mathrm{K}^{\hat{\mu}}(\mathrm{St}^{\mathrm{aff}}/\mathfrak{M})[\mathbb{L}^{-1/2}]$. This we may describe as follows. Given an element in

$$\mathrm{K}^{\hat{\mu}}(\mathrm{Var}/\mathfrak{M})[\mathbb{L}^{-1/2}, [\mathrm{GL}_{\mathbb{C}}(n)]^{-1}, n \in \mathbb{N}]$$

one obtains a nonunique element P of $\mathrm{K}^{\hat{\mu}}(\mathrm{Var}/\mathfrak{M})[\![u]\!][u^{-1}]$ after substituting instances of $[1-\mathbb{L}^n]^{-t}$ out for their power series expansions and then substituting[4] $\mathbb{L}^{1/2} \mapsto -u$. It is not hard to verify that the set of formal power series obtained in this way is closed under taking σ^i for each i if we extend σ^i to the ring of Laurent series by $\sigma^i(au^j) = \sigma^i(a) * u^{i\cdot j}$. One may then take $\sigma^i(P)$, followed by the substitution $u \mapsto -\mathbb{L}^{1/2}$ to arrive at a (unique) element of $\mathrm{K}^{\hat{\mu}}(\mathrm{Var}/\mathfrak{M})[\mathbb{L}^{-1/2}, [\mathrm{GL}_{\mathbb{C}}(n)]^{-1}, n \in \mathbb{N}]$. See [Davison and Meinhardt 2015] for details.

Definition 3.5. A power structure on a ring R is a map $(1 + T\cdot R[\![T]\!]) \times R \to (1 + T\cdot R[\![T]\!])$, written $(A(T), m) \mapsto A(T)^m$, satisfying

- $A(T)^0 = 1$,
- $A(T)^1 = A(T)$,
- $(A(T)\cdot B(T))^m = A(T)^m \cdot B(T)^m$,
- $A(T)^{m+n} = A(T)^m \cdot A(T)^n$ and $A(T)^{m\cdot n} = (A(T)^m)^n$,
- $(1+T)^m$ is equal to $1 + m\cdot T$ modulo $T^2\cdot R[\![T]\!]$,
- $A(T^a)^m = A(T)^m|_{T\mapsto T^a}$.

We assume all power structures are continuous with respect to the T-adic topology on $R[\![T]\!]$.

Given a power series $A(T) \in 1 + T\cdot R[\![T]\!]$ with R a pre-λ-ring, we may write $A(T)$ uniquely as an expression

$$A(T) = \mathrm{Sym}\biggl(\sum_{n\geq 1} a_n T^n\biggr). \tag{10}$$

It follows that there is a one to one correspondence between continuous power structures and pre-λ-ring structures: given a pre-λ-ring structure we write

$$A(T)^m = \mathrm{Sym}\biggl(\sum_{n\geq 1} m a_n T^n\biggr),$$

with a_n defined by (10), and given a power structure on R we may write $\sigma(m) = (1-T)^{-m}$. For R a ring, and R' a quotient ring of R, power structures on R

[4]There is a potentially confusing choice of sign here, especially since either choice of sign gives a formal square root of $\mathbb{L}$. We justify our choice by noting that $\mathbb{L}$ is supposed to be the motive of $\mathrm{H}_c(\mathbb{A}^1, \mathbb{Q})$, the tensor square root of which has odd cohomological degree.

descending to power structures on R' are exactly the power structures such that the associated pre-λ-ring structure descends to R'.

Proposition 3.6. *The power structure on* $\mathrm{K}^{\mathbb{G}_m,n}(\mathrm{Var}/\mathbb{A}^1_{\mathfrak{M}})$ *inducing the pre-λ-ring structure of Proposition 3.3 is defined by*

$$\Big(\sum_{n\geq 0}[A_n\to\mathbb{A}^1_{\mathfrak{M}}]\cdot T^n\Big)^{[B\xrightarrow{g}\mathbb{A}^1_{\mathfrak{M}}]}$$
$$:=\sum_{\pi}\bigg[\Big(\prod_i(B^{\pi_i}\times A_i^{\pi_i})\,/\,S_{\pi_i}\Big)\setminus\Delta\xrightarrow{+\circ\prod_i g^{\pi_i}\times f_i^{\pi_i}}\mathbb{A}^1_{\mathfrak{M}}\bigg]\cdot T^{\sum_i i\pi_i},\qquad(11)$$

where the sum is over all $\pi:\mathbb{N}\to\mathbb{N}$ *with finite support and* Δ *is the preimage of the* big *diagonal in* $\prod_i B^{\pi_i}/S_{\pi_i}$ *with respect to the obvious projection. This power structure descends to* $\mathrm{K}^{\mu_n}(\mathrm{Var}/\mathfrak{M})$, *is preserved by the embeddings* $\mathrm{K}^{\mathbb{G}_m,n}(\mathrm{Var}/\mathbb{A}^1_{\mathfrak{M}})\to\mathrm{K}^{\mathbb{G}_m,an}(\mathrm{Var}/\mathbb{A}^1_{\mathfrak{M}})$, *and induces a power structure on* $\mathrm{K}^{\hat{\mu}}(\mathrm{Var}/\mathfrak{M})$.

Given π one should think of $\big(\prod_i(B^{\pi_i}\times A_i^{\pi_i})\,/\,S_{\pi_i}\big)\setminus\Delta$ as being the configuration space of pairs (K,ϕ), where K is a finite subset of B of cardinality $\sum_i\pi_i$ and $\phi:K\longrightarrow\coprod_i A_i$ is a map sending π_i points to A_i.

Proof. The given power structure on $\mathrm{K}^{\mathbb{G}_m,n}(\mathrm{Var}/\mathbb{A}^1_{\mathfrak{M}})$ can be checked to be a power structure inducing the given pre-λ-ring structure as in [Gusein-Zade et al. 2004]. The statements regarding the preservation of power structures under embeddings and their descent to the quotient are true due to the correspondence between power structures and pre-λ-rings, and the truth of the corresponding statements on the side of pre-λ-rings — this is just Proposition 3.3 again. □

Remark 3.7. This power structure extends to $\mathrm{K}^{\mathbb{G}_m,n}(\mathrm{St}^{\mathrm{aff}}/\mathbb{A}^1_{\mathfrak{M}})$ inducing a power structure on $\mathrm{K}^{\hat{\mu}}(\mathrm{St}^{\mathrm{aff}}/\mathfrak{M})$ which corresponds to the pre-λ-ring structure discussed in Remark 3.4. In order to do this, we have to replace $[B\xrightarrow{g}\mathbb{A}^1_{\mathfrak{M}}]$ and $[A_i\xrightarrow{f_i}\mathbb{A}^1_{\mathfrak{M}}]$ with formal power series

$$B=\sum_j[B_j\xrightarrow{g_j}\mathbb{A}^1_{\mathfrak{M}}]u^j\quad\text{and}\quad A_i=\sum_k[A_{ik}\xrightarrow{f_{ik}}\mathbb{A}^1_{\mathfrak{M}}]u^k$$

in u with coefficients in $\mathrm{K}^{\mathbb{G}_m,n}(\mathrm{Var}/\mathbb{A}^1_{\mathfrak{M}})$. To get the correct formula, we should think of these series as being the motives of $\coprod_j B_j\longrightarrow\mathbb{A}^1_{\mathfrak{M}}$ and $\coprod_k A_{ik}\longrightarrow\mathbb{A}^1_{\mathfrak{M}}$, respectively. The configuration spaces decompose accordingly. If π_{ijk} denotes the number of points in B_j mapped into A_{ik}, then the correct form of the right hand side of the formula in (11) is

$$\sum_{\pi}\bigg[\Big(\prod_{i,j,k}(B_j^{\pi_{ijk}}\times A_{ik}^{\pi_{ijk}})/S_{\pi_{ijk}}\Big)\setminus\Delta\xrightarrow{+\circ\prod_{i,j,k}g_j^{\pi_{ijk}}\times f_{ik}^{\pi_{ijk}}}\mathbb{A}^1_{\mathfrak{M}}\bigg]\cdot u^{\sum_{i,j,k}(j+k)\pi_{ijk}}T^{\sum_{i,j,k}i\pi_{ijk}},$$

where the sum is now taken over all functions $\pi : \mathbb{N}^3 \to \mathbb{N}$ with compact support and Δ denotes the preimage of the big diagonal in $\prod_{i,j,k} B_j^{\pi_{ijk}} / S_{\pi_{ijk}}$ with respect to the obvious projection.

3B. ***Motivic Hall algebras.*** We recall the definition of the motivic Hall algebra for the stack of finite-dimensional $A_{Q,W}$-modules, for $A_{Q,W}$ a Jacobi algebra as in Definition 2.1. For Q a finite quiver and $\boldsymbol{n} \in \mathbb{N}^{\mathrm{V}(Q)}$ a dimension vector, we define the moduli stack

$$\mathcal{Y}_{Q,\boldsymbol{n}} := \prod_{a \in \mathrm{E}(Q)} \mathrm{Hom}(\mathbb{C}^{\boldsymbol{n}(t(a))}, \mathbb{C}^{\boldsymbol{n}(s(a))}) \Big/ \prod_{i \in \mathrm{V}(Q)} \mathrm{GL}_{\mathbb{C}}(\boldsymbol{n}(i)),$$

where $s(a)$ is the source of the arrow a and $t(a)$ is the target[5], and $\mathrm{GL}_{\mathbb{C}}(\boldsymbol{n}(i))$ acts by change of basis of $\mathbb{C}^{\boldsymbol{n}(i)}$. We define

$$\mathcal{Y}_Q := \coprod_{\boldsymbol{n} \in \mathbb{N}^{\mathrm{V}(Q)}} \mathcal{Y}_{Q,\boldsymbol{n}}.$$

If $W \in \mathbb{C}Q / [\mathbb{C}Q, \mathbb{C}Q]$ is a superpotential we define $\mathcal{X}_{Q,W,\boldsymbol{n}}$ to be the Zariski closed subscheme of $\mathcal{Y}_{Q,\boldsymbol{n}}$ cut out by the matrix valued equations given by the noncommutative partial differentials (as defined by (9) and the line following it) of W. We define

$$\mathcal{X}_{Q,W} := \coprod_{\boldsymbol{n} \in \mathbb{N}^{\mathrm{V}(Q)}} \mathcal{X}_{Q,W,\boldsymbol{n}},$$

the moduli stack of finite-dimensional modules for $A_{Q,W}$, the Jacobi algebra for (Q, W). Denote by

$$\mathcal{X}_{Q,W}^{\mathrm{nilp}} \subset \mathcal{X}_{Q,W} \quad \text{and} \quad \mathcal{Y}_Q^{\mathrm{nilp}} \subset \mathcal{Y}_Q \tag{12}$$

the stacks[6] of finite-dimensional nilpotent right modules for $A_{Q,W}$ and $\mathbb{C}Q$, respectively, cut out by the equations $\mathrm{tr}(\rho(c)) = 0$ for all cyclic paths c.

The abelian groups $\mathrm{K}(\mathrm{St}^{\mathrm{aff}}/\mathcal{Y}_Q)$, $\mathrm{K}(\mathrm{St}^{\mathrm{aff}}/\mathcal{Y}_Q^{\mathrm{nilp}})$, $\mathrm{K}(\mathrm{St}^{\mathrm{aff}}/\mathcal{X}_{Q,W})$, and $\mathrm{K}(\mathrm{St}^{\mathrm{aff}}/\mathcal{X}_{Q,W}^{\mathrm{nilp}})$ carry Hall algebra products for which the comprehensive reference is the series of papers by Dominic Joyce; see [2006a; 2006b; 2007a; 2007d] or also Bridgeland's summary [Bridgeland 2012]. For completeness we recall the definition.

We fix our attention on $\mathrm{K}(\mathrm{St}^{\mathrm{aff}}/\mathcal{Y}_Q)$ for now. Let $[X_i \xrightarrow{f_i} \mathcal{Y}_Q]$ be two effective classes, for $i = 0, 1$. The ring $\mathrm{K}(\mathrm{St}^{\mathrm{aff}}/\mathcal{Y}_Q)$ is isomorphic to the inverse limit of the

[5] In algebraic contexts (as in Section 4) it is generally better to work with right modules, which is why our homomorphisms go from the vector space labelled by the target of the arrow to the vector space labelled by the source.

[6] Note that these stacks do not represent the functor sending a ring A to the groupoid of nilpotent $A_{Q,W} \otimes A$ or $\mathbb{C}Q \otimes A$-modules, flat over A.

quotients

$$\mathfrak{Q}_t := \mathrm{K}(\mathrm{St}^{\mathrm{aff}}/\mathcal{Y}_Q) \,/\, \mathrm{K}\Big(\mathrm{St}^{\mathrm{aff}}/ \coprod_{\substack{\boldsymbol{n}\in\mathbb{N}^{\mathrm{V}(Q)}\\ |\boldsymbol{n}|\geq t}} \mathcal{Y}_{Q,\boldsymbol{n}}\Big),$$

by convention (2). Note that each stack

$$\coprod_{\substack{\boldsymbol{n}\in\mathbb{N}^{\mathrm{V}(Q)}\\ |\boldsymbol{n}|< t}} \mathcal{Y}_{Q,\boldsymbol{n}}$$

is of finite type. Since the product is linear, we may assume that each morphism f_i factors through an inclusion $\mathcal{Y}_{Q,\boldsymbol{n}_i} \hookrightarrow \mathcal{Y}_Q$. For $a, b \in \mathbb{N}$, denote by $\mathrm{GL}_{\mathbb{C}}(a, b)$ the Borel subgroup of $\mathrm{GL}_{\mathbb{C}}(a+b)$ preserving the standard flag $0 = \mathbb{C}^0 \subset \mathbb{C}^a \subset \mathbb{C}^{a+b}$. Let

$$\mathbb{A}_{Q,\boldsymbol{n}_0,\boldsymbol{n}_1} \subset \mathbb{A}_{Q,\boldsymbol{n}_0+\boldsymbol{n}_1} = \prod_{a\in\mathrm{E}(Q)} \mathrm{Hom}(\mathbb{C}^{\boldsymbol{n}_0(t(a))+\boldsymbol{n}_1(t(a))}, \mathbb{C}^{\boldsymbol{n}_0(s(a))+\boldsymbol{n}_1(s(a))})$$

be the subspace of points corresponding to linear maps preserving the standard flag

$$0 = \bigoplus_{i\in\mathrm{V}(Q)} \mathbb{C}^0 \subset \bigoplus_{i\in\mathrm{V}(Q)} \mathbb{C}^{\boldsymbol{n}_0(i)} \subset \bigoplus_{i\in\mathrm{V}(Q)} \mathbb{C}^{\boldsymbol{n}_0(i)+\boldsymbol{n}_1(i)},$$

and let

$$\mathcal{Y}_{Q,\boldsymbol{n}_0,\boldsymbol{n}_1} = \mathbb{A}_{Q,\boldsymbol{n}_0,\boldsymbol{n}_1} \,/\, \prod_{i\in\mathrm{V}(Q)} \mathrm{GL}_{\mathbb{C}}(\boldsymbol{n}_0(i), \boldsymbol{n}_1(i))$$

be the stack-theoretic quotient. Then there are three natural morphisms of stacks

$$\begin{cases} \pi_1 : \mathcal{Y}_{Q,\boldsymbol{n}_0,\boldsymbol{n}_1} \to \mathcal{Y}_{Q,\boldsymbol{n}_0} \\ \pi_2 : \mathcal{Y}_{Q,\boldsymbol{n}_0,\boldsymbol{n}_1} \to \mathcal{Y}_{Q,\boldsymbol{n}_0+\boldsymbol{n}_1} \\ \pi_3 : \mathcal{Y}_{Q,\boldsymbol{n}_0,\boldsymbol{n}_1} \to \mathcal{Y}_{Q,\boldsymbol{n}_1}, \end{cases}$$

and we define $[X_0 \xrightarrow{f_0} \mathcal{Y}_Q] \star [X_1 \xrightarrow{f_1} \mathcal{Y}_Q]$ to be the composition given by the top row of the following commutative diagram

$$\begin{tikzcd} X_2 \arrow[r] \arrow[d] & \mathcal{Y}_{Q,\boldsymbol{n}_0,\boldsymbol{n}_1} \arrow[r, "\pi_2"] \arrow[d, "\pi_1\times\pi_3"] & \mathcal{Y}_{Q,\boldsymbol{n}_0+\boldsymbol{n}_1} \arrow[r, hook] & \mathcal{Y}_Q \\ X_0 \times X_1 \arrow[r, "f_0\times f_1"] & \mathcal{Y}_{Q,\boldsymbol{n}_0} \times \mathcal{Y}_{Q,\boldsymbol{n}_1} & & \end{tikzcd}$$

where the leftmost square is Cartesian. This gives consistent well defined products on the quotients $\mathfrak{Q}_t$, and so it gives a well defined product on $\mathrm{K}(\mathrm{St}^{\mathrm{aff}}/\mathcal{Y}_Q)$. It is easy to see that under the Hall algebra product the group $\mathrm{K}(\mathrm{St}^{\mathrm{aff}}/\mathcal{Y}_Q^{\mathrm{nilp}})$ is a subalgebra.

Similarly, we define $[X_0 \xrightarrow{f_0} \mathcal{X}_{Q,W}] \star_{\mathrm{KS}} [X_1 \xrightarrow{f_1} \mathcal{X}_{Q,W}]$ via the diagram

$$\begin{tikzcd} X_2 \arrow[r] \arrow[d] & \mathcal{X}_{Q,W,\boldsymbol{n}_0,\boldsymbol{n}_1} \arrow[r, "\pi_2"] \arrow[d, "\pi_1\times\pi_3"] & \mathcal{X}_{Q,W,\boldsymbol{n}_0+\boldsymbol{n}_1} \arrow[r, hook] & \mathcal{X}_{Q,W} \\ X_0 \times X_1 \arrow[r, "f_0\times f_1"] & \mathcal{X}_{Q,W,\boldsymbol{n}_0} \times \mathcal{X}_{Q,W,\boldsymbol{n}_1}. \end{tikzcd}$$

Remark 3.8. Note that the group homomorphism $[X \to \mathcal{X}_{Q,W}] \mapsto [X \to \mathcal{Y}_Q]$ induced by the inclusion $\mathcal{X}_{Q,W} \subset \mathcal{Y}_Q$ is not an algebra homomorphism for these products — an extension of modules for the Jacobi algebra $A_{Q,W}$, considered as $\mathbb{C}Q$-modules, might not satisfy the relations required to be a $A_{Q,W}$-module. It is for this reason that we use different notation to distinguish the products $\star_{\mathrm{KS}}$ and $\star$.

3C. ***Motivic vanishing cycles.*** We present some of the ideas expanded upon in greater depth in [Looijenga 2002]. Let X be a smooth scheme over $\mathbb{C}$ and let $f : X \to \mathbb{A}^1_{\mathbb{C}}$ be a regular map. One defines $\mathcal{L}_n(X)$, the space of arcs in X of length n, to be the scheme representing the functor $Y \mapsto \mathrm{Hom}_{\mathrm{Sch}}(Y \times \mathrm{Spec}(\mathbb{C}[t]\,/\,t^{n+1}), X)$. Via the natural inclusion $\mathrm{Spec}(\mathbb{C}[t]\,/\,t) \to \mathrm{Spec}(\mathbb{C}[t]\,/\,t^{n+1})$ there is a map of schemes

$$p_n : \mathcal{L}_n(X) \to X.$$

We write $\mathcal{L}_n(X)|_{X_0} = p_n^{-1} f^{-1}(0)$. There is a natural morphism $f_* : \mathcal{L}_n(X) \to \mathcal{L}_n(\mathbb{A}^1_{\mathbb{C}})$ given by composition. An arc in $\mathbb{A}^1_{\mathbb{C}}$ is given by a polynomial $a_0 + \cdots + a_n t^n$, and so $\mathcal{L}_n(\mathbb{A}^1_{\mathbb{C}}) \cong \mathbb{A}^{n+1}_{\mathbb{C}}$ and the composition of f_* with the projection

$$\pi : \mathcal{L}_n(\mathbb{A}^1_{\mathbb{C}}) \to \mathbb{A}^1_{\mathbb{C}}, \quad a_0 + \cdots + a_n t^n \mapsto a_n$$

makes $\mathcal{L}_n(X)$ into a scheme over $\mathbb{A}^1_{\mathbb{C}}$. Moreover there is a $\mathbb{G}_m$-action on $\mathcal{L}_n(X)$ given by rescaling the coordinate t of $\mathbb{C}[t]\,/\,t^{n+1}$, and

$$[\mathcal{L}_n(X)|_{X_0} \xrightarrow{(\pi\circ f_*)\times p_n} \mathbb{A}^1_{X_0}] \in \mathrm{K}^{\mathbb{G}_m,n}(\mathrm{Var}\,/\mathbb{A}^1_{X_0}).$$

We consider the expression

$$Z^{\mathrm{eq}}_f(T) := \sum_{n\ge 1} \mathbb{L}^{-(n+1)\dim(X)/2} \cdot [\mathcal{L}_n(X)|_{X_0} \xrightarrow{(\pi\circ f_*)\times p_n} \mathbb{A}^1_{X_0}]T^n$$

as a formal power series with coefficients in $\mathrm{K}^{\hat{\mu}}(\mathrm{Var}\,/\,X_0)[\mathbb{L}^{-1/2}]$. In general (see [Denef and Loeser 1998, Theorem 2.2.1]) it makes sense to evaluate this function at infinity, and one defines

$$\phi_f = -Z^{\mathrm{eq}}_f(\infty) \in \mathrm{K}^{\hat{\mu}}(\mathrm{Var}\,/\,X_0)[\mathbb{L}^{-1/2}],$$

the motivic vanishing cycle of f. This definition differs by a factor of $(-\mathbb{L}^{1/2})^{\dim(X)}$ from the original definition of Denef and Loeser. This normalisation makes the motivic weights appearing in Donaldson–Thomas theory simpler; the principle is that in Donaldson–Thomas theory and elsewhere, it is best to work with the perverse sheaf of vanishing cycles, which is obtained from the complex of sheaves $\phi_f \mathbb{Q}_X$ by shifting by half the dimension of X.

The motivic vanishing cycle has the property that if $g : X_1 \to X_2$ is a smooth morphism of smooth schemes, and if $f : X_2 \to \mathbb{A}^1_{\mathbb{C}}$ is a regular function, then $\phi_{f\circ g} = \mathbb{L}^{-\dim(g)/2} \cdot g^* \phi_f$. Given an Artin stack $\mathcal{Z}$ that is a quotient stack $[Z/\mathrm{GL}_{\mathbb{C}}(m)]$ for smooth connected Z, and $f : \mathcal{Z} \to \mathbb{A}^1_{\mathbb{C}}$ a function, one defines[7]

$$\phi_f = \mathbb{L}^{m^2/2} \cdot [\mathrm{BGl}_{\mathbb{C}}(m)] \cdot \pi_* \phi_{f\circ\pi} \in \mathrm{K}^{\hat{\mu}}(\mathrm{St}^{\mathrm{aff}}/\mathcal{Z}),$$

where $\pi : Z \to \mathcal{Z}$ is the projection.

In studying 3-dimensional Calabi–Yau categories, one is often faced with the following situation, which necessitates the use of a *relative* version of motivic vanishing cycles. Firstly, let X be a finite type scheme, carrying a constructible vector bundle V, with a function $f : \mathrm{Tot}(V) \to \mathbb{C}$ vanishing on the zero fibre. By constructible vector bundle, we mean that there is a finite decomposition of X into locally closed subschemes $X = \coprod X_i$, and a vector bundle V_i on each of the X_i (we do not assume that these vector bundles are of the same rank). By a function on such an object we mean a function on each of the V_i, possibly after further decomposition. In full generality, one should consider formal functions on V, by which we mean a function on the formal neighbourhood of the zero section of each of the V_i. We would like to define a motivic vanishing cycle for such a function. This we do by defining $\mathcal{L}_n(V)$ to be the space of those arcs in $\mathrm{Tot}(V)$ that restrict to a single fibre of the projection $\pi : V \to X$. More precisely, we define $\mathcal{L}_n(V)$ via the Cartesian diagram

$$\begin{tikzcd}
\mathcal{L}_n(V) \arrow[r] \arrow[d] & \mathcal{L}_n(\mathrm{Tot}(V)) \arrow[d, "\tau_*"] \\
X \arrow[r, "\beta"] & \mathcal{L}_n(X)
\end{tikzcd}$$

where τ_* is induced by the projection $\tau : \mathrm{Tot}(V) \to X$, and the map β is the inclusion of constant arcs. We define $\mathcal{L}_n(V)|_X = p_n^{-1}(X)$ as before. Finally, define

$$Z_f^{\mathrm{eq}}(T) := \sum_{n\geq 1} \mathbb{L}^{-(n+1)\,\mathrm{rank}(V)/2} \cdot [\mathcal{L}_n(V)|_X \xrightarrow{(\pi\circ f_*)\times p_n} \mathbb{A}^1_X] T^n \tag{13}$$

in $\mathrm{K}^{\hat{\mu}}(\mathrm{Var}/X)[\mathbb{L}^{-1/2}]$. We claim that the definition

$$\phi_f^{\mathrm{rel}} := Z_f^{\mathrm{eq}}(\infty)$$

[7]Note that, by relation (1), $[\mathrm{BGl}_{\mathbb{C}}(m)] = [\mathrm{GL}_{\mathbb{C}}(m)]^{-1} \in \mathrm{K}(\mathrm{St}^{\mathrm{aff}}/\mathrm{Spec}(\mathbb{C}))$

makes sense, in other words, that the relative zeta function (13) can be evaluated at infinity. The claim is justified using Kontsevich's transformation formula (see [Looijenga 2002, Section 3]) in the same way as [Denef and Loeser 1998, Theorem 2.2.1]. In a little more detail, by Hironaka's theorem, we may find an embedded resolution $Y \xrightarrow{g} \mathrm{Tot}(V)$ of $f^{-1}(0)$, considered as a subvariety of $\mathrm{Tot}(V)$, blowing up along smooth centres $H_1, \dots, H_n$. That is, we have that $(fg)^{-1}(0)$ is a normal crossings divisor. After replacing X by a Zariski open subvariety $X' \subset X$, we may assume that each projection from H_i to X is smooth. Define $Y' = g^{-1}(X')$, then possibly after shrinking X' further, we may assume that $(fg)^{-1}(0)$ is a smooth family of normal crossing divisors. Now the claim (over X') follows by the proof of [Looijenga 2002, Theorem 5.4], and the discussion following it. Finally, we consider the complement $X \setminus X'$, which can be decomposed into finitely many smooth schemes $X' = \coprod X_i$ of dimension strictly less than $\dim(X)$ — the general result follows by Noetherian induction and the cut and paste relations.

4. Motivic Donaldson–Thomas theory

4A. ***Three-dimensional Calabi–Yau categories.*** We recall the essential ingredients of the theory of motivic Donaldson–Thomas invariants from [Kontsevich and Soibelman 2008]. We will begin with the data that one feeds into this machine. One starts with $\mathcal{C}$, a 3-Calabi–Yau category. By a 3-Calabi–Yau category $\mathcal{C}$ we mean a set of objects $\mathtt{ob}(\mathcal{C})$, between any two objects $x_i, x_j \in \mathtt{ob}(\mathcal{C})$ a $\mathbb{Z}$-graded vector space $\mathrm{Hom}_{\mathcal{C}}(x_i, x_j)$, and a countable collection of operations

$$b_{\mathcal{C},n} : \mathrm{Hom}_{\mathcal{C}}(x_{n-1}, x_n)[1] \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{C}}(x_0, x_1)[1] \to \mathrm{Hom}_{\mathcal{C}}(x_0, x_n)[1]$$

of degree 1, satisfying the condition

$$\sum_{\alpha+\beta+\gamma=n} b_{\mathcal{C},\alpha+1+\gamma} \circ (\mathbb{1}^{\otimes\alpha} \otimes b_{\mathcal{C},\beta} \otimes \mathbb{1}^{\otimes\gamma}) = 0.$$

See [Lefèvre-Hasegawa 2003] for a comprehensive guide to A_∞-categories, or [Kajiura 2007] for a similarly comprehensive guide to cyclic A_∞-categories, or [Keller 2001] for a gentle and concise reference for most of what follows. All these ideas are also covered in the notes [Kontsevich and Soibelman 2009]. The 3-Calabi–Yau condition consists of the extra data of a skewsymmetric nondegenerate bracket

$$\langle \cdot, \cdot \rangle_{\mathcal{C}} : \mathrm{Hom}_{\mathcal{C}}(x_i, x_j)[1] \otimes \mathrm{Hom}_{\mathcal{C}}(x_j, x_i)[1] \to \mathbb{C}$$

of degree -1, such that the functions $W_{\mathcal{C},n} := \langle b_{\mathcal{C},n-1}(\cdot, \dots, \cdot), \cdot \rangle$ are cyclically symmetric. One defines

$$W_{\mathcal{C}}(z) := \sum_{n \geq 2} \frac{1}{n} W_{\mathcal{C},n}(z, \dots, z),$$

a formal function on $\mathrm{Hom}^1_{\mathcal{C}}(x_i, x_i)$ for each $x_i \in \mathtt{ob}(\mathcal{C})$.

In this section we will recall the definition of a particular 3-Calabi–Yau category $\mathtt{tw}(\mathcal{D}(Q_{-2}, W_d))$, the *$A_\infty$-category of twisted complexes* over a certain 3-Calabi–Yau category $\mathcal{D}(Q_{-2}, W_d)$ built out of the same data (Q_{-2}, W_d) as A_{Q_{-2},W_d}. The category $\mathtt{tw}(\mathcal{D}(Q_{-2}, W_d))$ will be shown to be a 3-Calabi–Yau enrichment of the category $\mathrm{D}^{\mathrm{b}}_{\mathrm{exc}}(Y_d)$, the derived category of coherent sheaves on Y_d with bounded total cohomology, set-theoretically supported on the exceptional locus $C_d \subset Y_d$, in the sense that there is a composition of equivalences of categories, beginning with the homotopy category of $\mathtt{tw}(\mathcal{D}(Q_{-2}, W_d))$:

$$\begin{tikzcd}
 & \mathrm{D}_{\mathrm{f.d}}(\mathtt{Mod}\text{-}\hat{A}_{Q_{-2},W_d}) \arrow[r, "\simeq"] & \mathrm{D}_{\mathrm{nilp}}(\mathtt{Mod}\text{-}A_{Q_{-2},W_d}) \arrow[r, "\simeq"] & \mathrm{D}^{\mathrm{b}}_{\mathrm{exc}}(Y_d) \\
\mathrm{H}(\mathtt{tw}(\mathcal{D}(Q_{-2},W_d))) \arrow[r, "\simeq"] & \mathrm{D}_{\mathrm{f.d}}(\mathtt{Mod}\text{-}\Gamma(Q_{-2},W_d)) \arrow[u, "\simeq"] \arrow[r, "\simeq"] & \mathrm{D}_{\mathrm{nilp}}(\mathtt{Mod}\text{-}\Gamma(Q_{-2},W_d)) \arrow[u, "\simeq"] &
\end{tikzcd} \tag{14}$$

where the leftmost arrow is Koszul duality and the rightmost is VdB equivalence.

The algebra $\hat{A}_{Q_{-2},W_d}$ is the Jacobi algebra defined as in Definition 2.1, completed at the ideal generated by the arrows of Q_{-2}. The category $\mathrm{D}_{\mathrm{f.d}}(\mathtt{Mod}\text{-}\hat{A}_{Q_{-2},W_d})$ is the derived category of right $\hat{A}_{Q_{-2},W_d}$-modules with finite-dimensional total cohomology, and $\mathrm{D}_{\mathrm{nilp}}(\mathtt{Mod}\text{-}A_{Q_{-2},W_d})$ is the derived category of A_{Q_{-2},W_d}-modules with nilpotent finite-dimensional total cohomology. As diagram (14) indicates, the story starts with Koszul duality, so we start our exposition with the Koszul dual of $\mathcal{D}(Q_{-2}, W_d)$, which is the *Ginzburg differential graded category*.

Given the data of a quiver with potential (Q, W), Ginzburg [2006] defines the dg-category $\Gamma(Q, W)$. It is constructed as follows. The quiver Q defines a bimodule S for the semisimple ring $R := \mathbb{C}^{\mathrm{V}(Q)}$, where we set

$$\dim(e_i \cdot S \cdot e_j) := \#(\text{arrows from } j \text{ to } i).$$

The objects of the category $\Gamma(Q, W)$ are just the vertices of the quiver, i.e.,

$$\mathtt{ob}(\Gamma(Q, W)) := \mathrm{V}(Q),$$

and for two vertices x_i, x_j we put

$$\mathrm{Hom}_{\Gamma(Q,W)}(x_i, x_j) = e_j \cdot T_R(R[2] \oplus S^\vee[1] \oplus S) \cdot e_i,$$

where $e_i, e_j \in R$ are the idempotent elements corresponding to x_i and x_j, respectively, and $S^\vee$ is the dual of S in the category of R-bimodules. Moreover, $T_R(M)$ denotes the completion of the free unital algebra object generated by M in the category of R-bimodules. Composition in the category $\Gamma(Q, W)$ is given by the tensor product in the category of R-bimodules. We define a differential d of degree one on $T_R(R[2] \oplus S^\vee[1] \oplus S)$ satisfying the Leibniz rule and such that

$$\begin{cases} d(e_i[2]) = \sum_{a_k : x_i \to x_j} a_k^* a_k - \sum_{a_k : x_j \to x_i} a_k a_k^*, \\ d(a_k^*[1]) = \partial W / \partial a_k, \\ d(a_k) = 0, \end{cases}$$

where a_k runs through a basis of the vector space $e_j S e_i$, and a_k^* runs through a dual basis of $e_i S^\vee e_j = (e_j S e_i)^\vee$. This makes $\Gamma(Q, W)$ into a dg-category and hence into an A_∞-category. For certain choices of (Q, W), including our choice (Q_{-2}, W_d), the Ginzburg differential graded category $\Gamma(Q, W)$ has cohomology concentrated in degree zero. Moreover, for any choice of (Q, W), there is an isomorphism

$$\mathrm{H}^0(\Gamma(Q, W)) \cong \hat{A}_{Q,W}$$

where $\hat{A}_{Q,W}$ is the Jacobi algebra defined as in Definition 2.1, completed at the ideal generated by the arrows of Q, and considered in the usual way as a category whose objects are the idempotents e_i. There is a natural equivalence of categories between finite-dimensional modules over $\hat{A}_{Q,W}$ and nilpotent finite-dimensional modules over $A_{Q,W}$. Together, these facts provide the central commutative square of equivalences in (14).

As for $\Gamma(Q, W)$, the objects of the category $\mathcal{D}(Q, W)$ are defined to be the vertices of the quiver, i.e.,

$$\mathtt{ob}(\mathcal{D}(Q, W)) := \mathrm{V}(Q).$$

The homomorphism spaces between these objects are graded vector spaces concentrated in degrees between zero and three. One sets

$$\mathrm{Hom}^n_{\mathcal{D}(Q,W)}(x_i, x_j) := \begin{cases} \mathbb{C}^{\delta_{ij}} & \text{if } n = 0, \\ (e_i \cdot S \cdot e_j)^\vee & \text{if } n = 1, \\ (e_j \cdot S \cdot e_i) & \text{if } n = 2, \\ (\mathbb{C}^\vee)^{\delta_{ij}} & \text{if } n = 3, \end{cases} \tag{15}$$

where δ_{ij} is the Kronecker delta function and $\mathbb{C}^\vee \cong \mathbb{C}$ is the vector dual of the one dimensional complex vector space $\mathbb{C}$. The A_∞ operations on this category are given by first setting the natural generator $\mathbb{1}_i$ of $\mathrm{Hom}^0_{\mathcal{D}(Q,W)}(x_i, x_i)$ to be a strict unit for every $i \in Q_0$. This means that $b_2(f, \mathbb{1}_i) = f$ and[8] $b_2(\mathbb{1}_i, g) = -g$ for all $f \in \mathrm{Hom}_{\mathcal{D}(Q,W)}(x_i, x_j)$ and $g \in \mathrm{Hom}_{\mathcal{D}(Q,W)}(x_j, x_i)$, and any insertion of $\mathbb{1}_i$ into b_n for any $n \geq 3$ results in the zero function. We let $b_{\mathcal{D}(Q,W),2}(\theta, z) = -\theta(z)\mathbb{1}_j^*$ with $\mathbb{1}_j^* \in \mathrm{Hom}^3_{\mathcal{C}}(x_j, x_j)$ being the dual basis of $\mathbb{1}_j$, and $b_{\mathcal{D}(Q,W),2}(z, \theta) = \theta(z)\mathbb{1}_i^*$ for any

[8] The strange sign here is the price we pay for considering the maps $b_n : \mathrm{Hom}_{\mathcal{C}}(x_{n-1}, x_n)[1] \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{C}}(x_0, x_1)[1] \to \mathrm{Hom}_{\mathcal{C}}(x_0, x_n)[1]$ instead of $m_n : \mathrm{Hom}_{\mathcal{C}}(x_{n-1}, x_n) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{C}}(x_0, x_1) \to \mathrm{Hom}_{\mathcal{C}}(x_0, x_n)$. The payoff is that there are a lot fewer signs overall.

$\theta \in \mathrm{Hom}^1_{\mathcal{D}(Q,W)}(x_i, x_j)$ and $z \in \mathrm{Hom}^2_{\mathcal{D}(Q,W)}(x_j, x_i)$. Then for degree reasons all that is left is to define the degree one operations

$$b_{\mathcal{D}(Q,W),m} : \mathrm{Hom}^1_{\mathcal{D}(Q,W)}(x_{m-1}, x_m)[1] \otimes \cdots \otimes \mathrm{Hom}^1_{\mathcal{D}(Q,W)}(x_0, x_1)[1] \\ \to \mathrm{Hom}^2_{\mathcal{D}(Q,W)}(x_0, x_m)[1]$$

which are given by W_{m+1}, the $(m+1)$-th homogeneous part of W, via the natural pairing

$$(e_{m-1}\cdot S\cdot e_m)^\vee \otimes \cdots \otimes (e_0\cdot S\cdot e_1)^\vee \otimes e_0\cdot S\cdot e_1 \otimes \cdots \otimes e_{m-1}\cdot S\cdot e_m \otimes e_m\cdot S\cdot e_0 \to e_m\cdot S\cdot e_0.$$

Note that this definition results in the identity $W = W_{\mathcal{D}(Q,W)}|_{\mathrm{End}^1(\bigoplus_{i\in V(Q)} x_i)}$.

The category $\mathcal{D}(Q,W)$ has a natural inner product

$$\mathrm{Hom}_{\mathcal{D}(Q,W)}(x_i, x_j)[1] \otimes \mathrm{Hom}_{\mathcal{D}(Q,W)}(x_j, x_i)[1] \to \mathbb{C}[-1]$$

satisfying the cyclicity condition.

We now come to the connection between $\Gamma(Q,W)$ and $\mathcal{D}(Q,W)$. This is explained via Koszul duality for A_∞-algebras, for which an excellent reference is [Lu et al. 2008]. Using the projection to the degree zero part of $\mathcal{D}(Q,W)$, we can make $R_i := e_i R \cong \mathbb{C}$ into a (trivial) right $\mathcal{D}(Q,W)$-module, which we will denote $R_{i,\mathcal{D}(Q,W)}$, and we get an object in $\texttt{Mod-}(\mathcal{D}(Q,W)) := \mathrm{Fun}_\infty(\mathcal{D}(Q,W)^{\mathrm{op}}, \mathrm{Vect}^{\mathbb{Z}}_{\mathbb{C}})$, the dg-category of right A_∞-modules over $\mathcal{D}(Q,W)$ with finite-dimensional bounded cohomology. With the help of the bar construction one can show that there are quasi-isomorphisms

$$\mathrm{Hom}_{\texttt{Mod-}\mathcal{D}(Q,W)}(R_{i,\mathcal{D}(Q,W)}, R_{j,\mathcal{D}(Q,W)}) \simeq \mathrm{Hom}_{\Gamma(Q,W)}(x_j, x_i) \\ \simeq \mathrm{Hom}_{\texttt{Mod-}\Gamma(Q,W)}(x_i, x_j),$$

where we used the Yoneda embedding of $\Gamma(Q,W)$ into $\texttt{Mod-}\Gamma(Q,W)$ for the final quasi-isomorphism — in fact if one uses the (reduced) bar construction to demonstrate the first of these quasi-isomorphisms, it is an equality. This establishes that $\Gamma(Q,W)$ and $\mathcal{D}(Q,W)$ are Koszul dual, and so by [Lu et al. 2008, Theorem 2.4] we get similar quasi-isomorphisms after swapping them — i.e., there is a quasi-isomorphism

$$\mathrm{Hom}_{\texttt{Mod-}\Gamma(Q,W)}(R_{i,\Gamma(Q,W)}, R_{j,\Gamma(Q,W)}) \simeq \mathrm{Hom}_{\mathcal{D}(Q,W)}(x_i, x_j).$$

Hence the induced functor between homotopy categories

$$\mathrm{RHom}(\texttt{Mod-}R_{\Gamma(Q,W)}, \cdot) : \mathrm{D}(\texttt{Mod-}\Gamma(Q,W)) \to \mathrm{D}_\infty(\texttt{Mod-}\mathcal{D}(Q,W))$$

takes $R_{i,\Gamma(Q,W)}$ to a module quasi-isomorphic to x_i, considered as a $\mathcal{D}(Q,W)$ module, and restricting, we obtain the diagram of functors

$$\begin{tikzcd}
\mathrm{D}(\mathtt{Mod}\text{-}\Gamma(Q,W)) \arrow[r, "\mathrm{RHom}(R_{\Gamma(Q,W)},-)"] & \mathrm{D}_\infty(\mathtt{Mod}\text{-}\mathcal{D}(Q,W)) \\
\mathrm{D}(\langle R_{i,\Gamma(Q,W)}, i \in \mathrm{V}(Q)\rangle_{\mathrm{thick}}) \arrow[u, hook] \arrow[r, "\simeq"] & \mathrm{D}_\infty(\langle x_i \in \mathtt{Mod}\text{-}\mathcal{D}(Q,W), i \in \mathrm{V}(Q)\rangle_{\mathrm{thick}}) \arrow[u, hook] \\
\mathrm{D}(\langle R_{i,\Gamma(Q,W)}, i \in \mathrm{V}(Q)\rangle_{\mathrm{triang}}) \arrow[u, hook, "\alpha"', "\simeq"] \arrow[r, "\simeq"] & \mathrm{D}_\infty(\langle x_i \in \mathtt{Mod}\text{-}\mathcal{D}(Q,W), i \in \mathrm{V}(Q)\rangle_{\mathrm{triang}}) \arrow[u, hook, "\beta"', "\simeq"]
\end{tikzcd} \tag{16}$$

where for $S \subset \mathtt{ob}(\mathtt{Mod}\text{-}\Gamma(Q,W))$, the category $\mathrm{D}(\langle S\rangle_{\mathrm{triang}})$ is the full subcategory of the derived category of $\Gamma(Q,W)$-modules M that are quasi-isomorphic to objects in the closure of S under taking triangles and shifts, and $\mathrm{D}(\langle S\rangle_{\mathrm{thick}})$ is defined in the same way, except we take the closure under the operation of taking retracts too.

The lowest two horizontal functors in (16) are equivalences by Koszul duality for module categories [Lu et al. 2008, Theorem 5.4]. The inclusion α is a equivalence, since its source and target can both be seen to be the full subcategory of the derived category of $\Gamma(Q,W)$-modules consisting of dg-modules with finite-dimensional nilpotent total cohomology. In particular, $\mathrm{D}(\langle R_{i,\Gamma(Q,W)}, i \in \mathrm{V}(Q)\rangle_{\mathrm{triang}})$ is already closed under taking retracts. It follows that β is an equivalence too.

The point of introducing the category of twisted complexes $\mathtt{tw}(\mathcal{D}(Q,W))$ is that it is a category that is a natural 3-Calabi-Yau enrichment of

$$\mathrm{D}_\infty(\langle x_i \in \mathtt{Mod}\text{-}\mathcal{D}(Q,W), i \in \mathrm{V}(Q)\rangle_{\mathrm{triang}}),$$

which by (16) and (4) is equivalent to $\mathrm{D}^{\mathrm{b}}_{\mathrm{exc}}(Y_d)$ in the case $(Q,W) = (Q_{-2}, W_d)$. We refer to [Keller 2001, Section 7] for a comprehensive account of the category of twisted complexes, and here recall its main features.

Objects of $\mathtt{tw}(\mathcal{D}(Q,W))$ are given by pairs (T,α), where

$$T = \bigoplus_{i=1}^{n} x_{a_i}[b_i] \in \mathtt{Mod}\text{-}\mathcal{D}(Q,W)$$

is a finite direct sum of right $\mathcal{D}(Q,W)$-modules given by integer shifts of objects $x_{a_i} \in \mathtt{ob}(\mathcal{D}(Q,W))$ covariantly embedded via the Yoneda embedding, and α is an element of

$$\bigoplus_{i<j} \mathrm{Hom}^{b_j-b_i+1}_{\mathcal{D}(Q,W)}(x_{a_i}, x_{a_j}) \simeq \bigoplus_{i<j} \mathrm{Hom}^1_{\mathcal{D}(Q,W)}(x_{a_i}[b_i], x_{a_j}[b_j]) \subset \mathrm{Hom}_{\mathcal{D}(Q,W)}(T,T)$$

satisfying the Maurer–Cartan equation

$$\sum_{n\geq 1} b_{\mathcal{D}(Q,W),n}(\alpha,\dots,\alpha)=0.$$

Given two pairs (T_1,α_1) and (T_2,α_2), where

$$T_1=\bigoplus_{i\in I} x_{a_{1,i}}[b_{1,i}] \quad\text{and}\quad T_2=\bigoplus_{j\in J} x_{a_{2,j}}[b_{2,j}],$$

we define the graded vector space

$$\begin{aligned}\mathrm{Hom}_{\mathtt{tw}(\mathcal{D}(Q,W))}((T_1,\alpha_1),(T_2,\alpha_2)) &:= \bigoplus_{i,j}\mathrm{Hom}_{\mathcal{D}(Q,W)}(x_{a_{1,i}},x_{a_{2,j}})[b_{2,j}-b_{1,i}]\\ &\simeq \mathrm{Hom}_{\mathcal{D}(Q,W)}(T_1,T_2).\end{aligned}$$

Multiplication is twisted by setting

$$\begin{aligned}&b_{\mathtt{tw}(\mathcal{D}(Q,W))}(f_n,\dots,f_1)\\ &\qquad=\sum b_{\mathcal{D}(Q,W)}(\alpha_n,\dots,\alpha_n,f_n,\alpha_{n-1},\dots,\alpha_1,f_1,\alpha_0,\dots,\alpha_0)\end{aligned}$$

where $f_i\in\mathrm{Hom}_{\mathtt{tw}(\mathcal{D}(Q,W))}((T_{i-1},\alpha_{i-1}),(T_i,\alpha_i))$. One may check that this is again a 3-Calabi–Yau category. For

$$f\in\mathrm{Hom}_{\mathtt{tw}(\mathcal{D}(Q,W))}\left(\left(\bigoplus_{i\in I}x_{a_{1,i}}[b_{1,i}],\ \alpha_1\right),\left(\bigoplus_{j\in J}x_{a_{2,j}}[b_{2,j}],\ \alpha_2\right)\right)$$

and

$$g\in\mathrm{Hom}_{\mathtt{tw}(\mathcal{D}(Q,W))}\left(\left(\bigoplus_{j\in J}x_{a_{2,j}}[b_{2,j}],\ \alpha_2\right),\left(\bigoplus_{i\in I}x_{a_{1,i}}[b_{1,i}],\ \alpha_1\right)\right),$$

one sets

$$\langle f,g\rangle:=\sum_{\substack{i\in I\\ j\in J}}\langle f_{ij},g_{ji}\rangle,$$

where we denote by f_{ij} the degree $(b_{2,j}-b_{1,i})$ morphism $x_{a_{1,i}}\to x_{a_{2,j}}$ induced by f, and define g_{ji} similarly.

In fact we will only be interested in the category $\mathtt{tw}_0(\mathcal{D}(Q,W))$, which we define to be the full subcategory of $\mathtt{tw}(\mathcal{D}(Q,W))$ with objects given by pairs (T,α), with T isomorphic to a finite direct sum of *unshifted* copies of the right modules $x_i\in\mathtt{ob}(\mathcal{D}(Q,W))$. Under $\mathrm{RHom}_{\mathtt{Mod}\text{-}\Gamma(Q,W)}(R_{\Gamma(Q,W)},-)$ this in turn is an enrichment of the Abelian category of finite-dimensional nilpotent modules over the Jacobi algebra $A_{Q,W}$.

Let $\mathfrak{TW}_{\boldsymbol{n}}$ be the moduli functor on finite type schemes defined as follows: for each X, $\mathfrak{TW}_{\boldsymbol{n}}(X)$ is the set of pairs of vector bundles $\bigoplus_{i\in\mathrm{V}(Q)}T_{\boldsymbol{n}(i)}$ on X of rank

$\sum_{i\in \mathrm{V}(Q)} \boldsymbol{n}(i)$ and elements

$$\alpha \in \bigoplus_{i,j\in \mathrm{V}(Q)} \mathrm{Hom}^1_{\mathcal{D}(Q,W)}(x_i, x_j) \otimes T^*_{\boldsymbol{n}(i)} \otimes T_{\boldsymbol{n}(j)}$$

such that $\sum_{n\geq 1} b_{\mathcal{D}(Q,W),n}(\alpha,\ldots,\alpha) = 0$.

This moduli functor takes schemes over $\mathbb{C}$ to sets of families of objects in $\mathtt{tw}_0(\mathcal{D}(Q,W))$. This is naturally made into a groupoid valued moduli functor, where the morphisms are defined via the conjugation action of $\prod_{i\in\mathrm{V}(Q)} \mathrm{GL}_{\mathbb{C}}(\boldsymbol{n}(i))$. There is a natural isomorphism of moduli functors $\mathfrak{TW}_{\boldsymbol{n}} \to \mathfrak{nilp}_{\boldsymbol{n}}$, where $\mathfrak{nilp}_{\boldsymbol{n}}(X)$ consists of the set of vector bundles T on X with a $\mathcal{O}_X \otimes \Gamma(Q, W)$-action, nilpotent with respect to the $\Gamma(Q, W)$ factor, such that for all $i \in \mathrm{V}(Q)$, $T \cdot e_i$ is a rank $\boldsymbol{n}(i)$ vector bundle.

The moduli functor $\mathfrak{nilp}_{\boldsymbol{n}}$ is again a groupoid valued functor with morphisms given by conjugation, and its groupoid of geometric points is the same as for the stack $\mathcal{X}^{\mathrm{nilp}}_{Q,W,\boldsymbol{n}}$.

4B. ***Orientation data.*** There is one extra piece of data, aside from the 3-Calabi–Yau category $\mathtt{tw}_0(\mathcal{D}(Q_{-2}, W_d))$, that we need before we can apply the machinery of [Kontsevich and Soibelman 2008] to define and compute motivic Donaldson–Thomas invariants of (−2)-curves, which is the data $(\mathcal{L}, \phi)$ of an ind-constructible super (i.e., $\mathbb{Z}_2$-graded) line bundle $\mathcal{L}$ on $\mathcal{X}^{\mathrm{nilp}}_{Q_{-2},W_d}$ along with a chosen trivialisation of the tensor square

$$\phi : \mathcal{L}^{\otimes 2} \cong \mathbb{1}_{\mathcal{X}^{\mathrm{nilp}}_{Q_{-2},W_d}}.$$

Note that every constructible super line bundle $\mathcal{L}$ on a scheme X has trivial tensor square, since up to constructible decomposition of the base X we can write

$$\mathcal{L} \cong \mathbb{1}_{X_{\mathrm{even}}} \oplus \mathbb{1}_{X_{\mathrm{odd}}}[1],$$

where $X = X_{\mathrm{even}} \sqcup X_{\mathrm{odd}}$ is the constructible decomposition of X defined by the constructible function on X provided by taking the parity of $\mathcal{L}$. So all the data here is in this choice of trivialisation (and the parity of the super line bundle $\mathcal{L}$). Such data is required to satisfy a cocycle condition (see [Kontsevich and Soibelman 2008, Section 5.2]), ensuring that the integration map defined with respect to it (see (18)) is a $\mathrm{K}(\mathrm{St}^{\mathrm{aff}}/\operatorname{Spec}(\mathbb{C}))$-algebra homomorphism, and is called orientation data in [op. cit.]. An isomorphism of orientation data is just an isomorphism of the underlying constructible super line bundles commuting with the trivialisations of the squares. Isomorphic choices will give rise to the same integration map (18). In fact for Q, W a finite quiver with arbitrary potential, $\mathcal{X}^{\mathrm{nilp}}_{Q,W}$ comes with a natural choice of orientation data, which we briefly describe; more details can be found in [Davison 2010, Section 7.1].

Given an object $\eta = (T, \alpha)$ of $\mathtt{tw}_0(\mathcal{D}(Q, W))$, there is an explicit model of the cyclic A_∞-algebra $\mathrm{End}_{\mathtt{tw}(\mathcal{D}(Q,W))}(\eta)$, coming from the definition of $\mathtt{tw}(\mathcal{D}(Q, W))$. In particular, there is a differential[9] $b_{\alpha,1}$ on $\mathrm{End}^{\bullet}_{\mathtt{tw}(\mathcal{D}(Q,W)))}(\eta)$, and a nondegenerate inner product on $\mathrm{End}^1_{\mathtt{tw}(\mathcal{D}(Q,W))}(T) \,/\, \mathrm{Ker}(b_{\alpha,1})$ given by $\langle b_{\alpha,1}(\,\cdot\,), \cdot\,\rangle$. Across the family of possible α in the pair (T, α), given by solutions to the Maurer–Cartan equation, we obtain a constructible vector bundle

$$\mathrm{End}^1_{\mathtt{tw}(\mathcal{D}(Q,W))}(T) \,/\, \mathrm{Ker}(b_{\mathtt{tw}(D(Q,W)),1}) \tag{17}$$

with nondegenerate quadratic form which we will denote by $\bar{Q}$. It is only a constructible vector bundle since the dimension of (17) jumps, due to the dependency of $b_{\mathtt{tw}(\mathcal{D}(Q,W)),1}$ on α. Given a constructible super vector bundle $\mathcal{V}$ on a stack $\mathfrak{M}$, one defines the superdeterminant

$$\mathtt{sDet}(\mathcal{V}) := \prod^{\dim(\mathcal{V})} \Big(\bigwedge^{\mathrm{top}} \mathcal{V}_{\mathrm{even}} \otimes \bigwedge^{\mathrm{top}} \mathcal{V}^*_{\mathrm{odd}} \Big),$$

where here $\prod$ denotes the change of parity functor. Say now $\mathcal{V}$ has nondegenerate quadratic form $\bar{Q}_{\mathcal{V}}$, then we obtain a trivialisation of $\mathtt{sDet}(\mathcal{V})^{\otimes 2}$ since $\bar{Q}_{\mathcal{V}}$ establishes an isomorphism $\mathtt{sDet}(\mathcal{V}) \cong \mathtt{sDet}(\mathcal{V})^*$. In the present situation, orientation data on $\mathcal{X}^{\mathrm{nilp}}_{Q_{-2}, W_d}$, considered as the moduli space of objects in $\mathtt{tw}_0(\mathcal{D}(Q_{-2}, W_d))$, is provided by the superdeterminant of (17), with the trivialisation of the tensor square provided by the nondegenerate inner product $\langle b_{\mathtt{tw}(\mathcal{D}(Q_{-2},W_d)),1}(\,\cdot\,), \cdot\,\rangle$. We will denote this choice of orientation data by τ_{Q_{-2},W_d}.

Remark 4.1 [Davison 2010, Theorem 8.3.1]. There are in general several choices for the orientation data of a 3-Calabi–Yau category. However in the case of the category $\mathtt{tw}_0(\mathcal{D}(Q, W))$ this range of choices is quite small, due to the constraint that the orientation data must satisfy the cocycle condition from [Kontsevich and Soibelman 2008]. In fact the orientation data is determined up to isomorphism entirely by its restriction to the simple modules x_i, for $i \in \mathrm{V}(Q)$, and so one deduces that there are $2^{\mathrm{V}(Q)}$ isomorphism classes of choices, giving rise to $2^{\mathrm{V}(Q)}$ distinct integration maps, defined as in Theorem 4.6.

Definition 4.2. A constructible 3-Calabi–Yau vector bundle on a scheme X is a constructible $\mathbb{Z}$-graded vector bundle V, along with degree one morphisms $b_n(V[1])^{\otimes n} \to V[1]$ and a degree minus one morphism $\langle\,\cdot\,,\cdot\,\rangle : V[1] \otimes V[1] \to \mathbb{1}_X$ satisfying the same conditions as a 3-Calabi–Yau category.

We recall the definition of a morphism of cyclic A_∞-objects in the case of a constructible 3-Calabi–Yau vector bundle.

[9] Recall that $\mathrm{End}^{\bullet}_{\mathtt{tw}(\mathcal{D}(Q,W)))}(\eta) \cong \mathrm{End}^{\bullet}_{\mathcal{D}(Q,W)}(T)$ as graded vector spaces, and does not depend on α, so in fact we obtain a family of differentials as we vary α.

Definition 4.3. A morphism $f : V \to V'$ of constructible 3-Calabi–Yau vector bundles is a countable collection of morphisms of constructible vector bundles $f_n : V[1]^{\otimes n} \to V'[1]$ satisfying the conditions

$$\sum_{\alpha+\beta+\gamma=n} f_{\alpha+1+\gamma}(\mathbb{1}^{\otimes\alpha} \otimes b_\beta \otimes \mathbb{1}^{\otimes\gamma}) = \sum_{n=\alpha_1+\cdots+\alpha_s} b'_s(f_{\alpha_1} \otimes \cdots \otimes f_{\alpha_s}),$$

for all n as well as the extra conditions that $\langle\cdot,\cdot\rangle_{V'} \circ f_1 \otimes f_1 = \langle\cdot,\cdot\rangle_V$ and $\sum_{a+b=n} \langle\cdot,\cdot\rangle_{V'} \circ f_a \otimes f_b = 0$ for all $n \geq 3$.

To complete the definition of the integration map of [Kontsevich and Soibelman 2008] we need the following proposition.

Proposition 4.4 [Kajiura 2007, Theorem 5.15]. *There is a locally constructible formal isomorphism of cyclic A_∞-vector bundles*

$$(\mathrm{End}^\bullet_{\mathtt{tw}(\mathcal{D}(Q_{-2},W_d))},\ b_{\mathtt{tw}(\mathcal{D}(Q_{-2},W_d))}) \cong (\mathrm{Ext}^\bullet_{\mathtt{tw}(\mathcal{D}(Q_{-2},W_d))} \oplus V^\bullet,\ b')$$

on the stack $\mathcal{X}^{\mathrm{nilp}}_{Q_{-2},W_d}$ such that b'_1 factors via a map $V^\bullet \to V^\bullet$, b'_i factors via a map

$$(\mathrm{Ext}^\bullet_{\mathtt{tw}(\mathcal{D}(Q_{-2},W_d))})^{\otimes i} \to \mathrm{Ext}^\bullet_{\mathtt{tw}(\mathcal{D}(Q_{-2},W_d))}$$

for $i \geq 2$, and $(V^\bullet, b'_1)$ is an acyclic complex. This splitting is unique up to isomorphisms of cyclic A_∞-vector bundles.

Note that even though one starts with the data of a cyclic A_∞-vector bundle, the splitting will only take place in the category of locally constructible cyclic A_∞-vector bundles, since the dimension of the kernel of b_1 will jump in families. The reference [Kajiura 2007] demonstrates this splitting in the case of cyclic A_∞-categories; which for us corresponds to the case in which the base of the cyclic A_∞-vector bundle is a point. In fact the proof produces a *canonical* decomposition, once a *choice* of contracting homotopy is made. Since after constructible decomposition we can construct a contracting homotopy for $b_{\mathtt{tw}(\mathcal{D}(Q_{-2},W_d)),1}$, the version of the proposition stated above is indeed a consequence of [Kajiura 2007, Theorem 5.15].

Given a constructible 3-Calabi–Yau vector bundle V, we define the function $W_{\min}$ as follows. First, let E be a cyclic minimal model for V. Next, consider the Artin stack E^1 / E^0 over X, given over a point $x \in X$ by taking the stack theoretic quotient of the trivial action of E^0_x on E^1_x (this is an example of a cone stack; see, e.g., [Behrend and Fantechi 1997], where they arise in a similar context). We define $W_{\min}$ to be the function on this stack defined by W_E, the potential for the minimal part E. This potential is strictly speaking only defined up to a formal automorphism, which will not matter when it comes to considering motivic vanishing cycles, as a result of the following proposition.

Proposition 4.5. *Let V be a vector bundle on a scheme X, and let f be a formal function on V with trivial constant coefficient* (*i.e.*, *f vanishes on X*, *considered* *as the zero section of V) such that $\phi^{\mathrm{rel}}_f|_X$ is well defined. Let g be another formal function on V vanishing on X, such that there exists a formal change of coordinates q on the vector bundle V around X such that $f = g \circ q$, considered as functions on a formal neighbourhood of X. Then $\phi^{\mathrm{rel}}_g|_X$ is well defined and $\phi^{\mathrm{rel}}_f|_X = \phi^{\mathrm{rel}}_g|_X$.*

Note that since we are dealing here with functions defined on vector bundles, we use the relative version of motivic vanishing cycles introduced at the end of Section 3C.

Proof. The proposition follows directly from the definition, since q induces $\mathbb{G}_m$-equivariant isomorphisms on arc spaces making the following diagram commute

$$\begin{tikzcd}
\mathcal{L}_n(V)|_X \arrow[rr, "q_*"] \arrow[dr, "\pi\circ f_*"'] & & \mathcal{L}_n(V)|_X \arrow[dl, "\pi\circ g_*"] \\
& \mathbb{A}^1_{\mathbb{C}} &
\end{tikzcd}$$

where π is as in Section 3C, and $\mathcal{L}_n(V)$ is as at the end of the same section. □

As in [Kontsevich and Soibelman 2008] we would like to identify the relative motivic vanishing cycles $\phi^{\mathrm{rel}}_{Q_1}$, $\phi^{\mathrm{rel}}_{Q_2}$ of quadratic functions Q_1, Q_2 on constructible vector bundles V_1, V_2 under the conditions that taking the parity of V_1 or V_2 gives the same element of $\Gamma_{\mathrm{constr}}(X, \mathbb{Z}_2)$, the group of constructible $\mathbb{Z}_2$-valued function on X, and taking the determinant of Q_1 or Q_2 gives the same element of $\Gamma_{\mathrm{constr}}(X, \mathbb{C}^*) / (\Gamma_{\mathrm{constr}}(X, \mathbb{C}^*))^2$. The reason this is desirable is that it means that we only have to keep track of these two pieces of data for the pair V, Q to know what the relative motivic vanishing cycle ϕ^{rel}_Q is, and the reason this identification is justifiable is that this identification becomes trivial after taking realisations of motives (for example Hodge polynomials, etc.). This we achieve as follows: we impose the extra relation in $\mathrm{K}^G(\mathrm{Var}/\mathfrak{M})$ given by identifying

$$[X /_{\rho_1} H \xrightarrow{f/\rho_1} \mathfrak{M}] - [X /_{\rho_2} H \xrightarrow{f/\rho_2} \mathfrak{M}]$$

for all smooth X, for all H-actions ρ_1, ρ_2 for H a finite group satisfying the property that the G-equivariant function f is H-invariant, and that the induced H-actions on the cohomology of a fibre over x, for any $x \in \mathfrak{M}$, are the same. One may easily check that the pre-λ-ring structure on $\mathrm{K}^{\hat{\mu}}(\mathrm{St}^{\mathrm{aff}}/\mathbb{A}^1_{\mathfrak{M}})$ descends to a pre-λ-ring structure on $\overline{\mathrm{K}}^{\hat{\mu}}(\mathrm{St}^{\mathrm{aff}}/\mathbb{A}^1_{\mathfrak{M}})$.

Let Q, W be a quiver with polynomial potential. Given an element

$$[X \xrightarrow{f} \mathcal{X}^{\mathrm{nilp}}_{Q,W,\boldsymbol{n}}] \in \mathrm{K}(\mathrm{Var}/\mathcal{X}^{\mathrm{nilp}}_{Q,W,\boldsymbol{n}}),$$

Kontsevich and Soibelman define

$$\Phi_{Q,W}([X \xrightarrow{f} \mathcal{X}^{\mathrm{nilp}}_{Q,W,\boldsymbol{n}}]) = \left(\int_X f^*\phi^{\mathrm{rel}}_{W_{\min}\boxplus\bar{Q}_{\tau_{Q,W}}}\right)\hat{e}_{\boldsymbol{n}} \in \overline{\mathrm{K}}^{\hat{\mu}}(\mathrm{St}^{\mathrm{aff}}/\operatorname{Spec}(\mathbb{C}))[\mathbb{L}^{1/2}][\![\hat{e}_{\boldsymbol{m}} \mid \boldsymbol{m} \in \mathbb{N}^{\mathrm{V}(Q)}]\!] \quad (18)$$

where $\bar{Q}_{\tau_{Q,W}}$ is a function $\bar{Q}_{\tau_{Q,W}}(z) = \bar{Q}_{\tau_{Q,W}}(z,z)$ on V, for some pair of ind-constructible vector bundle V on $\mathcal{X}^{\mathrm{nilp}}_{Q,W,\boldsymbol{n}}$ and nondegenerate inner product $\bar{Q}_{\tau_{Q,W}}$ on V giving rise to the natural orientation data on $\mathcal{X}^{\mathrm{nilp}}_{Q,W}$ arising from its realisation as the moduli space of objects in $\mathtt{tw}_0(\mathcal{D}(Q,W))$. That is, under the natural identification $\mathtt{sDet}(V) \cong \mathtt{sDet}(V)^*$ induced by $\bar{Q}_{\tau_{Q,W}}$, we obtain the natural orientation data $\tau_{Q,W}$ on $\mathcal{X}^{\mathrm{nilp}}_{Q,W,\boldsymbol{n}}$ given by (17) with its natural nondegenerate product. The function, $W_{\min}$ is as defined after Proposition 4.4. The target is just the ring of formal power series in variables $\hat{e}_{\boldsymbol{m}}$, with the usual[10] multiplication $\hat{e}_{\boldsymbol{m}'}\cdot\hat{e}_{\boldsymbol{m}} = \hat{e}_{\boldsymbol{m}'+\boldsymbol{m}}$. One extends to a map

$$\Phi_{Q,W} : \mathrm{K}(\mathrm{St}^{\mathrm{aff}}/\mathcal{X}^{\mathrm{nilp}}_{Q,W}) \to \overline{\mathrm{K}}^{\hat{\mu}}(\mathrm{St}^{\mathrm{aff}}/\operatorname{Spec}(\mathbb{C}))[\mathbb{L}^{1/2}][\![\hat{e}_{\boldsymbol{m}} \mid \boldsymbol{m} \in \mathbb{N}^{\mathrm{V}(Q)}]\!]$$

by $\mathrm{K}(\mathrm{St}^{\mathrm{aff}}/\operatorname{Spec}(\mathbb{C}))$-linearity and Proposition 3.1.

For general 3-Calabi–Yau categories the following is only a theorem if one is able to work with motivic vanishing cycles of formal functions and prove the motivic integral identity of [Kontsevich and Soibelman 2008]. For the former, see the comment immediately following the theorem. For the latter, see [Maulik 2013] or [Lê 2015].

Theorem 4.6 [Kontsevich and Soibelman 2008]. *The morphism*

$$\Phi_{Q,W} : \mathrm{K}(\mathrm{St}^{\mathrm{aff}}/\mathcal{X}^{\mathrm{nilp}}_{Q,W}) \to \overline{\mathrm{K}}^{\hat{\mu}}(\mathrm{St}^{\mathrm{aff}}/\operatorname{Spec}(\mathbb{C}))[\mathbb{L}^{1/2}][\![\hat{e}_{\boldsymbol{n}} \mid \boldsymbol{n} \in \mathbb{N}^{\mathrm{V}(Q)}]\!]$$

is a $\mathrm{K}(\mathrm{St}^{\mathrm{aff}}/\operatorname{Spec}(\mathbb{C}))$*-algebra homomorphism.*

The issue with formal functions is not a serious one in our case. Let $X/\operatorname{Spec}(\mathbb{C})$ be a finite type scheme. There is a multiplication $\star_X$ on $\mathrm{K}^{\mathbb{G}_m,n}(\mathrm{St}^{\mathrm{aff}}/\mathbb{A}^1_X)$ given by

$$[Y_1 \xrightarrow{f_1} \mathbb{A}^1_X] \star_X [Y_2 \xrightarrow{f_2} \mathbb{A}^1_X] = [Y_1 \times_X Y_2 \xrightarrow{p} \mathbb{A}^1_X],$$

where the fibre product is with respect to the morphisms $\pi_X \circ f_1$ and $\pi_X \circ f_2$, and the map p is defined by $(y_1, y_2) \mapsto (\pi_{\mathbb{A}^1_{\mathbb{C}}} \circ f_1(y_1) + \pi_{\mathbb{A}^1_{\mathbb{C}}} \circ f_2(y_2),\ \pi_X \circ f_1(y_1))$. This multiplication descends to $\mathrm{K}^{\hat{\mu}}(\mathrm{St}^{\mathrm{aff}}/X)$. We make the following definition.

Proposition 4.7. *Let V be a vector bundle on the scheme X, and let f be a formal function on V. Furthermore, assume that there exists a vector bundle V' on X and a quadratic form $\bar{Q}$ on V', such that there is a formal change of coordinates on*

[10] In fact usually one would twist the multiplication by some power of $-\mathbb{L}^{1/2}$, but we escape this necessity as we only work with symmetric quivers.

$V \oplus V'$ taking $f \boxplus \overline{Q}$ to a polynomial function g (here we abuse notation and write $\overline{Q}(z) = \overline{Q}(z, z)$). Then ϕ_f^{rel} is well defined, and

$$\phi_f^{\mathrm{rel}}|_X = \phi_g^{\mathrm{rel}}|_X \star_X \phi_{\overline{Q}}^{\mathrm{rel}}|_X.$$

Proof. It is easy to show that

$$[\mathcal{L}_n(V' \oplus V')|_X \xrightarrow{(\pi \circ (\overline{Q} \boxplus \overline{Q})_*) \times p_n} \mathbb{A}^1_X] = [X \xrightarrow{(0, \mathrm{id}_X)} \mathbb{A}^1_X] = 1 \in \mathrm{K}^{\mathbb{G}_m, n}(\mathrm{St}^{\mathrm{aff}}/\mathbb{A}^1_X)$$

for all n. It follows that there are equalities of relative motivic zeta functions

$$\begin{aligned} Z_f(T)^{\mathrm{eq}} &= Z^{\mathrm{eq}}_{f \boxplus \overline{Q} \boxplus \overline{Q}}(T) \\ &= Z^{\mathrm{eq}}_{g \boxplus \overline{Q}}(T) \end{aligned}$$

and the result follows by the Thom–Sebastiani theorem. □

If one works with the minimal potentials of objects in the category of modules over a Ginzburg differential graded algebra for a quiver with polynomial potential, one only needs to deal with formal functions f satisfying the conditions of Proposition 4.7.

There is a more down to earth way to define the integration map for the Hall algebra $\mathrm{K}(\mathrm{St}^{\mathrm{aff}}/\mathcal{X}^{\mathrm{nilp}}_{Q,W})$. In fact this second way extends without any effort to an integration map

$$\Phi_{\mathrm{BBS},Q,W} : \mathrm{K}(\mathrm{St}^{\mathrm{aff}}/\mathcal{X}_{Q,W}) \to \mathrm{K}^{\hat{\mu}}(\mathrm{St}^{\mathrm{aff}}/\operatorname{Spec}(\mathbb{C}))[\mathbb{L}^{-1/2}][\![\hat{e}_{\boldsymbol{n}} \mid \boldsymbol{n} \in \mathbb{N}^{\mathrm{V}(Q)}]\!] \quad (19)$$

exploited by Behrend, Bryan and Szendrői [Behrend et al. 2013] to define and calculate motivic Donaldson–Thomas counts for Hilbert schemes of points on $\mathbb{C}^3$. The Hodge theoretic version of this construction is a part of [Kontsevich and Soibelman 2011]; see also [Dimca and Szendrői 2009]. One defines, similarly to the Kontsevich–Soibelman integration map,

$$\Phi_{\mathrm{BBS},Q,W} : [X \xrightarrow{f} \mathcal{X}_{Q,W,\boldsymbol{n}}] \mapsto \int_X f^* \phi_{\mathrm{tr}(W)} \hat{e}_{\boldsymbol{n}}.$$

Let

$$\begin{aligned} \bar{q} : \mathrm{K}^{\hat{\mu}}(\mathrm{St}^{\mathrm{aff}}/\operatorname{Spec}(\mathbb{C}))[\mathbb{L}^{-1/2}][\![\hat{e}_{\boldsymbol{n}} \mid \boldsymbol{n} \in \mathbb{N}^{\mathrm{V}(Q)}]\!] \\ \to \overline{\mathrm{K}}^{\hat{\mu}}(\mathrm{St}^{\mathrm{aff}}/\operatorname{Spec}(\mathbb{C}))[\mathbb{L}^{-1/2}][\![\hat{e}_{\boldsymbol{n}} \mid \boldsymbol{n} \in \mathbb{N}^{\mathrm{V}(Q)}]\!] \end{aligned}$$

be the natural quotient map. The following comparison theorem will be used in the proof of Propositions 5.2 and 5.3.

Proposition 4.8 [Davison 2010, Theorem 7.1.3]. *There is an equality of maps $\bar{q} \circ \Phi_{\mathrm{BBS},Q,W}|_{\mathrm{K}(\mathrm{St}^{\mathrm{aff}}/\mathcal{X}^{\mathrm{nilp}}_{Q,W})} = \Phi_{Q,W}$.*

5. Motivic Donaldson–Thomas invariants of (-2)-curves

5A. ***The calculation of the invariants.*** We are finally able to calculate the motivic Donaldson–Thomas invariants of the category of nilpotent modules over A_{Q_{-2},W_d}, the noncommutative crepant resolution of X_d defined in (5) and explicitly described by (8). First, we pick a stability condition, which for us is just an additive map

$$\zeta : \mathbb{N}^{\mathrm{V}(Q_{-2})} \setminus 0 \to \mathbb{H}_+,$$

where $\mathbb{H}_+$ is the set $\{r \cdot e^{i\theta} \mid r \in \mathbb{R}_{>0}, \theta \in (0, \pi]\}$ and $\mathrm{V}(Q_{-2})$ is the set of vertices of the quiver Q_{-2} of Figure 1. We make the genericity assumption that ζ does not map the whole of $\mathbb{N}^{\mathrm{V}(Q_{-2})} \setminus 0$ onto the same ray in $\mathbb{C}$. As a result of the fact that Q_{-2} is symmetric, the invariants we calculate will not depend on which stability condition ζ we pick. We recall, regardless, that a module M of slope $\theta := \arg(\zeta(\mathbf{dim}(M)))$ is called *stable* if for all proper submodules $N \subset M$, $\arg(\zeta(\mathbf{dim}(N))) < \arg(\zeta(\mathbf{dim}(M)))$. Similarly, M is called *semistable* if we only require weak inequality between the arguments.

Lemma 5.1. *Let M be a semistable nilpotent A_{Q_{-2},W_d}-module with slope θ. Then M is given by repeated extension by stable modules M_α of slope θ, such that $M_\alpha \cdot (X+Y) = 0$.*

Proof. The fact that M admits a filtration with subquotients given by stable modules of slope θ is the statement of the existence of Jordan–Hölder filtrations. For the second part, note that $X + Y \in A_{Q_{-2},W_d}$ is central, and so acts via module endomorphisms on all A_{Q_{-2},W_d}-modules. This endomorphism is nilpotent for M, by assumption. Define

$$F^m M = \mathrm{Ker}(\,\cdot\,(X+Y)^m : M \to M)$$

to be the filtration of M by the nilpotence degree of this endomorphism. then each $F^m M$ is semistable of slope θ, since we have the short exact sequence

$$0 \to F^m M \to M \to \mathrm{Image}(\,\cdot\,(X+Y)^m : M \to M) \to 0$$

where the middle term is semistable of slope θ, and both the first and last terms have slope no greater than θ, from which it follows that they have slope equal to θ. It follows that each subquotient $F^m M \,/\, F^{m-1} M$ is semistable of slope θ, and the subquotients occurring in a refinement of the filtration $F^\bullet M$ to a Jordan–Hölder filtration of M are all acted on by zero by $\cdot(X+Y)$, since each $F^m M / F^{m-1} M$ is. □

The data of a module over A_{Q_{-2},W_d} is just the data of a module M over $A_{Q_{\mathrm{con}},W_{\mathrm{con}}}$, the Jacobi algebra for the noncommutative conifold (see Remark 2.2), along with an endomorphism $\upsilon : M \to M$ given by the action of $X+Y$, satisfying

$$\upsilon^d = (a \mapsto a \cdot (AC + CA - BD - DB)).$$

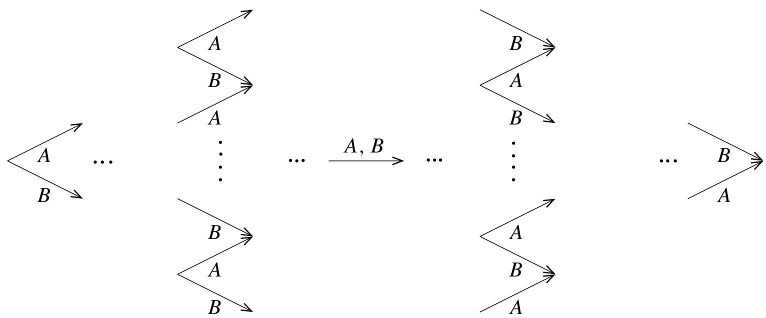

Figure 2. Stable representations for $A_{Q_{\mathrm{con}}}$ (or $A_{Q_{\mathrm{Kron}}}$) of dimension vector $(1, 2)$, $(n, n{+}1)$, $\ldots$, $(n{+}1, n)$, $(2, 1)$. The vertices represent a set of basis elements for the underlying vector space, while the labelled arrows represent the action of the homomorphism, labelled by those arrows, on this basis. The $(1, 1)$-dimensional representation in the centre of the figure lies in a family parametrised by $\mathbb{P}^1$.

By Lemma 5.1 and Remark 2.2, the semistable nilpotent modules of A_{Q_{-2},W_d} are given by iterated extension of stable $A_{Q_{\mathrm{con}},W_{\mathrm{con}}}$-modules, considered as A_{Q_{-2},W_d} modules by extension by zero. The stable nilpotent modules for $A_{Q_{\mathrm{con}},W_{\mathrm{con}}}$ are classified in [Nagao and Nakajima 2011, Theorem 3.5]. We have drawn a few of them in Figure 2. There is one stable nilpotent module for each slope equal to $\zeta((n, n+1))$ or $\zeta((n+1, n))$, for $n \in \mathbb{N}$ — consider the vertices in Figure 2 as a basis, then the arrows demonstrate the action of the morphisms assigned to A and B on this basis. These stable modules have dimension vector $(n, n+1)$ or $(n+1, n)$ respectively. For the slope $\zeta((1, 1))$, the stable nilpotent modules are all of dimension vector $(1, 1)$, and are parametrised by $\mathbb{P}^1$.

Recall from (12) that we denote by $\mathcal{X}^{\mathrm{nilp}}_{Q_{-2},W_d}$ the substack of finite-dimensional A_{Q_{-2},W_d}-modules cut out by the equations $\mathrm{tr}(\rho(c)) = 0$ for all cyclic paths $c \in \mathbb{C}Q_{-2}$. The isomorphism classes of closed points of this stack are in bijection with the isomorphism classes of nilpotent A_{Q_{-2},W_d}-modules. We use the familiar identity[11] in the Hall algebra $\mathrm{K}(\mathrm{St}^{\mathrm{aff}}/\mathcal{X}^{\mathrm{nilp}}_{Q_{-2},W_d})$ defined in Section 3B, where we abuse notation by omitting the obvious inclusion morphisms into $\mathcal{X}^{\mathrm{nilp}}_{Q_{-2},W_d}$,

$$\prod_{\text{decreasing slope } \theta} [\mathcal{X}^{\mathrm{nilp},\zeta-\mathrm{ss}}_{Q_{-2},W_d,\theta}] = [\mathcal{X}^{\mathrm{nilp}}_{Q_{-2},W_d}]. \tag{20}$$

Here $\mathcal{X}^{\mathrm{nilp},\zeta-\mathrm{ss}}_{Q_{-2},W_d,\theta}$ is the moduli stack of semistable nilpotent modules with slope θ, and the product is the Hall algebra product defined by Kontsevich and Soibelman on $\mathrm{K}(\mathrm{St}^{\mathrm{aff}}/\mathcal{X}^{\mathrm{nilp}}_{Q_{-2},W_d})$ (see Section 3B, and especially Remark 3.8).

[11] This identity is just a fancy way of stating the existence and uniqueness of Harder–Narasimhan filtrations.

Proposition 5.2. *In* $\overline{\mathrm{K}}^{\hat{\mu}}(\mathrm{St}^{\mathrm{aff}}/\operatorname{Spec}(\mathbb{C}))[\mathbb{L}^{1/2}][\![\hat{e}_{\boldsymbol{n}} \mid \boldsymbol{n} \in \mathbb{N}^{\mathrm{V}(Q_{-2})}]\!]$ *the equation of generating series*

$$\Phi_{Q_{-2},W_d}([\mathcal{X}^{\mathrm{nilp},\zeta-\mathrm{ss}}_{Q_{-2},W_d,\theta}]) = \operatorname{Sym}\left(\frac{\mathbb{L}^{-1/2}(1-[\mu_{d+1}])}{\mathbb{L}^{1/2}-\mathbb{L}^{-1/2}}\,\hat{e}_{\boldsymbol{n}}\right) \tag{21}$$

holds for $\arg(\zeta(\boldsymbol{n})) = \theta$, *and* $\boldsymbol{n} = (n, n+1)$ *or* $\boldsymbol{n} = (n+1, n)$ *with* $n \in \mathbb{N}$.

In (21) we consider $[\mu_{d+1}]$ as a μ_{d+1}-equivariant motive via the multiplication action.

Proof. The statement reduces to the computation of $W_{\min}$ and the orientation data above the unique $\boldsymbol{n}$-dimensional semistable module, for each $\boldsymbol{n}$ of slope θ. Note that on the geometric side of the derived equivalence (4) the unique stable module M of slope θ is given by $\mathcal{O}_C(a)[b]$ for some $a, b \in \mathbb{Z}$. Since there is a derived autoequivalence of $\mathrm{D}^{\mathrm{b}}(\operatorname{Coh}(Y_d))$ taking $\mathcal{O}_C(a)[b]$ to $\mathcal{O}_C$, we deduce that

$$\dim(\operatorname{Ext}^1(M, M)) = \begin{cases} 1 & \text{if } d \geq 2, \\ 0 & \text{otherwise,} \end{cases}$$

and in the case $d \geq 2$, $W_{\min}$ is given by x^{d+1}. Both of these facts follow since the universal deformation of $\mathcal{O}_C$ is over the Artinian ring $\mathbb{C}[x]/(x^d)$. The 3-Calabi–Yau category of semistable A_{Q_{-2},W_d}-modules of slope θ is, then, quasi-isomorphic as a cyclic A_∞-category to the category $\mathtt{tw}_0(Q^1_L, X^{d+1})$, where for $a \in \mathbb{N}$, Q^a_L is the quiver with one vertex and a loops.

We claim that the orientation data τ_{Q_{-2},W_d} over M, considered as a point in $\mathcal{X}^{\mathrm{nilp}}_{Q_{-2},W_d}$, is trivial if and only if $d \geq 2$. For this, note that there are precisely two isomorphism classes of orientation data over a point; given two super line bundles $\mathcal{V}_1$ and $\mathcal{V}_2$ over a point, i.e., vector spaces with parity, and isomorphisms $\mathcal{V}_i^{\otimes 2} \xrightarrow{\eta_i} \mathbb{C}$, there is an isomorphism $\mathcal{V}_1 \xrightarrow{f} \mathcal{V}_2$ such that $\eta_2 \circ (f \otimes f) = \eta_1$ if and only if the parity of $\mathcal{V}_1$ is the same as that of $\mathcal{V}_2$. It is sufficient, then, to show that $\dim(\operatorname{End}^1_{\mathtt{tw}(\mathcal{D}(Q_{-2},W_d))}(M)/\operatorname{Ker}(b_{\mathtt{tw}(\mathcal{D}(Q_{-2},W_d)),1}))$ is even if and only if $d \geq 2$. This follows from the following congruences modulo 2:

$$\dim \operatorname{Hom}^0_{\mathtt{tw}(\mathcal{D}(Q_{-2},W_d))}(M, M) \equiv \boldsymbol{n}(0)^2 + \boldsymbol{n}(1)^2 \equiv 1, \tag{22}$$

$$\dim \operatorname{Hom}^1_{\mathtt{tw}(\mathcal{D}(Q_{-2},W_d))}(M, M) \equiv 4\boldsymbol{n}(0)\boldsymbol{n}(1) + \boldsymbol{n}(0)^2 + \boldsymbol{n}(1)^2 \equiv 1, \tag{23}$$

$$\dim \operatorname{Ext}^0_{\mathtt{tw}(\mathcal{D}(Q_{-2},W_d))}(M, M) \equiv 1. \tag{24}$$

The first two identities follow from the definitions of homomorphism spaces in $\mathtt{tw}(\mathcal{D}(Q_{-2}, W_d))$, and the identity (24) follows from the fact that M is stable and hence simple. We then calculate, modulo 2:

$$\begin{aligned}
1 &\equiv \dim \mathrm{Hom}^1_{\mathtt{tw}(\mathcal{D}(Q_{-2},W_d))}(M,M)\\
&\equiv \dim \mathrm{Image}(b_{\mathtt{tw}(\mathcal{D}(Q_{-2},W_d)),0}) + \dim \mathrm{Ext}^1_{\mathtt{tw}(\mathcal{D}(Q_{-2},W_d))}(M,M)\\
&\qquad + \dim\bigl(\mathrm{End}^1_{\mathtt{tw}(\mathcal{D}(Q_{-2},W_d))}(M) \,/\, \mathrm{Ker}(b_{\mathtt{tw}(\mathcal{D}(Q_{-2},W_d)),1})\bigr)\\
&\equiv \dim \mathrm{Hom}^0_{\mathtt{tw}(\mathcal{D}(Q_{-2},W_d))}(M,M) - \dim \mathrm{Ext}^0_{\mathtt{tw}(\mathcal{D}(Q_{-2},W_d))}(M,M)\\
&\qquad + \dim \mathrm{Ext}^1_{\mathtt{tw}(\mathcal{D}(Q_{-2},W_d))}(M,M)\\
&\qquad + \dim\bigl(\mathrm{End}^1_{\mathtt{tw}(\mathcal{D}(Q_{-2},W_d))}(M) \,/\, \mathrm{Ker}(b_{\mathtt{tw}(\mathcal{D}(Q_{-2},W_d)),1})\bigr)\\
&\equiv \dim \mathrm{Ext}^1_{\mathtt{tw}(\mathcal{D}(Q_{-2},W_d))}(M,M)\\
&\qquad + \dim\bigl(\mathrm{End}^1_{\mathtt{tw}(\mathcal{D}(Q_{-2},W_d))}(M) \,/\, \mathrm{Ker}(b_{\mathtt{tw}(\mathcal{D}(Q_{-2},W_d)),1})\bigr).
\end{aligned}$$

Thus,

$$\dim\bigl(\mathrm{End}^1_{\mathtt{tw}(\mathcal{D}(Q_{-2},W_d))}(M) \,/\, \mathrm{Ker}(b_{\mathtt{tw}(\mathcal{D}(Q_{-2},W_d)),1})\bigr) \equiv \begin{cases} 0 & \text{if } d \geq 2,\\ 1 & \text{otherwise.}\end{cases}$$

Now one can see directly that the orientation data assigned to the unique simple object s_0 of the category $\mathtt{tw}_0(\mathcal{D}(Q^1_L, X^{d+1}))$ is trivial if and only if $d \geq 2$, since

$$b_1 : \mathrm{End}^1_{\mathtt{tw}(\mathcal{D}(Q^1_L))}(s_0) \to \mathrm{End}^2_{\mathtt{tw}(\mathcal{D}(Q^1_L))}(s_0)$$

is trivial if $d \geq 2$; otherwise this differential is an isomorphism. So as well as having a cyclic A_∞-isomorphism Ξ from the subcategory of $\mathtt{tw}_0(Q_{-2}, W_d)$ generated by M under extensions to the category $\mathtt{tw}_0(Q^1_L, W^{d+1})$, we have an isomorphism of orientation data $\Xi^*(\tau_{Q^1_L, X^{d+1}}) \cong \tau_{Q_{-2},W_d}$. It follows that

$$\begin{aligned}
\Phi_{Q_{-2},W_d}([\mathcal{X}^{\mathrm{nilp},\zeta-\mathrm{ss}}_{Q_{-2},W_d,\theta}]) &= \Phi_{Q^1_L,X^{d+1}}([\mathcal{X}^{\mathrm{nilp}}_{Q^1_L,X^{d+1}}])|_{\hat{e}_a\mapsto\hat{e}_{an}}\\
&= \Phi_{\mathtt{BBS},Q^1_L,X^{d+1}}([\mathcal{X}^{\mathrm{nilp}}_{Q^1_L,X^{d+1}}])|_{\hat{e}_a\mapsto\hat{e}_{an}}\\
&= \Phi_{\mathtt{BBS},Q^1_L,X^{d+1}}([\mathcal{X}_{Q^1_L,X^{d+1}}])|_{\hat{e}_a\mapsto\hat{e}_{an}}
\end{aligned}$$

where for the penultimate equation we used Proposition 4.8, and for the final equation we use the fact that all finite-dimensional $A_{Q^1_L,X^{d+1}}$-modules are nilpotent. The desired equality is then [Davison and Meinhardt 2015, Theorem 6.2]. □

Proposition 5.3. *In* $\overline{\mathrm{K}}^{\hat{\mu}}(\mathrm{St}^{\mathrm{aff}}/\,\mathrm{Spec}(\mathbb{C}))[\mathbb{L}^{1/2}][\![\hat{e}_{\boldsymbol{n}} \mid \boldsymbol{n} \in \mathbb{N}^{\mathrm{V}(Q_{-2})}]\!]$ *the equation of generating series*

$$\Phi_{Q_{-2},W_d}([\mathcal{X}^{\mathrm{nilp},\zeta-\mathrm{ss}}_{Q_{-2},W_d,\theta}]) = \mathrm{Sym}\biggl(\sum_{n\geq 1} \frac{\mathbb{L}^{-1/2}+\mathbb{L}^{-3/2}}{\mathbb{L}^{1/2}-\mathbb{L}^{-1/2}}\,\hat{e}_{(n,n)}\biggr)$$

holds for $\arg(\zeta((1,1))) = \theta$.

Proof. The simple stable nilpotent modules M with dimension vector $(1, 1)$ are given by choosing two linear maps $M(A)$ and $M(B)$, from $\mathbb{C}$ to $\mathbb{C}$, not both equal to zero. These modules correspond to the structure sheaves of points on the exceptional curve $C_d \subset Y_d$ under the derived equivalence (4). Let $\mathbb{A}_n^\circ$ be the subscheme of $\prod_{a\in \mathrm{E}(Q_{-2})} \mathrm{Hom}(\mathbb{C}^{\boldsymbol{n}(t(a))}, \mathbb{C}^{\boldsymbol{n}(s(a))})$, for $\boldsymbol{n} = (n, n)$, the points of which satisfy the condition that the linear map assigned to A is an isomorphism, and the Harder–Narasimhan filtration of the associated module only contains modules with dimension vector $(1, 1)$. The action of $\mathrm{GL}_{\mathbb{C}}(\boldsymbol{n}(1))$ on $\mathbb{A}_n^\circ$ is free. Taking the quotient by this action corresponds to forgetting the data of the isomorphism A, and identifying the two vertices of the quiver Q_{-2}, and so

$$\mathbb{A}_n^\circ \,/\, (\mathrm{GL}_{\mathbb{C}}(n) \times \mathrm{GL}_{\mathbb{C}}(n)) \cong \mathcal{Y}_{Q_L^5, n},$$

where Q_L^5 is the five loop quiver, with loops labelled B, C, D, X, Y. Furthermore, under the open inclusion $\mathcal{Y}_{Q_L^5,n} \hookrightarrow \mathcal{Y}_{Q_{-2},(n,n)}$, the function $\mathrm{tr}(W_d)$ pulls back to the function $\mathrm{tr}(W_d^\circ)$, where W_d° is the superpotential

$$W_d^\circ = \frac{1}{d+1} X^{d+1} - \frac{1}{d+1} Y^{d+1} - XC + XDB + YC - YBD.$$

If we define $\mathcal{X}^{\mathrm{nilp},\zeta-\mathrm{ss},\circ}_{Q_{-2},W_d,\boldsymbol{n}}$ to be the substack of $\mathcal{X}_{Q_{-2},W_d,\boldsymbol{n}}$, the points of which are ζ-semistable nilpotent A_{Q_{-2},W_d}-modules M such that $\theta(A)$ is an isomorphism, then we have shown that the stack $\mathcal{X}^{\mathrm{nilp},\zeta-\mathrm{ss},\circ}_{Q_{-2},W_d,\boldsymbol{n}}$ is naturally a substack of $\mathcal{X}_{Q_L^5,W_d^\circ,n} \subset \mathcal{Y}_{Q_L^5,n}$, and we identify it as the stack of n-dimensional representations for the Jacobi algebra associated to (Q_L^5, W_d°) such that all loops apart from B act via nilpotent linear maps. We denote

$$\mathcal{X}^{\mathrm{nilp},\zeta-\mathrm{ss},\circ}_{Q_{-2},W_d} = \coprod_{n\in\mathbb{N}} \mathcal{X}^{\mathrm{nilp},\zeta-\mathrm{ss},\circ}_{Q_{-2},W_d,(n,n)}. \tag{25}$$

Under the derived equivalence (4), the stack (25) is the substack of coherent sheaves supported on the exceptional curve $C_d \subset Y_d$, away from a fixed point. We will denote this point by p.

Denote by $\mathcal{X}^{\mathrm{nilp},\zeta-\mathrm{ss},p}_{Q_{-2},W_d}$ the stack of modules for A_{Q_{-2},W_d} which are supported at the point p under the derived equivalence (4). Then since every sheaf that is scheme-theoretically supported on C_d with zero-dimensional support splits uniquely as a direct sum of a coherent sheaf supported at p and a coherent sheaf supported away from p, there is an identity in the motivic Hall algebra $(\mathrm{K}(\mathrm{St}^{\mathrm{aff}}/\mathcal{X}_{Q_{-2},W_d}), \star_{\mathrm{KS}})$

$$[\mathcal{X}^{\mathrm{nilp},\zeta-\mathrm{ss}}_{Q_{-2},W_d,\theta}] = [\mathcal{X}^{\mathrm{nilp},\zeta-\mathrm{ss},\circ}_{Q_{-2},W_d}] \star_{\mathrm{KS}} [\mathcal{X}^{\mathrm{nilp},\zeta-\mathrm{ss},p}_{Q_{-2},W_d}]. \tag{26}$$

Now note that there is a splitting

$$W_d^\circ = XDB - XBD + (X - Y)\Big(BD - C + \frac{1}{d+1}(X^d + X^{d-1}Y + \cdots + Y^d)\Big).$$

We deduce that after giving $Y_{Q_L^5,n}$ the coordinates X, D, B, $Y'=X-Y$, and

$$C' = BD - C + \frac{1}{d+1}\Big(X^d + X^{d-1}Y + \cdots + Y^d\Big),$$

we have

$$\begin{aligned}
&\Phi_{Q_{-2},W_d}(\mathcal{X}^{\mathrm{nilp},\zeta-\mathrm{ss},\circ}_{Q_{-2},W_d}) \\
&\quad = \sum_{n\geq 0}\bigg(\int_{\mathcal{X}^{\mathrm{nilp},\zeta-\mathrm{ss},\circ}_{Q_{-2},W_d,(n,n)}\subset\mathcal{Y}_{Q_{-2},(n,n)}} \phi_{\mathrm{tr}(W_d)}\bigg)\hat{e}_{(n,n)} \qquad (27)\\
&\quad = \sum_{n\geq 0}[\mathrm{GL}_{\mathbb{C}}(n)]^{-1}\mathbb{L}^{n^2/2}\bigg(\int_{\{X,Y',C',D\text{ nilpotent}\}\subset\mathbb{A}_{Q_L^5,n}} \phi_{\mathrm{tr}(XDB-XBD)\boxplus\mathrm{tr}(Y'C')}\bigg)\hat{e}_{(n,n)} \\
&\quad = \sum_{n\geq 0}[\mathrm{GL}_{\mathbb{C}}(n)]^{-1}\mathbb{L}^{n^2/2}\bigg(\int_{\{X\text{ and }D\text{ nilpotent}\}\subset\mathbb{A}_{Q_L^3,n}} \phi_{\mathrm{tr}(XDB-XBD)}\bigg)\hat{e}_{(n,n)}. \qquad (28)
\end{aligned}$$

Here (27) comes from the comparison theorem (Proposition 4.8), and (28) comes from applying the motivic Thom–Sebastiani theorem. Now, giving the coordinates X, D, B weights 0, 0, and 1, respectively, and applying the weight 1 version of Conjecture 5.5 (which is a theorem), with Z' the scheme of pairs of matrices labelled by X and D, we obtain

$$\begin{aligned}
&\Phi_{Q_{-2},W_d}(\mathcal{X}^{\mathrm{nilp},\zeta-\mathrm{ss},\circ}_{Q_{-2},W_d}) \qquad (29)\\
&\quad = \sum_{n\geq 0}[\mathrm{GL}_{\mathbb{C}}(n)]^{-1}\mathbb{L}^{-n^2}\big(\{X\text{ and }D\text{ nilpotent, }XD\neq DX\}(\mathbb{L}^{n^2-1}-\mathbb{L}^{n^2-1}) \\
&\qquad\qquad - \{X\text{ and }D\text{ nilpotent, }XD=DX\}\mathbb{L}^{n^2}\big)\hat{e}_{(n,n)} \\
&\quad = \sum_{n\geq 0}[\mathrm{GL}_{\mathbb{C}}(n)]^{-1}\,\mathrm{C}_{n,\mathrm{nilp}}\,\hat{e}_{(n,n)} \\
&\quad = \sum_{n\geq 0}\mathcal{C}_{n,\mathrm{nilp}}\hat{e}_{(n,n)} \\
&\quad = \bigg(\sum_{n\geq 0}\mathcal{C}_n\hat{e}_{(n,n)}\bigg)^{-\mathbb{L}^2} \qquad (30)\\
&\quad = \mathrm{Sym}\bigg(\sum_{n\geq 1}\frac{\mathbb{L}^{-1/2}}{\mathbb{L}^{1/2}-\mathbb{L}^{-1/2}}\hat{e}_{(n,n)}\bigg) \qquad (31)
\end{aligned}$$

where C_n is the variety of pairs of commuting $n\times n$ matrices, and $\mathcal{C}_n$ is its quotient under the conjugation action of $\mathrm{GL}_{\mathbb{C}}(n)$. One can think of this stack as the stack of length n coherent sheaves on $\mathbb{C}^2$. Above, $\mathrm{C}_{n,\mathrm{nilp}}$ and $\mathcal{C}_{n,\mathrm{nilp}}$ are the variety and stack, respectively, of nilpotent commuting matrices, and one should think of $\mathcal{C}_{n,\mathrm{nilp}}$ as the stack of coherent sheaves on $\mathbb{C}^2$ scheme-theoretically supported at the origin. Then (30) follows from the definition of the power structure in Section 3A, and (31) follows from the main result of [Feit and Fine 1960], as in [Behrend et al. 2013, Proposition 1.1]. Similarly one deduces that

$$\Phi_{Q_{-2},W_d}(\mathcal{X}^{\mathrm{nilp},\zeta-\mathrm{ss},p}_{Q_{-2},W_d}) = \mathrm{Sym}\left(\sum_{n\geq 1}\frac{\mathbb{L}^{-3/2}}{\mathbb{L}^{1/2}-\mathbb{L}^{-1/2}}\hat{e}_{(n,n)}\right)$$

and now the result follows from applying the integration map to the Hall algebra identity (26). □

The following theorem now follows from applying the integration map to the Harder–Narasimhan identity (20) in $\mathrm{K}(\mathrm{St}^{\mathrm{aff}}/\mathcal{X}^{\mathrm{nilp}}_{Q_{-2},W_d})$.

Theorem 5.4. *There is an equality in* $\overline{\mathrm{K}}^{\hat{\mu}}(\mathrm{St}^{\mathrm{aff}}/\operatorname{Spec}(\mathbb{C}))[\mathbb{L}^{1/2}][\![\hat{e}_{\boldsymbol{n}} \mid \boldsymbol{n}\in\mathbb{N}^{\mathrm{V}(Q_{-2})}]\!]$:

$$\begin{aligned}&\Phi_{Q_{-2},W_d}(\mathcal{X}^{\mathrm{nilp}}_{Q_{-2},W_d})\\ &\quad=\mathrm{Sym}\left(\sum_{n\geq 0}\frac{\mathbb{L}^{-1/2}(1-[\mu_{d+1}])}{\mathbb{L}^{1/2}-\mathbb{L}^{-1/2}}(\hat{e}_{(n,n+1)}+\hat{e}_{(n+1,n)})+\sum_{n\geq 1}\frac{\mathbb{L}^{-1/2}+\mathbb{L}^{-3/2}}{\mathbb{L}^{1/2}-\mathbb{L}^{-1/2}}\hat{e}_{(n,n)}\right).\end{aligned}$$

In particular, the motivic Donaldson–Thomas invariants $\Omega^{\mathrm{nilp}}_{\zeta}$ *counting nilpotent* A_{Q_{-2},W_d}*-modules (for any* ζ*) are given by*

$$\Omega^{\mathrm{nilp}}_{\zeta}(\boldsymbol{n})=\begin{cases}(1-[\mu_{d+1}])\cdot\mathbb{L}^{-1/2} & \text{if there exists } n\in\mathbb{N} \text{ such that}\\ & \qquad \boldsymbol{n}=(n,n+1) \text{ or } \boldsymbol{n}=(n+1,n),\\ \mathbb{P}^1\cdot\mathbb{L}^{-3/2} & \text{if there exists } n\in\mathbb{N} \text{ such that } \boldsymbol{n}=(n,n).\end{cases}$$

5B. *Calculation using equivariant vanishing cycles.* We repeat the above calculations, but this time the other side of the comparison theorem (Proposition 4.8). It is more natural there to work out the Donaldson–Thomas invariants for the category of finite-dimensional A_{Q_{-2},W_d}-modules, not just the nilpotent ones. First we recall the following conjecture from [Davison and Meinhardt 2015].

Conjecture 5.5. Let Z' be a smooth scheme with trivial $\mathbb{G}_m$-action, and let $\mathbb{A}^n_{\mathbb{C}}$ carry a $\mathbb{G}_m$-action with nonnegative weights. Let $Z=Z'\times\mathbb{A}^n_{\mathbb{C}}$ with the induced $\mathbb{G}_m$-action, and let $f:Z\to\mathbb{A}^1_{\mathbb{C}}$ be a $\mathbb{G}_m$-equivariant function, with $\mathbb{G}_m$ acting on the target with weight $s>0$. Then there is an equality in $\mathrm{K}^{\hat{\mu}}(\mathrm{St}^{\mathrm{aff}}/Z')$

$$q_*\phi_f=\mathbb{L}^{-\dim(Z)/2}([f^{-1}(0)]-[f^{-1}(1)]),\tag{32}$$

where $f^{-1}(0)$ carries the trivial $\hat{\mu}$-action, and the $\hat{\mu}$-action on $f^{-1}(1)$ is given by the natural μ_s-action, and both are considered as varieties over Z' via the projection $q:Z\to Z'$. Equivalently, there is an identity in $\lim_{n\to\infty}\mathrm{K}^{\mathbb{G}_m,n}(\mathrm{St}^{\mathrm{aff}}/\mathbb{A}^1_{Z'})$

$$q_*\phi_f=\mathbb{L}^{-\dim(Z)/2}[Z\xrightarrow{f\times q}\mathbb{A}^1_{Z'}].\tag{33}$$

This conjecture follows from the proof of Theorem 5.9 in [Davison and Meinhardt 2015], under the assumption that the weights on $\mathbb{A}^n_{\mathbb{C}}$ are all at most 1. If, in addition

$s=1$, then [Davison and Meinhardt 2015, Theorem 5.9] is a result of [Behrend et al. 2013, Propoposition 1.11]. While this paper was being prepared for publication, we were informed that Johannes Nicaise and Sam Payne have a strategy for proving the general case based on tropical geometry and Hrushovski–Kazhdan motivic integration.

Theorem 5.6. *For $d \leq 2$, there is an identity*

$$\begin{aligned}&\Phi_{\mathrm{BBS},Q_{-2},W_d}([\mathcal{X}_{Q_{-2},W_d}])\\ &\quad= \mathrm{Sym}\Bigg(\sum_{n\geq 0}\frac{\mathbb{L}^{-1/2}(1-[\mu_{d+1}])}{\mathbb{L}^{1/2}-\mathbb{L}^{-1/2}}(\hat{e}_{(n,n+1)}+\hat{e}_{(n+1,n)})+\sum_{n\geq 1}\frac{\mathbb{L}^{3/2}+\mathbb{L}^{1/2}}{\mathbb{L}^{1/2}-\mathbb{L}^{-1/2}}\hat{e}_{(n,n)}\Bigg)\end{aligned}$$

in $\mathrm{K}^{\hat{\mu}}(\mathrm{St}^{\mathrm{aff}}/\operatorname{Spec}(\mathbb{C}))[\mathbb{L}^{1/2}][\![\hat{e}_{\boldsymbol{n}} \mid \boldsymbol{n} \in \mathbb{N}^{\mathrm{V}(Q_{-2})}]\!]$. Assuming Conjecture 5.5 this identity holds for all d. It follows that the motivic Donaldson–Thomas invariants $\Omega_\zeta(\boldsymbol{n})$ (which do not depend on ζ) are given by

$$\Omega_\zeta(\boldsymbol{n}) = \begin{cases}(1-[\mu_{d+1}])\cdot\mathbb{L}^{-1/2} & \text{if there exists } n\in\mathbb{N} \text{ such that}\\ & \qquad \boldsymbol{n}=(n,n+1) \text{ or } \boldsymbol{n}=(n+1,n),\\ [Y_d]_{\mathrm{virt}} := \mathbb{L}^{-\dim(Y_d)/2}\cdot[Y_d] & \text{if there exists } n\in\mathbb{N} \text{ such that } \boldsymbol{n}=(n,n).\end{cases}$$

The explicit description given in Section 2 shows that Y_d is a Zariski locally trivial fibre bundle over the exceptional curve $C_d \cong \mathbb{P}^1_{\mathbb{C}}$ with fibre $\mathbb{A}^2_{\mathbb{C}}$, and so

$$[Y_d] = (\mathbb{L}^1+1)\mathbb{L}^2 \quad \text{and} \quad [Y_d]_{\mathrm{virt}} = \mathbb{L}^{3/2}+\mathbb{L}^{1/2}$$

in $\mathrm{K}^{\hat{\mu}}(\mathrm{St}^{\mathrm{aff}}/\operatorname{Spec}(\mathbb{C}))$. The transition functions are linear only for $d=1$. In particular, although $Y_d \not\cong Y_{d'}$ for $d \neq d'$, the classes $[Y_d]$ and $[Y_d]_{\mathrm{virt}}$ do not depend on d.

Proof. For $\beta = (\beta_n, \dots, \beta_1)$ a path in a quiver Q, and for

$$M \in \mathbb{A}_{Q,\boldsymbol{n}} := \prod_{a\in \mathrm{E}(Q)} \mathrm{Hom}(\mathbb{C}^{\boldsymbol{n}(t(a))}, \mathbb{C}^{\boldsymbol{n}(s(a))})$$

we write $M(\beta) = M(\beta_n)\circ\cdots\circ M(\beta_1)$. We let $\mathbb{G}_m$ act on $\mathbb{A}_{Q_{-2},\boldsymbol{n}}$ via

$$(z\cdot M)(E) = z^{\iota(E)}\cdot M(E),$$

where $\iota(E)=1$ if $E = X, Y, A, B$, and $\iota(E) = d-1$ if $E = C, D$. Then

$$\mathrm{tr}(W_d) : Y_{Q_{-2},\boldsymbol{n}} \to \mathbb{A}^1_{\mathbb{C}}$$

is $\mathbb{G}_m$-equivariant, after giving $\mathbb{A}^1_{\mathbb{C}}$ the weight $(d+1)$-action. It follows from our assumption $d \leq 2$, or from Conjecture 5.5, for general d, that

$$\begin{aligned}\int_{\mathcal{X}_{Q_{-2},W_d},\boldsymbol{n}} \phi_{\mathrm{tr}(W_d)} &= \int_{\mathcal{Y}_{Q_{-2}},\boldsymbol{n}} \phi_{\mathrm{tr}(W_d)}\\ &= [\mathbb{A}_{Q_{-2},\boldsymbol{n}} \xrightarrow{\mathrm{tr}(W_d)} \mathbb{A}^1_{\mathbb{C}}]\cdot\mathbb{L}^{-2\boldsymbol{n}(0)\boldsymbol{n}(1)}\cdot[\mathrm{GL}_{\mathbb{C}}(\boldsymbol{n}(0))\times\mathrm{GL}_{\mathbb{C}}(\boldsymbol{n}(1))]^{-1} \quad (34)\end{aligned}$$

in $\mathrm{K}^{\mathbb{G}_m,d+1}(\mathrm{St}^{\mathrm{aff}}/\mathbb{A}^1_{\mathbb{C}})$ (in fact, all subsequent calculations will take place in this ring). The first of these equalities follows from the fact that the motive $\phi_{\mathrm{tr}(W_d)}$ is supported on the critical locus of $\mathrm{tr}(W_d)$, which is just $\mathcal{X}_{Q_{-2},W_d,\boldsymbol{n}}$.

For a set of edges $E' \subset E(Q)$ let $Q \setminus E'$ be the quiver obtained by deleting the edges of E' (this quiver has the same vertex set as Q). If W is a potential on $\mathbb{C}Q$, we denote by $W \setminus E'$ the potential on $Q \setminus E'$ obtained by changing the coefficient of any term in W containing any edge of E' to zero. By abuse of notation we will often denote the potential $W \setminus E'$ on $Q \setminus E'$ by W. There is a natural projection

$$\pi_C : \mathbb{A}_{Q_{-2},\boldsymbol{n}} \to \mathbb{A}_{Q_{-2}\setminus\{C\},\boldsymbol{n}}$$

given by forgetting the data $M(C)$. We consider this as the projection from the total space of a rank $\boldsymbol{n}(0)\cdot\boldsymbol{n}(1)$ vector bundle which we denote $\tilde{C}$. There is an obvious equality

$$\begin{aligned}[\mathbb{A}_{Q_{-2},\boldsymbol{n}} \xrightarrow{\mathrm{tr}(W_d)} \mathbb{A}^1_{\mathbb{C}}] = [\pi_C^{-1}\mathbb{A}_{Q_{-2}\setminus\{C\},\boldsymbol{n},M(AX)=M(YA)} \xrightarrow{\mathrm{tr}(W_d)} \mathbb{A}^1_{\mathbb{C}}] \\ + [\pi_C^{-1}\mathbb{A}_{Q_{-2}\setminus\{C\},\boldsymbol{n},M(AX)\neq M(YA)} \xrightarrow{\mathrm{tr}(W_d)} \mathbb{A}^1_{\mathbb{C}}].\end{aligned}$$

The restriction of the vector bundle $\tilde{C}$ to $\mathbb{A}_{Q_{-2}\setminus\{C\},\boldsymbol{n},M(AX)\neq M(YA)}$ has a subbundle $\tilde{C}_0$ of rank $(\boldsymbol{n}(0)\cdot\boldsymbol{n}(1)-1)$, given by those $M(C)$ such that $\mathrm{tr}(M(CAX)-M(CYA))=0$. The action of $\mathbb{G}_m$ on $\mathbb{A}_{Q_{-2}\setminus\{C\},\boldsymbol{n},M(AX)\neq M(YA)}$ is free, and from the corresponding nonequivariant statement on the quotient we deduce that after $\mathbb{G}_m$-equivariant constructible decomposition of the base $\mathbb{A}_{Q_{-2}\setminus\{C\},\boldsymbol{n},M(AX)\neq M(YA)}$, the inclusion $\tilde{C}_0 \subset \tilde{C}|_{\mathbb{A}_{Q_{-2}\setminus\{C\},\boldsymbol{n},M(AX)\neq M(YA)}}$ splits, and we may write

$$[\pi_C^{-1}\mathbb{A}_{Q_{-2}\setminus\{C\},\boldsymbol{n},M(AX)\neq M(YA)} \xrightarrow{\mathrm{tr}(W_d)} \mathbb{A}^1_{\mathbb{C}}] = [\tilde{C}_0 \times \mathbb{A}^1_{\mathbb{C}} \xrightarrow{\mathrm{tr}(W_d\setminus C)+\pi_{\mathbb{A}^1_{\mathbb{C}}}} \mathbb{A}^1_{\mathbb{C}}] \qquad (35)$$

where we have abused notation by identifying $\tilde{C}_0$ with its constructible decomposition. After a change of coordinates we may write the right hand side of (35) as

$$[\tilde{C}_0 \times \mathbb{A}^1_{\mathbb{C}} \xrightarrow{\pi_{\mathbb{A}^1_{\mathbb{C}}}} \mathbb{A}^1_{\mathbb{C}}].$$

Clearly this belongs to $\mathfrak{I}_{d+1}$. We deduce that

$$\begin{aligned}[\mathbb{A}_{Q_{-2},\boldsymbol{n}} \xrightarrow{\mathrm{tr}(W_d)} \mathbb{A}^1_{\mathbb{C}}] &= [\pi_C^{-1}\mathbb{A}_{Q_{-2}\setminus\{C\},\boldsymbol{n},M(AX)=M(YA)} \xrightarrow{\mathrm{tr}(W_d)} \mathbb{A}^1_{\mathbb{C}}] \\ &= \mathbb{L}^{\boldsymbol{n}(0)\cdot\boldsymbol{n}(1)} \cdot [\mathbb{A}_{Q_{-2}\setminus\{C\},\boldsymbol{n},M(AX)=M(YA)} \xrightarrow{\mathrm{tr}(W_d\setminus C)} \mathbb{A}^1_{\mathbb{C}}]\end{aligned}$$

and similarly

$$[\mathbb{A}_{Q_{-2},\boldsymbol{n}} \xrightarrow{\mathrm{tr}(W_d)} \mathbb{A}^1_{\mathbb{C}}] = \mathbb{L}^{2\boldsymbol{n}(0)\boldsymbol{n}(1)} \cdot [E_{Q_{\mathrm{Kron}},\boldsymbol{n}} \xrightarrow{\mathrm{tr}(X^{d+1}-Y^{d+1})} \mathbb{A}^1_{\mathbb{C}}], \qquad (36)$$

where we define

$$E_{Q_{\mathrm{Kron}},\boldsymbol{n}} := (\mathbb{A}_{Q_{-2}\setminus\{C,D\},\boldsymbol{n}})\Big|_{\substack{M(AX)=M(YA)\\ M(BX)=M(YB)}}$$

and define the stacks

$$\begin{aligned}\mathcal{E}_{Q_{\mathrm{Kron}},\boldsymbol{n}} &:= E_{Q_{\mathrm{Kron}},\boldsymbol{n}} \,/\, (\mathrm{GL}_{\mathbb{C}}(\boldsymbol{n}(0)) \times \mathrm{GL}_{\mathbb{C}}(\boldsymbol{n}(1)))\\ \mathcal{E}_{Q_{\mathrm{Kron}}} &:= \coprod_{\boldsymbol{n}\in\mathbb{N}^2} \mathcal{E}_{Q_{\mathrm{Kron}},\boldsymbol{n}}.\end{aligned}$$

These stacks represent pairs (M, ξ), where M is a right $A_{Q_{\mathrm{Kron}}}$-module, and $\xi = X + Y$ is an endomorphism of M, where Q_{Kron} is the Kronecker quiver with two vertices x_0 and x_1, and two arrows A and B, both going from x_0 to x_1. In other words, $\mathcal{E}_{Q_{\mathrm{Kron}}}$ is the stack of finite-dimensional $\mathcal{B} := A_{Q_{\mathrm{Kron}}}[z]$-modules. By Beilinson's theorem $\mathrm{D}^{\mathrm{b}}(\mathtt{Mod}\text{-}A_{Q_{\mathrm{Kron}}})$ is derived equivalent to the category of coherent sheaves on $\mathbb{P}^1$ via the derived equivalence $\mathrm{RHom}(E, -)$, where

$$E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1).$$

Similarly, $\mathrm{D}^{\mathrm{b}}(\mathtt{Mod}\text{-}\mathcal{B})$ is derived equivalent to the category of coherent sheaves on $\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1})$ via the derived equivalence $\mathrm{RHom}(\pi^*E, -)$, where

$$\pi : \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}) \to \mathbb{P}^1$$

is the projection. We claim that for M_α and M_β two semistable $\mathcal{B}$-modules with the slope of M_α lower than that of M_β, the following group vanishes

$$\mathrm{Ext}^2_{\mathtt{Mod}\text{-}\mathcal{B}}(M_\alpha, M_\beta) = 0.$$

By the five lemma and the existence of Jordan–Hölder filtrations it is enough to prove the claim in the case in which both M_α and M_β are stable. By Serre duality, and the above derived equivalence, this is equivalent to showing that

$$\mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2(-2)) = 0$$

for $\mathcal{F}_1$ occurring before $\mathcal{F}_2$ in the ordered collection of objects of $\mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(\mathbb{P}^1))$:

$$\mathcal{O}_{\mathbb{P}^1}(-1)[1], \mathcal{O}_{\mathbb{P}^1}(-2)[1], \ldots, \mathcal{O}_{\mathrm{pt}}, \ldots, \mathcal{O}_{\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1},$$

which is clear.

Let $\aleph_\zeta$ be the set of all possible Harder–Narasimhan types of finite-dimensional $\mathcal{B}$-modules. We could equally have defined $\aleph_\zeta$ as the set of all possible Harder–Narasimhan types of $A_{Q_{\mathrm{Kron}}}$-modules, since the endomorphism z in the definition of $\mathcal{B}$ preserves the Harder–Narasimhan filtration of the underlying $A_{Q_{\mathrm{Kron}}}$-module. Let $\gamma = (\boldsymbol{n}^1, \ldots, \boldsymbol{n}^s) \in \aleph_\zeta$, let

$$\mathcal{E}_{Q_{\mathrm{Kron}},\gamma}$$

be the stack of $\mathcal{B}$-modules of Harder–Narasimhan type γ, and let

$$\mathtt{JH}:\mathcal{E}_{Q_{\mathrm{Kron}},\gamma}\to\prod_{i\le s}\mathcal{E}^{\zeta-\mathrm{ss}}_{Q_{\mathrm{Kron}},\boldsymbol{n}^i}$$

be the map taking a module to its Jordan–Hölder filtration. Then above a module $M=M_1\oplus\cdots\oplus M_s$, the fibre of $\mathtt{JH}$ is given by a stack $[\mathbb{A}^m\,/\,\mathbb{A}^n]$, where

$$\begin{aligned}n&=\sum_{1\le i<j\le s}\dim(\mathrm{Hom}(M_j,M_i)),\\ m&=\sum_{1\le i<j\le s}\mathrm{Ext}^1(\mathrm{Hom}(M_j,M_i)).\end{aligned}$$

On the other hand, by the vanishing of Ext^2 groups, and the fact that the Euler form on $\mathrm{Coh}_{\mathrm{cpct}}(\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}))$ vanishes, each of the differences

$$\dim(\mathrm{Hom}(M_j,M_i))-\mathrm{Ext}^1(\mathrm{Hom}(M_j,M_i))$$

vanishes, and so $n=m$. We deduce from relation (1) that

$$[\mathcal{E}_{Q_{\mathrm{Kron}},\gamma}\xrightarrow{\mathrm{tr}(W_d)}\mathbb{A}^1_{\mathbb{C}}]=\prod_{0\le i\le s}[\mathcal{E}^{\zeta-\mathrm{ss}}_{Q_{\mathrm{Kron}},\boldsymbol{n}^i}\xrightarrow{\mathrm{tr}(W_d)}\mathbb{A}^1_{\mathbb{C}}].\tag{37}$$

If $\boldsymbol{n}^i$ is equal to $a\cdot(n,n\pm1)$ then

$$\begin{aligned}[\mathcal{E}^{\zeta-\mathrm{ss}}_{Q_{\mathrm{Kron}},\boldsymbol{n}^i}\xrightarrow{\mathrm{tr}(W_d)}\mathbb{A}^1_{\mathbb{C}}]&=[\mathrm{Mat}_{a\times a}(\mathbb{C})\,/\,\mathrm{GL}_{\mathbb{C}}(a)\xrightarrow{\mathrm{tr}(nX^{d+1}-(n\pm1)X^{d+1})}\mathbb{A}^1_{\mathbb{C}}]\\ &=[\mathrm{Mat}_{a\times a}(\mathbb{C})\,/\,\mathrm{GL}_{\mathbb{C}}(a)\xrightarrow{\mathrm{tr}(X^{d+1})}\mathbb{A}^1_{\mathbb{C}}].\end{aligned}$$

Similarly, if $\boldsymbol{n}^i=a\cdot(1,1)$, then the function $\mathrm{tr}(W_d)$ is zero, restricted to $\mathcal{E}^{\zeta-\mathrm{ss}}_{Q_{\mathrm{Kron}},\boldsymbol{n}^i}$. This is just the stack of length a coherent zero-dimensional sheaves on $\mathbb{P}^1$ with an endomorphism. It follows that

$$\sum_{a\ge0}[\mathcal{E}^{\zeta-\mathrm{ss}}_{Q_{\mathrm{Kron}},((a,a))}\to\mathbb{A}^1_{\mathbb{C}}]\hat{e}_{(a,a)}=\Bigl(\sum_{a\ge0}[\mathcal{E}^{\mathbb{C}}_{Q_{\mathrm{Kron}},((a,a))}\to\mathbb{A}^1_{\mathbb{C}}]\hat{e}_{(a,a)}\Bigr)^{\mathbb{P}^1},$$

where $\mathcal{E}^{\mathbb{C}}_{Q_{\mathrm{Kron}},((a,a))}$ is the stack of pairs (M,ξ), where M is a coherent $\mathcal{O}_{\mathbb{P}^1}$-module supported at zero, and ξ is an endomorphism of M. This is just the stack of pairs of commuting matrices N_1 and N_2 such that N_1 is nilpotent, which in turn is the stack of coherent sheaves on $\mathbb{C}^2$ scheme-theoretically supported on zero dimensional subschemes of a fixed coordinate line. As in [Kontsevich and Soibelman 2011, Section 5.6] one deduces that

$$\sum_{a\ge0}[\mathcal{E}^{\zeta-\mathrm{ss}}_{Q_{\mathrm{Kron}},((a,a))}\to\mathbb{A}^1_{\mathbb{C}}]\hat{e}_{(a,a)}=\mathrm{Sym}\Bigl(\sum_{n\ge1}\frac{\mathbb{L}^{3/2}+\mathbb{L}^{1/2}}{\mathbb{L}^{1/2}-\mathbb{L}^{-1/2}}\hat{e}_{(n,n)}\Bigr).\tag{38}$$

Finally, putting all this together, we have

$$\begin{aligned}
&\Phi_{\mathtt{BBS},Q_{-2},W_d}([\mathcal{X}_{Q_{-2},W_d}])\\
&\quad= \sum_{\boldsymbol{n}\in\mathbb{N}^{V(Q_{-2})}} \int_{\mathcal{Y}_{Q_{-2},\boldsymbol{n}}} \phi_{\mathrm{tr}(W_d)}\,\hat{e}_{\boldsymbol{n}}\\
&\quad= \sum_{\boldsymbol{n}\in\mathbb{N}^{V(Q_{-2})}} \mathbb{L}^{-2\boldsymbol{n}(0)\cdot\boldsymbol{n}(1)}[\mathbb{A}_{Q_{-2},\boldsymbol{n}} \xrightarrow{\mathrm{tr}(W_d)} \mathbb{A}^1_{\mathbb{C}}]\cdot[\mathrm{GL}_{\mathbb{C}}(\boldsymbol{n}(0))\times\mathrm{GL}_{\mathbb{C}}(\boldsymbol{n}(1))]^{-1}\hat{e}_{\boldsymbol{n}}\\
&\quad= \sum_{\boldsymbol{n}\in\mathbb{N}^{V(Q_{-2})}} [\mathcal{E}_{Q_{\mathrm{Kron}},\boldsymbol{n}} \xrightarrow{\mathrm{tr}(W_d)} \mathbb{A}^1_{\mathbb{C}}]\hat{e}_{\boldsymbol{n}}\\
&\quad= \sum_{\gamma\in\aleph_\zeta} [\mathcal{E}_{Q_{\mathrm{Kron}},\gamma} \xrightarrow{\mathrm{tr}(W_d)} \mathbb{A}^1_{\mathbb{C}}]\hat{e}_{\boldsymbol{n}}\\
&\quad= \prod_{\boldsymbol{n}=(n,n\pm1)} \Bigl(\sum_{a\geq 0}[\mathrm{Mat}_{a\times a}(\mathbb{C})\,/\,\mathrm{GL}_{\mathbb{C}}(a) \xrightarrow{\mathrm{tr}(X^{d+1})} \mathbb{A}^1_{\mathbb{C}}]\hat{e}_{a\boldsymbol{n}}\Bigr)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\cdot\mathrm{Sym}\Bigl(\sum_{n\geq 1}\frac{\mathbb{L}^{3/2}+\mathbb{L}^{1/2}}{\mathbb{L}^{1/2}-\mathbb{L}^{-1/2}}\hat{e}_{(n,n)}\Bigr)\\
&\quad= \mathrm{Sym}\Bigl(\sum_{\boldsymbol{n}=(n,n\pm1)}[\mathbb{A}^1_{\mathbb{C}} \xrightarrow{z\mapsto z^{d+1}} \mathbb{A}^1_{\mathbb{C}}]\cdot\mathbb{L}^{-1/2}(\mathbb{L}^{1/2}-\mathbb{L}^{-1/2})^{-1}\hat{e}_{\boldsymbol{n}}\Bigr)\cdot\\
&\qquad\qquad\qquad\qquad\qquad\qquad\cdot\mathrm{Sym}\Bigl(\sum_{n\geq 1}\frac{\mathbb{L}^{3/2}+\mathbb{L}^{1/2}}{\mathbb{L}^{1/2}-\mathbb{L}^{-1/2}}\hat{e}_{(n,n)}\Bigr),
\end{aligned}$$

where for the final equality we have again used the calculation of the motivic Donaldson–Thomas invariants for the one loop quiver with homogeneous potential from [Davison and Meinhardt 2015, Theorem 6.2], and we are done. □

Remark 5.7. It is possible to give the category of (not necessarily nilpotent) A_{Q_{-2},W_d}-modules the structure of a cyclic A_∞-category, and prove the above result in this framework, for arbitrary d.

Remark 5.8. We return to the case $d=1$ and the comparison with the calculations of [Szendrői 2008; Morrison et al. 2012]. There, an alternative quiver with potential $(Q_{\mathrm{con}}, W_{\mathrm{con}})$ is used to give a Jacobi algebra presentation of the noncommutative resolution of X_d, where Q_{con} is the subquiver of Q_{-2} depicted in Figure 3 and

$$W_{\mathrm{con}} = CADB - CBDA.$$

Then it is not hard to see that the inclusion of algebras $\mathbb{C}Q_{\mathrm{con}} \hookrightarrow \mathbb{C}Q_{-2}$ induced by the inclusion of quivers $Q_{\mathrm{con}} \subset Q_{-2}$ induces an isomorphism of algebras $A_{Q_{\mathrm{con}},W_{\mathrm{con}}} \cong A_{Q_{-2},W_1}$, and so we have *two* derived equivalences

$$\mathrm{D^b}(\mathtt{Mod}\text{-}A_{Q_{-2},W_1}) \xrightarrow{\Delta_1} \mathrm{D^b}(\mathrm{Coh}(Y_d)) \xleftarrow{\Delta_2} \mathrm{D^b}(\mathtt{Mod}\text{-}A_{Q_{\mathrm{con}},W_{\mathrm{con}}}).$$

Consider the sheaf $\mathcal{O}_{C_1}$, via either derived equivalence it corresponds to the unique simple object of dimension vector $(1,0)$. The motivic DT invariant for this dimension vector, calculated in the category $\mathtt{Mod}\text{-}A_{Q_{-2},W_1}$, with the natural orientation data coming from the presentation of this algebra as a Jacobi algebra, is $(1-[\mu_2])\mathbb{L}^{-1/2}$. On the other hand, the motivic DT invariant for this dimension vector, calculated in the category $A_{Q_{\mathrm{con}},W_{\mathrm{con}}}$, is given by calculating the motivic vanishing cycle of the function $\mathrm{tr}(W_{\mathrm{con}})$ on the space pt — the space parametrising $(1,0)$-dimensional representations of the quiver Q_{con}. This function is zero, and so we have $[\phi_{\mathrm{tr}(W_{\mathrm{con}})|_{\mathrm{pt}}}]=[\mathrm{pt}]=1\neq(1-[\mu_2])\mathbb{L}^{-1/2}$. The difference is accounted for by the difference in natural orientation data coming from the two presentations of the noncommutative resolution as a Jacobi algebra.

Remark 5.9. Restricting the derived equivalence (4), there are derived equivalences

$$\begin{aligned}\mathrm{D}_{\mathrm{nilp}}(\mathtt{Mod}\text{-}A_{Q_{-2},W_d})&\cong \mathrm{D}^{\mathrm{b}}_{\mathrm{exc}}(\mathrm{Coh}(Y_d)),\\ \mathrm{D}_{\mathrm{f.d}}(\mathtt{Mod}\text{-}A_{Q_{-2},W_d})&\cong \mathrm{D}_{\mathrm{cpct}}(\mathrm{Coh}(Y_d)),\end{aligned}$$

where $\mathrm{D}_{\mathrm{cpct}}(\mathrm{Coh}(Y_d))$ is the subcategory of the bounded derived category of coherent sheaves on Y_d with compactly supported total cohomology, while $\mathrm{D}^{\mathrm{b}}_{\mathrm{exc}}(\mathrm{Coh}(Y_d))$ is the subcategory of coherent sheaves with set-theoretic support contained in the exceptional curve C_d. This is the explanation for the fact that

$$\Omega^{\mathrm{nilp}}_{\zeta}(\boldsymbol{n})=\Omega_{\zeta}(\boldsymbol{n})$$

for all $\boldsymbol{n}$ not counting point sheaves under the derived equivalence (4), as C_d is the only proper subvariety of Y_d of dimension greater than zero.

Remark 5.10. Let $\boldsymbol{n}=(m,n)$ with $m,n\in\mathbb{N}$ and $m=n\pm 1$. Then the numerical DT invariant $\omega_\zeta(\boldsymbol{n})$ is extracted from our motivic DT invariant by first taking the Hodge spectrum

$$\begin{aligned}\mathrm{sp}(\Omega_\zeta(\boldsymbol{n}))&=\biggl(\sum_{l=1,d}u^{l/(d+1)}v^{(d+1-l)/(d+1)}\biggr)u^{-1/2}v^{-1/2}\\ &=\sum_{l=1,d}u^{(2l-d-1)/(2d+2)}v^{(d+1-2l)/(2d+2)}\end{aligned}$$

and then replacing u and v with q and setting $q^{1/2}=1$. The Hodge spectrum is as defined and discussed in [Kontsevich and Soibelman 2008, Section 4.3]; in general for a μ_{d+1} equivariant variety X, the expression $\mathrm{sp}([X])$ is given by taking the usual mixed Hodge polynomial of the compactly supported cohomology of X, and multiplying the summand with eigenvalue $\exp(\alpha i/(d+1))$ under the monodromy action by $u^{\alpha/(d+1)}v^{(d+1-\alpha)/(d+1)}$ if $\alpha\neq 0$, and by 1 otherwise. In particular, the operation of taking the Hodge spectrum of a $\hat{\mu}$-equivariant motive X and setting

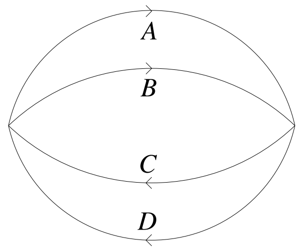

Figure 3. The quiver Q_{con}.

$u = v = q^{1/2} = 1$ is the same as taking the Euler characteristic of X (forgetting monodromy).

We deduce that the numerical BPS contribution from the curve C_d is precisely d, in agreement with the BPS contribution as defined and calculated in [Bryan et al. 2001, Theorem 1.5] in the context of Gromov–Witten theory. There, the crucial tool is the deformation invariance of Gromov–Witten invariants. Probably one could derive the numerical version of our result on the contribution of the curve C_d by deforming it to d $(-1,-1)$-curves as in [Bryan et al. 2001] and interpreting the calculation of [Young 2009] (see [Szendrői 2008, Theorem 2.7.2]) as stating that the numerical specialization of the motivic contribution of a $(-1,-1)$-curve is 1. On the other hand, again with reference to [op. cit., Theorem.7.2], we see that deformation invariance of the BPS contribution fails at the motivic level, and even at the level of the Hodge spectrum, as all of the invariants of [ibid] lie inside the subring $\overline{\mathrm{K}}(\mathrm{St}^{\mathrm{aff}}/\operatorname{Spec}(\mathbb{C}))[\mathbb{L}^{1/2}] \subset \overline{\mathrm{K}}^{\hat{\mu}}(\mathrm{St}^{\mathrm{aff}}/\operatorname{Spec}(\mathbb{C}))[\mathbb{L}^{1/2}]$ of monodromy-free motives, while our calculation of $\Omega_\zeta(\boldsymbol{n})$ has nontrivial monodromy.

References

[Aspinwall and Katz 2006] P. S. Aspinwall and S. Katz, "Computation of superpotentials for D-branes", *Comm. Math. Phys.* **264**:1 (2006), 227–253. MR Zbl

[Behrend and Fantechi 1997] K. Behrend and B. Fantechi, "The intrinsic normal cone", *Invent. Math.* **128**:1 (1997), 45–88. MR Zbl

[Behrend et al. 2013] K. Behrend, J. Bryan, and B. Szendrői, "Motivic degree zero Donaldson–Thomas invariants", *Invent. Math.* **192**:1 (2013), 111–160. MR Zbl

[van den Bergh 2004] M. van den Bergh, "Non-commutative crepant resolutions", pp. 749–770 in *The legacy of Niels Henrik Abel*, edited by O. A. Laudal and R. Piene, Springer, Berlin, 2004. MR Zbl

[Bridgeland 2012] T. Bridgeland, "An introduction to motivic Hall algebras", *Adv. Math.* **229**:1 (2012), 102–138. MR Zbl

[Bryan et al. 2001] J. Bryan, S. Katz, and N. C. Leung, "Multiple covers and the integrality conjecture for rational curves in Calabi–Yau threefolds", *J. Alg. Geom.* **10**:3 (2001), 549–568. MR Zbl

[Clemens et al. 1988] H. Clemens, J. Kollár, and S. Mori, *Higher-dimensional complex geometry*, Astérisque **166**, Société Mathématique de France, Paris, 1988. MR Zbl

[Davison 2010] B. Davison, "Invariance of orientation data for ind-constructible Calabi–Yau A_∞ categories under derived equivalence", preprint, 2010. arXiv

[Davison and Meinhardt 2015] B. Davison and S. Meinhardt, "Motivic Donaldson–Thomas invariants for the one-loop quiver with potential", *Geom. Topol.* **19**:5 (2015), 2535–2555. MR Zbl

[Denef and Loeser 1998] J. Denef and F. Loeser, "Motivic Igusa zeta functions", *J. Alg. Geom.* **7**:3 (1998), 505–537. MR Zbl arXiv

[Dimca and Szendrői 2009] A. Dimca and B. Szendrői, "The Milnor fibre of the Pfaffian and the Hilbert scheme of four points on $\mathbb{C}^3$", *Math. Res. Lett.* **16**:6 (2009), 1037–1055. MR Zbl

[Ekedahl 2009] T. Ekedahl, "The Grothendieck group of algebraic stacks", preprint, 2009. arXiv

[Feit and Fine 1960] W. Feit and N. J. Fine, "Pairs of commuting matrices over a finite field", *Duke Math. J* **27** (1960), 91–94. MR Zbl

[Ginzburg 2006] V. Ginzburg, "Calabi–Yau algebras", preprint, 2006. arXiv

[Guibert et al. 2006] G. Guibert, F. Loeser, and M. Merle, "Iterated vanishing cycles, convolution, and a motivic analogue of a conjecture of Steenbrink", *Duke Math. J.* **132**:3 (2006), 409–457. MR Zbl

[Gusein-Zade et al. 2004] S. M. Gusein-Zade, I. Luengo, and A. Melle-Hernández, "A power structure over the Grothendieck ring of varieties", *Math. Res. Lett.* **11**:1 (2004), 49–57. MR Zbl

[Joyce 2006a] D. Joyce, "Configurations in abelian categories, I: Basic properties and moduli stacks", *Adv. Math.* **203**:1 (2006), 194–255. MR Zbl

[Joyce 2006b] D. Joyce, "Constructible functions on Artin stacks", *J. London Math. Soc.* (2) **74**:3 (2006), 583–606. MR Zbl

[Joyce 2007a] D. Joyce, "Configurations in abelian categories, II: Ringel–Hall algebras", *Adv. Math.* **210**:2 (2007), 635–706. MR Zbl

[Joyce 2007b] D. Joyce, "Configurations in abelian categories, III: Stability conditions and identities", *Adv. Math.* **215**:1 (2007), 153–219. MR Zbl

[Joyce 2007c] D. Joyce, "Holomorphic generating functions for invariants counting coherent sheaves on Calabi–Yau 3-folds", *Geom. Topol.* **11** (2007), 667–725. MR Zbl

[Joyce 2007d] D. Joyce, "Motivic invariants of Artin stacks and 'stack functions'", *Q. J. Math.* **58**:3 (2007), 345–392. MR Zbl

[Joyce 2008] D. Joyce, "Configurations in abelian categories, IV: Invariants and changing stability conditions", *Adv. Math.* **217**:1 (2008), 125–204. MR Zbl

[Joyce and Song 2012] D. Joyce and Y. Song, *A theory of generalized Donaldson–Thomas invariants*, Mem. Amer. Math. Soc. **1020**, American Mathematical Society, Providence, RI, 2012. MR Zbl

[Kajiura 2007] H. Kajiura, "Noncommutative homotopy algebras associated with open strings", *Rev. Math. Phys.* **19**:1 (2007), 1–99. MR Zbl

[Katz and Morrison 1992] S. Katz and D. R. Morrison, "Gorenstein threefold singularities with small resolutions via invariant theory for Weyl groups", *J. Alg. Geom.* **1**:3 (1992), 449–530. MR Zbl

[Keller 2001] B. Keller, "Introduction to A-infinity algebras and modules", *Homology Homotopy Appl.* **3**:1 (2001), 1–35. MR Zbl

[Kollár 1989] J. Kollár, "Flops", *Nagoya Math. J.* **113** (1989), 15–36. MR Zbl

[Kontsevich and Soibelman 2008] M. Kontsevich and Y. Soibelman, "Stability structures, motivic Donaldson–Thomas invariants and cluster transformations", preprint, 2008. arXiv

[Kontsevich and Soibelman 2009] M. Kontsevich and Y. Soibelman, "Notes on A_∞-algebras, A_∞-categories and non-commutative geometry", pp. 153–219 in *Homological mirror symmetry*, edited by A. Kapustin et al., Lecture Notes in Phys. **757**, Springer, Berlin, 2009. MR Zbl

[Kontsevich and Soibelman 2011] M. Kontsevich and Y. Soibelman, "Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson–Thomas invariants", *Commun. Number Theory Phys.* **5**:2 (2011), 231–352. MR Zbl

[Laufer 1981] H. B. Laufer, "On $\mathbb{C}P^1$ as an exceptional set", pp. 261–275 in *Recent developments in several complex variables* (Princeton, 1979), edited by J. E. Fornaess, Ann. of Math. Stud. **100**, Princeton Univ. Press, 1981. MR Zbl

[Lê 2015] Q. T. Lê, "Proofs of the integral identity conjecture over algebraically closed fields", *Duke Math. J.* **164**:1 (2015), 157–194. MR Zbl

[Lefèvre-Hasegawa 2003] K. Lefèvre-Hasegawa, *Sur les A_∞-catégories*, Ph.D. thesis, Université Paris Diderot, 2003. arXiv

[Looijenga 2002] E. Looijenga, "Motivic measures", exposé 874, pp. 267–297 in *Séminaire Bourbaki,* 1999/2000, Astérisque **276**, Soc. Mat. de France, Paris, 2002. MR

[Lu et al. 2008] D. M. Lu, J. H. Palmieri, Q. S. Wu, and J. J. Zhang, "Koszul equivalences in A_∞-algebras", *New York J. Math.* **14** (2008), 325–378. MR Zbl

[Maulik 2013] D. Maulik, "Motivic residues and Donaldson–Thomas theory", preprint, 2013.

[Morrison and Nagao 2015] A. Morrison and K. Nagao, "Motivic Donaldson–Thomas invariants of small crepant resolutions", *Algebra Number Theory* **9**:4 (2015), 767–813. MR Zbl

[Morrison et al. 2012] A. Morrison, S. Mozgovoy, K. Nagao, and B. Szendrői, "Motivic Donaldson–Thomas invariants of the conifold and the refined topological vertex", *Adv. Math.* **230**:4-6 (2012), 2065–2093. MR Zbl

[Nagao and Nakajima 2011] K. Nagao and H. Nakajima, "Counting invariant of perverse coherent sheaves and its wall-crossing", *Int. Math. Res. Not.* **2011**:17 (2011), 3885–3938. MR Zbl

[Reid 1983] M. Reid, "Minimal models of canonical 3-folds", pp. 131–180 in *Algebraic varieties and analytic varieties* (Tokyo, 1981), edited by S. Iitaka, Adv. Stud. Pure Math. **1**, North-Holland, Amsterdam, 1983. MR Zbl

[Szendrői 2008] B. Szendrői, "Non-commutative Donaldson–Thomas invariants and the conifold", *Geom. Topol.* **12**:2 (2008), 1171–1202. MR Zbl

[Thomas 2000] R. P. Thomas, "A holomorphic Casson invariant for Calabi–Yau 3-folds, and bundles on $K3$ fibrations", *J. Differential Geom.* **54**:2 (2000), 367–438. MR Zbl

[Toda 2007] Y. Toda, "On a certain generalization of spherical twists", *Bull. Soc. Math. France* **135**:1 (2007), 119–134. MR Zbl

[Young 2009] B. Young, "Computing a pyramid partition generating function with dimer shuffling", *J. Combin. Theory Ser. A* **116**:2 (2009), 334–350. MR Zbl

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Classifying tilting complexes over preprojective algebras of Dynkin type

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We study tilting complexes over preprojective algebras of Dynkin type. We classify all tilting complexes by giving a bijection between tilting complexes and the braid group of the corresponding folded graph. In particular, we determine the derived equivalence class of the algebra. For the results, we develop the theory of silting-discrete triangulated categories and give a criterion for silting-discreteness.

1. Introduction

1A. ***Background and motivation.*** Derived categories are nowadays considered as a fundamental object in many branches of mathematics including representation theory and algebraic geometry. Among others, one of the most important problems is to understand their equivalences. Derived equivalences provide a lot of interesting connections between various different objects and they are also quite useful to study structures of the categories.

It is known that derived equivalences are controlled by tilting objects (complexes) [Rickard 1989; Keller 1994] and therefore these constructions have been extensively studied. As a tool for studying tilting objects, Keller and Vossieck [1988] introduced the notion of silting objects (Definition 2.1), which is a generalization of tilting

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objects. After that, it was shown that their mutation properties are much better than tilting ones and they yield a nice combinatorial description [Aihara and Iyama 2012] (see Definition 2.3). Furthermore, silting objects have turned out to have deep connections with several important objects such as cluster tilting objects and t-structures, for example [Adachi et al. 2014; Buan et al. 2011; Koenig and Yang 2014; Brüstle and Yang 2013; Iyama et al. 2014; Qiu and Woolf 2014; Broomhead et al. 2016].

One of the aims of this paper is to give a further development of the mutation theory of silting objects. In particular, we study a criterion when a triangulated category is *silting-discrete* (Definition 2.2). A remarkable property of this class is that all silting objects are connected to each other by iterated mutation and this fact allows us to achieve a comprehensive understanding of the categories.

Another aim of the paper is, by applying this technique, to classify all tilting complexes of *preprojective algebras* of Dynkin type. Since preprojective algebras were introduced in [Gel'fand and Ponomarev 1979; Dlab and Ringel 1980; Baer et al. 1987], it turned out that they have fundamental importance in representation theory as well as algebraic and differential geometry. We refer to [Ringel 1998] for quiver representations, [Lusztig 1991; 2000; Kashiwara and Saito 1997] for quantum groups, [Auslander and Reiten 1996; Crawley-Boevey 2000] for Kleinian singularities, [Nakajima 1994; 1998; 2001] for quiver varieties, and [Geiß et al. 2006; 2011] for cluster algebras.

For the case of preprojective algebras of non-Dynkin type, its tilting theory has been extensively studied in [Buan et al. 2009; Iyama and Reiten 2008]. In particular, they show that certain ideals parametrized by the Coxeter group (see Theorem 4.1) give tilting modules over the preprojective algebra and this fact provides a method for studying the derived category. On the other hand, in the case of Dynkin type, they are no longer tilting modules. Moreover, there is no spherical objects in this case and a similar nice theory had never been observed. In this paper, via a new strategy, we succeed in classifying all tilting complexes as described below.

1B. ***Our results.*** To explain our results, we give the following set-up. Let Δ be a Dynkin graph and Λ the preprojective algebra of Δ.

First we study two-term tilting complexes of Λ. For this purpose, we use *τ-tilting theory*. Mizuno [2014] showed that the above ideals are support τ-tilting Λ-modules (Theorem 4.1). Then, combining the results of [Adachi et al. 2014], we obtain a bijection between two-term silting complexes of Λ and the Weyl group (Theorem 4.1). Moreover we analyze this connection in more detail and we can give a classification of two-term tilting complexes of Λ using the *folded graph* Δ^{f} of Δ (Definition 3.2) given by the following correspondences.

Δ	$\mathbb{A}_{2n-1}, \mathbb{A}_{2n}$	$\mathbb{D}_{2n}$	$\mathbb{D}_{2n+1}$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$
Δ^{f}	$\mathbb{B}_n$	$\mathbb{D}_{2n}$	$\mathbb{B}_{2n}$	$\mathbb{F}_4$	$\mathbb{E}_7$	$\mathbb{E}_8$

Then our first result is summarized as follows.

Theorem 1.1 (Theorem 4.2). *Let $W_{\Delta^{\mathrm{f}}}$ be the Weyl group of Δ^{f} and* 2-tilt Λ *the set of isomorphism classes of basic two-term tilting complexes of* $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$. *Then we have a bijection*

$$W_{\Delta^{\mathrm{f}}} \longleftrightarrow \text{2-tilt}\,\Lambda.$$

We remark that we can give not only a bijection but also an explicit description of all two-term tilting complexes (Theorem 4.1). On the other hand, we study an important relationship between two-term silting complexes and silting-discrete categories. More precisely, we give the following criterion of silting-discreteness (tilting-discreteness).

Theorem 1.2 (Theorem 2.4, Corollary 2.11). *Let A be a finite dimensional algebra* (*respectively*, *finite dimensional self-injective algebra*). *The following are equivalent.*

(a) $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}A)$ *is silting-discrete* (*respectively*, *tilting-discrete*).

(b) 2-silt$_P$ A (*respectively*, 2-tilt$_P$ A) *is a finite set for any silting* (*respectively*, *tilting*) *complex P.*

(c) 2-silt$_P$ A (*respectively*, 2-tilt$_P$ A) *is a finite set for any silting* (*respectively*, *tilting*) *complex P which is given by iterated irreducible left silting* (*respectively*, *tilting*) *mutation from A.*

Here 2-silt$_P$ A (respectively, 2-tilt$_P$ A) denotes the subset of silting (respectively, tilting) objects T in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}A)$ such that $P \geq T \geq P[1]$ (Definition 2.2). An advantage of this theorem is that we can understand the condition of all silting (respectively, tilting) objects by studying a certain special class of silting (respectively, tilting) objects. Then, we can apply Theorem 1.2 and obtain the following result.

Theorem 1.3 (Theorem 5.1, Proposition 5.4). *The endomorphism algebra of any irreducible left tilting mutation* (Definition 2.3) *of Λ is isomorphic to Λ. In particular*, *the condition* (b) *of Theorem 1.2 is satisfied and hence* $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$ *is tilting-discrete.*

Then Theorem 1.3 implies that any tilting complexes are obtained from Λ by iterated irreducible mutation. As a consequence of this result, we determine the derived equivalence class of Λ as follows.

Corollary 1.4 (Theorem 5.1). *Any basic tilting complex T of Λ satisfies*

$$\mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)}(T) \cong \Lambda.$$

In particular, *the derived and Morita equivalence classes coincide.*

In fact, we give a more detailed description about tilting complexes. Indeed, using Theorem 1.1 and Corollary 1.4, we can show that irreducible tilting mutation satisfy braid relations (Proposition 6.1), which provide a nice relationship between the braid group and tilting complexes (see [Brav and Thomas 2011; Seidel and Thomas 2001; Grant 2013; Khovanov and Seidel 2002]).

Recall that the braid group $B_{\Delta^{\mathrm{f}}}$ is defined by generators a_i $(i \in \Delta^{\mathrm{f}}_0)$ with relations $(a_i a_j)^{m(i,j)} = 1$ for $i \neq j$ (see Section 3B for $m(i, j)$), that is, the difference with $W_{\Delta^{\mathrm{f}}}$ is that we do not require the relations $a_i^2 = 1$ for $i \in \Delta^{\mathrm{f}}_0$. We denote by $\boldsymbol{\mu}_i^+$ (respectively, $\boldsymbol{\mu}_i^-$) the irreducible left (respectively, right) tilting mutation associated with $i \in \Delta^{\mathrm{f}}_0$.

Then we can define the map from the braid group to tilting complexes and it gives a classification of tilting complexes as follows.

Theorem 1.5 (Theorem 6.6). *Let $B_{\Delta^{\mathrm{f}}}$ be the braid group of Δ^{f} and* tilt Λ *the set of isomorphism classes of basic tilting complexes of Λ. Then we have a bijection*

$$B_{\Delta^{\mathrm{f}}} \to \operatorname{tilt} \Lambda, \quad a = a_{i_1}^{\epsilon_{i_1}} \cdots a_{i_k}^{\epsilon_{i_k}} \mapsto \boldsymbol{\mu}_a(\Lambda) := \boldsymbol{\mu}_{i_1}^{\epsilon_{i_1}} \circ \cdots \circ \boldsymbol{\mu}_{i_k}^{\epsilon_{i_k}}(\Lambda).$$

We now describe the organization of this paper.

In Section 2, we deal with triangulated categories and study some properties of silting-discrete categories. In particular, we give a criterion of silting-discreteness. We also investigate a Bongartz-type lemma for silting objects. In Section 3, we recall definitions and some results related to preprojective algebras. In Section 4, we explain a connection between two-term silting complexes and the Weyl group. In particular, we characterize two-term tilting complexes in terms of the subgroup of the Weyl group and this observation is crucial in this paper. In Section 5, we show that preprojective algebras of Dynkin type are tilting-discrete. It implies that any tilting complex is obtained by iterated mutation from an arbitrary tilting complex. In Section 6, we show that there exists a map from the braid group to tilting complexes and we prove that it is a bijection.

Notation. Throughout this paper, let K be an algebraically closed field and $D := \operatorname{Hom}_K(-, K)$. For a finite dimensional algebra Λ over K, we denote by $\mathsf{mod}\Lambda$ the category of finitely generated right Λ-modules and by $\mathsf{proj}\Lambda$ the category of finitely generated projective Λ-modules. We denote by $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\Lambda)$ the bounded derived category of $\mathsf{mod}\Lambda$ and by $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$ the bounded homotopy category of $\mathsf{proj}\Lambda$.

2. Silting-discrete triangulated categories

In this section, we study silting-discrete triangulated categories. In particular, we give a criterion for silting-discreteness. Moreover we apply this theory for tilting-discrete categories for self-injective algebras. We also study a relationship between silting-discrete categories and a Bongartz-type lemma.

Throughout this section, let $\mathcal{T}$ be a Krull–Schmidt triangulated category and assume that it satisfies the following property:

- For any object X of $\mathcal{T}$, the additive closure $\operatorname{add} X$ is functorially finite in $\mathcal{T}$.

For example, it is satisfied if $\mathcal{T}$ is the homotopy category of bounded complexes of finitely generated projective modules over a finite dimensional algebra, which is a main object in this paper.

2A. ***Criteria for silting-discreteness.*** Let us start with recalling the definition of silting objects [Aihara and Iyama 2012; Buan et al. 2011; Keller and Vossieck 1988].

Definition 2.1. (a) We say an object P in $\mathcal{T}$ is *presilting* (respectively, *pretilting*) if it satisfies $\operatorname{Hom}_{\mathcal{T}}(P, P[i]) = 0$ for any $i > 0$ (respectively, $i \neq 0$).

(b) We call an object P in $\mathcal{T}$ *silting* (respectively, *tilting*) if it is presilting (respectively, pretilting) and the smallest thick subcategory containing P is $\mathcal{T}$.

We denote by $\operatorname{silt}\mathcal{T}$ (respectively, $\operatorname{tilt}\mathcal{T}$) the set of isomorphism classes of basic silting objects (respectively, tilting objects) in $\mathcal{T}$.

It is known that the number of nonisomorphic indecomposable summands of a silting object does not depend on the choice of silting objects [Aihara and Iyama 2012, Corollary 2.28]. Moreover, for objects P and Q of $\mathcal{T}$, we write $P \geq Q$ if $\operatorname{Hom}_{\mathcal{T}}(P, Q[i]) = 0$ for any $i > 0$, which gives a partial order on $\operatorname{silt}\mathcal{T}$ [Aihara and Iyama 2012, Theorem 2.11].

Then we give the definition of silting-discrete triangulated categories as follows.

Definition 2.2. (a) We call a triangulated category $\mathcal{T}$ *silting-discrete* if for any $P \in \operatorname{silt}\mathcal{T}$ and any $\ell > 0$, the set

$$\{T \in \operatorname{silt}\mathcal{T} \mid P \geq T \geq P[\ell]\}$$

is finite. Note that the property of being silting-discrete does not depend on the choice of silting objects [Aihara 2013, Proposition 3.8]. Hence it is equivalent to say that, for a silting object $A \in \mathcal{T}$ and any $\ell > 0$, the set

$$\{T \in \operatorname{silt}\mathcal{T} \mid A \geq T \geq A[\ell]\}$$

is finite. Similarly, we call $\mathcal{T}$ *tilting-discrete* if, for a tilting object $A \in \mathcal{T}$ and any $\ell > 0$, the set $\{T \in \operatorname{tilt}\mathcal{T} \mid A \geq T \geq A[\ell]\}$ is finite.

(b) For a silting object P of $\mathcal{T}$, we denote by $2\text{-}\operatorname{silt}_P\mathcal{T}$ the subset U of $\operatorname{silt}\mathcal{T}$ such that $P \geq U \geq P[1]$. We call $\mathcal{T}$ *2-silting-finite* if $2\text{-}\operatorname{silt}_P\mathcal{T}$ is a finite set for any silting object P of $\mathcal{T}$. Note that the finiteness of $2\text{-}\operatorname{silt}_P\mathcal{T}$ depends on a silting object P in general. Similarly, we denote by $2\text{-}\operatorname{tilt}_P\mathcal{T}$ the subset U of $\operatorname{tilt}\mathcal{T}$ such that $P \geq U \geq P[1]$.

Moreover we recall mutation for silting objects [Aihara and Iyama 2012, Theorem 2.31].

Definition 2.3. Let P be a basic silting object of $\mathcal{T}$ and decompose it as $P = X \oplus M$. We take a triangle

$$X \xrightarrow{f} M' \longrightarrow Y \longrightarrow X[1]$$

with a minimal left $(\mathsf{add}\, M)$-approximation f of X. Then $\mu_X^+(P) := Y \oplus M$ is again a silting object, and we call it the *left mutation* of P with respect to X. Dually, we define the right mutation $\mu_X^-(P)$.[1] Mutation will mean either left or right mutation. If X is indecomposable, then we say that mutation is *irreducible*. In this case, we have $P > \mu_X^+(P)$ and there is no silting object Q satisfying $P > Q > \mu_X^+(P)$ [Aihara and Iyama 2012, Theorem 2.35].

Moreover, if P and $\mu_X^+(P)$ are tilting objects, then we call it the (left) *tilting mutation*. In this case, if there exists no nontrivial direct summand X' of X such that $\mu_{X'}^+(T)$ is tilting, then we say that tilting mutation is *irreducible* ([Chan et al. 2015, Definition 5.3]).

We remark that all silting objects of a silting-discrete category are reachable by iterated irreducible mutation [Aihara 2013, Corollary 3.9].

Our first aim is to show the following theorem.

Theorem 2.4. *The following are equivalent.*

(a) *$\mathcal{T}$ is silting-discrete.*

(b) *$\mathcal{T}$ is 2-silting-finite.*

(c) *For a silting object $A \in \mathcal{T}$, $2\text{-silt}_P\, \mathcal{T}$ is a finite set for any silting object P which is given by iterated irreducible left mutation from A.*

We note that the theorem is different from [Qiu and Woolf 2014, Lemma 2.14], where the partial order is defined by a finite sequence of tilts; our partial order is valid for any silting objects.

Now we give some examples of silting-discrete categories.

Example 2.5. Let Λ be a finite dimensional algebra. Then $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$ is silting-discrete if:

(a) Λ is a path algebra of Dynkin type, which immediately follows from the definition.

(b) Λ is a local algebra [Aihara and Iyama 2012, Corollary 2.43].

(c) Λ is a representation-finite symmetric algebra [Aihara 2013, Theorem 5.6], which is also tilting-discrete.

(d) Λ is a derived discrete algebra of finite global dimension [Broomhead et al. 2016, Proposition 6.8].

[1] The convention of μ^+ and μ^- is different from [Mizuno 2014] in which the converse notation is used.

(e) Λ is a Brauer graph algebra whose Brauer graph contains at most one cycle of odd length and no cycle of even length [Adachi et al. 2015], which is also tilting-discrete.

For a proof of Theorem 2.4, we will introduce the following terminology.

Definition 2.6. We define a subset of silt $\mathcal{T}$

$$\nabla_A(T) := \{U \in \operatorname{silt}\mathcal{T} \mid A \geq U \geq A[1] \text{ and } U \geq T\},$$

where A is a silting object and T is a presilting object in $\mathcal{T}$ satisfying $A \geq T$. Note that we have $T \geq A[\ell]$ for some $\ell \geq 0$ [Aihara and Iyama 2012, Proposition 2.4].

Moreover, we say that a silting object P is *minimal* in $\nabla_A(T)$ if it is a minimal element in the partially ordered set $\nabla_A(T)$.

To keep this notation, we will make the following assumption.

Assumption 2.7. In the rest of this section, we always assume that $\mathcal{T}$ admits a silting object A and a presilting object T satisfying $A \geq T$.

Then we give the following key proposition.

Proposition 2.8. *If a silting object P is minimal in $\nabla_A(T)$ and $T \geq A[\ell]$ for some $\ell > 0$, then we have $T \geq P[\ell - 1]$.*

For a proof, we recall the following proposition. See [Aihara and Iyama 2012, Proposition 2.23, 2.24, 2.36] and [Aihara 2013, Proposition 2.12].

Proposition 2.9. *Let P be a silting object of $\mathcal{T}$. Then the following hold.*

(a) *There exists $\ell \geq 0$ such that $P \geq T \geq P[\ell]$ if and only if there exist triangles*

$$\begin{array}{ccccccc}
T_1 & \longrightarrow & P_0 & \xrightarrow{f_0} & T_0 := T & \longrightarrow & T_1[1], \\
\vdots & & \vdots & & \vdots & & \vdots \\
T_{\ell-1} & \longrightarrow & P_{\ell-2} & \xrightarrow{f_{\ell-2}} & T_{\ell-2} & \longrightarrow & T_{\ell-1}[1], \\
T_\ell & \longrightarrow & P_{\ell-1} & \xrightarrow{f_{\ell-1}} & T_{\ell-1} & \longrightarrow & P_\ell[1], \\
0 & \longrightarrow & P_\ell & \xrightarrow{f_\ell} & T_\ell & \longrightarrow & 0,
\end{array}$$

where f_i is a minimal right $(\mathsf{add}\, P)$*-approximation of T_i for $0 \leq i \leq \ell$.*

(b) *In the situation of* (a), *if $\ell \neq 0$, then there is a nonzero direct summand $X \in \mathsf{add}(P_\ell)$ such that the irreducible left mutation $\mu_X^+(P) \geq T$.*

Using Proposition 2.9, we give a proof of Proposition 2.8.

Proof of Proposition 2.8. Since P is minimal in $\nabla_A(T)$, we have

$$P \geq T \geq A[\ell] \geq P[\ell].$$

Then, by Proposition 2.9(a), there exist triangles

$$\begin{array}{ccccccc}
T_1 & \longrightarrow & P_0 & \xrightarrow{f_0} & T_0 := T & \longrightarrow & T_1[1], \\
\vdots & & \vdots & & \vdots & & \vdots \\
T_{\ell-1} & \longrightarrow & P_{\ell-2} & \xrightarrow{f_{\ell-2}} & T_{\ell-2} & \longrightarrow & T_{\ell-1}[1], \\
T_\ell & \longrightarrow & P_{\ell-1} & \xrightarrow{f_{\ell-1}} & T_{\ell-1} & \longrightarrow & P_\ell[1], \\
0 & \longrightarrow & P_\ell & \xrightarrow{f_\ell} & T_\ell & \longrightarrow & 0,
\end{array}$$

where f_i is a minimal right $(\mathsf{add}\, P)$-approximation of T_i for $0 \leq i \leq \ell$.

Similarly, since we have $P \geq A[1] \geq P[1]$, there is a triangle

$$Q_1 \longrightarrow Q_0 \xrightarrow{f} A[1] \longrightarrow Q_1[1], \tag{2-1}$$

where f is a minimal right $(\mathsf{add}\, P)$-approximation of $A[1]$ and $Q_1 \in \mathsf{add}\, P$.

(i) We show that P_ℓ belongs to $\mathsf{add}\, Q_1$. First, we have $\mathrm{Hom}_{\mathcal{T}}(T, A[1+\ell]) = 0$ by the definition of $T \geq A[\ell]$. Hence it follows from [Aihara and Iyama 2012, Lemma 2.25] that $(\mathsf{add}\, P_\ell) \cap (\mathsf{add}\, Q_0) = 0$.

On the other hand, since $A[1]$ is a silting object, we find that $Q_0 \oplus Q_1$ is also a silting object by the sequence (2-1). From [Aihara and Iyama 2012, Theorem 2.18], it is observed that $\mathsf{add}\, P = \mathsf{add}(Q_0 \oplus Q_1)$ and hence P_ℓ belongs to $\mathsf{add}\, Q_1$.

(ii) We show that $T \geq P[\ell - 1]$. Suppose that $P_\ell \neq 0$. Then we can take a direct summand $X \neq 0$ of P_ℓ such that $\mu_X^+(P) \geq T$ from Proposition 2.9(b).

On the other hand, (i) implies that X belongs to $\mathsf{add}\, Q_1$. Since $P \geq A[1] \geq P[1]$, by applying Proposition 2.9(b) to the sequence (2-1), we see that $\mu_X^+(P) \geq A[1]$. Thus, one gets a silting object $\mu_X^+(P)$ such that $P > \mu_X^+(P) \geq A[1]$ satisfying $\mu_X^+(P) \geq T$, which is a contradiction to the minimality of P. Therefore, we conclude that $P_\ell = 0$. Hence we get $T \geq P[\ell - 1]$ by Proposition 2.9(a). □

On the other hand, we can easily check the following lemma.

Lemma 2.10. *Let A be a silting object. If* $2\text{-silt}_A\, \mathcal{T}$ *is a finite set, then there exists a minimal element in* $\nabla_A(T)$.

Then we give a proof of Theorem 2.4, which provides a criterion of silting-discreteness.

Proof of Theorem 2.4. It is obvious that the implications (a) ⇒ (b) ⇒ (c) hold.

We show that the implication (c) $\Rightarrow$ (a) holds. Let T be a silting object such that $A \geq T \geq A[\ell]$ for some $\ell > 0$. Since 2-silt$_A\, \mathcal{T}$ is a finite set, there exists a minimal object P in $\nabla_A(T)$. Hence we get $P \geq T \geq P[\ell - 1]$ by Proposition 2.9.

Thus, one obtains

$$\{T \in \operatorname{silt} \mathcal{T} \mid A \geq T \geq A[\ell]\} \subseteq \bigcup_{P \in \text{2-silt}_A \mathcal{T}} \{U \in \operatorname{silt} \mathcal{T} \mid P \geq U \geq P[\ell - 1]\}.$$

By [Aihara 2013, Theorem 3.5], the finiteness of 2-silt$_A\, \mathcal{T}$ implies that P can be obtained from A by iterated irreducible left mutation. Therefore, our assumption yields that 2-silt$_P\, \mathcal{T}$ is also a finite set. Repeating this argument leads to the assertion. □

Moreover, using a statement analogous to Proposition 2.9 (see [Chan et al. 2015, Section 5]), we give a criterion for tilting-discreteness for self-injective algebras as follows.

Corollary 2.11. *Let Λ be a basic finite dimensional self-injective algebra and $\mathcal{T} := \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)$. Then the following are equivalent.*

(a) *$\mathcal{T}$ is tilting-discrete.*

(b) *$\mathcal{T}$ is 2-tilting-finite.*

(c) *2-tilt$_P\, \mathcal{T}$ is a finite set for any tilting object P which is given by iterated irreducible left tilting mutation from Λ.*

Proof. It is obvious that the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) hold.

We show that the implication (c) $\Rightarrow$ (a) holds. Let T be a tilting object such that $\Lambda \geq T \geq \Lambda[\ell]$ for some $\ell > 0$. Since 2-tilt$_\Lambda\, \mathcal{T}$ is a finite set, there exists a minimal tilting object P in $\nabla_\Lambda(T)$. Then, by [Chan et al. 2015, Proposition 5.10, Theorem 5.11], an argument the same as that of Proposition 2.9 works for tilting objects and irreducible tilting mutation. Hence we obtain Proposition 2.8 for tilting objects and one can get $P \geq T \geq P[\ell - 1]$.

Thus, one obtains

$$\{T \in \operatorname{tilt} \mathcal{T} \mid \Lambda \geq T \geq \Lambda[\ell]\} \subseteq \bigcup_{P \in \text{2-tilt}_\Lambda \mathcal{T}} \{U \in \operatorname{tilt} \mathcal{T} \mid P \geq U \geq P[\ell - 1]\}.$$

By [Chan et al. 2015, Theorem 5.11], the finiteness of 2-tilt$_\Lambda\, \mathcal{T}$ implies that P can be obtained from Λ by iterated irreducible left tilting mutation. Therefore, our assumption yields that 2-tilt$_P\, \mathcal{T}$ is also a finite set. Repeating this argument leads to the assertion. □

Finally, as an application of Theorem 2.4, we show that silting-discrete categories satisfy a Bongartz-type lemma. For this purpose, we give the following definition.

Definition 2.12. We call a presilting object T in $\mathcal{T}$ *partial silting* if it is a direct summand of some silting object, that is, there exists an object T' such that $T \oplus T'$ is a silting object.

One of the important questions is if any presilting object is partial silting or not [Brüstle and Yang 2013, Question 3.13]. We will show that it has a positive answer in the case of silting-discrete categories.

Let us recall the following result.

Proposition 2.13 [Aihara 2013, Proposition 2.16]. *Let T be a presilting object in $\mathcal{T}$. If $A \geq T \geq A[1]$, then T is partial silting.*

Then we can improve Proposition 2.13 as follows.

Proposition 2.14. *Let T be a presilting object in $\mathcal{T}$ such that $A \geq T$. Assume that for any silting object B in $\mathcal{T}$ such that $A \geq B \geq T$, there exists a minimal object in $\nabla_B(T)$.*

Then there exists a silting object P in $\mathcal{T}$ satisfying $P \geq T \geq P[1]$. In particular, T is partial silting.

Proof. We can take $\ell \geq 0$ such that $A \geq T \geq A[\ell]$ by [Aihara and Iyama 2012, Proposition 2.4]. It is enough to show the statement for $\ell \geq 2$. Since there is a minimal silting object in $\nabla_A(T)$, which we denote by A_1, we have $A_1 \geq T \geq A_1[\ell-1]$ by Proposition 2.8. By our assumption, we can repeat this argument and we obtain a sequence

$$A = A_0 \geq A_1 \geq \cdots \geq A_{\ell-1} \geq T \geq A_{\ell-1}[1] \geq \cdots \geq A_1[\ell-1] \geq A[\ell],$$

where A_{i+1} is a minimal object in $\nabla_{A_i}(T)$ for $0 \leq i \leq \ell-2$. Thus, we get the desired silting object $P := A_{\ell-1}$.

The second assertion immediately follows from the first one and Proposition 2.13. □

As a consequence, we obtain the following theorem.

Theorem 2.15. *If $\mathcal{T}$ is silting-discrete, then any presilting object is partial silting.*

Proof. Take a presilting object T in $\mathcal{T}$. If T is presilting, then so is $T[i]$ for any i. Hence we can assume that $A \geq T$. Then, by Theorem 2.4 and Lemma 2.10, $\mathcal{T}$ satisfies the assumption of Proposition 2.14 and hence we obtain the conclusion. □

We remark that in [Broomhead et al. 2016, Secion 5] the authors also discuss the Bongartz completion using a different type of the partial order.

3. Basic properties of preprojective algebras of Dynkin type

In this section, we review some definitions and results we will use in the rest of this paper.

3A. ***Preprojective algebras.*** Let Q be a finite connected acyclic quiver. We denote by Q_0 vertices of Q and by Q_1 arrows of Q. We denote by $\overline{Q}$ the double quiver of Q, which is obtained by adding an arrow $a^* : j \to i$ for each arrow $a : i \to j$ in Q_1. The *preprojective algebra* $\Lambda_Q = \Lambda$ associated to Q is the algebra $K\overline{Q}/I$, where I is the ideal in the path algebra $K\overline{Q}$ generated by the relation of the form

$$\sum_{a \in Q_1} (aa^* - a^*a).$$

We remark that Λ does not depend on the orientation of Q. Hence, for a graph Δ, we define the preprojective algebra by $\Lambda_\Delta = \Lambda_Q$, where Q is a quiver whose underlying graph is Δ. We denote by Δ_0 vertices of Δ.

Let Δ be a Dynkin graph (by Dynkin graph we always mean the one of type ADE). The preprojective algebra of Δ is finite dimensional and self-injective [Brenner et al. 2002, Theorem 4.8]. Without loss of generality, we may suppose that vertices are given as in Figure 1 and let e_i be the primitive idempotent of Λ associated with $i \in \Delta_0$. We denote the Nakayama permutation of Λ by $\iota : \Delta_0 \to \Delta_0$ (i.e., $D(\Lambda e_{\iota(i)}) \cong e_i\Lambda$). Then, one can check that we have $\iota = \mathrm{id}$ if Δ is type $\mathbb{D}_{2n}$, $\mathbb{E}_7$ and $\mathbb{E}_8$. Otherwise, we have $\iota^2 = \mathrm{id}$ and it is given as follows.

$$\begin{aligned}
&\iota(1) = 1 \text{ and } \iota(i) = i + n - 1 \text{ for } i \in \{2, \cdots, n\} && \text{if } \mathbb{A}_{2n-1},\\
&\iota(i) = i + n \text{ for } i \in \{1, \cdots, n\} && \text{if } \mathbb{A}_{2n},\\
&\iota(1) = n \text{ and } \iota(i) = i \text{ for } i \notin \{1, n\} && \text{if } \mathbb{D}_{2n+1},\\
&\iota(3) = 5, \iota(4) = 6 \text{ and } \iota(i) = i \text{ for } i \in \{1, 2\} && \text{if } \mathbb{E}_6.
\end{aligned}$$

3B. ***Weyl group.*** Let Δ be a graph from Figure 1. The *Weyl group* W_Δ associated to Δ is defined by the generators s_i and relations $(s_i s_j)^{m(i,j)} = 1$, where

$$m(i, j) := \begin{cases} 1 & \text{if } i = j, \\ 2 & \text{if no edge between } i \text{ and } j \text{ in } \Delta, \\ 3 & \text{if there is an edge } \ i \text{ ——— } j \text{ in } \Delta, \\ 4 & \text{if there is an edge } \ i \overset{4}{\text{———}} j \text{ in } \Delta. \end{cases}$$

For $w \in W_\Delta$, we denote by $\ell(w)$ the length of w.

Let Δ be a Dynkin graph, Λ the preprojective algebra and ι the Nakayama permutation of Λ. Then ι acts on an element of the Weyl group W_Δ by $\iota(w) := s_{\iota(i_1)}s_{\iota(i_2)} \cdots s_{\iota(i_k)}$ for $w = s_{i_1}s_{i_2} \cdots s_{i_k} \in W_\Delta$. We define the subgroup W_Δ^ι of W_Δ by

$$W_\Delta^\iota := \{w \in W \mid \iota(w) = w\}.$$

$\mathbb{A}_{2n-1}$: $\quad n \text{ --- } \cdots \text{ --- } 2 \text{ --- } 1 \text{ --- } (n+1) \text{ --- } \cdots \text{ --- } (2n-1)$

$\mathbb{A}_{2n}$: $\quad n \text{ --- } \cdots \text{ --- } 2 \text{ --- } 1 \text{ --- } (n+1) \text{ --- } \cdots \text{ --- } 2n$

$\mathbb{B}_n$ $(n \geq 1)$: $\quad 1 \overset{4}{\text{ --- }} 2 \text{ --- } \cdots \text{ --- } (n-1) \text{ --- } n$

$\mathbb{D}_n$ $(n \geq 4)$: $\quad$ 1 and n both joined to 2; $\; 2 \text{ --- } 3 \text{ --- } \cdots \text{ --- } (n-1)$

$\mathbb{E}_n$ $(n = 6, 7, 8)$: $\quad$ 1 joined to 2; $\; 4 \text{ --- } 3 \text{ --- } 2 \text{ --- } 5 \text{ --- } \cdots \text{ --- } n$

$\mathbb{F}_4$: $\quad 1 \text{ --- } 2 \overset{4}{\text{ --- }} 3 \text{ --- } 4$

Figure 1. Dynkin graphs with vertex labels.

Let w_0 be the longest element of W_Δ. Note that we have $w_0 w w_0 = \iota(w)$ for $w \in W_\Delta$ [Erdmann and Snashall 1998]. In particular we have $w_0 w = w w_0$ for any W_Δ^ι.

Theorem 3.1. *Let Δ be a Dynkin (ADE) graph whose vertices are given as in Figure 1 and W_Δ the Weyl group of Δ. Let Δ^{f} be a graph given by the following type.*

Δ	$\mathbb{A}_{2n-1}, \mathbb{A}_{2n}$	$\mathbb{D}_{2n}$	$\mathbb{D}_{2n+1}$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$
Δ^{f}	$\mathbb{B}_n$	$\mathbb{D}_{2n}$	$\mathbb{B}_{2n}$	$\mathbb{F}_4$	$\mathbb{E}_7$	$\mathbb{E}_8$

Then we have $W_\Delta^\iota = \langle t_i \mid i \in \Delta_0^{\mathrm{f}} \rangle$, where

$$t_i := \begin{cases} s_i & \text{if } i = \iota(i) \text{ in } \Delta, \\ s_i s_{\iota(i)} s_i & \text{if there is an edge } i \text{ --- } \iota(i) \text{ in } \Delta, \\ s_i s_{\iota(i)} & \text{if no edge between } i \text{ and } \iota(i) \text{ in } \Delta, \end{cases} \tag{T}$$

and W_Δ^ι is isomorphic to $W_{\Delta^{\mathrm{f}}}$.

Proof. This follows from the above property of the Nakayama permutation and [Carter 1989, Chapter 13]. □

Definition 3.2. We call the graph Δ^{f} given in Theorem 3.1 the *folded graph* of Δ.

Example 3.3. (a) Let Δ be a graph of type $\mathbb{A}_5$. Then one can check that W_Δ^ι is given by $\langle s_1, s_2 s_4, s_3 s_5 \rangle$ and this group is isomorphic to $W_{\Delta^{\mathrm{f}}}$, where Δ^{f} is a graph of type $\mathbb{B}_3$.

(b) Let Δ be a graph of type $\mathbb{A}_6$. Then one can check that W_Δ^ι is given by $\langle s_1 s_4 s_1, s_2 s_5, s_3 s_6\rangle$ and this group is isomorphic to $W_{\Delta^{\mathrm{f}}}$, where Δ^{f} is a graph of type $\mathbb{B}_3$.

(c) Let Δ be a graph of type $\mathbb{D}_5$. Then one can check that W_Δ^ι is given by $\langle s_1 s_5, s_2, s_3, s_4\rangle$ and this group is isomorphic to $W_{\Delta^{\mathrm{f}}}$, where Δ^{f} is a graph of type $\mathbb{B}_4$.

(d) Let Δ be a graph of type $\mathbb{E}_6$. Then one can check that W_Δ^ι is given by $\langle s_1, s_2, s_3 s_5, s_4 s_6\rangle$ and this group is isomorphic to $W_{\Delta^{\mathrm{f}}}$, where Δ^{f} is a graph of type $\mathbb{F}_4$.

3C. *Support τ-tilting modules and two-term silting complexes.* In this subsection, we briefly recall the notion of support τ-tilting modules introduced in [Adachi et al. 2014], and its relationship with silting complexes. We refer to [Adachi et al. 2014; Iyama and Reiten 2014] for a background of support τ-tilting modules.

Let Λ be a finite dimensional algebra and we denote by τ the AR translation [Auslander et al. 1995].

Definition 3.4. (a) We call X in $\mathsf{mod}\Lambda$ *τ-rigid* if $\mathrm{Hom}_\Lambda(X, \tau X) = 0$.

(b) We call X in $\mathsf{mod}\Lambda$ *τ-tilting* if X is τ-rigid and $|X| = |\Lambda|$, where $|X|$ denotes the number of nonisomorphic indecomposable direct summands of X.

(c) We call X in $\mathsf{mod}\Lambda$ *support τ-tilting* if there exists an idempotent e of Λ such that X is a τ-tilting $(\Lambda/\langle e\rangle)$-module.

We can also describe these notions as pairs as follows.

(d) We call a pair (X, P) of $X \in \mathsf{mod}\Lambda$ and $P \in \mathsf{proj}\Lambda$ *τ-rigid* if X is τ-rigid and $\mathrm{Hom}_\Lambda(P, X) = 0$.

(e) We call a τ-rigid pair (X, P) a *support τ-tilting* (respectively, *almost complete support τ-tilting*) pair if $|X| + |P| = |\Lambda|$ (respectively, $|X| + |P| = |\Lambda| - 1$).

We say that (X, P) is *basic* if X and P are basic, and we say that (X, P) is a *direct summand* of (X', P') if X is a direct summand of X' and P is a direct summand of P'. Note that a basic support τ-tilting module X determines a basic support τ-tilting pair (X, P) uniquely [Adachi et al. 2014, Proposition 2.3]. Hence we can identify basic support τ-tilting modules with basic support τ-tilting pairs. We denote by $\mathrm{s}\tau\text{-tilt}\,\Lambda$ the set of isomorphism classes of basic support τ-tilting Λ-modules.

Finally we recall an important relationship between support τ-tilting modules and two-term silting complexes. We write $\mathrm{silt}\,\Lambda := \mathrm{silt}\,\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$ and $\mathrm{tilt}\,\Lambda := \mathrm{tilt}\,\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$ for simplicity. We denote by $2\text{-silt}\,\Lambda$ (respectively, $2\text{-tilt}\,\Lambda$) the subset of $\mathrm{silt}\,\Lambda$ (respectively, $\mathrm{tilt}\,\Lambda$) consisting of two-term (i.e., it is concentrated in the degree 0 and -1) complexes. Note that a complex T is two-term if and only if $\Lambda \geq T \geq \Lambda[1]$.

Then we have the following nice correspondence.

Theorem 3.5 [Adachi et al. 2014, Theorem 3.2, Corollary 3.9]. *Let Λ be a finite dimensional algebra. There exists a bijection $\Psi : \mathrm{s}\tau\text{-tilt}\,\Lambda \to 2\text{-silt}\,\Lambda$,*

$$(X, P) \mapsto \Psi(X, P) := \begin{cases} \overset{-1}{P_X^1} \xrightarrow{f} \overset{0}{P_X^0} \\ \qquad \oplus \\ P \end{cases} \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda),$$

where $P_X^1 \xrightarrow{f} P_X^0 \to X \to 0$ is a minimal projective presentation of X. Moreover, it gives an isomorphism of the partially ordered sets between $\mathrm{s}\tau\text{-tilt}\,\Lambda$ and $2\text{-silt}\,\Lambda$.

By the above correspondence, we can give a description of two-term silting complexes by calculating support τ-tilting modules, which is much simpler than calculations of two-term silting complexes.

4. Two-term tilting complexes and Weyl groups

In this section, we characterize 2-term tilting complexes in terms of the Weyl group. In particular, we provide a complete description of 2-term tilting complexes.

Throughout this section, let Δ be a Dynkin (ADE) graph with $\Delta_0 = \{1, \dots, n\}$, Λ the preprojective algebra of Δ and $I_i := \Lambda(1 - e_i)\Lambda$, where e_i is the primitive idempotent of Λ associated with $i \in \Delta_0$. We denote by $\langle I_1, \dots, I_n \rangle$ the set of ideals of Λ which can be written as

$$I_{i_1} I_{i_2} \cdots I_{i_k}$$

for some $k \geq 0$ and $i_1, \dots, i_k \in \Delta_0$. Note that it has recently been understood that these ideals play an important role in several situations, for example [Iyama and Reiten 2008; Buan et al. 2009; Geiß et al. 2011; Oppermann et al. 2015; Baumann and Kamnitzer 2012; Baumann et al. 2014].

Then we use the following important results.

Theorem 4.1. (a) *There exists a bijection $W_\Delta \to \langle I_1, \dots, I_n \rangle$, which is given by $w \mapsto I_w = I_{i_1} I_{i_2} \cdots I_{i_k}$ for any reduced expression $w = s_{i_1} \cdots s_{i_k}$.*

(b) *There exist bijections*

$$W_\Delta \to \mathrm{s}\tau\text{-tilt}\,\Lambda \ \to 2\text{-silt}\,\Lambda,$$
$$w \mapsto (I_w, P_w) \mapsto S_w := \Psi(I_w, P_w).$$

(c) *The Weyl group W_Δ acts transitively and faithfully on $2\text{-silt}\,\Lambda$ by*

$$s_i \cdot (S_w) := \mu_i(S_w) \cong S_{s_i w},$$

where μ_i is the silting mutation associated with $i \in \Delta_0$.

Proof. (a) This follows from [Mizuno 2014, Theorem 2.14; Buan et al. 2009, III.1.9].

(b) This follows from [Mizuno 2014, Theorem 2.21] and Theorem 3.5.

(c) By [Mizuno 2014, Theorem 2.16], W_Δ acts transitively and faithfully on $\mathrm{s}\tau\text{-tilt}\,\Lambda$ by mutation of support τ-tilting pairs (see [Adachi et al. 2014, Theorems 2.18, 2.28] for mutation of support τ-tilting pairs). On the other hand, [Adachi et al. 2014, Corollary 3.9] implies that the bijection (b) gives the compatibility of mutation of support τ-tilting pairs and two-term silting complexes. Hence we get the conclusion. □

Now, the aim of this section is to show the following result.

Theorem 4.2. *Let Δ be a Dynkin graph, Λ the preprojective algebra of Δ and ι the Nakayama permutation of Λ.*

(a) *Let ν be the Nakayama functor of Λ. Then $\nu(I_w) \cong I_w$ if and only if $\iota(w) = w$.*

(b) *We have a bijection*

$$W_\Delta^\iota \to 2\text{-tilt}\,\Lambda, \quad w \mapsto S_w.$$

(c) *Let Δ^{f} be the folded graph of Δ* (Definition 3.2) *and define $\langle t_i \mid i \in \Delta_0^{\mathrm{f}} \rangle$ by* (T) *of Theorem 3.1. Then $\langle t_i \mid i \in \Delta_0^{\mathrm{f}} \rangle$ acts transitively and faithfully on* $2\text{-tilt}\,\Lambda$.

For a proof, we recall the notion of g-vectors of support τ-tilting modules. See [Mizuno 2014, Section 3; Adachi et al. 2014, Section 5] for details.

Let $K_0(\mathrm{proj}\,\Lambda)$ be the Grothendieck group of the additive category $\mathrm{proj}\,\Lambda$, which is isomorphic to the free abelian group $\mathbb{Z}^n$, and we identify the set of isomorphism classes of projective Λ-modules with the canonical basis $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ of $\mathbb{Z}^n$.

For a Λ-module X, take a minimal projective presentation

$$P_X^1 \longrightarrow P_X^0 \longrightarrow X \longrightarrow 0$$

and let $g(X) = (g_1(X), \cdots, g_n(X))^t := [P_X^0] - [P_X^1] \in \mathbb{Z}^n$. Then, for any $w \in W_\Delta$ and $i \in \Delta_0$, we define a *g-vector* by

$$\mathbb{Z}^n \ni g^i(w) = \begin{cases} g(e_i I_w) & \text{if } e_i I_w \neq 0, \\ -\boldsymbol{e}_{\iota(i)} & \text{if } e_i I_w = 0. \end{cases}$$

Then we define a *g-matrix* of a support τ-tilting Λ-module I_w by

$$g(w) := (g^1(w), \dots, g^n(w)) \in \mathrm{GL}_n(\mathbb{Z}).$$

Note that the g-vectors form a basis of $\mathbb{Z}^n$ [Adachi et al. 2014, Theorem 5.1].

On the other hand, we define a matrix $M_\iota := (\boldsymbol{e}_{\iota(1)}, \dots, \boldsymbol{e}_{\iota(n)}) \in \mathrm{GL}_n(\mathbb{Z})$ and, for $X \in \mathrm{GL}_n(\mathbb{Z})$, we define

$$\iota(X) := M_\iota \cdot X \cdot M_\iota.$$

Clearly the left multiplication (respectively, right multiplication) of M_ι to X gives a permutation of X from j-th to $\iota(j)$-th rows (respectively, columns) for any $j \in \Delta_0$ and $M_\iota^2 = \mathrm{id}$.

Moreover, we recall the following definition (see [Mizuno 2014, Definition 3.5]).

Definition 4.3 [Björner and Brenti 2005]. The *contragradient* $r : W_\Delta \to \mathrm{GL}_n(\mathbb{Z})$ *of the geometric representation* is defined by

$$r(s_i)(\boldsymbol{e}_j) = r_i(\boldsymbol{e}_j) = \begin{cases} \boldsymbol{e}_j, & i \neq j, \\ -\boldsymbol{e}_i + \sum\limits_{k - i} \boldsymbol{e}_k, & i = j, \end{cases}$$

where the sum is taken over all edges of i in Δ. We regard r_i as a matrix of $\mathrm{GL}_n(\mathbb{Z})$ and this extends to a group homomorphism.

Lemma 4.4. *For any $i \in \Delta_0$, we have*

$$\iota(r_i) = r_{\iota(i)}.$$

Proof. Since the left multiplication (respectively, right multiplication) of M_ι gives a permutation of rows (respectively, columns) from j-th to $\iota(j)$-th for any $j \in \Delta_0$, this follows from the definition of r_i and $r_{\iota(i)}$. □

Lemma 4.5. *For any $w \in W_\Delta$, we have*

$$\iota(g(w)) = g(\iota(w)).$$

Proof. Let $w = s_{i_1} \cdots s_{i_k}$ be an expression of w. Then, by [Mizuno 2014, Proposition 3.6], we conclude

$$g(w) = r_{i_k} \cdots r_{i_1}.$$

Hence we have

$$\begin{aligned} \iota(g(w)) &= M_\iota(r_{i_k} \dots r_{i_1})M_\iota \\ &= (M_\iota r_{i_k} M_\iota) \cdots (M_\iota r_{i_1} M_\iota) \quad && (M_\iota^2 = \mathrm{id}) \\ &= r_{\iota(i_k)} \dots r_{\iota(i_1)} && (\text{Lemma 4.4}) \\ &= g(\iota(w)). \end{aligned}$$

□

Moreover, we give the following lemma.

Lemma 4.6. *Let $w \in W_\Delta$.*

(a) *$\nu(I_w)$ is also a support τ-tilting Λ-module. In particular, there exists some $w' \in W_\Delta$ such that $\nu(I_w) \cong I_{w'}$.*

(b) *For the above w', we have*

$$g(w') = \iota(g(w)).$$

Proof. (a) Let (I_w, P_w) be a basic support τ-tilting pair of Λ, where P_w is the corresponding projective Λ-module. By Theorem 3.5, we have the two-term silting complex in $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\Lambda)$ by

$$S_w := (P^1_{I_w} \xrightarrow{f} P^0_{I_w}) \oplus P_w[1] \in \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\Lambda),$$

where

$$P^1_{I_w} \xrightarrow{f} P^0_{I_w} \longrightarrow I_w \longrightarrow 0$$

is a minimal projective presentation of I_w.

Then $\nu(S_w) = (\nu(P^1_{I_w}) \to \nu(P^0_{I_w})) \oplus \nu(P_w)[1] \in \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\Lambda)$ is clearly a two-term silting complex. Hence, by Theorem 3.5, $(\nu(I_w), \nu(P_w))$ is also a basic support τ-tilting pair of Λ. Thus, by Theorem 4.1, there exists $w' \in W_\Delta$ such that $\nu(I_w) \cong I_{w'}$.

(b) Take $i \in \Delta_0$. First assume that $e_i I_w \neq 0$ and take a minimal projective presentation of $e_i I_w$

$$P^1 \to P^0 \to e_i I_w \to 0.$$

By applying ν to this sequence, we have

$$\nu(P^1) \to \nu(P^0) \to \nu(e_i I_w) \to 0.$$

Because $[\nu(e_j\Lambda)] = [e_{\iota(j)}\Lambda] = M_\iota[e_j\Lambda]$ for any $j \in \Delta_0$, we have

$$[(\nu(P^0)] - [(\nu(P^1)] = M_\iota([P^0] - [P^1]) = M_\iota(g^i(w)).$$

Then, since we have $\nu(e_i I_w) \cong e_{\iota(i)} I_{w'}$, we obtain $g^{\iota(i)}(w') = M_\iota(g^i(w))$.

Next assume that $e_i I_w = 0$. Then we have $g^i(w) = -\boldsymbol{e}_{\iota(i)}$ by the definition. Because $\nu(e_j\Lambda) \cong e_{\iota(j)}\Lambda$ for any $j \in \Delta_0$, we obtain $g^{\iota(i)}(w') = -\boldsymbol{e}_i = M_\iota(g^i(w))$.

Consequently, we have

$$\begin{aligned} g(w') &= (g^1(w'), \dots, g^n(w')) \\ &= (g^{\iota(1)}(w'), \dots, g^{\iota(n)}(w')) \cdot M_\iota \\ &= (M_\iota(g^1(w)), \dots, M_\iota(g^n(w))) \cdot M_\iota \\ &= M_\iota \cdot (g^1(w), \dots, g^n(w)) \cdot M_\iota \\ &= \iota(g(w)). \end{aligned}$$

This finishes the proof. □

For the proof of Theorem 4.2 we recall the following nice property.

Theorem 4.7 [Adachi et al. 2014, Theorem 5.5]. *The map $X \to g(X)$ induces an injection from the set of isomorphism classes of τ-rigid pairs for Λ to $K_0(\mathsf{proj}\Lambda)$.*

Proof of Theorem 4.2. (a) We have the following equivalent conditions

$$\begin{aligned} \nu(I_w) \cong I_w \Leftrightarrow \iota(g(w)) = g(w) & \quad \text{(Lemma 4.6 and Theorem 4.7)} \\ \Leftrightarrow g(\iota(w)) = g(w) & \quad \text{(Lemma 4.5)} \\ \Leftrightarrow I_{\iota(w)} \cong I_w & \quad \text{(Theorem 4.7)} \\ \Leftrightarrow \iota(w) = w & \quad \text{(Theorem 4.1).} \end{aligned}$$

Thus we get the desired result.

(b) A silting complex S_w is a tilting complex if and only if $\nu(S_w) \cong S_w$ (see [Aihara 2013, Appendix]). Hence (a) implies that it is equivalent to say that $\iota(w) = w$. This proves our claim.

(c) By (b) and Theorem 3.1, the action of Theorem 4.1 induces the action of $\langle t_i \mid i \in \Delta_0^{\mathrm{f}} \rangle$ on 2-tilt Λ. □

Example 4.8. Let Δ be a graph of type $\mathbb{A}_3$ and Λ the preprojective algebra of Δ. Then the *support τ-tilting quiver* of Λ [Adachi et al. 2014, Definition 2.29] is given in Figure 2.

The framed modules indicate *ν-stable* modules [Mizuno 2015] (i.e., $I_w \cong \nu(I_w)$), which is equivalent to say that $\iota(w) = w$. Hence Theorems 3.1 and 4.2 imply that these modules are in bijection with the elements of the subgroup $W_\Delta^\iota = \langle s_1 s_3, s_2 \rangle$ and this group is isomorphic to the Weyl group of type $\mathbb{B}_2$.

5. Preprojective algebras are tilting-discrete

In this section, we show that preprojective algebras of Dynkin type are tilting-discrete. It implies that all tilting complexes are connected to each other by successive tilting mutation [Chan et al. 2015, Theorem 5.14; Aihara 2013, Theorem 3.5]. From this result, we can determine the derived equivalence class of the algebra.

Throughout this section, let Δ be a Dynkin graph with $\Delta_0 = \{1, \dots, n\}$, Λ the preprojective algebra of Δ, e_i the primitive idempotent of Λ associated with $i \in \Delta_0$ and Δ^{f} the folded graph of Δ. We also keep the notation of previous sections.

The aim of this section is to show the following theorem.

Theorem 5.1. *Let Λ be a preprojective algebra of Dynkin type.*

(a) $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$ *is tilting-discrete.*

(b) *Any basic tilting complex T of Λ satisfies* $\mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)}(T) \cong \Lambda$. *In particular, the derived equivalence class coincides with the Morita equivalence class.*

Notation. Let $\widetilde{\Delta}$ be an extended Dynkin graph obtained from Δ by adding a vertex 0 (i.e., $\widetilde{\Delta}_0 = \{0\} \cup \Delta_0$) with the associated arrows. Since

$$W_\Delta = \langle s_1, \dots, s_n \rangle \subset W_{\widetilde{\Delta}} = \langle s_1, \dots, s_n, s_0 \rangle,$$

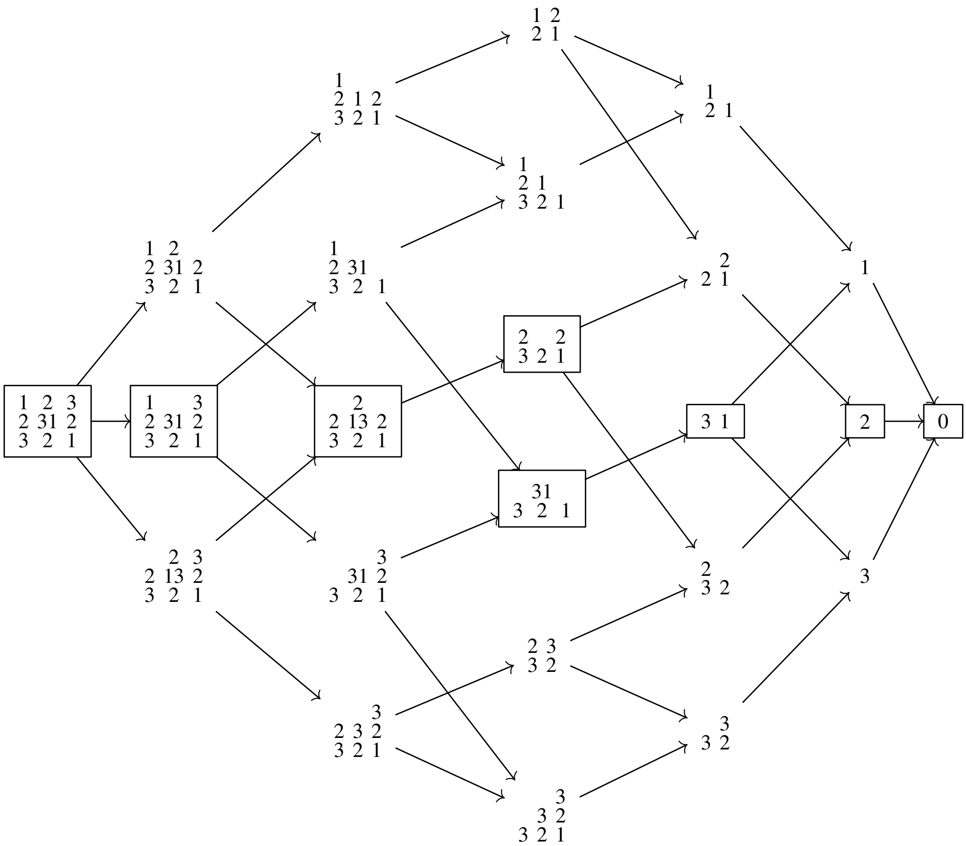

Figure 2. Diagram for Example 4.8.

we can regard elements of W_Δ as those of $W_{\widetilde{\Delta}}$. We denote by $\widetilde{\Lambda}$ the $\mathfrak{m}$-adic completion of the preprojective algebra of $\widetilde{\Delta}$, where $\mathfrak{m}$ is the ideal generated by all arrows. It implies that the Krull–Schmidt theorem holds for finitely generated projective $\widetilde{\Lambda}$-modules. Moreover we denote $\tilde{I}_i := \widetilde{\Lambda}(1-e_i)\widetilde{\Lambda}$, where e_i is the primitive idempotent of $\widetilde{\Lambda}$ associated with $i \in \widetilde{\Delta}_0$.

Recall that, by Theorem 4.1, we have a bijection between $W_{\widetilde{\Delta}}$ and $\langle \tilde{I}_1, \dots, \tilde{I}_n, \tilde{I}_0 \rangle$ [Buan et al. 2009, III.1.9] and hence for each element $w \in W_{\widetilde{\Delta}}$, we can define $\tilde{I}_w := \tilde{I}_{i_1} \cdots \tilde{I}_{i_k}$, where $w = s_{i_1} \cdots s_{i_k}$ is a reduced expression. Furthermore, it is shown that $\tilde{I}_w$ is a tilting $\widetilde{\Lambda}$-module [Buan et al. 2009, Theorem III.1.6].

Note that if $i \neq 0 \in \widetilde{\Delta}_0$, then we have

$$\Lambda = \widetilde{\Lambda}/\langle e_0 \rangle \quad \text{and} \quad I_i = \tilde{I}_i/\langle e_0 \rangle.$$

In particular, for $w \in W_\Delta$, we have $\tilde{I}_w/\langle e_0 \rangle = I_w$ and hence $\widetilde{\Lambda}/\tilde{I}_w \cong \Lambda/I_w$.

Recall that we can describe the two-term silting complex of $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\Lambda)$ by

$$S_w := \begin{cases} P^1_{I_w} \xrightarrow{f} P^0_{I_w} \\ \qquad\quad \oplus \\ P_w \end{cases}$$

where $P^1_{I_w} \xrightarrow{f} P^0_{I_w} \to I_w \to 0$ is a minimal projective presentation of I_w.

Then we show that $\tilde{I}_w \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \Lambda$ gives a two-term silting complex S_w.

Proposition 5.2. *For $w \in W_\Delta$, $\tilde{I}_w \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \Lambda$ is isomorphic to S_w in $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\Lambda)$.*

Proof. Since $\tilde{I}_w$ is a tilting $\widetilde{\Lambda}$-module, we have a minimal projective resolution

$$0 \longrightarrow \widetilde{P}_1 \xrightarrow{g} \widetilde{P}_0 \longrightarrow \tilde{I}_w \longrightarrow 0.$$

By applying the functor $- \otimes_{\widetilde{\Lambda}} \Lambda$, we have the following exact sequence [Mizuno 2014, Proposition 3.2]

$$0 \to \nu^{-1}(\Lambda/I_w) \to \widetilde{P}_1 \otimes_{\widetilde{\Lambda}} \Lambda \xrightarrow{g\otimes\Lambda} \widetilde{P}_0 \otimes_{\widetilde{\Lambda}} \Lambda \to \tilde{I}_w \otimes_{\widetilde{\Lambda}} \Lambda \to 0.$$

Because we have an isomorphism in $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\widetilde{\Lambda})$

$$\tilde{I}_w \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \Lambda \cong \big(\cdots \to 0 \to \widetilde{P}_1 \otimes_{\widetilde{\Lambda}} \Lambda \xrightarrow{g\otimes\Lambda} \widetilde{P}_0 \otimes_{\widetilde{\Lambda}} \Lambda \to 0 \to \cdots\big),$$

one can check that $\tilde{I}_w \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \Lambda$ is isomorphic to S_w (Theorem 3.5). □

For $w \in W_\Delta$, we denote the inclusion by $\boldsymbol{i} : \tilde{I}_w \hookrightarrow \widetilde{\Lambda}$. Then we show the following lemma.

Lemma 5.3. *Let w_0 be the longest element of W_Δ. For $w \in W^\iota_\Delta$, we have isomorphisms $p : \tilde{I}_w \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \tilde{I}_{w_0} \to \tilde{I}_{w_0} \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \tilde{I}_w$ and $q : \tilde{I}_w \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \widetilde{\Lambda} \to \widetilde{\Lambda} \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \tilde{I}_w$, which make the following diagram commutative*

$$\begin{tikzcd} \tilde{I}_w \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \tilde{I}_{w_0} \arrow[r, "\mathrm{id}\otimes\boldsymbol{i}"] \arrow[d, "\cong"', "p"] & \tilde{I}_w \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \widetilde{\Lambda} \arrow[d, "\cong"', "q"] \\ \tilde{I}_{w_0} \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \tilde{I}_w \arrow[r, "\boldsymbol{i}\otimes\mathrm{id}"] & \widetilde{\Lambda} \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \tilde{I}_w \end{tikzcd}$$

Proof. Because $\ell(w_0 w^{-1}) + \ell(w) = \ell(w_0)$, [Buan et al. 2009, Propositions II.1.5(a), II.1.10] gives the following commutative diagram:

$$\begin{tikzcd} \tilde{I}_{w_0} \arrow[r, "\boldsymbol{i}"] & \widetilde{\Lambda} \arrow[d, "\cong"'] \\ \tilde{I}_{w_0 w^{-1}} \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \tilde{I}_w \arrow[u, "\cong"] \arrow[r, "\boldsymbol{i}\otimes\boldsymbol{i}"] & \widetilde{\Lambda} \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \widetilde{\Lambda} \end{tikzcd}$$

Hence we have

$$\begin{tikzcd} \tilde{I}_w \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \tilde{I}_{w_0} \arrow[r, "\mathrm{id}\otimes \boldsymbol{i}"] & \tilde{I}_w \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \widetilde{\Lambda} \arrow[r, "\boldsymbol{i}\otimes \mathrm{id}"] & \widetilde{\Lambda} \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \widetilde{\Lambda} \arrow[d, "\cong"'] \\ \tilde{I}_w \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \tilde{I}_{w_0 w^{-1}} \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \tilde{I}_w \arrow[u, "\cong"] \arrow[rr, "\boldsymbol{i}\otimes\boldsymbol{i}\otimes\boldsymbol{i}"] & & \widetilde{\Lambda} \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \widetilde{\Lambda} \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \widetilde{\Lambda} \end{tikzcd}$$

Since $w \in W^{\iota}_{\Delta}$, we have $w_0 w = w w_0$ (Section 3B) and hence $\tilde{I}_{w_0 w^{-1}} = \tilde{I}_{w^{-1} w_0}$. Then similarly we have the following commutative diagram

$$\begin{tikzcd} \tilde{I}_w \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \tilde{I}_{w^{-1} w_0} \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \tilde{I}_w \arrow[rr, "\boldsymbol{i}\otimes\boldsymbol{i}\otimes\boldsymbol{i}"] \arrow[d, "\cong"'] & & \widetilde{\Lambda} \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \widetilde{\Lambda} \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \widetilde{\Lambda} \\ \tilde{I}_{w_0} \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \tilde{I}_w \arrow[r, "\boldsymbol{i}\otimes \mathrm{id}"] & \widetilde{\Lambda} \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \tilde{I}_w \arrow[r, "\mathrm{id}\otimes \boldsymbol{i}"] & \widetilde{\Lambda} \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \widetilde{\Lambda} \arrow[u, "\cong"'] \end{tikzcd}$$

Moreover we have the following commutative diagram

$$\begin{tikzcd} \tilde{I}_w \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \widetilde{\Lambda} \arrow[r, "\boldsymbol{i}\otimes \mathrm{id}"] \arrow[d, "\cong"'] & \widetilde{\Lambda} \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \widetilde{\Lambda} \arrow[d, "\cong"'] \\ \widetilde{\Lambda} \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \tilde{I}_w \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \widetilde{\Lambda} \arrow[r, "\mathrm{id}\otimes\boldsymbol{i}\otimes\mathrm{id}"] & \widetilde{\Lambda} \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \widetilde{\Lambda} \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \widetilde{\Lambda} \\ \widetilde{\Lambda} \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \tilde{I}_w \arrow[r, "\mathrm{id}\otimes \boldsymbol{i}"] \arrow[u, "\cong"] & \widetilde{\Lambda} \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \widetilde{\Lambda} \arrow[u, "\cong"] \end{tikzcd}$$

Put $L := \tilde{I}_w \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \tilde{I}_{w^{-1} w_0} \otimes^{\boldsymbol{L}}_{\widetilde{\Lambda}} \tilde{I}_w$. Consider a morphism $u : L \to \tilde{I}_w$ and the triangle

$$\cdots \longrightarrow \widetilde{\Lambda}/\tilde{I}_w[-1] \longrightarrow \tilde{I}_w \xrightarrow{\boldsymbol{i}} \widetilde{\Lambda} \longrightarrow \widetilde{\Lambda}/\tilde{I}_w \longrightarrow \cdots.$$

If $\boldsymbol{i} \circ u = 0$, then there exists a map $v : L \to \widetilde{\Lambda}/\tilde{I}_w[-1]$ which makes commutative the diagram

$$\begin{tikzcd} & & L \arrow[dll, dashed, "v"'] \arrow[d, "u"] & & & \\ \cdots \arrow[r] & \widetilde{\Lambda}/\tilde{I}_w[-1] \arrow[r] & \tilde{I}_w \arrow[r, "\boldsymbol{i}"] & \widetilde{\Lambda} \arrow[r] & \widetilde{\Lambda}/\tilde{I}_w \arrow[r] & \cdots \end{tikzcd}$$

Because $H^i(L) = 0$ for any $i > 0$, we get $v = 0$ and hence $u = 0$. Thus the above diagrams provide the required morphisms. $\square$

From the above results, we have the following nice consequence.

Proposition 5.4. *For any $w \in W_\Delta^\iota$, we have an isomorphism*

$$\operatorname{End}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj}\Lambda)}(\tilde{I}_w \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \Lambda) \cong \Lambda.$$

In particular, the endomorphism algebra of any basic two-term tilting complex is isomorphic to Λ.

Proof. Let w_0 be the longest element of W_Δ. Since $\tilde{I}_{w_0} = \langle e_0 \rangle$, we have the following exact sequence

$$0 \longrightarrow \tilde{I}_{w_0} \longrightarrow \widetilde{\Lambda} \longrightarrow \Lambda \longrightarrow 0.$$

Then applying the functor $\tilde{I}_w \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} -$ to the exact sequence, we have the triangle

$$\tilde{I}_w \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{I}_{w_0} \longrightarrow \tilde{I}_w \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \widetilde{\Lambda} \longrightarrow \tilde{I}_w \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \Lambda \longrightarrow \tilde{I}_w \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{I}_{w_0}[1].$$

Similarly, applying the functor $- \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{I}_w$ to the first exact sequence, we have the triangle

$$\tilde{I}_{w_0} \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{I}_w \longrightarrow \widetilde{\Lambda} \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{I}_w \longrightarrow \Lambda \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{I}_w \longrightarrow \tilde{I}_{w_0} \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{I}_w[1].$$

By Lemma 5.3, we have the following commutative diagram

$$\begin{tikzcd}
\tilde{I}_w \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{I}_{w_0} \arrow[r] \arrow[d, "p", "\cong"'] & \tilde{I}_w \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \widetilde{\Lambda} \arrow[r] \arrow[d, "q", "\cong"'] & \tilde{I}_w \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \Lambda \arrow[r] \arrow[d, dotted, "r"] & \tilde{I}_w \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{I}_{w_0}[1] \arrow[d] \\
\tilde{I}_{w_0} \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{I}_w \arrow[r] & \widetilde{\Lambda} \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{I}_w \arrow[r] & \Lambda \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{I}_w \arrow[r] & \tilde{I}_{w_0} \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{I}_w[1]
\end{tikzcd}$$

and the isomorphism r. Because $\tilde{I}_w$ is a tilting module [Buan et al. 2009, Theorem III.1.6] and we have $\widetilde{\Lambda} \cong \operatorname{Hom}_{\tilde{\Lambda}}(\tilde{I}_w, \tilde{I}_w)$ [Buan et al. 2009, Proposition II.1.4], we obtain

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_\Lambda(\tilde{I}_w \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \Lambda, \tilde{I}_w \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \Lambda) &\cong \mathbf{R}\mathrm{Hom}_{\tilde{\Lambda}}(\tilde{I}_w, \tilde{I}_w \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \Lambda) \\
&\cong \mathbf{R}\mathrm{Hom}_{\tilde{\Lambda}}(\tilde{I}_w, \Lambda \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{I}_w) \\
&\cong \Lambda \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \mathbf{R}\mathrm{Hom}_{\tilde{\Lambda}}(\tilde{I}_w, \tilde{I}_w) \\
&\cong \Lambda \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \widetilde{\Lambda} \\
&\cong \Lambda.
\end{aligned}$$

Then by taking the 0th part, we get the assertion. The second statement immediately follows from the first one, Theorem 4.2 and Proposition 5.2. $\square$

Corollary 5.5. *Let T be a tilting complex which is given by iterated irreducible left tilting mutation from Λ. Then we have*

$$\mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)}(T) \cong \Lambda.$$

Proof. Let $T = \boldsymbol{\mu}^{+}_{(\ell)} \circ \cdots \circ \boldsymbol{\mu}^{+}_{(1)}(\Lambda)$, where $\boldsymbol{\mu}$ denotes irreducible left tilting mutation. We proceed by induction on ℓ. Assume that, for $T' = \boldsymbol{\mu}^{+}_{(\ell-1)} \circ \cdots \circ \boldsymbol{\mu}^{+}_{(1)}(\Lambda)$, we have $\mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)}(T') \cong \Lambda$. Then we have an equivalence $F : \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda) \to \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$ such that $F(T') \cong \Lambda$ [Rickard 1989]. Therefore we have $\mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)}(\boldsymbol{\mu}^{+}_{(\ell)}(T')) \cong \mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)}(\boldsymbol{\mu}^{+}_{(\ell)}(\Lambda))$ and hence it is isomorphic to Λ by Proposition 5.4. □

Now we are ready to give a proof of Theorem 5.1.

Proof of Theorem 5.1. (a) We will check the condition (c) of Corollary 2.11.

Recall that $2\text{-tilt}_T\,\Lambda := \{U \in \mathrm{tilt}\,\Lambda \mid T \geq U \geq T[1]\}$. We denote by $\sharp\,2\text{-tilt}_T\,\Lambda$ the number of $2\text{-tilt}_T\,\Lambda$.

By Theorem 4.2, the set $2\text{-tilt}_\Lambda\,\Lambda = 2\text{-tilt}\,\Lambda$ is finite. Let T be a tilting complex which is given by iterated irreducible left tilting mutation from Λ. Then we have $\mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)}(T) \cong \Lambda$ from Corollary 5.5. Therefore, we have an equivalence

$$F : \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda) \to \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$$

such that $F(T) \cong \Lambda$ and hence we get

$$\sharp\big\{U \in \mathrm{tilt}\,\Lambda \;\big|\; T \geq U \geq T[1]\big\} = \sharp\big\{F(U) \in \mathrm{tilt}\,\Lambda \;\big|\; \Lambda \geq F(U) \geq \Lambda[1]\big\}.$$

Thus it is also finite and we obtain the statement.

(b) Let T be a basic tilting complex such that $\Lambda \geq T$. Since Λ is tilting-discrete, T is obtained by iterated irreducible left tilting mutation from Λ [Chan et al. 2015, Theorem 5.11; Aihara 2013, Theorem 3.5]. Thus the statement follows from Corollary 5.5. Because for any tilting complex T, we have $\Lambda \geq T[\ell]$ for some ℓ [Aihara and Iyama 2012, Proposition 2.4] and $\mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)}(T) \cong \mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)}(T[\ell])$, we get the conclusion from the above argument. □

6. Tilting complexes and braid groups

In this section, we show that irreducible mutation satisfy the braid relations and we give a bijection from the elements of the braid group to the set of tilting complexes.

We keep the notation of previous sections.

Define $W^{\iota}_{\Delta} = \langle t_i \mid i \in \Delta_0^{\mathrm{f}}\rangle$ by (T) of Theorem 3.1. By Theorems 4.1 and 4.2, we have $S_{t_i} = \boldsymbol{\mu}^{+}_{i}(\Lambda)$ $(i \in \Delta_0^{\mathrm{f}})$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\Lambda)$, where $\boldsymbol{\mu}^{+}_{i}$ is given as a composition of left silting mutation as follows

$$\boldsymbol{\mu}_i^+ := \begin{cases} \mu_i^+ & \text{if } i = \iota(i) \text{ in } \Delta, \\ \mu_i^+ \circ \mu_{\iota(i)}^+ \circ \mu_i^+ & \text{if there is an edge } i \text{ --- } \iota(i) \text{ in } \Delta, \\ \mu_i^+ \circ \mu_{\iota(i)}^+ & \text{if there is no edge between } i \text{ and } \iota(i) \text{ in } \Delta. \end{cases}$$

Moreover, we let

$$e_{t_i} := \begin{cases} e_i & \text{if } i = \iota(i) \text{ in } \Delta, \\ e_i + e_{\iota(i)} & \text{if } i \neq \iota(i) \text{ in } \Delta. \end{cases}$$

Then, it is easy to check that $\boldsymbol{\mu}_i^+(\Lambda) = \mu_{(e_{t_i}\Lambda)}^+(\Lambda)$ and hence we have

$$S_{t_i} = \begin{cases} \overset{-1}{e_{t_i}\Lambda} \xrightarrow{f} \overset{0}{R_{t_i}} \\ \qquad\quad \oplus \\ \qquad\qquad (1-e_{t_i})\Lambda \end{cases} \in \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\Lambda),$$

where f is a minimal left $(\mathsf{add}((1-e_{t_i})\Lambda))$-approximation.

Thus $\boldsymbol{\mu}_i^+$ is an irreducible left tilting mutation of Λ and any irreducible left tilting mutation of Λ is given as $\boldsymbol{\mu}_i^+$ for some $i \in \Delta_0^{\mathrm{f}}$. Dually, we define $\boldsymbol{\mu}_i^-$ so that $\boldsymbol{\mu}_i^- \circ \boldsymbol{\mu}_i^+ = \mathrm{id}$ [Aihara and Iyama 2012, Proposition 2.33].

Let $F_{\Delta^{\mathrm{f}}}$ be the free group generated by a_i $(i \in \Delta_0^{\mathrm{f}})$. Then we define the map

$$\begin{gathered} F_{\Delta^{\mathrm{f}}} \to \operatorname{tilt}\Lambda, \\ a = a_{i_1}^{\epsilon_{i_1}} \cdots a_{i_k}^{\epsilon_{i_k}} \mapsto \boldsymbol{\mu}_a(\Lambda) := \boldsymbol{\mu}_{i_1}^{\epsilon_{i_1}} \circ \cdots \circ \boldsymbol{\mu}_{i_k}^{\epsilon_{i_k}}(\Lambda). \end{gathered}$$

Then we give the following proposition.

Proposition 6.1. *For any $a \in F_{\Delta^{\mathrm{f}}}$, we let $T := \boldsymbol{\mu}_a(\Lambda)$. Then we have the following braid relations in $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\Lambda)$:*

$$\begin{aligned} &\boldsymbol{\mu}_i^+ \circ \boldsymbol{\mu}_j^+(T) \cong \boldsymbol{\mu}_j^+ \circ \boldsymbol{\mu}_i^+(T) && \text{if } \nexists \text{ an edge between } i \text{ and } j \text{ in } \Delta^{\mathrm{f}}, \\ &\boldsymbol{\mu}_i^+ \circ \boldsymbol{\mu}_j^+ \circ \boldsymbol{\mu}_i^+(T) \cong \boldsymbol{\mu}_j^+ \circ \boldsymbol{\mu}_i^+ \circ \boldsymbol{\mu}_j^+(T) && \text{if } \exists \text{ an edge } i \text{ --- } j \text{ in } \Delta^{\mathrm{f}}, \\ &\boldsymbol{\mu}_i^+ \circ \boldsymbol{\mu}_j^+ \circ \boldsymbol{\mu}_i^+ \circ \boldsymbol{\mu}_j^+(T) \cong \boldsymbol{\mu}_j^+ \circ \boldsymbol{\mu}_i^+ \circ \boldsymbol{\mu}_j^+ \circ \boldsymbol{\mu}_i^+(T) && \text{if } \exists \text{ an edge } i \overset{4}{\text{---}} j \text{ in } \Delta^{\mathrm{f}}. \end{aligned}$$

Proof. By Theorem 4.2, the assertion holds for $T = \Lambda$. Moreover, by Theorem 5.1, T satisfies $\operatorname{End}_{\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\Lambda)}(T) \cong \Lambda$ and hence we have an equivalence $F : \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\Lambda) \to \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\Lambda)$ such that $F(T) \cong \Lambda$. Since mutation is preserved by an equivalence, the assertion holds for T. $\square$

Now we recall the following definition.

Definition 6.2. The braid group $B_{\Delta^{\mathrm{f}}}$ is defined by generators a_i $(i \in \Delta_0^{\mathrm{f}})$ and relations $(a_i a_j)^{m(i,j)} = 1$ for $i \neq j$ (i.e., the difference with $W_{\Delta^{\mathrm{f}}}$ is that we do not require the relations $a_i^2 = 1$ for $i \in \Delta_0^{\mathrm{f}}$). Moreover we denote the positive braid monoid by $B_{\Delta^{\mathrm{f}}}^+$.

As a consequence of the above results, we have the following proposition.

Proposition 6.3. *There is a map*

$$B_{\Delta^{\mathrm{f}}} \to \operatorname{tilt}\Lambda, \quad a \mapsto \boldsymbol{\mu}_a(\Lambda).$$

Moreover, it is surjective.

Proof. The first statement follows from Proposition 6.1. Since Λ is tilting-discrete, any tilting complex can be obtained from Λ by iterated irreducible tilting mutation [Chan et al. 2015, Theorem 5.11; Aihara and Iyama 2012, Theorem 3.5]. Thus the map is surjective. □

Finally, we will show that the map of Proposition 6.3 is injective.

Recall that $T > \boldsymbol{\mu}_a(T)$ for any $a \in B^+_{\Delta^{\mathrm{f}}}$ (Definition 2.3). Then we have the following result.

Lemma 6.4. *The map*

$$B^+_{\Delta^{\mathrm{f}}} \to \operatorname{tilt}\Lambda, \quad a \mapsto \boldsymbol{\mu}_a(\Lambda)$$

is injective.

Proof. We denote by $\ell(a)$ the length of $a \in B^+_{\Delta^{\mathrm{f}}}$, that is, the number of elements of the expression a. We show by induction on the length of $B^+_{\Delta^{\mathrm{f}}}$. Take $b, c \in B^+_{\Delta^{\mathrm{f}}}$ such that $\boldsymbol{\mu}_b(\Lambda) \cong \boldsymbol{\mu}_c(\Lambda)$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\Lambda)$. Without loss of generality, we can assume that $\ell(b) \le \ell(c)$.

If $\ell(b) = 0$, (or equivalently, $b = \mathrm{id}$), then $\boldsymbol{\mu}_b(\Lambda) = \Lambda$. Then we have $c = \mathrm{id}$ because otherwise $\Lambda > \boldsymbol{\mu}_c(\Lambda)$.

Next assume that $\ell(b) > 0$ and the statement holds for any element if the length is less than $\ell(b)$. We write $b = b'a_i$ and $c = c'a_j$ for some $b', c' \in B^+_{\Delta^{\mathrm{f}}}$ and $i, j \in \Delta^{\mathrm{f}}_0$. If $i = j$, then $\boldsymbol{\mu}_{b'}(\Lambda) \cong \boldsymbol{\mu}_{c'}(\Lambda)$ and the induction hypothesis implies that $b' = c'$ and hence $b = c$.

Hence assume that $i \neq j$. Then we define

$$a_{i,j} := \begin{cases} a_i a_j & \text{if no edge between } i \text{ and } j \text{ in } \Delta^{\mathrm{f}}, \\ a_i a_j a_i & \text{if there is an edge } i \,\text{—}\, j \text{ in } \Delta^{\mathrm{f}}, \\ a_i a_j a_i a_j & \text{if there is an edge } i \overset{4}{\text{—}} j \text{ in } \Delta^{\mathrm{f}}. \end{cases}$$

Then $\boldsymbol{\mu}_{a_{i,j}}(\Lambda)$ is a meet of $\boldsymbol{\mu}_{a_i}(\Lambda)$ and $\boldsymbol{\mu}_{a_j}(\Lambda)$ by Theorem 4.2, [Mizuno 2014, Theorem 2.30] and [Adachi et al. 2014, Corollary 3.9]. Therefore we get $\boldsymbol{\mu}_{a_{i,j}}(\Lambda) \ge \boldsymbol{\mu}_b(\Lambda)$ since $\boldsymbol{\mu}_{a_i}(\Lambda) \ge \boldsymbol{\mu}_b(\Lambda)$ and $\boldsymbol{\mu}_{a_j}(\Lambda) \ge \boldsymbol{\mu}_c(\Lambda) \cong \boldsymbol{\mu}_b(\Lambda)$.

Because Λ is tilting-discrete and $\Lambda > \boldsymbol{\mu}_{a_{i,j}}(\Lambda)$, there exists $d \in B^+_{\Delta^{\mathrm{f}}}$ such that $\boldsymbol{\mu}_d(\boldsymbol{\mu}_{a_{i,j}}(\Lambda)) = \boldsymbol{\mu}_{da_{i,j}}(\Lambda) \cong \boldsymbol{\mu}_b(\Lambda)$. Then we have $\boldsymbol{\mu}_{da_{i,j}a_i^{-1}}(\Lambda) \cong \boldsymbol{\mu}_{b'}(\Lambda)$. Since we have $da_{i,j}a_i^{-1} \in B^+_{\Delta^{\mathrm{f}}}$, the induction hypothesis implies that $da_{i,j}a_i^{-1} = b'$ and hence $da_{i,j} = b$. Similarly, we have $\boldsymbol{\mu}_{da_{i,j}a_j^{-1}}(\Lambda) \cong \boldsymbol{\mu}_{c'}(\Lambda)$ and we get $da_{i,j}a_j^{-1} = c'$. Therefore, we get $b = da_{i,j} = c'a_j = c$ and the assertion holds. □

As an immediate consequence, we obtain the following result (cf. [Brav and Thomas 2011, Lemma 2.3]).

Proposition 6.5. *The map*

$$B_{\Delta^{\mathrm{f}}} \to \operatorname{tilt} \Lambda, \quad a \mapsto \boldsymbol{\mu}_a(\Lambda)$$

is injective.

Proof. It is enough to show that $\boldsymbol{\mu}_a(\Lambda) \cong \Lambda$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda)$ implies $a = \mathrm{id}$. In fact, $\boldsymbol{\mu}_a(\Lambda) \cong \boldsymbol{\mu}_{a'}(\Lambda)$ implies $\boldsymbol{\mu}_{aa'^{-1}}(\Lambda) \cong \Lambda$. Then if $aa'^{-1} = \mathrm{id}$, then we get $a = a'$.

It is well-known that any element $a \in B_{\Delta^{\mathrm{f}}}$ is given by $a = b^{-1}c$ for some $b, c \in B^+_{\Delta^{\mathrm{f}}}$ [Kassel and Turaev 2008, Section 6.6]. Hence, $\boldsymbol{\mu}_a(\Lambda) \cong \Lambda$ is equivalent to saying that $\boldsymbol{\mu}_{b^{-1}c}(\Lambda) \cong \Lambda$. Then we have $\boldsymbol{\mu}_b(\Lambda) \cong \boldsymbol{\mu}_c(\Lambda)$ and Lemma 6.4 implies $b = c$. Thus we get the assertion. □

Consequently, we obtain the following conclusion.

Theorem 6.6. *There is a bijection*

$$B_{\Delta^{\mathrm{f}}} \to \operatorname{tilt} \Lambda, \quad a \mapsto \boldsymbol{\mu}_a(\Lambda).$$

Proof. The statement follows from Propositions 6.3 and 6.5. □

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References

[Adachi et al. 2014] T. Adachi, O. Iyama, and I. Reiten, "τ-tilting theory", *Compos. Math.* **150**:3 (2014), 415–452. MR Zbl

[Adachi et al. 2015] T. Adachi, T. Aihara, and A. Chan, "Classification of two-term tilting complexes over Brauer graph algebras", preprint, 2015. arXiv

[Aihara 2013] T. Aihara, "Tilting-connected symmetric algebras", *Algebr. Represent. Theory* **16**:3 (2013), 873–894. MR Zbl

[Aihara and Iyama 2012] T. Aihara and O. Iyama, "Silting mutation in triangulated categories", *J. Lond. Math. Soc.* (2) **85**:3 (2012), 633–668. MR Zbl

[Auslander and Reiten 1996] M. Auslander and I. Reiten, "*D*Tr-periodic modules and functors", pp. 39–50 in *Representation theory of algebras* (Cocoyoc, Mexico, 1994), edited by R. Bautista et al., CMS Conf. Proc. **18**, Amer. Math. Soc., Providence, RI, 1996. MR

[Auslander et al. 1995] M. Auslander, I. Reiten, and S. O. Smalø, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics **36**, Cambridge University Press, 1995. MR Zbl

[Baer et al. 1987] D. Baer, W. Geigle, and H. Lenzing, "The preprojective algebra of a tame hereditary Artin algebra", *Comm. Algebra* **15**:1-2 (1987), 425–457. MR Zbl

[Baumann and Kamnitzer 2012] P. Baumann and J. Kamnitzer, "Preprojective algebras and MV polytopes", *Represent. Theory* **16** (2012), 152–188. MR Zbl

[Baumann et al. 2014] P. Baumann, J. Kamnitzer, and P. Tingley, "Affine Mirković–Vilonen polytopes", *Publ. Math. Inst. Hautes Études Sci.* **120** (2014), 113–205. MR Zbl

[Björner and Brenti 2005] A. Björner and F. Brenti, *Combinatorics of Coxeter groups*, Graduate Texts in Mathematics **231**, Springer, 2005. MR Zbl

[Brav and Thomas 2011] C. Brav and H. Thomas, "Braid groups and Kleinian singularities", *Math. Ann.* **351**:4 (2011), 1005–1017. MR Zbl

[Brenner et al. 2002] S. Brenner, M. C. R. Butler, and A. D. King, "Periodic algebras which are almost Koszul", *Algebr. Represent. Theory* **5**:4 (2002), 331–367. MR Zbl

[Broomhead et al. 2016] N. Broomhead, D. Pauksztello, and D. Ploog, "Discrete derived categories, II: The silting pairs CW complex and the stability manifold", *J. Lond. Math. Soc.* (2) **93**:2 (2016), 273–300. MR Zbl

[Brüstle and Yang 2013] T. Brüstle and D. Yang, "Ordered exchange graphs", pp. 135–193 in *Advances in representation theory of algebras*, edited by D. J. Benson et al., Eur. Math. Soc., Zürich, 2013. MR Zbl

[Buan et al. 2009] A. B. Buan, O. Iyama, I. Reiten, and J. Scott, "Cluster structures for 2-Calabi–Yau categories and unipotent groups", *Compos. Math.* **145**:4 (2009), 1035–1079. MR Zbl

[Buan et al. 2011] A. B. Buan, I. Reiten, and H. Thomas, "Three kinds of mutation", *J. Algebra* **339** (2011), 97–113. MR Zbl

[Carter 1989] R. W. Carter, *Simple groups of Lie type*, Wiley, New York, 1989. MR Zbl

[Chan et al. 2015] A. Chan, S. Koenig, and Y. Liu, "Simple-minded systems, configurations and mutations for representation-finite self-injective algebras", *J. Pure Appl. Algebra* **219**:6 (2015), 1940–1961. MR Zbl

[Crawley-Boevey 2000] W. Crawley-Boevey, "On the exceptional fibres of Kleinian singularities", *Amer. J. Math.* **122**:5 (2000), 1027–1037. MR Zbl

[Dlab and Ringel 1980] V. Dlab and C. M. Ringel, "The preprojective algebra of a modulated graph", pp. 216–231 in *Representation theory, II* (Ottawa, 1979), edited by V. Dlab and P. Gabriel, Lecture Notes in Math. **832**, Springer, 1980. MR Zbl

[Erdmann and Snashall 1998] K. Erdmann and N. Snashall, "Preprojective algebras of Dynkin type, periodicity and the second Hochschild cohomology", pp. 183–193 in *Algebras and modules, II* (Geiranger, Norway, 1996), edited by I. Reiten et al., CMS Conf. Proc. **24**, Amer. Math. Soc., Providence, RI, 1998. MR Zbl

[Geiß et al. 2006] C. Geiß, B. Leclerc, and J. Schröer, "Rigid modules over preprojective algebras", *Invent. Math.* **165**:3 (2006), 589–632. MR Zbl

[Geiß et al. 2011] C. Geiß, B. Leclerc, and J. Schröer, "Kac–Moody groups and cluster algebras", *Adv. Math.* **228**:1 (2011), 329–433. MR Zbl

[Gel'fand and Ponomarev 1979] I. M. Gel'fand and V. A. Ponomarev, "Model algebras and representations of graphs", *Funktsional. Anal. i Prilozhen.* **13**:3 (1979), 1–12. In Russian; translated in *Functional Anal. Appl.* **13**:3 (1979), no. 3, 157–166. MR

[Grant 2013] J. Grant, "Derived autoequivalences from periodic algebras", *Proc. Lond. Math. Soc.* (3) **106**:2 (2013), 375–409. MR Zbl

[Iyama and Reiten 2008] O. Iyama and I. Reiten, "Fomin–Zelevinsky mutation and tilting modules over Calabi–Yau algebras", *Amer. J. Math.* **130**:4 (2008), 1087–1149. MR Zbl

[Iyama and Reiten 2014] O. Iyama and I. Reiten, "Introduction to τ-tilting theory", *Proc. Natl. Acad. Sci. USA* **111**:27 (2014), 9704–9711. MR Zbl

[Iyama et al. 2014] O. Iyama, P. Jørgensen, and D. Yang, "Intermediate co-t-structures, two-term silting objects, τ-tilting modules, and torsion classes", *Algebra Number Theory* **8**:10 (2014), 2413–2431. MR Zbl

[Kashiwara and Saito 1997] M. Kashiwara and Y. Saito, "Geometric construction of crystal bases", *Duke Math. J.* **89**:1 (1997), 9–36. MR Zbl

[Kassel and Turaev 2008] C. Kassel and V. Turaev, *Braid groups*, Graduate Texts in Mathematics **247**, Springer, 2008. MR Zbl

[Keller 1994] B. Keller, "Deriving DG categories", *Ann. Sci. École Norm. Sup.* (4) **27**:1 (1994), 63–102. MR Zbl

[Keller and Vossieck 1988] B. Keller and D. Vossieck, "Aisles in derived categories", pp. 239–253 in *Deuxième Contact Franco–Belge en Algèbre* (Faulx-les-Tombes, Belgium, 1987), edited by J.-P. Tignol and P. Van Praag, Bull. Soc. Math. Belg. Sér. A **2**, Société Mathématique de Belgique, Brussels, 1988. MR Zbl

[Khovanov and Seidel 2002] M. Khovanov and P. Seidel, "Quivers, Floer cohomology, and braid group actions", *J. Amer. Math. Soc.* **15**:1 (2002), 203–271. MR Zbl

[Koenig and Yang 2014] S. Koenig and D. Yang, "Silting objects, simple-minded collections, t-structures and co-t-structures for finite-dimensional algebras", *Doc. Math.* **19** (2014), 403–438. MR Zbl

[Lusztig 1991] G. Lusztig, "Quivers, perverse sheaves, and quantized enveloping algebras", *J. Amer. Math. Soc.* **4**:2 (1991), 365–421. MR Zbl

[Lusztig 2000] G. Lusztig, "Semicanonical bases arising from enveloping algebras", *Adv. Math.* **151**:2 (2000), 129–139. MR Zbl

[Mizuno 2014] Y. Mizuno, "Classifying τ-tilting modules over preprojective algebras of Dynkin type", *Math. Z.* **277**:3-4 (2014), 665–690. MR Zbl

[Mizuno 2015] Y. Mizuno, "ν-stable τ-tilting modules", *Comm. Algebra* **43**:4 (2015), 1654–1667. MR Zbl

[Nakajima 1994] H. Nakajima, "Instantons on ALE spaces, quiver varieties, and Kac–Moody algebras", *Duke Math. J.* **76**:2 (1994), 365–416. MR Zbl

[Nakajima 1998] H. Nakajima, "Quiver varieties and Kac–Moody algebras", *Duke Math. J.* **91**:3 (1998), 515–560. MR Zbl

[Nakajima 2001] H. Nakajima, "Quiver varieties and finite-dimensional representations of quantum affine algebras", *J. Amer. Math. Soc.* **14**:1 (2001), 145–238. MR Zbl

[Oppermann et al. 2015] S. Oppermann, I. Reiten, and H. Thomas, "Quotient closed subcategories of quiver representations", *Compos. Math.* **151**:3 (2015), 568–602. MR Zbl

[Qiu and Woolf 2014] Y. Qiu and J. Woolf, "Contractible stability spaces and faithful braid group actions", preprint, 2014. arXiv

[Rickard 1989] J. Rickard, "Morita theory for derived categories", *J. London Math. Soc.* (2) **39**:3 (1989), 436–456. MR Zbl

[Ringel 1998] C. M. Ringel, "The preprojective algebra of a quiver", pp. 467–480 in *Algebras and modules, II* (Geiranger, Norway, 1996), edited by I. Reiten et al., CMS Conf. Proc. **24**, Amer. Math. Soc., Providence, RI, 1998. MR Zbl

[Seidel and Thomas 2001] P. Seidel and R. Thomas, "Braid group actions on derived categories of coherent sheaves", *Duke Math. J.* **108**:1 (2001), 37–108. MR Zbl

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Distinguished-root formulas for generalized Calabi–Yau hypersurfaces

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By a "generalized Calabi–Yau hypersurface" we mean a hypersurface in $\mathbb{P}^n$ of degree d dividing $n+1$. The zeta function of a generic such hypersurface has a reciprocal root distinguished by minimal p-divisibility. We study the p-adic variation of that distinguished root in a family and show that it equals the product of an appropriate power of p times a product of special values of a certain p-adic analytic function $\mathcal{F}$. That function $\mathcal{F}$ is the p-adic analytic continuation of the ratio $F(\Lambda)/F(\Lambda^p)$, where $F(\Lambda)$ is a solution of the A-hypergeometric system of differential equations corresponding to the Picard–Fuchs equation of the family.

1. Introduction

Dwork [1963; 1969] was the first to obtain p-adic analytic formulas for eigenvalues of Frobenius. In [Dwork 1969, Section 6], he developed an analytic theory of Frobenius for families of hypersurfaces: Frobenius acts semilinearly on the space of local solutions of the Picard–Fuchs equation and preserves p-adic growth conditions. In particular, p-adically bounded local solutions and p-adic unit eigenvalues of Frobenius are closely related. In this article, we apply these ideas (with some modifications) to obtain p-adic analytic formulas for the unique eigenvalue of minimal p-divisibility for what we call generalized Calabi–Yau hypersurfaces.

The Legendre family of elliptic curves was the first case to be studied in detail. In characteristic zero the Picard–Fuchs equation is of order 2, but Igusa [1958] noted that in odd characteristic p it has only one series solution (up to p-th powers). The truncation of the unique series solution ${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \Lambda\right)$ in characteristic zero at the $(p-1)$-st term makes sense in characteristic p and is the unique solution in characteristic p. Furthermore, for the elliptic curve in characteristic p, the number of rational points is determined modulo p by this truncation. Dwork used the Frobenius action on local solutions of Picard–Fuchs to give a much more precise result, namely, a formula for the unit root of the zeta function of a nonsupersingular elliptic curve of the Legendre family in terms of special values of the p-adic analytic

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continuation of the ratio ${}_2F_1\big(\frac{1}{2},\frac{1}{2};1;\Lambda\big)/{}_2F_1\big(\frac{1}{2},\frac{1}{2};1;\Lambda^p\big)$ [Dwork 1969, (6.29)]. Similar formulas have been found as well for the Dwork family of hypersurfaces by Dwork [1969] and J.-D. Yu [2009], more general families of varieties by N. Katz [1985], and for families of toric exponential sums [Dwork 1974; Adolphson and Sperber 1984; 1987b; 2012].

Novel features of this work are that we obtain explicit formulas for very general families of generalized Calabi–Yau hypersurfaces where the defining form is subject only to condition (1.9) below. We avoid in particular any hypothesis of nonsingularity. Dwork had suggested this might in fact be possible in his 1962 International Congress talk [1963, Section 5]. This is achieved here in part by adopting the A-hypergeometric point of view, which makes it easy to write down the explicit solution (1.15) of the Picard–Fuchs equation satisfied by the differential form (1.10), and by avoiding any computations involving the cohomology of the hypersurfaces in the family.

In addition, we apply here the dual theory associated with Dwork's θ_∞-splitting function. While this is technically more complicated than the dual theory associated with the θ_1-splitting function used in [Dwork 1964], the advantage is that our results are valid for all primes rather than just all sufficiently large primes.

We proceed now to make precise the main results. Let

$$f_\lambda(x_0,\dots,x_n)=\sum_{j=1}^{N}\lambda_j x^{\boldsymbol{a}_j}\in\mathbb{F}_q^\times[x_0,\dots,x_n] \tag{1.1}$$

be a homogeneous polynomial of degree $d\geq 2$ over the finite field $\mathbb{F}_q$, $q=p^a$, p a prime. Let $\mathbb{N}$ denote the set of nonnegative integers. For each j we write $\boldsymbol{a}_j=(a_{0j},\dots,a_{nj})\in\mathbb{N}^{n+1}$ with $\sum_{i=0}^n a_{ij}=d$ and $x^{\boldsymbol{a}_j}=x_0^{a_{0j}}\cdots x_n^{a_{nj}}$. Let $X_\lambda\subseteq\mathbb{P}^n_{\mathbb{F}_q}$ be defined by the vanishing of f_λ and let $X'_\lambda\subseteq\mathbb{A}^{n+1}_{\mathbb{F}_q}$ be the affine cone over X_λ. By [Ax 1964] we have for all s

$$\operatorname{card} X'_\lambda(\mathbb{F}_{q^s})\equiv 0 \pmod{q^{\mu s}}, \tag{1.2}$$

where μ is the least nonnegative integer that is greater than or equal to $\frac{n+1}{d}-1$. Equivalently,

$$\operatorname{card} X_\lambda(\mathbb{F}_{q^s})\equiv\frac{1}{1-q^s} \pmod{q^{\mu s}} \tag{1.3}$$

for all s.

This latter congruence can be expressed in terms of the zeta function of X_λ. Define a function $P_\lambda(t)$ by

$$P_\lambda(t)=\big(Z(X_\lambda/\mathbb{F}_q,t)(1-t)(1-qt)\cdots(1-q^{n-1}t)\big)^{(-1)^n}.$$

When the fiber X_λ is smooth, $P_\lambda(t)$ is the characteristic polynomial of Frobenius acting on middle-dimensional primitive cohomology. In this case, $P_\lambda(t)$ has degree

$d^{-1}((d-1)^{n+1}+(-1)^{n+1}(d-1))$. In the general setting, we have only that $P_\lambda(t)$ is a rational function [Dwork 1960]. The congruence (1.3) is equivalent to the assertion that all reciprocal zeros ρ and reciprocal poles σ of $P_\lambda(t)$ satisfy

$$\operatorname{ord}_q \rho,\ \operatorname{ord}_q \sigma \geq \mu, \tag{1.4}$$

where ord_q is the p-adic valuation normalized by $\operatorname{ord}_q q = 1$ [Ax 1964; Katz 1971, Proposition 2.4].

The integer μ has Hodge-theoretic significance. Let $Y \subseteq \mathbb{P}^n_{\mathbb{C}}$ be a smooth hypersurface of degree d and let $\{h^{i,n-1-i}\}_{i=0}^{n-1}$ be the Hodge numbers of the primitive part of middle-dimensional cohomology of Y (the $h^{i,n-1-i}$ depend only on n and d). Then $i=\mu$ is the smallest value of i for which $h^{i,n-1-i}\neq 0$ and, as such, is referred to as the Hodge type of Y. Furthermore, for X_λ smooth over $\mathbb{F}_q$ the rational function $P_\lambda(t)$ is a polynomial and, by [Illusie 1990], the generic smooth X_λ has exactly $h^{\mu,n-1-\mu}$ reciprocal zeros ρ satisfying $\operatorname{ord}_q \rho = \mu$.

In this paper we focus our attention on cases where $h^{\mu,n-1-\mu}=1$, i.e., where the polynomial $P_\lambda(t)$ has a unique reciprocal zero ρ with smallest q-ordinal μ for generic smooth X_λ. By standard formulas for Hodge numbers — a convenient source, with references, is [Adolphson and Sperber 2006, (1.3)] — this occurs when d is a divisor of $n+1$. From the definition of μ, we then have

$$n+1 = d(\mu+1), \tag{1.5}$$

which we assume from now on. We refer to these varieties as generalized Calabi–Yau hypersurfaces. (The case $\mu=0$ is the classical case of projective Calabi–Yau hypersurfaces.) Assuming only this condition, one can refine the description of $P_\lambda(t)$.

For $j=1,\dots,N$, put

$$\boldsymbol{a}_j^+ = (\boldsymbol{a}_j, 1) = (a_{0j}, a_{1j}, \dots, a_{nj}, 1) \in \mathbb{N}^{n+2}.$$

Let $\Lambda_1,\dots,\Lambda_N$ be indeterminates and set

$$H(\Lambda) = \sum_{\substack{u=(u_1,\dots,u_N)\in\mathbb{N}^N \\ \sum_{j=1}^N u_j \boldsymbol{a}_j^+ = (p-1)(1,\dots,1,\mu+1)}} \frac{\Lambda_1^{u_1}\cdots\Lambda_N^{u_N}}{u_1!\cdots u_N!} \in (\mathbb{Q}\cap\mathbb{Z}_p)[\Lambda_1,\dots,\Lambda_N]. \tag{1.6}$$

Note that the conditions on the summation imply $0\leq u_j\leq p-1$ for $j=1,\dots,N$. We denote by $\bar{H}(\Lambda)\in\mathbb{F}_p[\Lambda_1,\dots,\Lambda_N]$ the reduction mod p of $H(\Lambda)$.

We express the rational function $P_\lambda(t)$ as a ratio $P_\lambda(t) = P_\lambda^{(1)}(t)/P_\lambda^{(2)}(t)$, where $P_\lambda^{(1)}(t)$ and $P_\lambda^{(2)}(t)$ are relatively prime polynomials with integer coefficients and constant term 1. By (1.4) we have

$$P_\lambda^{(1)}(q^{-\mu}t),\ P_\lambda^{(2)}(q^{-\mu}t) \in 1+t\mathbb{Z}[t].$$

We prove the following result in Section 7.

Proposition 1.7. *Let f_λ be as in* (1.1) *and suppose* (1.5) *holds. Let $\hat\lambda \in \mathbb{Q}_p(\zeta_{q-1})^N$ be the Teichmüller lifting of λ. Then $P^{(2)}_\lambda(q^{-\mu}t) \equiv 1 \pmod{q}$ and*

$$P^{(1)}_\lambda(q^{-\mu}t) \equiv 1 - t\prod_{i=0}^{a-1}((-1)^{\mu+1}H(\hat\lambda^{p^i})) \pmod{p}.$$

As an immediate consequence of Proposition 1.7, we get a criterion for the zeta function of a generalized Calabi–Yau hypersurface to have a reciprocal root distinguished by minimal p-divisibility.

Proposition 1.8. *Under the hypotheses of Proposition 1.7, the rational function $P_\lambda(t)$ has a unique reciprocal root of q-ordinal μ if and only if $\overline{H}(\lambda) \neq 0$. Furthermore, when $\overline{H}(\lambda) \neq 0$, that reciprocal root is a reciprocal zero, not a reciprocal pole, of $P_\lambda(t)$.*

When $\overline{H}(\lambda) \neq 0$, we denote by $\rho_{\min}(\lambda)$ the unique reciprocal root of $P_\lambda(t)$ having q-ordinal μ. Let $\overline{\mathbb{F}}_q$ denote an algebraic closure of $\mathbb{F}_q$. We call the set

$$\{\lambda \in \overline{\mathbb{F}}_q^N \mid \overline{H}(\lambda) \neq 0\}$$

the *Hasse domain* for the family.

It can happen that the sum defining $H(\Lambda)$ is empty, for example, if f_λ is the diagonal hypersurface of degree d dividing $n+1$ and $p \not\equiv 1 \pmod{d}$. To guarantee that for all primes p the polynomial $H(\Lambda)$ is not identically zero, we make the assumption that $\mu+1$ of the vectors $\{\boldsymbol{a}_j\}_{j=1}^N$ sum to the vector $(1,\dots,1)$, say,

$$\sum_{j=1}^{\mu+1} \boldsymbol{a}_j = (1,\dots,1). \tag{1.9}$$

The monomial $\prod_{j=1}^{\mu+1}(\Lambda_j^{p-1}/(p-1)!)$ then appears in $H(\Lambda)$ and, as a consequence, the subset of $(\mathbb{F}_q^\times)^N$ where $\overline{H}(\lambda) \neq 0$ is nonempty. Equation (1.9) is equivalent to the condition that $x^{\boldsymbol{a}_1}\cdots x^{\boldsymbol{a}_{\mu+1}} = x_0x_1\cdots x_n$. For example, in the case of Calabi–Yau hypersurfaces where $d = n+1$ and $\mu = 0$, this just says that $x_0x_1\cdots x_n$ must be one of the monomials that appear in f_λ. Our main goal in this paper is to give a p-adic analytic description of $\rho_{\min}(\lambda)$ in terms of A-hypergeometric functions when $\overline{H}(\lambda) \neq 0$.

Let $U \subseteq \mathbb{P}^n_{\mathbb{C}}$ be the open complement of a smooth hypersurface Y defined by a homogeneous polynomial g of degree d. Under the hypothesis (1.5), there is an n-form on U which can be expressed in homogeneous coordinates as

$$\frac{\sum_{i=0}^n (-1)^i x_i\, dx_0\cdots\widehat{dx_i}\cdots dx_n}{g^{\mu+1}}. \tag{1.10}$$

This n-form determines a cohomology class in $H^n_{\mathrm{DR}}(U)$, and also, by applying the residue map, a cohomology class in $H^{n-1}_{\mathrm{DR}}(Y)$. The one-dimensional space spanned by this cohomology class is the Hodge subspace of "colevel" μ. When Y varies in a family, this cohomology class satisfies a Picard–Fuchs equation. The A-hypergeometric equation that describes the variation of $\rho_{\min}(\lambda)$ when $\overline{H}(\lambda) \neq 0$ is the A-hypergeometric version of this Picard–Fuchs equation.

We describe the relevant A-hypergeometric system. Let $A = \{\boldsymbol{a}_j^+\}_{j=1}^N$ and let $L \subseteq \mathbb{Z}^N$ be the lattice of relations on the set A:

$$L = \left\{ l = (l_1, \ldots, l_N) \in \mathbb{Z}^N \;\middle|\; \sum_{j=1}^N l_j \boldsymbol{a}_j^+ = \boldsymbol{0} \right\}.$$

For each $l = (l_1, \ldots, l_N) \in L$, we define a partial differential operator $\Box_l$ in variables $\{\Lambda_j\}_{j=1}^N$ by

$$\Box_l = \prod_{l_j > 0} \left(\frac{\partial}{\partial \Lambda_j} \right)^{l_j} - \prod_{l_j < 0} \left(\frac{\partial}{\partial \Lambda_j} \right)^{-l_j}. \tag{1.11}$$

For $\beta = (\beta_0, \beta_1, \ldots, \beta_{n+1}) \in \mathbb{C}^{n+2}$, the corresponding Euler (or homogeneity) operators are defined by

$$Z_i = \sum_{j=1}^N a_{ij} \Lambda_j \frac{\partial}{\partial \Lambda_j} - \beta_i \tag{1.12}$$

for $i = 0, \ldots, n+1$. The *A-hypergeometric system with parameter β* consists of (1.11) for $l \in L$ and (1.12) for $i = 0, 1, \ldots, n+1$.

The A-hypergeometric system satisfied by the n-form (1.10) is obtained by taking the parameter β to be

$$\boldsymbol{b} := -\sum_{j=1}^{\mu+1} \boldsymbol{a}_j^+ = (-1, \ldots, -1, -\mu - 1) \in \mathbb{C}^{n+2} \tag{1.13}$$

(using (1.9) above). Let $v = (-1, \ldots, -1, 0, \ldots, 0) \in \mathbb{C}^N$ (-1 repeated $\mu + 1$ times followed by 0 repeated $N - \mu - 1$ times). Then

$$\sum_{j=1}^N v_j \boldsymbol{a}_j^+ = \boldsymbol{b} \tag{1.14}$$

and v has minimal negative support in the terminology of Saito–Sturmfels–Takayama [Saito et al. 2000], so by [Saito et al. 2000, Proposition 3.4.13] we get a series solution of this A-hypergeometric system. Let L' be the subset of L consisting of all $l = (l_1, \ldots, l_N)$ such that $l_j \leq 0$ for $j = 1, \ldots, \mu+1$ and $l_j \geq 0$ for $j = \mu+2, \ldots, N$.

The series solution is $(\Lambda_1 \cdots \Lambda_{\mu+1})^{-1} F(\Lambda)$, where

$$F(\Lambda) = \sum_{l \in L'} \frac{(-1)^{\sum_{j=1}^{\mu+1} l_j} \prod_{j=1}^{\mu+1} (-l_j)!}{\prod_{j=\mu+2}^{N} l_j!} \prod_{j=1}^{N} \Lambda_j^{l_j}. \tag{1.15}$$

Since the last coordinate of each $\boldsymbol{a}_j^+$ equals 1, the condition $l \in L$ implies that $\sum_{j=1}^{N} l_j = 0$, and hence that $F(\Lambda)$ is homogeneous of degree 0 in the Λ_j. For $j = 1, \dots, \mu+1$, the Λ_j occur to nonpositive powers in $F(\Lambda)$, while for $j = \mu+2, \dots, N$, the Λ_j occur to nonnegative powers in $F(\Lambda)$. The coefficients of the series $F(\Lambda)$ are integers by [Adolphson and Sperber 2013, Proposition 5.2] and it has constant term 1. Therefore it converges and assumes unit values on the set

$$\mathcal{D} = \bigl\{(\Lambda_1, \dots, \Lambda_N) \in \mathbb{C}_p^N \bigm| |\Lambda_j| > 1 \text{ for } 1 \le j \le \mu+1 \\ \text{and } |\Lambda_j| < 1 \text{ for } \mu+2 \le j \le N\bigr\}$$

(where $\mathbb{C}_p$ denotes the completion of an algebraic closure of $\mathbb{Q}_p$). Note that the Laurent polynomial $(\Lambda_1 \cdots \Lambda_{\mu+1})^{-(p-1)} H(\Lambda)$ has only nonpositive powers of Λ_j for $j = 1, \dots, \mu+1$, only nonnegative powers of Λ_j for $j = \mu+2, \dots, N$, and constant term $((p-1)!)^{-(\mu+1)}$. This implies that $(\Lambda_1 \cdots \Lambda_{\mu+1})^{-(p-1)} H(\Lambda)$ assumes unit values on $\mathcal{D}$. In particular, $F(\Lambda)/F(\Lambda^p)$ and $((\Lambda_1 \cdots \Lambda_{\mu+1})^{-(p-1)} H(\Lambda))^{-1}$ assume unit values on $\mathcal{D}$ and can be represented by convergent series there.

Note that $\mathcal{D}$ is a subset of

$$\mathcal{D}_+ := \bigl\{\Lambda \in \mathbb{C}_p^N \bigm| |\Lambda_j| \ge 1 \text{ for } 1 \le j \le \mu+1, |\Lambda_j| \le 1 \text{ for } \mu+2 \le j \le N, \\ \text{and } |(\Lambda_1 \cdots \Lambda_{\mu+1})^{-(p-1)} H(\Lambda)| = 1\bigr\}.$$

Let R' be the $\mathbb{C}_p$-vector space of uniform limits on $\mathcal{D}_+$ of rational functions whose numerators are polynomials in $\{\Lambda_j^{-1}\}_{j=1}^{\mu+1}$ and $\{\Lambda_j\}_{j=\mu+2}^{N}$ and whose denominators are powers of $(\Lambda_1 \cdots \Lambda_{\mu+1})^{-(p-1)} H(\Lambda)$. The elements of R' define functions on $\mathcal{D}_+$. Since $H(\Lambda)$ has coefficients in $\mathbb{Z}_p$, we have $H(\Lambda^p) \equiv H(\Lambda)^p \pmod{p}$. This implies that the set $\mathcal{D}_+$ is closed under the mapping $\Lambda \to \Lambda^p$, and that if $\xi(\Lambda) \in R'$ then $\xi(\Lambda^p) \in R'$ also.

Our main result is the following.

Theorem 1.16. *Under hypotheses* (1.5) *and* (1.9), *the ratio* $\mathcal{F}(\Lambda) := F(\Lambda)/F(\Lambda^p)$ *lies in* R'. *Let* $\lambda \in (\mathbb{F}_q^\times)^N$ *and let* $\hat{\lambda} \in \mathbb{Q}_p(\zeta_{q-1})^N$ *be its Teichmüller lifting. If* $\overline{H}(\lambda) \neq 0$, *then* $\hat{\lambda}^{p^i} \in \mathcal{D}_+$ *for* $i = 0, \dots, a-1$ *and*

$$\rho_{\min}(\lambda) = q^\mu \prod_{i=0}^{a-1} \mathcal{F}(\hat{\lambda}^{p^i}).$$

Examples. (1) When $d = n+1$ and $\mu = 0$, Theorem 1.16 gives a unit root formula assuming only that $x_0 \cdots x_n$ is one of the monomials appearing in f_λ. If f_λ defines a smooth hypersurface, then $P_\lambda(t)$ is a polynomial and this is its unique unit root. Consider for instance the Dwork family of hypersurfaces:

$$f_\lambda(x_0, \ldots, x_n) = \lambda_1 x_0 \cdots x_n + \lambda_2 x_0^{n+1} + \lambda_3 x_1^{n+1} + \cdots + \lambda_{n+2} x_n^{n+1}.$$

One computes that $L' = \{(-(n+1)l, l, \ldots, l) \in \mathbb{Z}^{n+2} \mid l \in \mathbb{N}\}$ and

$$F(\Lambda) = \sum_{l=0}^{\infty} \frac{(-1)^{(n+1)l}((n+1)l)!}{(l!)^{n+1}} \left(\frac{\Lambda_2 \cdots \Lambda_{n+2}}{\Lambda_1^{n+1}} \right)^l.$$

By Theorem 1.16, the ratio $\mathcal{F}(\Lambda) = F(\Lambda)/F(\Lambda^p)$ defines a function on $\mathcal{D}_+$ and the product $\prod_{i=0}^{a-1} \mathcal{F}(\hat{\lambda}^{p^i})$ gives the unit reciprocal zero of $P_\lambda(t)$ when $\overline{H}(\lambda) \neq 0$.

The more usual way of normalizing the Dwork family is

$$x_0^{n+1} + \cdots + x_n^{n+1} - (n+1)\Lambda^{-1/(n+1)} x_0 \cdots x_n,$$

which we can recover from the specialization $\Lambda_1 \mapsto -(n+1)\Lambda^{-1/(n+1)}$ and $\Lambda_j \mapsto 1$ for $j = 2, \ldots, n+2$, giving

$$\begin{aligned} F(-(n+1)\Lambda^{-1/(n+1)}, 1, \ldots, 1) &= \sum_{l=0}^{\infty} \frac{((n+1)l)!}{(l!)^{n+1}(n+1)^{(n+1)l}} \Lambda^l \\ &= {}_nF_{n-1}(1/(n+1), \ldots, n/(n+1); 1, \ldots, 1; \Lambda). \end{aligned}$$

The assertion of Theorem 1.16 for this normalization of the Dwork family was recently proved by Yu [2009].

(2) Let

$$f_\lambda(x_0, \ldots, x_5) = \lambda_1 x_0 x_1 x_2 + \lambda_2 x_3 x_4 x_5 + \sum_{i=0}^{5} \lambda_{i+3} x_i^3.$$

One computes that

$$L' = \{l_1(-3, 0, 1, 1, 1, 0, 0, 0) + l_2(0, -3, 0, 0, 0, 1, 1, 1) \mid l_1, l_2 \in \mathbb{N}\},$$

and hence

$$F(\Lambda) = \sum_{l_1, l_2 = 0}^{\infty} \frac{(-1)^{l_1+l_2}(3l_1)!(3l_2)!}{(l_1!)^3(l_2!)^3} \frac{(\Lambda_3\Lambda_4\Lambda_5)^{l_1}(\Lambda_6\Lambda_7\Lambda_8)^{l_2}}{\Lambda_1^{3l_1}\Lambda_2^{3l_2}}.$$

By Theorem 1.16, the ratio $\mathcal{F}(\Lambda) = F(\Lambda)/F(\Lambda^p)$ defines a function on $\mathcal{D}_+$ and $q\prod_{i=0}^{a-1} \mathcal{F}(\hat{\lambda}^{p^i})$ equals the reciprocal zero $\rho_{\min}(\lambda)$ of $P_\lambda(t)$ with $\mathrm{ord}_q\, \rho_{\min}(\lambda) = 1$ when $\overline{H}(\lambda) \neq 0$.

Remark. Even when there is no choice of $\mu+1$ elements of the set $\{\boldsymbol{a}_j\}_{j=1}^N$ satisfying (1.9), results similar to Theorem 1.16 may be true. For example, suppose that $p \equiv 1 \pmod{d}$ and that

$$\boldsymbol{a}_j = (0, \dots, 0, d, 0, \dots, 0) \quad \text{for } j = 1, \dots, n+1,$$

where the d occurs in the $(j-1)$-st coordinate (i.e., the polynomial f_λ is a deformation of the diagonal hypersurface). Equation (1.14) remains valid if we choose

$$v = (-1/d, \dots, -1/d, 0, \dots, 0),$$

where the $-1/d$ is repeated $n+1$ times. Since this vector v has minimal negative support, there is a corresponding series solution of the A-hypergeometric system with parameter $\boldsymbol{b}$ given by [Saito et al. 2000, Proposition 3.4.13]. And by [Adolphson and Sperber 2013, Corollary 3.6], this series has p-integral coefficients for $p \equiv 1 \pmod{d}$. Arguments similar to those of this article then show that an analogue of Theorem 1.16 is true for this series solution when $p \equiv 1 \pmod{d}$.

This paper is organized as follows. In Section 2 we collect some notation that is used throughout the paper. In Section 3 we recall some estimates from [Dwork 1962] that play a key role in what follows. In Section 4 we show that Theorem 1.16 is equivalent to the same statement with $F(\Lambda)$ replaced by a related series $G(\Lambda)$. The series $G(\Lambda)$ depends on the prime p but satisfies better p-adic estimates than $F(\Lambda)$. (Without introducing $G(\Lambda)$, we would only be able to prove Theorem 1.16 for almost all primes.) We use these estimates in Sections 5 and 6 to prove that $G(\Lambda)/G(\Lambda^p)$ and some related series are elements of R'. Finally, in Section 7, we prove Proposition 1.7 and derive the formula for $\rho_{\min}(\lambda)$ in terms of special values of $G(\Lambda)/G(\Lambda^p)$ at Teichmüller points.

In a future work, we hope to treat as well the case in which the first nonvanishing Hodge number $h := h^{\mu,n-1-\mu}$ is > 1. In this case, the (higher) Hasse–Witt matrix is $h \times h$ and, as in the case $h = 1$, its entries may be described in terms of power series solutions of appropriate A-hypergeometric systems.

2. Notation

For the convenience of the reader we collect in this section some notation that will be used throughout the paper.

Let $\mathbb{N}A \subseteq \mathbb{Z}^{n+2}$ be the semigroup generated by A and let $M \subseteq \mathbb{Z}^{n+2}$ be the abelian group generated by A. Note that M lies in the hyperplane $\sum_{i=0}^n u_i = du_{n+1}$ in $\mathbb{R}^{n+2}$. Set $M_- = M \cap (\mathbb{Z}_{<0})^{n+2}$, $M_+ = M \cap \mathbb{N}^{n+2}$. We denote by δ_- the truncation operator on formal Laurent series in variables $x_0, \dots, x_{n+1}$ that preserves only those terms having all exponents negative:

$$\delta_-\biggl(\sum_{k\in\mathbb{Z}^{n+2}} c_k x^k\biggr) = \sum_{k\in(\mathbb{Z}_{<0})^{n+2}} c_k x^k.$$

We use the same notation for formal Laurent series in a single variable t:

$$\delta_-\biggl(\sum_{k=-\infty}^{\infty} c_k t^k\biggr) = \sum_{k=-\infty}^{-1} c_k t^k.$$

It is convenient to note that if ξ_1 and ξ_2 are two series for which the product $\xi_1\xi_2$ is defined and for which $\delta_-(\xi_2)=0$, then $\delta_-\bigl(\delta_-(\xi_1)\xi_2\bigr) = \delta_-(\xi_1\xi_2)$.

Let $E \subseteq \mathbb{Z}^N$ be the set

$$E = \{(l_1,\dots,l_N) \mid l_j \le 0 \text{ for } 1 \le j \le \mu+1 \text{ and } l_j \ge 0 \text{ for } \mu+2 \le j \le N\}.$$

Note that, in the notation of Section 1, $L' = L \cap E$. We need to consider series in the Λ_j that, like $F(\Lambda)$ in (1.15), have exponents lying in E. For $u \in \mathbb{N}A$, put

$$E_u = \biggl\{(\nu_1,\dots,\nu_N) \in E \biggm| \sum_{j=1}^N \nu_j \boldsymbol{a}_j^+ = u\biggr\}.$$

Let $\mathbb{C}_p$ be the completion of an algebraic closure of $\mathbb{Q}_p$. For each $u \in M$, put

$$R_u = \biggl\{\xi(\Lambda) = \sum_{\nu\in E_u} c_\nu \prod_{j=1}^N \Lambda_j^{\nu_j} \biggm| c_\nu \in \mathbb{C}_p \text{ and } \{|c_\nu|\}_\nu \text{ is bounded}\biggr\}.$$

We define the *degree* of a monomial Λ^ν to be $\sum_{j=1}^N \nu_j \boldsymbol{a}_j^+ \in M$. The series in R_u are convergent and bounded on $\mathcal{D}$ and are homogeneous of degree u.

For each $u \in M$, let R'_u be the space of uniform limits on $\mathcal{D}_+$ of sequences of rational functions of the form $h(\Lambda)/((\Lambda_1\cdots\Lambda_{\mu+1})^{-(p-1)}H(\Lambda))^k$, where $h(\Lambda) \in R_u$ is a Laurent polynomial and $k \in \mathbb{N}$. The elements of R'_u define functions on $\mathcal{D}_+$. Since $((\Lambda_1\cdots\Lambda_{\mu+1})^{-(p-1)}H(\Lambda))^{-1}$ lies in R'_0, we have $R'_u \subseteq R_u$.

The set R_0 is a ring, R_u is a module over R_0, R'_0 is a subring of R_0, and R'_u is a module over R'_0. We define a norm on R_u by setting, for $\xi(\Lambda) = \sum_{\nu\in E_u} c_\nu \prod_{j=1}^N \Lambda_j^{\nu_j}$,

$$|\xi| = \sup_\nu |c_\nu|.$$

Note that for $\xi(\Lambda) \in R_u$, we have $|\xi| = \sup_{\Lambda\in\mathcal{D}} |\xi(\Lambda)|$ (for example, apply the argument of [Dwork 1962, Lemma 1.2]). Furthermore, if $\xi(\Lambda) \in R'_u$, then

$$|\xi| = \sup_{\Lambda\in\mathcal{D}} |\xi(\Lambda)| = \sup_{\Lambda\in\mathcal{D}_+} |\xi(\Lambda)|$$

since this equality holds for Laurent polynomials in R'_u. Both R_u and R'_u are complete in this norm.

From the discussion in Section 1 we see that $F(\Lambda)/F(\Lambda^p) \in R_0$. To prove the first assertion of Theorem 1.16 we need to show that $F(\Lambda)/F(\Lambda^p) \in R'_0$. In Section 4, we show that this is equivalent to the same assertion for a related function $G(\Lambda)$, for which the desired assertion is proved in Corollary 5.17.

Let γ_0 be a zero of the series $\sum_{i=0}^{\infty} t^{p^i}/p^i$ having $\operatorname{ord}_p \gamma_0 = 1/(p-1)$, where ord_p is the p-adic valuation normalized by $\operatorname{ord}_p p = 1$ (the role of γ_0 is discussed more fully in the next section). Define S to be the $\mathbb{C}_p$-vector space of formal series

$$S = \left\{ \xi(\Lambda, x) = \sum_{u \in M_-} \xi_u(\Lambda) \gamma_0^{u_{n+1}} x^u \;\middle|\; \xi_u(\Lambda) \in R_u \text{ and } \{|\xi_u|\}_u \text{ is bounded} \right\}.$$

Let S' be defined analogously with the condition "$\xi_u(\Lambda) \in R_u$" being replaced by "$\xi_u(\Lambda) \in R'_u$". Define a norm on S by setting

$$|\xi(\Lambda, x)| = \sup_u \{|\xi_u|\}.$$

Both S and S' are complete under this norm.

3. Some p-adic estimates

We begin by recording some basic p-adic estimates from [Dwork 1962, Section 4] that will play a role in what follows. Let $\mathrm{AH}(t) = \exp\bigl(\sum_{i=0}^{\infty} t^{p^i}/p^i\bigr)$ be the Artin–Hasse series, a power series in t that has p-integral coefficients, and set

$$\theta(t) = \mathrm{AH}(\gamma_0 t) = \sum_{i=0}^{\infty} \theta_i t^i.$$

We then have

$$\operatorname{ord}_p \theta_i \geq \frac{i}{p-1}. \tag{3.1}$$

We define $\hat{\theta}(t) = \prod_{j=0}^{\infty} \theta(t^{p^j})$, which gives $\theta(t) = \hat{\theta}(t)/\hat{\theta}(t^p)$. If we set

$$\gamma_j = \sum_{i=0}^{j} \frac{\gamma_0^{p^i}}{p^i}, \tag{3.2}$$

then

$$\hat{\theta}(t) = \exp\left(\sum_{j=0}^{\infty} \gamma_j t^{p^j}\right) = \prod_{j=0}^{\infty} \exp(\gamma_j t^{p^j}). \tag{3.3}$$

Since $(p^i/(p-1)) - i$ is an increasing function of i for $i \geq 1$, we have from the definition of γ_0 that

$$\operatorname{ord}_p \gamma_j = \frac{p^{j+1}}{p-1} - (j+1). \tag{3.4}$$

We estimate each of the series $\exp(\gamma_j t^{p^j}) = \sum_{k=0}^{\infty} (\gamma_j t^{p^j})^k/k!$. We have

$$\begin{aligned} \operatorname{ord}_p \frac{\gamma_j^k}{k!} &= k\left(\frac{p^{j+1}}{p-1} - (j+1)\right) - \frac{k-s_k}{p-1} \\ &= k(p^j + p^{j-1} + \cdots + p - j) + \frac{s_k}{p-1}, \end{aligned} \tag{3.5}$$

where s_k denotes the sum of the digits in the p-adic expansion of k. It follows that if $\exp(\gamma_j t^{p^j}) = \sum_{i=0}^{\infty} a_i^{(j)} t^i$, then $a_i^{(j)} = 0$ if $p^j \nmid i$, while if $i = p^j k$ then we have

$$\begin{aligned} \operatorname{ord}_p a_i^{(j)} &= \frac{i}{p^j}(p^j + p^{j-1} + \cdots + p - j) + \frac{s_i}{p-1} \\ &= i\left(1 + \frac{1}{p} + \cdots + \frac{1}{p^{j-1}} - \frac{j}{p^j}\right) + \frac{s_i}{p-1} \end{aligned} \tag{3.6}$$

(using $s_i = s_k$). This equation implies that $\operatorname{ord}_p a_i^{(j_1)} \geq \operatorname{ord}_p a_i^{(j_2)}$ if $j_1 \geq j_2$. It follows that for all $j \geq 1$,

$$\operatorname{ord}_p a_i^{(j)} \geq \operatorname{ord}_p a_i^{(1)} \geq \frac{i(p-1)}{p} + \frac{s_i}{p-1} \geq \frac{s_i}{p-1} = \operatorname{ord}_p a_i^{(0)}. \tag{3.7}$$

If we write $\hat{\theta}(t) = \sum_{i=0}^{\infty} \hat{\theta}_i (\gamma_0 t)^i / i!$, then (3.3) and (3.7) imply

$$\operatorname{ord}_p \hat{\theta}_i \geq 0. \tag{3.8}$$

We also need the series

$$\hat{\theta}_1(t) = \prod_{j=1}^{\infty} \exp(\gamma_j t^{p^j}) =: \sum_{i=0}^{\infty} \frac{\hat{\theta}_{1,i}}{i!} (\gamma_0 t)^i. \tag{3.9}$$

Note that $\hat{\theta}(t) = \exp(\gamma_0 t)\hat{\theta}_1(t)$. Using the relation $s_{i_1} + s_{i_2} \geq s_{i_1+i_2}$, (3.7) implies

$$\operatorname{ord}_p \hat{\theta}_{1,i} \geq \frac{i(p-1)}{p}. \tag{3.10}$$

Define a series $\hat{\theta}_1(\Lambda, x)$ by the formula

$$\hat{\theta}_1(\Lambda, x) = \prod_{j=1}^{N} \hat{\theta}_1(\Lambda_j x^{\boldsymbol{a}_j^+}). \tag{3.11}$$

Expanding the product (3.11) according to powers of x, we get

$$\hat{\theta}_1(\Lambda, x) = \sum_{u=(u_0,\ldots,u_{n+1}) \in \mathbb{N}A} \hat{\theta}_{1,u}(\Lambda) \gamma_0^{u_{n+1}} x^u, \tag{3.12}$$

where

$$\hat{\theta}_{1,u}(\Lambda) = \sum_{\substack{k_1,\ldots,k_N \in \mathbb{N} \\ \sum_{j=1}^{N} k_j \boldsymbol{a}_j^+ = u}} \left(\prod_{j=1}^{N} \frac{\hat{\theta}_{1,k_j}}{k_j!}\right) \Lambda_1^{k_1} \cdots \Lambda_N^{k_N}. \tag{3.13}$$

We have similar results for the reciprocal power series

$$\hat{\theta}_1(t)^{-1} = \prod_{j=1}^{\infty} \exp(-\gamma_j t^{p^j}).$$

If we write

$$\hat{\theta}_1(t)^{-1} = \sum_{i=0}^{\infty} \frac{\hat{\theta}'_{1,i}}{i!}(\gamma_0 t)^i, \tag{3.14}$$

then the coefficients satisfy

$$\operatorname{ord}_p \hat{\theta}'_{1,i} \geq \frac{i(p-1)}{p}. \tag{3.15}$$

We also have

$$\hat{\theta}_1(\Lambda, x)^{-1} = \prod_{j=1}^{N} \hat{\theta}_1(\Lambda_j x^{\boldsymbol{a}_j^+})^{-1}, \tag{3.16}$$

which we again expand in powers of x as

$$\hat{\theta}_1(\Lambda, x)^{-1} = \sum_{u=(u_0,\dots,u_{n+1})\in\mathbb{N}A} \hat{\theta}'_{1,u}(\Lambda)\gamma_0^{u_{n+1}} x^u \tag{3.17}$$

with

$$\hat{\theta}'_{1,u}(\Lambda) = \sum_{\substack{k_1,\dots,k_N\in\mathbb{N} \\ \sum_{j=1}^N k_j \boldsymbol{a}_j^+ = u}} \left(\prod_{j=1}^{N} \frac{\hat{\theta}'_{1,k_j}}{k_j!}\right)\Lambda_1^{k_1}\cdots\Lambda_N^{k_N}. \tag{3.18}$$

We also define

$$\theta(\Lambda, x) = \prod_{j=1}^{N} \theta(\Lambda_j x^{\boldsymbol{a}_j^+}). \tag{3.19}$$

Expanding the right-hand side in powers of x, we have

$$\theta(\Lambda, x) = \sum_{u\in\mathbb{N}A} \theta_u(\Lambda) x^u, \tag{3.20}$$

where

$$\theta_u(\Lambda) = \sum_{\nu\in\mathbb{N}^N} \theta_\nu^{(u)} \Lambda^\nu \tag{3.21}$$

and

$$\theta_\nu^{(u)} = \begin{cases} \prod_{j=1}^N \theta_{\nu_j} & \text{if } \sum_{j=1}^N \nu_j \boldsymbol{a}_j^+ = u, \\ 0 & \text{if } \sum_{j=1}^N \nu_j \boldsymbol{a}_j^+ \neq u, \end{cases} \tag{3.22}$$

so $\theta_u(\Lambda)$ is homogeneous of degree u. The equation $\sum_{j=1}^N \nu_j \boldsymbol{a}_j^+ = u$ has only finitely many solutions $\nu \in \mathbb{N}^N$, so $\theta_u(\Lambda)$ is a polynomial in the Λ_j. Equations (3.1) and (3.22) show that

$$\operatorname{ord}_p \theta_\nu^{(u)} \geq \frac{\sum_{j=1}^N \nu_j}{p-1} = \frac{u_{n+1}}{p-1}. \tag{3.23}$$

We observe one congruence that will allow us to simplify some later formulas. From (3.2) and (3.4) with $j = 1$ we have

$$\gamma_0 + \frac{\gamma_0^p}{p} \equiv 0 \pmod{\gamma_0 p^{p-1}}.$$

Multiplying this congruence by p/γ_0 gives $\gamma_0^{p-1} \equiv -p \pmod{p^p}$, so, a fortiori,

$$\gamma_0^{p-1} \equiv -p \pmod{p^2} \quad \text{for all primes } p. \tag{3.24}$$

4. Generating series for A-hypergeometric functions

In Dwork's theory, hypergeometric functions often appear in contiguous families as coefficients of a generating series. We describe the relevant generating series that will appear in our situation.

Consider the formal series $\zeta(t)$ defined by

$$\zeta(t) = \sum_{l=0}^{\infty} (-1)^l l! t^{-l-1}. \tag{4.1}$$

We note that the series $\zeta(t)$ shares a property with the exponential series $\exp t$: differentiating a term of the series with respect to t equals the term of the series involving the next lower power of t.

We define the formal generating series $F(\Lambda, x)$ by the formula

$$F(\Lambda, x) = \delta_-\biggl(\prod_{j=1}^{\mu+1} \zeta(\gamma_0 \Lambda_j x^{\boldsymbol{a}_j^+}) \prod_{j=\mu+2}^{N} \exp(\gamma_0 \Lambda_j x^{\boldsymbol{a}_j^+})\biggr), \tag{4.2}$$

where δ_- is as defined in Section 2. A straightforward calculation shows that

$$F(\Lambda, x) = \sum_{u \in M_-} F_u(\Lambda) \gamma_0^{u_{n+1}} x^u, \tag{4.3}$$

where

$$F_u(\Lambda) = (\Lambda_1 \cdots \Lambda_{\mu+1})^{-1} \sum_{\substack{l \in E \\ \boldsymbol{b} + \sum_{j=1}^N l_j \boldsymbol{a}_j^+ = u}} (-1)^{\sum_{j=1}^{\mu+1} l_j} \frac{\prod_{j=1}^{\mu+1} (-l_j)!}{\prod_{j=\mu+2}^{N} l_j!} \prod_{j=1}^{N} \Lambda_j^{l_j}. \tag{4.4}$$

It follows from the definition of $\zeta(t)$ that for $j = 1, \ldots, \mu + 1$,

$$\frac{\partial}{\partial \Lambda_j} \zeta(\gamma_0 \Lambda_j x^{\boldsymbol{a}_j^+}) = \gamma_0 x^{\boldsymbol{a}_j^+} \zeta(\gamma_0 \Lambda_j x^{\boldsymbol{a}_j^+}) - \frac{1}{\Lambda_j}.$$

A straightforward calculation then gives

$$\frac{\partial}{\partial \Lambda_j} F(\Lambda, x) = \delta_-(\gamma_0 x^{\boldsymbol{a}_j^+} F(\Lambda, x)) \tag{4.5}$$

for $j = 1, \dots, \mu+1$. Equivalently, for $u \in M_-$ we have by (4.3)

$$\frac{\partial}{\partial \Lambda_j} F_u(\Lambda) = F_{u-\boldsymbol{a}_j^+}(\Lambda). \tag{4.6}$$

More generally, if $l_1, \dots, l_N$ are nonnegative integers, then

$$\prod_{j=1}^{N} \Big(\frac{\partial}{\partial \Lambda_j}\Big)^{l_j} F_u(\Lambda) = F_{u-\sum_{j=1}^N l_j \boldsymbol{a}_j^+}(\Lambda). \tag{4.7}$$

In particular we have, from the definition of the box operators,

$$\Box_l(F_u(\Lambda)) = 0 \quad \text{for all } l \in L \text{ and all } u \in M_-. \tag{4.8}$$

It is immediate from (4.4) that $F_u(\Lambda)$ satisfies the Euler operators (1.12) with $\beta = u$, hence by (4.8) the series $F_u(\Lambda)$ satisfies the A-hypergeometric system with parameter $\beta = u$.

Comparing (4.4) with (1.15), one sees that

$$F_{\boldsymbol{b}}(\Lambda) = (\Lambda_1 \cdots \Lambda_{\mu+1})^{-1} F(\Lambda), \tag{4.9}$$

a series which we noted in Section 1 has integer coefficients.

Lemma 4.10. *For all $u \in M_-$, the series $F_u(\Lambda)$ given by* (4.4) *has integer coefficients.*

Proof. Enlarge the set $\{x^{\boldsymbol{a}_j}\}_{j=1}^N$ by adding additional monomials $\{x^{\boldsymbol{a}_j}\}_{j=N+1}^{\widetilde{N}}$, so that $\{x^{\boldsymbol{a}_j}\}_{j=1}^{\widetilde{N}}$ consists of all monomials of degree d in $x_0, \dots, x_n$. As in (4.2) and (4.3), we define

$$\widetilde{F}(\Lambda, x) = \delta_-\bigg(\prod_{j=1}^{\mu+1} \zeta(\gamma_0 \Lambda_j x^{\boldsymbol{a}_j^+}) \prod_{j=\mu+2}^{\widetilde{N}} \exp(\gamma_0 \Lambda_j x^{\boldsymbol{a}_j^+})\bigg)$$

and set

$$\widetilde{F}(\Lambda, x) = \sum_{u \in \widetilde{M}_-} \widetilde{F}_u(\Lambda) \gamma_0^{u_{n+1}} x^u,$$

where $\widetilde{M} \subseteq \mathbb{Z}^{n+2}$ denotes the abelian group generated by the set $\{(\boldsymbol{a}_j, 1)\}_{j=1}^{\widetilde{N}}$ and $\widetilde{M}_- = \widetilde{M} \cap (\mathbb{Z}_{<0})^{n+2}$. The same argument that proved (4.7) shows that if $l_1, \dots, l_{\widetilde{N}}$ are nonnegative integers, then

$$\prod_{j=1}^{\widetilde{N}} \Big(\frac{\partial}{\partial \Lambda_j}\Big)^{l_j} \widetilde{F}_u(\Lambda) = \widetilde{F}_{u-\sum_{j=1}^N l_j \boldsymbol{a}_j^+}(\Lambda).$$

Note that for $u \in M_-$, the series $F_u(\Lambda)$ is obtained from the series $\widetilde{F}_u(\Lambda)$ by setting $\Lambda_j = 0$ for $j = N+1, \dots, \widetilde{N}$. To prove the lemma, it thus suffices to prove that $\widetilde{F}_u(\Lambda)$ has integer coefficients for all $u \in \widetilde{M}_-$.

Every monomial in $x_0, \dots, x_n$ of degree divisible by d is a product of monomials of degree d. In particular, if x^v is such a monomial which is divisible by $x_0 \cdots x_n$, then one can write

$$x^v = x^{\boldsymbol{a}_1} \cdots x^{\boldsymbol{a}_{\mu+1}} \prod_{j=1}^{\widetilde{N}} x^{l_j \boldsymbol{a}_j}$$

for some nonnegative integers $l_1, \dots, l_{\widetilde{N}}$. It follows from this that every $u \in \widetilde{M}_-$ can be written in the form

$$u = \boldsymbol{b} - \sum_{j=1}^{\widetilde{N}} l_j \boldsymbol{a}_j^+$$

for some nonnegative integers $l_1, \dots, l_{\widetilde{N}}$. We thus have

$$\prod_{j=1}^{\widetilde{N}} \Big(\frac{\partial}{\partial \Lambda_j}\Big)^{l_j} \widetilde{F}_{\boldsymbol{b}}(\Lambda) = \widetilde{F}_u(\Lambda). \tag{4.11}$$

The series $\widetilde{F}_{\boldsymbol{b}}(\Lambda)$ has integer coefficients by [Adolphson and Sperber 2013, Proposition 5.2]. It now follows from (4.11) that $\widetilde{F}_u(\Lambda)$ also has integer coefficients. □

We can improve the conclusion of Lemma 4.10. Fix $u \in M_-$. There are finitely many N-tuples $(k_1, \dots, k_N) \in \mathbb{N}^N$ such that

$$u + \sum_{j=1}^{N} k_j \boldsymbol{a}_j^+ \in M_-. \tag{4.12}$$

Define K_u to be the least common multiple of the integers $\prod_{j=1}^N k_j!$ over all $(k_1, \dots, k_N) \in \mathbb{N}^N$ satisfying (4.12).

Lemma 4.13. *For $u \in M_-$, all coefficients of the series $F_u(\Lambda)$ are divisible by K_u.*

Proof. Let $(k_1, \dots, k_N) \in \mathbb{N}^N$ satisfy (4.12) and put

$$w = u + \sum_{j=1}^{N} k_j \boldsymbol{a}_j^+ \in M_-.$$

It follows from (4.7) that

$$\prod_{j=1}^{N} \Big(\frac{\partial}{\partial \Lambda_j}\Big)^{k_j} F_w(\Lambda) = F_u(\Lambda).$$

By Lemma 4.10, $F_w(\Lambda)$ has integer coefficients, so an elementary calculation shows that the coefficients of $F_u(\Lambda)$ are divisible by $\prod_{j=1}^N k_j!$. □

Although the relevant hypergeometric functions appear as coefficients in the series $F(\Lambda, x)$, it is necessary for our proof of Theorem 1.16 to work with a related series which satisfies better p-adic estimates. Define $G(\Lambda, x)$ to be

$$\begin{aligned} G(\Lambda, x) &= \delta_-(F(\Lambda, x)\hat{\theta}_1(\Lambda, x)) \\ &= \delta_-\biggl(\biggl(\prod_{j=1}^{\mu+1} \zeta(\gamma_0\Lambda_j x^{\boldsymbol{a}_j^+})\hat{\theta}_1(\Lambda_j x^{\boldsymbol{a}_j^+})\biggr)\biggl(\prod_{j=\mu+2}^{N} \hat{\theta}(\Lambda_j x^{\boldsymbol{a}_j^+})\biggr)\biggr). \end{aligned} \tag{4.14}$$

If we set

$$G(\Lambda, x) = \sum_{u\in M_-} G_u(\Lambda)\gamma_0^{u_{n+1}} x^u, \tag{4.15}$$

then we have from (3.12) and (4.3) that

$$G_u(\Lambda) = \sum_{\substack{u^{(1)}\in M_-,u^{(2)}\in\mathbb{N}A \\ u^{(1)}+u^{(2)}=u}} F_{u^{(1)}}(\Lambda)\hat{\theta}_{1,u^{(2)}}(\Lambda). \tag{4.16}$$

Let $K_{u^{(1)}}$ be defined as in Lemma 4.13. By (3.13) we have

$$\begin{aligned} G_u(\Lambda) = \sum_{\substack{u^{(1)}\in M_-,u^{(2)}\in\mathbb{N}A \\ u^{(1)}+u^{(2)}=u}} & K_{u^{(1)}}^{-1} F_{u^{(1)}}(\Lambda) \\ & \cdot \sum_{\substack{k_1,\dots,k_N\in\mathbb{N} \\ \sum_{j=1}^N k_j\boldsymbol{a}_j^+=u^{(2)}}} \biggl(\prod_{j=1}^N \hat{\theta}_{1,k_j}\biggr)\frac{K_{u^{(1)}}}{\prod_{j=1}^N k_j!}\Lambda_1^{k_1}\cdots\Lambda_N^{k_N}. \end{aligned} \tag{4.17}$$

The series $K_{u^{(1)}}^{-1}F_{u^{(1)}}(\Lambda)$ has integral coefficients by Lemma 4.13, and the ratio $K_{u^{(1)}}/\prod_{j=1}^N k_j!$ is an integer by the definition of $K_{u^{(1)}}$. For each $u^{(2)} \in \mathbb{N}A$ in the inner sum on the right-hand side of (4.17) we have

$$\operatorname{ord}_p \prod_{j=1}^N \hat{\theta}_{1,k_j} \geq \frac{1}{p}\biggl(\sum_{j=1}^N k_j(p-1)\biggr) = \frac{1}{p}(u_{n+1}^{(2)}(p-1)) \tag{4.18}$$

by (3.10). This implies that the series on the right-hand side of (4.17) converges to a series with p-integral coefficients, and hence

$$|G_u(\Lambda)| \leq 1 \quad \text{for all } u \in M_-. \tag{4.19}$$

To simplify notation, for $u, u^{(1)} \in M_-$ set

$$C_{u,u^{(1)}} = \sum_{\substack{k_1,\dots,k_N\in\mathbb{N} \\ \sum_{j=1}^N k_j\boldsymbol{a}_j^+=u-u^{(1)}}} \biggl(\prod_{j=1}^N \hat{\theta}_{1,k_j}\biggr)\frac{K_{u^{(1)}}}{\prod_{j=1}^N k_j!}\Lambda_1^{k_1}\cdots\Lambda_N^{k_N}$$

(a finite sum). Note that $C_{u,u^{(1)}}$ is p-integral by the definition of $K_{u^{(1)}}$, $C_{u,u} = 1$, and $\operatorname{ord}_p C_{u,u^{(1)}} > 0$ for $u \neq u^{(1)}$ by (4.18). Then (4.17) becomes

$$G_u(\Lambda) = F_u(\Lambda) + \sum_{\substack{u^{(1)}\in M_- \\ u^{(1)}\neq u}} C_{u,u^{(1)}} K_{u^{(1)}}^{-1} F_{u^{(1)}}(\Lambda). \tag{4.20}$$

Furthermore, the estimate (4.18) implies that $C_{u,u^{(1)}} \to 0$ as $u^{(1)} \to \infty$, in the sense that for any $\kappa > 0$, the estimate $\operatorname{ord}_p C_{u,u^{(1)}} > \kappa$ holds for all but finitely many $u^{(1)}$.

By analogy with (4.9) we define $G(\Lambda) \in R_0$ by

$$G_{\boldsymbol{b}}(\Lambda) = (\Lambda_1 \cdots \Lambda_{\mu+1})^{-1} G(\Lambda). \tag{4.21}$$

Lemma 4.22. *We have $G(\Lambda, x) \in S$, $|G(\Lambda, x)| = |G_{\boldsymbol{b}}(\Lambda)| = 1$, and $G(\Lambda)$ assumes unit values on $\mathcal{D}$.*

Proof. The preceding calculation shows that $G(\Lambda, x) \in S$ and $|G(\Lambda, x)| \leq 1$. Equation (4.20) shows that

$$G(\Lambda) \equiv F(\Lambda) \pmod{\gamma_0}.$$

We noted in Section 1 that $F(\Lambda)$ assumes unit values on $\mathcal{D}$, hence the same is true of $G(\Lambda)$. It then follows from (4.21) that $|G_{\boldsymbol{b}}(\Lambda)| = 1$. □

Remark. The congruence $G(\Lambda) \equiv F(\Lambda) \pmod{\gamma_0}$ shows that the constant term of $G(\Lambda)$ is a p-adic unit and that the series $G(\Lambda) \in R_0$ has p-integral coefficients. This implies that the reciprocal series $G(\Lambda)^{-1}$ also has constant term a p-adic unit and p-integral coefficients.

Before proceeding to the main result of this section, we show that the $G_u(\Lambda)$ satisfy the analogue of Lemma 4.13.

Lemma 4.23. *For $u \in M_-$, the coefficients of the series $K_u^{-1} G_u(\Lambda)$ are p-integral.*

Proof. By (4.17), it suffices to prove that the coefficients of $F_{u^{(1)}}(\Lambda) / \prod_{j=1}^N k_j!$ are divisible by K_u whenever $k_1, \dots, k_N \in \mathbb{N}$ satisfy

$$u^{(1)} + \sum_{j=1}^N k_j \boldsymbol{a}_j^+ = u. \tag{4.24}$$

By the definition of K_u, this is equivalent to showing that if $l_1, \dots, l_N \in \mathbb{N}$ satisfy

$$u + \sum_{j=1}^N l_j \boldsymbol{a}_j^+ \in M_-, \tag{4.25}$$

then the coefficients of $F_{u^{(1)}}(\Lambda) / \prod_{j=1}^N k_j!$ are divisible by $\prod_{j=1}^N l_j!$. The equations (4.24) and (4.25) imply that

$$u^{(1)} + \sum_{j=1}^N (k_j + l_j) \boldsymbol{a}_j^+ \in M_-, \tag{4.26}$$

so by Lemma 4.13 the coefficients of $F_{u^{(1)}}(\Lambda)$ are divisible by $\prod_{j=1}^N (k_j + l_j)!$. Since $(k_j + l_j)!$ is divisible by $k_j! l_j!$, the result follows. □

Theorem 4.27. (a) *The ratio $F_u(\Lambda)/F(\Lambda)$ lies in R'_u for all $u \in M_-$ if and only if the ratio $G_u(\Lambda)/G(\Lambda)$ lies in R'_u for all $u \in M_-$. When either of these equivalent conditions is satisfied, the ratios $F_u(\Lambda)/G(\Lambda)$ and $G_u(\Lambda)/F(\Lambda)$ also lie in R'_u for all $u \in M_-$.*

(b) *If either of the equivalent conditions of part (a) is satisfied, then the ratio $\mathcal{F}(\Lambda) := F(\Lambda)/F(\Lambda^p)$ lies in R'_0 if and only if the ratio $\mathcal{G}(\Lambda) := G(\Lambda)/G(\Lambda^p)$ lies in R'_0. Furthermore, if this is the case, then for any $\lambda \in (\mathbb{F}_q^\times)^N$ with $\overline{H}(\lambda) \neq 0$, we have*

$$\prod_{i=0}^{a-1} \mathcal{F}(\hat{\lambda}^{p^i}) = \prod_{i=0}^{a-1} \mathcal{G}(\hat{\lambda}^{p^i}),$$

where $\hat{\lambda} \in \mathbb{Q}_p(\zeta_{q-1})^N$ denotes the Teichmüller lifting of λ.

Proof. Suppose that the ratios $F_u(\Lambda)/F(\Lambda)$ lie in R'_u for all $u \in M_-$. Divide (4.20) by $F(\Lambda)$:

$$\frac{G_u(\Lambda)}{F(\Lambda)} = \frac{F_u(\Lambda)}{F(\Lambda)} + \sum_{\substack{u^{(1)} \in M_- \\ u^{(1)} \neq u}} C_{u,u^{(1)}} K_{u^{(1)}}^{-1} \frac{F_{u^{(1)}}(\Lambda)}{F(\Lambda)}. \tag{4.28}$$

Since $F(\Lambda)$ assumes unit values and $|F_u(\Lambda)| \leq 1$ on $\mathcal{D}$, we have $|F_u(\Lambda)/F(\Lambda)| \leq 1$ on $\mathcal{D}_+$. Our earlier observation that $C_{u,u^{(1)}} \to 0$ as $u^{(1)} \to \infty$ then shows that this series converges to an element of R'_u that is bounded by 1.

Taking $u = \boldsymbol{b}$ in (4.28) and multiplying both sides by $\Lambda_1 \cdots \Lambda_{\mu+1}$ gives

$$\frac{G(\Lambda)}{F(\Lambda)} = 1 + \sum_{\substack{u^{(1)} \in M_- \\ u^{(1)} \neq \boldsymbol{b}}} \Lambda_1 \cdots \Lambda_{\mu+1} C_{\boldsymbol{b},u^{(1)}} K_{u^{(1)}}^{-1} \frac{F_{u^{(1)}}(\Lambda)}{F(\Lambda)}.$$

Thus $G(\Lambda)/F(\Lambda) \in R'_0$ and it assumes unit values on $\mathcal{D}_+$. This equation also shows that $|G(\Lambda)/F(\Lambda) - 1| < 1$, so the reciprocal of $G(\Lambda)/F(\Lambda)$ can be written as a geometric series to give

$$\frac{F(\Lambda)}{G(\Lambda)} = 1 + \sum_{\substack{u^{(1)} \in M_- \\ u^{(1)} \neq \boldsymbol{b}}} \Lambda_1 \cdots \Lambda_{\mu+1} C'_{\boldsymbol{b},u^{(1)}} K_{u^{(1)}}^{-1} \frac{F_{u^{(1)}}(\Lambda)}{F(\Lambda)}$$

for some polynomials $C'_{\boldsymbol{b},u^{(1)}}$ whose coefficients have positive p-ordinal and approach 0 as $u^{(1)} \to \infty$. Thus the ratio $F(\Lambda)/G(\Lambda)$ also lies in R'_0 and assumes unit values on $\mathcal{D}_+$. It now follows that the product

$$\frac{G_u(\Lambda)}{G(\Lambda)} = \frac{G_u(\Lambda)}{F(\Lambda)} \frac{F(\Lambda)}{G(\Lambda)}$$

lies in R'_0. This proves one direction of part (a).

For the other direction, suppose that the ratios $G_u(\Lambda)/G(\Lambda)$ lie in R'_u. It follows from (4.14) that

$$F(\Lambda, x) = \delta_-(G(\Lambda, x)\hat{\theta}_1(\Lambda, x)^{-1}). \tag{4.29}$$

This leads to the analogue of (4.17):

$$F_u(\Lambda) = \sum_{\substack{u^{(1)}\in M_-, u^{(2)}\in\mathbb{N}A \\ u^{(1)}+u^{(2)}=u}} K_{u^{(1)}}^{-1} G_{u^{(1)}}(\Lambda) \cdot \sum_{\substack{k_1,\dots,k_N\in\mathbb{N} \\ \sum_{j=1}^N k_j \boldsymbol{a}_j^+ = u^{(2)}}} \left(\prod_{j=1}^N \hat{\theta}'_{1,k_j}\right) \frac{K_{u^{(1)}}}{\prod_{j=1}^N k_j!} \Lambda_1^{k_1}\cdots\Lambda_N^{k_N}, \tag{4.30}$$

where the θ'_{1,k_j} are defined by (3.14) and Lemma 4.23 tells us that the $K_{u^{(1)}}^{-1} G_{u^{(1)}}(\Lambda)$ have p-integral coefficients. One can then argue as before since the $\hat{\theta}'_{1,k_j}$ also satisfy the estimate (4.18) (see (3.15)). This completes the proof of part (a).

When the equivalent conditions of part (a) are satisfied, we showed in the proof of part (a) that the ratio $\mathcal{H}(\Lambda) := G(\Lambda)/F(\Lambda)$ lies in R'_0 and assumes unit values there. The same assertions are true for its reciprocal. The first assertion of part (b) then follows from the equation

$$\frac{G(\Lambda)}{G(\Lambda^p)} = \frac{F(\Lambda)}{F(\Lambda^p)} \frac{\mathcal{H}(\Lambda)}{\mathcal{H}(\Lambda^p)} \tag{4.31}$$

on $\mathcal{D}$. Since $\mathcal{H}$ is a function on $\mathcal{D}_+$, we have $\mathcal{H}(\hat{\lambda}^{p^a}) = \mathcal{H}(\hat{\lambda})$ when $\hat{\lambda}^{p^a} = \hat{\lambda}$, so

$$\prod_{i=0}^{a-1} \frac{\mathcal{H}(\hat{\lambda}^{p^i})}{\mathcal{H}(\hat{\lambda}^{p^{i+1}})} = 1.$$

The second assertion of part (b) now follows from (4.31). □

Once we establish one of the equivalent conditions of part (a) of Theorem 4.27, part (b) implies that Theorem 1.16 is equivalent to the following statement (the assertion of Theorem 1.16 with $F(\Lambda)$ replaced by $G(\Lambda)$).

Theorem 4.32. *Under hypotheses* (1.5) *and* (1.9), *the ratio* $\mathcal{G}(\Lambda) := G(\Lambda)/G(\Lambda^p)$ *lies in* R'_0. *Let* $\lambda \in (\mathbb{F}_q^\times)^N$ *and let* $\hat{\lambda} \in \mathbb{Q}_p(\zeta_{q-1})^N$ *be its Teichmüller lifting. If* $\overline{H}(\lambda) \neq 0$, *then* $\hat{\lambda}^{p^i} \in \mathcal{D}_+$ *for* $i = 0, \dots, a-1$ *and*

$$\rho_{\min}(\lambda) = q^\mu \prod_{i=0}^{a-1} \mathcal{G}(\hat{\lambda}^{p^i}).$$

Sections 5 and 6 are devoted to establishing the conditions of Theorem 4.27(a). In Section 7 we prove Proposition 1.7 and Theorem 4.32.

5. Contraction mapping

We construct a map ϕ on a certain space of formal series whose coefficients are p-adic series. Hypothesis (1.5) will then imply that ϕ is a contraction mapping.

Let
$$\xi(\Lambda, x) = \sum_{\nu\in M_-} \xi_\nu(\Lambda)\gamma_0^{\nu_{n+1}} x^\nu \in S.$$

We claim that the product $\theta(\Lambda, x)\xi(\Lambda^p, x^p)$ is well defined as a formal series in x. Formally, we have
$$\theta(\Lambda, x)\xi(\Lambda^p, x^p) = \sum_{\rho\in M} \zeta_\rho(\Lambda)x^\rho,$$
where
$$\zeta_\rho(\Lambda) = \sum_{\substack{u\in\mathbb{N}A,\ \nu\in M_- \\ u+p\nu=\rho}} \gamma_0^{\nu_{n+1}}\theta_u(\Lambda)\xi_\nu(\Lambda^p). \tag{5.1}$$

Since $\theta_u(\Lambda)$ is a polynomial, the product $\theta_u(\Lambda)\xi_\nu(\Lambda^p)$ is clearly well defined. It follows from (3.21), (3.23), and the equality $u + p\nu = \rho$ that the coefficients of $\gamma_0^{\nu_{n+1}}\theta_u(\Lambda)$ all have p-ordinal at least $(\rho_{n+1}/(p-1)) - \nu_{n+1}$. Since $|\xi_\nu(\Lambda)|$ is bounded independently of ν and there are only finitely many terms on the right-hand side of (5.1) with a given value of ν_{n+1}, the series (5.1) converges to an element of R_ρ. This estimate also shows that if $\xi(\Lambda, x) \in S'$, then $\zeta_\rho(\Lambda) \in R'_\rho$.

For $\xi(\Lambda, x) \in S$, define
$$\begin{aligned}\alpha^*(\xi(\Lambda, x)) &= \delta_-(\theta(\Lambda, x)\xi(\Lambda^p, x^p)) \\ &= \sum_{\rho\in M_-} \zeta_\rho(\Lambda)x^\rho.\end{aligned}$$

For $\rho \in M_-$, put $\eta_\rho(\Lambda) = \gamma_0^{-\rho_{n+1}}\zeta_\rho(\Lambda)$, so that
$$\alpha^*(\xi(\Lambda, x)) = \sum_{\rho\in M_-} \eta_\rho(\Lambda)\gamma_0^{\rho_{n+1}} x^\rho \tag{5.2}$$
with (by (5.1))
$$\eta_\rho(\Lambda) = \sum_{\substack{u\in\mathbb{N}A,\ \nu\in M_- \\ u+p\nu=\rho}} \gamma_0^{-\rho_{n+1}+\nu_{n+1}}\theta_u(\Lambda)\xi_\nu(\Lambda^p). \tag{5.3}$$

Proposition 5.4. *The map α^* is an endomorphism of S and S', and for $\xi(\Lambda, x) \in S$ we have*
$$|\alpha^*(\xi(\Lambda, x))| \le |p^{\mu+1}\xi(\Lambda, x)|. \tag{5.5}$$

Proof. By (5.2), the proposition follows from the estimate
$$|\eta_\rho(\Lambda)| \le |p^{\mu+1}\xi(\Lambda, x)| \quad \text{for all } \rho \in M_-.$$

Using (5.3), we see that this estimate follows in turn from the estimate

$$|\gamma_0^{-\rho_{n+1}+\nu_{n+1}}\theta_u(\Lambda)| \le |p^{\mu+1}|$$

for all $u \in \mathbb{N}A$ and $\nu \in M_-$ with $u + p\nu = \rho$. From (3.21) and (3.23) we see that all coefficients of $\gamma_0^{-\rho_{n+1}+\nu_{n+1}}\theta_u(\Lambda)$ have p-ordinal greater than or equal to

$$\frac{-\rho_{n+1} + \nu_{n+1} + u_{n+1}}{p-1}.$$

Since $u + p\nu = \rho$, this expression simplifies to $-\nu_{n+1}$, which is greater than or equal to $\mu + 1$ because $\nu \in M_-$. □

Note that the equality $-\nu_{n+1} = \mu + 1$ occurs for only one point $\nu \in M_-$, namely, $\nu = (-1, \ldots, -1, -\mu - 1)$ $(= \boldsymbol{b})$. The following corollary is then an immediate consequence of the proof of Proposition 5.4.

Corollary 5.6. *If* $\xi_{\boldsymbol{b}}(\Lambda) = 0$, *then* $|\alpha^*(\xi(\Lambda, x))| \le |p^{\mu+2}\xi(\Lambda, x)|$.

We examine the polynomial $\theta_{-(p-1)\boldsymbol{b}}(\Lambda)$ to determine its relation to $H(\Lambda)$. Let

$$V = \left\{ v = (v_1, \ldots, v_N) \in \mathbb{N}^N \;\Big|\; \sum_{j=1}^{N} v_j \boldsymbol{a}_j^+ = -(p-1)\boldsymbol{b} \right\}.$$

From (3.21) and (3.22) we have

$$\theta_{-(p-1)\boldsymbol{b}}(\Lambda) = \sum_{v\in V} \left(\prod_{j=1}^{N} \theta_{v_j} \right) \Lambda_1^{v_1} \cdots \Lambda_N^{v_N}.$$

Clearly $v_j \le p-1$ for all j, so $\theta_{v_j} = \gamma_0^{v_j}/v_j!$. Furthermore, $\sum_{j=1}^N v_j = (p-1)(\mu+1)$, so this formula can be written

$$\theta_{-(p-1)\boldsymbol{b}}(\Lambda) = \gamma_0^{(p-1)(\mu+1)} \sum_{v\in V} \frac{\Lambda_1^{v_1} \cdots \Lambda_N^{v_N}}{v_1! \cdots v_N!}.$$

It now follows from (3.24) that

$$(-p)^{\mu+1} H(\Lambda) \equiv \theta_{-(p-1)\boldsymbol{b}}(\Lambda) \pmod{p^{\mu+2}}. \tag{5.7}$$

Corollary 5.8. *The Laurent polynomial* $(\Lambda_1 \cdots \Lambda_{\mu+1})^{-(p-1)}\theta_{-(p-1)\boldsymbol{b}}(\Lambda)$ *is an invertible element of* R_0' *with*

$$|(\Lambda_1 \cdots \Lambda_{\mu+1})^{-(p-1)}\theta_{-(p-1)\boldsymbol{b}}(\Lambda)| = |p^{\mu+1}|.$$

Proof. It is clear that $(\Lambda_1 \cdots \Lambda_{\mu+1})^{-1} H(\Lambda)$ is an invertible element of R_0' of norm 1. The assertion of the corollary then follows from (5.7). □

Let $\xi(\Lambda, x) \in S$ and let $\eta(\Lambda, x) = \alpha^*(\xi(\Lambda, x))$. Then $\eta(\Lambda, x)$ is given by the right-hand side of (5.2), and by (5.3) we have

$$\begin{aligned}\eta_{\boldsymbol{b}}(\Lambda) &= \sum_{\substack{u\in\mathbb{N}A,\ v\in M_-\\ u+pv=\boldsymbol{b}}} \gamma_0^{\mu+1+v_{n+1}}\theta_u(\Lambda)\xi_v(\Lambda^p)\\ &= \theta_{-(p-1)\boldsymbol{b}}(\Lambda)\xi_{\boldsymbol{b}}(\Lambda^p) + \sum_{\substack{u\in\mathbb{N}A,\ v\in M_-\\ u+pv=\boldsymbol{b}\\ -v_{n+1}\geq\mu+2}} \gamma_0^{\mu+1+v_{n+1}}\theta_u(\Lambda)\xi_v(\Lambda^p). \end{aligned}\tag{5.9}$$

Lemma 5.10. *Let $\xi(\Lambda, x) \in S$ (resp. $\xi(\Lambda, x) \in S'$) with $\big(\prod_{i=1}^{\mu+1}\Lambda_i\big)\xi_{\boldsymbol{b}}(\Lambda)$ an invertible element of R_0 (resp. R_0') and $|\xi_{\boldsymbol{b}}(\Lambda)| = |\xi(\Lambda, x)|$. Then $\big(\prod_{i=1}^{\mu+1}\Lambda_i\big)\eta_{\boldsymbol{b}}(\Lambda)$ is also an invertible element of R_0 (resp. R_0') and*

$$|\eta(\Lambda, x)| = |\eta_{\boldsymbol{b}}(\Lambda)| = |p^{\mu+1}\xi_{\boldsymbol{b}}(\Lambda)|.$$

Proof. First note that

$$\begin{aligned}&\Big(\prod_{j=1}^{\mu+1}\Lambda_j\Big)\theta_{-(p-1)\boldsymbol{b}}(\Lambda)\xi_{\boldsymbol{b}}(\Lambda^p)\\ &\qquad= \bigg(\Big(\prod_{j=1}^{\mu+1}\Lambda_j\Big)^{-(p-1)}\theta_{-(p-1)\boldsymbol{b}}(\Lambda)\bigg)\cdot\bigg(\Big(\prod_{j=1}^{\mu+1}\Lambda_j\Big)^{p}\xi_{\boldsymbol{b}}(\Lambda^p)\bigg),\end{aligned}$$

where the right-hand side is a product of two invertible elements by Corollary 5.8 and our hypothesis. Also by Corollary 5.8, it has norm

$$|p^{\mu+1}\xi_{\boldsymbol{b}}(\Lambda)| = |p^{\mu+1}\xi(\Lambda, x)|. \tag{5.11}$$

Equation (5.9) gives

$$\begin{aligned}&\frac{\big(\prod_{j=1}^{\mu+1}\Lambda_j\big)\eta_{\boldsymbol{b}}(\Lambda)}{\big(\prod_{j=1}^{\mu+1}\Lambda_j\big)\theta_{-(p-1)\boldsymbol{b}}(\Lambda)\xi_{\boldsymbol{b}}(\Lambda^p)}\\ &\qquad= 1 + \sum_{\substack{u\in\mathbb{N}A,\ v\in M_-\\ u+pv=\boldsymbol{b}\\ -v_{n+1}\geq\mu+2}} \frac{\gamma_0^{\mu+1+v_{n+1}}\big(\prod_{j=1}^{\mu+1}\Lambda_j\big)\theta_u(\Lambda)\xi_v(\Lambda^p)}{\big(\prod_{j=1}^{\mu+1}\Lambda_j\big)\theta_{-(p-1)\boldsymbol{b}}(\Lambda)\xi_{\boldsymbol{b}}(\Lambda^p)}.\end{aligned}\tag{5.12}$$

From (3.21), (3.23), and the condition $u + pv = \boldsymbol{b}$ it follows that each term in $\gamma_0^{\mu+1+v_{n+1}}\theta_u(\Lambda)$ has p-ordinal greater than or equal to

$$\frac{\mu+1+v_{n+1}}{p-1} + \frac{-pv_{n+1}-\mu-1}{p-1} = -v_{n+1} \geq \mu+2.$$

Corollary 5.8 and our hypothesis then imply that each term in the summation on the right-hand side of (5.12) has norm < 1 and that this norm approaches 0 as $v \to \infty$, in the sense that for any $\kappa > 0$ this norm is $< \kappa$ for all but finitely many ν. This proves that the right-hand side of (5.12) is invertible and has norm equal to 1. The assertions of the lemma now follow from (5.12) and the relations

$$|\eta_{\boldsymbol{b}}(\Lambda)| \leq |\eta(\Lambda, x)| \leq |p^{\mu+1}\xi(\Lambda, x)| = |p^{\mu+1}\xi_{\boldsymbol{b}}(\Lambda)|,$$

where the second inequality follows from Proposition 5.4 and the equality holds by hypothesis. $\square$

Put

$$T = \{\xi(\Lambda, x) \in S \mid \xi_{\boldsymbol{b}}(\Lambda) = (\Lambda_1 \cdots \Lambda_{\mu+1})^{-1} \text{ and } |\xi(\Lambda, x)| = 1\}$$

and $T' = T \cap S'$. It follows from Lemma 5.10 that if $\xi(\Lambda, x) \in T$ (resp. $\xi(\Lambda, x) \in T'$), then $\Lambda_1 \cdots \Lambda_{\mu+1}\eta_{\boldsymbol{b}}(\Lambda)$ is invertible in R_0 (resp. in R'_0). We may thus define

$$\phi(\xi(\Lambda, x)) = \frac{\alpha^*(\xi(\Lambda, x))}{\Lambda_1 \cdots \Lambda_{\mu+1}\eta_{\boldsymbol{b}}(\Lambda)}.$$

Lemma 5.10 also implies that

$$\left| \frac{\alpha^*(\xi(\Lambda, x))}{\Lambda_1 \cdots \Lambda_{\mu+1}\eta_{\boldsymbol{b}}(\Lambda)} \right| = 1,$$

so $\phi(T) \subseteq T$ and $\phi(T') \subseteq T'$.

Proposition 5.13. *The operator ϕ is a contraction mapping on the complete metric space T. More precisely, if $\xi^{(1)}(\Lambda, x), \xi^{(2)}(\Lambda, x) \in T$, then*

$$\left|\phi(\xi^{(1)}(\Lambda, x)) - \phi(\xi^{(2)}(\Lambda, x))\right| \leq |p| \cdot \left|\xi^{(1)}(\Lambda, x) - \xi^{(2)}(\Lambda, x)\right|.$$

Proof. We have (in the obvious notation)

$$\begin{aligned}
&\phi(\xi^{(1)}(\Lambda, x)) - \phi(\xi^{(2)}(\Lambda, x)) \\
&\quad = \frac{\alpha^*(\xi^{(1)}(\Lambda, x))}{\Lambda_1 \cdots \Lambda_{\mu+1}\eta^{(1)}_{\boldsymbol{b}}(\Lambda)} - \frac{\alpha^*(\xi^{(2)}(\Lambda, x))}{\Lambda_1 \cdots \Lambda_{\mu+1}\eta^{(2)}_{\boldsymbol{b}}(\Lambda)} \\
&\quad = \frac{\alpha^*(\xi^{(1)}(\Lambda, x) - \xi^{(2)}(\Lambda, x))}{\Lambda_1 \cdots \Lambda_{\mu+1}\eta^{(1)}_{\boldsymbol{b}}(\Lambda)} - \alpha^*(\xi^{(2)}(\Lambda, x)) \frac{\eta^{(1)}_{\boldsymbol{b}}(\Lambda) - \eta^{(2)}_{\boldsymbol{b}}(\Lambda)}{\Lambda_1 \cdots \Lambda_{\mu+1}\eta^{(1)}_{\boldsymbol{b}}(\Lambda)\eta^{(2)}_{\boldsymbol{b}}(\Lambda)}.
\end{aligned}$$

By Corollary 5.6 and Lemma 5.10 we have

$$\left| \frac{\alpha^*(\xi^{(1)}(\Lambda, x) - \xi^{(2)}(\Lambda, x))}{\Lambda_1 \cdots \Lambda_{\mu+1}\eta^{(1)}_{\boldsymbol{b}}(\Lambda)} \right| \leq |p| \cdot \left|\xi^{(1)}(\Lambda, x) - \xi^{(2)}(\Lambda, x)\right|.$$

Since $\eta_{\boldsymbol{b}}^{(1)}(\Lambda) - \eta_{\boldsymbol{b}}^{(2)}(\Lambda)$ is the coefficient of $x^{\boldsymbol{b}}$ in $\alpha^*(\xi^{(1)}(\Lambda, x) - \xi^{(2)}(\Lambda, x))$, we have

$$\begin{aligned}|\eta_{\boldsymbol{b}}^{(1)}(\Lambda) - \eta_{\boldsymbol{b}}^{(2)}(\Lambda)| &\leq \left|\alpha^*(\xi^{(1)}(\Lambda, x) - \xi^{(2)}(\Lambda, x))\right| \\ &\leq |p^{\mu+2}| \cdot \left|\xi^{(1)}(\Lambda, x) - \xi^{(2)}(\Lambda, x)\right|\end{aligned}$$

by Corollary 5.6. We have $|\eta_{\boldsymbol{b}}^{(1)}(\Lambda)\eta_{\boldsymbol{b}}^{(2)}(\Lambda)| = |p^{2\mu+2}|$ by Lemma 5.10, so by (5.5)

$$\left|\alpha^*(\xi^{(2)}(\Lambda, x))\frac{\eta_{\boldsymbol{b}}^{(1)}(\Lambda) - \eta_{\boldsymbol{b}}^{(2)}(\Lambda)}{\Lambda_1 \cdots \Lambda_{\mu+1}\eta_{\boldsymbol{b}}^{(1)}(\Lambda)\eta_{\boldsymbol{b}}^{(2)}(\Lambda)}\right| \leq |p| \cdot \left|\xi^{(1)}(\Lambda, x) - \xi^{(2)}(\Lambda, x)\right|.$$

This establishes the proposition. □

By a well-known theorem, Proposition 5.13 implies that ϕ has a unique fixed point in T. And since ϕ is stable on T', that fixed point must lie in T'. This fixed point of ϕ is related to a certain eigenvector of α^*.

Theorem 5.14. *We have* $\alpha^*(G(\Lambda, x)) = p^{\mu+1}G(\Lambda, x)$.

The proof of Theorem 5.14 will be given in the next section. In the remainder of this section, we use Proposition 5.13 and Theorem 5.14 to prove that $G(\Lambda)/G(\Lambda^p)$ lies in R'_0. This establishes the first sentence of Theorem 4.32. Note that $G(\Lambda, x)/G(\Lambda) \in T$ by the remark following Lemma 4.22.

Proposition 5.15. *The unique fixed point of* ϕ *in* T *is* $G(\Lambda, x)/G(\Lambda)$; *hence* $G(\Lambda, x)/G(\Lambda) \in T'$. *In particular, for each* $u \in M_-$, *the ratio* $G_u(\Lambda)/G(\Lambda)$ *lies in* R'_u.

Proof. We have

$$\alpha^*\left(\frac{G(\Lambda, x)}{G(\Lambda)}\right) = \frac{\alpha^*(G(\Lambda, x))}{G(\Lambda^p)} = \left(\frac{p^{\mu+1}G(\Lambda)}{G(\Lambda^p)}\right)\frac{G(\Lambda, x)}{G(\Lambda)}, \tag{5.16}$$

where the second equality follows from Theorem 5.14. By the definition of ϕ, this implies the result. □

Corollary 5.17. *With the above notation,* $G(\Lambda)/G(\Lambda^p)$ *lies in* R'_0.

Proof. Since α^* is stable on S', Proposition 5.15 implies that the right-hand side of (5.16) lies in S'. Since the coefficient of $\gamma_0^{-\mu-1}x^{\boldsymbol{b}}$ on the right-hand side of (5.16) is $p^{\mu+1}(\Lambda_1 \cdots \Lambda_{\mu+1})^{-1}G(\Lambda)/G(\Lambda^p)$, the result follows. □

6. Proof of Theorem 5.14

Consider the space of formal series

$$C = \left\{\xi = \sum_{i=0}^{\infty} c_i i! \gamma_0^{-i-1} t^{-i-1} \;\middle|\; \{c_i\}_{i=0}^{\infty} \text{ is bounded}\right\}.$$

Recall that δ_- is the truncation operator on series:

$$\delta_-\bigg(\sum_{i=-\infty}^{\infty} d_i t^{-i-1}\bigg) = \sum_{i=0}^{\infty} d_i t^{-i-1}.$$

Lemma 6.1. *The map $\delta_- \circ \hat{\theta}_1(t)$ is an isomorphism of C with itself. The inverse isomorphism is $\delta_- \circ \hat{\theta}_1(t)^{-1}$. (We use $\hat{\theta}_1(t)$ as an operator to mean multiplication by $\hat{\theta}_1(t)$, and likewise $\hat{\theta}_1(t)^{-1}$.)*

Proof. Let $\xi = \sum_{j=0}^{\infty} c_j j! \gamma_0^{-j-1} t^{-j-1} \in C$ and let k be a nonnegative integer. To simplify the estimate, assume that the c_j are bounded by 1. The coefficient of t^{-k-1} in the product $\hat{\theta}_1(t)\xi$ is

$$\sum_{i-j-1=-k-1} c_j j! \gamma_0^{-j-1} \frac{\hat{\theta}_{1,i}}{i!} \gamma_0^i = \bigg(\sum_{i=0}^{\infty} \hat{\theta}_{1,i} c_{i+k} \frac{(i+k)!}{i!k!}\bigg) k! \gamma_0^{-k-1}.$$

We have, by (3.10),

$$\operatorname{ord}_p \hat{\theta}_{1,i} c_{i+k} \frac{(i+k)!}{i!k!} \geq \frac{i(p-1)}{p} + \frac{-s_{i+k} + s_i + s_k}{p-1} \geq \frac{i(p-1)}{p}.$$

This shows that the series $\sum_{i=0}^{\infty} \hat{\theta}_{1,i} c_{i+k} (i+k)!/(i!k!)$ converges and is bounded by 1. Hence $\delta_- \circ \hat{\theta}_1(t)$ maps C into itself. Since the coefficients of the reciprocal power series $\hat{\theta}_1(t)^{-1} = \prod_{j=1}^{\infty} \exp(-\gamma_j t^{p^j})$ satisfy the same estimate (3.15), the same argument shows that $\delta_- \circ \hat{\theta}_1(t)^{-1}$ also maps C into itself and hence is the inverse of $\delta_- \circ \hat{\theta}_1(t)$. □

Define an operator D' on C by

$$D' = \delta_- \circ \bigg(t\frac{d}{dt} - \sum_{j=0}^{\infty} \gamma_j p^j t^{p^j}\bigg) = \delta_- \circ \hat{\theta}(t) \circ t\frac{d}{dt} \circ \hat{\theta}(t)^{-1}. \tag{6.2}$$

Proposition 6.3. *The operator D' has a one-dimensional (over $\mathbb{C}_p$) kernel as an operator on the space C.*

Proof. If $\xi \in C$ is a solution of D', then $\delta_-(\hat{\theta}_1(t)^{-1}\xi)$ lies in C by Lemma 6.1 and is a solution of the operator

$$\delta_- \circ \bigg(t\frac{d}{dt} - \gamma_0 t\bigg) = \delta_- \circ \exp(\gamma_0 t) \circ t\frac{d}{dt} \circ \exp(-\gamma_0 t). \tag{6.4}$$

Conversely, if $\xi \in C$ is a solution of (6.4), then $\delta_-(\hat{\theta}_1(t)\xi)$ lies in C and is a solution of D'. Thus it suffices to show that (6.4) has a unique solution (up to scalars) in C. Applying the operator (6.4) to $\xi = \sum_{i=0}^{\infty} c_i i! \gamma_0^{-i-1} t^{-i-1} \in C$ gives

$$\sum_{i=0}^{\infty} (-c_i - c_{i+1})(i+1)! \gamma_0^{-i-1} t^{-i-1},$$

from which it is clear that the solutions of (6.4) in C are scalar multiples of

$$q(t) := \sum_{i=0}^{\infty} (-1)^i i! \gamma_0^{-i-1} t^{-i-1}. \tag{6.5}$$

This completes the proof. □

Define

$$Q(t) = \delta_-(\hat{\theta}_1(t) q(t)) = \sum_{i=0}^{\infty} Q_i i! \gamma_0^{-i-1} t^{-i-1}. \tag{6.6}$$

From Lemma 6.1 we have $Q(t) \in C$; the proof of Lemma 6.1 shows that the Q_i are p-integral. From the proof of Proposition 6.3 we get the following corollary.

Corollary 6.7. *The solutions of D' in C are the scalar multiples of $Q(t)$.*

For $\xi(t) = \sum_{i=0}^{\infty} c_i i! \gamma_0^{-i-1} t^{-i-1} \in C$, define $\alpha'(\xi)$ to be

$$\alpha'(\xi) = \delta_-(\theta(t)\xi(t^p)).$$

Proposition 6.8. *The operator α' maps C into itself.*

Proof. For $k \geq 0$, the coefficient of t^{-k-1} in $\theta(t)\xi(t^p)$ is

$$\sum_{\substack{i,j \geq 0 \\ j-pi-p=-k-1}} \theta_j c_i i! \gamma_0^{-i-1}.$$

We may assume the c_i to be p-integral, in which case we have the estimate

$$\operatorname{ord}_p \theta_j c_i i! \gamma_0^{-i-1} \geq \frac{j}{p-1} + \frac{i-s_i}{p-1} - \frac{i+1}{p-1} = \frac{j-s_i-1}{p-1}.$$

Since i is a linear function of j (k is fixed) and s_i is bounded above by a positive multiple of $\log i$, this estimate shows that the series converges. The condition $j - pi - p = -k - 1$ gives $j + k = pi + (p-1)$, which implies

$$s_{j+k} = s_i + (p-1).$$

Since $s_j + s_k \geq s_{j+k}$, we get the estimate

$$\operatorname{ord}_p \theta_j c_i i! \gamma_0^{-i-1} \geq \frac{j - s_j + (p-1)}{p-1} - \frac{s_k+1}{p-1}.$$

The first term on the right-hand side is always ≥ 1, which implies that we can write

$$\sum_{\substack{i,j \geq 0 \\ j-pi-p=-k-1}} \theta_j c_i i! \gamma_0^{-i-1} = p d_k k! \gamma_0^{-k-1}$$

for some d_k which is p-integral. This proves the proposition. □

Proposition 6.9. *We have $D' \circ \alpha' = p\alpha' \circ D'$ as operators on C.*

Proof. Let $\xi(t)=\sum_{i=0}^{\infty} c_i i!\gamma_0^{-i-1}t^{-i-1}\in C$. The proof of Proposition 6.8 shows that

$$\alpha'(\xi(t))=\sum_{i=0}^{\infty} c_i i!\gamma_0^{-i-1}\alpha'(t^{-i-1}).$$

From the definition of D', it is clear that

$$D'(\xi(t))=\sum_{i=0}^{\infty} c_i i!\gamma_0^{-i-1}D'(t^{-i-1}),$$

so to prove the commutativity relation of the proposition it suffices to verify it on the t^{-i-1}. If we let Φ be the map that sends an element $\xi(t)\in C$ to $\xi(t^p)$, then the formal factorizations of α' as

$$\alpha'=\delta_-\circ\hat{\theta}(t)\circ\Phi\circ\hat{\theta}(t)^{-1}$$

and D' in (6.2) may be used to compute the actions on the t^{-i-1}. This reduces the assertion of the proposition to the obvious equality

$$t\frac{d}{dt}\circ\Phi=p\Phi\circ t\frac{d}{dt}. \qquad \square$$

It follows from Corollary 6.7 and Proposition 6.9 that $Q(t)$ is an eigenvector of α'. More precisely, we have the following result.

Proposition 6.10. $\alpha'(Q(t))=pQ(t).$

Proof. Let C^* be the space of series

$$C^*=\left\{\eta(t)=\sum_{i=0}^{\infty} c_i\gamma_0^i t^i \;\middle|\; \{c_i\}\text{ is bounded}\right\}$$

and let C_0^* be the subset consisting of those series $\eta\in C^*$ with $c_0=0$. The differential operator $D:=td/dt+\sum_{j=0}^{\infty}\gamma_j p^j t^{p^j}$ acts on C^*, and by [Adolphson and Sperber 2000, Theorem 3.8] the map $D:C^*\to C_0^*$ is an isomorphism.

Define $\psi:C^*\to C^*$ by $\psi\big(\sum_{i=0}^{\infty} c_i\gamma_0^i t^i\big)=\sum_{i=0}^{\infty} c_{pi}\gamma_0^{pi}t^i$ and let $\alpha:C^*\to C^*$ be the composition $\psi\circ\theta(t)$. A calculation analogous to the proof of Proposition 6.9 shows that as operators on C^*,

$$\alpha\circ D=pD\circ\alpha. \tag{6.11}$$

We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_0^* & \longrightarrow & C^* & \longrightarrow & \mathbb{C}_p & \longrightarrow & 0\\
\big\downarrow & & \big\downarrow{\scriptstyle D} & & \big\downarrow{\scriptstyle D} & & \big\downarrow & & \big\downarrow\\
0 & \longrightarrow & C_0^* & \xrightarrow{\text{id}} & C_0^* & \longrightarrow & 0 & \longrightarrow & 0
\end{array} \tag{6.12}$$

where $C_0^* \to C^*$ is the inclusion, $C_0^* \to C_0^*$ is the identity, and $C^* \to \mathbb{C}_p$ is the map defined by setting $t = 0$. Since $D : C^* \to C_0^*$ is an isomorphism, the long-exact cohomology sequence associated to (6.12) implies that there is an isomorphism $\mathbb{C}_p \cong C_0^*/DC_0^*$ which identifies $1 \in \mathbb{C}_p$ with the class $D(1) + DC_0^* \in C_0^*/DC_0^*$. It is easily seen that $\alpha(1) \in 1 + C_0^*$, so (6.11) implies

$$\alpha(D(1)) = pD(\alpha(1)) \equiv pD(1) \pmod{DC_0^*}. \tag{6.13}$$

It follows that the induced action of α on $\mathbb{C}_p \cong C_0^*/DC_0^*$ is multiplication by p.

Define a pairing between the spaces C and C_0^*: for $\xi = \sum_{i=0}^{\infty} c_i i! \gamma_0^{-i-1} t^{-i-1} \in C$ and $\eta = \sum_{i=0}^{\infty} b_i \gamma_0^{i+1} t^{i+1} \in C_0^*$, put

$$\langle \xi, \eta \rangle = \sum_{i=0}^{\infty} b_i c_i i!.$$

The series on the right-hand side converges because the $\{c_i\}$ and $\{b_i\}$ are bounded and $i! \to 0$ as $i \to \infty$. Note that if $u \in \mathbb{Z}_{>0}$ and $v \in \mathbb{Z}_{<0}$, then

$$\langle t^v, D(t^u) \rangle = -\langle D'(t^v), t^u \rangle = \begin{cases} u & \text{if } u + v = 0, \\ \gamma_j p^j & \text{if } u + v = -p^j \text{ for some } j, \\ 0 & \text{otherwise,} \end{cases}$$

which implies that

$$\langle D'(\xi), \eta \rangle = -\langle \xi, D(\eta) \rangle \tag{6.14}$$

for $\xi \in C$ and $\eta \in C_0^*$. A direct calculation also shows that

$$\langle \alpha'(t^v), t^u \rangle = \langle t^v, \alpha(t^u) \rangle = \theta_{-pv-u},$$

which implies that

$$\langle \alpha'(\xi), \eta \rangle = \langle \xi, \alpha(\eta) \rangle \tag{6.15}$$

for $\xi \in C$ and $\eta \in C_0^*$. We then have

$$\langle \alpha'(Q(t)), D(1) \rangle = \langle Q(t), \alpha(D(1)) \rangle = \langle Q(t), pD(1) + \eta \rangle$$

for some $\eta \in DC_0^*$ by (6.13). But $\langle Q(t), DC_0^* \rangle = 0$ by (6.14) and Corollary 6.7, so we get

$$\langle \alpha'(Q(t)), D(1) \rangle = p \langle Q(t), D(1) \rangle.$$

Since we already know that $\alpha'(Q(t))$ is a scalar multiple of $Q(t)$, the proposition will follow from this equality once we have checked that $\langle Q(t), D(1) \rangle \neq 0$.

We have $D(1) = \sum_{j=0}^{\infty} \gamma_j p^j t^{p^j}$ and $Q(t) = \sum_{i=0}^{\infty} Q_i i! \gamma_0^{-i-1} t^{-i-1}$, so

$$\langle Q(t), D(1) \rangle = \sum_{j=0}^{\infty} \gamma_j p^j Q_{p^j-1} (p^j - 1)! \gamma_0^{-p^j}. \tag{6.16}$$

We have, by (3.4) and the p-integrality of the Q_i,

$$\operatorname{ord}_p \gamma_j p^j Q_{p^j-1}(p^j-1)!\gamma^{-p^j} \geq \frac{p^{j+1}}{p-1}-(j+1)+j+\frac{p^j-1-j(p-1)}{p-1}-\frac{p^j}{p-1},$$

which simplifies to

$$\operatorname{ord}_p \gamma_j p^j Q_{p^j-1}(p^j-1)!\gamma^{-p^j} \geq \sum_{i=0}^{j}(p^i-1).$$

The right-hand side of this inequality is an increasing function of j, positive for $j>0$, so to prove the expression (6.16) is not zero, it suffices to show that Q_0, the contribution to the sum on the right-hand side of (6.16) for $j=0$, is a unit. From the definition (6.6) we compute

$$Q_0 = \sum_{i=0}^{\infty}(-1)^i\hat{\theta}_{1,i}.$$

The desired assertion about Q_0 then follows from (3.10) and the fact that $\hat{\theta}_{1,0}=1$. □

Proposition 6.10 implies that

$$\theta(t)Q(t^p) = A(t) + pQ(t)$$

for some series $A(t)$ in nonnegative powers of t. Replacing t in this equation by $\Lambda_i x^{\boldsymbol{a}_i^+}$ for $i=1,\dots,\mu+1$ and multiplying gives

$$\prod_{i=1}^{\mu+1}\theta(\Lambda_i x^{\boldsymbol{a}_i^+})Q(\Lambda_i^p x^{p\boldsymbol{a}_i^+}) = \prod_{i=1}^{\mu+1}\Big(A(\Lambda_i x^{\boldsymbol{a}_i^+}) + pQ(\Lambda_i x^{\boldsymbol{a}_i^+})\Big), \tag{6.17}$$

where $A(\Lambda_i x^{\boldsymbol{a}_i^+})$ is a series in nonnegative powers of $x^{\boldsymbol{a}_i^+}$. Our choice of the set $\{\boldsymbol{a}_i^+\}_{i=1}^{\mu+1}$ implies that an integral linear combination $\sum_{i=1}^{\mu+1} l_i\boldsymbol{a}_i^+$ lies in M_- only if $l_i<0$ for $i=1,\dots,\mu+1$. It follows that when the product on the right-hand side of (6.17) is expanded, all terms except for $\prod_{i=1}^{\mu+1} pQ(\Lambda_i x^{\boldsymbol{a}_i^+})$ are annihilated by δ_-, so we get

$$\delta_-\bigg(\prod_{i=1}^{\mu+1}\theta(\Lambda_i x^{\boldsymbol{a}_i^+})Q(\Lambda_i^p x^{p\boldsymbol{a}_i^+})\bigg) = \delta_-\bigg(\prod_{i=1}^{\mu+1} pQ(\Lambda_i x^{\boldsymbol{a}_i^+})\bigg).$$

But $\delta_-\big(\prod_{i=1}^{\mu+1} pQ(\Lambda_i x^{\boldsymbol{a}_i^+})\big) = \prod_{i=1}^{\mu+1} pQ(\Lambda_i x^{\boldsymbol{a}_i^+})$, giving finally

$$\delta_-\bigg(\prod_{i=1}^{\mu+1}\theta(\Lambda_i x^{\boldsymbol{a}_i^+})Q(\Lambda_i^p x^{p\boldsymbol{a}_i^+})\bigg) = p^{\mu+1}\prod_{i=1}^{\mu+1} Q(\Lambda_i x^{\boldsymbol{a}_i^+}). \tag{6.18}$$

Lemma 6.19. *We have*

$$G(\Lambda,x) = \delta_-\bigg(\bigg(\prod_{j=1}^{\mu+1} Q(\Lambda_i x^{\boldsymbol{a}_i^+})\bigg)\bigg(\prod_{j=\mu+2}^{N}\hat{\theta}(\Lambda_j x^{\boldsymbol{a}_j^+})\bigg)\bigg).$$

Proof. From the definitions of $F(\Lambda, x)$ and $q(t)$ we have

$$F(\Lambda, x) = \delta_-\biggl(\prod_{j=1}^{\mu+1} q(\Lambda_j x^{\boldsymbol{a}_j^+}) \prod_{j=\mu+2}^{N} \exp(\gamma_0 \Lambda_j x^{\boldsymbol{a}_j^+})\biggr).$$

From the definitions of $G(\Lambda, x)$ and $\hat{\theta}_1(\Lambda, x)$ (see (4.14) and (3.11)), we get

$$G(\Lambda, x) = \delta_-\biggl(\prod_{j=1}^{\mu+1} q\bigl(\Lambda_j x^{\boldsymbol{a}_j^+}\bigr) \prod_{j=\mu+2}^{N} \exp\bigl(\gamma_0 \Lambda_j x^{\boldsymbol{a}_j^+}\bigr) \prod_{j=1}^{N} \hat{\theta}_1\bigl(\Lambda_j x^{\boldsymbol{a}_j^+}\bigr)\biggr).$$

Using the definitions of $\hat{\theta}(t)$ and $\hat{\theta}_1(t)$ (see (3.3) and (3.9)), this equation may be rewritten as

$$G(\Lambda, x) = \delta_-\biggl(\prod_{j=1}^{\mu+1} (q(\Lambda_j x^{\boldsymbol{a}_j^+}) \hat{\theta}_1(\Lambda_j x^{\boldsymbol{a}_j^+})) \prod_{j=\mu+2}^{N} \hat{\theta}(\Lambda_j x^{\boldsymbol{a}_j^+})\biggr).$$

The assertion now follows from the definition of $Q(t)$ (see (6.6)). $\square$

We can now prove Theorem 5.14. First note that since $\theta(t) = \hat{\theta}(t)/\hat{\theta}(t^p)$, we have

$$\prod_{j=\mu+2}^{N} \theta\bigl(\Lambda_j x^{\boldsymbol{a}_j^+}\bigr) \prod_{j=\mu+2}^{N} \hat{\theta}\bigl(\Lambda_j^p x^{p\boldsymbol{a}_j^+}\bigr) = \prod_{j=\mu+2}^{N} \hat{\theta}\bigl(\Lambda_j x^{\boldsymbol{a}_j^+}\bigr). \tag{6.20}$$

We now compute:

$$\begin{aligned}
&\alpha^*(G(\Lambda, x)) \\
&\quad = \delta_-\biggl(\prod_{j=1}^{N} \theta(\Lambda_j x^{\boldsymbol{a}_j^+}) \delta_-\biggl(\biggl(\prod_{i=1}^{\mu+1} Q(\Lambda_i^p x^{p\boldsymbol{a}_i^+})\biggr)\biggl(\prod_{j=\mu+2}^{N} \hat{\theta}(\Lambda_j^p x^{p\boldsymbol{a}_j^+})\biggr)\biggr)\biggr) \\
&\quad = \delta_-\biggl(\biggl(\prod_{i=1}^{\mu+1} \theta(\Lambda_i x^{\boldsymbol{a}_i^+}) Q(\Lambda_i^p x^{p\boldsymbol{a}_i^+})\biggr)\biggl(\prod_{j=\mu+2}^{N} \theta(\Lambda_j x^{\boldsymbol{a}_j^+}) \prod_{j=\mu+2}^{N} \hat{\theta}(\Lambda_j^p x^{p\boldsymbol{a}_j^+})\biggr)\biggr) \\
&\quad = p^{\mu+1} \delta_-\biggl(\biggl(\prod_{i=1}^{\mu+1} Q(\Lambda_i x^{\boldsymbol{a}_i^+})\biggr)\biggl(\prod_{j=\mu+2}^{N} \hat{\theta}(\Lambda_j x^{\boldsymbol{a}_j^+})\biggr)\biggr) = p^{\mu+1} G(\Lambda, x),
\end{aligned}$$

where the first equality follows from Lemma 6.19, the next-to-last equality follows from (6.18) and (6.20), and the last equality follows from Lemma 6.19.

7. Zeta functions

Let $f_\lambda(x_0, \dots, x_n)$ be as defined in Section 1. We associate to f_λ exponential sums

$$S_\lambda(m) = \sum_{x \in \mathbb{A}^{n+2}(\mathbb{F}_{q^m})} \Psi\bigl(\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_p}(x_{n+1} f_\lambda(x_0, \dots, x_n))\bigr),$$

where $\Psi : \mathbb{F}_p \to \mathbb{Q}_p(\zeta_p)^\times$ is the additive character satisfying

$$\Psi(1) \equiv 1 + \gamma_0 \pmod{\gamma_0^2}.$$

We denote the corresponding L-function by $L_\lambda(t)$:

$$L_\lambda(t) = \exp\biggl(\sum_{m=1}^{\infty} S_\lambda(m) \frac{t^m}{m}\biggr).$$

Recall the relationship [Adolphson and Sperber 2008, (2.3)] between $L_\lambda(t)$ and the rational function $P_\lambda(t)$ defined in Section 1:

$$L_\lambda(t)^{(-1)^{n+1}} = (1 - q^{n+1}t)^{(-1)^n} \frac{P_\lambda(qt)}{P_\lambda(q^2 t)}. \tag{7.1}$$

We first prove Proposition 1.7 and then prove the last assertion of Theorem 4.32. We begin by reviewing the expression for $L_\lambda(t)$ that comes from Dwork's trace formula [Adolphson and Sperber 2008, Section 2]. For $s \in \mathbb{Z}$, let L_s be the space of series

$$L_s = \biggl\{ \sum_{u \in \mathbb{N}^{n+2}} c_u \gamma_0^{p u_{n+1}} x^u \biggm| \sum_{i=0}^{n} u_i - d u_{n+1} = s,\ c_u \in \mathbb{C}_p, \text{ and } \{c_u\} \text{ is bounded} \biggr\}.$$

For a subset $I = \{i_1, \dots, i_k\} \subseteq \{0, \dots, n+1\}$, define

$$L_I = \begin{cases} L_{-k} & \text{if } n+1 \notin I, \\ L_{d-k+1} & \text{if } n+1 \in I. \end{cases}$$

We construct a de Rham-type complex as follows. For $k = 0, \dots, n+1$, let

$$\Omega^k = \bigoplus_{0 \le i_1 < \cdots < i_k \le n+1} L_{\{i_1,\dots,i_k\}}\, dx_{i_1} \cdots dx_{i_k}.$$

Define $d : \Omega^k \to \Omega^{k+1}$ by

$$d(\xi\, dx_{i_1} \cdots dx_{i_k}) = \sum_{i=0}^{n+1} \frac{\partial \xi}{\partial x_i}\, dx_i\, dx_{i_1} \cdots dx_{i_k}$$

for $\xi \in L_{\{i_1,\dots,i_k\}}$. Define $\hat{f}_\lambda$ to be the Teichmüller lifting of $x_{n+1} f_\lambda$:

$$\hat{f}_\lambda(x_0, \dots, x_{n+1}) = \sum_{j=1}^{N} \hat{\lambda}_j x^{\boldsymbol{a}_j^+} \in \mathbb{Q}_p(\zeta_{q-1})[x_0, \dots, x_{n+1}].$$

Set

$$h = \sum_{j=0}^{\infty} \gamma_j x_{n+1}^{p^j} \hat{f}^{\sigma^j}(x^{p^j}),$$

where
$$\hat{f}^{\sigma}(x^p) = \sum_{j=1}^{N} \hat{\lambda}_j^p x^{p\boldsymbol{a}_j^+},$$

and note that $dh \in \Omega^1$. We observe that in general, if $\omega_1 \in \Omega^{k_1}$ and $\omega_2 \in \Omega^{k_2}$, then $\omega_1 \wedge \omega_2 \in \Omega^{k_1+k_2}$. Let $D : \Omega^k \to \Omega^{k+1}$ be defined by
$$D(\omega) = d\omega + dh \wedge \omega.$$

This gives a complex $(\Omega^\bullet, D)$.

We define the Frobenius operator on this complex. From (3.19) we have
$$\theta(\hat{\lambda}, x) = \prod_{j=1}^{N} \theta(\hat{\lambda}_j x^{\boldsymbol{a}_j^+}). \tag{7.2}$$

We also need to consider the series $\theta_0(\hat{\lambda}, x)$ defined by
$$\theta_0(\hat{\lambda}, x) = \prod_{i=0}^{a-1} \prod_{j=1}^{N} \theta((\hat{\lambda}_j x^{\boldsymbol{a}_j^+})^{p^i}) = \prod_{i=0}^{a-1} \theta(\hat{\lambda}^{p^i}, x^{p^i}). \tag{7.3}$$

Define an operator ψ on formal power series by
$$\psi\biggl(\sum_{u \in \mathbb{N}^{n+2}} c_u x^u\biggr) = \sum_{u \in \mathbb{N}^{n+2}} c_{pu} x^u. \tag{7.4}$$

Denote by $\alpha_{\hat{\lambda}}$ the composition
$$\alpha_{\hat{\lambda}} := \psi^a \circ \theta_0(\hat{\lambda}, x),$$

where $\theta_0(\hat{\lambda}, x)$ is used as an operator to represent multiplication by $\theta_0(\hat{\lambda}, x)$.

We define a map $\alpha_{\hat{\lambda},\bullet} : \Omega^\bullet \to \Omega^\bullet$ by additivity and the formula
$$\alpha_{\hat{\lambda},k}(\xi \, dx_{i_1} \cdots dx_{i_k}) = \frac{q^{n+2-k}}{x_{i_1} \cdots x_{i_k}} \alpha_{\hat{\lambda}}(x_{i_1} \cdots x_{i_k} \xi) \, dx_{i_1} \cdots dx_{i_k}, \tag{7.5}$$

when $\xi \in L_{\{i_1,\dots,i_k\}}$. Note that in this case $x_{i_1} \cdots x_{i_k} \xi$ and $\alpha_{\hat{\lambda}}(x_{i_1} \cdots x_{i_k} \xi)$ lie in L_0. The map $\alpha_{\hat{\lambda},\bullet}$ is a map of complexes and by the Dwork trace formula (as formulated by Robba; see [Adolphson and Sperber 2008, Section 2]) we have
$$L_\lambda(t) = \prod_{k=0}^{n+2} \det(I - t\alpha_{\hat{\lambda},k} \mid \Omega^k)^{(-1)^{k+1}}. \tag{7.6}$$

The factors on the right-hand side of (7.6) are p-adic entire functions.

We now combine (7.1) and (7.6) to get a formula for $P_\lambda(qt)$. First of all, for $I = \{i_1, \dots, i_k\} \subseteq \{0, 1, \dots, n+1\}$, let $L_0^I \subseteq L_0$ be the image of $L_I \, dx_{i_1} \cdots dx_{i_k}$

under the map ϕ defined by

$$\xi\, dx_{i_1}\cdots dx_{i_k} \to x_{i_1}\cdots x_{i_k}\xi.$$

We have a commutative diagram

$$\begin{array}{ccc} L_I\, dx_{i_1}\cdots dx_{i_k} & \xrightarrow{\phi} & L_0^I \\ {\scriptstyle \alpha_{\hat{\lambda},k}}\big\downarrow & & \big\downarrow{\scriptstyle q^{n+2-k}\alpha_{\hat{\lambda}}} \\ L_I\, dx_{i_1}\cdots dx_{i_k} & \xrightarrow{\phi} & L_0^I \end{array}$$

in which the horizontal arrows are isomorphisms, hence there is a product decomposition

$$\det(I - t\alpha_{\hat{\lambda},k} \mid \Omega^k) = \prod_{|I|=k} \det(I - q^{n+2-k} t\alpha_{\hat{\lambda}} \mid L_0^I). \tag{7.7}$$

Combining this with (7.6) gives

$$L_\lambda(t) = \prod_{I\subseteq\{0,1,\ldots,n+1\}} \det(I - q^{n+2-|I|} t\alpha_{\hat{\lambda}} \mid L_0^I)^{(-1)^{|I|+1}}. \tag{7.8}$$

Note that $x^u \in L_0^I$ if and only if $\sum_{i=0}^n u_i = du_{n+1}$ and $u_i > 0$ for $i \in I$. Suppose $I \subseteq \{0, 1, \ldots, n\}$ and $I \neq \varnothing$. If $x^u \in L_0^I$ then $u_{n+1} > 0$ also, and hence $L_0^I = L_0^{I\cup\{n+1\}}$. It follows that for such I we have

$$\det(I - q^{n+1-|I|} t\alpha_{\hat{\lambda}} \mid L_0^I) = \det\bigl(I - q^{n+2-|I\cup\{n+1\}|} t\alpha_{\hat{\lambda}} \mid L_0^{I\cup\{n+1\}}\bigr). \tag{7.9}$$

We can therefore rewrite (7.8) as

$$L_\lambda(t) = \frac{\det\bigl(I - q^{n+1} t\alpha_{\hat{\lambda}} \mid L_0^{\{n+1\}}\bigr)}{\det(I - q^{n+2} t\alpha_{\hat{\lambda}} \mid L_0^{\varnothing})} \cdot \prod_{\varnothing\neq I\subseteq\{0,1,\ldots,n\}} \left(\frac{\det(I - q^{n+2-|I|} t\alpha_{\hat{\lambda}} \mid L_0^I)}{\det(I - q^{n+1-|I|} t\alpha_{\hat{\lambda}} \mid L_0^I)}\right)^{(-1)^{|I|+1}}. \tag{7.10}$$

We examine the first quotient on the right-hand side of (7.10) more closely. It is easy to see that the quotient $L_0^{\varnothing}/L_0^{\{n+1\}}$ is one-dimensional, spanned by the constant 1, and that $\alpha_{\hat{\lambda}}$ acts on this quotient as the identity map. We therefore have

$$\det(I - q^{n+1} t\alpha_{\hat{\lambda}} \mid L_0^{\varnothing}) = (1 - q^{n+1}t)\det\bigl(I - q^{n+1} t\alpha_{\hat{\lambda}} \mid L_0^{\{n+1\}}\bigr).$$

Thus (7.10) implies

$$L_\lambda(t)^{(-1)^{n+1}} = (1 - q^{n+1}t)^{(-1)^n} \cdot \frac{\prod_{I\subseteq\{0,1,\ldots,n\}} \det(I - q^{n+2-|I|} t\alpha_{\hat{\lambda}} \mid L_0^I)^{(-1)^{n+|I|}}}{\prod_{I\subseteq\{0,1,\ldots,n\}} \det(I - q^{n+1-|I|} t\alpha_{\hat{\lambda}} \mid L_0^I)^{(-1)^{n+|I|}}}. \tag{7.11}$$

Comparing (7.1) and (7.11) now gives the desired formula:

$$P_\lambda(qt) = \prod_{I\subseteq\{0,1,\dots,n\}} \det(I - q^{n+1-|I|} t\alpha_{\hat\lambda} \mid L_0^I)^{(-1)^{n+1+|I|}}. \tag{7.12}$$

For notational convenience, we set $\Gamma = \{0, 1, \dots, n\}$.

Proposition 7.13. (a) *The entire function* $\det(I - t\alpha_{\hat\lambda} \mid L_0^\Gamma)$ *has at most one reciprocal zero of q-ordinal equal to $\mu+1$; all other reciprocal zeros have q-ordinal $> \mu+1$. If it has a reciprocal zero of q-ordinal equal to $\mu+1$, then all other reciprocal zeros have q-ordinal $\geq \mu+2$.*

(b) *The reciprocal zeros of* $\det(I - q^{n+1-|I|} t\alpha_{\hat\lambda} \mid L_0^I)$ *all have q-ordinal $\geq \mu+2$ for $I \subsetneq \{0, 1, \dots, n\}$.*

Proof. Consider first the case $I = \varnothing$, i.e., the entire function $\det(I - q^{n+1} t\alpha_{\hat\lambda} \mid L_0^\varnothing)$. All reciprocal zeros are divisible by q^{n+1} and $n+1 \geq \mu+2$ since $n+1 = d(\mu+1)$ and we are assuming $d \geq 2$.

Now suppose that $I \neq \varnothing$ and let

$$\omega(I) = \min\{u_{n+1} \mid x^u \in L_0^I\}.$$

Since $x^u \in L_0^I$ if and only if $\sum_{i=0}^n u_i = du_{n+1}$ and $u_i > 0$ for $i \in I$, we have $\omega(I) = \lceil |I|/d \rceil$, where $\lceil z \rceil$ denotes the least integer that is $\geq z$.

It follows from [Adolphson and Sperber 1987a, Proposition 4.2] that the first side of the Newton polygon of $\deg(I - t\alpha_{\hat\lambda} \mid L_0^I)$ has slope $\geq \omega(I)$. Hence all reciprocal zeros of the entire function $\det(I - q^{n+1-|I|} t\alpha_{\hat\lambda} \mid L_0^I)$ have q-ordinal greater than or equal to

$$n + 1 - |I| + \lceil |I|/d \rceil. \tag{7.14}$$

First take $I = \Gamma$, i.e., $|I| = n+1$. In this case the hypothesis that $n+1 = d(\mu+1)$ reduces the expression (7.14) to $\mu+1$. Furthermore, since $(1, \dots, 1, \mu+1)$ is the unique element u with $x^u \in L_0^\Gamma$ and $u_{n+1} = \mu+1$, it follows from [Adolphson and Sperber 1987a, Proposition 4.2] that the Newton polygon of $\det(I - t\alpha_{\hat\lambda} \mid L_0^\Gamma)$ has a lower bound whose first side has slope $\mu+1$ and length 1. This implies that $\det(I - t\alpha_{\hat\lambda} \mid L_0^\Gamma)$ has at most one reciprocal zero of q-ordinal equal to $\mu+1$ and all other reciprocal zeros have q-ordinal $> \mu+1$. This proves the first sentence of part (a). If $\det(I - t\alpha_{\hat\lambda} \mid L_0^\Gamma)$ has a reciprocal zero of q-ordinal equal to $\mu+1$, then by [Adolphson and Sperber 1987a, Proposition 4.2] the second side of its Newton polygon has slope $\geq \mu+2$. This proves the second sentence of (a).

Next take $|I| = n$. The expression (7.14) reduces to

$$1 + \left\lceil \frac{n}{d} \right\rceil = 1 + \left\lceil \mu + 1 - \frac{1}{d} \right\rceil = \mu + 2$$

since $d \geq 2$. Furthermore, (7.14) cannot decrease when $|I|$ decreases, which proves part (b) of the proposition. $\square$

Recall from Section 1 that we write $P_\lambda(t) = P_\lambda^{(1)}(t)/P_\lambda^{(2)}(t)$, where $P_\lambda^{(1)}(t)$ and $P_\lambda^{(2)}(t)$ are relatively prime polynomials with integer coefficients and constant term 1 which satisfy

$$P_\lambda^{(1)}(q^{-\mu}t),\ \ P_\lambda^{(2)}(q^{-\mu}t) \in 1 + t\mathbb{Z}[t].$$

Proposition 7.13, together with (7.12), shows that

$$P_\lambda^{(2)}(q^{-\mu}t) \equiv 1 \pmod q$$

and that $P_\lambda^{(1)}(q^{-\mu}t) \pmod p$ has degree at most 1 in t. To complete the proof of Proposition 1.7 it suffices, by Proposition 7.13(a), to show that

$$\mathrm{Tr}(\alpha_{\hat\lambda} \mid L_0^\Gamma) \equiv q^{\mu+1} \prod_{i=0}^{a-1} ((-1)^{\mu+1} H(\hat\lambda^{p^i})) \pmod{pq^{\mu+1}}. \tag{7.15}$$

Using (5.7), one sees that (7.15) is equivalent to the following assertion.

Proposition 7.16. *For $\lambda \in (\mathbb{F}_q^\times)^N$, we have*

$$\mathrm{Tr}(\alpha_{\hat\lambda} \mid L_0^\Gamma) \equiv \prod_{i=0}^{a-1} \theta_{-(p-1)\boldsymbol{b}}(\hat\lambda^{p^i}) \pmod{pq^{\mu+1}}.$$

Proof. Consider the series

$$\theta_0(\hat\lambda, x) = \sum_{w\in\mathbb{N}A} \theta_{0,w}(\hat\lambda)x^w.$$

By (7.3) we have

$$\theta_{0,w}(\hat\lambda) \ = \sum_{\substack{u^{(0)},\dots,u^{(a-1)}\in\mathbb{N}A \\ \sum_{i=0}^{a-1} p^i u^{(i)} = w}} \ \prod_{i=0}^{a-1} \theta_{u^{(i)}}(\hat\lambda^{p^i}). \tag{7.17}$$

Let $U \subseteq \mathbb{N}^{n+2}$ be the set of all exponents u such that $x^u \in L_0^\Gamma$. For $w \in U$, a direct calculation shows that

$$\alpha_{\hat\lambda}(x^w) = \sum_{u\in U} \theta_{0,qu-w}(\hat\lambda)x^u. \tag{7.18}$$

It then follows from the Dwork trace formula that

$$\mathrm{Tr}(\alpha_{\hat\lambda} \mid L_0^\Gamma) = \sum_{w\in U} \theta_{0,(q-1)w}(\hat\lambda). \tag{7.19}$$

Equation (7.17) gives

$$\theta_{0,(q-1)w}(\hat\lambda) \ = \sum_{\substack{u^{(0)},\dots,u^{(a-1)}\in\mathbb{N}A \\ \sum_{i=0}^{a-1} p^i u^{(i)} = (q-1)w}} \ \prod_{i=0}^{a-1} \theta_{u^{(i)}}(\hat\lambda^{p^i}). \tag{7.20}$$

It follows from (3.21) and (3.23) that

$$\begin{aligned}&\operatorname{ord}_p \theta_{0,(q-1)w}(\hat{\lambda})\\ &\quad\geq \min\left\{\sum_{i=0}^{a-1}\frac{u^{(i)}_{n+1}}{p-1}\;\middle|\; u^{(0)},\dots,u^{(a-1)}\in\mathbb{N}A \text{ and } \sum_{i=0}^{a-1}p^i u^{(i)} = (q-1)w\right\}.\end{aligned}\tag{7.21}$$

We prove Proposition 7.16 by studying this estimate for $w\in U$.

Fix $u^{(0)},\dots,u^{(a-1)}\in\mathbb{N}A$ with

$$\sum_{i=0}^{a-1}p^i u^{(i)} = (q-1)w \tag{7.22}$$

and $w\in U$. We define inductively a sequence $w^{(0)},\dots,w^{(a)}\in U$ such that

$$u^{(i)} = pw^{(i+1)} - w^{(i)} \quad \text{for } i=0,\dots,a-1. \tag{7.23}$$

First of all, take $w^{(0)}=w$. Then (7.22) shows that $u^{(0)}+w^{(0)}=pw^{(1)}$ for some $w^{(1)}\in\mathbb{Z}^{n+2}$; since $u^{(0)}\in\mathbb{N}A$ and $w^{(0)}\in U$ we conclude that $w^{(1)}\in U$. Suppose that for some $0<k\leq a-1$ we have defined $w^{(0)},\dots,w^{(k)}\in U$ satisfying (7.23) for $i=0,\dots,k-1$. Substituting $pw^{(i+1)}-w^{(i)}$ for $u^{(i)}$ for $i=0,\dots,k-1$ in (7.22) gives

$$-w^{(0)} + p^k w^{(k)} + \sum_{i=k}^{a-1} p^i u^{(i)} = p^a w - w. \tag{7.24}$$

Since $w^{(0)}=w$, we can divide this equation by p^k to get $w^{(k)}+u^{(k)}=pw^{(k+1)}$ for some $w^{(k+1)}\in\mathbb{Z}^{n+2}$. Since $u^{(k)}\in\mathbb{N}A$ and (by induction) $w^{(k)}\in U$, we conclude that $w^{(k+1)}\in U$. This completes the inductive construction. Note that in the special case $k=a-1$, this computation gives $w^{(a)}=w$.

Summing (7.23) over $i=0,\dots,a-1$ and using $w^{(0)}=w^{(a)}=w$ gives

$$\sum_{i=0}^{a-1} u^{(i)} = (p-1)\sum_{i=0}^{a-1} w^{(i)}. \tag{7.25}$$

Hence

$$\sum_{i=0}^{a-1}\frac{u^{(i)}_{n+1}}{p-1} = \sum_{i=0}^{a-1} w^{(i)}_{n+1}. \tag{7.26}$$

Since $w^{(i)}\in U$, we have

$$\begin{cases} w^{(i)}_{n+1} = \mu+1 & \text{if } w^{(i)} = (1,\dots,1,\mu+1),\\ w^{(i)}_{n+1} \geq \mu+2 & \text{if } w^{(i)} \neq (1,\dots,1,\mu+1).\end{cases} \tag{7.27}$$

It now follows from (7.26) that

$$\sum_{i=0}^{a-1} \frac{u_{n+1}^{(i)}}{p-1} \begin{cases} = a(\mu+1) & \text{if } w^{(i)} = (1, \dots, 1, \mu+1) \text{ for } i = 0, \dots, a-1, \\ \geq a(\mu+1)+1 & \text{otherwise.} \end{cases} \tag{7.28}$$

Therefore, by (7.23), $\sum_{i=0}^{a-1} u_{n+1}^{(i)}/(p-1) = a(\mu+1)$ if and only if for all i, $u^{(i)} = (p-1)(1, \dots, 1, \mu+1)$.

By (7.21), this implies that if $w \neq (1, \dots, 1, \mu+1)$, then

$$\theta_{0,(q-1)w}(\hat{\lambda}) \equiv 0 \pmod{pq^{\mu+1}}.$$

If $w = (1, \dots, 1, \mu+1)$, this implies by (7.20) that

$$\theta_{0,(q-1)(1,\dots,1,\mu+1)}(\hat{\lambda}) \equiv \prod_{i=0}^{a-1} \theta_{(p-1)(1,\dots,1,\mu+1)}(\hat{\lambda}^{p^i}) \pmod{pq^{\mu+1}}.$$

Since $-\boldsymbol{b} = (1, \dots, 1, \mu+1)$, (7.19) now implies the proposition. □

Let $\lambda \in (\mathbb{F}_q^\times)^N$. In the course of proving Proposition 1.7, we have shown that $\overline{H}(\lambda) \neq 0$ is a necessary and sufficient condition for $\det(I - t\alpha_{\hat{\lambda}} \mid L_0^\Gamma)$ to have a unique reciprocal zero of q-ordinal equal to $\mu+1$. To prove the last assertion of Theorem 4.32, it suffices by (7.12) and Proposition 7.13 to prove the following result.

Theorem 7.29. *If $\lambda \in (\mathbb{F}_q^\times)^N$ and $\overline{H}(\lambda) \neq 0$, then $q^{\mu+1} \prod_{i=0}^{a-1} \mathcal{G}(\hat{\lambda}^{p^i})$ is an eigenvalue of $\alpha_{\hat{\lambda}}$ on L_0^Γ.*

Before beginning the proof of Theorem 7.29, we give an alternate description of $\det(I - t\alpha_{\hat{\lambda}} \mid L_0^\Gamma)$. Let

$$\hat{M}_- = \left\{ u = (u_0, \dots, u_{n+1}) \in (\mathbb{Z}_{<0})^{n+2} \;\middle|\; \sum_{i=0}^{n} u_i = du_{n+1} \right\},$$
$$\hat{M}_+ = \left\{ u = (u_0, \dots, u_{n+1}) \in (\mathbb{Z}_{>0})^{n+2} \;\middle|\; \sum_{i=0}^{n} u_i = du_{n+1} \right\}.$$

Set

$$B = \left\{ \xi^* = \sum_{u \in \hat{M}_-} c_u^* \gamma_0^{pu_{n+1}} x^u \;\middle|\; c_u^* \to 0 \text{ as } u \to -\infty \right\},$$

a p-adic Banach space with norm $|\xi^*| = \sup_{u \in \hat{M}_-} \{|c_u^*|\}$. We define a pairing $\langle\ ,\ \rangle : B \times L_0^\Gamma \to \mathbb{C}_p$ as follows. If

$$\xi = \sum_{u \in \hat{M}_+} c_u \gamma_0^{pu_{n+1}} x^u \in L_0^\Gamma \quad \text{and} \quad \xi^* = \sum_{u \in \hat{M}_-} c_u^* \gamma_0^{pu_{n+1}} x^u \in B,$$

define

$$\langle \xi^*, \xi \rangle = \sum_{u \in \hat{M}_+} c_u c_{-u}^*,$$

the constant term of the product $\xi^*\xi$. This pairing identifies B with the dual space of L_0^Γ, the space of continuous linear mappings from L_0^Γ to $\mathbb{C}_p$; see [Serre 1962, Proposition 3]. We extend the definition of the mapping Φ defined in the proof of Proposition 6.9 by setting

$$\Phi\biggl(\sum_{u\in\mathbb{Z}^n} c_u x^u\biggr) = \sum_{u\in\mathbb{Z}^n} c_u x^{pu}.$$

Consider the formal composition $\alpha^*_{\hat\lambda} = \delta_- \circ \theta_0(\hat\lambda, x) \circ \Phi^a$, where again $\theta_0(\hat\lambda, x)$ represents multiplication by $\theta_0(\hat\lambda, x)$.

Proposition 7.30. *The operator $\alpha^*_{\hat\lambda}$ is an endomorphism of B which is adjoint to $\alpha_{\hat\lambda} : L_0^\Gamma \to L_0^\Gamma$.*

Proof. Since $\alpha^*_{\hat\lambda}$ is the a-fold composition of the operators $\delta_- \circ \theta(\hat\lambda^{p^i}, x) \circ \Phi$ and $\alpha_{\hat\lambda}$ the a-fold composition of the operators $\psi \circ \theta(\hat\lambda^{p^i}, x)$ for $i = 0, \dots, a-1$, it suffices to check that $\delta_- \circ \theta(\hat\lambda, x) \circ \Phi$ is an endomorphism of B adjoint to $\psi \circ \theta(\hat\lambda, x) : L_0^\Gamma \to L_0^\Gamma$. Let $\xi^*(x) = \sum_{v\in\hat M_-} c^*_v \gamma_0^{pv_{n+1}} x^v \in B$. The proof that the product $\theta(\hat\lambda, x)\xi^*(x^p)$ is well defined is analogous to the proof of convergence of (5.1). We have

$$\delta_-(\theta(\hat\lambda, x)\xi^*(x^p)) = \sum_{u\in\hat M_-} C^*_u \gamma_0^{pu_{n+1}} x^u,$$

where

$$C^*_u = \sum_{w+pv=u} \theta_w(\hat\lambda) c^*_v \gamma_0^{p(v_{n+1}-u_{n+1})}. \tag{7.31}$$

Note that by (3.23),

$$\operatorname{ord}_p \theta_w(\hat\lambda)\gamma_0^{p(v_{n+1}-u_{n+1})} \geq \frac{w_{n+1}}{p-1} + \frac{pv_{n+1}}{p-1} - \frac{pu_{n+1}}{p-1} = -u_{n+1} \tag{7.32}$$

since $w + pv = u$. Since $c^*_v \to 0$ as $v \to -\infty$, this implies that the series on the right-hand side of (7.31) converges. Furthermore, the estimate (7.32) then shows that $C^*_u \to 0$ as $u \to -\infty$. We conclude that $\delta_-(\theta(\hat\lambda, x)\xi^*(x^p)) \in B$. In fact, (7.32) implies

$$|\delta_-(\theta(\hat\lambda, x)\xi^*(x^p))| \leq |p^{\mu+1}\xi^*(x)|,$$

since $u_{n+1} \leq -(\mu+1)$ for all $u \in M_-$. □

Proof of Theorem 7.29. From Proposition 7.30, it follows by [Serre 1962, Proposition 15] that

$$\det(I - t\alpha_{\hat\lambda} \mid L_0^\Gamma) = \det(I - t\alpha^*_{\hat\lambda} \mid B), \tag{7.33}$$

so to complete the proof of Theorem 7.29 it suffices to show that if $\overline{H}(\lambda) \neq 0$, then $\alpha^*_{\hat\lambda}$ has an eigenvector in B with eigenvalue $q^{\mu+1}\prod_{i=0}^{a-1} \mathcal{G}(\hat\lambda^{p^i})$. From (5.16) we have

$$\alpha^*\biggl(\frac{G(\Lambda, x)}{G(\Lambda)}\biggr) = p^{\mu+1}\mathcal{G}(\Lambda)\frac{G(\Lambda, x)}{G(\Lambda)}.$$

It follows by iteration that for $m \geq 0$,

$$(\alpha^*)^m \left(\frac{G(\Lambda, x)}{G(\Lambda)} \right) = p^{m(\mu+1)} \left(\prod_{i=0}^{m-1} \mathcal{G}(\Lambda^{p^i}) \right) \frac{G(\Lambda, x)}{G(\Lambda)}. \tag{7.34}$$

From (4.15) we have

$$\frac{G(\Lambda, x)}{G(\Lambda)} = \sum_{u \in M_-} \left(\gamma_0^{-(p-1)u_{n+1}} \frac{G_u(\Lambda)}{G(\Lambda)} \right) \gamma_0^{pu_{n+1}} x^u.$$

By Proposition 5.15, the ratio $\mathcal{G}_u(\Lambda) := G_u(\Lambda)/G(\Lambda)$ lies in R'_u. We may therefore evaluate the $\mathcal{G}_u(\Lambda)$ at $\Lambda = \hat{\lambda}$:

$$\frac{G(\Lambda, x)}{G(\Lambda)} \bigg|_{\Lambda=\hat{\lambda}} = \sum_{u \in M_-} \big(\gamma_0^{-(p-1)u_{n+1}} \mathcal{G}_u(\hat{\lambda}) \big) \gamma_0^{pu_{n+1}} x^u.$$

Since $\gamma_0^{-(p-1)u_{n+1}} \to 0$ as $u \to \infty$, this expression lies in B. It is straightforward to check that the specialization of the left-hand side of (7.34) with $m = a$ at $\Lambda = \hat{\lambda}$ is exactly $\alpha^*_{\hat{\lambda}}\big(G(\Lambda, x)/G(\Lambda)|_{\Lambda=\hat{\lambda}}\big)$, so specializing (7.34) with $m = a$ at $\Lambda = \hat{\lambda}$ gives

$$\begin{aligned} \alpha^*_{\hat{\lambda}} \Bigg(\sum_{u \in M_-} \big(\gamma_0^{-(p-1)u_{n+1}} \mathcal{G}_u(\hat{\lambda}) \big) \gamma_0^{pu_{n+1}} x^u \Bigg) \\ = q^{\mu+1} \left(\prod_{i=0}^{a-1} \mathcal{G}(\hat{\lambda}^{p^i}) \right) \Bigg(\sum_{u \in M_-} \big(\gamma_0^{-(p-1)u_{n+1}} \mathcal{G}_u(\hat{\lambda}) \big) \gamma_0^{pu_{n+1}} x^u \Bigg). \end{aligned} \tag{7.35}$$

This equation shows that $q^{\mu+1} \prod_{i=0}^{a-1} \mathcal{G}(\hat{\lambda}^{p^i})$ is an eigenvalue of $\alpha^*_{\hat{\lambda}}$. □

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References

[Adolphson and Sperber 1984] A. Adolphson and S. Sperber, "Twisted Kloosterman sums and p-adic Bessel functions", *Amer. J. Math.* **106**:3 (1984), 549–591. MR Zbl

[Adolphson and Sperber 1987a] A. Adolphson and S. Sperber, "p-adic estimates for exponential sums and the theorem of Chevalley–Warning", *Ann. Sci. École Norm. Sup.* (4) **20**:4 (1987), 545–556. MR Zbl

[Adolphson and Sperber 1987b] A. Adolphson and S. Sperber, "Twisted Kloosterman sums and p-adic Bessel functions, II: Newton polygons and analytic continuation", *Amer. J. Math.* **109**:4 (1987), 723–764. MR Zbl

[Adolphson and Sperber 2000] A. Adolphson and S. Sperber, "Exponential sums on $\mathbb{A}^n$", *Israel J. Math.* **120**:3 (2000), 3–21. MR Zbl

[Adolphson and Sperber 2006] A. Adolphson and S. Sperber, "On the Jacobian ring of a complete intersection", *J. Algebra* **304**:2 (2006), 1193–1227. MR Zbl

[Adolphson and Sperber 2008] A. Adolphson and S. Sperber, "On the zeta function of a projective complete intersection", *Illinois J. Math.* **52**:2 (2008), 389–417. MR Zbl

[Adolphson and Sperber 2012] A. Adolphson and S. Sperber, "On unit root formulas for toric exponential sums", *Algebra Number Theory* **6**:3 (2012), 573–585. MR Zbl

[Adolphson and Sperber 2013] A. Adolphson and S. Sperber, "On the p-integrality of A-hypergeometric series", preprint, 2013. arXiv

[Ax 1964] J. Ax, "Zeroes of polynomials over finite fields", *Amer. J. Math.* **86** (1964), 255–261. MR Zbl

[Dwork 1960] B. Dwork, "On the rationality of the zeta function of an algebraic variety", *Amer. J. Math.* **82** (1960), 631–648. MR Zbl

[Dwork 1962] B. Dwork, "On the zeta function of a hypersurface", *Inst. Hautes Études Sci. Publ. Math.* **12** (1962), 5–68. MR Zbl

[Dwork 1963] B. Dwork, "A deformation theory for the zeta function of a hypersurface", pp. 247–259 in *Proceedings of the International Congress of Mathematicians* (Stockholm, 1962), Inst. Mittag-Leffler, Djursholm, Sweden, 1963. MR Zbl

[Dwork 1964] B. Dwork, "On the zeta function of a hypersurface, II", *Ann. of Math.* (2) **80** (1964), 227–299. MR Zbl

[Dwork 1969] B. Dwork, "p-adic cycles", *Inst. Hautes Études Sci. Publ. Math.* **37** (1969), 27–115. MR Zbl

[Dwork 1974] B. Dwork, "Bessel functions as p-adic functions of the argument", *Duke Math. J.* **41** (1974), 711–738. MR Zbl

[Igusa 1958] J.-I. Igusa, "Class number of a definite quaternion with prime discriminant", *Proc. Nat. Acad. Sci. U.S.A.* **44** (1958), 312–314. MR Zbl

[Illusie 1990] L. Illusie, "Ordinarité des intersections complètes générales", pp. 376–405 in *The Grothendieck Festschrift*, vol. 2, edited by P. Cartier et al., Progr. Math. **87**, Birkhäuser, Boston, 1990. MR Zbl

[Katz 1971] N. M. Katz, "On a theorem of Ax", *Amer. J. Math.* **93** (1971), 485–499. MR Zbl

[Katz 1985] N. M. Katz, "Internal reconstruction of unit-root F-crystals via expansion-coefficients", *Ann. Sci. École Norm. Sup.* (4) **18**:2 (1985), 245–285. MR Zbl

[Saito et al. 2000] M. Saito, B. Sturmfels, and N. Takayama, *Gröbner deformations of hypergeometric differential equations*, Algorithms and Computation in Mathematics **6**, Springer, 2000. MR Zbl

[Serre 1962] J.-P. Serre, "Endomorphismes complètement continus des espaces de Banach p-adiques", *Inst. Hautes Études Sci. Publ. Math.* **12** (1962), 69–85. MR Zbl

[Yu 2009] J.-D. Yu, "Variation of the unit root along the Dwork family of Calabi–Yau varieties", *Math. Ann.* **343**:1 (2009), 53–78. MR Zbl

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On defects of characters and decomposition numbers

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We propose upper bounds for the number of modular constituents of the restriction modulo p of a complex irreducible character of a finite group, and for its decomposition numbers, in certain cases.

1. Introduction

Let G be a finite group and let p be a prime. Richard Brauer, in a fundamental paper [1941], studied the irreducible complex characters $\chi \in \mathrm{Irr}(G)$ such that $\chi(1)_p = |G|_p/p$, where here n_p denotes the largest power of p dividing the integer n. This was the birth of what later became the cyclic defect theory, developed by E. C. Dade [1966], building upon work of J. A. Green and J. G. Thompson on vertices and sources. Of course, Brauer and Nesbitt had studied before the *defect zero* characters (those $\chi \in \mathrm{Irr}(G)$ with $\chi(1)_p = |G|_p$) proving that they lift irreducible modular characters in characteristic p. A constant in Brauer's work was to analyse the decomposition of the complex irreducible characters χ into modular characters:

$$\chi^0 = \sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi\varphi}\varphi,$$

where here χ^0 is the restriction of χ to the elements of G of order prime to p, and where we have chosen a set $\mathrm{IBr}(G)$ of irreducible p-Brauer characters of G. To better understand the *decomposition numbers* $d_{\chi\varphi}$ remains one of the challenges in representation theory.

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If $\chi \in \mathrm{Irr}(G)$ let us write $\mathrm{IBr}(\chi^0) = \{\varphi \in \mathrm{IBr}(G) \mid d_{\chi\varphi} \neq 0\}$, and recall that the *defect* of χ is the integer d_χ with

$$p^{d_\chi} \chi(1)_p = |G|_p .$$

For $\chi \in \mathrm{Irr}(G)$ of defect one Brauer proved that all decomposition numbers $d_{\chi\varphi}$ are less than or equal to 1, and implicitly, that there are less than p characters $\varphi \in \mathrm{IBr}(G)$ occurring with multiplicity $d_{\chi\varphi} \neq 0$.

In order to gain insight into decomposition numbers in general, it is natural from this perspective to next study characters of defect two. This step looks innocent, but it deepens things in such a way that at present we can only guess what might be happening in general:

Conjecture A. *Let G be a finite group, p a prime. Let $\chi \in \mathrm{Irr}(G)$ with $|G|_p = p^2 \cdot \chi(1)_p$. Then*:

(1) $|\mathrm{IBr}(\chi^0)| \leq p^2 - 1$.

(2) $d_{\chi\varphi} \leq p$ *for all* $\varphi \in \mathrm{IBr}(G)$.

It is remarkable that, as we shall prove below (Theorem 3.2), Conjecture A follows from the Alperin–McKay conjecture together with the work of K. Erdmann, but only for the prime $p = 2$. We do prove below Conjecture A for p-solvable groups (see Theorem 2.5) and for certain classes of quasisimple and almost simple groups (see Theorem 4.2, Propositions 4.3 and 4.4 and Theorem 5.4). On the other hand, Conjecture A is wide open for example for groups of Lie type in nondefining characteristic (see Example 5.5, but also Proposition 5.9). As far as we are aware, no bounds for the number $|\mathrm{IBr}(\chi^0)|$ have been proposed before, and therefore, we move into unexplored territory here.

If Conjecture A is true then both of our bounds are sharp. Of course, the irreducible characters of degree p in a nonabelian group of order p^3 have decomposition numbers equal to p. Also, the irreducible character χ of degree $p^2 - 1$ in the semidirect product of $C_p \times C_p$ with a cyclic group of order $p^2 - 1$ acting faithfully satisfies $|\mathrm{IBr}(\chi^0)| = p^2 - 1$.

In view of Brauer's analysis of characters of defect one and our Conjecture A, it is tempting to guess that whenever $\chi(1)_p = |G|_p/p^3$, then $d_{\chi\varphi} \leq p^2$ and $|\mathrm{IBr}(\chi^0)| \leq p^3 - 1$. While we are not aware of any p-solvable counter-examples, still this is not true in general: For $p = 2$, $G = 3.J_3$ has a character χ such that $\chi(1)_p = |G|_p/p^3$, having 8 irreducible Brauer constituents. For $p = 3$, $G = Co_3$ has an irreducible character such that $\chi(1)_p = |G|_p/p^3$ with 13 occurring as a decomposition number. Perhaps other bounds are possible.

While studying Conjecture A, we came across a remarkable inequality, to which we have not yet found a counterexample.

Conjecture B. *Let G be a finite group, p a prime, and $\chi \in \mathrm{Irr}(G)$. Then*

$$|\mathrm{IBr}(\chi^0)| \cdot \chi(1)_p \leq |G|_p.$$

For characters of defect 1, this is the cited result of Brauer, and for defect 2 it follows from Conjecture A(1).

We shall prove below that Conjecture B is satisfied in symmetric groups and for simple groups of Lie type in defining characteristic (see Proposition 4.1 and Corollary 5.2). Our bound seems exactly the right bound for these classes of groups, and somehow this makes us think that Conjecture B points in the right direction. On the other hand, we do not know what is happening in other important classes of groups, like solvable groups or groups of Lie type in nondefining characteristic, for instance. Whether Conjecture B is true or false, we believe that the relation between $|\mathrm{IBr}(\chi^0)|$ and $|G|_p/\chi(1)_p$ is worth exploring. Of course, if $|G|_p/\chi(1)_p = 1$, then $|\mathrm{IBr}(\chi^0)| = 1$ by the Brauer–Nesbitt theorem.

So far we have not mentioned blocks, heights, or defect groups. These are, of course, related to Conjectures A and B. If $\chi \in \mathrm{Irr}(G)$, then $\chi \in \mathrm{Irr}(B)$ for a unique p-block B of G, and $\chi(1)_p = p^{a-d+h}$, where $|G|_p = p^a$, $|D| = p^d$ is the order of a defect group D for B, d is the defect of the block B, and $h \geq 0$ is the height of χ. Hence, the defect d_χ of χ is

$$d_\chi = d - h.$$

Brauer's famous $k(B)$-conjecture asserts that $k(B) := |\mathrm{Irr}(B)| \leq |D| = p^d$. Since obviously

$$|\mathrm{IBr}(\chi^0)| \leq |\mathrm{IBr}(B)| =: l(B) < |\mathrm{Irr}(B)| \quad (\text{if } d > 0),$$

it follows that for characters of height zero both Conjectures A(1) and B are implied by Brauer's $k(B)$-conjecture. (In particular, by Kessar and Malle's solution [2013] of one implication of the height zero conjecture, Conjectures A(1) and B follow from the $k(B)$-conjecture for characters in blocks with abelian defect groups.) It is worth mentioning that $l(B)$ is bounded by $\frac{1}{4}p^{2d} + 1$, using a well-known bound of $k(B)$ by Brauer and Feit.

A recent conjecture, formulated by Malle and Robinson [2017], is also related to the present work. Malle and Robinson have proposed that $l(B) \leq p^{s(D)}$, where $s(D)$ is the so called p-sectional rank of the group D. Hence Conjecture B follows from the Malle–Robinson conjecture for those irreducible characters whose height h is such that

$$s(D) + h \leq d. \tag{1}$$

Observe that the stronger inequality $l(B) \cdot \chi(1)_p \leq |G|_p$ is not always true. For example, let G be the central product of n copies of $\mathrm{SL}_2(3)$ where the centres of order 2 are identified. Then the principal 2-block B of G has a normal defect group and satisfies $l(B) = 3^n$, but there is an irreducible character $\chi \in \mathrm{Irr}(B)$ (deflated from the direct product of $\mathrm{SL}_2(3)$'s) such that $|G|_2/\chi(1)_2 = 2^{n+1}$. Other examples are the principal 2-block of J_3 or the principal 3-block of $\mathrm{SL}_6(2)$.

Finally, we come back to characters of defect 2 for small primes, but from the perspective of their relationship with their blocks and defect groups. Here we prove:

Theorem C. (a) *If $p = 2$ then the Alperin–McKay conjecture implies Conjecture A.*

(b) *If $p = 3$ then Robinson's ordinary weight conjecture implies $l(B) \leq 10$ for every block B containing a character χ as in Conjecture A.*

If there is an upper bound for $l(B)$ in Theorem C for arbitrary primes, we have not been able to find it. In the situation of Theorem C for p-solvable groups, we shall prove below that either $|D| \leq p^3$ or $|D| = p^4$ and $p \leq 3$, and the possible defect groups are classified. For non p-solvable groups, however, $|D|$ is unbounded, as shown by $\mathrm{SL}_2(q)$ with q odd and $p = 2$.

2. p-solvable groups

We start with the proof of Conjecture A for p-solvable groups. In fact, we prove something more general (which will include certain p-constrained groups). Our notation for complex characters follows [Isaacs 1976], and for Brauer characters [Navarro 1998]. If G is a finite group, $N \lhd G$, and $\theta \in \mathrm{Irr}(N)$, then $\mathrm{Irr}(G|\theta)$ is the set of complex irreducible characters $\chi \in \mathrm{Irr}(G)$ such that the restriction χ_N contains θ as a constituent. Notice that if $\psi^G = \chi \in \mathrm{Irr}(G)$ for ψ an irreducible character of the inertia group T of θ, then $|G : T|_p \psi(1)_p = \chi(1)_p$, and therefore $|G|_p/\chi(1)_p = |T|_p/\psi(1)_p$. Hence, $d_\chi = d_\psi$.

First we collect some rather well-known results.

Lemma 2.1. *Let p be a prime, and let U be a subgroup of* $\mathrm{GL}_2(p)$.

(a) *Assume that U has order divisible by p. Then either U has a normal Sylow p-subgroup or* $\mathrm{SL}_2(p) \subseteq U$.

(b) *If $W \subseteq \mathrm{SL}_2(p) \subseteq U \subseteq \mathrm{GL}_2(p)$, then W and U have trivial Schur multiplier.*

(c) *If $W \subseteq \mathrm{SL}_2(p)$ is a p'-subgroup, then either W is cyclic of order a divisor of $p-1$ or $p+1$, W has a normal cyclic subgroup of index* 2, *or* $W = \mathrm{SL}_2(3)$, `SmallGroup(48, 28)` *or* $\mathrm{SL}_2(5)$.

(d) *Suppose that $G = (C_p \times C_p) \rtimes U$ in natural action, where* $\mathrm{SL}_2(p) \subseteq U \subseteq \mathrm{GL}_2(p)$. *If $\chi \in \mathrm{Irr}(G)$, and p divides $\chi(1)$, then $\chi(1) = p$.*

(e) *Suppose that L is an extra-special group of order p^3 and exponent p. Then we have that* $\mathrm{Aut}(L)/\mathrm{Inn}(L) \cong \mathrm{GL}_2(p)$. *If fact, if $Z = Z(L)$, $A = \mathrm{Aut}(L)$ and $I = \mathrm{Inn}(L)$, then $A = C_A(Z) \rtimes \langle\sigma\rangle$, where σ has order $p-1$, and* $C_A(Z)/I = \mathrm{SL}_2(p)$.

Proof. (a) This part follows from [Kurzweil and Stellmacher 2004, 8.6.7].

(b) Checking $p=3$ directly, we may assume $p\geq 5$. Suppose by way of contradiction that U has a proper covering group S with $1\neq Z\leq Z(S)\cap S'$ and $S/Z\cong U$. Let $N\trianglelefteq S$ be the preimage of $\mathrm{SL}_2(p)$. It is a well-known fact that the Sylow subgroups of $\mathrm{SL}_2(p)$ are cyclic or quaternion groups. This implies that $\mathrm{SL}_2(p)$ (and any of its subgroups) has trivial Schur multiplier. In particular, N is not a covering group of $\mathrm{SL}_2(p)$. Since $\mathrm{SL}_2(p)$ is perfect for $p\geq 5$, we must have $N=N'Z$ and $Z\nsubseteq N'$. Since $S/N\cong U/\mathrm{SL}_2(p)$ is cyclic, it follows that $S'=N'$. But this gives the contradiction $Z\nsubseteq S'$.

(c) We follow the well-known classification of the subgroups of $\mathrm{L}_2(p)$. Notice that S has a unique involution, so the 2-subgroups of S are cyclic or quaternion. Set $Z:=Z(S)$. Now, if WZ/Z is cyclic of order a divisor of $(p\pm 1)/2$, then it follows that WZ is abelian with cyclic Sylow subgroups. Thus WZ (and W) are cyclic of order dividing $p\pm 1$. Suppose now that H/Z is dihedral of order $2\cdot(p\pm 1)$, where H is a p'-subgroup of S. Then H/Z has a cyclic subgroup of index 2. Thus H has a cyclic subgroup of index 2. Hence if W is a subgroup of H, it has a cyclic normal 2-complement Q, and a Sylow 2-subgroup P such that $|Q:C_Q(P)|\leq 2$. Since Q is cyclic, or generalised quaternion, it follows that W has a cyclic normal subgroup of index 2. Suppose next that $WZ/Z=\mathfrak{A}_4$. Then WZ has order 24, centre Z of order 2, and a unique involution. Hence WZ is $\mathrm{SL}_2(3)$. The proper subgroups of $\mathrm{SL}_2(3)$ are cyclic or quaternion of order 8. Suppose now that $WZ/Z=\mathfrak{S}_4$. Then WZ has order 48, centre Z of order 2 and a unique involution. Thus $WZ=\texttt{SmallGroup}(48,28)$. The proper subgroups of this group are cyclic, have a cyclic normal subgroup of index 2, or are isomorphic to $\mathrm{SL}_2(3)$. Finally, if $WZ/Z=\mathfrak{A}_5$, then $WZ=\mathrm{SL}_2(5)$. The proper subgroups of $\mathrm{SL}_2(5)$ are already on our list.

(d) Write $V=C_p\times C_p$. Note that $\mathrm{Irr}(V)$ is then the dual of the natural module for U, and $\mathrm{SL}_2(p)\leq U$ acts transitively on $\mathrm{Irr}(V)\setminus\{1\}$. Let $\theta\in\mathrm{Irr}(V)$. If $\theta=1$ then it extends to G and the constituents of θ^G are the inflations of characters of U. The ones of degree divisible by p are thus the extensions of the Steinberg character of $\mathrm{SL}_2(p)$ to U, all of degree p. Now assume that $\theta\neq 1$. Then up to conjugation the inertia group T of θ in U contains all elements $\left(\begin{smallmatrix}1&*\\0&*\end{smallmatrix}\right)$ in U, hence T is metacyclic with abelian Sylow subgroups and normal Sylow p-subgroup. So θ extends to $V\rtimes T$, and the characters above θ have degrees prime to p. Since $|U:T|$ is prime to p, all characters in $\mathrm{Irr}(G|\theta)$ are of degree prime to p.

(e) Consider the canonical map $F:\mathrm{Aut}(L)\to\mathrm{Aut}(L/L')$. Then $\mathrm{Inn}(L)$ lies in the kernel of F. Assume conversely that f lies in $\ker(F)$. If L is generated by x and y, we have at most p^2 possibilities for $f(x)$ in xL' and $f(y)$ in yL'. On the other hand, $|\mathrm{Inn}(L)|=p^2$. Hence, $\ker(F)=\mathrm{Inn}(L)$ and F induces an embedding

$\mathrm{Out}(L)$ in $\mathrm{GL}_2(p)$. It is shown in [Winter 1972] that $|\mathrm{Out}(L)| = |\mathrm{GL}_2(p)|$. The rest follows from the main theorem in the same work. □

Lemma 2.2. *Let $S = \mathrm{SL}_2(p)$, where p is an odd prime, and let $\alpha, \beta \in \mathrm{Irr}(S)$ of degrees $\frac{p-1}{2}$ and $\frac{p+1}{2}$ such that $|\alpha(x) + \beta(x)|^2 = |C_V(x)|$ for all $x \in S$, where $V = C_p \times C_p$ is the natural module for S. Let $\Psi = \alpha + \beta$.*

(a) *We have that $\Psi(x) = \pm 1$ for p-regular nontrivial $x \in S$. Furthermore, the values of Ψ on p-regular elements do not depend of the choices of α and β.*

(b) *Let $\gamma \in \mathrm{Irr}(S)$ of degree p. Let $\Delta = \gamma\Psi$. Assume that $p \geq 7$. Then the irreducible p-Brauer constituents of Δ^0 appear with multiplicity less than or equal to $(p-1)/2$, and $|\mathrm{IBr}(\Delta^0)| \leq p$. This remains true when $p = 5$, except that the multiplicity of the unique $\mu \in \mathrm{IBr}(S)$ of degree 3 is 3.*

(c) *Suppose that $1 < W$ is a subgroup of S of order not divisible by p. Suppose that ψ is a character of W such that $\psi(1) = p$ and $\psi(x) = \pm 1$ for $1 \neq x \in W$. Let $\theta, \delta \in \mathrm{Irr}(W)$. Then*
$$[\psi, \theta\delta] \leq \frac{p}{2}$$
unless $W = Q_8$ and $\theta = \delta$ has degree 2, or $|W| = 2$. In these cases,
$$[\psi, \theta\delta] \leq \frac{p+1}{2}.$$

Proof. Part (a) follows immediately from the well-known character table of $\mathrm{SL}_2(p)$.

(b) From the ordinary character table it can be worked out easily that $\gamma\Psi$ is multiplicity free (and does not involve the trivial character nor the characters of degree $\frac{1}{2}(p-1)$). The Brauer trees of the p-blocks of S of positive defect have an exceptional node of multiplicity 2, so any Brauer character occurs in at most three ordinary characters. This shows the first claim for $p \geq 7$. For $p = 5$, direct calculation suffices. The second assertion is immediate, as $l(S) = p$.

(c) If W has order 2, our assertion easily follows because $\psi(1) = p$, and $\psi(w) = \pm 1$ if $1 \neq w \in W$. Suppose that W is a p'-subgroup of S with order $|W| > 2$. We have checked (c) with GAP for primes $3 \leq p \leq 23$. Hence, we may assume that $p > 23$, if necessary.

Now W is one of the groups in Lemma 2.1(c). Let $\theta, \delta \in \mathrm{Irr}(W)$. Then
$$[\psi, \theta\delta] = \frac{1}{|W|} \sum_{w \in W} \psi(w)\theta(w^{-1})\delta(w^{-1}) \leq \frac{\theta(1)\delta(1)(|W| - 1 + p)}{|W|},$$
using that $|\theta(w)| \leq \theta(1)$ for $\theta \in \mathrm{Irr}(W)$.

Suppose first that θ and δ are linear. Then we see that $[\psi, \theta\delta] \leq p/2$ if $p > 3$. Thus we may assume that W is nonabelian.

Suppose now that W has a normal abelian subgroup of index 2. If $\theta, \delta \in \mathrm{Irr}(W)$, then $\theta(1)\delta(1) \le 4$. We see, assuming that $p \ge 23$, that $[\psi_W, \theta\delta] \le p/2$ if $|W| \ge 12$. There is only one nonabelian group of order less than 12 with a unique involution, which is Q_8. If $\theta(1)\delta(1) \le 2$, then $[\psi, \theta\delta] \le p/2$. So we assume that $\theta = \delta$ has degree 2. Let $Z = Z(W)$. Then using that θ is zero off Z, we have that

$$[\psi, \theta\delta] = \tfrac{1}{8}(4p \pm 4) = \frac{p \pm 1}{2}.$$

Suppose now that $W = \mathrm{SL}_2(3)$. The largest character degree of W is 3, and there is a unique character θ with that degree. Assuming that $p \ge 23$, we have that $[\psi, \theta\delta] \le p/2$, if $\theta(1)\delta(1) \le 6$. Assume now that $\theta = \delta$ has degree 3. This character has Z in its kernel, and otherwise takes value 0 except on the unique conjugacy class of elements of order 4. On these six elements, θ has value -1. Hence

$$[\psi, \theta\delta] \le \tfrac{1}{24}(9p + 9 + 6) \le p/2$$

(if $p \ge 5$, which we are assuming).

Suppose now that $W = \texttt{SmallGroup}(48, 28)$. This group has a unique character $\theta \in \mathrm{Irr}(W)$ of degree 4. By using the values of this character, we have that

$$[\psi, \theta^2] \le \tfrac{1}{48}(16p + 32) \le p/2$$

for $p \ge 5$. If $\theta(1)\delta(1) = 12$, then $\theta\delta$ is zero except on $Z(W)$, and the inequality is clear. If $\theta(1)\delta(1) = 9$, we again use the character values to check the inequality. Finally, if $\theta(1)\delta(1) \le 6$, then we do not need to use the character values. The case $W = \mathrm{SL}_2(5)$ is done similarly. □

The character Ψ in Lemma 2.2 is relevant in the character theory of fully ramified sections, as we shall see.

Lemma 2.3. *Suppose that $L \lhd G$ is an extra-special group of order p^3 and exponent p, where $p \ge 5$ is odd. Let $Z = Z(L) \subseteq Z(G)$, and assume that $G/L \cong \mathrm{SL}_2(p)$ and that $C_G(L) = Z$. Let $\alpha, \beta \in \mathrm{Irr}(G/L)$ of degrees $\frac{p-1}{2}$ and $\frac{p+1}{2}$ such that $|\alpha(x) + \beta(x)|^2 = |C_V(x)|$ for all $x \in S$, where V is the natural module for S. Let $1 \ne \lambda \in \mathrm{Irr}(Z)$, and write $\lambda^L = p\eta$, where $\eta \in \mathrm{Irr}(L)$. Then η has a unique extension $\hat{\eta} \in \mathrm{Irr}(G)$ and $\hat{\eta}^0 = \alpha^0 + \beta^0$.*

Proof. Let $\Psi = \alpha + \beta$. If K/L is the centre of G/L, and $Q \in \mathrm{Syl}_2(K)$, then N/Z is the unique (up to G-conjugacy) complement of L/Z in G/Z, where $N = N_G(Q)$. This follows by the Frattini argument and the fact that Q acts on L/Z with no nontrivial fixed points. Now, N is a central extension of $\mathrm{SL}_2(p)$ so $N = Z \times N'$, where $N' \cong \mathrm{SL}_2(p)$. Since the Schur multiplier of G/L is trivial (by Lemma 2.1(b)), it follows that η extends to G by [Isaacs 1976, Theorem 11.7]. Since G/L is perfect, there is a unique extension $\hat{\eta} \in \mathrm{Irr}(G)$ by Gallagher's theorem. Now, by [Isaacs 1973, Theorem 9.1], there is a character ψ of degree p of G/L such that

$$\hat{\eta}_N = \psi\nu$$

for some linear character ν of N over λ. Since N' is perfect, we have that $\nu = \lambda \times 1_{N'}$. By the proof of [Isaacs 1973, Theorem 4.8], we have that $\psi = \Psi$, and therefore $\hat{\eta}^0 = \Psi^0$. (Notice that there are two choices of Ψ, but Ψ^0 is uniquely determined by Lemma 2.2(a).) $\square$

Lemma 2.4. *Suppose that $H \le G$ are finite groups. Let $\chi \in \mathrm{Irr}(G)$ and $\theta \in \mathrm{Irr}(H)$. Then we have $[\chi_H, \theta]^2 \le |G : H|$.*

Proof. Write $\chi_H = e\theta + \Delta$, where Δ is a character of H or zero, and $[\Delta, \theta] = 0$. Then $\chi(1) \ge e\theta(1)$. By Frobenius reciprocity, we have that $\theta^G = e\chi + \Xi$, where Ξ is a character of G or zero. Hence $|G : H|\theta(1) \ge e\chi(1) \ge e^2\theta(1)$, and the proof is complete. $\square$

Theorem 2.5. *Suppose that G is a finite group and let $\chi \in \mathrm{Irr}(G)$ be such that $\chi(1)_p = |G|_p/p^2$. Let $N = \boldsymbol{O}_{p'}(G)$ and $L = \boldsymbol{O}_p(G)$. Suppose that $N \subseteq Z(G)$, and that $C_G(L) \subseteq LN$. Then $|\mathrm{IBr}(\chi^0)| \le p^2 - 1$ and $d_{\chi\varphi} \le p$ for all $\varphi \in \mathrm{IBr}(G)$.*

Proof. If $\chi^0 \in \mathrm{IBr}(G)$, then $d_{\chi\varphi} \le 1$ and $|\mathrm{IBr}(\chi^0)| = 1$. We may clearly assume that $\chi(1) > p$. Let $\theta \in \mathrm{Irr}(N)$ be the irreducible constituent of the restriction χ_N.

By hypothesis, we have that $C_G(L) = Z(L) \times N$. In particular, $L > 1$ (because otherwise G is a p'-group and $\chi^0 \in \mathrm{IBr}(G)$). Also, we have that $G/C_G(L) \cong U \subseteq \mathrm{Aut}(L)$. Since N is central, then $\boldsymbol{O}_p(G/N) = LN/N$, and thus $\boldsymbol{O}_p(U) \cong L/Z(L)$.

Let $\eta \in \mathrm{Irr}(L)$ be under χ. Then $\chi(1)_p/\eta(1)$ divides $|G|_p/|L|$ by [Isaacs 1976, Corollary 11.29]. Using the hypothesis, we have that $|L|/\eta(1) \le p^2$. Since $\eta(1) < |L|^{1/2}$ because L is a nontrivial p-group, we deduce that $|L| \le p^3$. Also, $\eta(1) \le p$.

If G/LN is a p-group, then $G = N \times L$. Then $\chi = \theta \times \eta$. Thus $\chi^0 = \eta(1)\theta$. Hence $d_{\chi\varphi} = \eta(1) \le p$, and $|\mathrm{IBr}(\chi^0)| = 1$, so the theorem is true in this case too.

Suppose that θ extends to some $\gamma \in \mathrm{Irr}(G)$. Note that $\gamma^0 \in \mathrm{IBr}(G)$ also extends θ. By a result of Gallagher [Isaacs 1976, Corollary 6.17], we know that $\chi = \beta\gamma$ for some $\beta \in \mathrm{Irr}(G/N)$. If $\beta^0 = d_1\tau_1 + \cdots + d_s\tau_s$, where $\tau_i \in \mathrm{IBr}(G/N)$ are distinct, then we have that

$$\chi^0 = d_1\tau_1\gamma^0 + \cdots + d_s\tau_s\gamma^0$$

and that the $\tau_i\gamma^0$ are also distinct and irreducible (using that γ^0 is linear). In particular, we see that if θ extends to G, then the theorem holds for G if it holds for G/N.

(a) Assume first that $\eta(1) = 1$. We have then that $|L| \le p^2$, by the third paragraph of this proof. In particular, L is abelian. In this case, $G/LN \cong U \subseteq \mathrm{Aut}(L)$ and $\boldsymbol{O}_p(U) = 1$.

(a1) If L is a Sylow p-subgroup of G, then by [Navarro 1998, Theorem 10.20], we have that $\mathrm{Irr}(G|\theta)$ is a block of G, which has defect group L. By the $k(GV)$-theorem [Gluck et al. 2004], we have that $|\mathrm{Irr}(G|\theta)| \le p^2$. Hence $|\mathrm{IBr}(G|\theta)| \le p^2 - 1$ by [Navarro 1998, Theorem 3.18]. Thus $|\mathrm{IBr}(\chi^0)| \le p^2 - 1$. Also, notice that the decomposition numbers are

$$d = [\chi_H, \mu]$$

for $\mu \in \mathrm{Irr}(H)$, where H is a p-complement of G. By Lemma 2.4, $d^2 \le |G:H| = p^2$, and the theorem is true in this case.

(a2) We assume that L is not a Sylow p-subgroup of G. Suppose first that L is cyclic. Notice that L cannot have order p, since p^2 divides the order of G (and $|\mathrm{Aut}(C_p)|$ is not divisible by p). Suppose that $L = C_{p^2}$. Then G/LN is cyclic. Since $\boldsymbol{O}_p(G/LN) = 1$, we conclude that G/LN is cyclic of order dividing $p-1$. Thus L is a normal Sylow p-subgroup of G, a contradiction.

Assume now that $L = C_p \times C_p$. In this case $G/LN \cong U \subseteq \mathrm{GL}_2(p)$. Thus $|U|_p = p$. Recall that $\boldsymbol{O}_p(U) = 1$. By Lemma 2.1(a), we conclude that $\mathrm{SL}_2(p) \subseteq U \subseteq \mathrm{GL}_2(p)$. Therefore U has trivial Schur multiplier by Lemma 2.1(b). Now, consider $\hat{\theta} = 1_L \times \theta$. By [Isaacs 1976, Theorem 11.7], we have that $\hat{\theta}$ extends to G. In particular, so does θ. Again, by the fifth paragraph of this proof, we may assume in this case that $N = 1$. If $p = 2$, then $G = \mathfrak{S}_4$ and $\chi(1) = 2$, so the theorem is true in this case. If p is odd, let $Z/L \subseteq Z(G/L)$ of order 2, and let $Q \in \mathrm{Syl}_2(Z)$. Since Q acts on L as the minus identity matrix, we have that $C_L(Q) = 1$. Therefore G is the semidirect product of L with $N_G(Q)$. In this case, by Lemma 2.1(d), we have that $\chi(1) = p$, and we are also done in this case.

(b) Assume now that $\eta(1) = p$. Hence L is extraspecial of order p^3 and exponent p or p^2. Write $Z = Z(L)$. We have that $G/ZN \cong U \subseteq \mathrm{Aut}(L)$ and $\boldsymbol{O}_p(U) = C_p \times C_p$. Also, write $\eta_Z = \eta(1)\lambda$, where $1 \ne \lambda \in \mathrm{Irr}(Z)$.

(b1) Suppose first that $p = 2$. Then $L = D_8$ or Q_8. If $L = D_8$, then $\mathrm{Aut}(L)$ is a 2-group, and then $G = L \times N$, and the theorem is true in this case. If $L = Q_8$, then $\mathrm{Aut}(Q_8)$ is $\mathfrak{S}_4$. Then $G/(L \times N)$ is a subgroup of $\mathfrak{S}_3$. Then $\theta \times 1_L$ (and therefore θ) extends to G. By the fifth paragraph of this proof, we may therefore assume that $N = 1$. Thus $|G| \le 48$. In this case, $G = \mathrm{SL}_2(3)$, $\mathrm{GL}_2(3)$ or the *fake* $\mathrm{GL}_2(3)$ (`SmallGroup(48,28)`). If $G = \mathrm{SL}_2(3)$, then $\chi(1) = 2$, and we are done. In the remaining cases, $\chi(1) = 4$, $|\mathrm{IBr}(\chi^0)| = 2$ and $d_{\chi\varphi} = 1$ or 2. Hence, we may assume that p is odd.

(b2) Suppose first that L is extra-special of exponent p^2, p odd. By [Winter 1972], we have that $\mathrm{Aut}(L) = X \rtimes C_{p-1}$, were $|X| = p^3$ has as a normal subgroup $\mathrm{Inn}(L)$, and C_{p-1} acts Frobenius on Z. Thus $G/(L \times N)$ has cyclic Sylow subgroups. It follows that $1_L \times \theta$ (and therefore θ) extends to G. Recall that $\boldsymbol{O}_p(G/Z) = L/Z$.

Since G/L is a subgroup of $C_p \rtimes C_{p-1}$ and this group has a normal Sylow p-subgroup, it follows that G/L is cyclic of order dividing $p-1$. Now, G is the semidirect product of L with a cyclic group C of order h dividing $p-1$ that acts Frobenius on Z. Since C acts Frobenius on Z, it follows that the stabiliser $I_G(\lambda) = L$. Since $\lambda^L = p\eta$, it follows that $I_G(\eta) = L$. Hence $\chi = \eta^G$. Then $\chi_C = p\rho$, where ρ is the regular character of C. Thus $d_{\chi\varphi} = p$ for every $\varphi \in \mathrm{IBr}(G)$ and $|\mathrm{IBr}(\chi^0)| = h \leq p-1$. The theorem follows in this case too.

(b3) So finally assume that L is extra-special of exponent p, p odd. We have that $\mathrm{Aut}(L)/\mathrm{Inn}(L) = \mathrm{GL}_2(p)$ by Lemma 2.1(e). Thus G/LN is isomorphic to a subgroup W of $\mathrm{GL}_2(p)$ with $\boldsymbol{O}_p(W) = 1$. Write $C = C_G(Z)$. Notice that C/LN maps into $\mathrm{SL}_2(p)$ by [Winter 1972], and therefore this group has at most one involution. Also, G/C is a cyclic p'-group (because $\mathrm{GL}_2(p)/\mathrm{SL}_2(p)$ is) that acts Frobenius on Z. Thus, our group W has a normal subgroup that has at most one involution and with cyclic p'-quotient. Also notice that C is the stabiliser of λ (and of η) in G. Furthermore, we claim that the stabiliser of any $\gamma \in \mathrm{Irr}(C/LN)$ in G has index at most 2 in G. This is because $\mathrm{GL}_2(p)$ has a centre of order $p-1$ that intersects with $\mathrm{SL}_2(p)$ in its unique subgroup of order 2. Write $\chi = \mu^G$, where $\mu \in \mathrm{Irr}(C)$.

(b.3.1) If L is a Sylow p-subgroup of G, we claim that $|\mathrm{IBr}(\chi^0)| \leq p^2 - 1$. Let H be a p-complement of G. Then H/N is isomorphic to a p'-subgroup of $\mathrm{GL}_2(p)$. By the $k(GV)$-theorem applied to $\Gamma = (C_p \times C_p) \rtimes H/N$, we have that $k(\Gamma) \leq p^2$. Since Γ has a unique p-block, it follows that $k(H/N) \leq p^2 - 1$. Now, $|\mathrm{IBr}(\chi^0)| \leq |\mathrm{Irr}(H|\theta)| \leq k(H/N)$, and the claim is proved. (The last inequality follows from Problem 11.10 of [Isaacs 1976].)

(b.3.2) If L is a Sylow p-subgroup of G, then $d_{\chi\varphi} \leq p$:

Let H be a p-complement of G. Write $C \cap H = Q$. Also, $U = HZ \cap C$ is the unique complement of L/Z in C/Z up to C-conjugacy. By Lemma 2.1(b) (and [Isaacs 1976, Theorem 11.7]), we have that $1_L \times \theta$ has a (linear) extension $\tilde{\theta} \in \mathrm{Irr}(C)$. Therefore $\mu = \beta\tilde{\theta}$, where $\beta \in \mathrm{Irr}(C/N)$. Notice that β lies over η. By [Isaacs 1973, Theorem 9.1], we have that

$$\beta_U = \psi\beta_0$$

for some $\beta_0 \in \mathrm{Irr}(U/N)$, where ψ is a character of C/N of degree p. Since $C_{L/Z}(x)$ is trivial for nontrivial p-regular xN, it follows that $\psi(xN) = \pm 1$ for p-regular $Nx \neq N$. (The values of ψ are given in page 619 of [Isaacs 1973], and it can be checked that ψ is the restriction of the character Ψ of $\mathrm{SL}_2(p)$ given in Lemma 2.2(a) under suitable identification.) Notice that $(\beta_0)_Q \in \mathrm{Irr}(Q/N)$ because Q is central in $U = ZQ$. Now, let $\nu \in \mathrm{Irr}(Q|\theta)$. By Gallagher, we have that $\nu = \tilde{\theta}_Q\tau$ for some

$\tau \in \mathrm{Irr}(Q/N)$. Then

$$[\mu_Q, \nu] = [\beta_Q \tilde{\theta}_Q, \tilde{\theta}_Q \tau] = [\psi_Q (\beta_0)_Q, \tau]\,.$$

Now, by Lemma 2.2(c) applied in Q/N this number is less than $p/2$, except if $Q/N = Q_8$ and τ is the unique character of degree 2. In this case, this number is less than $\frac{p+1}{2}$, and τ extends to H because H/Q is cyclic. Hence, if $\rho \in \mathrm{Irr}(H|\theta)$, then

$$[\chi_H, \rho] = [\mu_Q, \rho_Q]\,.$$

If ρ_Q is irreducible, then we are done. Let $\tau \in \mathrm{Irr}(Q|\theta)$ be under ρ. We know that the stabiliser I of τ in H has index at most 2. If $I = H$, then ρ_Q is irreducible (because H/Q is cyclic). We conclude that $|H : I| = 2$. Using that I/Q is cyclic, we have that $\rho_Q = \tau + \tau^x$, where $x \in H \setminus I$. In this case,

$$[\chi_H, \rho] = [\mu_Q, \tau + \tau^x] \leq p/2 + p/2 = p\,.$$

(b.3.3) We may assume that $p > 3$: Else we have that G/LN is a subgroup of $\mathrm{GL}_2(3)$. All subgroups of $\mathrm{GL}_2(3)$ have trivial Schur multiplier except for $C_2 \times C_2$, D_8 and D_{12}. By the requirements in (b3), only $C_2 \times C_2$ can occur. But then, L is a normal Sylow p-subgroup, and again the theorem holds in this case.

(b.3.4) Suppose finally that p divides $|W|$, and that $p > 3$. Then $\mathrm{SL}_2(p) \subseteq W \subseteq \mathrm{GL}_2(p)$ by Lemma 2.1(a). Now, by considering the character $1_L \times \theta$ and using Lemma 2.1(b), we may again assume that θ extends to G, and therefore that $N = 1$ in this case. Now, $C = C_G(Z)$ is such that $\mathrm{SL}_2(p) \cong C/L \lhd G/L$. By Lemma 2.3, we have that η has a unique extension $\hat{\eta} \in \mathrm{Irr}(C)$. Hence, μ, the Clifford correspondent of χ over η is such that $\mu = \gamma\hat{\eta}$ for a unique $\gamma \in \mathrm{Irr}(C/L)$ of degree p. Also, by Lemma 2.3, we know that $\hat{\eta}^0 = \Psi^0$. Therefore, by Lemma 2.2(b), we have that

$$\mu^0 = d_1\varphi_1 + \cdots + d_k\varphi_k$$

for some distinct $\varphi_i \in \mathrm{IBr}(C/L)$, with $k \leq p$, and $d_i < p/2$, except in the case where $p = 5$. In this latter case, we still have that $k \leq p$ and that $d_i \leq p/2$, except for the unique irreducible p-Brauer character of degree 3, call it φ_1, which is such that $d_1 = 3$. Now, since G/C is cyclic of p'-order and $|Z(\mathrm{GL}_2(p))| = p - 1$, it follows that the stabiliser T_i of φ_i in G has index at most 2. Also φ_i extends to T_i [Navarro 1998, Corollary 8.12] and the irreducible constituents of $(\varphi_i)^{T_i}$ all appear with multiplicity 1, by [Navarro 1998, Theorem 8.7] and Gallagher's theorem for Brauer characters [Navarro 1998, Theorem 8.20]. Also, they all induce irreducibly to G by the Clifford correspondence for Brauer characters. Hence, the number of Brauer irreducible constituents of $\chi^0 = (\mu^0)^G$ is less than or equal to $k \cdot (p-1) \leq p^2 - p$, and the decomposition numbers are at most $2d_i \leq p$, except if $p = 5$ and $i = 1$. In this case, φ_1 is G-invariant, because it is the unique irreducible p-Brauer character of $\mathrm{SL}_2(5)$ of degree 3. Hence, for $j > 1$ the Brauer character φ_j^G does not contain any irreducible constituent of φ_1^G. So the irreducible constituents of φ_1^G appear in χ^0 with multiplicity $3 < 5 = p$. □

Now, we can finally prove the p-solvable case of Conjecture A.

Corollary 2.6. *Suppose that G is a finite p-solvable group and let $\chi \in \mathrm{Irr}(G)$ be such that $\chi(1)_p = |G|_p/p^2$. Then $|\mathrm{IBr}(\chi^0)| \le p^2 - 1$ and $d_{\chi\varphi} \le p$ for all $\varphi \in \mathrm{IBr}(G)$.*

Proof. Write $\boldsymbol{O}_{p'}(G) = N$. We argue by induction on $|G : N|$. Let $\theta \in \mathrm{Irr}(N)$ be under χ.

Let T be the inertia group of θ in G and let $\psi \in \mathrm{Irr}(T|\theta)$ be the Clifford correspondent of χ over θ. Since $\psi^G = \chi$, then we know that $d_\psi = 2$. Now we apply the Fong–Reynolds theorem [Navarro 1998, Theorem 9.14] to conclude that we may assume that θ is G-invariant.

By using ordinary/modular character triples (see Problem 8.13 of [Navarro 1998]), we may replace (G, N, θ) by some other triple (Γ, M, λ), where

$$M = \boldsymbol{O}_{p'}(\Gamma) \subseteq Z(\Gamma).$$

Hence, by working now in Γ, it is no loss to assume that $N \subseteq Z(G)$. Now, if $L = \boldsymbol{O}_p(G)$, then we have that $C_G(L) \subseteq LN$, and we may apply Theorem 2.5 to conclude. □

3. Proof of Theorem C

A well-known theorem by Taussky asserts that a nonabelian 2-group P has maximal (nilpotency) class if and only if $|P/P'| = 4$ (see [Huppert 1967, Satz III.11.9]). In this case P is a dihedral group, a semidihedral group or a quaternion group. Of course, $|P/P'|$ is the number of linear characters of P. Our next result indicates that Taussky's theorem holds for blocks, assuming the Alperin–McKay conjecture. For this we need:

Lemma 3.1. *Let B be a 2-block of G with defect group $D \trianglelefteq G$. Then $k_0(B) = 4$ if and only if D is dihedral (including Klein four), semidihedral, quaternion or cyclic of order 4.*

Proof. If D is one of the listed groups, then $k_0(B) = 4$ by work of Brauer and Olsson (see [Sambale 2014, Theorem 8.1]). Now suppose conversely that $k_0(B) = 4$. By [Reynolds 1963], we may also assume that D is a Sylow 2-subgroup of G. By [Külshammer 1987, Theorem 6], B dominates a block $\overline{B}$ of G/D' with defect group $\overline{D} := D/D'$ and $k(\overline{B}) = k_0(B) = 4$. Using Taussky's theorem, it suffices to show $|\overline{D}| = 4$.

By way of contradiction, suppose that $2^d := |\overline{D}| > 4$. Let $(1, \overline{B}) = (x_1, b_1), \dots, (x_r, b_r)$ be a set of representatives for the conjugacy classes of $\overline{B}$-subsections. Then

$$\sum_{i=1}^{r} l(b_i) = k(\overline{B}) = 4.$$

By [Külshammer 1984, Theorem A], we have $l(\overline{B}) \geq 2$ and $r \leq 3$. Let I be the inertial quotient of $\overline{B}$. Then I has odd order and so is solvable by Feit–Thompson. The case $r = 3$ is impossible, since 2^d is the sum of I-orbit lengths. Hence, $r = 2$ and $\overline{D}$ is elementary abelian.

Since $\overline{D} \rtimes I$ is a solvable 2-transitive group, [Huppert 1957] implies that I lies in the semilinear group $\Gamma L_1(2^d) \cong C_{2^d-1} \rtimes C_d$. Let $N \trianglelefteq I$ with $N \leq C_{2^d-1}$ and $I/N \leq C_d$. Then N acts semiregularly on $\overline{D} \setminus \{1\}$. Hence, $C_I(x_2)$ is cyclic (as a subgroup of I/N). On the other hand, $C_I(x_2)$ is the inertial quotient of b_2. It follows that $l(b_2) = |C_I(x)|$ is odd and therefore $l(b_2) = 1$. Consequently, I acts regularly on $\overline{D} \setminus \{1\}$. This implies that all Sylow subgroups of I are cyclic. Hence, by a theorem of Külshammer (see [Sambale 2014, Theorem 1.19]), $\overline{B}$ is Morita equivalent to the group algebra of $\overline{D} \rtimes I$. We obtain the contradiction $l(\overline{B}) = k(I) > 3$. □

Our next result includes the first part of Theorem C.

Theorem 3.2. *Let B be a 2-block of G satisfying the Alperin–McKay conjecture. If there exists a character $\chi \in \mathrm{Irr}(B)$ such that $\chi(1)_2 = |G|_2/4$, then $|\mathrm{IBr}(\chi^0)| \leq l(B) \leq 3$ and $d_{\chi\varphi} \leq 2$ for all $\varphi \in \mathrm{IBr}(B)$.*

Proof. By a result of Landrock (see [Sambale 2014, Proposition 1.31]), $k_0(B) = 4$. Hence, Lemma 3.1 applies. If B has defect 2, then B is Morita equivalent to the principal block of a defect group D of B, of $\mathfrak{A}_4$ or $\mathfrak{A}_5$. The claim follows easily in this case. Thus, we may assume that B has defect at least 3. Then by work of Brauer and Olsson (see [Sambale 2014, Theorem 8.1]), $l(B) \leq 3$. The claim about the decomposition numbers follows from the tables at the end of [Erdmann 1990]. □

Remark 3.3. For every defect $d \geq 2$ there are 2-blocks with defect d containing an irreducible character χ such that $\chi(1)_2 = |G|_2/4$. This is clear for $d = 2$ and for $d \geq 3$ one can take the principal block of $\mathrm{SL}_2(q)$ where q is a suitable odd prime power. These blocks have quaternion defect groups. Similarly, the principal 2-block of $\mathrm{GL}_2(q)$ where $q \equiv 3 \pmod 4$ gives an example with semidihedral defect group. On the other hand, Brauer showed that there are no examples with dihedral defect group of order at least 16 (see [Sambale 2014, Theorem 8.1]).

In order to say something about odd primes, we need to invoke a stronger conjecture known as Robinson's *ordinary weight conjecture* (see [Sambale 2014, Conjecture 2.7]). Robinson gave the following consequence of his conjecture which is relevant to our work.

Lemma 3.4 [Robinson 2008, Lemma 4.7]. *Let B be a p-block of G with defect group D satisfying the ordinary weight conjecture. Assume that there exists $\chi \in \mathrm{Irr}(B)$ such that $\chi(1)_p = |G|_p/p^2$. Then $|D| = p^2$ or D has maximal class. Let $\mathcal{F}$ be the fusion system of B. If $|D| \geq p^4$, then D contains an $\mathcal{F}$-radical, $\mathcal{F}$-centric subgroup Q of order p^3 such that $\mathrm{SL}_2(p) \leq \mathrm{Out}_{\mathcal{F}}(Q) \leq \mathrm{GL}_2(p)$. In particular, $Qd(p)$ is involved in G. If $p = 2$, then $Q \cong Q_8$ and if $p > 2$, then Q is the extraspecial group p_+^{1+2} with exponent p.*

Conversely, if B is any block (satisfying the ordinary weight conjecture) with an $\mathcal{F}$-radical, $\mathcal{F}$-centric subgroup Q as above, then $\mathrm{Irr}(B)$ contains a character χ with $\chi(1)_p = |G|_p/p^2$.

Observe that a p-group P of order $|P| \geq p^3$ has maximal class if and only if there exists $x \in P$ with $|C_P(x)| = p^2$ (see [Huppert 1967, Satz III.14.23]). We will verify that latter condition in a special case in Proposition 5.7.

Lemma 3.4 implies for instance that the group $p_+^{1+2} \rtimes \mathrm{SL}_2(p)$ contains irreducible characters χ with $\chi(1)_p = p^2$. Hence, for every prime p there are p-blocks of defect 4 with characters of defect 2. As another consequence we conditionally extend Landrock's result mentioned in the proof of Theorem 3.2 to odd primes.

Proposition 3.5. *Let B be a p-block of G satisfying the ordinary weight conjecture. If there exists $\chi \in \mathrm{Irr}(B)$ such that $\chi(1)_p = |G|_p/p^2$, then $k_0(B) \leq p^2$.*

Proof. By Lemma 3.4, a defect group D of B has maximal class or $|D| = p^2$. In any case $|D/D'| = p^2$. The ordinary weight conjecture implies the Alperin–McKay conjecture (blockwise) and by [Külshammer 1987], the Alperin–McKay conjecture implies Olsson's conjecture, $k_0(B) \leq |D/D'| = p^2$. □

If we also assume the Eaton–Moretó conjecture [Eaton and Moretó 2014] for B, it follows that $k_1(B) > 0$ in the situation of Proposition 3.5. This is because a p-group P of maximal class has a (unique) normal subgroup N such that P/N is nonabelian of order p^3. Hence, P has an irreducible character of degree p.

Proposition 3.6. *Let B be a p-block of a p-solvable group G with $\chi \in \mathrm{Irr}(B)$ such that $\chi(1)_p = |G|_p/p^2$. Then we are in one of the following cases (all three of which occur):*

(1) *B has defect* 2 *or* 3.

(2) *$p = 2 = l(B)$ and B has defect group Q_{16} or SD_{16}. Both cases occur.*

(3) *$p = 3$ and B has defect group* `SmallGroup`$(3^4, a)$ *with $a \in \{7, 8, 9\}$.*

Proof. By [Haggarty 1977], B has defect at most 4. Moreover, if B has defect 4, then $p \leq 3$. Let D be a defect group of B. Since the ordinary weight conjecture holds for p-solvable groups, Lemma 3.4 implies that D has maximal class. If $p=2$, then the fusion system $\mathcal{F}$ of B contains an $\mathcal{F}$-radical, $\mathcal{F}$-centric subgroup isomorphic to Q_8. Hence, $D \in \{Q_{16}, SD_{16}\}$. On the other hand, every fusion system of a block of a p-solvable group is constrained. This implies that there is only one $\mathcal{F}$-radical, $\mathcal{F}$-centric subgroup in D. It follows from [Sambale 2014, Theorem 8.1] that $l(B)=2$. Examples are given by the two double covers of $\mathfrak{S}_4$.

Now suppose that $p=3$. According to [GAP 2016], there are four possibilities for D: $\texttt{SmallGroup}(3^4, a)$ with $a \in \{7, 8, 9, 10\}$. In case $a=10$, D has no extraspecial subgroup of order 27 and exponent 3. Hence, Lemma 3.4 excludes this case. Conversely, examples for the remaining three cases are given by the (solvable) groups $\texttt{SmallGroup}(3^4 \cdot 8, b)$ with $b \in \{531, 532, 533\}$. □

In the following we (conditionally) classify the possible defect groups in case $p=3$. This relies ultimately on Blackburn's classification of the 3-groups of maximal class. Unfortunately, there is no such classification for $p>3$.

Proposition 3.7. *Let B be a 3-block of G satisfying the ordinary weight conjecture. Suppose that there exists $\chi \in \mathrm{Irr}(B)$ such that $\chi(1)_3 = |G|_3/9$. If B has defect $d \geq 4$, then are at most three possible defect groups of order 3^d up to isomorphism. If d is even, they all occur, and if d is odd, only one of them occurs. In particular, we have examples for every defect $d \geq 2$.*

Proof. In case $d=4$ we can argue as in Proposition 3.6. Thus, suppose that $d \geq 5$. By Lemma 3.4, a defect group D of B has maximal class and contains a radical, centric subgroup $Q \cong 3^{1+2}_{+}$. In particular, B is not a controlled block. Since $d \geq 5$, it is known that D has 3-rank 2 (see [Díaz et al. 2007, Theorem A.1]). Hence, the possible fusion systems $\mathcal{F}$ of B are described in [Díaz et al. 2007, Theorem 5.10]. It turns out that D is one of the groups $B(3, d; 0, \gamma, 0)$ with $\gamma \in \{0, 1, 2\}$. If d is odd, then $\gamma = 0$. In all these cases examples are given such that $\mathrm{Out}_{\mathcal{F}}(Q) \cong \mathrm{SL}_2(3)$. We can pick for instance the principal 3-blocks of $3.\mathrm{PGL}_3(q)$ and ${}^2F_4(q)$ for a suitable prime power q. An inspection of the character tables in [Steinberg 1951; Malle 1990] shows the existence of χ. □

Now we are in a position to cover the second part of Theorem C (recall that the ordinary weight conjecture for all blocks of all finite groups implies Alperin's weight conjecture).

Corollary 3.8. *Let B be a 3-block of G with defect d satisfying the ordinary weight conjecture and Alperin's weight conjecture. If there exists $\chi \in \mathrm{Irr}(B)$ such that $\chi(1)_3 = |G|_3/9$, then*

$$l(B) \leq \begin{cases} 8 & \text{if } 4 \neq d \equiv 0 \pmod 2, \\ 9 & \text{if } d \equiv 1 \pmod 2, \\ 10 & \text{if } d = 4. \end{cases}$$

Proof. Let D be a defect group of B. We may assume that $|D| \geq 27$. Then D has maximal class. In case $|D| = 27$ and $\exp(D) = 9$, Watanabe has shown that $l(B) \leq 2$ without invoking any conjecture (see [Sambale 2014, Theorems 1.33 and 8.8]). Now assume that $D \cong 3^{1+2}_{+}$. Then the possible fusion systems $\mathcal{F}$ of B are given in [Ruiz and Viruel 2004]. To compute $l(B)$ we use Alperin's weight conjecture in the form [Sambale 2014, Conjecture 2.6]. Let $Q \leq D$ be $\mathcal{F}$-radical and $\mathcal{F}$-centric. For $Q = D$ we have $\mathrm{Out}_{\mathcal{F}}(D) \leq SD_{16}$. Hence, regardless of the Külshammer–Puig class, D contributes at most 7 to $l(B)$ (for a definition of the Külshammer–Puig class see [Sambale 2014, Theorem 7.3]). For $Q < D$ we have $\mathrm{Out}_{\mathcal{F}}(Q) \in \{\mathrm{SL}_2(3), \mathrm{GL}_2(3)\}$. The groups $\mathrm{SL}_2(3)$ and $\mathrm{GL}_2(3)$ have trivial Schur multiplier and exactly one respectively two irreducible characters of 3-defect 0. Hence, each $\mathcal{F}$-conjugacy class of such a subgroup Q contributes at most 2 to $l(B)$. There are at most two such subgroups up to conjugation. Now an examination of the tables in [Ruiz and Viruel 2004] yields the claim for $d = 3$. Note that $l(B) = 9$ only occurs for the exceptional fusion systems on ${}^2F_4(2)'$ and ${}^2F_4(2)$.

Now let $d \geq 5$. As in Proposition 3.7 there are at most three possibilities for D and the possible fusion systems $\mathcal{F}$ are listed in [Díaz et al. 2007, Theorem 5.10]. We are only interested in those cases where there exists an $\mathcal{F}$-radical, $\mathcal{F}$-centric, extraspecial subgroup of order 27. We have $\mathrm{Out}_{\mathcal{F}}(D) \leq C_2 \times C_2$. Hence, D contributes at most 4 to $l(B)$. Now assume that $Q < D$ is $\mathcal{F}$-radical and $\mathcal{F}$-centric (i.e., $\mathcal{F}$-Alperin in the notation of [Díaz et al. 2007]). Then as above, $\mathrm{Out}_{\mathcal{F}}(Q) \in \{\mathrm{SL}_2(3), \mathrm{GL}_2(3)\}$. There are at most three such subgroups up to conjugation. The claim $l(B) \leq 9$ follows easily. In case $l(B) = 9$, $\mathcal{F}$ is the fusion system of ${}^2F_4(q^2)$ for some 2-power q^2 or $\mathcal{F}$ is exotic. In both cases d is odd. The principal block of ${}^2F_4(q^2)$ shows that $l(B) = 9$ really occurs (see [Malle 1990]).

It remains to deal with the case $d = 4$. By the results of [Díaz et al. 2007], we may assume that $D \cong C_3 \wr C_3$. The fusion systems on this group seem to be unknown. Therefore, we have to analyse the structure of D by hand. Up to conjugation, D has the following candidates of $\mathcal{F}$-radical, $\mathcal{F}$-centric subgroups:

$$Q_1 \cong C_3 \times C_3, \quad Q_2 \cong 3^{1+2}_{+}, \quad Q_3 \cong C_3 \times C_3 \times C_3 \quad \text{and} \quad Q_4 = D.$$

We may assume that $Q_1 \leq Q_2$. As before, Q_2 must be $\mathcal{F}$-radical and $\mathrm{Out}_{\mathcal{F}}(Q_2) \in \{\mathrm{SL}_2(3), \mathrm{GL}_2(3)\}$. Hence, Q_1 is conjugate to D' under $\mathrm{Aut}_{\mathcal{F}}(Q_2)$. Since $C_D(D') = Q_3$, we conclude that Q_1 is not $\mathcal{F}$-centric. Now let $\alpha \in \mathrm{Aut}_{\mathcal{F}}(Q_2)$ the automorphism inverting the elements of $Q_2/Z(Q_2)$. Then α acts trivially on $Z(Q_2)$. By the saturation property of fusion systems, α extends to D. Since Q_3 is the only abelian maximal subgroup of D, the extension of α restricts to Q_3. Since $Z(Q_2) \leq Q_3$, it follows from a GAP computation that $\mathrm{Out}_{\mathcal{F}}(Q_3) \in \{\mathfrak{S}_4, \mathfrak{S}_4 \times C_2\}$. Using similar arguments we end up with two configurations:

(i) $\mathrm{Out}_{\mathcal{F}}(Q_2) \cong \mathrm{SL}_2(3)$, $\mathrm{Out}_{\mathcal{F}}(D) \cong C_2$ and $\mathrm{Out}_{\mathcal{F}}(Q_3) \cong \mathfrak{S}_4$.

(ii) $\mathrm{Out}_{\mathcal{F}}(Q_2) \cong \mathrm{GL}_2(3),\ \mathrm{Out}_{\mathcal{F}}(D) \cong C_2 \times C_2$ and $\mathrm{Out}_{\mathcal{F}}(Q_3) \cong \mathfrak{S}_4 \times C_2$.

In the first case we have $l(B) \leq 5$ (occurs for the principal block of $\mathrm{L}_4(4)$) and in the second case $l(B) \leq 10$ (occurs for the principal block of $\mathrm{L}_6(2)$). □

Concerning the primes $p > 3$ we note that for example the principal 5-block of $\mathrm{U}_6(4)$ has defect 6 and an irreducible character of defect 2. However, we do not know if for any prime $p \geq 5$ and any $d \geq 5$ there are p-blocks of defect d with irreducible characters of defect 2.

4. Symmetric, alternating and sporadic groups

In this section we discuss the validity of our conjectures for alternating, symmetric and sporadic groups.

4A. The irreducible characters of the symmetric group $\mathfrak{S}_n$ are parametrised by partitions λ of n, and we shall write $\chi_\lambda \in \mathrm{Irr}(\mathfrak{S}_n)$ for the character labelled by λ. Its degree is given by the well-known hook formula. We first address Conjecture B.

Proposition 4.1. *Conjecture B holds for the alternating and symmetric groups at any prime.*

Proof. First consider $G = \mathfrak{S}_n$. Let p be a prime. For $\chi = \chi_\lambda \in \mathrm{Irr}(\mathfrak{S}_n)$ let $w \geq 0$ be its defect, i.e., such that $p^w \chi(1)_p = |\mathfrak{S}_n|_p$. It is immediate from the hook formula that this can only happen if there are at most w ways to move a bead upwards on its respective ruler in the p-abacus diagram of λ. But then clearly the p-core of λ can be reached by removing at most w p-hooks, so λ lies in a p-block of weight at most w. But it is well-known that any such block B has less than p^w modular irreducible characters (see, e.g., [Malle and Robinson 2017, Proposition 5.2]).

We now consider the alternating groups where we first assume that p is odd. Clearly $\chi \in \mathrm{Irr}(\mathfrak{A}_n)$ satisfies our hypothesis if and only if it lies below a character χ_λ of $\mathfrak{S}_n$ which does. But then χ_λ lies in a p-block B of $\mathfrak{S}_n$ of weight at most w, as shown before. If B is not self-associate, that is, if the parametrising p-core is not self-dual, then B and its conjugate B' both lie over a block B_0 of $\mathfrak{A}_n$, namely the one containing χ, with the same invariants. So we are done by the case of $\mathfrak{S}_n$. If B is self-associate then an easy estimate shows that again $l(B) < p^w$ (see the proof of the cited proposition).

Finally consider $p = 2$ for alternating groups. Here the number of modular irreducibles in a 2-block B of weight w is $\pi(w)$ if w is odd, respectively $\pi(w) + \pi(w/2)$ if w is even, with $\pi(w)$ denoting the number of partitions of w. Now $\pi(w) \leq 2^{w-1}$, and moreover $\pi(w) + \pi(w/2) \leq 2^{w-1}$ for even $w \geq 4$, so for $w \geq 3$ we have $l(B) \leq 2^{w-1}$. On the other hand the difference $d(B) - \mathrm{ht}(\chi)$ is at most one smaller for $\chi \in \mathrm{Irr}(\mathfrak{A}_n)$ than for a character $\tilde{\chi}$ of $\mathfrak{S}_n$ lying above χ. Thus for all $w \geq 3$ we have $l(B) \leq 2^{w-1}$ is at most $2^{d(B)-\mathrm{ht}(\chi)}$ for all $\chi \in \mathrm{Irr}(B)$, as required. For $w = 2$ the defect groups of B are abelian and the claim is easily verified. □

The next statement follows essentially from a result of Scopes:

Theorem 4.2. *Conjecture A holds for the alternating and symmetric groups at any prime.*

Proof. We first consider symmetric groups. Let $\chi = \chi_\lambda \in \mathrm{Irr}(\mathfrak{S}_n)$ be such that $p^2\chi(1)_p = |\mathfrak{S}_n|_p$ for some prime p. By the hook formula this happens only if exactly two beads can be moved in exactly one way on their respective ruler in the p-abacus diagram of λ, or if one bead can be moved in two ways. In either case the p-core of λ can be reached by removing two p-hooks, so λ lies in a p-block of weight 2.

If B is a p-block of weight 2 of $\mathfrak{S}_n$, with p odd, then all decomposition numbers are either 0 or 1 by [Scopes 1995, Theorem I], and furthermore any row in the decomposition matrix has at most 5 nonzero entries. This proves our claim for $\mathfrak{S}_n$ and odd primes. For $p = 2$ the defect groups of B of weight 2 are dihedral of order 8, and our claim also follows.

We now consider the alternating groups. First assume that p is odd. Then $\chi \in \mathrm{Irr}(\mathfrak{A}_n)$ satisfies our hypothesis if and only if it lies below a character χ_λ of $\mathfrak{S}_n$ which does. But then χ_λ lies in a p-block of B of $\mathfrak{S}_n$ of weight 2, as shown before. If B is not self-associate then B and its conjugate B' both lie over a block B_0 of $\mathfrak{A}_n$ with the same invariants, in particular, with the same decomposition numbers. So we are done by the case of $\mathfrak{S}_n$. If B is self-associate then it is easy to count that B contains $(p-1)/2$ self-associate characters and $(p+1)^2/2$ that are not. That is, $(p+1)^2/2$ characters in B restrict irreducibly, while $(p-1)/2$ of them split. It is clear from the $\mathfrak{S}_n$-result that the decomposition numbers are at most two, and $l(B) \leq (p^2+6p-3)/4 < p^2$, so the conjecture holds.

Finally the case $p = 2$ for $\mathfrak{A}_n$ follows by Theorem 3.2, as the Alperin–McKay conjecture is known to hold for all blocks of $\mathfrak{A}_n$, see [Michler and Olsson 1990]. □

The case of faithful blocks for the double covering groups of $\mathfrak{A}_n$ and $\mathfrak{S}_n$ seems considerably harder to investigate, at least in as far as decomposition numbers are concerned, due to the missing analogue of the theorem of Scopes for this situation.

Proposition 4.3. *Conjecture B holds for the* 2*-fold covering groups of alternating and symmetric groups at any odd prime*, *and Conjecture A holds for these groups at* $p = 2$.

Proof. As the Alperin–McKay conjecture has been verified for all blocks of the covering groups of alternating and symmetric groups [Michler and Olsson 1990], our claim for the prime $p = 2$ follows from Theorem 3.2.

G	p	$d(B)$	max ht	$l(B)$	$\chi(1)$	Conjecture A(1)
Ly	2	8	5	9	4 997 664	ok
Co_1	3	9	6	29	469 945 476	
Co_1	5	4	2	29	210 974 400	ok
$2.Co_1$	5	4	2	29	1 021 620 600	
J_4	3	3	1	9	5 chars in 2 blks	

Table 1. Blocks in sporadic groups.

So now assume that p is an odd prime, and first consider $G = 2.\mathfrak{S}_n$, a 2-fold covering group of $\mathfrak{S}_n$, with $n \geq 5$. By the hook formula for spin characters [Olsson 1993, (7.2)], the p-defect of any spin character $\chi \in \mathrm{Irr}(G)$ is at least the weight of the corresponding p-block B. On the other hand, by [Malle and Robinson 2017, Proposition 5.2] the blocks of G satisfy the $l(B)$-conjecture, so Conjecture B holds. Now for any p-block of $2.\mathfrak{A}_n$ there exists a height and defect group preserving bijection to a p-block of a suitable $2.\mathfrak{S}_m$, so the claim for $2.\mathfrak{A}_n$ also follows. □

4B. We now turn to verifying our conjectures for the sporadic quasisimple groups.

Proposition 4.4. *Let B be a p-block of a covering group of a sporadic simple group or of ${}^2F_4(2)'$. Then*:

(a) *B satisfies Conjecture B, unless possibly when B is as in the first four lines of Table 1.*

(b) *B satisfies Conjecture A unless possibly when B is as in the last four lines of Table 1.*

Proof. For most blocks of sporadic groups, the inequality (1) can be checked using the known character tables and Brauer tables; the only remaining cases are listed in Table 1, where the first four lines contain cases in which Conjecture B might fail, while the last four lines are those cases where Conjecture A might fail.

In two of these remaining cases we can show that at least Conjecture A(1) holds. For Ly, the tensor product of the 2-defect 0 characters of degree 120064 with the irreducible character of degree 2480 is projective, has nontrivial restriction to the principal block, but does not contain the (unique) character χ of defect 2 of degree 4 997 664. Thus $|\mathrm{IBr}(\chi^0)| \leq 8$.

For Co_1 the tensor products of irreducible 5-defect zero characters with irreducible characters, restricted to the principal block, span a 7-dimensional space of projective characters not containing the unique defect 2 character χ of degree 210 974 400, so $|\mathrm{IBr}(\chi^0)| \leq 22$. □

Φ	A_n	B_n, C_n	D_n $(n \geq 4)$	G_2	F_4	E_6	E_7	E_8
$N(\Phi)$	$\binom{n+1}{2}$	n^2	n^2-n	6	24	36	63	120
$\max N(\Psi)$	$\binom{n}{2}$	n^2-n	n^2-3n+2	2	16	20	36	64
$N(\Phi)-N(\Psi)$	n	n	$2n-2$	4	8	16	27	56

Table 2. Maximal subsystems.

5. Groups of Lie type

In this section we consider our conjectures for quasisimple groups of Lie type G. We prove both Conjectures A and B when p is the defining characteristic of G. On the other hand, we only treat one series of examples in the case of nondefining characteristic.

5A. *Defining characteristic.* We need an auxiliary result about root systems:

Lemma 5.1. *Let Φ be an indecomposable root system and denote by $N(\Phi)$ its number of positive roots. If $\Psi \subset \Phi$ is any proper subsystem of Φ then $N(\Phi)-N(\Psi) \geq n$, where n is the rank of Φ.*

In fact, our result is more precise: we determine the minimum of $N(\Phi)-N(\Psi)$ for each type; see Table 2.

Proof. The values of $N(\Phi)$ for indecomposable root systems Φ are given as in Table 2 (see, e.g., [Malle and Testerman 2011, Table 24.1]).

The possible proper subsystems can be determined by the algorithm of Borel–de Siebenthal (see [Malle and Testerman 2011, §13.2]). For Φ of type A_n the largest proper subsystem Ψ has type A_{n-1}, and then $N(\Phi)-N(\Psi) = \binom{n+1}{2} - \binom{n}{2}$. For Φ of type B_n, we need to consider subsystems of types D_n and $B_{n-1}B_1$, of which the second always has the larger number of positive roots. Next, for type D_n with $n \geq 4$, the largest subsystems are those of types A_{n-1} and D_{n-1}, which lead to the entries in our table. Finally, it is straightforward to handle the possible subsystems for Φ of exceptional type. □

Corollary 5.2. *Let G be a finite quasisimple group of Lie type in characteristic p. Let $\chi \in \mathrm{Irr}(G)$ lie in the p-block B. Then $l(B) < |G_p|/\chi(1)_p$. In particular Conjecture B holds for G at the prime p.*

Proof. The faithful p-blocks of the finitely many exceptional covering groups can be seen to satisfy Conjecture B by inspection using the Atlas [Conway et al. 1985]. Note that the nonexceptional Schur multiplier of a simple group of Lie type has order prime to the characteristic (see, e.g., [Malle and Testerman 2011, Table 24.2]). Thus we may assume that G is the universal nonexceptional covering group of its simple quotient S. Hence, G can be obtained as the group of fixed points $\boldsymbol{G}^F$ of a simple simply connected linear algebraic group $\boldsymbol{G}$ over an algebraic closure of $\mathbb{F}_p$ under a Steinberg endomorphism $F : \boldsymbol{G} \to \boldsymbol{G}$.

Now the order formula [Malle and Testerman 2011, Corollary 24.6] shows that $|G|_p = q^N$, where q is the underlying power of p defining G, and $N = N(\Phi) = |\Phi^+|$ denotes the number of positive roots of the root system Φ of $\boldsymbol{G}$. According to Lusztig's Jordan decomposition of ordinary irreducible characters of G, any $\chi \in \mathrm{Irr}(G)$ lies in some Lusztig series $\mathcal{E}(G, s)$, for s a semisimple element in the dual group G^*, and

$$\chi(1) = |G^* : C_{G^*}(s)|_{p'}\ \psi(1)$$

for some unipotent character ψ of $C_{G^*}(s)$. In particular, $\chi(1)_p = \psi(1)_p$ is at most the p-part in $|C_{G^*}(s)|$, hence at most q^M for M the number of positive roots of the connected reductive group $C^\circ_{\boldsymbol{G}^*}(s)$. (Observe that $|C_{G^*}(s) : C^\circ_{G^*}(s)|$ is prime to p by [Malle and Testerman 2011, Proposition 14.20].) Now first assume that $s \neq 1$, so s is not central in G^* (which is of adjoint type by our assumption on $\boldsymbol{G}$). Then the character(s) in $\mathcal{E}(G, s)$ with maximal p-part correspond to the Steinberg character of $C^\circ_{\boldsymbol{G}^*}(s)$, of degree $q^{N(\Psi)}$, where Ψ is the root system of $C^\circ_{\boldsymbol{G}^*}(s)$, a proper subsystem of Φ. In particular, $|G|_p/\chi(1)_p \geq q^n$ by Lemma 5.1, with n denoting the Lie rank of $\boldsymbol{G}$.

We next deal with the Lusztig series of $s = 1$, that is, the unipotent characters of G. If $\chi \in \mathrm{Irr}(G)$ is unipotent then its degree is given by a polynomial in q, $\chi(1) = q^{a_\chi} f_\chi(q)$, where $f_\chi(X)$ has constant term ± 1, so that $\chi(1)_p = q^{a_\chi}$ (see [Carter 1985, 13.8 and 13.9]). We let A_χ denote the degree of the polynomial $X^{a_\chi} f_\chi(X)$. Let $D(\chi)$ denote the Alvis–Curtis dual of χ. Then $D(\chi)(1) = q^{N-A_\chi} f'_\chi(q)$, for some polynomial f'_χ of the same degree as f_χ, so the degree polynomial of $D(\chi)$ is of degree $N - a_\chi$. We are interested in unipotent characters χ with large a_χ, that is, those for which the degree polynomial of the Alvis–Curtis dual has small degree $N - a_\chi$. The Alvis–Curtis dual of the trivial character is the Steinberg character, whose degree is just the full p-power q^N of $|G|$. The smallest possible degrees of degree polynomials of nontrivial unipotent characters for simple groups of Lie type are easily read off from the explicit formulas in [Carter 1985, 13.8 and 13.9], they are given as follows:

Φ	A_n	B_n, C_n	D_n $(n \geq 4)$	G_2	F_4	E_6	E_7	E_8
$N-a_\chi$	n	$2n-1$	$2n-3$	5	11	11	17	29

It transpires that again $|G|_p/\chi(1)_p \geq q^n$ in all cases.

On the other hand, the p-modular irreducibles of G are parametrised by q-restricted weights of $\boldsymbol{G}$, so there are q^n of them, one of which is the Steinberg character, of defect zero. Thus $l(B) < q^n$ for all p-blocks B of G of positive defect which shows Conjecture B. $\square$

We now turn to Conjecture A. In view of Corollary 5.2 only its second assertion remains to be considered. The following result shows that the assumptions of Conjecture A are hardly ever satisfied:

Proposition 5.3. *Let G be a quasisimple group of Lie type in characteristic p. Assume that G has an irreducible character $\chi \in \mathrm{Irr}(G)$ such that $|G|_p = p^2\,\chi(1)_p$. Then G is a central quotient of* $\mathrm{SL}_2(p^2)$, $\mathrm{SL}_3(p)$, $\mathrm{SU}_3(p)$ *or* $\mathrm{Sp}_4(p)$.

Proof. Again the finitely many exceptional covering groups can be handled by inspection using the Atlas [Conway et al. 1985] and so as in the proof of Corollary 5.2 we may assume that G is the universal nonexceptional covering group of its simple quotient S and so can be obtained as the group of fixed points of a simply connected simple algebraic group under a Steinberg map.

Using Lusztig's Jordan decomposition we see that our question boils down to

(1) determining the irreducible root systems Φ possessing a proper root subsystem Ψ such that $N(\Psi) \geq N(\Phi) - 2$, and

(2) finding the unipotent characters χ of G such that $p^2\chi(1)_p \geq |G|_p$.

The first issue can easily be answered using Lemma 5.1. According to Table 2 only $\boldsymbol{G}$ of types A_1, A_2 or B_2 are candidates, the first with $q = p^2$, that is, $G = \mathrm{SL}_2(p^2)$, and the other two only for $q = p$. For type A_2 this leads to $\mathrm{SL}_3(p)$ and $\mathrm{SU}_3(p)$, in the case of type B_2 we have $|\Phi(B_2)| - |\Phi(B_1^2)| = 2$, which only leads to $G = \mathrm{Sp}_4(p)$. Indeed, for the twisted Suzuki groups ${}^2B_2(q^2)$, where q^2 is even, there is no centraliser of root system A_1^2 in the dual group.

Issue (2) has already been partly discussed in the proof of Corollary 5.2. Using the list of maximal q-powers occurring in unipotent character degrees given there it follows that examples can only possibly arise if the root system of $\boldsymbol{G}$ is of type A_1 or A_2. □

Theorem 5.4. *Let G be a quasisimple group of Lie type in characteristic p. Then Conjecture A holds for G and the prime p.*

Proof. By Proposition 5.3 we only need to consider the groups $\mathrm{SL}_2(p^2)$, $\mathrm{SL}_3(p)$, $\mathrm{SU}_3(p)$ and $\mathrm{Sp}_4(p)$. The ordinary character tables of all these groups are known and available for example in the Chevie system [Geck et al. 1996].

The p-modular irreducibles of $G = \mathrm{SL}_2(p^2)$ are indexed by p^2-restricted dominant weights, so there are exactly p^2 of them. One of them, the Steinberg representation, is of defect 0, so any p-block of G contains at most $p^2 - 1$ irreducible Brauer characters. All decomposition numbers are equal to 0 or 1 by Srinivasan [Srinivasan 1964], which deals with this case.

For $G = \mathrm{SL}_3(p)$ the only characters χ satisfying the assumptions of Conjecture A are the unipotent character χ_u of degree $p(p+1)$ and the regular characters χ_r of degree $p(p^2+p+1)$. The p-modular irreducibles of G are parametrised by

p-restricted weights, so since G is of rank 2 there are p^2 of them, one of which is the Steinberg character, of defect zero. Thus $l(B) < p^2$ for all blocks of positive defect which already shows (1). Now assume that $p \geq 11$. By [Humphreys 1981, Table 1 and §4] χ_u^0 has just two modular composition factors. Furthermore, projective indecomposables of G have degree at most $12p^3$ (see [Humphreys 1981]), so any decomposition number occurring for χ_r is at most $12p^2/(p^2+p+1)$, which is smaller or equal to p for $p \geq 11$. The decomposition numbers for $p \leq 7$ are contained in [GAP 2016].

The arguments for $G = \mathrm{SU}_3(p)$ are entirely similar, using [Humphreys 1990] for decomposition numbers when $p \geq 11$. The cases with $3 \leq p \leq 7$ are contained in [GAP 2016].

Finally consider $G = \mathrm{Sp}_4(p)$. Here the only relevant characters χ are those of degree $\frac{1}{2}p^2(p^2 \pm 1)$ parametrised by involutions in the dual group with centraliser of type A_1^2; they only exist when p is odd, which we now assume. Then G has two p-blocks of positive defect both containing $(p^2-1)/2$ modular irreducibles. This already proves (1). The second claim follows from the explicit lists of decomposition numbers provided in Appendix 3 of [Jantzen 1987]: indeed, all relevant decomposition numbers are bounded above by 2. □

5B. ***Nondefining characteristic.*** The current knowledge about decomposition numbers for blocks of groups of Lie type in nondefining characteristic does not seem sufficient to prove even Conjecture A, not even for unipotent blocks in general. We hence just make some preliminary observations.

Example 5.5. This example shows that characters in blocks of groups of Lie type in nondefining characteristic can have rather large heights. Let $G = \mathrm{GL}_n(q)$, and ℓ a prime dividing $q-1$. Then all unipotent characters lie in the principal ℓ-block B_0 of G, and so do all characters in Lusztig series $\mathcal{E}(G,s)$ for any ℓ-element $t \in G^* \cong G$. Now assume that $n = \ell^a$ for some $a \geq 1$. Then $|G|_\ell = \ell^{cn+(n-1)/(\ell-1)}$ where ℓ^c is the precise power of ℓ dividing $q-1$. Let $T \leq \mathrm{GL}_n(q)$ be a Coxeter torus, of order q^n-1, and $t \in T$ an element of maximal ℓ-power order. It can easily be seen that then $o(t) = (q-1)_\ell \ell^a = \ell^{c+a}$ and t is regular, so $\mathcal{E}(G,t)$ consists of a single character χ, say. Now $\chi(1) = |G : C_G(t)|$ and hence

$$\chi(1)_\ell = |G|_\ell/|T|_\ell = \ell^{cn+(n-1)/(l-1)-c-a},$$

so χ has height $c(n-1)+(n-1)/(l-1)-a$ and defect $c+a$. So the height can become arbitrarily large by varying q for fixed n and in particular it is not bounded in terms of the (relative) Weyl group. Moreover, the unipotent characters form a basic set for B_0. As they are in bijection with $\mathrm{Irr}(\mathfrak{S}_n)$, $l(B_0)$ is the number of partitions of n and so grows exponentially in n.

Choosing $a = c = 1$, that is, $n = \ell$ and $\ell \| (q-1)$ we obtain principal ℓ-blocks of defect $\ell + 1$ containing a character of defect 2. These examples also give rise to similar blocks for the quasisimple groups $\mathrm{SL}_n(q)$. We will deal with them in Proposition 5.9 below.

Now let G be the group of fixed points of a connected reductive linear algebraic group $\boldsymbol{G}$ under a Frobenius endomorphism F defining an $\mathbb{F}_q$-structure. Let ℓ be a prime different from the defining characteristic of G and set $e = e_\ell(q)$, the order of q modulo ℓ if $\ell > 2$, respectively the order of q modulo 4 if $\ell = 2$. We write $a_G(e)$ for the precise power of Φ_e dividing the order polynomial of the derived subgroup $[\boldsymbol{G}, \boldsymbol{G}]$, that is, since $\boldsymbol{G} = [\boldsymbol{G}, \boldsymbol{G}]Z(\boldsymbol{G})$, the precise power of Φ_e dividing the order polynomial of $\boldsymbol{G}/Z(\boldsymbol{G})$. If e is clear from the context, we will just denote it by a_G.

We start with an observation on defects of characters in unipotent e-Harish-Chandra series of G.

Lemma 5.6. *Let $\boldsymbol{G}$, F, q be as above. Let χ be a unipotent character of $G = \boldsymbol{G}^F$ lying in the e-Harish-Chandra series of the e-cuspidal pair $(\boldsymbol{L}, \lambda)$. Let ℓ be a prime with $e = e_\ell(q)$ and set $\ell^c \| \Phi_e(q)$. If χ has defect 2, then $c(a_G - a_L) \leq 2$. Moreover, if $c(a_G - a_L) = 2$ then χ is $e\ell^a$-cuspidal for all $a > 0$.*

Proof. As the unipotent character χ lies in the e-Harish-Chandra series of $(\boldsymbol{L}, \lambda)$, the Φ_e-part of its degree polynomial agrees with the one of λ (see [Broué et al. 1993]). As λ is e-cuspidal, the Φ_e-part in its degree polynomial equals a_L [Broué et al. 1993, Proposition 2.4]. So $|G|/\chi(1)$ is divisible by at least $\Phi_e^{a_G - a_L}$ and hence by $\ell^{c(a_G - a_L)}$. If χ has defect 2 this implies that $c(a_G - a_L) \leq 2$. If $c(a_G - a_L) = 2$ then, since $\ell | \Phi_{e\ell^a}$, $\chi(1)$ must be divisible by the same power of $\Phi_{e\ell^a}$ as $|G|$ for all $a > 0$, that is, χ must be $e\ell^a$-cuspidal. □

Now assume that $\ell \geq 5$ is a prime that is good for $\boldsymbol{G}$. By the main result of [Cabanes and Enguehard 1994] the unipotent ℓ-blocks of G are then parametrised by e-cuspidal unipotent pairs $(\boldsymbol{L}, \lambda)$ up to conjugation. Let $B = B_G(\boldsymbol{L}, \lambda)$ be a unipotent ℓ-block of G. Then according to [Cabanes and Enguehard 1994, Theorem] the irreducible characters in B are the constituents of $R_{G_t}^G(\hat{t}\chi_t)$ where t runs over ℓ-elements in G^*, $\boldsymbol{G}_t$ is a Levi subgroup of $\boldsymbol{G}$ dual to $C^\circ_{\boldsymbol{G}^*}(t)$, $\hat{t}$ is the linear character of G_t dual to t, and χ_t lies in the unipotent ℓ-block $B_{G_t}(\boldsymbol{L}_t, \lambda_t)$, where $(\boldsymbol{L}_t, \lambda_t)$ is a unipotent e-cuspidal pair of $\boldsymbol{G}_t$ such that $[\boldsymbol{L}, \boldsymbol{L}] = [\boldsymbol{L}_t, \boldsymbol{L}_t]$ and λ, λ_t have the same restriction to $[\boldsymbol{L}, \boldsymbol{L}]^F$. Now as $\boldsymbol{G}_t^* = C_{\boldsymbol{G}^*}(t)$, $R_{G_t}^G$ induces a bijection (with signs) $\mathcal{E}(G_t, \hat{t}) \to \mathcal{E}(G, \hat{t})$ such that degrees are multiplied by $|G : G_t|_{p'}$. In particular this shows that χ and χ_t have the same ℓ-defect.

Proposition 5.7. *Let $\boldsymbol{G}$, F, q, ℓ be as above. Assume that the principal ℓ-block of G contains a character of ℓ-defect 2. Then a Sylow ℓ-subgroup S of G^* contains an element t with centraliser $|C_S(t)| = \ell^2$.*

Proof. The principal ℓ-block B_0 of G is the ℓ-block above the e-cuspidal pair $(\boldsymbol{L}, 1_L)$, where $\boldsymbol{L}$ is the centraliser of a Sylow e-torus of $\boldsymbol{G}$. In particular $a_L = 0$. Now let $\chi \in \mathrm{Irr}(B_0)$ of defect 2. By the result of Cabanes and Enguehard cited above, there is an ℓ-element $t \in G^*$ and a unipotent character $\chi_t \in \mathrm{Irr}(G_t)$ of defect 2 in the e-Harish-Chandra series of $(\boldsymbol{L}_t, 1_{L_t})$ with $[\boldsymbol{L}_t, \boldsymbol{L}_t] = [\boldsymbol{L}, \boldsymbol{L}]$. By Lemma 5.6 this implies that $ca_{G_t} \leq 2$. In particular Φ_e divides the order polynomial of $[\boldsymbol{G}_t, \boldsymbol{G}_t]$ at most twice, and so the same statement holds for the dual group $\boldsymbol{C} := C_{\boldsymbol{G}^*}(t)$. An inspection of the order formulas of the finite reductive groups shows that then the Sylow ℓ-subgroups of C are abelian (using that $\ell > 3$), and hence contained in a Sylow e-torus of $\boldsymbol{C}$. For χ_t to have defect 2 this forces $|C|_\ell = \ell^2$, so t is as claimed. □

We will not attempt to classify the cases when Sylow ℓ-subgroups of finite reductive groups contain ℓ-elements with this property, even though that seems possible, since in most of these cases our knowledge on decomposition numbers would not suffice to settle Conjecture A anyway. We just discuss one particular case.

Lemma 5.8. *Let $G = \mathrm{PGL}_n(q)$ with $n \geq 4$ and ℓ a prime dividing $q - 1$. If G contains an ℓ-element t with $|C_G(t)|_\ell = \ell^2$ then we have one of*

(1) $n = \ell$ *and* $C \cong \mathrm{GL}_1(q^\ell)$,

(2) $n = \ell + 1$ *and* $C \cong \mathrm{GL}_\ell(q) \times \mathrm{GL}_1(q)$, *or*

(3) $n = \ell^2$ *and* $C \cong \mathrm{GL}_1(q^{\ell^2})$,

where C denotes the centraliser in $\mathrm{GL}_n(q)$ *of a preimage of t under the natural map.*

Proof. Let $\hat{t}$ be a preimage of t of ℓ-power order in $\hat{G} := \mathrm{GL}_n(q)$ under the natural surjection. Then $|C_{\hat{G}}(\hat{t})|_\ell \leq \ell^{2+c}$ where ℓ^c is the precise power of ℓ dividing $q - 1$. Now

$$C := C_{\hat{G}}(\hat{t}) \cong \mathrm{GL}_{n_1}(q^{a_1}) \times \cdots \times \mathrm{GL}_{n_r}(q^{a_r})$$

for suitable $n_i, a_i \geq 1$ with $\sum n_i a_i = n$. Let s_i denote the projection of $\hat{s}$ into the i-th factor of C. Then $s_i \in Z(\mathrm{GL}_{n_i}(q^{a_i})) \cong \mathbb{F}_{q^{a_i}}^\times$ is an ℓ-element generating the field $\mathbb{F}_{q^{a_i}}$. In particular, ℓ divides the cyclotomic polynomial $\Phi_{a_i}(q)$, and so $a_i = \ell^{f_i}$ for some $f_i \geq 0$ (see, e.g., [Malle and Testerman 2011, Lemma 25.13]). Then $|C|_\ell \geq \sum_{i=1}^r (c + f_i) n_i$. Our assumption thus implies that $\sum n_i \leq 3$.

If $n_1 = 3$ then $c = 1$, $f_1 = 0$, so $a_1 = 1$ and $n = \sum n_i = 3$, which was excluded. Now assume that $\sum n_i = 2$. If $n_1 = 2$ then $C \cong \mathrm{GL}_2(q^{n/2})$ and $f_1 = 0$, whence $n = 2$. If $n_1 = n_2 = 1$ then $C \cong \mathrm{GL}_1(q^{a_1}) \times \mathrm{GL}_1(q^{a_2})$ and $2c + f_1 + f_2 \leq 2 + c$, so $c = 1$ and $f_1 + f_2 \leq 1$. This leads to $C \cong \mathrm{GL}_1(q)^2 \leq \mathrm{GL}_2(q)$ or to $C \cong \mathrm{GL}_\ell(q) \times \mathrm{GL}_1(q) \leq \mathrm{GL}_{\ell+1}(q)$, with $\ell \| (q - 1)$. Finally assume that $\sum n_i = 1$, so

$n_1 = 1$ and $C \cong \mathrm{GL}_1(q^n)$ with $n = \ell^f$ and $c + f \le 3$. If $c = 3$ then $f = 0$ and $n = 1$; if $c = 2$ then we find $C \cong \mathrm{GL}_1(q^\ell) \le \mathrm{GL}_\ell(q)$, and if $c = 1$ then we also could have $C \cong \mathrm{GL}_1(q^{\ell^2}) \le \mathrm{GL}_{\ell^2}(q)$. □

Proposition 5.9. *Conjecture A holds for all characters in the principal ℓ-block of $\mathrm{SL}_n(q)$ when $\ell|(q-1)$.*

Proof. Assume that the principal ℓ-block B_0 of $G := \mathrm{SL}_n(q)$ contains a character $\chi \in \mathrm{Irr}(B_0)$ of ℓ-defect 2. Then $\chi \in \mathcal{E}(G,t)$ for some ℓ-element $t \in G^* = \mathrm{PGL}_n(q)$ such that $|C_{G^*}(t)|_\ell = \ell^2$ by Proposition 5.7. According to Lemma 5.8 then either $n \le 3$ or $n \in \{\ell, \ell+1, \ell^2\}$. For $n \le 3$ the claim is easily checked from the known decomposition matrices. We consider the remaining cases in turn.

Assume that $n = \ell^f$ with $f \in \{1,2\}$. Then $C_{G^*}(t)$ is a Coxeter torus of PGL_n by Lemma 5.8(1) and (3), and t is a regular element. Its centraliser is disconnected, with $|C_{G^*}(t)| = \ell^f(q^{\ell^f}-1)/(q-1)$, so the characters in these Lusztig series can only have defect 2 if $f = 1$, that is, $n = \ell$, which we assume from now on. The ℓ characters in $\mathcal{E}(G,t)$ are the constituents of the restriction to G of the unique character ψ in $\mathcal{E}(\tilde{G},\hat{t})$, where $\tilde{G} = \mathrm{GL}_n(q)$. Let $\tilde{T}$ be a Coxeter torus of $\tilde{G}$. Then $\psi = \mathrm{R}_{\tilde{T}}^{\tilde{G}}(\theta)$ for an ℓ-element $\theta \in \mathrm{Irr}(\tilde{T})$ in duality with $\hat{t}$. The reduction of $\mathrm{R}_{\tilde{T}}^{\tilde{G}}(\theta)$ modulo ℓ thus coincides with the reduction of $\mathrm{R}_{\tilde{T}}^{\tilde{G}}(1)$. The latter decomposes as $\sum_{i=1}^{\ell}(-1)^{i-1}\chi_i$, where $\chi_i \in \mathrm{Irr}(B_0)$ is the unipotent character of $\tilde{G}$ parametrised by the hook partition $(i, 1^{\ell-i})$. Now the decomposition numbers of the unipotent characters in B_0 are known: the ℓ-decomposition matrix of the Iwahori–Hecke algebra of type $A_{\ell-1}$, that is, of the symmetric group $\mathfrak{S}_\ell$ embeds into that of G. Since ℓ divides $|\mathfrak{S}_\ell|$ just once, we obtain a Brauer tree with $\chi_i + \chi_{i+1}$ being projective for $i = 1, \ldots, \ell-1$. Adding up we see that ψ is irreducible modulo ℓ, hence the same is true for $\chi \in \mathcal{E}(G,t)$ and our claim is proved in this case.

Assume next that $n = \ell+1$. Then by Lemma 5.8(2) the centraliser of t is a maximal torus with $|C_{G^*}(t)| = q^\ell - 1$. So again t is regular, but now with connected centraliser, hence $\mathcal{E}(G,t) = \{\chi\}$ consists of just one character. Again the principal block of $\mathfrak{S}_{\ell+1}$ has cyclic defect, and a computation as in the previous case shows that χ is in fact irreducible. The validity of Conjecture A follows. □

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References

[Brauer 1941] R. Brauer, "Investigations on group characters", *Ann. of Math.* (2) **42** (1941), 936–958. MR Zbl

[Broué et al. 1993] M. Broué, G. Malle, and J. Michel, "Generic blocks of finite reductive groups", pp. 7–92 in *Représentations unipotentes génériques et blocs des groupes réductifs finis*, Astérisque **212**, Société Mathématique de France, Paris, 1993. MR Zbl

[Cabanes and Enguehard 1994] M. Cabanes and M. Enguehard, "On unipotent blocks and their ordinary characters", *Invent. Math.* **117**:1 (1994), 149–164. MR Zbl

[Carter 1985] R. W. Carter, *Finite groups of Lie type: conjugacy classes and complex characters*, Wiley, New York, 1985. MR Zbl

[Conway et al. 1985] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *Atlas of finite groups: maximal subgroups and ordinary characters for simple groups*, Oxford University Press, 1985. MR Zbl

[Dade 1966] E. C. Dade, "Blocks with cyclic defect groups", *Ann. of Math.* (2) **84** (1966), 20–48. MR Zbl

[Díaz et al. 2007] A. Díaz, A. Ruiz, and A. Viruel, "All p-local finite groups of rank two for odd prime p", *Trans. Amer. Math. Soc.* **359**:4 (2007), 1725–1764. MR Zbl

[Eaton and Moretó 2014] C. W. Eaton and A. Moretó, "Extending Brauer's height zero conjecture to blocks with nonabelian defect groups", *Int. Math. Res. Not.* **2014**:20 (2014), 5581–5601. MR Zbl

[Erdmann 1990] K. Erdmann, *Blocks of tame representation type and related algebras*, Lecture Notes in Mathematics **1428**, Springer, Berlin, 1990. MR Zbl

[GAP 2016] GAP — *Groups, Algorithms, and Programming*, The GAP Group, 2016, Available at http://www.gap-system.org. version 4.8.4.

[Geck et al. 1996] M. Geck, G. Hiss, F. Lübeck, G. Malle, and G. Pfeiffer, "CHEVIE — a system for computing and processing generic character tables", *Appl. Algebra Engrg. Comm. Comput.* **7**:3 (1996), 175–210. MR Zbl

[Gluck et al. 2004] D. Gluck, K. Magaard, U. Riese, and P. Schmid, "The solution of the $k(GV)$-problem", *J. Algebra* **279**:2 (2004), 694–719. MR

[Haggarty 1977] R. J. Haggarty, "On the heights of group characters", *Proc. Amer. Math. Soc.* **63**:2 (1977), 213–216. MR Zbl

[Humphreys 1981] J. E. Humphreys, "Ordinary and modular characters of SL(3, p)", *J. Algebra* **72**:1 (1981), 8–16. MR Zbl

[Humphreys 1990] J. E. Humphreys, "Ordinary and modular characters of SU(3, p)", *J. Algebra* **131**:2 (1990), 425–431. MR Zbl

[Huppert 1957] B. Huppert, "Zweifach transitive, auflösbare Permutationsgruppen", *Math. Z.* **68** (1957), 126–150. MR Zbl

[Huppert 1967] B. Huppert, *Endliche Gruppen, I*, Die Grundlehren der Math. Wissenschaften **134**, Springer, Berlin, 1967. MR Zbl

[Isaacs 1973] I. M. Isaacs, "Characters of solvable and symplectic groups", *Amer. J. Math.* **95** (1973), 594–635. MR Zbl

[Isaacs 1976] I. M. Isaacs, *Character theory of finite groups*, Pure and Applied Mathematics **69**, Academic Press, New York, 1976. MR Zbl

[Jantzen 1987] J. C. Jantzen, "Representations of Chevalley groups in their own characteristic", pp. 127–146 in *Representations of finite groups* (Arcata, CA, 1986), edited by P. Fong, Proc. Sympos. Pure Math. **47**, Amer. Math. Soc., Providence, RI, 1987. MR Zbl

[Kessar and Malle 2013] R. Kessar and G. Malle, "Quasi-isolated blocks and Brauer's height zero conjecture", *Ann. of Math.* (2) **178**:1 (2013), 321–384. MR Zbl

[Külshammer 1984] B. Külshammer, “Symmetric local algebras and small blocks of finite groups”, *J. Algebra* **88**:1 (1984), 190–195. MR Zbl

[Külshammer 1987] B. Külshammer, “A remark on conjectures in modular representation theory”, *Arch. Math.* (*Basel*) **49**:5 (1987), 396–399. MR Zbl

[Kurzweil and Stellmacher 2004] H. Kurzweil and B. Stellmacher, *The theory of finite groups: an introduction*, Springer, New York, 2004. MR Zbl

[Malle 1990] G. Malle, “Die unipotenten Charaktere von ${}^2F_4(q^2)$”, *Comm. Algebra* **18**:7 (1990), 2361–2381. MR Zbl

[Malle and Robinson 2017] G. Malle and G. R. Robinson, “On the number of simple modules in a block of a finite group”, *J. Algebra* **475** (2017), 423–438. MR Zbl

[Malle and Testerman 2011] G. Malle and D. Testerman, *Linear algebraic groups and finite groups of Lie type*, Cambridge Studies in Advanced Mathematics **133**, Cambridge University Press, 2011. MR Zbl

[Michler and Olsson 1990] G. O. Michler and J. B. Olsson, “The Alperin–McKay conjecture holds in the covering groups of symmetric and alternating groups, $p \neq 2$”, *J. Reine Angew. Math.* **405** (1990), 78–111. MR Zbl

[Navarro 1998] G. Navarro, *Characters and blocks of finite groups*, London Mathematical Society Lecture Note Series **250**, Cambridge University Press, 1998. MR Zbl

[Olsson 1993] J. B. Olsson, *Combinatorics and representations of finite groups*, Vorlesungen aus dem Fachbereich Mathematik der Universität GH Essen **20**, Universität Essen, 1993. MR Zbl

[Reynolds 1963] W. F. Reynolds, “Blocks and normal subgroups of finite groups”, *Nagoya Math. J.* **22** (1963), 15–32. MR Zbl

[Robinson 2008] G. R. Robinson, “Large character heights, $Qd(p)$, and the ordinary weight conjecture”, *J. Algebra* **319**:2 (2008), 657–679. MR Zbl

[Ruiz and Viruel 2004] A. Ruiz and A. Viruel, “The classification of p-local finite groups over the extraspecial group of order p^3 and exponent p”, *Math. Z.* **248**:1 (2004), 45–65. MR Zbl

[Sambale 2014] B. Sambale, *Blocks of finite groups and their invariants*, Lecture Notes in Mathematics **2127**, Springer, Cham, 2014. MR Zbl

[Scopes 1995] J. Scopes, “Symmetric group blocks of defect two”, *Quart. J. Math. Oxford Ser.* (2) **46**:182 (1995), 201–234. MR Zbl

[Srinivasan 1964] B. Srinivasan, “On the modular characters of the special linear group SL(2, p^n)”, *Proc. London Math. Soc.* (3) **14** (1964), 101–114. MR Zbl

[Steinberg 1951] R. Steinberg, “The representations of GL(3, q), GL(4, q), PGL(3, q), and PGL(4, q)”, *Canadian J. Math.* **3** (1951), 225–235. MR Zbl

[Winter 1972] D. L. Winter, “The automorphism group of an extraspecial p-group”, *Rocky Mountain J. Math.* **2**:2 (1972), 159–168. MR Zbl

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Slicing the stars: counting algebraic numbers, integers, and units by degree and height

Robert Grizzard and Joseph Gunther

Masser and Vaaler have given an asymptotic formula for the number of algebraic numbers of given degree d and increasing height. This problem was solved by counting lattice points (which correspond to minimal polynomials over $\mathbb{Z}$) in a homogeneously expanding star body in $\mathbb{R}^{d+1}$. The volume of this star body was computed by Chern and Vaaler, who also computed the volume of the codimension-one "slice" corresponding to monic polynomials; this led to results of Barroero on counting algebraic integers. We show how to estimate the volume of higher-codimension slices, which allows us to count units, algebraic integers of given norm, trace, norm and trace, and more. We also refine the lattice point-counting arguments of Chern-Vaaler to obtain explicit error terms with better power savings, which lead to explicit versions of some results of Masser–Vaaler and Barroero.

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1. Introduction

A classical theorem of Northcott states that there are only finitely many elements of $\overline{\mathbb{Q}}$ of bounded degree and height. It's then natural to ask, for interesting subsets $\mathcal{S} \subset \overline{\mathbb{Q}}$ of bounded degree, how the number of elements of bounded height grows as we let the height bound increase. More precisely, one considers the asymptotics of

$$N(\mathcal{S}, \mathcal{H}) = \#\{x \in \mathcal{S} \mid H(x) \leq \mathcal{H}\},$$

where $H(x)$ is the absolute multiplicative Weil height of x; see, for example, [Bombieri and Gubler 2006, p. 16].

Many of the oldest instances of such asymptotic statements concern elements of a fixed number field. Schanuel [1979, Corollary] proved that, for any number field K, as $\mathcal{H}$ grows,

$$N(K, \mathcal{H}) = c_K \cdot \mathcal{H}^{2[K:\mathbb{Q}]} + O(\mathcal{H}^{2[K:\mathbb{Q}]-1} \log \mathcal{H}),$$

where the constant c_K involves all the classical invariants of the number field K, and the $\log \mathcal{H}$ factor disappears for $K \neq \mathbb{Q}$.

Lang [1983, Chapter 3, Theorem 5.2] states analogous asymptotics for the ring of integers $\mathcal{O}_K$ and its unit group $\mathcal{O}_K^*$:

$$\begin{aligned} N(\mathcal{O}_K, \mathcal{H}) &= \gamma_K \cdot \mathcal{H}^{[K:\mathbb{Q}]}(\log \mathcal{H})^r + O(\mathcal{H}^{[K:\mathbb{Q}]}(\log \mathcal{H})^{r-1}), \\ N(\mathcal{O}_K^*, \mathcal{H}) &= \gamma_K^* \cdot (\log \mathcal{H})^r + O((\log \mathcal{H})^{r-1}), \end{aligned}$$

where r is the rank of $\mathcal{O}_K^*$ and γ_K and γ_K^* are unspecified constants. That first count was later refined to a multiterm asymptotic by Widmer [2016, Theorem 1.1].

More recently, natural subsets that aren't contained within a single number field have been examined. Masser and Vaaler [2008, Theorem] determined the asymptotic for the entire set $\overline{\mathbb{Q}}_d = \{x \in \overline{\mathbb{Q}} \mid [\mathbb{Q}(x) : \mathbb{Q}] = d\}$:

$$N(\overline{\mathbb{Q}}_d, \mathcal{H}) = \frac{d \cdot V_d}{2\zeta(d+1)} \cdot \mathcal{H}^{d(d+1)} + O(\mathcal{H}^{d^2}(\log \mathcal{H})), \tag{1-1}$$

where the $\log \mathcal{H}$ factor disappears for $d \geq 3$, and V_d is an explicit positive constant that we'll define shortly.

This asymptotic was deduced from results of Chern and Vaaler [2001] (discussed at length in Section 2), which also imply an asymptotic for the set $\mathcal{O}_d$ of all algebraic integers of degree d, as noted in [Widmer 2016, (1.2)]. It was sharpened by Barroero [2014, Theorem 1.1, case $k = \mathbb{Q}$]:

$$N(\mathcal{O}_d, \mathcal{H}) = d \cdot V_{d-1} \cdot \mathcal{H}^{d^2} + O(\mathcal{H}^{d(d-1)}(\log \mathcal{H})), \tag{1-2}$$

where again the $\log \mathcal{H}$ factor disappears for $d \geq 3$.

After algebraic numbers and integers, it's natural to turn to the problem of counting units and other interesting sets of algebraic numbers. It's also desirable to obtain versions of these estimates with explicit error terms. These are the two purposes of this paper.

We establish counts of units, algebraic integers of given norm, given trace, and given norm and trace in Corollaries 1.2 through 1.5, which follow from the more general Theorem 1.1 stated below. As for explicit error bounds, we have made several improvements to the existing literature. The lack of explicit error terms in the results (1-1) and (1-2) is inherited from results of Chern and Vaaler [2001] on counting polynomials. Specifically, on p. 6 they mention that it would be of interest to make the implied constant in their Theorem 3 explicit, but they were unable to do so. In this paper we are able to make this constant explicit (Theorem 7.1 below), and we also prove an analogous result for monic polynomials (Theorem 8.1). We use these to obtain versions of (1-1) and (1-2) that are uniform in both $\mathcal{H}$ and d. These, along with an explicit version of our result on counting units, are summarized below in Theorem 1.10.

Results. Throughout the paper, we will understand the *minimal polynomial* of an algebraic number to be its minimal polynomial over $\mathbb{Z}$; we obtain this by multiplying the minimal monic polynomial over $\mathbb{Q}$ by the smallest positive integer such that all its coefficients become integers.

Counting algebraic integers, as in (1-2), is equivalent to counting only those algebraic numbers whose minimal polynomials have leading coefficient 1. Our primary goal in this paper is to count algebraic numbers of fixed degree and bounded height subject to specifying *any* number of the leftmost and rightmost coefficients of their minimal polynomials. Besides specializing to the cases of algebraic numbers and algebraic integers above, this will allow us to count units, algebraic integers with given norm, algebraic integers with given trace, and algebraic integers with given norm and trace.

To state our theorem, we need a little notation. Our asymptotic counts will involve the Chern-Vaaler constants

$$V_d = 2^{d+1}(d+1)^s \prod_{j=1}^{s} \frac{(2j)^{d-2j}}{(2j+1)^{d+1-2j}}, \tag{1-3}$$

where $s = \lfloor (d-1)/2 \rfloor$. These constants are volumes of certain star bodies discussed later.

For integers m, n, and d with $0 < m,\ 0 \le n$, and $m+n \le d$, and integer vectors $\vec{\ell} \in \mathbb{Z}^m$ and $\vec{r} \in \mathbb{Z}^n$, we write $\mathcal{N}(d, \vec{\ell}, \vec{r}, \mathcal{H})$ for the number of algebraic numbers of degree d and height at most $\mathcal{H}$, whose minimal polynomials are of the form

$$f(z) = \ell_0 z^d + \cdots + \ell_{m-1} z^{d-(m-1)} + x_m z^{d-m} + \cdots + x_{d-n} z^n + r_{d-n+1} z^{n-1} + \cdots + r_d.$$

Lastly, we set $g = d - m - n$. In the statements below, the implied constants depend on all parameters stated other than $\mathcal{H}$.

Theorem 1.1. *Fix d, $\vec{\ell} \in \mathbb{Z}^m$, and $\vec{r} \in \mathbb{Z}^n$ as above. Assume that $\ell_0 > 0$, that*

$$\gcd(\ell_0, \ldots, \ell_{m-1}, r_{d-n+1}, \ldots, r_d) = 1,$$

and that $r_d \neq 0$ if $n > 0$. Then, as $\mathcal{H} \to \infty$,

$$\mathcal{N}(d, \vec{\ell}, \vec{r}, \mathcal{H}) = d \cdot V_g \cdot \mathcal{H}^{d(g+1)} + O(\mathcal{H}^{d(g+\frac{1}{2})} \log \mathcal{H}).$$

This generalizes the situation one faces when counting algebraic integers, whose minimal polynomials are monic ($m = 1,\ n = 0,\ \vec{\ell} = (1)$). Certain special cases are of particular interest, and we prove stronger power savings terms for them.

Corollary 1.2. *Let $d \geq 2$, and let $N(\mathcal{O}_d^*, \mathcal{H})$ denote the number of units in the algebraic integers of height at most $\mathcal{H}$ and degree d over $\mathbb{Q}$. Then, as $\mathcal{H} \to \infty$,*

$$N(\mathcal{O}_d^*, \mathcal{H}) = 2d \cdot V_{d-2} \cdot \mathcal{H}^{d(d-1)} + O(\mathcal{H}^{d(d-2)}).$$

Corollary 1.3. *Let $\nu \neq 0$ be an integer, $d \geq 2$, and let $\mathcal{N}_{\mathrm{Nm}=\nu}(d, \mathcal{H})$ denote the number of algebraic integers with norm ν, of height at most $\mathcal{H}$ and degree d over $\mathbb{Q}$. Then, as $\mathcal{H} \to \infty$,*

$$\mathcal{N}_{\mathrm{Nm}=\nu}(d, \mathcal{H}) = d \cdot V_{d-2} \cdot \mathcal{H}^{d(d-1)} + O(\mathcal{H}^{d(d-2)}).$$

Corollary 1.4. *Let τ be an integer, $d \geq 2$, and let $\mathcal{N}_{\mathrm{Tr}=\tau}(d, \mathcal{H})$ denote the number of algebraic integers with trace τ, of height at most $\mathcal{H}$ and degree d over $\mathbb{Q}$. Then, as $\mathcal{H} \to \infty$,*

$$\mathcal{N}_{\mathrm{Tr}=\tau}(d, \mathcal{H}) = d \cdot V_{d-2} \cdot \mathcal{H}^{d(d-1)} + \begin{cases} O(\mathcal{H}), & \text{if } d = 2, \\ O(\mathcal{H}^3 \log \mathcal{H}), & \text{if } d = 3, \\ O(\mathcal{H}^{d(d-2)}), & \text{if } d \geq 4. \end{cases}$$

Corollary 1.5. *Let $\nu \neq 0$ and τ be integers, $d \geq 3$, and let $\mathcal{N}_{\mathrm{Nm}=\nu,\mathrm{Tr}=\tau}(d, \mathcal{H})$ denote the number of algebraic integers with norm ν, trace τ, of height at most $\mathcal{H}$ and degree d over $\mathbb{Q}$. Then, as $\mathcal{H} \to \infty$,*

$$\mathcal{N}_{\mathrm{Nm}=\nu,\mathrm{Tr}=\tau}(d, \mathcal{H}) = d \cdot V_{d-3} \cdot \mathcal{H}^{d(d-2)} + O(\mathcal{H}^{d(d-3)}).$$

Remark 1.6. For two real-valued functions f and g with the same domain, we write $f = O(g)$ to mean there exist positive constants C and C' such that $|f(x)| \leq C|g(x)|$ for all $x > C'$. In Theorem 1.1, the implied constants depend on d, $\vec{\ell}$, and $\vec{r}$; in Corollary 1.2 on d; in Corollary 1.3 on d and ν; in Corollary 1.4 on d and τ; and in Corollary 1.5 on d, ν, and τ.

Remark 1.7. In Corollaries 1.3 through 1.5, the main term of the asymptotic doesn't depend on the specific coefficients being enforced. Thus these may be interpreted as results on the equidistribution of norms and traces.

Remark 1.8. The type of counts found in this paper are related to Manin's conjecture, which addresses the asymptotic number of rational points of bounded height on Fano varieties. Counting points of degree d and bounded height in $\overline{\mathbb{Q}}$, or equivalently, on $\mathbb{P}^1$, can be transferred to a question of counting rational points of bounded height on the d-th symmetric product of $\mathbb{P}^1$, which is $\mathbb{P}^d$. This is what Masser and Vaaler implicitly do when they count algebraic numbers by counting their minimal polynomials (as does this paper; see the Methods section below). However, one needs to use a nonstandard height on $\mathbb{P}^d$; Le Rudulier [2014, Théorème 1.1] takes this approach explicitly, thereby reproving and generalizing (the main term of) the result of Masser and Vaaler. It should be noted, though, that while the shape of the main term — a constant times the appropriate power of the height — follows from known results on Manin's conjecture, *explicitly* determining the constant in front relies ultimately on an archimedean volume calculation of Chern and Vaaler.

Barroero's count of algebraic integers of degree d corresponds to counting rational points on $\mathbb{P}^d$ that are integral with respect to the hyperplane at infinity. As noted in [Le Rudulier 2014, Remarque 5.3], the shape of his count's main term then follows from general results on counting integral points of bounded height on equivariant compactifications of affine spaces [Chambert-Loir and Tschinkel 2012, Theorem 3.5.6].

Our own units count corresponds to counting points on $\mathbb{P}^d$ integral with respect to *two* hyperplanes, and does not appear to follow from any results currently in the literature.

Remark 1.9. The algebraic number and integer counts of (1-1) and (1-2) have also been extended to arbitrary base number fields [Masser and Vaaler 2007; Barroero 2014] and to vectors of algebraic numbers [Schmidt 1995; Gao 1995; Widmer 2009; 2016; Guignard 2017]. We expect there should be extensions of our new counts to these contexts as well.

The second goal of this paper is to give explicit error terms, which we feel is especially justified in this context, beyond general principles of error-term morality. Namely, it's natural to ask questions about properties of "random algebraic numbers" (or random algebraic integers, random units, etc.). For example: "What's the probability that a random element of $\overline{\mathbb{Q}}$ generates a Galois extension of $\mathbb{Q}$?"

How to make sense of a question like this? There are models from other arithmetic contexts; for example, if we're asked "What's the probability that a random positive integer is square-free?" we know what to do: count the number of square-free integers from 1 to N, divide that by N, and ask if that proportion has a limit as

N grows. (Answer: Yes, $6/\pi^2$). Note that the easiest part is dividing by N, the number of elements in your finite box. In order to make sense of probabilistic statements in the context of $\overline{\mathbb{Q}}$, one would like to first take a box of bounded height and degree (which will have only finitely many algebraic numbers by Northcott), determine the relevant proportion within that finite box, and then let the box size grow. But now the denominator in question is far from trivial; unlike counting the number of integers from 1 to N, estimating how many algebraic numbers are in a height-degree box is a more delicate matter.

In the context of $\overline{\mathbb{Q}}$, where there are *two* natural parameters to increase (the height and the degree), the gold standard for a "probabilistic" result would be that it holds for any increasing set of height-degree boxes such that the minimum of the height and degree goes to infinity. To prove results that even approach this standard (e.g., one might require that the height of the boxes grows at least as fast as some function of the degree), one likely needs good estimates for how many numbers are in a height-degree box to begin with. Without an estimate that holds uniformly in both $\mathcal{H}$ and d, one would be justified in making statements about random elements in $\overline{\mathbb{Q}}$ of fixed degree d, but not random elements of $\overline{\mathbb{Q}}$ overall. Thus controlling the error terms in the theorems above is crucial.

To this end, in this paper we give explicit error bounds for the algebraic number counts of Masser and Vaaler, the algebraic integer counts of Barroero, and our own unit counts. Below $p_d(T)$ is a polynomial defined in Section 2 whose leading term is $V_{d-1}T^d$, so our result is consistent with (1-2).

Theorem 1.10. *Let $\overline{\mathbb{Q}}_d$ denote the set of algebraic numbers of degree d over $\mathbb{Q}$, let $\mathcal{O}_d$ denote the set of algebraic integers of degree d over $\mathbb{Q}$, and let $\mathcal{O}_d^*$ denote the set of units of degree d over $\mathbb{Q}$ in the ring of all algebraic integers. For all $d \geq 3$,*

(i) $$\left|N(\overline{\mathbb{Q}}_d,\mathcal{H}) - \frac{d\cdot V_d}{2\zeta(d+1)}\mathcal{H}^{d(d+1)}\right| \leq 3.37\cdot(15.01)^{d^2}\cdot\mathcal{H}^{d^2} \quad \textit{for } \mathcal{H}\geq 1;$$

(ii) $$|N(\mathcal{O}_d,\mathcal{H}) - dp_d(\mathcal{H}^d)| \leq 1.13\cdot 4^d d^d 2^{d^2}\cdot\mathcal{H}^{d(d-1)} \quad \textit{for } \mathcal{H}\geq 1; \textit{ and}$$

(iii) $$|N(\mathcal{O}_d^*,\mathcal{H}) - 2dV_{d-2}\cdot\mathcal{H}^{d(d-1)}| \leq 0.0000126\cdot d^3 4^d(15.01)^{d^2}\cdot\mathcal{H}^{d(d-1)-1} \quad \textit{for } \mathcal{H}\geq d2^{d+1/d}.$$

Methods. The starting point of all our proofs is the relationship between the height of an algebraic number and the Mahler measure of its minimal polynomial. Recall that the Mahler measure $\mu(f)$ of a polynomial with complex coefficients

$$f(z) = w_0z^d + w_1z^{d-1} + \cdots + w_d = w_0(z-\alpha_1)\cdots(z-\alpha_d) \in \mathbb{C}[z],$$

with $w_0 \neq 0$, is defined by

$$\mu(f) = |w_0|\prod_{i=1}^{d}\max\{1, |\alpha_i|\},$$

and $\mu(0)$ is defined to be zero. It's immediate that the Mahler measure is multiplicative: $\mu(f_1 f_2) = \mu(f_1)\mu(f_2)$.

Crucially for our purposes, if $f(z)$ is the minimal polynomial of an algebraic number α, then (see, for example, [Bombieri and Gubler 2006, Proposition 1.6.6])

$$\mu(f) = H(\alpha)^d.$$

Thus, in order to count degree d algebraic numbers of height at most $\mathcal{H}$, we can instead count minimal integer polynomials of Mahler measure at most $\mathcal{H}^d$.

We identify a polynomial with its vector of coefficients, so that counting integer polynomials amounts to counting lattice points. To do this we employ techniques from the geometry of numbers, which make rigorous the idea that, for a reasonable subset of Euclidean space, the number of integer lattice points in the set should be approximated by its volume. So for example, the number of integer polynomials with degree at most d and Mahler measure at most T should be roughly the volume of the set of such *real* polynomials

$$\{f \in \mathbb{R}[z]_{\deg \leq d} \mid \mu(f) \leq T\} \subset \mathbb{R}^{d+1}.$$

Note that by multiplicativity of the Mahler measure, this set is the same as $T\mathcal{U}_d$, where

$$\mathcal{U}_d := \{f \in \mathbb{R}[z]_{\deg \leq d} \mid \mu(f) \leq 1\}.$$

The set $\mathcal{U}_d$ will be our primary object of study. It is a closed, compact "star body," i.e., a subset of euclidean space closed under scaling by numbers in [0, 1]. Chern and Vaaler [2001, Corollary 2] explicitly determined the volume of $\mathcal{U}_d$. In a rather heroic calculation, they showed that $V_d := \mathrm{vol}_{d+1}(\mathcal{U}_d)$ is given by the positive rational number in (1-3).* Thus by geometry of numbers, and noting that $\mathrm{vol}(T\mathcal{U}_d) = T^{d+1} \cdot \mathrm{vol}(\mathcal{U}_d)$, one expects the number of integer polynomials of degree at most d and Mahler measure at most T to be approximately $T^{d+1} \cdot V_d$. Chern and Vaaler proved this is indeed the case. Masser and Vaaler then showed how to refine this count of all such polynomials to just minimal polynomials, which let them prove the algebraic number count in (1-1).

What if you only want to count algebraic integers? Again, the above approach suggests you should do that by counting their minimal polynomials. Algebraic integers are characterized by having *monic* minimal polynomials. Thus one is naturally led to seek the volume of the "monic slice" of $T\mathcal{U}_d$ consisting of those real polynomials with leading coefficient 1. However, these slices are no longer dilations of each other, so their volumes aren't determined by knowing the volume of one such slice. Still, Chern and Vaaler were able to compute the volumes of

*Our $\mathcal{U}_d$ is the same as what would be denoted by $\mathcal{S}_{d+1}$ in the notation of [Chern and Vaaler 2001], and our V_d matches their V_{d+1}. Our subscripts correspond to the degree of the polynomials being counted rather than the dimension of the space.

monic slices of $T\mathcal{U}_d$; rather than a constant times a power of T, they are given by a polynomial in T, whose leading term is $V_{d-1}T^d$. Geometry of numbers can then be applied again to obtain the algebraic integer count in (1-2).

In order to count units of degree d, or algebraic integers with given norm and/or trace, one needs to take higher-codimension slices. For example, the minimal polynomial of a unit will have leading coefficient 1 and constant coefficient ± 1. But one quickly discovers that these higher-dimensional slices have volumes that are, in general, no longer polynomial in T. Rather than trying to explicitly calculate these volumes, we depart from the methods of earlier works, and instead approximate the volumes of such slices.

When we cut a dilate $T\mathcal{U}_d$ by a certain kind of linear space, then as T grows the slices look more and more like a lower-dimensional unit star body; this will be explained in Section 4. This explains the appearance of the volume V_d in all of our asymptotic counts. We also use a careful analysis of the boundary of $\mathcal{U}_d$ to show that the above convergence happens relatively fast; this makes our approximations precise enough to obtain algebraic number counts with good power-saving error terms.

We state here our main result on counting polynomials. For nonnegative integers m, n, and d with $0 < m + n \leq d$, and integer vectors $\vec{\ell} \in \mathbb{Z}^m$ and $\vec{r} \in \mathbb{Z}^n$, let $\mathcal{M}(d, \vec{\ell}, \vec{r}, T)$ denote the number of polynomials f of the form

$$f(z) = \ell_0 z^d + \cdots + \ell_{m-1} z^{d-(m-1)} + x_m z^{d-m} + \cdots + x_{d-n} z^n + r_{d-n+1} z^{n-1} + \cdots + r_d$$

with Mahler measure at most T, where $x_m, \ldots, x_{d-n}$ are integers. Let $g = d - m - n$.

Combining our volume estimates with a counting principle of Davenport, we obtain the following.

Theorem 1.11. *For all $0 < m + n \leq d$, $\vec{\ell} \in \mathbb{Z}^m$, and $\vec{r} \in \mathbb{Z}^n$, as $T \to \infty$,*

$$\mathcal{M}(d, \vec{\ell}, \vec{r}, T) = V_g \cdot T^{g+1} + O(T^g).$$

Here the implied constant depends on d, $\vec{\ell}$, and $\vec{r}$.

Now we briefly discuss the methods used in the second half of the paper to prove our explicit results, and how these results fit in with the literature. [Chern and Vaaler 2001, Theorem 3], the main ingredient in (1-1), gives an asymptotic count of the number of integer polynomials of given degree d and Mahler measure at most T. The error term in this result contains a full power savings — order T^d against a main term of order T^{d+1} — but the implied constant in the error term is not made explicit. They do produce an explicit error term of order $T^{d+1-1/d}$ in [op. cit., Theorem 5] using [op. cit., Theorem 4], which is a quantitative statement on the continuity of the Mahler measure.

Our Theorem 7.1 below makes the constant in the error term explicit in [op. cit., Theorem 3], using a careful study of the boundary of $\mathcal{U}_d$. We apply the classical

Lipschitz counting principle in place of the Davenport principle; the latter is not very amenable to producing explicit bounds. Theorem 8.1 is the analogous result to Theorem 7.1 for monic polynomials, and is obtained in a similar manner. However, the application of the Lipschitz principle is more delicate in this case. We also prove an explicit version of our Theorem 1.11 counting polynomials with specified coefficients (Theorem 9.3). For this result we also apply [op. cit., Theorem 4], and, reminiscent of Chern and Vaaler's application, this method yields an inferior power savings.

We now describe the organization of the paper. In Section 2 we collect key facts about the unit star body $\mathcal{U}_d$, including a detailed discussion of its boundary. In Section 3 we describe the counting principles we use to estimate the difference between the number of lattice points in a set and the set's volume. In Section 4 we estimate the volume of the sets in which we must count lattice points to prove Theorem 1.11; this theorem is then proved in Section 5. In Section 6 we transfer our counts for polynomials to counts for various kinds of algebraic numbers, thereby proving Theorem 1.1 and Corollaries 1.2 through 1.5. This involves using a version of Hilbert's irreducibility theorem to account for reducible polynomials.

The rest of the paper is devoted to obtaining explicit versions of these counts. In Section 7 we prove the aforementioned explicit version of [op. cit., Theorem 3] on counting polynomials of given degree and bounded Mahler measure, and in Section 8 we do the same for the count of monic polynomials. Section 9 contains a version of the general Theorem 1.11 with an explicit error term, at the cost of weaker power savings. In Section 10 we begin to convert our explicit counts of polynomials to explicit counts of minimal polynomials. The main piece of this is showing that the reducible polynomials are negligible. We follow the techniques for this used by Masser and Vaaler (sharper than the more general Hilbert irreducibility method described above), obtaining explicit bounds. In Section 11 we prove our final explicit results on counting algebraic numbers, including explicit versions of Masser and Vaaler's result (1-1), Barroero's result (1-2), and Corollaries 1.2 and 1.3. Finally, we include an appendix with some estimates for various expressions involving binomial coefficients which occur in our explicit error terms throughout the paper.

2. The unit star body

In this section we discuss some properties of the unit star body

$$\mathcal{U}_d := \{\vec{w} \in \mathbb{R}^{d+1} \mid \mu(\vec{w}) \leq 1\}.$$

Since for all $f \in \mathbb{R}[x]$ and $t \in \mathbb{R}$ we have

$$\mu(tf) = |t|\mu(f), \tag{2-1}$$

it's easy to see that $\mathcal{U}_d$ is in fact a (symmetric) star body. Furthermore, $\mathcal{U}_d$ is compact; it is closed because μ is continuous [Mahler 1961, Lemma 1], and we can see it is bounded by classical results that bound the coefficients of a polynomial in terms of its Mahler measure, for example the following (see [Mahler 1976, p. 7] and [Bombieri and Gubler 2006, Lemma 1.6.7 and its proof]).

Lemma 2.1 (Mahler). *Every polynomial*

$$f(z) = w_0 z^d + w_1 z^{d-1} + \cdots + w_0 \in \mathbb{C}[z]$$

has coefficients satisfying

$$|w_i| \le \binom{d}{i} \mu(f), \;\; i = 0, \ldots, d. \tag{2-2}$$

Furthermore, we have the following double inequality comparing Mahler measure with the sup-norm of coefficients:

$$\binom{d}{\lfloor d/2 \rfloor}^{-1} \|\vec{w}\|_\infty \le \mu(\vec{w}) \le \sqrt{d+1}\, \|\vec{w}\|_\infty, \quad \textit{for all} \;\; \vec{w} \in \mathbb{R}^{d+1}. \tag{2-3}$$

Volumes. As mentioned in the introduction, the exact volume of $\mathcal{U}_d$ was determined by Chern and Vaaler [2001, Corollary 2]:

$$V_d := \mathrm{vol}_{d+1}(\mathcal{U}_d) = 2^{d+1}(d+1)^s \prod_{j=1}^{s} \frac{(2j)^{d-2j}}{(2j+1)^{d+1-2j}},$$

where $s = \lfloor (d-1)/2 \rfloor$. (Here vol_N denotes Lebesgue measure on $\mathbb{R}^N$.)

We record some numerical information about the volume of $\mathcal{U}_d$. We note that a result like Lemma 2.2 below would follow quite easily from the asymptotic formula for $\log V_d$ given in [op. cit., (1.31)]. However, this formula was given without proof and contains an error. The correct version of that formula is apparently (using our notation):

$$\log V_d = -\frac{1}{2} d \log d + \left(\frac{1}{2}\log 2\pi + 1\right) d - \frac{5}{4}\log d + \left(3\zeta'(-1) + \frac{1}{2} + \frac{1}{3}\log 2\right) + \frac{19\theta_2}{12d},$$

where $|\theta_2| \le 1$. In this corrected version, the constant term differs from what was printed in [op. cit.] by $\log 2$. Since in this paper we are mainly interested in the maximum of V_d, we settle for the following simpler result that can be proved quickly.

Lemma 2.2. *We have*

$$\begin{aligned} V_d \le V_{15} &= \frac{2658455991569831745807614120560689152}{139048725878708489575791571230468 75} \\ &= \frac{2^{121}}{3^{20} \cdot 5^9 \cdot 7^9 \cdot 11^6 \cdot 13^4} \approx 191.1888 \end{aligned}$$

for all $d \ge 0$, *and*

$$\lim_{d\to\infty} V_d = 0.$$

Proof. Note using Stirling's estimates (see (A-1) in the appendix) that for any positive integer s,

$$\begin{aligned}\prod_{j=1}^{s}\left\{\frac{2j}{2j+1}\right\} &= \frac{2^s s!}{(2s+1)!/(2^s s!)} = \frac{4^s s!^2}{(2s+1)!}\\ &\le \frac{4^s(e^{1-s}s^{s+1/2})^2}{\sqrt{2\pi}e^{-2s-1}(2s+1)^{2s+3/2}} \le \frac{4^s(e^{2-2s}s^{2s+1})}{\sqrt{2\pi}e^{-2s-1}(2s)^{2s+3/2}}\\ &\le \frac{e^3 4^s s^{2s+1}}{\sqrt{2\pi}4^s 2^{3/2}s^{2s+1}\sqrt{s}} \le \frac{e^3}{4\sqrt{\pi s}}.\end{aligned}$$

Suppose that d is odd, so we may take $s = \left\lfloor \frac{d-1}{2}\right\rfloor = \left\lfloor \frac{(d+1)-1}{2}\right\rfloor$. Then

$$\begin{aligned}\frac{V_{d+1}}{V_d} &= \frac{2^{d+2}(d+2)^s}{2^{d+1}(d+1)^s}\prod_{j=1}^{s}\left\{\frac{(2j)^{d+1-2j}}{(2j)^{d-2j}}\right\}\prod_{j=1}^{s}\left\{\frac{(2j+1)^{d+1-2j}}{(2j+1)^{d+2-2j}}\right\}\\ &= 2\left(\frac{d+2}{d+1}\right)^s\prod_{j=1}^{s}\left\{\frac{2j}{2j+1}\right\} \le \left(\frac{d+2}{d+1}\right)^s\cdot\frac{e^3}{2\sqrt{\pi s}}.\end{aligned}$$

If d is even and $s = \left\lfloor \frac{d-1}{2}\right\rfloor = \frac{d}{2}-1$, then $\left\lfloor \frac{(d+1)-1}{2}\right\rfloor = s+1$. Then

$$\begin{aligned}\frac{V_{d+1}}{V_d} &= \frac{2^{d+2}(d+2)^{s+1}}{2^{d+1}(d+1)^s}\cdot\frac{d}{(d+1)^2}\prod_{j=1}^{s}\left\{\frac{(2j)^{d+1-2j}}{(2j)^{d-2j}}\right\}\prod_{j=1}^{s}\left\{\frac{(2j+1)^{d+1-2j}}{(2j+1)^{d+2-2j}}\right\}\\ &= 2\frac{(d+2)^s}{(d+1)^s}\cdot\frac{d^2+2d}{d^2+2d+1}\prod_{j=1}^{s}\left\{\frac{2j}{2j+1}\right\} \le \left(\frac{d+2}{d+1}\right)^s\cdot\frac{e^3}{2\sqrt{\pi s}}.\end{aligned}$$

In either case, the ratio of successive terms tends to zero, so in fact V_d decays to zero faster than exponentially, proving the second claim of our lemma. For the first claim, it suffices to compute enough values of V_d. We see the maximum is attained at $d = 15$, as advertised. □

For any $T \ge 0$, by (2-1)

$$\mathrm{vol}_{d+1}(\{\vec{w}\in\mathbb{R}^{d+1} \mid \mu(\vec{w}) \le T\}) = \mathrm{vol}_{d+1}(T\mathcal{U}_d) = V_d\cdot T^{d+1}.$$

Chern and Vaaler (see [2001, (1.16)], corrected as in [Barroero 2014, footnote on p. 38]) also computed the volume of the "monic slice"

$$\mathcal{W}_{d,T} := \{(w_0,\ldots,w_d)\in T\mathcal{U}_d \mid w_0 = 1\}. \tag{2-4}$$

They showed:

$$\mathrm{vol}_d(\mathcal{W}_{d,T}) = p_d(T) := \mathcal{C}_d 2^{-s}\{s!\}^{-1}\sum_{m=0}^{s}(-1)^m(d-2m)^s\binom{s}{m}T^{d-2m}, \tag{2-5}$$

where again

$$s = \left\lfloor \frac{d-1}{2} \right\rfloor \quad \text{and} \quad \mathcal{C}_d = 2^d \prod_{j=1}^{s} \left(\frac{2j}{2j+1} \right)^{d-2j}.$$

Note that, since $p_d(T)$ is a polynomial in T, we automatically have (carefully inspecting the leading term):

$$\mathrm{vol}_d(\mathcal{W}_{d,T}) = V_{d-1} \cdot T^d + O(T^{d-1}).$$

For other slices besides the monic one, we will have to work harder (in Section 4) to obtain such power savings. Along the way, it will become clear why the leading coefficient takes the form it does.

Remark 2.3. Above, and throughout the paper, for a measurable set $S \subset \mathbb{R}^N$ and $n < N$, we will sometimes write $\mathrm{vol}_n(S)$. In this case, S will always be a subset contained in an affine space defined by fixing $N-n$ coordinates of $\mathbb{R}^N$, and then $\mathrm{vol}_n(S)$ will always denote the Lebesgue measure of the projection of S to $\mathbb{R}^n$ given by simply forgetting the fixed coordinates. For ease of notation, we will sometimes drop the subscript when it is clear from context.

Semialgebraicity. Next we establish a qualitative result we will need in proving Theorem 1.11. A (real) *semialgebraic set* is a subset of euclidean space which is cut out by finitely many polynomial equations and/or inequalities, or a finite union of such subsets. Recall that the class of semialgebraic sets is closed under finite unions and intersections, and also closed under projections by the Tarski–Seidenberg theorem [Bierstone and Milman 1988, Theorem 1.5].

Lemma 2.4. *The set $\mathcal{U}_d \subset \mathbb{R}^{d+1}$ is semialgebraic.*

Proof. Our proof is similar to that of [Barroero 2014, Lemma 4.1]. For each $j = 0, \ldots, d$, we wish to define a semialgebraic set $S_j \subset \mathbb{R}^{d+1}$ corresponding to degree j polynomials in $\mathcal{U}_d$. We start by constructing auxiliary subsets of $\mathbb{R}^{d+1} \times \mathbb{C}^j$ corresponding to the polynomials' coefficients and roots, where $\mathbb{C}$ is identified with $\mathbb{R}^2$ in the obvious way. We define

$$\begin{aligned} S_j^0 = \big\{ (0, \ldots, 0, w_{d-j}, \ldots, w_d, \alpha_1, \ldots, \alpha_j) \in \mathbb{R}^{d+1} \times \mathbb{C}^j \;\big|\; w_{d-j} \neq 0 \text{ and } \\ w_{d-j} z^j + w_{d-j+1} z^{j-1} + \cdots + w_d = w_{d-j}(z-\alpha_1) \cdots (z-\alpha_j) \big\}, \end{aligned}$$

where the equalities defining the set are given by equating the real part of each elementary symmetric function in the roots $\alpha_1, \ldots, \alpha_j$ with the corresponding coefficient w_i, and setting the imaginary part to zero. To enforce the inequality $\mu((0, \ldots, 0, w_{d-j}, \ldots, w_d)) \leq 1$, we define S_j^1 to comprise those elements of S_j^0 such that all products of subsets of $\{\alpha_1, \ldots, \alpha_j\}$ are less than or equal to $1/|w_{d-j}|$

in absolute value. Finally, we let S_j be the projection of S_j^1 onto $\mathbb{R}^{d+1}$. Now simply note that

$$\mathcal{U}_d = \{0\} \cup \bigcup_{j=0}^{d} S_j. \qquad \square$$

Remark 2.5. Note that for any $T > 0$ the dilation $T\mathcal{U}_d$ is also semialgebraic, and is defined by the same number of polynomials (and of the same degrees) as is $\mathcal{U}_d$.

Boundary parametrizations. Next we describe the parametrization of the boundary of $\mathcal{U}_d$, which consists of vectors corresponding to polynomials with Mahler measure exactly 1. The simple idea behind the parametrization is that such a polynomial is the product of a *monic* polynomial with all its roots inside (or on) the unit circle, and a polynomial with constant coefficient ± 1 and all its roots outside (or on) the unit circle. Recall that $\mathcal{U}_d$ is a compact, symmetric star body in $\mathbb{R}^{d+1}$. The parametrization is described in [Chern and Vaaler 2001, Section 10]. We briefly summarize the key points here. The boundary $\partial\mathcal{U}_d$ is the union of $2d+2$ "patches" $\mathcal{P}_{k,d}^{\varepsilon}$, for $k = 0, \ldots, d$, $\varepsilon = \pm 1$. The patch $\mathcal{P}_{k,d}^{\varepsilon}$ is the image of a certain compact set $\mathcal{J}_{k,d}^{\varepsilon}$ under the map

$$b_{k,d}^{\varepsilon} : \mathbb{R}^k \times \mathbb{R}^{d-k} \to \mathbb{R}^{d+1},$$

defined by

$$b_{k,d}^{\varepsilon}\big((x_1, \ldots, x_k), (y_0, \ldots, y_{d-k-1})\big) = B_{k,d}\big((1, x_1, \ldots, x_k), (y_0, \ldots, y_{d-k-1}, \varepsilon)\big), \tag{2-6}$$

and

$$B_{k,d}\big((x_0, x_1, \ldots, x_k), (y_0, \ldots, y_{d-k})\big) = (w_0, \ldots, w_d),$$

with

$$w_i = \sum_{l=0}^{k} \sum_{\substack{m=0 \\ l+m=i}}^{d-k} x_l y_m, \qquad i = 0, \ldots, d. \tag{2-7}$$

Note that this simply corresponds to the polynomial factorization

$$w_0 z^d + \cdots + w_d = (x_0 z^k + \cdots + x_k) \cdot (y_0 z^{d-k} + \cdots + y_{d-k}).$$

The sets $\mathcal{J}_{k,d}^{\varepsilon}$ are given by

$$\mathcal{J}_{k,d}^{\varepsilon} = J_k \times K_{d-k}^{\varepsilon} \subseteq \mathbb{R}^k \times \mathbb{R}^{d-k},$$

where

$$J_k = \{\vec{x} \in \mathbb{R}^k \mid \mu(1, \vec{x}) = 1\} \tag{2-8}$$

and

$$K_{d-k}^{\varepsilon} = \{\vec{y} \in \mathbb{R}^{d-k} \mid \mu(\vec{y}, \varepsilon) = 1\}.$$

It will also be useful in Section 8 to have a parametrization of $\partial\mathcal{W}_{d,T}$, the boundary of a monic slice (see (2-4)), along the lines of that given for $\partial\mathcal{U}_d$ above. Consider a monic polynomial

$$f(z) = z^d + w_1 z^{d-1} + \cdots + w_d \in \mathbb{R}[z],$$

having Mahler measure equal to $T \geq 1$ and roots $\alpha_1, \ldots, \alpha_d \in \mathbb{C}$. We note that such a polynomial can be factored as $f(z) = g_1(z)g_2(z)$, where g_1 and $g_2 \in \mathbb{R}[z]$ are monic, $\mu(g_1) = 1$ (forcing $\mu(g_2) = T$), the constant coefficient of g_2 is $\pm T$, and $\deg(g_1) = k \in \{0, \ldots, d-1\}$. To do this, we simply let

$$g_1(z) = \prod_{|\alpha_i|<1} (z - \alpha_i) \quad \text{and} \quad g_2(z) = \prod_{|\alpha_i|\geq 1} (z - \alpha_i).$$

It is easy to check that g_1 and g_2 have the desired properties. For $k = 0, \ldots, d-1$, we let J_k be as in (2-8), and let

$$Y_{d-k}^{\varepsilon T} = \{\vec{y} \in \mathbb{R}^{d-k-1} \mid \mu(1, \vec{y}, \varepsilon T) = T\}$$

and

$$\mathcal{L}_{k,d}^{\varepsilon T} = J_k \times Y_{d-k}^{\varepsilon T} \subseteq \mathbb{R}^k \times \mathbb{R}^{d-k-1},$$

for each $k = 0, \ldots, d-1$ and $\varepsilon = \pm 1$. We also define

$$\begin{aligned} \beta_{k,d}^{\varepsilon T}&\big((x_1, \ldots, x_k), (y_1, \ldots, y_{d-k-1})\big) \\ &= B_{k,d}^{\varepsilon}\big((1, x_1, \ldots, x_k), (1, y_1, \ldots, y_{d-k-1}, \varepsilon T)\big), \end{aligned} \tag{2-9}$$

similarly to (2-6).

We have that $\partial\mathcal{W}_{d,T}$ is covered by the $2d$ "patches"

$$\beta_{k,d}^{\varepsilon T}(\mathcal{L}_{k,d}^{\varepsilon T}). \tag{2-10}$$

3. Counting principles

We'll need a counting principle of Davenport to estimate the number of lattice points in semialgebraic sets.

Theorem 3.1 (Davenport). *Let S be a compact, semialgebraic subset of $\mathbb{R}^n$ defined by at most k polynomial equalities and inequalities of degree at most l. Then the number of integer lattice points contained in S is equal to*

$$\mathrm{vol}_n(S) + O(\max\{\overline{\mathrm{vol}}(S), 1\}),$$

where $\overline{\mathrm{vol}}(S)$ denotes the maximum, for $m = 1, \ldots, n-1$, of the volume of the projection of S onto the m-dimensional coordinate space given by setting any $n-m$ coordinates equal to zero. The implicit constant in the error term depends only on k, l, and n.

Remark 3.2. This follows from the main theorem of [Davenport 1951], as described immediately after its statement. (The argument for this reduction was corrected in [Davenport 1964].) Davenport's principle has been generalized in a couple directions, to allow for lattices other than the standard integer lattice [Barroero and Widmer 2014, (1.2)], and to apply to sets definable in any o-minimal structure [op. cit., Theorem 1.3], of which semialgebraic sets are but one example. However, the above version will suffice for our purposes.

For our explicit error estimates we will use a different counting principle, namely a refinement of the classical Lipschitz counting principle due to Spain [1995]. The classical principle allows one to estimate the difference between the number of lattice points in a set and the set's volume: one uses that the boundary is parametrized by finitely many Lipschitz maps, and that a Lipschitz map sends a cube in the domain into a cube in the codomain. In our case it will be convenient to use "tiles" other than cubes in the domain. This could be achieved by precomposing the maps with other maps which cover our tiles with the images of cubes, but we feel the following alternative formulation is intuitive and less awkward in application.

Theorem 3.3. *Let $S \subset \mathbb{R}^n$ be a set whose boundary ∂S is contained in the images of finitely many maps $\phi_i : J_i \to \mathbb{R}^n$, where $\mathcal{I}$ is a finite set of indices and each J_i is a set. For each $i \in \mathcal{I}$, assume that J_i can be covered by m_i sets $T_{i,1}, \dots, T_{i,m_i}$, with the property that for each j the image $\phi_i(T_{i,j})$ is contained in a translate of $[0, 1]^n$ inside $\mathbb{R}^n$. Then*

$$|\#(S \cap \mathbb{Z}^n) - \mathrm{vol}_n(S)| \leq 2^n \sum_{i \in \mathcal{I}} m_i.$$

Proof. We follow the "every other tile" approach of [Spain 1995]. The number of lattice points in S differs from the volume of S by at most the number of integer vector translates of the half-open unit tile $[0, 1)^n \subseteq \mathbb{R}^n$ that meet the boundary ∂S. Consider the set $\mathcal{E}$ of tiles which are *even* integer vector translates of $[0, 1)^n$; it is clear that any translate of $[0, 1]^n$ meets exactly one such tile. Since ∂S is contained in at most $\sum_{i \in \mathcal{I}} m_i$ translates of $[0, 1]^n$, this means that at most that many tiles from $\mathcal{E}$ meet ∂S. But $\mathbb{R}^n$ is partitioned by 2^n sets of tiles which, like $\mathcal{E}$, are made up of "every other tile." (Explicitly, these sets are of the form $\mathcal{E} + \vec{v}$, where $\vec{v}$ is a vector with entries only 0 and 1.) The bound claimed in the theorem follows. □

4. Volumes of slices of star bodies

We keep all the notation established just before Theorem 1.11 in the introduction, so $d, m, n, \vec{\ell} = (\ell_0, \dots, \ell_{m-1}) \in \mathbb{Z}^m$, and $\vec{r} = (r_{d-n+1}, \dots, r_d) \in \mathbb{Z}^n$[†] are fixed, and

[†]For this section we could take $\vec{\ell}$ and $\vec{r}$ to be *real* vectors, but this will not be important for our results.

again we set $g = d - m - n$. Let T be a positive real number. We continue to use the volume convention of Remark 2.3. The primary step in proving Theorem 1.11 is to estimate the volume of the slice

$$\mathcal{S}(T) = \mathcal{S}_{\vec{\ell},\vec{r}}(T) := \left\{ \vec{w} = (w_0, \ldots, w_d) \in \mathbb{R}^{d+1} \;\middle|\; \substack{\mu(\vec{w}) \le T, \\ w_i = \ell_i \text{ for } i = 0,\ldots,m-1 \\ w_j = r_j \text{ for } j = d-n+1,\ldots,d} \right\} \tag{4-1}$$

as T grows. Specifically, we show the following.

Theorem 4.1. *We have*

$$\mathrm{vol}_{g+1}(\mathcal{S}(T)) = V_g T^{g+1} + O(T^g), \quad \textit{as } T \to \infty.$$

We won't obtain an explicit error estimate of this strength, but in Section 9 we will discuss how to obtain an explicit error term of order $T^{g+1-1/d}$.

The idea of the proof of Theorem 4.1 is as follows. Because $\mu(T\vec{w}) = T\mu(\vec{w})$ for all $T \ge 0$, and all $\vec{w} \in \mathbb{R}^{d+1}$,

$$\{\vec{w} \in \mathbb{R}^{d+1} \mid \mu(\vec{w}) \le T\} = T\{\vec{w} \in \mathbb{R}^{d+1} \mid \mu(\vec{w}) \le 1\} = T\,\mathcal{U}_d.$$

Let

$$\vec{v} = (\ell_0, \ldots, \ell_{m-1}, 0, \ldots, 0, r_{d-n+1}, \ldots, r_d) \in \mathbb{R}^{d+1},$$

and for each $t \in [0, \infty)$, set

$$W_t := t\vec{v} + \mathrm{Span}\{e_m, e_{m+1}, \ldots, e_{d-n}\} \subset \mathbb{R}^{d+1}, \tag{4-2}$$

where $e_0, e_1, \ldots, e_d$ are standard basis vectors for $\mathbb{R}^{d+1}$. Then for $T > 0$,

$$\mathcal{S}(T) = W_1 \cap T\,\mathcal{U}_d = T(W_{1/T} \cap \mathcal{U}_d), \tag{4-3}$$

and since $W_{1/T}$ is $(g+1)$-dimensional, this means

$$\mathrm{vol}_{g+1}(\mathcal{S}(T)) = T^{g+1}\,\mathrm{vol}_{g+1}(W_{1/T} \cap \mathcal{U}_d). \tag{4-4}$$

Letting $t = 1/T$, we should expect that

$$\mathrm{vol}_{g+1}(W_{1/T} \cap \mathcal{U}_d) = \mathrm{vol}_{g+1}(\mathcal{U}_d \cap (W_0 + t\vec{v})) \to \mathrm{vol}_{g+1}(\mathcal{U}_d \cap W_0) \quad \text{as } t \to 0,$$

unless the boundary of $\mathcal{U}_d$ were to intersect with W_0 in an unusual way; for example, if $\mathcal{U}_d$ were a cube and W_0 was a plane containing one of the faces. This basic idea of using continuity of volumes of slices appears in the proof of [Sinclair 2008, Theorem 1.5]. We will show below that $\mathrm{vol}_{g+1}(\mathcal{U}_d \cap W_0) = V_g$, whence the main term in the statement of Theorem 4.1. We'll obtain a full power savings by showing that the boundary of $\mathcal{U}_d$ is never tangent to W_0.[‡]

[‡] As an exercise to see why tangency is a problem, consider the length of cross-sections of a disk as the cross-sections slide toward a tangent line.

Proposition 4.2. *Let $S \subset \mathbb{R} \times \mathbb{R}^N$ be a compact set bounded by finitely many smooth hypersurfaces H_i, $i = 1, \dots, m$. Assume each boundary component $H_i \cap \partial S$ has smooth intersection with (i.e., is not tangent to) the hyperplane $\{0\} \times \mathbb{R}^N$, and that these boundary components $H_i \cap \partial S$ have pairwise disjoint interiors. Then*

$$V(t) := \mathrm{vol}_N(S \cap (\{t\} \times \mathbb{R}^N))$$

satisfies

$$V(t) = V(0) + O(t), \quad \text{as } t \to 0^+.$$

Proof. We denote points in $\mathbb{R} \times \mathbb{R}^N$ by $(x, y_1, \dots, y_N)$. For each $t \geq 0$, let $S_{[0,t]} = S \cap ([0, t] \times \mathbb{R}^N)$, and let $S_t = S \cap (\{t\} \times \mathbb{R}^N)$. Let F denote the constant vector field $(1, 0, \dots, 0)$ on $\mathbb{R} \times \mathbb{R}^N$. By the divergence theorem,

$$\oint_{\partial S_{[0,t]}} F \cdot d\vec{s} = \int_{S_{[0,t]}} \nabla \cdot F \, d\mathrm{vol}_{N+1} = \int_{S_{[0,t]}} 0 \, d\mathrm{vol}_{N+1} = 0,$$

where the first integral is with respect to the surface measure with outward normal. Note that our assumption that $\{0\} \times \mathbb{R}^N$ is not tangent to any of the H_i means that neither is the parallel hyperplane $\{t\} \times \mathbb{R}^N$ for t sufficiently small. Now let $R_t = ([0, t] \times \mathbb{R}^N) \cap \partial S$, and note that, as long as t is small enough to avoid the aforementioned tangencies, the boundary of $S_{[0,t]}$ decomposes into three pieces with disjoint interiors as follows:

$$\partial S_{[0,t]} = S_0 \cup S_t \cup R_t.$$

and so

$$\begin{aligned} 0 = \oint_{\partial S_{[0,t]}} F \cdot d\vec{s} &= \int_{S_0} F \cdot d\vec{s} + \int_{S_t} F \cdot d\vec{s} + \int_{R_t} F \cdot d\vec{s} \\ &= -V(0) + V(t) + \int_{R_t} F \cdot d\vec{s}, \end{aligned}$$

where

$$\int_{R_t} F \cdot d\vec{s} = \sum_i \int_{H_i \cap R_t} F \cdot d\vec{s}.$$

Now we must show that

$$|V(t) - V(0)| = \left| \int_{R_t} F \cdot d\vec{s} \right| = O(t). \tag{4-5}$$

Since S is compact, the set R_t is contained in a "pizza box" $[0, t] \times [-M, M]^N$ for some positive number M independent of t. Fix $i \in \{1, \dots, m\}$. By assumption, $H_i \cap \partial S$ is not tangent to the hyperplane $\{x = 0\}$, but since H_i is smooth and we're working in a compact set, we know $H_i \cap \partial S$ is not tangent to $\{x = t\}$ for any t sufficiently small. This means that, by the implicit function theorem, for t sufficiently small and any point $P \in H_i \cap R_t$, we have that H_i coincides in an open subset

$U \subseteq H_i \cap R_t$ containing P with the graph of a function $y_r = f(x, y_1, \ldots, \hat{y}_r, \ldots, y_N)$ for some $r \in \{1, \ldots, N\}$ which depends on P. So we have $f : V \to [-M, M]$, where V is an open subset of $[0, t] \times [-M, M]^{N-1}$. Letting $\vec{n}$ denote the outward unit normal,

$$\int_U F \cdot d\vec{s} = \int_U F \cdot \vec{n}\, ds = \int \cdots \int_V \mp \frac{\partial f}{\partial x}\, dx\, dy_1 \cdots d\hat{y}_r \cdots dy_N, \tag{4-6}$$

where the sign in the final integral is $-$ or $+$ depending on whether $\vec{n}$ is an upward or downward normal to the graph of f, respectively.

By our nontangency assumption again, the partial derivative $\partial f/\partial x$ is bounded in absolute value inside our pizza box by a constant K which does not depend on U, i, or t as $t \to 0$. By compactness, finitely many of these neighborhoods U cover $H_i \cap R_t$, and the number of neighborhoods required — call this number n — can be chosen independent of t or i. Using (4-6), we estimate the integral in (4-5) as follows:

$$\begin{aligned}\left|\int_{R_t} F \cdot d\vec{s}\right| &\le \sum_{i=1}^{m} \left|\int_{H_i \cap R_t} F \cdot d\vec{s}\right| \le \sum_{i=1}^{m} \int_{H_i \cap R_t} |F \cdot \vec{n}|\, ds \le \sum_{i=1}^{m} \sum_{U} \int_U |F \cdot \vec{n}|\, ds \\ &\le \sum_{i=1}^{m} \sum_{U} \int_{-M}^{M} \cdots \int_{-M}^{M} \int_0^t \left|\frac{\partial f}{\partial x}\right| dx\, dy_1 \cdots d\hat{y}_r \cdots dy_N \\ &\le m \cdot n \cdot [(2M)^{N-1} t] K = O(t). \qquad \square\end{aligned}$$

Now we verify that the boundary of $\mathcal{U}_d$ satisfies the hypotheses of Proposition 4.2. We refer to the parametrization of said boundary described in Section 2, and follow that notation. As noted in [Chern and Vaaler 2001, Section 10], the condition of the boundary components having disjoint interiors is satisfied here — this can be readily verified directly from the description of the parametrization. Let $H = H^{\varepsilon}_{k,d}$ be one of the hypersurfaces which bound $\mathcal{U}_d$. The hypersurface H is the image of $\mathbb{R}^k \times \mathbb{R}^{d-k}$ under the map $b = b^{\varepsilon}_{k,d}$ described in (2-6).

Proposition 4.3. *Let* $\vec{v} = (\ell_0, \ldots, \ell_{m-1}, 0, \ldots, 0, r_{d-n+1}, \ldots, r_d) \in \mathbb{R}^{d+1}$, *and let*

$$W_0 = \operatorname{Span}\{e_m, e_{m+1}, \ldots, e_{d-n}\} \quad \text{and} \quad W = \operatorname{Span}\{\vec{v}, e_m, e_{m+1}, \ldots, e_{d-n}\},$$

where $e_0, e_1, \ldots, e_d$ *are standard basis vectors for* $\mathbb{R}^{d+1}$. *Then* W_0 *is not tangent to* $H \cap W$ *at any point.*

We will break up the proof of this proposition into three lemmas.

Lemma 4.4. *The subspace* W_0 *does not meet* H *unless*

$$n \le k \le d - m.$$

If those inequalities hold and $P = (w_0, \dots, w_d) = b(x_1, \dots, x_k, y_0, \dots, y_{d-k-1})$ *is a point in* $H \cap W_0$, *then*

$$y_0 = \cdots = y_{m-1} = x_{k-n+1} = \cdots = x_k = 0. \tag{4-7}$$

Proof. Suppose that the inequalities are satisfied. We'll prove vanishing of the parameters y_i, by induction on $0 \le i \le m-1$. If $m = 0$, there's nothing to prove. Otherwise, for the base case $i = 0$, by the definition of W_0 we have $w_0 = 0$, but also $w_0 = y_0$ by the definition of b in (2-6). For arbitrary i, we again have $w_i = 0$, while by the definition of b, every summand in the formula for w_i is of the form $x_{i-j} y_j$ for $j < i$, except for the summand y_i. Thus we're done by induction. Essentially the same proof works for the vanishing of $x_{k-n+1}, \dots, x_k$.

However, if $n > k$, then the above argument would imply that $x_0 = 0$, but we know $x_0 = 1$, a contradiction. Similarly, if $k > d - m$, the above would give $0 = y_{d-k} = \varepsilon$, also a contradiction. □

Lemma 4.5. *The tangent space* $T_P(H)$ *of* H *at* P *is the row space of the following* $d \times (d+1)$ *matrix, where the first* $(d-k)$ *rows represent the tangent vectors* $(\partial w_0/\partial y_j, \dots, \partial w_d/\partial y_j)$, $j = 0, \dots, d-k-1$, *and the last* k *rows represent the tangent vectors* $(\partial w_0/\partial x_i, \dots, \partial w_d/\partial x_i)$, $i = 1, \dots, k$. *Let* $q = d-k-1$ *for ease of reading:*

$$(Db)^T = \begin{bmatrix} 1 & x_1 & x_2 & \cdots & \cdots & x_k & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & x_1 & x_2 & \cdots & \cdots & x_k & 0 & \cdots & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & & \ddots & & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & & & \ddots & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & x_1 & x_2 & \cdots & \cdots & x_k & 0 \\ 0 & y_0 & y_1 & \cdots & \cdots & y_q & \varepsilon & 0 & \cdots & \cdots & 0 \\ 0 & 0 & y_0 & y_1 & \cdots & \cdots & y_q & \varepsilon & 0 & \cdots & 0 \\ \vdots & & & \ddots & \ddots & & & \ddots & \ddots & & \vdots \\ \vdots & & & & \ddots & \ddots & & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & 0 & y_0 & y_1 & \cdots & \cdots & y_q & \varepsilon \end{bmatrix}.$$

Lemma 4.6. *The projection of* $T_P(H)$ *onto* $W_0^\perp$ *is surjective.*

Proof. Using Lemma 4.4, the image of that projection contains the row space (in appropriate coordinates) of the following matrix, obtained by taking the first m columns and first m rows of the above matrix, as well as its last n columns and last n rows:

$$C := \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix},$$

where

$$A = \begin{bmatrix} 1 & x_1 & x_2 & \cdots & x_{m-1} \\ 0 & 1 & x_1 & \cdots & x_{m-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & x_1 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

is an $m \times m$-matrix, and

$$B = \begin{bmatrix} \varepsilon & 0 & \cdots & \cdots & 0 \\ y_q & \varepsilon & \ddots & & \vdots \\ \vdots & \ddots & \varepsilon & \ddots & \vdots \\ y_{q-n+3} & \ddots & \ddots & \ddots & 0 \\ y_{q-n+2} & \cdots & y_{q-1} & y_q & \varepsilon \end{bmatrix}$$

is an $n \times n$-matrix.

Thus C is a block diagonal matrix (we've used the vanishing of parameters described in (4-7) here) with determinant $\varepsilon^n \neq 0$, so its row space is all of $W_0^{\perp}$. □

Proof of Proposition 4.3. We seek a tangent vector to H at P which is contained in $W \setminus W_0$. By Lemma 4.6, $T_P(H)$ surjects onto the positive-dimensional space $W_0^{\perp}$. Since its kernel under this map is exactly W_0, a vector must exist as desired. □

Proof of Theorem 4.1. We begin by noting that we may identify $\mathcal{U}_d \cap W_0 \subseteq \mathbb{R}^{d+1}$ with $\mathcal{U}_g \subseteq \mathbb{R}^{g+1}$ as follows.

Define a map $\tau : \mathbb{R}^{g+1} \to \mathbb{R}^{d+1}$ by

$$\tau(w_m, \dots, w_{d-n}) = (\underbrace{0, \dots, 0}_{m}, w_m, \dots, w_{d-n}, \underbrace{0, \dots, 0}_{n}) \in W_0,$$

which corresponds to multiplying the polynomial corresponding to the input by z^n. Notice that this operation preserves the Mahler measure. It's also clear that τ maps $\mathcal{U}_g$ isometrically onto $\mathcal{U}_d \cap W_0$, so we conclude that

$$\operatorname{vol}_{g+1}(\mathcal{U}_d \cap W_0) = \operatorname{vol}_{g+1}(\mathcal{U}_g) = V_g. \tag{4-8}$$

Using Proposition 4.3, we can apply Proposition 4.2 to the set $\mathcal{S} = \mathcal{U}_d \cap W$, considered as a subset of $W \cong \mathbb{R} \times \mathbb{R}^{g+1}$ (so we are setting $N = g+1$). Here for $t \geq 0$,

$$\mathcal{S} \cap (\{t\} \times \mathbb{R}^{g+1}) = \mathcal{U}_d \cap W_t.$$

Then Proposition 4.2 gives

$$\operatorname{vol}_{g+1}(\mathcal{U}_d \cap W_{1/T}) = \operatorname{vol}_{g+1}(\mathcal{U}_d \cap W_0) + O(1/T).$$

Now by (4-4) and (4-8),

$$\begin{aligned}\mathrm{vol}_{g+1}(S(T)) &= \bigl(\mathrm{vol}_{g+1}(\mathcal{U}_d \cap W_0) + O(1/T)\bigr)\cdot T^{g+1}\\ &= V_g \cdot T^{g+1} + O(T^g),\end{aligned}$$

completing our proof. □

5. Lattice points in slices: proof of Theorem 1.11

Now that we have an estimate for the volume of $\mathcal{S}(T)$, we want to in turn estimate the number of integer lattice points in $\mathcal{S}(T)$, via Theorem 3.1. Note that this is the same as the number of integer lattice points of $S'(T)$, which will denote the projection of $\mathcal{S}(T)$ on $W_0 \cong \mathbb{R}^{g+1}$. Note that $\mathrm{vol}(\mathcal{S}(T)) = \mathrm{vol}(S'(T))$.

Since $\mathcal{U}_d$ is semialgebraic by Lemma 2.4 (and thus $T\cdot\mathcal{U}_d$ as well), it is clear that the number and degrees of the polynomial inequalities and equalities needed to define $S'(T)$ are independent of T. Thus to apply Theorem 3.1, it remains only to bound the volumes of projections of $S'(T)$ on coordinate planes.

For $\vec{w} \in S'(T)$, by (2-3),

$$\|\vec{w}\|_\infty \le \|(\vec{\ell},\vec{w},\vec{r})\|_\infty \le \binom{d}{\lfloor d/2\rfloor}\mu(\vec{\ell},\vec{w},\vec{r}) \le \binom{d}{\lfloor d/2\rfloor}T,$$

so $S'(T)$ is contained inside a cube of side length $2\binom{d}{\lfloor d/2\rfloor}T$ in $\mathbb{R}^{g+1}$. Thus for $j = 1,\dots,g$, any projection of $S'(T)$ on a j-dimensional coordinate plane is contained inside a cube of side length $2\binom{d}{\lfloor d/2\rfloor}T$ in $\mathbb{R}^j$, and thus has volume at most

$$\Bigl(2\binom{d}{\lfloor d/2\rfloor}T\Bigr)^j,$$

which is certainly $O(T^g)$ for $j = 1,\dots,g$.

By Theorem 3.1, we now get

$$\mathcal{M}(d,\vec{\ell},\vec{r},T) = \mathrm{vol}(S'(T)) + O(T^g),$$

and so by Theorem 4.1,

$$\mathcal{M}(d,\vec{\ell},\vec{r},T) = V_g\cdot T^{g+1} + O(T^g).$$

6. Proofs of Theorem 1.1 and corollaries

In this section we transfer our counts for degree d polynomials in Theorem 1.11 to the counts for degree d algebraic numbers in Theorem 1.1. This only requires estimating the number of reducible polynomials, because the hypotheses of Theorem 1.1 (fixing a positive number of coefficients which must be coprime) ensure that the only irreducible polynomials we count are actually *minimal* polynomials of degree d.

We'll apply a version of Hilbert's irreducibility theorem to achieve the most general result, which is the last ingredient needed to prove Theorem 1.1. However, in various special cases we work a little harder to improve the power savings, which will prove the sharper results of Corollaries 1.2 through 1.5.

We keep the notation and hypotheses of Theorem 1.1, fixing $d, m, n, \vec{\ell} \in \mathbb{Z}^m$, and $\vec{r} \in \mathbb{Z}^n$. Furthermore, we let $\mathcal{M}^{\text{red}}(d, \vec{\ell}, \vec{r}, T)$ denote the number of *reducible* integer polynomials of the form

$$f(z) = \ell_0 z^d + \cdots + \ell_{m-1} z^{d-(m-1)} + x_m z^{d-m} + \cdots + x_{d-n} z^n + r_{d-n+1} z^{n-1} + \cdots + r_d,$$

and as before we set $g = d - m - n$.

Proposition 6.1. *We have*

$$\mathcal{M}^{\text{red}}(d, \vec{\ell}, \vec{r}, T) = O(T^{g+1/2} \log T). \tag{6-1}$$

Proof. One of our hypotheses is that, if $n > 0$, then $r_d \neq 0$; that is, we don't want $f(z)$ to be divisible by z. It's not hard to see that, under this hypothesis, the "generic polynomial" $f(x_m, \dots, x_{d-n}, z)$ defined above is irreducible in $\mathbb{Z}[x_m, \dots, x_{d-n}, z]$, by the following argument. Suppose f factors nontrivially as $f = f_1 f_2$. Since f has degree 1 in x_m, without loss of generality f_1 has degree 1 in x_m and f_2 has degree 0 in x_m. Let $f_1 = g_1 x_m + g_2$, where g_1 and g_2 are in $\mathbb{Z}[x_{m+1}, \dots, x_{d-n}, z]$, so we have $f = f_2 g_1 x_m + f_2 g_2$, which means that $f_2 g_1 = z^{d-m}$. We discover that f_2 is (plus or minus) a power of z, and so f was divisible by z all along.

Now our proposition follows immediately from a quantitative form of Hilbert's irreducibility theorem [Cohen 1981, Theorem 2.5]. In the notation of the cited theorem, we are setting $r = 1$ and $s = g + 1$. Cohen uses the ℓ_∞ norm on polynomials rather than Mahler measure, but these are directly comparable by (2-3). It's worth noting that, as can be inferred from Section 2 of that reference, the implied constant in (6-1) depends only on d, g, and $\|(\vec{\ell}, \vec{r})\|_\infty$, and could in principle be effectively computed. □

In the situations of Corollaries 1.2 through 1.5, we can obtain stronger bounds.

Proposition 6.2. *For $d \geq 2$ and $r \in \mathbb{Z} \setminus \{0\}$,*

$$\mathcal{M}^{\text{red}}(d, (1), (r), T) = O(T^{d-2}).$$

For $d \geq 3$, $t \in \mathbb{Z}$, and $r \in \mathbb{Z} \setminus \{0\}$,

$$\mathcal{M}^{\text{red}}(d, (1, t), (r), T) = O(T^{d-3}).$$

For $d \geq 2$, $T \geq 1$, and $t \in \mathbb{Z}$,

$$\mathcal{M}^{\text{red}}(d, (1, t), (\,), T) = \begin{cases} O(\sqrt{T}) & \text{if } d = 2, \\ O(T \log T) & \text{if } d = 3, \\ O(T^{d-2}) & \text{if } d > 3. \end{cases}$$

We postpone the proof until Section 10, where we'll prove it with explicit constants. For now, we show how Theorem 1.1 and Corollaries 1.2 through 1.5 follow from our results so far.

Proof of Theorem 1.1 and Corollaries 1.2 through 1.5. By Theorem 1.11 we have

$$\mathcal{M}(d, \vec{\ell}, \vec{r}, T) = V_g \cdot T^{g+1} + O(T^g). \tag{6-2}$$

We write $\mathcal{M}^{\text{irr}}(d, \vec{\ell}, \vec{r}, T)$ for the corresponding number of *irreducible* degree d polynomials with specified coefficients. Since $\vec{\ell}$ is nonempty and $\ell_0 \neq 0$,

$$\mathcal{M}^{\text{irr}}(d, \vec{\ell}, \vec{r}, T) = \mathcal{M}(d, \vec{\ell}, \vec{r}, T) - \mathcal{M}^{\text{red}}(d, \vec{\ell}, \vec{r}, T). \tag{6-3}$$

Applying Theorem 1.11 and Proposition 6.1, we see that

$$\mathcal{M}^{\text{irr}}(d, \vec{\ell}, \vec{r}, T) = V_g \cdot T^{g+1} + O(T^{g+1/2} \log T). \tag{6-4}$$

By our assumption that the specified coefficients had no common factor, and that $\ell_0 > 0$, any irreducible polynomial counted will be a minimal polynomial. Thus each of the degree d irreducible polynomials f we count corresponds to exactly d algebraic numbers $\alpha_1, \dots, \alpha_d$ of degree d and height at most $\mathcal{H}$, where $\mathcal{H}^d = T$, since $\mu(f) = H(\alpha_i)^d$ for $i = 1, \dots, d$. In other words,

$$\mathcal{N}(d, \vec{\ell}, \vec{r}, \mathcal{H}) = d\mathcal{M}^{\text{irr}}(d, \vec{\ell}, \vec{r}, \mathcal{H}^d). \tag{6-5}$$

Now Theorem 1.1 follows from (6-4).

Corollaries 1.3, 1.4, and 1.5 follow similarly, by replacing the general upper bound for reducible polynomials in Proposition 6.1 with the sharper bounds in Proposition 6.2. The count for units in Corollary 1.2 follows immediately from Corollary 1.3, since an algebraic number is a unit exactly if it is an algebraic integer with norm ± 1. $\square$

7. Counting polynomials: explicit bounds

Let $\mathcal{M}(\leq d, T)$ denote the number of polynomials in $\mathbb{Z}[z]$ of degree at most d and Mahler measure at most T. The following is an explicit version of [Chern and Vaaler 2001, Theorem 3]. To condense notation, we define for each $d \geq 0$ the constants

$$P(d) = \prod_{j=0}^{d} \binom{d}{j}, \tag{7-1}$$

and

$$A(d) = \sum_{k=0}^{d} P(k) P(d-k).$$

Theorem 7.1. *For $d \geq 1$ and $T \geq 1$,*

$$|\mathcal{M}(\leq d, T) - \mathrm{vol}(\mathcal{U}_d)T^{d+1}| \leq \kappa_0(d)T^d,$$

where

$$\begin{aligned}\kappa_0(d) &= 4^{d+1}A(d)\Big(d\binom{d}{\lfloor d/2 \rfloor}+1\Big)^d \\ &\leq 40\sqrt[4]{2}\pi^{3/4}e^{-3}\cdot d^{-1/4}\cdot(4\sqrt{2}e^{3/2}\pi^{-3/2})^d\cdot(2\sqrt{e})^{d^2} \\ &\leq 5.59\cdot(15.01)^{d^2}.\end{aligned}$$

Proof. We refer to the parametrization of the boundary of $\mathcal{U}_d$ detailed on page 1397. The boundary $\partial(T\,\mathcal{U}_d)$ is parametrized by $2d+2$ maps of the form

$$\begin{aligned}&Tb^{\varepsilon}_{k,d} : \mathcal{J}^{\varepsilon}_{k,d} \to \partial(T\,\mathcal{U}_d) \subseteq \mathbb{R}^{d+1}, \\ &Tb^{\varepsilon}_{k,d}(\vec{x},\vec{y}) = \big(Tf_0(\vec{x},\vec{y}),\ldots,Tf_d(\vec{x},\vec{y})\big),\end{aligned}$$

where

$$f_i(\vec{x},\vec{y}) := w_i\big((1,\vec{x}),(\vec{y},\varepsilon)\big), \quad \text{for } i = 0,\ldots,d,$$

and w_i is as in (2-7).

Fix for the moment $k \in \{0,\ldots,d\}$ and $\varepsilon \in \{\pm 1\}$. If $(\vec{x},\vec{y})$ lies in any $\mathcal{J}^{\varepsilon}_{k,d}$, then $\mu(1,\vec{x}) = \mu(\vec{y},\varepsilon) = 1$, and so by (2-2), $\|(\vec{x},\vec{y})\|_\infty \leq \binom{d}{\lfloor d/2 \rfloor}$, and so

$$\|(\vec{x},\vec{y})\|_2 \leq \sqrt{d}\,\|(\vec{x},\vec{y})\|_\infty \leq \sqrt{d}\cdot\binom{d}{\lfloor d/2 \rfloor}. \tag{7-2}$$

Also, for any $i \in \{0,\ldots,d\}$, by (2-7),

$$\|\nabla f_i(\vec{x},\vec{y})\|_\infty \leq \max\{1, \|(\vec{x},\vec{y})\|_\infty\}. \tag{7-3}$$

Now for any $i \in \{0,\ldots,d\}$ and for any $(\vec{x}_1,\vec{y}_1),(\vec{x}_2,\vec{y}_2) \in \mathcal{J}^{\varepsilon}_{k,d}$, using (7-2) and (7-3),

$$\begin{aligned}&|Tf_i(\vec{x}_1,\vec{y}_1) - Tf_i(\vec{x}_2,\vec{y}_2)| \\ &\qquad = T\,|f_i(\vec{x}_1,\vec{y}_1) - f_i(\vec{x}_2,\vec{y}_2)| \\ &\qquad \leq T\cdot \sup_{(\vec{x},\vec{y})\in\mathcal{J}} \|\nabla f_i(\vec{x},\vec{y})\|_2 \cdot \|(\vec{x}_1,\vec{y}_1) - (\vec{x}_2,\vec{y}_2)\|_2 \\ &\qquad \leq T\cdot\sqrt{d}\cdot \sup_{(\vec{x},\vec{y})\in\mathcal{J}} \|(\vec{x},\vec{y})\|_\infty \cdot\sqrt{d}\cdot \|(\vec{x}_1,\vec{y}_1) - (\vec{x}_2,\vec{y}_2)\|_\infty \\ &\qquad \leq T\cdot\sqrt{d}\cdot\binom{d}{\lfloor d/2 \rfloor}\cdot\sqrt{d}\cdot\|(\vec{x}_1,\vec{y}_1) - (\vec{x}_2,\vec{y}_2)\|_\infty \\ &\qquad = d\cdot\binom{d}{\lfloor d/2 \rfloor}\cdot T\cdot\|(\vec{x}_1,\vec{y}_1) - (\vec{x}_2,\vec{y}_2)\|_\infty.\end{aligned}$$

We obtain the Lipschitz estimate

$$\|Tb^{\varepsilon}_{k,d}(\vec{x}_1,\vec{y}_1)-Tb^{\varepsilon}_{k,d}(\vec{x}_2,\vec{y}_2)\|_\infty \le KT\cdot\|(\vec{x}_1,\vec{y}_1)-(\vec{x}_2,\vec{y}_2)\|_\infty, \tag{7-4}$$

where

$$K=K(d):=d\cdot\binom{d}{\lfloor d/2\rfloor}\le\sqrt{d}\cdot 2^d.$$

We now apply the Lipschitz counting principle from Section 3. Fix $T\ge 1$, so

$$\lceil KT\rceil\le KT+1\le (K+1)T.$$

Since $Tb^{\varepsilon}_{k,d}$ satisfies the Lipschitz estimate (7-4), the image under $Tb^{\varepsilon}_{k,d}$ of any translate of $[0,1/\lceil KT\rceil]^d$ is contained in a unit cube in $\mathbb{R}^{d+1}$.

Let $Q^{\varepsilon}_{k,d}(T)$ denote the number of d-cubes of side length $1/\lceil KT\rceil$ required to cover $\mathcal{J}^{\varepsilon}_{k,d}$. The easiest way to get an estimate for this quantity would be to note that each $\mathcal{J}$ is contained in a cube of side length $2\cdot\binom{d}{\lfloor d/2\rfloor}$. However, we can do significantly better than this without too much effort, using the bounds on the individual coordinates (coefficients) from Lemma 2.1.

Using (2-2), we see that $\mathcal{J}^{\varepsilon}_{k,d}$ is contained in the cuboid

$$\left\{(x_1,\dots,x_k,y_0,\dots,y_{d-k-1})\in\mathbb{R}^k\times\mathbb{R}^{d-k}\;\middle|\;|x_\ell|\le\binom{k}{\ell},\ |y_m|\le\binom{d-k}{m},\ \forall\ell,m\right\},$$

and therefore $\mathcal{J}^{\varepsilon}_{k,d}$ can be covered by

$$\prod_{\ell=1}^{k}2\binom{k}{\ell}\cdot\prod_{m=0}^{d-k-1}2\binom{d-k}{m}=2^dP(k)\cdot P(d-k)$$

unit d-cubes. Hence surely,

$$Q^{\varepsilon}_{k,d}(T)\le 2^dP(k)P(d-k)\lceil KT\rceil^d\le 2^dP(k)P(d-k)((K+1)T)^d. \tag{7-5}$$

Using Theorem 3.3 we conclude that

$$\begin{aligned}|\mathcal{M}(\le d,T)-\mathrm{vol}(\mathcal{U}_d)T^{d+1}| &\le 2^{d+1}\sum_{k,\varepsilon}Q^{\varepsilon}_{k,d}(T)\\ &\le 2^{d+1}\cdot 2\sum_{k=0}^{d}2^dP(k)P(d-k)(K+1)^dT^d\\ &=4^{d+1}A(d)(K+1)^dT^d=\kappa_0(d)T^d.\end{aligned}$$

We now estimate $\kappa_0(d)$ as in the statement of the theorem, using Lemma A.1 from the appendix:

$$\begin{aligned}\kappa_0(d) &= 4^{d+1}A(d)\Big(d\binom{d}{\lfloor d/2\rfloor}+1\Big)^d\\ &\le 4^{d+1}A(d)\Big(2d\binom{d}{\lfloor d/2\rfloor}\Big)^d\\ &\le 4^{d+1}A(d)\Big(\frac{2e}{\pi}\sqrt{d}\,2^d\Big)^d\\ &\le \big(40\sqrt[4]{2}\pi^{3/4}e^{-3}\big)d^{-1/4}(4\sqrt{2}e^{3/2}\pi^{-3/2})^d(2\sqrt{e})^{d^2}\\ &= a\frac{b^d c^{d^2}}{\sqrt[4]{d}} \le a(bc)^{d^2} = 40\sqrt[4]{2}\pi^{3/4}e^{-3}\cdot(8\sqrt{2}\pi^{-3/2}e^2)^{d^2}\\ &\le 5.59\cdot(15.01)^{d^2},\end{aligned}$$

where $a = 40\sqrt[4]{2}\pi^{3/4}e^{-3}$, $b = 4\sqrt{2}e^{3/2}\pi^{-3/2}$, and $c = 2\sqrt{e}$. □

Remark 7.2. As each $\mathcal{J}^{\varepsilon}_{k,d}$ is measurable, it follows that for each d,

$$Q^{\varepsilon}_{k,d}(T) \sim \mathrm{vol}(\mathcal{J}^{\varepsilon}_{k,d})\cdot((K+1)T)^d, \quad \text{as } T\to\infty. \tag{7-6}$$

Notice that

$$\mathrm{vol}(\mathcal{J}^{\varepsilon}_{k,d}) = p_k(1)\cdot p_{d-k}(1),$$

where $p_d(T)$ is as defined in (2-5). The sharpest way to proceed would be to explicitly estimate the error in (7-6). Comparing (7-6) with (7-5): how much does $\mathrm{vol}(\mathcal{J}^{\varepsilon}_{k,d})$ differ from $2^d P(k)P(d-k)$?

8. Counting monic polynomials: explicit bounds

Let $\mathcal{W}_{d,T}$ denote the subset of $\mathbb{R}^d$ corresponding to monic polynomials of degree d in $\mathbb{R}[z]$ with Mahler measure at most T, i.e.,

$$\mathcal{W}_{d,T} = \{\vec{w} = (w_1,\dots,w_d)\in\mathbb{R}^d \mid \mu(1,\vec{w})\le T\}.$$

We want to estimate the number of lattice points $\mathcal{M}_1(d,T)$ in this region. Note that, in the notation of the introduction, we have $\mathcal{M}_1(d,T) = \mathcal{M}(d,(1),(),T)$. Recall that the volume of $\mathcal{W}_{d,T}$ is given by the Chern-Vaaler polynomial $p_d(T)$, as defined in (2-5).

We define, for d a nonnegative integer,

$$B(d) = \sum_{k=0}^{d-1} P(k)P(d-k)\gamma(k)^{d-k-1}\gamma(d-k)^k,$$

where P is as defined in (7-1), and $\gamma(k) := \binom{k}{\lfloor k/2\rfloor}$.

Theorem 8.1. *For all $d \geq 2$ and $T \geq 1$,*

$$|\mathcal{M}_1(d,T) - p_d(T)| \leq \kappa_1(d) T^{d-1},$$

where

$$\kappa_1(d) = 4^d d^{d-1} B(d) \leq 4^d d^{d-1} 2^{d^2}.$$

Proof. Our starting point is the parametrization of the boundary $\partial W_{d,T}$ given in Section 2, which consists of the patches described in (2-9) and (2-10). As opposed to the previous proof, we'll need to be a bit more careful in our application of Theorem 3.3. Instead of a Lipschitz estimate of the form

$$\|\text{output}_1 - \text{output}_2\|_\infty \leq [\text{constant}] \cdot \|\text{input}_1 - \text{input}_2\|_\infty,$$

we'll estimate each component of the parametrization separately, which will lead to an argument where the parameter space is tiled by "rectangles" instead of "squares." We fix $k \in \{0, \ldots, d-1\}$ and $\varepsilon \in \{\pm 1\}$, and set $\mathcal{L} = \mathcal{L}_{k,d}^{\varepsilon T}$. We write

$$\beta_{k,d}^{\varepsilon T}(\vec{x}, \vec{y}) = (1, g_1(\vec{x}, \vec{y}), \ldots, g_d(\vec{x}, \vec{y})).$$

We have

$$\begin{aligned}|g_i(\vec{x}_1, \vec{y}_1) - g_i(\vec{x}_2, \vec{y}_2)| &\leq \sup_{(\vec{x},\vec{y}) \in \mathcal{L}} \bigl|\nabla g_i(\vec{x}, \vec{y}) \cdot ((\vec{x}_1, \vec{y}_1) - (\vec{x}_2, \vec{y}_2))\bigr| \\ &\leq \sup_{(\vec{x},\vec{y}) \in \mathcal{L}} \left(\sum_{\ell=1}^{k} \left| \frac{\partial g_i}{\partial x_\ell}(\vec{x}, \vec{y}) \right| |x_{1,\ell} - x_{2,\ell}| + \sum_{m=1}^{d-k-1} \left| \frac{\partial g_i}{\partial y_m}(\vec{x}, \vec{y}) \right| |y_{1,m} - y_{2,m}| \right).\end{aligned}$$

By (2-2), if $(\vec{x}, \vec{y}) \in \mathcal{L}$, then we must have $|x_\ell| \leq \binom{k}{\ell} \leq \gamma(k)$, for each $\ell = 1, \ldots, k$, and $|y_m| \leq T\binom{d-k}{m}$, for each $m = 1, \ldots, d-k-1$. Now notice that each partial derivative $\partial g_i / \partial x_\ell$, as a function, is either equal to 1, εT, or $y_{i-\ell}$, and thus has absolute value at most $T\binom{d-k}{i-\ell} \leq T\gamma(d-k)$. By the same token, each $\partial g_i / \partial y_m$ is equal to either 1 or x_{i-m}, and thus has absolute value at most $\binom{k}{i-m} \leq \gamma(k)$. Applying this to the inequality above gives

$$\begin{aligned}&|g_i(\vec{x}_1, \vec{y}_1) - g_i(\vec{x}_2, \vec{y}_2)| \\ &\qquad \leq k\gamma(d-k)T\|\vec{x}_1 - \vec{x}_2\|_\infty + (d-k-1)\gamma(k)\|\vec{y}_1 - \vec{y}_2\|_\infty. \qquad (8\text{-}1)\end{aligned}$$

Suppose for the moment that $0 < k < d-1$. Now if $\frac{1}{p} + \frac{1}{q} = 1$, and if

$$\|\vec{x}_1 - \vec{x}_2\|_\infty \leq \frac{1}{pk\gamma(d-k)T}$$

and

$$\|\vec{y}_1 - \vec{y}_2\|_\infty \leq \frac{1}{q(d-k-1)\gamma(k)},$$

then (8-1) will give

$$|g_i(\vec{x}_1, \vec{y}_1) - g_i(\vec{x}_2, \vec{y}_2)| \le 1.$$

So, if $\mathcal{P}$ is a cube in $\mathbb{R}^k$ with sides parallel to the axes and side length

$$\frac{1}{\lceil p\gamma(d-k)kT\rceil}, \tag{8-2}$$

and if $\mathcal{Q}$ is a cube in $\mathbb{R}^{d-k-1}$ with sides parallel to the axes and side length

$$\frac{1}{\lceil q(d-k-1)\gamma(k)\rceil}, \tag{8-3}$$

then $\beta^{\varepsilon T}_{k,d}(\mathcal{P} \times \mathcal{Q})$ is contained in a unit d-cube with sides parallel to the axes in $\mathbb{R}^d$. If $k = 0$, we take $q = 1$ in (8-3), and $\beta^{\varepsilon T}_{k,d}(\mathcal{Q})$ is contained in a unit d-cube with sides parallel to the axes in $\mathbb{R}^d$. Similarly, if $k = d-1$, then we take $p = 1$ in (8-2), and we have the same result for $\beta^{\varepsilon T}_{k,d}(\mathcal{P})$.

This is the first part of preparing to apply Theorem 3.3. We let $R^{\varepsilon}_{k,d}(T)$ denote the minimum number of such "rectangles" $\mathcal{P} \times \mathcal{Q}$ required to cover $\mathcal{L}$. As we argued in the previous section for the sets $\mathcal{J}^{\varepsilon}_{k,d}$, we see that $\mathcal{L}$ can be covered by

$$\prod_{\ell=1}^{k} 2\binom{k}{\ell} \cdot \prod_{m=1}^{d-k-1} 2T\binom{d-k}{m} = 2^{d-1}P(k)P(d-k)\cdot T^{d-k-1}$$

unit cubes. Since each unit cube can be covered by

$$\lceil pk\gamma(d-k)T\rceil^k \cdot \lceil q(d-k-1)\gamma(k)\rceil^{d-k-1}$$

of our rectangles,

$$R^{\varepsilon}_{k,d}(T) \le 2^{d-1}P(k)P(d-k)\lceil pk\gamma(d-k)T\rceil^k \cdot \lceil q(d-k-1)\gamma(k)\rceil^{d-k-1}T^{d-k-1},$$

for $0 < k < d-1$. Similarly, when $k = 0$,

$$R^{\varepsilon}_{k,d}(T) \le 2^{d-1}P(k)P(d-k)\cdot[(d-k-1)\gamma(k)]^{d-k-1}T^{d-k-1},$$

and when $k = d-1$,

$$R^{\varepsilon}_{k,d}(T) \le 2^{d-1}P(k)P(d-k)\cdot[k\gamma(d-k)T]^k T^{d-k-1}.$$

Following the proof in the previous section, by Theorem 3.3,

$$\begin{aligned}
&|\mathcal{M}_1(d,T)-p_d(T)| \\
&\quad \le \sum_{k,\varepsilon} 2^d R^{\varepsilon}_{k,d}(T) \\
&\quad \le 2^d \cdot 2 \sum_{k=0}^{d-1} 2^{d-1} P(k)P(d-k)\lceil pk\gamma(d-k)T\rceil^k \cdot \lceil q(d-k-1)\gamma(k)\rceil^{d-k-1} T^{d-k-1} \\
&\quad = 4^d \sum_{k=0}^{d-1} P(k)P(d-k)\lceil pk\gamma(d-k)T\rceil^k \cdot \lceil q(d-k-1)\gamma(k)\rceil^{d-k-1} T^{d-k-1},
\end{aligned}$$

where we understand that

$$\begin{cases} \lceil pk\gamma(d-k)T\rceil^k = 1 & \text{when } k=0, \\ \lceil q(d-k-1)\gamma(k)\rceil^{d-k-1} = 1 & \text{when } k=d-1, \end{cases}$$

and similarly below.

It will now be convenient to set

$$p = \frac{d-1}{k} \quad \text{and} \quad q = \frac{d-1}{d-k-1}.$$

Note that if $k=0$ we have $q=1$, and p does not appear; similarly if $k=d-1$ we have $p=1$, and q does not appear. We conclude our proof, assuming $T \ge 1$:

$$\begin{aligned}
&|\mathcal{M}_1(d,T)-p_d(T)| \\
&\quad \le 4^d \sum_{k=0}^{d-1} P(k)P(d-k)(pk+1)^k (q(d-k-1)+1)^{d-k-1}\gamma(k)^{d-k-1}\gamma(d-k)^k T^{d-1} \\
&\quad = 4^d \sum_{k=0}^{d-1} P(k)P(d-k)d^k d^{d-k-1}\gamma(k)^{d-k-1}\gamma(d-k)^k T^{d-1} \\
&\quad = 4^d d^{d-1} B(d) T^{d-1} = \kappa_1(d) T^{d-1}.
\end{aligned}$$

Finally, we note that $B(d) \le 2^{d^2}$ by Lemma A.2 from the appendix. □

9. Lattice points in slices: explicit bounds

The goal of this section is to prove a version of the lattice point-counting result Theorem 1.11 with an explicit error term, albeit with worse power savings — Theorem 9.3 stated below. As a byproduct of the proof, we also obtain an explicit version of our volume estimate Theorem 4.1. Our explicit version of Theorem 1.11 makes it possible to estimate the quantities in Corollaries 1.2 through 1.5 with explicit error terms.

We start with some notation. Fix $d, m, n, \vec{\ell}, \vec{r}$, and $T > 0$ as in Section 1, and again set $g = d - m - n$. Let $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^{g+1}$ denote the projection forgetting the first m and last n coordinates, given by

$$\pi(w_0, \ldots, w_d) = (w_m, \ldots, w_{d-n}).$$

Let $S(T)$ be as defined in (4-1). For $t \in [0, \infty)$, define W_t as in (4-2), and set

$$B_t := \pi(W_t \cap \mathcal{U}_d).$$

By (4-3),

$$\pi(S(T)) = \pi(T(W_{1/T} \cap \mathcal{U}_d)) = T\pi((W_{1/T} \cap \mathcal{U}_d)) = TB_{1/T}. \tag{9-1}$$

Also note that by (4-8),

$$\mathrm{vol}(B_0) = \mathrm{vol}_{g+1}(\mathcal{U}_d \cap W_0) = V_g. \tag{9-2}$$

For subsets A and A' of a common set, we use the usual notation for a symmetric difference $A \triangle A' = (A \cup A') \setminus (A \cap A')$. Note that for $T > 0$,

$$T(A \triangle A') = (TA) \triangle (TA'),$$

for any two subsets A and A' of a common euclidean space.

The following lemma is the main tool of this section. We postpone its proof until the end.

Lemma 9.1. *Let*

$$k_1 = k_1(d, \vec{\ell}, \vec{r}) := 2^{d^2} d^d (m+n) \|(\vec{\ell}, \vec{r})\|_\infty \quad \text{and} \quad \delta_T := (k_1/T)^{1/d}.$$

If $T \geq k_1$, then

$$\begin{aligned} B_0 \triangle B_{1/T} &\subseteq \{\vec{x} \in \mathbb{R}^{g+1} \mid 1 - \delta_T \leq \mu(\vec{x}) \leq 1 + \delta_T\} \\ &= [(1+\delta_T)\mathcal{U}_g] \setminus [(1-\delta_T)\mathcal{U}_g]. \end{aligned} \tag{9-3}$$

Using this result we take a brief detour to make the advertised explicit volume estimate. Compare the following with Theorem 4.1, in which we obtain a better power-savings in the error term, though in that theorem the error term is not made explicit.

Theorem 9.2. *Let $S(T) = S_{\vec{\ell}, \vec{r}}(T)$. If $T \geq k_1$, then*

$$|\mathrm{vol}_{g+1}(S(T)) - V_g T^{g+1}| \leq c T^{g+1-1/d},$$

where

$$c = c(d, \vec{\ell}, \vec{r}) = 2^{d+1}\big(((m+n)\|(\vec{\ell}, \vec{r})\|_\infty)^{1/d} \cdot d \cdot V_g\big).$$

Proof. Using (9-1) and (9-2),

$$\begin{aligned}\left|\frac{\mathrm{vol}_{g+1}(S(T))}{T^{g+1}} - V_g\right| & \\ &= |\mathrm{vol}(B_{1/T}) - \mathrm{vol}(B_0)| \le \mathrm{vol}(B_0 \triangle B_{1/T}) \\ &\le \mathrm{vol}(\{\vec{x} \in \mathbb{R}^{g+1} \mid 1-\delta_T \le \mu(\vec{x}) \le 1+\delta_T\}) \qquad \text{(by Lemma 9.1)} \\ &= 2\delta_T V_g = \frac{c}{T^{1/d}}. \qquad \square\end{aligned}$$

In Section 4 we estimated the volume of $S(T)$ in order to estimate the number of lattice points in that set. Here, by contrast, we actually don't require a volume estimate; Lemma 9.1 allows us to directly estimate the number of lattice points in $S(T)$, which we have denoted $\mathcal{M}(d, \vec{\ell}, \vec{r}, T)$, as follows.

Theorem 9.3. *Let $k_1 = k_1(d, \vec{\ell}, \vec{r})$ be as in Lemma 9.1, and κ_0 as defined in Theorem 7.1. For all $T \ge k_1$,*

$$|\mathcal{M}(d, \vec{\ell}, \vec{r}, T) - V_g \cdot T^{g+1}| \le \kappa(d, \vec{\ell}, \vec{r})(T^{g+1-1/d}),$$

where

$$\kappa(d, \vec{\ell}, \vec{r}) = (g+1)2^{g+1}k_1^{1/d} V_g + (g2^g k_1^{1/d} + 1)\kappa_0(g).$$

We note for later that $V_g \le 2 \cdot 15^{g^2}$ for all $g \ge 0$, and so

$$\begin{aligned}\kappa(d, \vec{\ell}, \vec{r}) &\le (g+1)2^{g+1}k_1^{1/d}(V_g + \kappa_0(g)) \\ &\le d(g+1)2^{d+g+1}(m+n)^{1/d}\|\vec{\ell}, \vec{r}\|_\infty (V_g + \kappa_0(g)) \qquad (9\text{-}4) \\ &\le (2+a)d(g+1)2^{d+g+1}(m+n)^{1/d}\|\vec{\ell}, \vec{r}\|_\infty (bc)^{g^2},\end{aligned}$$

where a, b, and c are the constants appearing in the end of the proof of Theorem 7.1 (note that $bc > 15$).

Proof. We let $Z(\Omega)$ denote the number integer lattice points in a subset Ω of euclidean space. Again applying (9-1),

$$\mathcal{M}(d, \vec{\ell}, \vec{r}, T) = Z(S(T)) = Z(\pi(S(T)) = Z(TB_{1/T}).$$

Also note that

$$Z(TB_0) = \mathcal{M}(\le g, T),$$

which we estimated in Section 7. Therefore, using the triangle inequality and Theorem 7.1,

$$
\begin{aligned}
|\mathcal{M}(d, \vec{\ell}, \vec{r}, T) - V_g \cdot T^{g+1}| \\
&= |Z(TB_{1/T}) - V_g \cdot T^{g+1}| \\
&\leq |Z(TB_{1/T}) - Z(TB_0)| + |Z(TB_0) - V_g \cdot T^{g+1}| \\
&\leq |Z(TB_{1/T}) - Z(TB_0)| + \kappa_0(g)T^g. \qquad (9\text{-}5)
\end{aligned}
$$

Clearly,

$$|Z(TB_{1/T}) - Z(TB_0)| \leq Z((TB_{1/T})\triangle(TB_0)) = Z(T(B_{1/T}\triangle B_0)),$$

and by Lemma 9.1,

$$T(B_{1/T}\triangle B_0) \subseteq [(T + T\delta_T)\mathcal{U}_g] \setminus [(T - T\delta_T)\mathcal{U}_g].$$

Hence, applying Theorem 7.1 a second time and using an elementary estimate from the mean value theorem, we find that

$$
\begin{aligned}
|Z(TB_{1/T}) - Z(TB_0)| \\
&\leq Z((T + T\delta_T)\mathcal{U}_g) - Z((T - T\delta_T)\mathcal{U}_g) \\
&\leq V_g[(T + T\delta_T)^{g+1} - (T - T\delta_T)^{g+1}]\kappa_0(g)[(T + T\delta_T)^g - (T - T\delta_T)^g] \\
&\leq V_g(g+1)(T + T\delta_T)^g(2T\delta_T) + \kappa_0(g)g(T + T\delta_T)^{g-1}(2T\delta_T).
\end{aligned}
$$

Recall that $\delta_T = k_1^{1/d}T^{-1/d}$. Assuming $T \geq k_1$ means that $\delta_T \leq 1$. Combining the estimate just obtained with (9-5), we achieve

$$
\begin{aligned}
|\mathcal{M}(d, \vec{\ell}, \vec{r}, T) - V_g \cdot T^{g+1}| \\
&\leq V_g(g+1)(2T)^g \cdot 2T^{1-1/d} \cdot k_1^{1/d} \\
&\quad + g\kappa_0(g)(2T)^{g-1} \cdot 2T^{1-1/d} \cdot k_1^{1/d} + \kappa_0(g)T^g \\
&\leq [(g+1)2^{g+1}k_1^{1/d}V_g + (g2^g k_1^{1/d} + 1)\kappa_0(g)]T^{g+1-1/d}. \qquad \square
\end{aligned}
$$

Proof of Lemma 9.1. We will require the following Lipschitz-type estimate for the Mahler measure [Chern and Vaaler 2001, Theorem 4], which is a quantitative form of the continuity of Mahler measure:

Theorem 9.4 (Chern–Vaaler). *For any $\vec{w}_1, \vec{w}_2 \in \mathbb{R}^{d+1}$,*

$$|\mu(\vec{w}_1)^{1/d} - \mu(\vec{w}_2)^{1/d}| \leq 2\|\vec{w}_1 - \vec{w}_2\|_1^{1/d}, \qquad (9\text{-}6)$$

where $\|\vec{w}\|_1 = \sum_{i=0}^{d}|w_i|$ is the usual ℓ^1-norm of a vector $\vec{w} = (w_0, \ldots, w_d) \in \mathbb{R}^{d+1}$.

If $\mu(\vec{w}_1)$ and $\mu(\vec{w}_2)$ are both less than some constant k, then (9-6) yields

$$\begin{aligned}|\mu(\vec{w}_1)-\mu(\vec{w}_2)| &= |\mu(\vec{w}_1)^{1/d}-\mu(\vec{w}_2)^{1/d}|\cdot\sum_{i=1}^{d}(\mu(\vec{w}_1)^{(d-i)/d}\mu(\vec{w}_2)^{(i-1)/d}) \\ &\le 2\|\vec{w}_1-\vec{w}_2\|_1^{1/d}\cdot dk^{(d-1)/d}. \end{aligned} \tag{9-7}$$

We will shortly apply this observation with $k=2^d$. We assume $T\ge k_1$.

Let $\vec{x}$ be a vector in $B_0\triangle B_{1/T}$, and write

$$\vec{x}_0=\tau(\vec{x})=(\vec{0}_m,\vec{x},\vec{0}_n)\in\mathbb{R}^{d+1} \quad\text{and}\quad \vec{x}_T=\left(\frac{\vec{\ell}}{T},\vec{x},\frac{\vec{r}}{T}\right)\in\mathbb{R}^{d+1}.$$

Notice that $\mu(\vec{x}_0)=\mu(\vec{x})$ because τ preserves Mahler measure, as noted in the proof of Theorem 4.1.

Since $\vec{x}\in B_0\triangle B_{1/T}$, it's clear that either

$$\mu(\vec{x}_0)\le 1<\mu(\vec{x}_T). \tag{9-8}$$

or

$$\mu(\vec{x}_T)\le 1<\mu(\vec{x}_0). \tag{9-9}$$

must hold. In either case,

$$1-|\mu(\vec{x}_0)-\mu(\vec{x}_T)|\le\mu(\vec{x}_0)\le 1+|\mu(\vec{x}_0)-\mu(\vec{x}_T)| \tag{9-10}$$

First, suppose $\vec{x}$ is in B_0, but not in $B_{1/T}$, so (9-8) holds. Then, by (2-3) and our assumption that $T\ge k_1$,

$$\begin{aligned}\mu(\vec{x}_T) &\le \|\vec{x}_T\|_\infty\sqrt{d+1}\le\max\{\|\vec{x}_0\|_\infty,1\}\sqrt{d+1} \\ &\le\binom{d}{\lfloor d/2\rfloor}\sqrt{d+1}\max\{\mu(\vec{x}_0),1\}\le 2^d, \end{aligned} \tag{9-11}$$

as in the statement of the proposition. Here we have used that $\binom{d}{\lfloor d/2\rfloor}\sqrt{d+1}\le 2^d$; see, for example, [Bombieri and Gubler 2006, Lemma 1.6.12]. Note that the second inequality in (9-11) follows because $T\ge\|(\vec{\ell},\vec{r})\|_\infty$. On the other hand, if $\vec{x}$ is in $B_{1/T}$, but not in B_0, so that (9-9) holds, then by applying (2-3) again, we have, in the same fashion as before:

$$\mu(\vec{x}_0)\le\|\vec{x}\|_\infty\sqrt{g+1}\le\max\{\|\vec{x}_T\|_\infty,1\}\sqrt{d+1}\le\max\{\mu(\vec{x}_T),1\}\le 2^d.$$

Since in either case we have that both $\mu(\vec{x}_0)$ and $\mu(\vec{x}_T)$ are at most 2^d, we may apply (9-7) to achieve

$$|\mu(\vec{x}_0)-\mu(\vec{x}_T)|\le 2\|\vec{x}_0-\vec{x}_T\|_1^{1/d}\cdot d(2^d)^{(d-1)/d}. \tag{9-12}$$

Note that

$$\|\vec{x}_0 - \vec{x}_T\|_1 = \sum_{i=0}^{m-1} |\ell_i|/T + \sum_{i=d-n+1}^{d} |r_i|/T \le (m+n)\|(\vec{\ell},\vec{r})\|_\infty/T,$$

which, combined with (9-12), yields

$$|\mu(\vec{x}_0) - \mu(\vec{x}_T)| \le \delta_T.$$

Now we combine with (9-10), and conclude that $1-\delta_T \le \mu(\vec{x}) \le 1+\delta_T$. This completes our justification of (9-3), which concludes our proof of Lemma 9.1. □

10. Reducible and imprimitive polynomials

In this section we begin to transfer our explicit counts for polynomials of degree at most d to explicit counts for algebraic numbers of degree d, by counting their minimal polynomials. In most cases, this simply means bounding the number of reducible polynomials, because the hypotheses imposed in Theorem 1.1 don't allow for any irreducible polynomials to be counted other than minimal polynomials of degree d. We'll apply a version of Hilbert's irreducibility theorem to achieve the most general bound, which will finish off the proof of Theorem 1.1. However, in various special cases we work a little harder to improve the power savings.

In the one case we consider outside the hypotheses of Theorem 1.1, namely polynomials with no coefficients fixed, we must also address the presence of imprimitive degree d polynomials and lower-degree polynomials.

Several times in our arguments we use the following estimate: if $a \ge 2$, then

$$\sum_{k=1}^{K} a^k = \frac{a^{K+1}-a}{a-1} \le \frac{a^{K+1}}{a/2} = 2a^K. \tag{10-1}$$

We write

$$P(d) := \prod_{j=0}^{d} \binom{d}{j}, \quad \text{for } d \ge 0,$$

and

$$C_{m,n}(d) := \prod_{j=m}^{d-n} \left(2\binom{d}{j}+1\right), \quad \text{for } 0 \le m+n \le d.$$

All polynomials. Let $\mathcal{M}(d,T)$ denote the number of integer polynomials of degree *exactly* d and Mahler measure at most T, and let $\mathcal{M}^{\text{red}}(d,T)$ denote the number of such polynomials that are reducible. Recall that $\mathcal{M}(\le d,T)$ denotes the number of integer polynomials of degree *at most* d and Mahler measure at most T. By (2-2), for all $d \geq 0$ and $T > 0$,

$$\mathcal{M}(d,T) \leq \mathcal{M}(\leq d, T) \leq C_{0,0}(d)T^{d+1} \leq c_0 2^{d+1} P(d) T^{d+1}, \tag{10-2}$$

where $c_0 = 3159/1024$, using Lemma A.3 from the appendix.

Proposition 10.1. *We have*

$$\mathcal{M}^{\mathrm{red}}(d,T) \leq \begin{cases} 1758 \cdot T^2 \log T, & \text{if } d = 2 \text{ and } T \geq 2, \\ 16c_0^2 4^d P(d-1) \cdot T^d, & \text{if } d \geq 3 \text{ and } T \geq 1. \end{cases}$$

Proof. For a reducible polynomial f of degree d and Mahler measure at most T, there exist $1 \leq d_2 \leq d_1 \leq d-1$ such that $f = f_1 f_2$, where each f_i is an integer polynomial with $\deg(f_i) = d_i$. Of course we have $d = d_1 + d_2$. Let k be the unique integer such that $2^{k-1} \leq \mu(f_1) < 2^k$. We have $1 \leq k \leq K$, where $K = \lfloor \log T / \log 2 \rfloor + 1$, and $\mu(f_2) \leq 2^{1-k} T$.

Given such a pair (d_1, d_2), by (10-2) there are at most $c_0 2^{d_1+1} P(d_1) 2^{k(d_1+1)}$ choices of such an f_1, and at most $c_0 2^{d_2+1} P(d_2)(2^{1-k}T)^{d_2+1}$ choices for f_2. First assume that $d_1 > d_2$. We'll use below that $P(d_1)P(d_2)$ is always at most $P(d-1)$, by Lemma A.4 in the appendix. Summing over all possible k and applying (10-1), the number of pairs of polynomials is at most

$$\begin{aligned}
&\sum_{k=1}^{K} c_0 2^{d_1+1} P(d_1) c_0 2^{d_2+1} P(d_2) 2^{k(d_1+1)} (2^{1-k}T)^{d_2+1} \\
&\qquad = 4c_0^2 2^d P(d_1) P(d_2) (2T)^{d_2+1} \sum_{k=1}^{K} 2^{k(d_1-d_2)} \\
&\qquad \leq 4c_0^2 2^d P(d-1)(2T)^{d_2+1} [2 \cdot 2^{K(d_1-d_2)}] \\
&\qquad \leq 8c_0^2 2^d P(d-1)(2T)^{d_1+1} \\
&\qquad \leq 16c_0^2 2^d 2^{d_1} P(d-1) T^d.
\end{aligned}$$

If instead $d_1 = d_2 = d/2$, (so in particular d is even), then the first line above is at most

$$4c_0^2 2^d P(d-1)(2T)^{d_1+1} K.$$

In the case $d = 2$, note that for $T \geq 2$ we have $K \leq (2/\log 2)\log T$, and so

$$\begin{aligned}
\mathcal{M}^{\mathrm{red}}(2,T) &\leq 4c_0^2 2^2 P(1)(2T)^{1+1} K \leq 64 c_0^2 T^2 \frac{2}{\log 2} \log T \\
&= \frac{128 c_0^2}{\log 2} \cdot T^2 \log T \leq 1758 \cdot T^2 \log T.
\end{aligned}$$

Whenever $T \geq 1$ we have $K \leq 2T$, and thus for even $d \geq 4$,

$$\begin{aligned} 4c_0^2 2^d P(d-1)(2T)^{d_1+1} K &\leq 8c_0^2 2^d 2^{d_1} P(d-1) T^{d/2+1} \cdot 2T \\ &\leq 16 c_0^2 2^d 2^{d_1} P(d-1) T^d, \end{aligned}$$

so we have the same bound we had when we assumed $d_2 < d_1$.

Finally, for any $d \geq 3$, summing over the possible values of d_1 gives that

$$\begin{aligned} \mathcal{M}^{\mathrm{red}}(d,T) &\leq \sum_{d_1=\lceil d/2 \rceil}^{d-1} 16 c_0^2 2^d 2^{d_1} P(d-1) T^d \\ &\leq 16 c_0^2 2^d P(d-1) T^d \sum_{d_1=1}^{d-1} 2^{d_1} \\ &= 16 c_0^2 2^d P(d-1) T^d (2^d - 2) \\ &\leq 16 c_0^2 4^d P(d-1) \cdot T^d. \end{aligned}$$ □

We follow the proof of [Masser and Vaaler 2008, Lemma 2] in counting primitive polynomials, but we'll keep track of implied constants. For $n = 1, 2, \ldots,$ let $\mathcal{M}^n(\leq d, T)$ denote the number of *nonzero* integer polynomials of degree at most d and Mahler measure at most T, such that the greatest common divisor of the coefficients is n. We let $\mathcal{M}^n(d, T)$ denote the corresponding number of polynomials with degree *exactly* d, so $\mathcal{M}^1(d, T)$ is the number of primitive polynomials of degree d and Mahler measure at most T. Recall that $\kappa_0(d)$ is a function of d appearing in Theorem 7.1.

Theorem 10.2. *For all $d \geq 2$ and $T \geq 1$,*

$$\left| \mathcal{M}^1(d,T) - \frac{V_d}{\zeta(d+1)} T^{d+1} \right| \leq \left(\frac{V_d}{d} + 1 \right) T + \left(C_{0,0}(d-1) + \zeta(d)\kappa_0(d) \right) T^d,$$

where ζ is the Riemann zeta-function.

Proof. Being careful to account for the zero polynomial,

$$\mathcal{M}(\leq d, T) - 1 = \sum_{1 \leq n \leq T} \mathcal{M}^n(\leq d, T) = \sum_{1 \leq n \leq T} \mathcal{M}^1(\leq d, T/n).$$

By Möbius inversion (below we commit a sin of notation overloading and let μ denote the Möbius function), this tells us that

$$\mathcal{M}^1(\leq d, T) = \sum_{1 \leq n \leq T} \mu(n) [\mathcal{M}(\leq d, T/n) - 1].$$

Combining this with Theorem 7.1 and (10-2),

$$\begin{aligned}
&\Bigl|\mathcal{M}^1(d,T) - V_d T^{d+1} \sum_{1\le n\le T} \frac{\mu(n)}{n^{d+1}}\Bigr| \\
&\quad = \Bigl|\mathcal{M}^1(d,T) - \mathcal{M}^1(\le d,T) + \sum_{n=1}^{T} \mu(n)[\mathcal{M}(\le d, T/n) - 1] - V_d T^{d+1} \sum_{n=1}^{T} \frac{\mu(n)}{n^{d+1}}\Bigr| \\
&\quad \le \mathcal{M}^1(\le d-1,T) + \sum_{n=1}^{T} |\mu(n)| + \sum_{n=1}^{T} |\mathcal{M}(\le d, T/n) - V_d (T/n)^{d+1}| \\
&\quad \le \mathcal{M}(\le d-1,T) + T + \sum_{n=1}^{T} \kappa_0(d)(T/n)^d \\
&\quad \le C_{0,0}(d-1)T^d + T + \kappa_0(d)T^d \sum_{n=1}^{T} \frac{1}{n^d} \\
&\quad \le T + \bigl(C_{0,0}(d-1) + \zeta(d)\kappa_0(d)\bigr)T^d.
\end{aligned}$$

This in turn gives

$$\begin{aligned}
&\Bigl|\mathcal{M}^1(d,T) - \frac{V_d}{\zeta(d+1)} T^{d+1}\Bigr| \\
&\qquad\qquad \le V_d T^{d+1} \sum_{n=T+1}^{\infty} n^{-(d+1)} + T + \bigl(C_{0,0}(d-1) + \zeta(d)\kappa_0(d)\bigr)T^d \\
&\qquad\qquad \le \Bigl(\frac{V_d}{d} + 1\Bigr)T + \bigl(C_{0,0}(d-1) + \zeta(d)\kappa_0(d)\bigr)T^d,
\end{aligned}$$

by applying the integral estimate

$$\sum_{n=T+1}^{\infty} n^{-(d+1)} \le d^{-1}T^{-d}.$$

This establishes the theorem. □

Monic polynomials. Next, let $\mathcal{M}_1(d,T)$ denote the number of monic integer polynomials of degree d and Mahler measure at most T, and let $\mathcal{M}_1^{\mathrm{red}}(d,T)$ denote the number of such polynomials that are reducible. Using (2-2), for all $d \ge 0$ and $T > 0$,

$$\mathcal{M}_1(d,T) \le C_{1,0}(d)T^d \le c_1 2^d P(d) T^d,$$

where $c_1 = \frac{1053}{512}$, from Lemma A.3 in the appendix.

We'll assume $d \ge 2$. In estimating the number of reducible monic polynomials, we follow the pattern of the proof of Proposition 10.1, noting that if a *monic* polynomial is reducible, its factors can be chosen to be monic. Using the same notation as in that proof, we have that the number of pairs of monic polynomials of degree d_1 and d_2, with $d_1 > d_2$, is at most

$$\begin{aligned}\sum_{k=1}^{K} c_1 2^{d_1} P(d_1) c_1 2^{d_2} P(d_2) 2^{kd_1} (2^{1-k}T)^{d_2} &= c_1^2 2^d P(d_1) P(d_2) (2T)^{d_2} \sum_{k=1}^{K} 2^{k(d_1-d_2)} \\ &\le 2c_1^2 2^d 2^{d_1} P(d-1) T^{d-1}.\end{aligned}$$

Noting that

$$\frac{16c_1^2}{\log 2} < 98,$$

we continue almost exactly as in Proposition 10.1 and obtain the following.

Proposition 10.3. *We have*

$$\mathcal{M}_1^{\mathrm{red}}(d,T) \le \begin{cases} 98 \cdot T \log T, & \text{if } d = 2 \text{ and } T \ge 2, \\ 2c_1^2 4^d P(d-1) \cdot T^{d-1}, & \text{if } d \ge 3 \text{ and } T \ge 1. \end{cases}$$

Monic polynomials with given final coefficient. Next we want to bound the number of reducible, monic, integer polynomials with fixed constant coefficient. For r a nonzero integer, let $\mathcal{M}^{\mathrm{red}}(d, (1), (r), T)$ denote the number of reducible monic polynomials with constant coefficient r, degree d, and Mahler measure at most T. Using (2-2), we have for all $d \ge 0$ and $T > 0$ that

$$\mathcal{M}(d, (1), (r), T) \le C_{1,1}(d) T^{d-1} \le c_2 2^{d-1} P(d) T^{d-1},$$

where $c_2 = \frac{351}{256}$, from Lemma A.3 in the appendix.

Let $\omega(r)$ denote the number of positive divisors of r. We'll assume $d > 2$; if $d = 2$, we easily have the constant bound $\mathcal{M}^{\mathrm{red}}(d, (1), (r), T) \le \omega(r) + 1$.

For a polynomial f counted by $\mathcal{M}^{\mathrm{red}}(d, (1), (r), T)$, there exist $1 \le d_2 \le d_1 \le d-1$ such that $f = f_1 f_2$, where each f_i is an integer polynomial with $\deg(f_i) = d_i$, and of course the constant coefficient of f is the product of those of f_1 and f_2. Define k as in the previous two cases. Given such a pair (d_1, d_2), summing over the $2\omega(r)$ possibilities for the final coefficient of f_1 there are at most $2\omega(r) c_2 2^{d_1-1} P(d_1) 2^{k(d_1-1)}$ choices of such an f_1, and then at most $c_2 2^{d_2-1} P(d_2) (2^{1-k}T)^{d_2-1}$ choices for f_2. The rest proceeds essentially as before, and we find that:

Proposition 10.4. *For $T \ge 1$,*

$$\mathcal{M}^{\mathrm{red}}(d, (1), (r), T) \le \begin{cases} \omega(r) + 1, & \text{if } d = 2, \\ \frac{1}{2}\omega(r) c_2^2 4^d P(d-1) \cdot T^{d-2}, & \text{if } d \ge 3. \end{cases}$$

Monic polynomials with a given second coefficient. For our next case, we want to bound the number of reducible, monic, integer polynomials with a given second leading coefficient. Let $\mathcal{M}^{\mathrm{red}}(d, (1, t), (), T)$ denote the number of reducible monic polynomials of degree $d \geq 3$ (we'll treat $d = 2$ separately at the end) with integer coefficients, second leading coefficient equal to t, and Mahler measure at most T.

Proposition 10.5. *For all $t \in \mathbb{Z}$,*

$$\mathcal{M}^{\mathrm{red}}(d, (1, t), (\,), T) \leq \begin{cases} \frac{1}{2}\sqrt{t^2 + 4T} + 1, & \text{if } d = 2 \text{ and } T \geq 1, \\ \dfrac{96}{\log 2} \cdot T \log T, & \text{if } d = 3 \text{ and } T \geq 2, \\ d2^{2d-1} P(d-1) \cdot T^{d-2}, & \text{if } d \geq 4 \text{ and } T \geq 1. \end{cases}$$

Proof. As before, we write such a polynomial as $f = f_1 f_2$, with

$$f_1(z) = z^{d_1} + x_1 z^{d_1 - 1} + \cdots x_{d_1} \quad \text{and} \quad f_2(z) = z^{d_2} + y_1 z^{d_2 - 1} + \cdots y_{d_2}.$$

Also as before, we enforce $1 \leq d_2 \leq d_1 \leq d - 1$ to avoid double-counting, and we define k as in the previous three cases. For $1 \leq i \leq d_1$ and $1 \leq j \leq d_2$,

$$|x_i| \leq \binom{d_1}{i} 2^k \quad \text{and} \quad |y_j| \leq \binom{d_2}{j} 2^{1-k} T. \tag{10-3}$$

We also, of course, have

$$x_1 + y_1 = t. \tag{10-4}$$

First assume $d_1 > d_2 + 1$. Observe that the number of integer lattice points (x_1, y_1) in $[-M_1, M_1] \times [-M_2, M_2]$ such that $x_1 + y_1 = t$ is at most $2\min\{M_1, M_2\} + 1$. So the number of $(x_1, \ldots, x_{d_1}, y_1, \ldots, y_{d_2})$ satisfying (10-3) and (10-4) is at most

$$\begin{aligned}
\bigl(2\min\{d_1 2^k, d_2 2^{1-k} T\} + 1\bigr) &\prod_{j=2}^{d_1} \Bigl[2\binom{d_1}{j} 2^k + 1\Bigr] \cdot \prod_{j=2}^{d_2} \Bigl[2\binom{d_2}{j} 2^{1-k} T + 1\Bigr] \\
&\leq \bigl(2\min\{d_1 2^k, d_2 2^{1-k} T\} + 1\bigr) \cdot C_{2,0}(d_1) 2^{k(d_1 - 1)} \cdot C_{2,0}(d_2)(2^{1-k} T)^{d_2 - 1} \\
&\leq (2d \cdot 2^{1-k} T)(2T)^{d_2 - 1} 2^{k(d_1 - d_2)} \cdot 2^{d_1 - 1} P(d_1) \cdot 2^{d_2 - 1} P(d_2) \\
&\leq d2^{d-1} P(d-1)(2T)^{d_2} 2^{k(d_1 - d_2 - 1)},
\end{aligned} \tag{10-5}$$

using Lemma A.3. Summing over all the possibilities $1 \leq k \leq K$, the number of possible pairs f_1 and f_2 of degrees d_1 and d_2, respectively, is at most

$$\begin{aligned}
d2^{d-1} P(d-1)(2T)^{d_2} \sum_{k=1}^{K} 2^{(d_1 - d_2 - 1)k} &\leq d2^{d-1} 2^{d_2} P(d-1) T^{d_2} [2 \cdot 2^{K(d_1 - d_2 - 1)}] \\
&\leq d2^{d-1} 2^{d_1} P(d-1) T^{d-2}.
\end{aligned} \tag{10-6}$$

Now, if $d_1 = d_2 = d/2$ (in this case d must be even), then the geometric sum above becomes $\sum_{k=1}^{K} 2^{-k} \leq 1$. So for $d \geq 4$ again we obtain the estimate (10-6) we achieved assuming $d_1 > d_2 + 1$. If $d_1 = d_2 + 1$ (so d is odd), then the number of possible pairs is at most $d2^{d-1}P(d-1)(2T)^{d_2}K$, which does not exceed (10-6) for $d \geq 5$, and for $d=3$, $T \geq 2$ is at most

$$3 \cdot 2^{3-1}P(2)(2T)^1 \frac{2\log T}{\log 2} = \frac{96}{\log 2} \cdot T \log T,$$

which gives us the $d=3$ case of the proposition. Finally, for $d \geq 4$ we sum over the at most $d/2$ possibilities for (d_1, d_2), yielding

$$\mathcal{M}^{\mathrm{red}}(d, (1, t), (\,), T) \leq d2^{2d-1}P(d-1)T^{d-2}.$$

For the case $d=2$, we'll see that the error term is on the order of $\sqrt{T}$. Note that we are simply counting integers c such that the polynomial

$$f(z) = (z^2 + tz + c) = (z + x_1)(z + y_1)$$

has Mahler measure at most T. Since we know $|c| \leq T$, it suffices to control the size of $\{x_1 \in \mathbb{Z} \mid |x_1(t - x_1)| \leq T\}$, which is itself bounded by the size of $\{x_1 \in \mathbb{Z} \mid x_1^2 - tx_1 \leq T\}$. By the quadratic formula, that last set is simply

$$\left\{x_1 \in \mathbb{Z} \;\middle|\; \frac{t-\sqrt{t^2+4T}}{2} \leq x_1 \leq \frac{t+\sqrt{t^2+4T}}{2}\right\},$$

which has size at most $\sqrt{t^2+4T}+1$. To better bound the number of c of the form $x_1(t-x_1)$, note that such a c can be written in this form for exactly two values of x_1, except for at most one value of c for which x_1 is unique (this occurs when t is even). So overall, the number of such c with $|c| \leq T$ is at most $\frac{1}{2}\sqrt{t^2+4T}+1$. □

Monic polynomials with given second and final coefficient. For our final case, we want to bound the number of monic, reducible polynomials with a given second leading coefficient $t \in \mathbb{Z}$ and given constant coefficient $0 \neq r \in \mathbb{Z}$. We can clearly assume that $d \geq 3$ since we're imposing three coefficient conditions. We write $\mathcal{M}^{\mathrm{red}}(d, (1, t), (r), T)$ for the number of reducible monic polynomials of degree d with integer coefficients, second leading coefficient equal to t, and constant coefficient equal to r. We'll show this is $O(T^{d-3})$ in all cases. While we don't write an explicit bound for the error term, it should be clear from our proof that this is possible.

Proposition 10.6. *For all $d \geq 3$, $t \in \mathbb{Z}$, and $r \in \mathbb{Z} \setminus \{0\}$,*

$$\mathcal{M}^{\mathrm{red}}(d, (1, t), (r), T) = O(T^{d-3}).$$

Proof. As before, we write such a polynomial as $f = f_1 f_2$, with

$$f_1(z) = z^{d_1} + x_1 z^{d_1-1} + \cdots x_{d_1} \quad \text{and} \quad f_2(z) = z^{d_2} + y_1 z^{d_2-1} + \cdots y_{d_2}.$$

We always enforce $1 \leq d_2 \leq d_1 \leq d-1$ to avoid double-counting. We'll consider the count in several different cases. First, if $d_2 = 1$, then $f_2 = z + y_{d_2}$, so we must have $y_{d_2} \mid r$ and $y_{d_2} + x_1 = t$. Thus there are only $2\omega(r)$ possible choices of f_2;

each choice will in turn determine x_{d_1} and x_1, so we have $O(T^{d_1-2}) = O(T^{d-3})$ choices of f_1 altogether, by Theorem 1.11. Note that this completely covers the case $d = 3$.

Now assume $d_2 \geq 2$, so $d \geq 4$. There are again only $2\omega(r)$ possible choices of y_{d_2}, and each one will determine what x_{d_1} is (they must multiply to give r). Fix a choice of y_{d_2} for now.

Assume first that $d_1 > d_2 + 1$. Again take k between 1 and $K = \lfloor \log T / \log 2 \rfloor + 1$, and assume that $2^{k-1} \leq \mu(f_1) \leq 2^k$, so $\mu(f_2) \leq 2^{1-k}T$. Almost exactly as in (10-5), we get that the number of $(x_1, \dots, x_{d_1-1}, y_1, \dots, y_{d_2-1})$ contributing to $\mathcal{M}^{\mathrm{red}}(d, (1, t), (r), T)$ is at most

$$\begin{aligned}
\bigl(2\min\{d_1 2^k, d_2 2^{1-k}T\} + 1\bigr) \cdot \prod_{i=2}^{d_1-1} \Bigl[2\binom{d_1}{i}2^k + 1\Bigr] \cdot \prod_{j=2}^{d_2-1} \Bigl[2\binom{d_2}{j}(2^{1-k}T) + 1\Bigr]& \\
\leq (2d \cdot 2^{1-k}T) \cdot 2^{k(d_1-2)} C_{2,1}(d_1) \cdot (2^{1-k}T)^{d_2-2} C_{2,1}(d_2)& \\
= d 2^{d_2} C_{2,1}(d_1) C_{2,1}(d_2) T^{d_2-1} 2^{(d_1-d_2-1)k}& \\
\leq \frac{1}{64} d 2^d 2^{d_2} P(d-1) T^{d_2-1} 2^{(d_1-d_2-1)k}&,
\end{aligned}$$

using Lemmas A.3 and A.4. Summing over all the possibilities $1 \leq k \leq K$, the number of possible pairs f_1 and f_2 of degrees d_1 and d_2, respectively, is at most

$$\begin{aligned}
\frac{1}{64} d 2^d 2^{d_2} P(d-1) T^{d_2-1} \sum_{k=1}^{K} 2^{(d_1-d_2-1)k} &\leq \frac{1}{32} d 2^d 2^{d_1} P(d-1) T^{d_1-2} \\
&\leq \frac{1}{32} d 2^d 2^{d_1} P(d-1) T^{d-3}, \qquad \text{(10-7)}
\end{aligned}$$

which is certainly $O(T^{d-3})$.

Next, if $d_1 = d_2 = d/2$ (in this case d must be even), then the expression in (10-7), which contains a partial geometric sum that's bounded by 1, is at most

$$\frac{1}{64} d 2^d 2^{d_2} P(d-1) T^{d/2-1},$$

which is certainly $O(T^{d-3})$ since $d \geq 4$. Lastly, if $d_1 = d_2 + 1$, (so $d \geq 5$), then $d_2 \leq d - 3$, and (using $K \leq 2T$) the expression in (10-7) is at most

$$\begin{aligned}
\frac{1}{64} d 2^d 2^{d_2} P(d-1) T^{d_2-1} K &\leq \frac{1}{32} d 2^d 2^{d_2} P(d-1) T^{d_2} \\
&\leq \frac{1}{32} d 2^d 2^{d_2} P(d-1) T^{d-3},
\end{aligned}$$

which is $O(T^{d-3})$. Finally, we sum over the $2\omega(r)$ possibilities for y_{d_2} and the at most $d/2$ possibilities for (d_1, d_2) and obtain overall that

$$\mathcal{M}^{\text{red}}(d, (1,t), (r), T) = O(T^{d-3}). \qquad \square$$

11. Explicit results

Let $N(\overline{\mathbb{Q}}_d, \mathcal{H})$ denote the number of algebraic numbers of degree d over $\mathbb{Q}$ and height at most $\mathcal{H}$. We give an explicit version of the main theorem of Masser and Vaaler [2008], which follows from Theorem 7.1, our explicit version of [Chern and Vaaler 2001, Theorem 3].

Theorem 11.1. *For all $d \geq 2$ and $\mathcal{H} \geq 1$,*

$$\left| N(\overline{\mathbb{Q}}_d, \mathcal{H}) - \frac{dV_d}{2\zeta(d+1)} \mathcal{H}^{d(d+1)} \right| \leq \begin{cases} 16690 \cdot \mathcal{H}^4 \log \mathcal{H}, & \text{if } d = 2 \text{ and } \mathcal{H} \geq \sqrt{2}, \\ 3.37 \cdot (15.01)^{d^2} \cdot \mathcal{H}^{d^2}, & \text{if } d \geq 3 \text{ and } \mathcal{H} \geq 1. \end{cases}$$

Proof. We combine Proposition 10.1 and Theorem 10.2 to estimate the number of irreducible, primitive (i.e., having relatively prime coefficients) polynomials of degree d and Mahler measure at most $\mathcal{H}^d$; we write $\mathcal{M}^{\text{irr, prim}}(d, \mathcal{H}^d)$ for this number. Each pair of such a polynomial and its opposite corresponds to d algebraic numbers of degree d and height at most $\mathcal{H}$ (the roots). So

$$N(\overline{\mathbb{Q}}_d, \mathcal{H}) = \frac{d}{2} \mathcal{M}^{\text{irr, prim}}(d, \mathcal{H}^d),$$

and

$$\begin{aligned} &\left| N(\overline{\mathbb{Q}}_d, \mathcal{H}) - \frac{dV_d}{2\zeta(d+1)} \mathcal{H}^{d(d+1)} \right| \\ &\quad \leq \left| \frac{d}{2} \mathcal{M}^{\text{irr, prim}}(d, \mathcal{H}^d) - \frac{d}{2} \mathcal{M}^1(d, \mathcal{H}^d) \right| + \left| \frac{d}{2} \mathcal{M}^1(d, \mathcal{H}^d) - \frac{dV_d}{2\zeta(d+1)} \mathcal{H}^{d(d+1)} \right| \\ &\quad \leq \frac{d}{2} \Big(\mathcal{M}^{\text{red}}(d, \mathcal{H}^d) + \left| \mathcal{M}^1(d, \mathcal{H}^d) - \frac{V_d}{\zeta(d+1)} \mathcal{H}^{d(d+1)} \right| \Big), \end{aligned}$$

and it follows from Proposition 10.1 and Theorem 10.2 that

$$\begin{aligned} \left(\frac{d}{2}\right)^{-1} &\left| N(\overline{\mathbb{Q}}_d, \mathcal{H}) - \frac{dV_d}{2\zeta(d+1)} \mathcal{H}^{d(d+1)} \right| \\ &\leq \left(\frac{V_d}{d} + 1 \right) \mathcal{H}^d + \big(C_{0,0}(d-1) + \zeta(d)\kappa_0(d) \big) \mathcal{H}^{d^2} \\ &\qquad + \begin{cases} 1758 \mathcal{H}^4 \log(\mathcal{H}^2), & \text{if } d = 2 \text{ and } \mathcal{H}^2 \geq 2, \\ 16 c_0^2 4^d P(d-1) \mathcal{H}^{d^2}, & \text{if } d \geq 3 \text{ and } \mathcal{H}^2 \geq 1. \end{cases} \end{aligned}$$

Here $\kappa_0(d)$ is the constant from Theorem 7.1, and $c_0 = 3159/1024$. The $d = 2$ case of our theorem follows immediately, as

$$\left(\frac{V_2}{2}+1\right)+C_{0,0}(1)+\zeta(2)\kappa_0(2)+2\cdot 1758=\left(\frac{8}{2}+1\right)+8000\zeta(2)+9+3516 < 16690.$$

We now turn to $d \geq 3$, where

$$\left|N(\overline{\mathbb{Q}}_d,\mathcal{H})-\frac{dV_d}{2\zeta(d+1)}\mathcal{H}^{d(d+1)}\right|\leq \theta_0(d)\cdot\mathcal{H}^{d^2},$$

with

$$\begin{aligned}\theta_0(d)&=\frac{d}{2}\left(1+\frac{V_d}{d}+\zeta(d)\kappa_0(d)+C_{0,0}(d-1)+16c_0^2 4^d P(d-1)\right)\\&=\left[\zeta(d)+\frac{1}{\kappa_0(d)}+\frac{V_d}{d\kappa_0(d)}+\frac{C_{0,0}(d-1)}{\kappa_0(d)}+\frac{16c_0^2 4^d P(d-1)}{\kappa_0(d)}\right]\frac{d\kappa_0(d)}{2}.\end{aligned}$$

Note that the quantity in brackets above decreases for $d \geq 3$ (for this it may be helpful to consult Lemma 2.2 and compute a few values of V_d) and so is no more than

$$\lambda_0 := \zeta(3)+\frac{1}{\kappa_0(3)}+\frac{V_3}{3\kappa_0(3)}+\frac{C_{0,0}(2)}{\kappa_0(3)}+\frac{16c_0^2 4^3 P(2)}{\kappa_0(3)}.$$

So, using the notation of the end of the proof of Theorem 7.1,

$$\begin{aligned}\left|N(\overline{\mathbb{Q}}_d,\mathcal{H})-\frac{dV_d}{2\zeta(d+1)}\mathcal{H}^{d(d+1)}\right|&\leq \theta_0(d)\cdot\mathcal{H}^{d^2}\leq\lambda_0\frac{d\kappa_0(d)}{2}\cdot\mathcal{H}^{d^2}\\&\leq\frac{\lambda_0}{2}ad^{3/4}b^d c^{d^2}\cdot\mathcal{H}^{d^2}\leq\frac{a\lambda_0}{2}(bc)^{d^2}\cdot\mathcal{H}^{d^2}\\&\leq 3.37\cdot(15.01)^{d^2}\cdot\mathcal{H}^{d^2}.\qquad\square\end{aligned}$$

Next, we record an explicit version of [Barroero 2014, Theorem 1.1] in the case $k = \mathbb{Q}$, i.e., an explicit estimate for the number of algebraic integers of bounded height and given degree over $\mathbb{Q}$. This explicit estimate follows from our Theorem 8.1, which improved the power savings of [Chern and Vaaler 2001, Theorem 6]. We write $N(\mathcal{O}_d, \mathcal{H})$ for the number of algebraic integers of degree d over $\mathbb{Q}$ and height at most $\mathcal{H}$.

Theorem 11.2. *We have*

$$\left|N(\mathcal{O}_d,\mathcal{H})-d\cdot p_d(\mathcal{H}^d)\right|\leq\begin{cases}584\cdot\mathcal{H}^2\log\mathcal{H}, & \text{if } d=2 \text{ and } \mathcal{H}\geq\sqrt{2},\\ 1.13\cdot 4^d d^d 2^{d^2}\cdot\mathcal{H}^{d(d-1)}, & \text{if } d\geq 3 \text{ and } \mathcal{H}\geq 1.\end{cases}$$

Proof. We follow the idea of the previous proof. Now that we require polynomials to be monic, we never count two irreducible polynomials with the same set of roots, and so combining Theorem 8.1 and Proposition 10.3 we obtain:

$$
\begin{aligned}
&d^{-1}\left|N(\mathcal{O}_d,\mathcal{H}) - d\cdot p_d(\mathcal{H}^d)\right| \\
&\qquad\qquad \le \kappa_1(d)\mathcal{H}^{d(d-1)} + \begin{cases} 98\mathcal{H}^2\log(\mathcal{H}^2), & \text{if } d=2 \text{ and } \mathcal{H}^2\ge 2,\\ 2c_1^2 4^d P(d-1)\mathcal{H}^{d(d-1)}, & \text{if } d\ge 3 \text{ and } \mathcal{H}^2 \ge 1,\end{cases}
\end{aligned}
$$

where $c_1 = 1053/512$. We immediately have the $d=2$ case of our theorem, as $\kappa_1(2)=96$. Assuming $d\ge 3$,

$$\left|N(\mathcal{O}_d,\mathcal{H}) - d\cdot p_d(\mathcal{H}^d)\right| \le \theta_1(d)\cdot\mathcal{H}^{d(d-1)},$$

where

$$\theta_1(d) = d\kappa_1(d) + 2c_1^2 d4^d P(d-1) = d\kappa_1(d)\left[1+\frac{2c_1^2 4^d P(d-1)}{\kappa_1(d)}\right].$$

The quantity in brackets decreases for $d\ge 3$, and so is no more than

$$\lambda_1 := 1+\frac{2c_1^2 4^3 P(2)}{\kappa_1(3)} \le 1.13,$$

and the result follows from the estimate for $\kappa_1(d)$ stated in Theorem 8.1. □

We can also prove an explicit version of our Corollary 1.3, albeit with worse power savings.

Theorem 11.3. *For each $d\ge 2$, ν a nonzero integer, and $\mathcal{H}\ge d\cdot 2^{d+1/d}|\nu|^{1/d}$,*

$$
\begin{aligned}
&\left|\mathcal{N}_{\mathrm{Nm}=\nu}(d,\mathcal{H}) - dV_{d-2}\cdot\mathcal{H}^{d(d-1)}\right| \\
&\qquad\qquad \le \begin{cases} (64\sqrt{2|\nu|}+8)\cdot\mathcal{H} + 2\omega(\nu)+2, & \textit{if } d=2,\\ 0.0000063|\nu|\omega(\nu)\cdot d^3 4^d (15.01)^{d^2}\cdot\mathcal{H}^{d(d-1)-1}, & \textit{if } d\ge 3,\end{cases}
\end{aligned}
$$

where $\omega(\nu)$ is the number of positive integer divisors of ν.

Proof. Our proof proceeds very similarly to the last two. Let $r=(-1)^d\nu$. Using Theorem 9.3 and Proposition 10.4, we have for all $\mathcal{H}\ge d\cdot 2^{d+1/d}|\nu|^{1/d}$:

$$
\begin{aligned}
&d^{-1}\left|\mathcal{N}_{\mathrm{Nm}=\nu}(d,\mathcal{H}) - d\cdot V_{d-2}\cdot\mathcal{H}^{d(d-1)}\right| \\
&\qquad\qquad \le \kappa\big(d,(1),(r)\big)\mathcal{H}^{d(d-1-1/d)} + \begin{cases} \omega(r)+1, & \text{if } d=2,\\ \dfrac{1}{2}\omega(r)c_2^2 4^d P(d-1)\cdot\mathcal{H}^{d(d-2)}, & \text{if } d\ge 3,\end{cases}
\end{aligned}
$$

where $\kappa(d,(1),(r))$ is as defined in Theorem 9.3, and

$$c_2 = \frac{351}{256}.$$

Consider the case $d=2$. By definition (stated in Theorem 9.3)

$$\begin{aligned}\kappa(2,(1),(r)) &= (0+1)2^{0+1}\big[2^4\cdot 2^2(1+1)|r|\big]^{1/2}V_0+(0+1)\kappa_0(0)\\ &= 32\sqrt{2|r|}+4,\end{aligned}$$

using $V_0=2$ and $\kappa_0(0)=4$. Therefore,

$$\begin{aligned}\big|\mathcal{N}_{\mathrm{Nm}=\nu}(2,\mathcal{H})-2\cdot V_0\cdot\mathcal{H}^2\big| &\le 2\big((32\sqrt{2|r|}+4)\mathcal{H}+\omega(r)+1\big)\\ &= (64\sqrt{2|r|}+8)\cdot\mathcal{H}+2\omega(r)+2.\end{aligned}$$

Now we assume $d\ge 3$, so

$$\big|\mathcal{N}_{\mathrm{Nm}=\nu}(d,\mathcal{H})-d\cdot V_{d-2}\cdot\mathcal{H}^{d^2-d}\big|\le\theta_2(d,r)\mathcal{H}^{d^2-d-1},$$

where, using (9-4) and letting a, b, and c be as in the end of the proof of Theorem 7.1,

$$\begin{aligned}\theta_2(d,r) &= d\big(\kappa(d,(1),(r))+\tfrac{1}{2}\omega(r)c_2^2 4^d P(d-1)\big)\\ &\le d\cdot(2+a)d(d-1)2^{2d-1+1/d}|r|(bc)^{(d-1)^2}+\frac{d}{2}\omega(r)c_2^2 4^d P(d-1)\\ &\le d^3 2^{2d-1}|r|\omega(r)(bc)^{d^2}\left[\frac{(2+a)d(d-1)2^{1/d}}{(bc)^{2d-1}\omega(r)d^2}+\frac{c_2^2P(d-1)}{d^2(bc)^{d^2}|r|}\right]\\ &\le d^3 2^{2d-1}|r|\omega(r)(bc)^{d^2}\left[\frac{(2+a)2^{1/d}}{(bc)^{2d-1}}+\frac{c_2^2P(d-1)}{d^2(bc)^{d^2}}\right].\end{aligned}$$

As the quantity in brackets just above decreases for $d\ge 3$, it does not exceed

$$\frac{(2+a)2^{1/3}}{(bc)^5}+\frac{c_2^2P(2)}{3^2(bc)^9}\le 0.0000126,$$

completing our proof. □

We can immediately state the following explicit unit count, since counting units amounts to counting algebraic integers of norm ± 1.

Theorem 11.4. *For each $d\ge 2$ and $\mathcal{H}\ge d\cdot 2^{d+1/d}$,*

$$\big|N(\mathcal{O}_d^*,\mathcal{H})-2dV_{d-2}\cdot\mathcal{H}^{d(d-1)}\big| \le \begin{cases}(128\sqrt{10})\mathcal{H}+8, & \text{if } d=2,\\ 0.0000126\cdot d^3 4^d(15.01)^{d^2}\cdot\mathcal{H}^{d(d-1)-1}, & \text{if } d\ge 3.\end{cases}$$

Finally, since Proposition 10.5 gives an explicit bound, it is also possible to obtain an explicit estimate for $\mathcal{N}_{\mathrm{Tr}=\tau}(d,\mathcal{H})$ similar to that of Theorem 11.3; we leave this to the interested reader.

Appendix: Combinatorial estimates

This appendix contains estimates for the combinatorial functions appearing in some of the constants in this paper. For any integer $d \geq 0$, define

$$\begin{aligned}
P(d) &:= \prod_{j=0}^{d} \binom{d}{j}, \\
C_{m,n}(d) &:= \prod_{j=m}^{d-n} \left(2\binom{d}{j}+1\right), \quad \text{for } 0 \leq m+n \leq d, \\
A(d) &:= \sum_{k=0}^{d} P(k)P(d-k), \\
B(d) &:= \sum_{k=0}^{d-1} P(k)P(d-k)\gamma(k)^{d-k-1}\gamma(d-k)^{k},
\end{aligned}$$

where $\gamma(k) := \binom{k}{\lfloor k/2 \rfloor}$.

Stirling's inequality is the following estimate for factorials, which we will use several times:

$$\sqrt{2\pi} \cdot k^{k+1/2}e^{-k} \leq k! \leq e \cdot k^{k+1/2}e^{-k}, \quad \text{for all } k \geq 1. \tag{A-1}$$

Using this we can easily see that

$$\gamma(k) \leq \frac{e \cdot 2^k}{\pi\sqrt{k}}. \tag{A-2}$$

Lemma A.1. *For all $d \geq 1$,*

$$A(d) \leq (10\sqrt[4]{2}\pi^{3/4}e^{-3})e^{d^2/2+d}(2\pi)^{-d/2}d^{-d/2-1/4}.$$

Proof. We write

$$\Phi(d) := \sqrt{\frac{e^{d^2+d}}{(2\pi)^d d!}}.$$

Note that of course the first and last factor appearing in the product $P(d)$ are 1, so they may be omitted when convenient. Also notice that

$$P(d) = \prod_{k=1}^{d} \frac{k^k}{k!}.$$

Using Stirling's inequality,

$$P(d) = \prod_{j=1}^{d} \frac{j^j}{j!} \leq \prod_{j=1}^{d} \frac{e^j}{\sqrt{2\pi j}} = \frac{\exp\left(\frac{1}{2}(d^2+d)\right)}{\sqrt{2\pi}^d \sqrt{d!}} = \sqrt{\frac{e^{d^2+d}}{(2\pi)^d d!}}. \tag{A-3}$$

Therefore,

$$P(d) \leq \Phi(d), \quad \text{for all } d \geq 0. \tag{A-4}$$

Now, for all $d \geq 1$,

$$\begin{aligned} A(d) &= \sum_{k=0}^{d} P(k)P(d-k) \leq \sum_{k=0}^{d} \Phi(k)\Phi(d-k) \\ &= \sum_{k=0}^{d} \sqrt{\frac{e^{k^2+k}}{(2\pi)^k k!}} \cdot \sqrt{\frac{e^{(d-k)^2+d-k}}{(2\pi)^{d-k}(d-k)!}} \\ &= \Phi(d) \sum_{k=0}^{d} \sqrt{\binom{d}{k}} e^{k^2-dk} = \Phi(d)\left(2 + \sum_{k=1}^{d-1} \sqrt{\binom{d}{k}} e^{k^2-dk}\right). \end{aligned} \tag{A-5}$$

Now, since $k^2 - dk = -k(d-k) \leq -(d-1)$ when $1 \leq k \leq d-1$, we can easily estimate the sum

$$\sum_{k=1}^{d-1} \sqrt{\binom{d}{k}} e^{k^2-dk} \leq 2^d \cdot e^{1-d} = e \cdot \left(\frac{2}{e}\right)^d. \tag{A-6}$$

The interested reader will easily verify that

$$\frac{A(d)}{\Phi(d)} \leq \frac{A(2)}{\Phi(2)} = 10\pi\sqrt{2}e^{-3} \approx 2.21198 \tag{A-7}$$

for $0 \leq d \leq 8$, and by (A-5) and (A-6), we can easily check that

$$\frac{A(d)}{\Phi(d)} \leq 2 + e \cdot \left(\frac{2}{e}\right)^d < 2.2$$

for $d \geq 9$.

Finally, we estimate $\Phi(d)$ using Stirling's inequality again:

$$\Phi(d) \leq \sqrt{\frac{e^{d^2+d}}{(2\pi)^d} \cdot \frac{e^d}{\sqrt{2\pi d} \cdot d^d}} = e^{d^2/2+d}(2\pi d)^{-d/2-1/4}. \tag{A-8}$$

Combining with (A-7) completes the proof. □

Lemma A.2. *For all $d \geq 0$,*

$$B(d) \leq 2^{d^2}.$$

Proof. We can readily verify the inequality for $d \le 3$, so we'll assume below that $d \ge 4$, and proceed by induction. Suppose that

$$B(d-1) \le 2^{(d-1)^2}.$$

Notice that

$$P(d) = \frac{d^d}{d!} P(d-1), \tag{A-9}$$

and also that $\gamma(d) \le 2\gamma(d-1)$ for all $d \ge 1$. We also easily have $P(d) \le e^{d^2/2+d}$ from the previous proof. Using these facts, we calculate:

$$\begin{aligned}
&B(d) - P(d-1) \\
&\quad = \sum_{k=0}^{d-2} P(k)P(d-k)\gamma(k)^{d-k-1}\gamma(d-k)^k \\
&\quad \le \sum_{k=0}^{d-2} P(k)\frac{(d-k)^{d-k}}{(d-k)!} P(d-k-1)\gamma(k)^{d-k-2}\gamma(k)2^k\gamma(d-k-1)^k \\
&\quad \le \sum_{k=0}^{d-2} \left[\frac{e^{d-k}2^k}{\sqrt{2\pi(d-k)}}\gamma(k+1)\right] P(k)P(d-k-1)\gamma(k)^{d-k-2}\gamma(d-k-1)^k \\
&\quad \le \sum_{k=0}^{d-2} \left[\frac{e^{d-k}2^k}{\sqrt{2\pi(d-k)}}\frac{e\cdot 2^{k+1}}{\pi\sqrt{k+1}}\right] P(k)P(d-k-1)\gamma(k)^{d-k-2}\gamma(d-k-1)^k \\
&\quad \le \sum_{k=0}^{d-2} \left[\frac{e\sqrt{2}}{\pi^{3/2}}\frac{e^d(4/e)^k}{\sqrt{(d-k)(k+1)}}\right] P(k)P(d-k-1)\gamma(k)^{d-k-2}\gamma(d-k-1)^k.
\end{aligned}$$

We note that $(d-k)(k+1) \ge d$ holds whenever $0 \le k \le d-2$, and continue the calculation:

$$\begin{aligned}
&B(d) - P(d-1) \\
&\qquad \le \left[\frac{e\sqrt{2}}{\pi^{3/2}} \cdot \frac{e^d(4/e)^d}{\sqrt{d}}\right] \sum_{k=0}^{d-2} P(k)P(d-1-k)\gamma(k)^{d-1-k-1}\gamma(d-1-k)^k \\
&\qquad = \left[\frac{e\sqrt{2}}{\pi^{3/2}} \cdot \frac{4^d}{\sqrt{d}}\right] B(d-1) \le \left[\frac{e\sqrt{2}}{\pi^{3/2}} \cdot \frac{4^d}{\sqrt{d}}\right] 2^{(d-1)^2} \\
&\qquad = \left[\frac{e\sqrt{2}}{\pi^{3/2}} \cdot \frac{4^d}{\sqrt{d}}\right] \frac{2}{4^d} 2^{d^2} = \left[\frac{e\cdot 2^{3/2}}{\pi^{3/2}\sqrt{d}}\right] 2^{d^2}
\end{aligned}$$

Therefore,

$$
\begin{aligned}
B(d) &\le P(d-1) + \left[\frac{e\cdot 2^{3/2}}{\pi^{3/2}\sqrt{d}}\right]2^{d^2} \\
&\le \left[\frac{P(d)d!}{d^d 2^{d^2}} + \frac{e\cdot 2^{3/2}}{\pi^{3/2}\sqrt{d}}\right]2^{d^2} \\
&\le \left[\frac{e^{\frac{1}{2}d^2+d}\cdot e\sqrt{d}}{e^d 2^{d^2}} + \frac{e\cdot 2^{3/2}}{\pi^{3/2}\sqrt{d}}\right]2^{d^2} \\
&= \left[e\sqrt{d}\left(\frac{\sqrt{e}}{2}\right)^{d^2} + \frac{e\cdot 2^{3/2}}{\pi^{3/2}\sqrt{d}}\right]2^{d^2} \\
&\le 2^{d^2},
\end{aligned}
$$

where the final inequality holds for any $d \ge 4$. □

Lemma A.3. *We have*

$$
\begin{aligned}
C_{0,0}(d) &\le \frac{3159}{1024}\cdot 2^{d+1}P(d), \quad \textit{for all } d \ge 0, && \text{(A-10)}\\
C_{1,0}(d) &\le \frac{1053}{512}\cdot 2^{d}P(d), \quad \textit{for all } d \ge 0,\\
C_{1,1}(d) &\le \frac{351}{256}\cdot 2^{d-1}P(d), \quad \textit{for all } d \ge 1,\\
C_{2,0}(d) &\le 2^{d-1}P(d), \quad \textit{for all } d \ge 1,\\
C_{2,1}(d) &\le \frac{1}{2}\cdot 2^{d-2}P(d), \quad \textit{for all } d \ge 2.
\end{aligned}
$$

Proof. We'll prove the bound for $C_{0,0}(d)$, and leave the other cases as exercises. The inequality (A-10) is easily verified for $d \le 3$, and we have equality for $d = 4$. If we set

$$
R(d) := \frac{C_{0,0}(d)}{2^{d+1}P(d)} = \prod_{j=0}^{d} \frac{2\binom{d}{j}+1}{2\binom{d}{j}},
$$

then to establish (A-10) it will suffice to show that

$$
\frac{R(d+1)}{R(d)} \le 1, \quad \text{for } d \ge 4.
$$

We'll use the standard identity

$$
\binom{d+1}{j} = \frac{d+1}{d+1-j}\binom{d}{j}.
$$

We have

$$
\begin{aligned}
\frac{R(d+1)}{R(d)} &= \left(\prod_{j=0}^{d+1} \frac{2\binom{d+1}{j}+1}{2\binom{d+1}{j}}\right) \Big/ \left(\prod_{j=0}^{d} \frac{2\binom{d}{j}+1}{2\binom{d}{j}}\right) \\
&= \frac{3}{2}\prod_{j=0}^{d} \frac{\binom{d}{j}}{\binom{d+1}{j}} \cdot \frac{2\binom{d+1}{j}+1}{2\binom{d}{j}+1} = \frac{3}{2}\prod_{j=0}^{d} \frac{d+1-j}{d+1} \cdot \frac{2\frac{d+1}{d+1-j}\binom{d}{j}+1}{2\binom{d}{j}+1} \\
&= \frac{3}{2}\prod_{j=0}^{d} \frac{2\binom{d}{j}+\frac{d+1-j}{d+1}}{2\binom{d}{j}+1} = \frac{3}{2}\prod_{j=0}^{d}\left[1-\frac{j}{(d+1)\big(2\binom{d}{j}+1\big)}\right] \\
&\le \frac{3}{2}\prod_{j=d-2}^{d}\left[1-\frac{j}{(d+1)\big(2\binom{d}{j}+1\big)}\right] \\
&= \frac{3}{2}\cdot\frac{4d^6+10d^5+6d^4+8d^3+20d^2+24d+18}{6d^6+15d^5+12d^4+9d^3+15d^2+12d+3} \\
&= \frac{2d^6+5d^5+3d^4+4d^3+10d^2+12d+9}{2d^6+5d^5+4d^4+3d^3+5d^2+4d+1} \le 1, \quad \text{for } d \ge 4. \qquad \square
\end{aligned}
$$

Lemma A.4. *If $d \ge 2$ and $1 \le k \le d-1$, then*

$$P(k)P(d-k) \le P(d-1).$$

Proof. We have

$$
\begin{aligned}
P(k)P(d-k) &= \prod_{j=0}^{k-1}\binom{k}{j}\prod_{i=0}^{d-k-1}\binom{d-k}{i} \le \prod_{j=0}^{k-1}\binom{d-1}{j}\prod_{i=0}^{d-k-1}\binom{d-1}{i} \\
&= \prod_{j=0}^{k-1}\binom{d-1}{j}\prod_{i=0}^{d-k-1}\binom{d-1}{d-1-i} = \prod_{j=0}^{k-1}\binom{d-1}{j}\prod_{j=k}^{d-1}\binom{d-1}{j} \\
&= P(d-1).
\end{aligned}
$$

We have equality if and only if $k=1$ or $k=d-1$. $\square$

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References

[Barroero 2014] F. Barroero, "Counting algebraic integers of fixed degree and bounded height", *Monatsh. Math.* **175**:1 (2014), 25–41. MR Zbl

[Barroero and Widmer 2014] F. Barroero and M. Widmer, "Counting lattice points and O-minimal structures", *Int. Math. Res. Not.* **2014**:18 (2014), 4932–4957. MR Zbl

[Bierstone and Milman 1988] E. Bierstone and P. D. Milman, "Semianalytic and subanalytic sets", *Inst. Hautes Études Sci. Publ. Math.* **1988**:67 (1988), 5–42. MR Zbl

[Bombieri and Gubler 2006] E. Bombieri and W. Gubler, *Heights in Diophantine geometry*, New Mathematical Monographs **4**, Cambridge University Press, Cambridge, 2006. MR Zbl

[Chambert-Loir and Tschinkel 2012] A. Chambert-Loir and Y. Tschinkel, "Integral points of bounded height on partial equivariant compactifications of vector groups", *Duke Math. J.* **161**:15 (2012), 2799–2836. MR Zbl

[Chern and Vaaler 2001] S.-J. Chern and J. D. Vaaler, "The distribution of values of Mahler's measure", *J. Reine Angew. Math.* **540** (2001), 1–47. MR Zbl

[Cohen 1981] S. D. Cohen, "The distribution of Galois groups and Hilbert's irreducibility theorem", *Proc. London Math. Soc.* (*3*) **43**:2 (1981), 227–250. MR Zbl

[Davenport 1951] H. Davenport, "On a principle of Lipschitz", *J. London Math. Soc.* **26** (1951), 179–183. MR Zbl

[Davenport 1964] H. Davenport, "Corrigendum: "On a principle of Lipschitz"", *J. London Math. Soc.* **39** (1964), 580. MR Zbl

[Gao 1995] X. Gao, "On Northcott's theorem", 1995, http://search.proquest.com/docview/304189207. MR

[Guignard 2017] Q. Guignard, "Counting algebraic points of bounded height on projective spaces", *J. Number Theory* **170** (2017), 103–141. MR Zbl

[Lang 1983] S. Lang, *Fundamentals of Diophantine geometry*, Springer, New York, 1983. MR Zbl

[Le Rudulier 2014] C. Le Rudulier, "Points algébriques de hauteur bornée sur la droite projective", *J. Théor. Nombres Bordeaux* **26**:3 (2014), 789–813. MR Zbl

[Mahler 1961] K. Mahler, "On the zeros of the derivative of a polynomial", *Proc. Roy. Soc. Ser. A* **264** (1961), 145–154. MR Zbl

[Mahler 1976] K. Mahler, *Lectures on transcendental numbers*, Lecture Notes in Mathematics **546**, Springer, Berlin, 1976. MR Zbl

[Masser and Vaaler 2007] D. Masser and J. D. Vaaler, "Counting algebraic numbers with large height, II", *Trans. Amer. Math. Soc.* **359**:1 (2007), 427–445. MR Zbl

[Masser and Vaaler 2008] D. Masser and J. D. Vaaler, "Counting algebraic numbers with large height, I", pp. 237–243 in *Diophantine approximation*, edited by H. P. Schlickewei et al., Dev. Math. **16**, Springer, Vienna, 2008. MR Zbl

[Schanuel 1979] S. H. Schanuel, "Heights in number fields", *Bull. Soc. Math. France* **107**:4 (1979), 433–449. MR Zbl

[Schmidt 1995] W. M. Schmidt, "Northcott's theorem on heights, II: The quadratic case", *Acta Arith.* **70**:4 (1995), 343–375. MR Zbl

[Sinclair 2008] C. D. Sinclair, "The range of multiplicative functions on $\mathbb{C}[x]$, $\mathbb{R}[x]$ and $\mathbb{Z}[x]$", *Proc. Lond. Math. Soc.* (*3*) **96**:3 (2008), 697–737. MR Zbl

[Spain 1995] P. G. Spain, "Lipschitz: a new version of an old principle", *Bull. London Math. Soc.* **27**:6 (1995), 565–566. MR Zbl

[Widmer 2009] M. Widmer, "Counting points of fixed degree and bounded height", *Acta Arith.* **140**:2 (2009), 145–168. MR Zbl

[Widmer 2016] M. Widmer, "Integral points of fixed degree and bounded height", *Int. Math. Res. Not.* **2016**:13 (2016), 3906–3943. MR

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Greatest common divisors of iterates of polynomials

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Following work of Bugeaud, Corvaja, and Zannier for integers, Ailon and Rudnick prove that for any multiplicatively independent polynomials, $a, b \in \mathbb{C}[x]$, there is a polynomial h such that for all n, we have

$$\gcd(a^n - 1, b^n - 1) \mid h$$

We prove a compositional analog of this theorem, namely that if $f, g \in \mathbb{C}[x]$ are compositionally independent polynomials and $c(x) \in \mathbb{C}[x]$, then there are at most finitely many λ with the property that there is an n such that $(x - \lambda)$ divides $\gcd(f^{\circ n}(x) - c(x), g^{\circ n}(x) - c(x))$.

Bugeaud, Corvaja, and Zannier [2003] obtained an upper bound for the greatest common divisors among two families of integer sequences. More precisely, let a and b be two positive integers that are multiplicatively independent and let $\epsilon > 0$ be given. Then for all n, we have $\gcd(a^n - 1, b^n - 1) \ll_\epsilon \exp(\epsilon n)$ where the implied constant is independent of n.

Since Bugeaud, Corvaja, and Zannier's paper appeared, there have been many extensions and generalizations of their results (see, for example, [Ailon and Rudnick 2004; Silverman 2004a; 2004b; 2005; Corvaja and Zannier 2005; 2008; 2013; Luca 2005; Murty and Murty 2011; Zannier 2012; Ghioca et al. 2017a; 2017b; Huang 2017]). In particular, Ailon and Rudnick [2004] obtained a stronger upper bound in the setting of function fields of characteristic zero. They showed that for two multiplicatively independent nonconstant polynomials $a, b \in \mathbb{C}[x]$, there is a polynomial $h \in \mathbb{C}[x]$, depending on a and b such that $\gcd(a^n - 1, b^n - 1) \mid h$ for all positive integer n. We note here that the result of Ailon and Rudnick also holds when one takes the greatest common divisors of $a^m - 1$ and $b^n - 1$ across all pairs of positive integers m and n (not merely those where $m = n$).

Instead of taking multiplicative powers of polynomials, one can consider iterated compositions of polynomials and look for an upper bound on the degrees of the

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greatest common divisors among two such sequences of polynomials as asked by A. Ostafe [2016, Problem 4.2]. In this paper, we prove a compositional analog of theorem of Ailon and Rudnick described above.

In the following, for a polynomial q, we let $q^{\circ n}$ denote the composition of q with itself n times. To state our theorem precisely, we need a definition of compositional independence.

Definition. We say two polynomials f and g are *compositionally independent* if the semigroup generated by f and g under composition is isomorphic to the free semigroup with two generators. This is equivalent to the property that whenever $i_1, \dots, i_s, j_1, \dots, j_s, \ell_1, \dots, \ell_t, m_1, \dots m_t$ are positive integers such that

$$f^{\circ i_1} \circ g^{\circ j_1} \circ \cdots \circ f^{\circ i_s} \circ g^{\circ j_s} = f^{\circ \ell_1} \circ g^{\circ m_1} \circ \cdots \circ f^{\circ \ell_t} \circ g^{\circ m_t},$$

we must have $s = t$, and $i_k = \ell_k$, $j_k = m_k$ for $k = 1, \dots, s$.

Under the compositional independence condition, our first result is the finiteness of the irreducible factors of $\gcd(f^{\circ m}(x) - c(x), g^{\circ n}(x) - c(x))$ where f, g and c are polynomials with complex coefficients. More precisely, we have the following theorem which answers Ostafe's question.

Theorem 1. *Let $f(x)$ and $g(x)$ be two compositionally independent polynomials in $\mathbb{C}[x]$, at least one of which has degree greater than one. Suppose that $c(x)$ is not a compositional power of f or g. Then there are at most finitely many $\lambda \in \mathbb{C}$ such that*

$$(x - \lambda) \mid \gcd(f^{\circ m}(x) - c(x), g^{\circ n}(x) - c(x))$$

for some positive integers m, n.

The restriction on the degrees of the two polynomials f and g in Theorem 1 is necessary. As the examples at the beginning of Section 3 demonstrate that Theorem 1 must be modified when f and g are both linear. If we restrict to the case $m = n$ in Theorem 1, then we still obtain a finiteness result when the two polynomials f and g are both linear.

Theorem 2. *Let f and g be two compositionally independent linear polynomials and let c be any polynomial. Then there is a polynomial $h \in \mathbb{C}[x]$ such that*

$$\gcd(f^{\circ n}(x) - c(x), g^{\circ n}(x) - c(x)) \mid h$$

for all positive integers n.

Putting Theorem 2 together with Theorem 1 under the condition that the composition power $m = n$, then for *any* polynomials $c(x)$ we have the same conclusion.

Theorem 3. *Let f and g be two compositionally independent polynomials. Then there are at most finitely many $\lambda \in \mathbb{C}$ such that*

$$(x-\lambda) \mid \gcd(f^{\circ n}(x)-c(x), g^{\circ n}(x)-c(x))$$

for some positive integer n.

We note that Theorem 2 is a compositional analog of Ailon and Rudnick's result for linear polynomials. To obtain a theorem that is parallel to their result for nonlinear polynomials, we need a bound for the multiplicity of each irreducible factor that divides the greatest common divisors. In general, one cannot expect such a bound exists. For instance, take $f(x)=x^3+x^2$, $g(x)=x^3+5x^2$ and $c=0$. Then, for any positive integer n, we have

$$x^{2^n} \mid \gcd(f^{\circ n}(x), g^{\circ n}(x))$$

Hence, in this case there does not exist a polynomial h divisible by all the greatest common divisors of the sequences in question. To get control on the bound of the multiplicities of irreducible factors dividing the greatest common multiples, we need one extra condition.

Definition. We say that $\xi \in \mathbb{C}$ is in a ramified cycle of a polynomial q if there is a positive integer i such that $q^{\circ i}(\xi)=\xi$ and $(q^{\circ i})'(\xi)=0$.

Once we exclude this sort of possibility, we are able to show that there exists a polynomial that is divisible by all the greatest common divisors of the compositional sequences formed by f and g.

Theorem 4. *Let $f(x)$ and $g(x)$ be two compositionally independent polynomials of degree greater than one in $\mathbb{C}[x]$. Suppose that $c(x)$ is not a compositional power of f or g. Suppose furthermore that $c(x)$ is not equal to a constant c that is in a ramified cycle of both f and g. Then there is a polynomial $h \in \mathbb{C}[x]$ such that*

$$\gcd(f^{\circ m}(x)-c(x), g^{\circ n}(x)-c(x)) \mid h$$

for all positive integers m and n.

Remark. (1) One might naturally ask if the theorems of this paper still hold when one replaces $\gcd(f^{\circ m}(x)-c(x), g^{\circ n}(x)-c(x))$ with

$$\gcd(f^{\circ m}(x)-c_1(x), g^{\circ n}(x)-c_2(x))$$

for any $c_1(x), c_2(x) \in \mathbb{C}[X]$ (with $c_1(x)$ and $c_2(x)$ not necessarily equal). The proof of Theorem 1 goes through without change to treat this generalization as long as $f(x), g(x), c_1(x), c_2(x) \in \overline{\mathbb{Q}}[x]$; we believe that the case of complex coefficients can probably also be treated with some additional work. We do not, however, have an idea of how to generalize Theorem 2 to treat the case where $c_1(x) \neq c_2(x)$.

(2) In the situation considered by Ailon and Rudnick, the number 1 is not in a ramified cycle of any powering map. In fact, any nonzero polynomial $c(x)$ is not in a ramified cycle of any powering map.

We give a brief description of the organization of our paper and explain the ideas of the proofs. In Section 1, we set up notations and provide some background about canonical height functions associated to rational maps on the projective line over a global field. After the preliminaries in Section 1, we begin to prove our results.

We prove Theorem 1 in Section 2. The proof is split into two parts. We first treat the case where neither f nor g is linear. This is done in Proposition 8. An additional ingredient is required for the case where one of f and g is linear; we treat this case separately in Proposition 9. Then Theorem 1 is just the combination of these two propositions. We sketch the proof of Proposition 8 here. Assume that the set of λ that are roots of

$$\gcd\bigl(f^{\circ m}(x) - c(x),\, g^{\circ n}(x) - c(x)\bigr)$$

is infinite as m and n run through all positive integers. Then these numbers have the property that the canonical heights $\hat{h}_f(\lambda)$ and $\hat{h}_g(\lambda)$ both converge to zero (see Lemma 6). Applying equidistribution theorems in arithmetic dynamics, following the pattern of [Ghioca and Tucker 2010; Baker and DeMarco 2011; Ghioca et al. 2015], we conclude that both polynomials f and g have the same Julia set in the complex plane. Then the work of Baker and Erëmenko [1987] and Schmidt and Steinmetz [1995] shows that a compositional relation between f and g exists. Thus we get a contradiction to the assumption that f and g are compositionally independent and finish the proof.

Section 3 is devoted to the proof of Theorem 2 and Theorem 3. The proof of Theorem 2 is quite different, as the tools used to prove Theorem 1 are no longer applicable to the case where both polynomials f and g are linear. The proof for this case relies heavily on diophantine methods, in particular an application of results from [Corvaja and Zannier 2005], Roth's theorem, and a lemma of Siegel. These results are used to prove the case where everything is defined over $\overline{\mathbb{Q}}$, in Proposition 15. The general case of Theorem 2 then follows via specialization. Theorem 3 follows easily by combining Theorem 1 and Theorem 2.

We prove Theorem 4 in Section 4. It is sufficient to bound the multiplicities of the roots of $\gcd(f^{\circ m}(x) - c(x), g^{\circ n}(x) - c(x))$ in Theorem 1 provided that $c(x)$ is not a constant in a ramified cycle of both f and g. The analysis on the bound of the multiplicity used here is similar to those used in [Morton and Silverman 1995, Lemma 3.4]. We provide such a bound in Lemma 16. Then, Theorem 4 follows from Theorem 1 coupled with Lemma 16. Finally, we end this paper by raising several questions for further study in Section 5.

1. Preliminaries

In this section, we set up some notations and recall facts from the theory of height functions that will be used in this paper.

Let K be a field of characteristic 0 equipped with a set of inequivalent absolute values (places) Ω_K, normalized so that the product formula holds. More precisely, for each $v \in \Omega_K$ there exists a positive integer N_v such that for all $\alpha \in K^*$ we have $\prod_{v\in\Omega} |\alpha|_v^{N_v} = 1$ where for $v \in \Omega_K$, the corresponding absolute value is denoted by $|\cdot|_v$. Examples of product formula fields (or *global fields*) are number fields and function fields of projective varieties which are regular in codimension 1 over another field k (see [Lang 1983, §2.3] or [Bombieri and Gubler 2006, §1.4.6]).

We let $\mathbb{C}_v$ be the completion of an algebraic closure of K_v, a completion of K with respect to $|\cdot|_v$. When v is an archimedean valuation, then $\mathbb{C}_v = \mathbb{C}$. We fix an extension of $|\cdot|_v$ to an absolute value of $\mathbb{C}_v$ which by abuse of notation, we still denote by $|\cdot|_v$.

If K is a number field, we let Ω_K be the set of all absolute values of K which extend the (usual) absolute values of $\mathbb{Q}$. For each $v \in \Omega_K$, we let v_0 denote the (unique) absolute value of $\mathbb{Q}$ such that $v|_{\mathbb{Q}} = v_0$ and we let $N_v := [K_v : \mathbb{Q}_{v_0}]$. If K is a function field of a projective normal variety $\mathcal{V}$ defined over a field k, then Ω_K is the set of all absolute values on K associated to the irreducible divisors of $\mathcal{V}$. Then there exist positive integers N_v (for each $v \in \Omega_K$) such that $\prod_{v\in\Omega_K} |x|_v^{N_v} = 1$ for each nonzero $x \in K$. (See [Lang 1983; Serre 1997] for more details).

Let L be a finite extension of K, and let Ω_L be the set of all absolute values of K which extend the absolute values in Ω_K. For each $w \in \Omega_L$ extending some $v \in \Omega_K$ we let $N_w := N_v \cdot [L_w : K_v]$. The (naive) Weil height of any point $x \in L$ is defined as

$$h(x) = \frac{1}{[L:K]} \sum_{w\in\Omega_L} N_w \cdot \log\max\{1, |x|_w\}.$$

To ease the notation, we set $\|x\|_v := |x|_v^{N_v}$ for $x \in K$.

Let $f \in K(x)$ be any rational map of degree $d \geq 2$. Then the *global canonical height* $\hat{h}_f(x)$ of $x \in \overline{K}$ associated to f is given by the limit

$$\hat{h}_f(x) = \lim_{n\to\infty} \frac{h(f^n(x))}{d^n}$$

(see [Call and Silverman 1993] for details). In addition, Call and Silverman proved that the global canonical height decomposes as a sum of the local canonical heights, i.e.,

$$\hat{h}_f(x) = \frac{1}{[K(x):K]} \sum_{\sigma:K(x)\to\overline{K}} \sum_{v\in\Omega_K} N_v \hat{h}_{f,v}(x^\sigma), \tag{4.1}$$

where σ runs through all embeddings of $K(x)$ into $\overline{K}$ and for each $v \in \Omega_K$ the function $\hat{h}_{f,v}$ is the local canonical height associated to f. For the existence and functorial property of the local canonical height see [Call and Silverman 1993, Theorem 2.1].

The following facts about height functions are well known.

Proposition 5. *Let $f \in K(x)$ be a rational function of degree $d \geq 2$ defined over K. There are constants c_1, c_2, c_3, and c_4, depending only on d, such that the following estimates hold for all $x \in \overline{K}$:*

(a) $\left|h(f(x)) - dh(x)\right| \leq c_1 h(f) + c_2$.

(b) $\left|\hat{h}_f(x) - h(x)\right| \leq c_3 h(f) + c_4$.

(c) $\hat{h}_f(f(x)) = d\hat{h}_f(x)$.

(d) *If K is a number field then $x \in \operatorname{PrePer}(f)$ if and only if $\hat{h}_f(x) = 0$.*

Here, $h(f)$ is the height of the polynomial f, see for example [Bombieri and Gubler 2006, §1.6] for the definition of $h(f)$.

Proof. See, for example, [Hindry and Silverman 2000, §B.2, §B.4] or [Silverman 2007, §3.4]. □

We use the following lemma (see also [Call and Silverman 1993; Ingram 2013] for more general techniques along these lines).

Lemma 6. *Let K be a global field. Let $(\lambda_n)_{n=1}^{\infty}$ be a sequence in $\overline{K}$ satisfying $f^{\circ n}(\lambda_n) = c(\lambda_n)$ for all n, where $f, c \in K[x]$ and $\deg f > 1$. Then*

$$\lim_{n\to\infty} \hat{h}_f(\lambda_n) = 0.$$

Proof. By Proposition 5 (b), the canonical height $\hat{h}_f(\cdot)$ associated to f is a height function on the projective line $\mathbb{P}^1$. It follows that

$$\hat{h}_f(c(\lambda)) = (\deg c)\hat{h}_f(\lambda) + O(1), \quad \text{for all } \lambda \in \overline{K}. \tag{6.1}$$

Since by assumption the sequence $(\lambda_n)_{n=1}^{\infty}$ satisfies $f^{\circ n}(\lambda_n) = c(\lambda_n)$ for all n, we have that $(\deg f)^n \hat{h}_f(\lambda_n) = \hat{h}_f(f^{\circ n}(\lambda_n)) = \hat{h}_f(c(\lambda_n))$ and thus

$$(\deg f)^n \hat{h}_f(\lambda_n) = (\deg c)\hat{h}_f(\lambda_n) + O(1), \quad \text{for all } n \in \mathbb{N}, \tag{6.2}$$

where the implied constant is independent of n.

Therefore, $((\deg f)^n - \deg c)\hat{h}_f(\lambda_n)$ is bounded by a constant independent of n. Since by assumption $\deg f > 1$, it's clear that $\hat{h}_f(\lambda_n)$ must go to zero as n goes to infinity. □

We now state a result about equalities of canonical heights.

Proposition 7. *Let K be a global field of characteristic zero and let $f, g \in K[x]$ be polynomials of degree greater than one. If there is an infinite nonrepeating sequence $(\lambda_i)_{i=1}^{\infty}$, where $\lambda_i \in \overline{K}$, such that*

$$\lim_{i\to\infty} \hat{h}_f(\lambda_i) = \lim_{i\to\infty} \hat{h}_g(\lambda_i) = 0,$$

then $\hat{h}_f = \hat{h}_g$.

Proof. In the case where K is a number field, this is proved in [Petsche et al. 2012, Theorem 3] and [Mimar 2013, Theorem 1.8]. The proof given in [Petsche et al. 2012] goes through for function fields without any changes. Proofs of similar equalities over function fields appear in [Baker and DeMarco 2011; 2013; Ghioca et al. 2011; 2015; Yuan and Zhang 2017], Thus, we only give a sketch here. The idea is to apply equidistribution results such as those in [Favre and Rivera-Letelier 2004; Baker and Rumely 2006; Chambert-Loir 2006], all of which hold over both number fields and function fields of characteristic 0. For each place v of K, the λ_i equidistribute with respect to the measures of maximal entropy $\mu_{f,v}$ and $\mu_{g,v}$ for f and g respectively at v. This implies that the local canonical heights $\hat{h}_{f,v}$ and $\hat{h}_{g,v}$ for f and g are equal to each other. By (4.1), the global canonical heights $\hat{h}_f$ and $\hat{h}_g$ are the sum of the corresponding local canonical heights. Therefore, $\hat{h}_f = \hat{h}_g$, as desired. □

2. Proof of Theorem 1

In this section we prove Theorem 1 by first treating the case where f and g both have degree greater than one.

Proposition 8. *Let $f(x)$ and $g(x)$ be two compositionally independent polynomials with complex coefficients of degree greater than one. Then there are at most finitely many $\lambda \in \mathbb{C}$ such that there are positive integers m and n with the following properties*:

(i) $f^{\circ m}(x) \neq c(x)$.

(ii) $g^{\circ n}(x) \neq c(x)$.

(iii) $(x-\lambda) \mid \gcd(f^{\circ m}(x) - c(x), g^{\circ n}(x) - c(x))$.

Proof. Let K be the field generated by all the coefficients of f, g, and c over $\mathbb{Q}$. Then either K is a number field or a function field of finite transcendence degree over $\overline{\mathbb{Q}}$. In the latter case, we let $k = K \cap \overline{\mathbb{Q}}$ be its field of constants.

We prove the proposition by contradiction. Suppose that there is an infinite nonrepeating sequence $(\lambda_i)_{i=1}^{\infty}$ such for every i, there is an m_i and n_i such that $f^{\circ m_i} \neq c$ and $g^{\circ n_i} \neq c$, and $(x-\lambda_i)$ divides both $f^{\circ m_i}(x) - c(x)$ and $g^{\circ n_i}(x) - c(x)$. We will show that the two polynomials f and g must be compositionally dependent.

Observe that for such m_i and n_i, the polynomials $f^{\circ m_i}(x)-c(x)$ and $g^{\circ n_i}(x)-c(x)$ have only finitely many roots, so m_i and n_i must both go to infinity as i goes to infinity. Then, by Lemma 6, we have

$$\lim_{i\to\infty} \hat{h}_f(\lambda_i) = \lim_{i\to\infty} \hat{h}_g(\lambda_i) = 0.$$

It follows from Proposition 7 that $\hat{h}_f = \hat{h}_g$.

Let $\Lambda_0 := \{\lambda \in \overline{K} \mid \hat{h}_f(\lambda) = 0\} = \{\lambda \in \overline{K} \mid \hat{h}_g(\lambda) = 0\}$. If K is a number field, then by Proposition 5 (d), we immediately conclude that f and g share the same set of preperiodic point. Likewise, if K is a function field and neither f nor g is isotrivial over k, then by [Benedetto 2005; Baker 2009], Proposition 5 (d) also holds and hence f and g also share the same set of preperiodic points.

Now assume that at least one of f and g is isotrivial. Without loss of generality, we assume that f is isotrivial. Since $\hat{h}_f = \hat{h}_g$, it follows from the weak Northcott property of [Baker 2009] that g is also isotrivial. Here, we provide an elementary proof of this fact as follows. Since f is isotrivial, there exists a linear polynomial $\sigma \in \overline{K}[x]$ such that $f^\sigma = \sigma \circ f \circ \sigma^{-1} \in \bar{k}[x]$. Then, the canonical height $\hat{h}_{f^\sigma}(x)$ associated to f^σ is equal to the Weil height $h(x)$ of $x \in \overline{K}$. On the other hand,

$$\begin{aligned}\hat{h}_{f^\sigma}(\sigma(x)) &= \lim_{n\to\infty} \frac{h((f^\sigma)^{\circ n}(\sigma x))}{d^n} = \lim_{n\to\infty} \frac{h(\sigma \circ f^{\circ n}(x))}{d^n} \\ &= \lim_{n\to\infty} \frac{h(f^{\circ n}(x))}{d^n} = \hat{h}_f(x).\end{aligned}$$

Thus, $\hat{h}_f(x) = 0$ if and only if $h(\sigma x) = \hat{h}_{f^\sigma}(\sigma x) = 0$. In other words, we have $\sigma(\Lambda_0) = \bar{k} = \overline{\mathbb{Q}}$. Note that $g^\sigma : \sigma(\Lambda_0) \to \sigma(\Lambda_0)$ (since $g : \Lambda_0 \to \Lambda_0$). We see that $g^\sigma(\alpha) \in \overline{\mathbb{Q}}$ for $\alpha \in \overline{\mathbb{Q}}$. It follows that $g^\sigma \in \overline{\mathbb{Q}}[x]$ as well. Then after conjugating by σ, we assume that both f and g are defined over $\overline{\mathbb{Q}}$. Note that since each λ_i is a solution to $f^{m_i}(\lambda_i) = g^{n_i}(\lambda_i)$, each λ_i must be in $\overline{\mathbb{Q}}$. Since $c(\lambda_i)$ is thus in $\overline{\mathbb{Q}}$ for each λ_i, and there are infinitely many λ_i, it follows that $c \in \overline{\mathbb{Q}}[x]$ as well.

We have reduced to the case where K is a number field, and we conclude that the set of preperiodic points of f and g are the same. This means that the Julia set $\mathcal{J}_f$ and $\mathcal{J}_g$ are equal. By [Baker and Erëmenko 1987; Schmidt and Steinmetz 1995], it follows that unless f and g are both conjugate to a multiple of a Chebychev polynomial or a multiple of powering map, then there is a polynomial q and a finite (compositional) order linear map τ such that any word in f and g is equal to $\tau^{\circ i} q^{\circ j}$ for some i, j. This means that f and g must be compositionally dependent.

Now, we are left with the case where f and g are both conjugate to either a multiple of a Chebychev polynomial or a multiple of a powering map. If f and g are conjugate to $\pm T_{d_1}$ and $\pm T_{d_2}$, respectively, where T_{d_i} is the monic Chebychev polynomial of degree d_i, then f and g are compositionally dependent (easy to check). If f and g are both conjugate to powering maps, then after conjugation we may write $f(x) = x^{d_1}$ and $g(x) = \gamma x^{d_2}$ for some $\gamma \in \overline{\mathbb{Q}}$. Note that both f and g have the same set of preperiodic points which are all the roots of unity in this case. In particular, $\gamma = g(1)$ is a root of unity. Therefore f and g must be compositionally dependent as well. $\square$

Next, we treat the case where exactly one of f and g is linear.

Proposition 9. *Let $f(x)$ and $g(x)$ be two polynomials of $\mathbb{C}[x]$ such that $\deg f > 1$ and $\deg g = 1$. Then there are at most finitely many $\lambda \in \mathbb{C}$ such that there are positive integers m and n with the following properties*:

(i) $f^{\circ m} \neq c(x)$.

(ii) $g^{\circ n} \neq c(x)$.

(iii) $(x - \lambda) \mid \gcd(f^{\circ m}(x) - c(x), g^{\circ n}(x) - c(x))$.

Proof. Let K be the field generated by the coefficients of f, g, and c. Since $g^{\circ n}(x) - c(x)$ is a polynomial of degree at most $\deg c + 1$, we see that every λ such that $g^{\circ n}(\lambda) - c(\lambda) = 0$ has degree at most $\deg c + 1$ over K. Note that for any nonrepeating infinite sequences $(\lambda_i)_{i=1}^{\infty}$ and $(n_i)_{i=1}^{\infty}$ such that $f^{\circ n_i}(\lambda_i) = c(\lambda_i)$ for all i, we have $\lim_{i\to\infty} \hat{h}_f(\lambda_i) = 0$ by Lemma 6. If K is a number field, then by the Northcott property we conclude that there are at most finitely many λ that satisfy properties (i) to (iii) given above. Hence, the proposition holds in this case.

Now, let's assume that K is a function field and that there is a nonrepeating infinite sequences $(\lambda_i)_{i=1}^{\infty}$ and $(n_i)_{i=1}^{\infty}$ such that $f^{\circ n_i}(\lambda_i) = c(\lambda_i)$ for all $i \in \mathbb{N}$. We note that as in the proof of Proposition 8, both m_i and n_i must go to infinity since $c(x)$ is not a compositional power of f or g.

By [Baker 2009], if there is an infinite sequence of $(\lambda_i)_{i=1}^{\infty}$ of bounded degree with $\hat{h}_f(\lambda_i) = 0$ then f must be isotrivial. Thus, after changing variables, we may assume that $f \in k[x]$ for some number field k. As a consequence, $\hat{h}_f(x) = h(x)$ the Weil height of x for all $x \in \overline{K}$. On the other hand, it follows from the definition of the Weil height that for $x \in \overline{K}$ with $h(x) > 0$ we must have $h(x) \geq 1/(\deg x)$. Now the sequence $(\lambda_i)_{i=1}^{\infty}$ has the property that all λ_i have degree bounded above by $\deg c + 1$ over K and that $\lim_{i\to\infty} h(\lambda_i) = 0$. Therefore we must have $h(\lambda_i) = 0$ for all but finitely many i. Also note that for $x \in \overline{K}$ we have $h(x) = 0$ if and only if $x \in \bar{k} = \overline{\mathbb{Q}}$. So, all but finitely many λ_i in the sequence $(\lambda_i)_{i=1}^{\infty}$ must be in $\overline{\mathbb{Q}}$.

We are left to treat the case where there are infinitely many λ in $\overline{\mathbb{Q}}$ such that $f^{\circ m}(\lambda) = c(\lambda) = g^{\circ n}(\lambda)$. We see in this case that c must have coefficients in $\overline{\mathbb{Q}}$ since there are infinitely many $\lambda \in \overline{\mathbb{Q}}$ such that $c(\lambda) \in \overline{\mathbb{Q}}$. Let k be the field generated by the coefficients of f and c over $\mathbb{Q}$, and let $g(x) = \alpha x + \beta$. Then all λ such that $f^{\circ m}(\lambda) = c(\lambda) = g^{\circ n}(\lambda)$ lie in extensions of $\overline{\mathbb{Q}} \cap k(\alpha, \beta)$ having degree at most $\deg c + 1$. Since $\overline{\mathbb{Q}} \cap k(\alpha, \beta)$ is a finitely generated extension of k, all such λ have bounded degree over $\mathbb{Q}$. Since the λ also have bounded height, again we have a contradiction by Northcott's theorem. $\square$

Proof of Theorem 1. If $\deg f, \deg g > 1$, then Theorem 1 follows immediately from Proposition 8. If $\max(\deg f, \deg g) > 1$ and $\min(\deg f, \deg g) = 1$, then we may assume without loss of generality that $\deg f > 1$ and $\deg g = 1$. Theorem 1 then follows from Proposition 9. $\square$

3. Proof of Theorem 2

When f and g are both linear, there may be infinitely many λ such that $(x-\lambda)$ divides $\gcd(f^{\circ m}(x) - c(x), g^{\circ n}(x) - c(x))$ for some m and n. Take for example, $c(x) = x^2$, with $f(x) = 2x$ and $g(x) = x + 1$. Then

$$f^{\circ n}(x) - c(x) = 2^n x - x^2 = -x(x - 2^n)$$

while if $m = 2^n(2^n - 1)$, then

$$g^{\circ m}(x) - c(x) = x + 2^n(2^n - 1) - x^2 = -(x + 2^n - 1)(x - 2^n),$$

so clearly there are infinitely many λ such that

$$(x - \lambda) \mid \gcd(f^{\circ m}(x) - c(x), g^{\circ n}(x) - c(x)),$$

for some positive integers m and n. On the other hand, if we restrict to the case where $m = n$, then we may obtain a suitable finiteness result.

The techniques in this section are mostly from diophantine geometry. We use these to prove Proposition 15 which treats the case where the coefficients of f, g, and c are algebraic. We then derive Theorem 2 using some simple specialization arguments. Theorem 3 then follows from Theorem 2 and Propositions 8 and 9.

3.1. *Results from diophantine geometry.* We will use the following version of Roth's Theorem (see [Lang 1983, Chapter 7, Theorem 1.1 and Remark (v)]).

Theorem 10. *Let k be a number field, let $\alpha_1, \dots, \alpha_n$ be distinct points in k, and let S be a finite set of places of k. Then for any $\epsilon > 0$, there are at most finitely many $\beta \in k$ such that*

$$\frac{1}{[k:\mathbb{Q}]}\left(\sum_{v\in S}\sum_{i=1}^{n} -\min(\log\|\alpha_i - \beta\|_v, 0) + \sum_{v\in S}\max(\log\|\beta\|_v, 0)\right) \geq (2+\epsilon)h(\beta) \quad (10.1)$$

The following is Siegel's well-known theorem on the set of integral points of curves of genus zero, which can be derived from Theorem 10 without difficulty. We refer the reader to [Lang 1983, Chapter 8, Theorem 5.1] for a proof.

Theorem 11. *Let k be a number field. Let C be a complete nonsingular curve of genus* 0, *defined over k, let S be a finite set of places of k containing all the archimedean places, and let ϕ be a nonconstant function in $k(C)$ with at least three distinct poles. Then there are at most finitely many $Q \in C(k)$ such that $\phi(Q)$ is an S-integer.*

As a corollary to Theorem 11, we have the following, which we will use to treat the case where the coefficients of the linear terms of f and g are multiplicatively dependent.

Proposition 12. *Let W be a one-dimensional subtorus in $\mathbb{G}_m^2$ defined over a number field k and let S be a finite set of places of k containing all the archimedean places. Let $\Phi(X, Y) = P(X, Y)/Q(X, Y)$ where $P, Q \in k[X, Y]$ are two relatively prime polynomials neither of which is divisible by X or Y. Assume that Φ restricts to a nonconstant rational function ϕ on W with at least a pole in $W(\bar{k})$. Let Γ be a finitely generated subgroup of $W(k)$. Then, there are at most finitely many points $Q \in \Gamma$ such that $\phi(Q)$ is an S-integer.*

Proof. Here, as usual, we consider $\mathbb{G}_m^2$ to be the open subset of $\mathbb{P}^2$ with coordinates $[x : y : z]$ defined by $x \neq 0$, $y \neq 0$, $z \neq 0$. The functions X and Y are equal to x/z and y/z with respect to these coordinates. By making a finite extension of k, we assume that the poles of ϕ are all k-rational points of W. Moreover, because Γ is finitely generated, we may assume, possibly after extending S to a larger finite set of places, that the coordinates of all of the elements of Γ and all of poles of ϕ are S-units. Let Γ^* be the union of Γ and the set of poles of ϕ.

Now, we fix a positive integer $m > 3$ and let $\mu_m : \mathbb{G}_m^2 \to \mathbb{G}_m^2$ be the m-th powering map. Namely, $\mu_m(X, Y) = (X^m, Y^m)$ for all $(X, Y) \in \mathbb{G}_m^2$. By Kummer theory, there exists a finite extension L over k such that the inverse image $\mu_m^{-1}(\Gamma^*)$ of Γ^* is contained in $W(L)$. Let S' denote the set of places of L that extend the places in S.

As $\mu_m : W \to W$ is an unramified map of degree m we see that the function $\phi_m := \phi \circ \mu_m$ is a rational function with at least m distinct poles on W. The subtorus W is viewed as an affine curve in the projective plane $\mathbb{P}_k^2$ and we denote its Zariski closure in $\mathbb{P}^2$ by $\overline{W}$. Note that ϕ_m extends to a rational function on $\overline{W}$ which we still denote by ϕ_m. Let $\pi : \widetilde{W} \to \overline{W}$ denote the normalization of $\overline{W}$. Then, $\widetilde{W}$ is a projective smooth curve of genus 0. Furthermore, the function $\psi_m := \phi_m \circ \pi$ is a rational function on $\widetilde{W}$ with at least m distinct poles. On the other hand, the set of L-rational points $W(L)$ lift to the set $\widetilde{W}(L)$.

Observe that for any point $Q \in \Gamma$ such that $\phi(Q)$ is an S-integer, then $\psi_m(Q')$ is an S'-integer where $Q' \in \widetilde{W}(L)$ is any point such that $(\mu_m \circ \pi)(Q') = Q$. On the other hand, since $m \geq 3$, there are at most finitely many $Q' \in \widetilde{W}(L)$ such that $\psi_m(Q')$ is an S'-integer by Theorem 11. Thus, there are at most finitely many Q such that $\phi(Q)$ is an S-integer. ☐

We will use the following lemma, due originally to Siegel [2014]. We provide a proof in modern language for the sake of completeness.

Lemma 13. *Let w be an element of a number field k, let y be a nonzero element of k, and let S be a finite set of places of k including all the archimedean places. Let $\epsilon > 0$. Then*

$$\frac{1}{[k:\mathbb{Q}]}\sum_{v\notin S} -\min(\log\|w^n - y\|_v, 0) \geq (1-\epsilon)nh(w), \tag{13.1}$$

for all sufficiently large n.

Proof. We may assume that S contains all the places v of k such that $\|w\|_v \neq 1$. Then applying Theorem 10, to the points 0 and y, we see that for any $\epsilon > 0$, we have

$$\frac{1}{[k:\mathbb{Q}]}\sum_{v\in S}(-\min(\log\|w^n - y\|_v, 0) - \min(\log\|w^n\|_v, 0) + \max(\log\|w^n\|_v, 0)) \\ \leq (2+\epsilon)nh(w) + O(1).$$

Since S contains all places such that $\|w\| \neq 1$, we have

$$\frac{1}{[k:\mathbb{Q}]}\sum_{v\in S}(-\min(\log\|w^n\|_v, 0) + \max(\log\|w^n\|_v, 0)) = 2nh(w).$$

Thus,

$$\frac{1}{[k:\mathbb{Q}]}\sum_{v\in S} -\min(\log\|w^n - y\|_v, 0) \leq \epsilon nh(w) + O(1). \tag{13.2}$$

Since

$$nh(w) \leq \frac{1}{[k:\mathbb{Q}]}\sum_{v\in \Omega_k} -\min(\log\|w^n - y\|_v, 0) + O(1),$$

we see that (13.1) must hold. □

The following lemma will be used to treat the case where the coefficients of the linear terms of f and g are multiplicatively independent.

Lemma 14. *Let w_1 and w_2 be two multiplicatively independent elements of a number field k, neither of which is a root of unity, and let y be a nonzero element of k. Let S be a finite set of places of k including all the archimedean places. Then for all sufficiently large n, there is a $v \notin S$ such that $|w_1^n - y|_v < |w_2^n - y|_v \leq 1$.*

Proof. We begin by showing that if w_1 and w_2 are multiplicatively independent, then w_1^n/y and w_2^n/y are multiplicatively independent for all but at most finitely many n. Note that if y is not in the multiplicative group generated by w_1 and w_2, then w_1^n/y and w_2^n/y are multiplicatively independent for all n. Otherwise, we have $y^{\ell_1} = w_1^{\ell_2} w_2^{\ell_3}$ for some integer $\ell_1 > 0$ and some integers ℓ_2 and ℓ_3. Since it suffices to prove our lemma for ℓ_1-th roots of w_1 and w_2 we may assume that we have $y = w_1^i w_2^j$ for some integers i and j. Now, if $w_1^n/(w_1^i w_2^j)$ and $w_2^n/(w_1^i w_2^j)$ are multiplicatively dependent, then we must have $(n-i)(n-j) = (-i)(-j)$, since w_1 and w_2 are multiplicatively independent. For all sufficiently large n, we clearly have $(n-i)(n-j) > (-i)(-j)$, so we are done.

By Theorem 1 and equation (1.2) of [Corvaja and Zannier 2005], we see that for any $\epsilon > 0$, there is a constant C_ϵ such that

$$\frac{1}{[k:\mathbb{Q}]}\sum_{v\in\Omega_k} -\log^{-}\max(\|w_1^n - y\|_v, \|w_2^n - y\|_v) < \epsilon n h(w_1) + C_\epsilon, \tag{14.1}$$

where $\log^{-}(\,\cdot\,) = \min(0, \log(\,\cdot\,))$. We may enlarge S to include the place v where $|w_1|_v > 1$ or $|y|_v > 1$. Suppose that for a positive integer n, inequalities $|w_1^n - y|_v \geq |w_2^n - y|_v$ hold for all $v \notin S$. Then, from (14.1) we have that

$$\begin{aligned}\epsilon n h(w_1) + C_\epsilon &\geq \frac{1}{[k:\mathbb{Q}]}\sum_{v\in\Omega_k}(-\min(0, \max\{\log\|w_1^n - y\|_v, \log\|w_2^n - y\|_v\}) \\ &\geq \frac{1}{[k:\mathbb{Q}]}\sum_{v\notin S} -\min(0, \log\|w_1^n - y\|_v) \\ &\geq (1-\epsilon) n h(w_1),\end{aligned}$$

where the last inequality follows from (13.1). Taking $\epsilon = \frac{1}{3}$, we see that there are only finitely many positive integers n such that the above inequality holds. Hence, for all sufficiently large n there is a $v \notin S$ such that $|w_1^n - y|_v < |w_2^n - y|_v \leq 1$, as desired. $\square$

3.2. *Proofs of Theorems 2 and 3.* We are now ready to treat the case where f and g are linear polynomials, and f, g, and c all have algebraic coefficients. The proof breaks into several cases. The first case is when c is constant; this case is already treated in [Ghioca et al. 2008]. The idea in all of the other cases is the same: to force certain quantities coming from any solutions to $f^{\circ n}(x) = c(x) = g^{\circ n}(x)$ to have poles outside a finite set and then derive contradictions from the existence of these poles to show that there are no solutions to $f^{\circ n}(x) = c(x) = g^{\circ n}(x)$ when n is sufficiently large.

Proposition 15. *Let $f(x) = \alpha x$ and $g(x) = \beta x + \gamma$ where α, β, and γ are nonzero algebraic numbers such that α is not a root of unity, $\alpha\beta$ is not a root of unity, β is not a root of unity other than 1, and $\gamma \neq 0$. Let $c(x)$ be any polynomial with coefficients in $\overline{\mathbb{Q}}$. Then for all but at most finitely many n, we have*

$$\gcd(f^{\circ n}(x) - c(x), g^{\circ n}(x) - c(x)) = 1. \tag{15.1}$$

Proof. Suppose that there are infinitely many n such that (15.1) does not hold. Let n be an integer such that $\gcd(f^{\circ n}(x) - c(x), g^{\circ n}(x) - c(x)) \neq 1$. Then there exists a $\lambda_n \in \overline{\mathbb{Q}}$ such that

$$(x - \lambda_n) \mid \gcd(f^{\circ n}(x) - c(x), g^{\circ n}(x) - c(x))$$

and thus, $f^{\circ n}(\lambda_n) = c(\lambda_n) = g^{\circ n}(\lambda_n)$.

In the following, we break the proof into four cases and show a contradiction in each case.

Case I. Suppose that c is a constant. Let θ be the compositional inverse of f and let τ be the compositional inverse of g. We observe that if $f^{\circ n}(\lambda) = g^{\circ n}(\lambda) = c$ then $\theta^{\circ n}(c) = \tau^{\circ n}(c)$. By [Ghioca et al. 2008, Proposition 5.4], this implies that either θ and τ have a common iterate or that c is periodic under both θ and τ. Since $\theta = \alpha^{-1}x$, we see that zero is the only periodic point of θ. Since $\tau = x/\beta - \gamma/\beta$, we see that the constant term of $\tau^{\circ n}$ is always nonzero, so 0 cannot be a periodic point of τ. Thus, there is an n such that $\theta^{\circ n} = \tau^{\circ n}$, which means that f and g have a common iterate. Since the constant term of $g^{\circ n}$ is nonzero for all n, we see that f and g cannot have a common iterate, which gives a contradiction.

In the following, we assume that $\deg c \geq 1$.

Case II. Assume that $\beta = 1$. Then

$$\lambda_n = \frac{n\gamma}{\alpha^n - 1}.$$

Let S be the set of places v that are archimedean or where α, γ, or a coefficient of c has v-adic absolute value not equal to 1. Assume that λ_n is an S-integer. Then,

$$\begin{aligned}
h(\lambda_n) &= \frac{1}{[K:\mathbb{Q}]} \sum_{v \in S} \max\left(0, \log \left\| \frac{n\gamma}{\alpha^n - 1} \right\|_v\right) \\
&\leq \frac{1}{[K:\mathbb{Q}]} \sum_{v \in S} \left\{ \max\left(0, \log \left\| \frac{1}{\alpha^n - 1} \right\|_v\right) + \max(0, \log\|n\gamma\|_v) \right\} \\
&= \frac{1}{[K:\mathbb{Q}]} \sum_{v \in S} \max\left(0, \log \left\| \frac{1}{\alpha^n - 1} \right\|_v\right) + h(n\gamma) \\
&\leq \frac{1}{[K:\mathbb{Q}]} \sum_{v \in S} \max\left(0, \log \left\| \frac{1}{\alpha^n - 1} \right\|_v\right) + \log n + O(1).
\end{aligned} \tag{15.2}$$

Let $\epsilon > 0$ be given. By (13.2), there exists a constant C_ϵ such that

$$\frac{1}{[K:\mathbb{Q}]} \sum_{v \in S} \max\left(0, \log \left\| \frac{1}{\alpha^n - 1} \right\|_v\right) \leq \epsilon n h(\alpha) + C_\epsilon. \tag{15.3}$$

On the other hand, there is a constant $D = D(\gamma)$ such that

$$h(\lambda_n) = h\left(\frac{n\gamma}{\alpha^n - 1}\right) \geq nh(\alpha) - h(n) - D = nh(\alpha) - \log n - D.$$

Fixing a positive $\epsilon < 1$ and combing (15.2) with (15.3), we see that λ_n cannot be an S-integer if n is large enough. Therefore, for n large there exists a place

v outside of S such that $|\lambda_n|_v > 1$. If $\deg c > 1$, then $|c(\lambda)|_v = |\lambda_v|^{\deg c}$ but $|f^{\circ n}(\lambda_n)| = |\alpha^n \lambda_n|_v = |\lambda_n|_v$. This gives a contradiction.

If $\deg c = 1$, then we write $c(x) = tx + u$ and note that since $f^{\circ n}(\lambda_n) = g^{\circ n}(\lambda_n) = c(\lambda_n)$, we must have

$$\lambda_n = \frac{u - n\gamma}{1 - t} = \frac{u}{\alpha^n - t}.$$

If $u \neq 0$ and n is large, then by enlarging S to contain the places v where $|1-t|_v \neq 1$, then $(u - n\gamma)/(1 - t)$ is an S-integer for all n. On the other hand, by taking $\Phi(X, Y) = u/(X - t)$ in Proposition 12, we see that $u/(\alpha^n - t)$ cannot be an S-integer for n sufficiently large. This gives a contradiction. If $u = 0$, then we have $\lambda_n = \alpha^n \lambda_n = t\lambda_n = g^{\circ n}(\lambda_n)$, which has no solutions when $\alpha^n \neq t$, and thus has a solution for at most one n, since α is not a root of unity. Thus the proof of this case is complete.

We assume in the following that $\beta \neq 1$. Note that when $\alpha^n = \beta^n$, there is no solution to $f^{\circ n}(x) = g^{\circ n}(x)$ and that when $\alpha^n \neq \beta^n$, the unique solution to $f^{\circ n}(x) = c(x) = g^{\circ n}(x)$ is given by

$$\lambda_n = \frac{(\beta^n - 1)\gamma}{(\beta - 1)(\alpha^n - \beta^n)}. \tag{15.4}$$

Case III. Suppose that α and β are multiplicatively dependent. Then, the point $P = (\alpha, \beta)$ is in a one-dimensional subtorus W of $\mathbb{G}_m^2$. Let S be the set of places v that are archimedean or where α, γ, $\beta - 1$, or a coefficient of c has v-adic absolute value not equal to 1. Then, by taking $\Phi(X, Y) = (Y - 1)/(X - Y)$ and Γ to be the group generated by P in Proposition 12, we see that for all sufficient large n there exists a place v outside of S such that

$$\left| \frac{\beta^n - 1}{\alpha^n - \beta^n} \right|_v > 1.$$

It follows that for such v we have $|\lambda_n|_v > 1$. Observe that on the one hand, $|f^{\circ n}(\lambda_n)|_v = |\alpha^n \lambda_n|_v = |\lambda_n|_v$ while on the other hand, we have $|f^{\circ n}(\lambda_n)|_v = |c(\lambda_n)|_v = |\lambda_n|_v^{\deg c}$. This gives a contradiction if $\deg c > 1$.

If $\deg c = 1$, we write $c(x) = tx + u$, $t \neq 0$. If $f^{\circ n}(\lambda_n) = c(\lambda_n) = g^{\circ n}(\lambda_n)$ then we have

$$\lambda_n = \frac{u - \gamma \frac{\beta^n - 1}{\beta - 1}}{\beta^n - t} = \frac{u}{\alpha^n - t}. \tag{15.5}$$

From this we deduce that

$$u\left(\frac{\beta^n - t}{\alpha^n - t}\right) = u - \left(\frac{\gamma}{\beta - 1}\right)(\beta^n - 1). \tag{15.6}$$

Note that the right-hand side of (15.6) is an S-integer. However, by taking $\Phi(X, Y) = (Y - t)/(X - t)$ in Proposition 12 we conclude that for n large enough the left-hand side of (15.6) is not an S-integer. This leads to a contradiction and completes the proof in this case.

Case IV. Suppose that α and β are multiplicatively independent. Let S be the set of places v that are archimedean or where α, γ, or a coefficient of c has v-adic absolute value not equal to 1.

Suppose that $\deg c > 1$. Then, applying Lemma 14 to $\beta^n - 1$ and $(\alpha/\beta)^n - 1$, we see that there is a place v outside of S such that $|\lambda_n|_v > 1$. Again, if $\deg c > 1$, this gives a contradiction since $|c(\lambda)|_v = |\lambda_v|^{\deg c}$ but $|f^{\circ n}(\lambda_n)| = |\alpha^n \lambda_n|_v = |\lambda_n|_v$.

Now suppose that $\deg c = 1$. Again, we write $c(x) = tx + u$. Then we also have

$$\lambda_n = \frac{u - \gamma \frac{\beta^n - 1}{\beta - 1}}{\beta^n - t} = \frac{u}{\alpha^n - t}. \tag{15.7}$$

This is equivalent to

$$1 - \frac{\gamma(\beta^n - 1)}{u(\beta - 1)} = \frac{\beta^n - t}{\alpha^n - t}. \tag{15.8}$$

We enlarge S to include all the places such that u or $\beta - 1$ are S-unit. Then applying Lemma 14, we see that for all sufficiently large n, there is a place $v \notin S$ such that $|\alpha^n - t|_v < |\beta^n - t|_v \le 1$. For this v, we see that the left-hand side of (15.8) is a v-adic integer while the right-hand side is not. Therefore, (15.7) cannot hold for n sufficiently large. □

Remark. To see that Proposition 15 does not hold in general if $\alpha\beta$ is a root of unity, consider the case where $f(x) = x/2$, $g(x) = 2x + 1$ and $c(x) = -(x + 1)$. Then for any n, the common root of $f^{\circ n}$ and $g^{\circ n}$ is

$$\frac{2^n - 1}{2^{-n} - 2^n} = -2^n \frac{2^n - 1}{2^{2n} - 1} = \frac{-2^n}{2^n + 1}.$$

while the common root of $f^{\circ n}$ and $c(x)$ is

$$\frac{-1}{\left(\frac{1}{2}\right)^n + 1} = \frac{-2^n}{2^n + 1}.$$

Thus, for every positive integer n, there is a λ_n such that

$$f^{\circ n}(\lambda_n) - c(\lambda_n) = g^{\circ n}(\lambda_n) - c(\lambda_n) = 0.$$

We can now prove Theorem 2 by specializing from $\mathbb{C}$ to a number field.

Proof of Theorem 2. First we note that any nonconstant affine map $x \mapsto ax + b$ has a fixed point unless $a = 1$. Any two monic linear polynomial must commute with each other. Thus, we may assume that at least one of f and g has a fixed point. Without loss of generality, we may assume that f has a fixed point. After a possible change of coordinates, we may then write $f(x) = \alpha x$ and $g(x) = \beta x + \gamma$.

If α is a root of unity, then f and g are not compositionally independent since f itself has finite order, so α must not be a root of unity. Similarly, if β is a root of unity other than one, then g has finite order so that f and g are not compositionally independent either. We may therefore assume that β is not a root of unity other than one. Finally, we see that if there are integers i and j such that $\alpha^i\beta^j = 1$, then the linear terms in $f^{\circ i}g^{\circ j}$ and $g^{\circ j}f^{\circ i}$ are both 1, which means that $f^{\circ i}g^{\circ j}$ and $g^{\circ j}f^{\circ i}$ commute. This would imply f and g are not compositionally dependent, so we may assume that there are no positive integer i and j such that $\alpha^i\beta^j = 1$.

As in the proof of Proposition 15, we assume that there are infinitely many n such that (15.1) does not hold. Let K be the field generated by α, β, and γ over $\mathbb{Q}$, and let R be the ring generated over $\mathbb{Z}$ by α, β, γ, and the coefficients of c. Observe that any solution λ_n to $f^{\circ n}(\lambda_n) = g^{\circ n}(\lambda_n) = c(\lambda_n)$ must lie in K. By our assumption, there are infinitely many such n, so c takes infinitely many values in K to other values in K so $c \in K[x]$. Hence, we may assume that $c \in K[x]$.

If α, β, and γ are in $\overline{\mathbb{Q}}$, then we are done by Proposition 15. If K has positive transcendence degree over $\overline{\mathbb{Q}}$, then there exists a specialization map t from R to $\overline{\mathbb{Q}}$ such that $\gamma_t \neq 0$ and α_t, β_t, $\alpha_t\beta_t$, and α_t/β_t are not roots of unity. We may prove this, for example, by induction on the transcendence degree of $\mathbb{Q}(\alpha, \beta, \gamma)$. If the transcendence degree is 0, there is nothing to prove. If it is n, take a subfield L of transcendence degree of $n-1$ in K. Then, by [Call and Silverman 1993, Theorem 4.1], for all specializations s from R to $\bar{L}$ of sufficiently large height, we have that $\gamma_s \neq 0$ and that α_s, β_s, $\alpha_s\beta_s$, and α_s/β_s are not roots of unity. We then apply the inductive hypothesis on the transcendence degree to $\mathbb{Q}(\alpha_s, \beta_s, \gamma_s)$.

Let $f_t = \alpha_t x$, $g_t = \beta_t x + \gamma_t$, and c_t be the polynomial obtained by specializing all the coefficient of c at t. Now, if $\gcd(f^{\circ n}(x) - c(x), g^{\circ n}(x) - c(x)) \neq 1$, then $\gcd(f_t^{\circ n}(x) - c_t(x), g_t^{\circ n}(x) - c_t(x)) \neq 1$. But there are at most finitely many n such that $\gcd(f_t^{\circ n}(x) - c_t(x), g_t^{\circ n}(x) - c_t(x)) \neq 1$, by Proposition 15, which gives a contradiction, and finishes our proof. □

Remark. We note that by Proposition 15, the condition needed for Theorem 2 is weaker than compositional dependency, since Proposition 15 holds unless the linear term of $f \circ g$ is a root of unity. Mike Zieve has shown us that something similar is true for polynomials of higher degree, namely that the sorts of compositional dependencies that may arise all take a specific form.

We now prove Theorem 3.

Proof of Theorem 3. The case where f and g are both linear is covered by Theorem 2, so we may assume that either both f and g are nonlinear or that g is linear and f is not.

By Propositions 8 and 9, there are at most finitely many λ such that $(x-\lambda)$ divides $\gcd(f^{\circ n}(x) - c(x), g^{\circ n}(x) - c(x))$ for some n such that $f^{\circ n} \neq c$ and $g^{\circ n} \neq c$. Let

$\mathcal{S}$ denote the set of such λ. Since f and g are compositionally independent, there is at most one N such that $f^{\circ N} = c$ or $g^{\circ N} = c$ exclusively. If such an N exists, let $\mathcal{T}$ denote the set of λ such that $(x-\lambda)$ divides $\gcd(f^{\circ N}(x)-c(x), g^{\circ N}(x)-c(x))$. We observe that $\mathcal{T}$ must be finite since otherwise we would have $f^{\circ N} - c = 0 = g^{\circ N} - c$. However, this cannot happen because f and g are compositionally independent. Any λ such that $(x-\lambda)$ divides $\gcd(f^{\circ n}(x) - c(x), g^{\circ n}(x) - c(x))$ is in $\mathcal{S} \cup \mathcal{T}$, so our proof is complete ☐

4. Proof of Theorem 4

Theorem 4 is now an easy consequence of the following lemma. To state the lemma, we introduce a small bit of new notation: for any nonzero polynomial $q(x)$ we let $v_\lambda(q)$ denote the largest positive integer e such that $(x-\lambda)^e$ divides q when $(x-\lambda) \mid q$ and let $v_\lambda(q) = 0$ if $(x-\lambda)$ does not divide q.

Lemma 16. *Let q be a polynomial in $\mathbb{C}[x]$ of degree greater than one and let $c(x) \in \mathbb{C}[x]$ be a polynomial that is not equal to a constant that is in a ramified cycle of f. Let $\lambda \in \mathbb{C}$. Then there is a constant $M_{\lambda,q}$ such that $v_\lambda(q^{\circ n}(x) - c(x)) \le M_{\lambda,q}$ for all n such that $q^{\circ n}(x) \ne c(x)$.*

Proof. We write $c(x) = \sum_{i=0}^{d_c} c_i (x-\lambda)^i$ as a polynomial in $(x-\lambda)$. If there are finitely many n such that $v_\lambda(q^{\circ n}(x) - c(x)) > 0$, then the proof is immediate. Thus, we assume that there are infinitely many n such that $v_\lambda(q^{\circ n}(x) - c(x)) > 0$. It follows that $q^{\circ n}(\lambda) = c_0$ for infinitely many n, so c_0 must be periodic under q. Let ℓ be the smallest positive integer such that $q^{\circ \ell}(\lambda) = c_0$ and let r be the smallest positive integer such that $q^{\circ r}(c_0) = c_0$. Then we see that $v_\lambda(q^{\circ n}(x) - c(x)) > 0$ if and only if n can be written as $\ell + kr$ for some k. We write $q^{\circ r}(x) = \sum_{i=0}^{d_r} a_i (x-c_0)^i$ and $q^{\circ \ell}(x) = \sum_{j=0}^{d_\ell} b_j (x-\lambda)^j$. Let e be the smallest positive integer such that $b_e \ne 0$.

Suppose now that $c(x) = c_0$ is a constant. By assumption, c_0 is not in a ramified cycle of q, thus $a_1 \ne 0$ in this case. Then by induction we find that

$$q^{\circ(\ell+rk)}(x) = c_0 + a_1^k b_e (x-\lambda)^e + \text{higher order terms in } (x-\lambda),$$

so $v_\lambda(q^{\circ(\ell+rk)}(x) - c) = e$ for all k.

Suppose now that $c(x)$ is not a constant. We may suppose that there are infinitely many n such that $v_\lambda(q^{\circ n}(x) - c(x)) > e$ since otherwise the lemma clearly holds. Note that it's possible that c_0 is in a ramified cycle of q. In any case, let u be the smallest integer such that $a_u \ne 0$.

We first assume that $u = 1$. Equivalently, c_0 is not in a ramified cycle of q. Then, we must have $a_1^k b_e = c_e$ for infinitely many k. Since $a_1 b_e \ne 0$, this means that a_1 must be a root of unity. Suppose that $a_1^s = 1$. Then we may write

$$q^{\circ rs}(x) = c_0 + (x-c_0) + \alpha_d (x-c_0)^d + O((x-c_0)^{d+1}),$$

for some $d > 0$ with $\alpha_d \neq 0$. It follows that for any k, we have

$$q^{\circ rsk}(x) = c_0 + (x - c_0) + k\alpha_d (x - c_0)^d + O((x - c_0)^{d+1}).$$

Now, let $g(x) = \sum_{i=0}^{\infty} \beta_i (x - \lambda)^i$ be any nonconstant polynomial in $(x - \lambda)$ such that $\beta_0 = c_0$. Let t be the smallest positive integer such that $\beta_t \neq 0$. Then, for any k, the coefficient of $(x - \lambda)^{td}$ in $q^{\circ rsk} \circ g$ is $k\alpha_d \beta_t^d + \beta_{td}$. Since $\alpha_d \neq 0$, there are in particular at most finitely many k such that the coefficient of $(x - \lambda)^{td}$ in $q^{\circ rsk} \circ g$ is equal to c_{td}. Thus, there are at most finitely many k such that $v_\lambda(q^{\circ rsk} \circ g(x) - c(x)) > td$, and hence $v_\lambda(q^{\circ rsk} \circ g(x) - c(x))$ is bounded for all k. Applying this to $g = q^{\circ y}$ for $y = \ell, \ell + r, \ldots, \ell + (s - 1)r$ completes our proof, since any number of the form $\ell + kr$ can be written as $y + krs$ for some such y.

Assume now that $u > 1$. Then, by induction

$$q^{\circ \ell + rk}(x) = c_0 + a_u^{(u^k - 1)/(u-1)} b_e^{u^k} (x - \lambda)^{eu^k} + O((x - \lambda)^{eu^k + 1}).$$

So, $v_\lambda(q^{\circ n}(x) - c(x)) \leq \deg c$ for all sufficiently large k. Hence, $v_\lambda(q^{\circ n}(x) - c(x))$ is bounded above by a constant depending on λ and q only. □

Remark. We note that in Lemma 16, if $v_\lambda(q^{\circ n}(x) - c(x)) > 0$ then the integer n is in a congruence class $\ell + r\,\mathbb{N}$ for some positive integer r. In fact, r is the least period of $c_0 = c(\lambda)$ under the action of q.

Proof of Theorem 4. We may assume without loss of generality that c is not in a ramified cycle of f. By Theorem 1, there are at most finitely many λ such that $(x - \lambda)$ divides $\gcd(f^{\circ m}(x) - c(x), g^{\circ n}(x) - c(x))$ for some m and n. Let $\mathcal{S}$ be the set of all such λ. By Lemma 16, there is an M_λ such that $v_\lambda(q^{\circ n}(x) - c(x)) \leq M_{\lambda,q}$ for all n, since c is not a compositional power of f. Then, if

$$h(x) = \prod_{\lambda \in \mathcal{S}} (x - \lambda)^{M_\lambda},$$

we see that

$$\gcd(f^{\circ m}(x) - c(x), g^{\circ n}(x) - c(x)) \mid h(x),$$

for all m and n, as desired. □

5. Further directions

Many of these techniques may work more generally. We close with several questions.

Silverman [2004a] showed that the characteristic p function field analog of the theorem of Bugeaud, Corvaja, and Zannier theorem is not true; in particular, one can find multiplicatively independent polynomials $a, b \in \mathbb{F}_q[x]$ (where $\mathbb{F}_q$ is as usual the finite field with q elements) and an $\epsilon > 0$ such that $\deg(\gcd(a^n - 1, b^n - 1)) > \epsilon n$ for infinitely many n. Similarly, we suspect that one can find compositionally independent polynomials $f, g \in \mathbb{F}_q[x]$, an $\epsilon > 0$, and a $c(x) \in \mathbb{F}_q[x]$ that is not a compositional power of f or g such that $\deg(\gcd(f^{\circ n}(x)-c(x), g^{\circ n}(x)-c(x))) > \epsilon n$ for infinitely many n. On the other hand, one might ask the following question in characteristic p. See also [Ghioca et al. 2017a, Corollary 17] for an answer to a slightly different question.

Question 17. *Let $F = \mathbb{F}_q[T]$ be the polynomial ring in one variable over the finite field with q elements. Let f and g be two compositionally independent nonisotrivial polynomials in $F[x]$, and let $c \in F[x]$. Is it true that there are at most finitely many $\lambda \in \overline{F}$ such that there is an n for which $(x-\lambda)$ divides $\gcd(f^{\circ n}(x)-c(x), g^{\circ n}(x)-c(x))$? Given an $\epsilon > 0$ and assuming that $c(x)$ is not in a ramified cycle of f and g, is it even true that*

$$\deg(\gcd(f^{\circ n}(x)-c(x), g^{\circ n}(x)-c(x))) < \epsilon n$$

for all but finitely many n?

We might also ask for characteristic 0 results in more general settings.

Question 18. *Let $\phi_1, \phi_2 : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ be two nonconstant, compositionally independent morphisms. Let $c : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ be any morphism. It is true that there must be at most finitely many $\lambda \in \mathbb{C}$ such that $\phi_1^{\circ n}(\lambda) = \phi_2^{\circ n}(\lambda) = c(\lambda)$?*

We should note that the counterexamples to the dynamical Manin–Mumford conjecture given in [Ghioca et al. 2011] do not yield counterexamples here in an obvious way, since the Lattés maps given there commute with each other and hence they are not compositionally independent.

For more general varieties, we ask the following.

Question 19. *Let V be a variety defined over $\mathbb{C}$ and let $\phi_1, \phi_2 : V \to V$ be two dominant compositionally independent morphisms. Let $c : V \to V$ be any morphism. Is it true that the set of $\lambda \in V(\mathbb{C})$ such that $\phi_1^{\circ n}(\lambda) = \phi_2^{\circ n}(\lambda) = c(\lambda)$ must be contained in a proper Zariski closed subset of V?*

In the case where V is projective and some iterates of ϕ_1 and ϕ_2 extend to maps on projective space of degree greater than one (the case where ϕ_1 and ϕ_2 are "polarizable" in the language of Zhang [2006]), it may be possible, using higher dimensional results such as those of [Yuan 2008; Gubler 2008; Yuan and Zhang 2017], to show that $h_{\phi_1} = h_{\phi_2}$ whenever the λ such that $\phi_1^n(\lambda) = \phi_2^n(\lambda) = c(\lambda)$ are Zariski dense. On the other hand, that may not imply a compositional dependence between ϕ_1 and ϕ_2. One natural place to look for counterexamples might be abelian varieties with quaternion endomorphism rings.

Finally, it is natural to ask for a result along the lines of [Bugeaud et al. 2003] where one considers iterates of integers under polynomial maps rather than simply powers of integers. More precisely, one might hope that for $a, b \in \mathbb{Z}$, two polynomials $f, g \in \mathbb{Z}[x]$ of degree $d > 1$, and an $\epsilon > 0$, the inequality

$$\gcd(f^{\circ n}(a), g^{\circ n}(b)) < \epsilon d^n$$

should hold for all but at most finitely many n, given reasonable conditions on f, g, a, and b. Huang [2017] has shown that such an inequality must indeed hold for all sufficiently large n whenever the sequence $(f^{\circ n}(a), g^{\circ n}(b))_n$ is Zariski dense in $\mathbb{A}^2$ if one assumes Vojta's conjecture for heights with respect to canonical divisors on surfaces (see [Vojta 1987, Conjecture 3.4.3]). The proof uses an idea of Silverman [2005], which relates the original results of [Bugeaud et al. 2003] to Vojta's conjecture.

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References

[Ailon and Rudnick 2004] N. Ailon and Z. Rudnick, "Torsion points on curves and common divisors of $a^k - 1$ and $b^k - 1$", *Acta Arith.* **113**:1 (2004), 31–38. MR Zbl

[Baker 2009] M. Baker, "A finiteness theorem for canonical heights attached to rational maps over function fields", *J. Reine Angew. Math.* **626** (2009), 205–233. MR Zbl

[Baker and De Marco 2013] M. Baker and L. De Marco, "Special curves and postcritically finite polynomials", *Forum Math. Pi* **1** (2013), e3, 35. MR Zbl

[Baker and DeMarco 2011] M. Baker and L. DeMarco, "Preperiodic points and unlikely intersections", *Duke Math. J.* **159**:1 (2011), 1–29. MR Zbl

[Baker and Erëmenko 1987] I. N. Baker and A. Erëmenko, "A problem on Julia sets", *Ann. Acad. Sci. Fenn. Ser. A I Math.* **12**:2 (1987), 229–236. MR Zbl

[Baker and Rumely 2006] M. H. Baker and R. Rumely, "Equidistribution of small points, rational dynamics, and potential theory", *Ann. Inst. Fourier* (*Grenoble*) **56**:3 (2006), 625–688. MR Zbl

[Benedetto 2005] R. L. Benedetto, "Heights and preperiodic points of polynomials over function fields", *Int. Math. Res. Not.* **2005**:62 (2005), 3855–3866. MR Zbl

[Bombieri and Gubler 2006] E. Bombieri and W. Gubler, *Heights in Diophantine geometry*, New Mathematical Monographs **4**, Cambridge University Press, 2006. MR Zbl

[Bugeaud et al. 2003] Y. Bugeaud, P. Corvaja, and U. Zannier, "An upper bound for the G.C.D. of $a^n - 1$ and $b^n - 1$", *Math. Z.* **243**:1 (2003), 79–84. MR Zbl

[Call and Silverman 1993] G. S. Call and J. H. Silverman, "Canonical heights on varieties with morphisms", *Compositio Math.* **89**:2 (1993), 163–205. MR Zbl

[Chambert-Loir 2006] A. Chambert-Loir, "Mesures et équidistribution sur les espaces de Berkovich", *J. Reine Angew. Math.* **595** (2006), 215–235. MR Zbl

[Corvaja and Zannier 2005] P. Corvaja and U. Zannier, "A lower bound for the height of a rational function at S-unit points", *Monatsh. Math.* **144**:3 (2005), 203–224. MR Zbl

[Corvaja and Zannier 2008] P. Corvaja and U. Zannier, "Some cases of Vojta's conjecture on integral points over function fields", *J. Algebraic Geom.* **17**:2 (2008), 295–333. MR Zbl

[Corvaja and Zannier 2013] P. Corvaja and U. Zannier, "Greatest common divisors of $u-1$, $v-1$ in positive characteristic and rational points on curves over finite fields", *J. Eur. Math. Soc.* (*JEMS*) **15**:5 (2013), 1927–1942. MR Zbl

[Favre and Rivera-Letelier 2004] C. Favre and J. Rivera-Letelier, "Théorème d'équidistribution de Brolin en dynamique p-adique", *C. R. Math. Acad. Sci. Paris* **339**:4 (2004), 271–276. MR Zbl

[Ghioca and Tucker 2010] D. Ghioca and T. J. Tucker, "Proof of a dynamical Bogomolov conjecture for lines under polynomial actions", *Proc. Amer. Math. Soc.* **138**:3 (2010), 937–942. MR Zbl

[Ghioca et al. 2008] D. Ghioca, T. J. Tucker, and M. E. Zieve, "Intersections of polynomials orbits, and a dynamical Mordell–Lang conjecture", *Invent. Math.* **171**:2 (2008), 463–483. MR Zbl

[Ghioca et al. 2011] D. Ghioca, T. J. Tucker, and S. Zhang, "Towards a dynamical Manin–Mumford conjecture", *Int. Math. Res. Not.* **2011**:22 (2011), 5109–5122. MR Zbl

[Ghioca et al. 2015] D. Ghioca, L.-C. Hsia, and T. J. Tucker, "Preperiodic points for families of rational maps", *Proc. Lond. Math. Soc.* (3) **110**:2 (2015), 395–427. MR Zbl

[Ghioca et al. 2017a] D. Ghioca, L.-C. Hsia, and T. J. Tucker, "On a variant of the Ailon–Rudnick theorem in finite characteristic", *New York J. Math.* **23** (2017), 213–225. MR Zbl

[Ghioca et al. 2017b] D. Ghioca, L.-C. Hsia, and T. J. Tucker, "A variant of a theorem by Ailon–Rudnick for elliptic curves", preprint, 2017. arXiv

[Gubler 2008] W. Gubler, "Equidistribution over function fields", *Manuscripta Math.* **127**:4 (2008), 485–510. MR Zbl

[Hindry and Silverman 2000] M. Hindry and J. H. Silverman, *Diophantine geometry*, Graduate Texts in Mathematics **201**, Springer, 2000. MR Zbl

[Huang 2017] K. Huang, "Generalized Greatest Common Divisors for the Orbits under Rational Functions", preprint, 2017. arXiv

[Ingram 2013] P. Ingram, "Variation of the canonical height for a family of polynomials", *J. Reine Angew. Math.* **685** (2013), 73–97. MR Zbl

[Lang 1983] S. Lang, *Fundamentals of Diophantine geometry*, Springer, 1983. MR Zbl

[Luca 2005] F. Luca, "On the greatest common divisor of $u-1$ and $v-1$ with u and v near S-units", *Monatsh. Math.* **146**:3 (2005), 239–256. MR Zbl

[Mimar 2013] A. Mimar, "On the preperiodic points of an endomorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ which lie on a curve", *Trans. Amer. Math. Soc.* **365**:1 (2013), 161–193. MR Zbl

[Morton and Silverman 1995] P. Morton and J. H. Silverman, "Periodic points, multiplicities, and dynamical units", *J. Reine Angew. Math.* **461** (1995), 81–122. MR Zbl

[Murty and Murty 2011] M. R. Murty and V. K. Murty, "On a problem of Ruderman", *Amer. Math. Monthly* **118**:7 (2011), 644–650. MR Zbl

[Ostafe 2016] A. Ostafe, "On some extensions of the Ailon–Rudnick theorem", *Monatsh. Math.* **181**:2 (2016), 451–471. MR Zbl

[Petsche et al. 2012] C. Petsche, L. Szpiro, and T. J. Tucker, "A dynamical pairing between two rational maps", *Trans. Amer. Math. Soc.* **364**:4 (2012), 1687–1710. MR Zbl

[Schmidt and Steinmetz 1995] W. Schmidt and N. Steinmetz, "The polynomials associated with a Julia set", *Bull. London Math. Soc.* **27**:3 (1995), 239–241. MR Zbl

[Serre 1997] J.-P. Serre, *Lectures on the Mordell–Weil theorem*, 3rd ed., Aspects of Mathematics, Friedr. Vieweg & Sohn, Braunschweig, 1997. MR Zbl

[Siegel 2014] C. L. Siegel, "Über einige Anwendungen diophantischer Approximationen [reprint of Abhandlungen der Preußischen Akademie der Wissenschaften. Physikalisch-mathematische Klasse 1929, Nr. 1]", pp. 81–138 in *On some applications of Diophantine approximations*, edited by U. Zannier, Quad./Monogr. **2**, Ed. Norm., Pisa, 2014. MR Zbl

[Silverman 2004a] J. H. Silverman, "Common divisors of $a^n - 1$ and $b^n - 1$ over function fields", *New York J. Math.* **10** (2004), 37–43. MR Zbl

[Silverman 2004b] J. H. Silverman, "Common divisors of elliptic divisibility sequences over function fields", *Manuscripta Math.* **114**:4 (2004), 431–446. MR Zbl

[Silverman 2005] J. H. Silverman, "Generalized greatest common divisors, divisibility sequences, and Vojta's conjecture for blowups", *Monatsh. Math.* **145**:4 (2005), 333–350. MR Zbl

[Silverman 2007] J. H. Silverman, *The arithmetic of dynamical systems*, Graduate Texts in Mathematics **241**, Springer, 2007. MR Zbl

[Vojta 1987] P. Vojta, *Diophantine approximations and value distribution theory*, Lecture Notes in Mathematics **1239**, Springer, 1987. MR Zbl

[Yuan 2008] X. Yuan, "Big line bundles over arithmetic varieties", *Invent. Math.* **173**:3 (2008), 603–649. MR Zbl

[Yuan and Zhang 2017] X. Yuan and S.-W. Zhang, "The arithmetic Hodge index theorem for adelic line bundles", *Math. Ann.* **367**:3-4 (2017), 1123–1171. MR Zbl

[Zannier 2012] U. Zannier, *Some problems of unlikely intersections in arithmetic and geometry*, Annals of Mathematics Studies **181**, Princeton University Press, 2012. MR Zbl

[Zhang 2006] S.-W. Zhang, "Distributions in algebraic dynamics", pp. 381–430 in *Surveys in differential geometry. Vol. X*, edited by S. T. Yau, Surv. Differ. Geom. **10**, International Press, Somerville, MA, 2006. MR Zbl

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The role of defect and splitting in finite generation of extensions of associated graded rings along a valuation

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Suppose that R is a 2-dimensional excellent local domain with quotient field K, K^* is a finite separable extension of K and S is a 2-dimensional local domain with quotient field K^* such that S dominates R. Suppose that ν^* is a valuation of K^* such that ν^* dominates S. Let ν be the restriction of ν^* to K. The associated graded ring $\mathrm{gr}_\nu(R)$ was introduced by Bernard Teissier. It plays an important role in local uniformization. We show that the extension $(K,\nu)\to(K^*,\nu^*)$ of valued fields is without defect if and only if there exist regular local rings R_1 and S_1 such that R_1 is a local ring of a blowup of R, S_1 is a local ring of a blowup of S, ν^* dominates S_1, S_1 dominates R_1 and the associated graded ring $\mathrm{gr}_{\nu^*}(S_1)$ is a finitely generated $\mathrm{gr}_\nu(R_1)$-algebra.

We also investigate the role of splitting of the valuation ν in K^* in finite generation of the extensions of associated graded rings along the valuation. We say that ν does not split in S if ν^* is the unique extension of ν to K^* which dominates S. We show that if R and S are regular local rings, ν^* has rational rank 1 and is not discrete and $\mathrm{gr}_{\nu^*}(S)$ is a finitely generated $\mathrm{gr}_\nu(R)$-algebra, then S is a localization of the integral closure of R in K^*, the extension $(K,\nu)\to(K^*,\nu^*)$ is without defect and ν does not split in S. We give examples showing that such a strong statement is not true when ν does not satisfy these assumptions. As a consequence, we deduce that if ν has rational rank 1 and is not discrete and if $R\to R'$ is a nontrivial sequence of quadratic transforms along ν, then $\mathrm{gr}_\nu(R')$ is not a finitely generated $\mathrm{gr}_\nu(R)$-algebra.

Suppose that K is a field. Associated to a valuation ν of K is a value group Φ_ν and a valuation ring V_ν with maximal ideal m_ν. Let R be a local domain with quotient field K. We say that ν dominates R if $R\subset V_\nu$ and $m_\nu\cap R=m_R$, where m_R is the maximal ideal of R. We have an associated semigroup $S^R(\nu)=\{\nu(f)\mid f\in R\}$,

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as well as the associated graded ring along the valuation

$$\mathrm{gr}_\nu(R)=\bigoplus_{\gamma\in\Phi_\nu}\mathcal{P}_\gamma(R)/\mathcal{P}^+_\gamma(R)=\bigoplus_{\gamma\in S^R(\nu)}\mathcal{P}_\gamma(R)/\mathcal{P}^+_\gamma(R), \tag{1}$$

which is defined in [Teissier 2003]. Here

$$\mathcal{P}_\gamma(R)=\{f\in R\mid \nu(f)\geq\gamma\}\quad\text{and}\quad \mathcal{P}^+_\gamma(R)=\{f\in R\mid \nu(f)>\gamma\}.$$

This ring plays an important role in local uniformization of singularities [Teissier 2003; 2014]. The ring $\mathrm{gr}_\nu(R)$ is a domain, but it is often not Noetherian, even when R is.

Suppose that $K\to K^*$ is a finite extension of fields and ν^* is a valuation which is an extension of ν to K^*. We have the classical indices

$$e(\nu^*/\nu)=[\Phi_{\nu^*}:\Phi_\nu]\quad\text{and}\quad f(\nu^*/\nu)=[V_{\nu^*}/m_{\nu^*}:V_\nu/m_\nu]$$

as well as the defect $\delta(\nu^*/\nu)$ of the extension. Ramification of valuations and the defect are discussed in [Zariski and Samuel 1960, Chapter VI; Endler 1972; Kuhlmann 2000; 2010]; a survey is given in [Cutkosky and Piltant 2004, Section 7.1]. By Ostrowski's lemma, if ν^* is the unique extension of ν to K^*, we have that

$$[K^*:K]=e(\nu^*/\nu)f(\nu^*/\nu)p^{\delta(\nu^*/\nu)}, \tag{2}$$

where p is the characteristic of the residue field V_ν/m_ν. From this formula, the defect can be computed using Galois theory in an arbitrary finite extension. If V_ν/m_ν has characteristic 0, then $\delta(\nu^*/\nu)=0$ and $p^{\delta(\nu^*/\nu)}=1$, so there is no defect. Further, if $\Phi_\nu=\mathbb{Z}$ and K^* is separable over K then there is no defect.

If K is an algebraic function field over a field k, then an algebraic local ring R of K is a local domain which is essentially of finite type over k and has K as its field of fractions. In [Cutkosky 1999], it is shown that if $K\to K^*$ is a finite extension of algebraic function fields over a field k of characteristic 0, ν^* is a valuation of K^* (which is trivial on k) with restriction ν to K and if $R\to S$ is an inclusion of algebraic regular local rings of K and K^* such that ν^* dominates S and S dominates R, then there exists a commutative diagram

$$\begin{array}{ccc} R_1 & \longrightarrow & S_1\\ \uparrow & & \uparrow\\ R & \longrightarrow & S\end{array} \tag{3}$$

where the vertical arrows are products of blowups of nonsingular subschemes along the valuation ν^* (monoidal transforms) and $R_1\to S_1$ is dominated by ν^* and is a monomial mapping; that is, there exist regular parameters $x_1,\ldots,x_n$ in R_1, regular parameters $y_1,\ldots,y_n$ in S_1, units $\delta_i\in S_1$, and a matrix $A=(a_{ij})$ of natural numbers with $\mathrm{Det}(A) \neq 0$ such that

$$x_i = \delta_i \prod_{j=1}^{n} y_j^{a_{ij}} \quad \text{for } 1 \leq i \leq n. \tag{4}$$

In [Cutkosky and Piltant 2004], it is shown that this theorem is true, giving a monomial form of the mapping (4) after appropriate blowing up (3) along the valuation, if $K \to K^*$ is a separable extension of two dimension algebraic function fields over an algebraically closed field, which has no defect. This result is generalized to the situation of this paper, namely when R is a 2-dimensional excellent local ring, in [Cutkosky 2016b]. However, it may be that such monomial forms do not exist, even after blowing up, if the extension has defect, as is shown by examples in [Cutkosky 2015].

In the case when k has characteristic 0 and for separable defectless extensions of 2-dimensional algebraic function fields in positive characteristic, it is further shown in [Cutkosky and Piltant 2004] that the expressions (3) and (4) are stable under further simple sequences of blowups along ν^* and the form of the matrix A stably reflects invariants of the valuation.

We always have an inclusion of graded domains $\mathrm{gr}_\nu(R) \to \mathrm{gr}_{\nu^*}(S)$ and the index of their quotient fields is

$$[\mathrm{QF}(\mathrm{gr}_{\nu^*}(S)) : \mathrm{QF}(\mathrm{gr}_\nu(R))] = e(\nu^*/\nu) f(\nu^*/\nu), \tag{5}$$

as shown in [Cutkosky 2016a, Proposition 3.3]. Comparing with Ostrowski's lemma (2), we see that the defect has disappeared in (5).

Even though $\mathrm{QF}(\mathrm{gr}_{\nu^*}(S))$ is finite over $\mathrm{QF}(\mathrm{gr}_\nu(R))$, it is possible for $\mathrm{gr}_{\nu^*}(S)$ to not be a finitely generated $\mathrm{gr}_\nu(R)$-algebra. Examples showing this for extensions $R \to S$ of 2-dimensional algebraic local rings over arbitrary algebraically closed fields are given in Example 9.4 of [Cutkosky and Vinh 2014].

It was shown by Ghezzi, Hà and Kashcheyeva [Ghezzi et al. 2006] for extensions of 2-dimensional algebraic function fields over an algebraically closed field k of characteristic 0, and later by Ghezzi and Kashcheyeva [2007] for defectless separable extensions of 2-dimensional algebraic functions fields over an algebraically closed field k of positive characteristic, that there exists a commutative diagram (3) such that $\mathrm{gr}_{\nu^*}(S_1)$ is a finitely generated $\mathrm{gr}_\nu(R_1)$-algebra. Further, this property is stable under further suitable sequences of blowups.

In [Cutkosky 2016a, Theorem 1.6], it is shown that for algebraic regular local rings of arbitrary dimension, if the ground field k is algebraically closed of characteristic 0 and the valuation has rank 1 and is 0-dimensional ($V_\nu/m_\nu = k$), then we can also construct a commutative diagram (3) such that $\mathrm{gr}_{\nu^*}(S_1)$ is a finitely generated $\mathrm{gr}_\nu(R_1)$-algebra, and this property is stable under further suitable sequences of blowups.

An example is given in [Cutkosky and Piltant 2004] of an inclusion $R \to S$ in a separable defect extension of 2-dimensional algebraic function fields such that $\mathrm{gr}_{\nu^*}(S_1)$ is stably not a finitely generated $\mathrm{gr}_\nu(R_1)$-algebra in diagram (3) under sequences of blowups. This raises the question of whether the existence of a finitely generated extension of associated graded rings along the valuation implies that K^* is a defectless extension of K.

We find that we must impose the condition that K^* is a separable extension of K to obtain a positive answer to this question, as there are simple examples of inseparable defect extensions such that $\mathrm{gr}_{\nu^*}(S)$ is a finitely generated $\mathrm{gr}_\nu(R)$-algebra, such as in the following example, which is Example 8.6 of [Kuhlmann 2000]. Let k be a field of characteristic $p > 0$ and $k((x))$ the field of formal power series over k with the x-adic valuation ν_x. Let $y \in k((x))$ be transcendental over $k(x)$ with $\nu_x(y) > 0$. Let $\tilde{y} = y^p$, and $K = k(x, \tilde{y}) \subset K^* = k(x, y)$. Let $\nu^* = \nu_x | K^*$ and $\nu = \nu_x | K$. Then we have equality of value groups $\Phi_\nu = \Phi_{\nu^*} = \nu(x)\mathbb{Z}$ and equality of residue fields of valuation rings $V_\nu/m_\nu = V_{\nu^*}/m_{\nu^*} = k$, so $e(\nu^*/\nu) = 1$ and $f(\nu^*/\nu) = 1$. We have that ν^* is the unique extension of ν to K^* since K^* is purely inseparable over K. By Ostrowski's lemma (2), the extension $(K, \nu) \to (K^*, \nu^*)$ is a defect extension with defect $\delta(\nu^*/\nu) = 1$. Let $R = k[x, \tilde{y}]_{(x,\tilde{y})} \to S = k[x, y]_{(x,y)}$. Then we have equality

$$\mathrm{gr}_\nu(R) = k[t] = \mathrm{gr}_{\nu^*}(S),$$

where t is the class of x.

In this paper we show that the question does have a positive answer for separable extensions in the following theorem.

Theorem 0.1. *Suppose that R is a 2-dimensional excellent local domain with quotient field K. Further suppose that K^* is a finite separable extension of K and S is a 2-dimensional local domain with quotient field K^* such that S dominates R. Suppose that ν^* is a valuation of K^* such that ν^* dominates S. Let ν be the restriction of ν^* to K. Then the extension $(K, \nu) \to (K^*, \nu^*)$ is without defect if and only if there exist regular local rings R_1 and S_1 such that R_1 is a local ring of a blowup of R, S_1 is a local ring of a blowup of S, ν^* dominates S_1, S_1 dominates R_1 and $\mathrm{gr}_{\nu^*}(S_1)$ is a finitely generated $\mathrm{gr}_\nu(R_1)$-algebra.*

We immediately obtain the following corollary for 2-dimensional algebraic function fields.

Corollary 0.2. *Suppose $K \to K^*$ is a finite separable extension of 2-dimensional algebraic function fields over a field k and ν^* is a valuation of K^* with restriction ν to K. Then the extension $(K, \nu) \to (K^*, \nu^*)$ is without defect if and only if there exist algebraic regular local rings R of K and S of K^* such that ν^* dominates S, S dominates R and $\mathrm{gr}_{\nu^*}(S)$ is a finitely generated $\mathrm{gr}_\nu(R)$-algebra.*

We see from Theorem 0.1 that the defect, which is completely lost in the extension of quotient fields of the associated graded rings along the valuation (5), can be recovered from knowledge of all extensions of associated graded rings along the valuation of regular local rings $R_1 \to S_1$ within the field extensions which dominate $R \to S$ and are dominated by the valuation.

The fact that there exists $R_1 \to S_1$ as in the conclusions of the theorem if the assumptions of the theorem hold and the extension is without defect is proven within 2-dimensional algebraic function fields over an algebraically closed field in [Ghezzi et al. 2006; Ghezzi and Kashcheyeva 2007], and in the generality of the assumptions of Theorem 0.1 in Theorems 4.3 and 4.4 of [Cutkosky 2016b]. Further, if the assumptions of the theorem hold and $\delta(\nu^*/\nu) \neq 0$, then the value group Φ_{ν^*} is not finitely generated by [Cutkosky and Piltant 2004, Theorem 7.3] in the case of algebraic function fields over an algebraically closed field. With the full generality of the hypothesis of Theorem 0.1, the defect is zero by [Endler 1972, Corollary 18.7] in the case of discrete rank 1 valuations and the defect is zero by [Cutkosky 2016b, Theorem 3.7] in the case of rational rank 2 valuations, so by Abhyankar's inequality [Abhyankar 1956, Proposition 2] or [Zariski and Samuel 1960, Appendix 2], if $\delta(\nu^*/\nu) \neq 0$, then the value group Φ_{ν^*} has rational rank 1 and is not discrete and V_{ν^*}/m_{ν^*} is algebraic over S/m_S. Thus, to prove Theorem 0.1, we have reduced to the following proposition, which we establish in this paper.

Proposition 0.3. *Suppose that R is a 2-dimensional excellent local domain with quotient field K. Further suppose that K^* is a finite separable extension of K and S is a 2-dimensional local domain with quotient field K^* such that S dominates R. Suppose that ν^* is a valuation of K^* such that ν^* dominates S. Let ν be the restriction of ν^* to K.*

Suppose that ν^* has rational rank 1 and ν^* is not discrete. Further suppose that there exist regular local rings R_1 and S_1 such that R_1 is a local ring of a blowup of R, S_1 is a local ring of a blowup of S, ν^* dominates S_1, S_1 dominates R_1 and $\mathrm{gr}_{\nu^*}(S_1)$ is a finitely generated $\mathrm{gr}_{\nu}(R_1)$-algebra. Then $\delta(\nu^*/\nu) = 0$.

Another factor in the question of finite generation of extensions of associated graded rings along a valuation is the splitting of ν in K^*. We say that ν does not split in S if ν^* is the unique extension of ν to K^* such that ν^* dominates S. After a little blowing up, we can always obtain nonsplitting, as the following lemma shows.

Lemma 0.4. *Given an extension $R \to S$ as in the hypotheses of Theorem 0.1, there exists a normal local ring R' which is a local ring of a blowup of R such that ν dominates R', and if*

$$\begin{array}{ccc} R_1 & \longrightarrow & S_1 \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

is a commutative diagram of normal local rings, where R_1 is a local ring of a blowup of R and S_1 is a local ring of a blowup of S, ν^* dominates S_1 and R_1 dominates R', then ν does not split in S_1.

Lemma 0.4 will be proven in Section 1. We have the following theorem.

Theorem 0.5. *Suppose that R is a 2-dimensional excellent regular local ring with quotient field K. Further suppose that K^* is a finite separable extension of K and S is a 2-dimensional regular local ring with quotient field K^* such that S dominates R. Suppose that ν^* is a valuation of K^* such that ν^* dominates S. Let ν be the restriction of ν^* to K. Further suppose that ν^* has rational rank 1 and ν^* is not discrete. Suppose that $\mathrm{gr}_{\nu^*}(S)$ is a finitely generated $\mathrm{gr}_\nu(R)$-algebra. Then S is a localization of the integral closure of R in K^*, $\delta(\nu^*/\nu)=0$ and ν^* does not split in S.*

We give examples showing that the condition that ν^* has rational rank 1 and is not discrete in Theorem 0.5 are necessary. As an immediate consequence of Theorem 0.5, we obtain the following corollary.

Corollary 0.6. *Suppose that R is a 2-dimensional excellent regular local ring with quotient field K. Suppose that ν is a valuation of K such that ν dominates R. Further suppose that ν has rational rank 1 and ν is not discrete. Suppose that $R \to R'$ is a nontrivial sequence of quadratic transforms along ν. Then $gr_\nu(R')$ is not a finitely generated $gr_\nu(R)$-algebra.*

Michel Vaquié [2007] extended Mac Lane's theory of key polynomials [Mac Lane 1936] to show that if $(K, \nu) \to (K^*, \nu^*)$ is a finite extension of valued fields with $\delta(\nu^*/\nu)=0$ and ν^* is the unique extension of ν to K^*, then ν^* can be constructed from ν by a finite sequence of augmented valuations. This suggests that a converse of Theorem 0.5 may be true.

1. Local degree and defect

We will use the following criterion to measure defect. This result is implicit in [Cutkosky and Piltant 2004] with the assumptions of Proposition 0.3.

Proposition 1.1 [Cutkosky 2016b, Proposition 3.4]. *Suppose R is a 2-dimensional excellent local domain with quotient field K. Further suppose that K^* is a finite separable extension of K and S is a 2-dimensional local domain with quotient field K^* such that S dominates R. Suppose that ν^* is a valuation of K^* such that ν^* dominates S, the residue field V_{ν^*}/m_{ν^*} of V_{ν^*} is algebraic over S/m_S and the value group Φ_{ν^*} of ν^* has rational rank 1. Let ν be the restriction of ν^* to K. There exists a local ring R' of K which is essentially of finite type over R, is dominated by ν and* *dominates R such that if we have a commutative diagram*

$$\begin{tikzcd} V_\nu \arrow[r] & V_{\nu^*} \\ R_1 \arrow[u] \arrow[r] & S_1 \arrow[u] \\ R' \arrow[u] & \\ R \arrow[u] \arrow[r] & S \arrow[uu] \end{tikzcd} \tag{6}$$

where

- *R_1 is a regular local ring of K which is essentially of finite type over R and dominates R,*
- *S_1 is a regular local ring of K^* which is essentially of finite type over S and dominates S, and*
- *R_1 has a regular system of parameters u, v and S_1 has a regular system of parameters x, y such that there is an expression*

$$u = \gamma x^a, \qquad v = x^b f,$$

where $a > 0$, $b \geq 0$, γ is a unit in S, $x \nmid f$ in S_1 and f is not a unit in S_1,

then

$$ad[S_1/m_{S_1} : R_1/m_{R_1}] = e(\nu^*/\nu) f(\nu^*/\nu) p^{\delta(\nu^*/\nu)}, \tag{7}$$

where $d = \bar{\nu}(f \text{ mod } x)$ with $\bar{\nu}$ being the natural valuation of the DVR S/xS.

Proof of Lemma 0.4. Let $\nu_1 = \nu^*, \nu_2, \ldots, \nu_r$ be the extensions of ν to K^*. Let T be the integral closure of V_ν in K^*. Then $T = V_{\nu_1} \cap \cdots \cap V_{\nu_r}$ is the integral closure of V_{ν^*} in K^* by [Abhyankar 1959, Propositions 2.36 and 2.38]. Let $m_i = m_{\nu_i} \cap T$ be the maximal ideals of T. By the Chinese remainder theorem, there exists $u \in T$ such that $u \in m_1$ and $u \notin m_i$ for $2 \leq i \leq r$. Let

$$u^n + a_1 u^{n-1} + \cdots + a_n = 0$$

be an equation of integral dependence of u over V_ν. Let A be the integral closure of $R[a_1, \ldots, a_n]$ in K and let $R' = A_{A \cap m_\nu}$. Let T' be the integral closure of R' in K^*. We have that $u \in T' \cap m_i$ if and only if $i = 1$. Let $S' = T'_{T' \cap m_1}$. Then ν does not split in S' and R' has the property of the conclusions of the lemma. $\square$

2. Generating sequences

Given an additive group G with $\lambda_0, \ldots, \lambda_r \in G$, let $G(\lambda_0, \ldots, \lambda_r)$ and $S(\lambda_0, \ldots, \lambda_r)$ denote the subgroup and the semigroup, respectively, generated by $\lambda_0, \ldots, \lambda_r$.

In this section, we suppose that R is a regular local ring of dimension 2, with maximal ideal m_R and residue field R/m_R. For $f \in R$, let $\bar{f}$ or $[f]$ denote the residue of f in R/m_R.

The following theorem is Theorem 4.2 of [Cutkosky and Vinh 2014], as interpreted by [Cutkosky and Vinh 2014, Remark 4.3].

Theorem 2.1. *Suppose that ν is a valuation of the quotient field of R dominating R. Let $L = V_\nu/m_\nu$ be the residue field of the valuation ring V_ν of ν. For $f \in V_\nu$, let $[f]$ denote the class of f in L. Suppose that x, y are regular parameters in R. Then there exist $\Omega \in \mathbb{Z}_+ \cup \{\infty\}$ and $P_i(\nu, R) \in m_R$ for $i \in \mathbb{Z}_+$ with $i < \min\{\Omega + 1, \infty\}$ such that $P_0(\nu, R) = x$, $P_1(\nu, R) = y$ and for $1 \le i < \Omega$, there is an expression*

$$P_{i+1}(\nu, R) = P_i(\nu, R)^{n_i(\nu,R)} + \sum_{k=1}^{\lambda_i} c_k P_0(\nu, R)^{\sigma_{i,0}(k)} P_1(\nu, R)^{\sigma_{i,1}(k)} \cdots P_i(\nu, R)^{\sigma_{i,i}(k)}, \quad (8)$$

where $n_i(\nu, R) \ge 1$, $\lambda_i \ge 1$, the c_k are nonzero units in R for $1 \le k \le \lambda_i$, $\sigma_{i,s}(k) \in \mathbb{N}$ for all s, k, and $0 \le \sigma_{i,s}(k) < n_s(\nu, R)$ for $s \ge 1$. Further,

$$n_i(\nu, R)\nu(P_i(\nu, R)) = \nu\big(P_0(\nu, R)^{\sigma_{i,0}(k)} P_1(\nu, R)^{\sigma_{i,1}(k)} \cdots P_i(\nu, R)^{\sigma_{i,i}(k)}\big) \quad (9)$$

for all k.

For all $i \in \mathbb{Z}_+$ with $i < \Omega$, the following are true:

(1) $\nu(P_{i+1}(\nu, R)) > n_i(\nu, R)\nu(P_i(\nu, R))$.

(2) *Suppose that $r \in \mathbb{N}$, $m \in \mathbb{Z}_+$, $j_k(l) \in \mathbb{N}$ for $1 \le l \le m$ and $0 \le j_k(l) < n_k(\nu, R)$ for $1 \le k \le r$ are such that $(j_0(l), j_1(l), \dots, j_r(l))$ are distinct for $1 \le l \le m$, and*

$$\nu\big(P_0(\nu, R)^{j_0(l)} P_1(\nu, R)^{j_1(l)} \cdots P_r(\nu, R)^{j_r(l)}\big) = \nu\big(P_0(\nu, R)^{j_0(1)} \cdots P_r(\nu, R)^{j_r(1)}\big)$$

for $1 \le l \le m$. Then

$$1, \left[\frac{P_0(\nu, R)^{j_0(2)} P_1(\nu, R)^{j_1(2)} \cdots P_r(\nu, R)^{j_r(2)}}{P_0(\nu, R)^{j_0(1)} P_1(\nu, R)^{j_1(1)} \cdots P_r(\nu, R)^{j_r(1)}}\right], \dots, \left[\frac{P_0(\nu, R)^{j_0(m)} P_1(\nu, R)^{j_1(m)} \cdots P_r(\nu, R)^{j_r(m)}}{P_0(\nu, R)^{j_0(1)} P_1(\nu, R)^{j_1(1)} \cdots P_r(\nu, R)^{j_r(1)}}\right]$$

are linearly independent over R/m_R.

(3) *Let*

$$\bar{n}_i(\nu, R) = \big[G\big(\nu(P_0(\nu, R)), \dots, \nu(P_i(\nu, R))\big) : G\big(\nu(P_0(\nu, R)), \dots, \nu(P_{i-1}(\nu, R))\big)\big].$$

Then $\bar{n}_i(\nu, R) \mid \sigma_{i,i}(k)$ for all k in (8). In particular, $n_i(\nu, R) = \bar{n}_i(\nu, R) d_i(\nu, R)$ with $d_i(\nu, R) \in \mathbb{Z}_+$.

(4) *There exists* $U_i(\nu,R)=P_0(\nu,R)^{w_0(i)}P_1(\nu,R)^{w_1(i)}\cdots P_{i-1}(\nu,R)^{w_{i-1}(i)}$ *for* $i\ge 1$ *with* $w_0(i),\ldots,w_{i-1}(i)\in\mathbb{N}$ *and* $0\le w_j(i)<n_j(\nu,R)$ *for* $1\le j\le i-1$ *such that* $\nu(P_i(\nu,R)^{\bar n_i})=\nu(U_i(\nu,R))$. *Setting*

$$\alpha_i(\nu,R)=\left[\frac{P_i(\nu,R)^{\bar n_i(\nu,R)}}{U_i(\nu,R)}\right],$$

we have

$$b_{i,t}=\left[\sum_{\sigma_{i,i}(k)=t\bar n_i(\nu,R)} c_k\frac{P_0(\nu,R)^{\sigma_{i,0}(k)}P_1(\nu,R)^{\sigma_{i,1}(k)}\cdots P_{i-1}(\nu,R)^{\sigma_{i,i-1}(k)}}{U_i(\nu,R)^{(d_i(\nu,R)-t)}}\right] \in R/m_R(\alpha_1(\nu,R),\ldots,\alpha_{i-1}(\nu,R))$$

for $0\le t\le d_i(\nu,R)-1$, *and*

$$f_i(u)=u^{d_i(\nu,R)}+b_{i,d_i(\nu,R)-1}u^{d_i(\nu,R)-1}+\cdots+b_{i,0}$$

is the minimal polynomial of $\alpha_i(\nu,R)$ *over* $R/m_R(\alpha_1(\nu,R),\ldots,\alpha_{i-1}(\nu,R))$.

The algorithm terminates with $\Omega<\infty$ *if and only if either*

$$\bar n_\Omega(\nu,R)=\big[G\big(\nu(P_0(\nu,R)),\ldots,\nu(P_\Omega(\nu,R))\big):G\big(\nu(P_0(\nu,R)),\ldots,\nu(P_{\Omega-1}(\nu,R))\big)\big]=\infty \tag{10}$$

or

$\bar n_\Omega(\nu,R)<\infty$ (*so that* $\alpha_\Omega(\nu,R)$ *is defined as in* (4)) *and*

$$d_\Omega(\nu,R)=\big[R/m_R(\alpha_1(\nu,R),\ldots,\alpha_\Omega(\nu,R)):R/m_R(\alpha_1(\nu,R),\ldots,\alpha_{\Omega-1}(\nu,R))\big]=\infty. \tag{11}$$

If $\bar n_\Omega(\nu,R)=\infty$, *set* $\alpha_\Omega(\nu,R)=1$.

Let notation be as in Theorem 2.1. The following formula is statement $B(i)$ on page 360 of [Cutkosky and Vinh 2014].

Suppose M *is a Laurent monomial in* $P_0(\nu,R),P_1(\nu,R),\ldots,P_i(\nu,R)$ *and* $\nu(M)=0$. *Then there exist* $s_i\in\mathbb{Z}$ *such that*

$$M=\prod_{j=1}^{i}\left[\frac{P_j(\nu,R)^{\bar n_j}}{U_j(\nu,R)}\right]^{s_j}, \tag{12}$$

so that

$$[M]\in R/m_R[\alpha_1(\nu,R),\ldots,\alpha_i(\nu,R)].$$

Define $\beta_i(\nu,R)=\nu(P_i(\nu,R))$ for $0\le i$.

Since ν is a valuation of the quotient field of R, we have that

$$\Phi_\nu=\bigcup_{i=1}^{\infty}G(\beta_0(\nu,R),\beta_1(\nu,R),\ldots,\beta_i(\nu,R)) \tag{13}$$

and

$$V_\nu/m_\nu = \bigcup_{i=1}^{\infty} R/m_R[\alpha_1(\nu, R), \ldots, \alpha_i(\nu, R)]. \tag{14}$$

The following is [Cutkosky and Vinh 2014, Theorem 4.10].

Theorem 2.2. *Suppose that ν is a valuation dominating R. Let*

$$P_0(\nu, R) = x, \quad P_1(\nu, R) = y, \quad P_2(\nu, R), \quad \ldots$$

be the sequence of elements of R constructed by Theorem 2.1. Suppose that $f \in R$ and there exists $n \in \mathbb{Z}_+$ such that $\nu(f) < n\nu(m_R)$. Then there exists an expansion

$$f = \sum_I a_I P_0(\nu, R)^{i_0} P_1(\nu, R)^{i_1} \cdots P_r(\nu, R)^{i_r} + \sum_J \varphi_J P_0(\nu, R)^{j_0} \cdots P_r(\nu, R)^{j_r} + h,$$

where $r \in \mathbb{N}$ and

- *$I = (i_0, \ldots, i_r) \in \mathbb{N}^{r+1}$ with $0 \le i_k < n_k(\nu, R)$ for $1 \le k \le r$, the a_I are units in R and $\nu(P_0(\nu, R)^{i_0} P_1(\nu, R)^{i_1} \cdots P_r(\nu, R)^{i_r}) = \nu(f)$ for all I in the first sum*;
- *$J = (j_0, \ldots, j_r) \in \mathbb{N}^{r+1}$, $\varphi_J \in R$ and $\nu(P_0(\nu, R)^{j_0} \cdots P_r(\nu, R)^{j_r}) > \nu(f)$ for all J in the second sum*; *and*
- *$h \in m_R^n$.*

The terms in the first sum are uniquely determined, *up to the choice of units a_I, whose residues in R/m_R are uniquely determined.*

Let $\sigma_0(\nu, R) = 0$ and inductively define

$$\sigma_{i+1}(\nu, R) = \min\{j > \sigma_i(\nu, R) \mid n_j(\nu, R) > 1\}. \tag{15}$$

In Theorem 2.2, we see that all of the monomials in the expansion of f are in terms of the P_{σ_i}.

We have that

$$S(\beta_0(\nu, R), \beta_1(\nu, R), \ldots, \beta_{\sigma_j(\nu,R)}(\nu, R)) \\ = S(\beta_{\sigma_0}(\nu, R), \beta_{\sigma_1(\nu,R)}(\nu, R), \ldots, \beta_{\sigma_j(\nu,R)}(\nu, R))$$

for all $j \ge 0$ and

$$R/m_R[\alpha_1(\nu, R), \alpha_2(\nu, R), \ldots, \alpha_{\sigma_j(\nu,R)}(\nu, R)] \\ = R/m_R[\alpha_{\sigma_1(\nu,R)}(\nu, R), \alpha_{\sigma_2(\nu,R)}(\nu, R), \ldots, \alpha_{\sigma_j(\nu,R)}(\nu, R)]$$

for all $j \ge 1$.

Suppose that R is a regular local ring of dimension 2 which is dominated by a valuation ν. The quadratic transform T_1 of R along ν is defined as follows. Let

u, v be a system of regular parameters in R. Then $R[v/u] \subset V_\nu$ if $\nu(u) \le \nu(v)$ and $R[u/v] \subset V_\nu$ if $\nu(u) \ge \nu(v)$. Let

$$T_1 = R\left[\frac{v}{u}\right]_{R[v/u]\cap m_\nu} \quad \text{or} \quad T_1 = R\left[\frac{u}{v}\right]_{R[u/v]\cap m_\nu},$$

depending on whether $\nu(u) \le \nu(v)$ or $\nu(u) > \nu(v)$. T_1 is a 2-dimensional regular local ring which is dominated by ν. Let

$$R \to T_1 \to T_2 \to \cdots \tag{16}$$

be the infinite sequence of quadratic transforms along ν, so that $V_\nu = \bigcup_{i\ge 1} T_i$ [Abhyankar 1959, Lemma 4.5] and $L = V_\nu/m_\nu = \bigcup_{i\ge 1} T_i/m_{T_i}$.

For $f \in R$ and $R \to R^*$ a sequence of quadratic transforms along ν, we define a strict transform of f in R^* to be f_1 if $f_1 \in R^*$ is a local equation of the strict transform in R^* of the subscheme $f = 0$ of R. In this way, a strict transform is only defined up to multiplication by a unit in R^*. This ambiguity will not be a difficulty in our proof. We will denote a strict transform of f in R^* by $\mathrm{st}_{R^*}(f)$.

We use the notation of Theorem 2.1 and its proof for R and the $P_i(\nu, R)$. Recall that $U_1 = U^{w_0(1)}$. Let $w = w_0(1)$. Since $\bar{n}_1(\nu, R)$ and w are relatively prime, there exist $a, b \in \mathbb{N}$ such that

$$\varepsilon := \bar{n}_1(\nu, R)b - wa = \pm 1.$$

Define elements of the quotient field of R by

$$x_1 = (x^b y^{-a})^\varepsilon, \qquad y_1 = (x^{-w} y^{\bar{n}_1(\nu,R)})^\varepsilon. \tag{17}$$

We have that

$$x = x_1^{\bar{n}_1(\nu,R)} y_1^a, \qquad y = x_1^w y_1^b. \tag{18}$$

Since $\bar{n}_1(\nu, R)\nu(y) = w\nu(x)$, it follows that

$$\bar{n}_1(\nu, R)\nu(x_1) = \nu(x) > 0 \quad \text{and} \quad \nu(y_1) = 0.$$

We further have that

$$\alpha_1(\nu, R) = [y_1]^\varepsilon \in V_\nu/m_\nu. \tag{19}$$

Let $A = R[x_1, y_1] \subset V_\nu$ and $m_A = m_\nu \cap A$.

Let $R_1 = A_{m_A}$. We have that R_1 is a regular local ring and the divisor of xy in R_1 has only one component ($x_1 = 0$). In particular, $R \to R_1$ is "free" [Cutkosky and Piltant 2004, Definition 7.5]. $R \to R_1$ factors (uniquely) as a product of quadratic transforms and the divisor of xy in R_1 has two distinct irreducible factors in all intermediate rings.

Theorem 2.3 [Cutkosky and Vinh 2014, Theorem 7.1]. *Let R be a 2-dimensional regular local ring with regular parameters x, y. Suppose that R is dominated by a valuation ν. Let $P_0(\nu, R) = x$, $P_1(\nu, R) = y$ and $\{P_i(\nu, R)\}$ be the sequence* *of elements of R constructed in Theorem 2.1. Suppose that $\Omega \geq 2$. Then there exists some smallest value i in the sequence* (16) *such that the divisor of xy in* $\mathrm{Spec}(T_i)$ *has only one component. Let $R_1 = T_i$. Then $R_1/m_{R_1} \cong R/m_R(\alpha_1(\nu, R))$, and there exists $x_1 \in R_1$ and $w \in \mathbb{Z}_+$ such that $x_1 = 0$ is a local equation of the exceptional divisor of* $\mathrm{Spec}(R_1) \to \mathrm{Spec}(R)$, *and $Q_0 = x_1$, $Q_1 = P_2/x_1^{wn_1}$ are regular parameters in R_1. We have that*

$$P_i(\nu, R_1) = \frac{P_{i+1}(\nu, R)}{P_0(\nu, R_1)^{wn_1(\nu,R)\cdots n_i(\nu,R)}}$$

for $1 \leq i < \max\{\Omega, \infty\}$ satisfy the conclusions of Theorem 2.1 for the ring R_1.

We have that

$$G(\beta_0(\nu, R_1), \ldots, \beta_i(\nu, R_1)) = G(\beta_0(\nu, R), \ldots, \beta_{i+1}(\nu, R))$$

for $i \geq 1$, so that

$$\bar{n}_i(\nu, R_1) = \bar{n}_{i+1}(\nu, R) \quad \text{for } i \geq 1$$

and

$$R_1/m_{R_1}[\alpha_1(\nu, R_1), \ldots, \alpha_i(\nu, R_1)] = R/m_R[\alpha_1(\nu, R), \ldots, \alpha_{i+1}(\nu, R)] \quad \text{for } i \geq 1,$$

giving

$$d_i(\nu, R_1) = d_{i+1}(\nu, R) \quad \text{and} \quad n_i(\nu, R_1) = n_{i+1}(\nu, R) \quad \text{for } i \geq 1.$$

Let $\sigma_0(\nu, R_1) = 0$ and inductively define

$$\sigma_{i+1}(\nu, R_1) = \min\{j > \sigma_i(1) \mid n_j(\nu, R_1) > 1\}.$$

We then have that $\sigma_0(\nu, R_1) = 0$ and for $i \geq 1$, $\sigma_i(\nu, R_1) = \sigma_{i+1}(\nu, R) - 1$ if $n_1(\nu, R) > 1$, and $\sigma_i(\nu, R_1) = \sigma_i(\nu, R) - 1$ if $n_1(\nu, R) = 1$. For all $j \geq 0$,

$$\begin{aligned} &S(\beta_0(\nu, R_1), \beta_1(\nu, R_1), \ldots, \beta_{\sigma_{j+1}(\nu,R_1)}(\nu, R_1)) \\ &\qquad = S(\beta_{\sigma_0(1)}(\nu, R_1), \beta_{\sigma_1(\nu,R_1)}, \ldots, \beta_{\sigma_j(\nu,R_1)}(\nu, R_1)). \end{aligned}$$

Iterating this construction, we produce a sequence of sequences of quadratic transforms along ν,

$$R \to R_1 \to \cdots \to R_{\sigma_1(\nu,R)}.$$

Now $x, \bar{y} = P_{\sigma_1(\nu,R)}$ are regular parameters in R. By (17) (with y replaced with $\bar{y}$) we have that $R_{\sigma_1(\nu,R)}$ has regular parameters

$$x_1 = (x^b \bar{y}^{-a})^{\varepsilon}, \quad y_1 = (x^{-\omega} \bar{y}^{\bar{n}_{\sigma_1(\nu,R)}(\nu,R)})^{\varepsilon}, \tag{20}$$

where $\omega, a, b \in \mathbb{N}$ satisfy $\varepsilon = \bar{n}_{\sigma_1(\nu,R)}(\nu, R)b - \omega a = \pm 1$. Furthermore, $R_{\sigma_1(\nu,R_1)}$

has regular parameters $x_{\sigma_1(\nu,R)}$, $y_{\sigma_1(\nu,R)}$, where

$$x = \delta x_{\sigma_1(\nu,R_1)}^{\bar{n}_{\sigma_1(\nu,R)}(\nu,R_1)} \quad \text{and} \quad y_{\sigma_1(\nu,R_1)} = \mathrm{st}_{R_{\sigma_1}(\nu,R_1)} P_{\sigma_1(\nu,R)}(\nu, R)$$

with $\delta \in R_{\sigma_1(\nu,R)}$ a unit.

For the remainder of this section, we suppose that R is a 2-dimensional regular local ring and ν is a nondiscrete rational rank 1 valuation of the quotient field of R with valuation ring V_ν, so that V_ν/m_ν is algebraic over R/m_R. Suppose that $f \in R$ and $\nu(f) = \gamma$. We denote the class of f in $\mathcal{P}_\gamma(R)/\mathcal{P}_\gamma^+(R) \subset \mathrm{gr}_\nu(R)$ by $\mathrm{in}_\nu(f)$. By Theorem 2.2, we have that $\mathrm{gr}_\nu(R)$ is generated by the initial forms of the $P_i(\nu, R)$ as an R/m_R-algebra. That is,

$$\begin{aligned}\mathrm{gr}_\nu(R) &= R/m_R\big[\mathrm{in}_\nu(P_0(\nu, R)), \mathrm{in}_\nu(P_1(\nu, R)), \ldots\big] \\ &= R/m_R\big[\mathrm{in}_\nu(P_{\sigma_0(\nu,R)}(\nu, R)), \mathrm{in}_\nu(P_{\sigma_1(\nu,R)}(\nu, R)), \ldots\big].\end{aligned}$$

Thus the semigroup $S^R(\nu) = \{\nu(f) \mid f \in R\}$ is equal to

$$S^R(\nu) = S(\beta_0(\nu, R), \beta_1(\nu, R), \ldots) = S(\beta_{\sigma_0(\nu,R)}(\nu, R), \beta_{\sigma_1(\nu,R)}(\nu, R), \ldots),$$

the value group Φ_ν is equal to

$$G(\beta_0(\nu, R), \beta_1(\nu, R), \ldots)$$

and the residue field of the valuation ring V_ν/m_ν is

$$R/m_R[\alpha_1(\nu, R), \alpha_2(\nu, R), \ldots] = R/m_R[\alpha_{\sigma_1}(\nu, R), \alpha_{\sigma_2}(\nu, R), \ldots].$$

By (1) of Theorem 2.1, every element $\beta \in S^R(\nu)$ has a unique expression

$$\beta = \sum_{i=0}^{r} a_i \beta_i(\nu, R)$$

for some r with $a_i \in \mathbb{N}$ for all i and $0 \le a_i < n_i(\nu, R)$ for $1 \le i$. In particular, if $a_i \neq 0$ in the expansion then $\beta_i(\nu, R) = \beta_{\sigma_j(\nu,R)}(\nu, R)$ for some j.

Lemma 2.4. *Let*

$$\begin{aligned}
\sigma_i &= \sigma_i(\nu, R), & \sigma_i(1) &= \sigma_i(\nu, R_{\sigma_1}),\\
\beta_i &= \beta_i(\nu, R), & \beta_i(1) &= \beta_i(\nu, R_{\sigma_1}),\\
P_i &= P_i(\nu, R), & P_i(1) &= P_i(\nu, R_{\sigma_1}),\\
n_i &= n_i(\nu, R), & n_i(1) &= n_i(\nu, R_{\sigma_1}),\\
\bar{n}_i &= \bar{n}_i(\nu, R), & \bar{n}_i(1) &= \bar{n}_i(\nu, R_{\sigma_1}).
\end{aligned}$$

Suppose that $i \in \mathbb{N}$, $r \in \mathbb{N}$ and $a_j \in \mathbb{N}$ for $j = 0, \ldots, r$, with $0 \le a_j < n_{\sigma_j}$ for $j \ge 1$, are such that

$$\nu(P_{\sigma_0}^{a_0} \cdots P_{\sigma_r}^{a_r}) > \nu(P_{\sigma_i})$$

or $r < i$ and

$$\nu(P_{\sigma_0}^{a_0}\cdots P_{\sigma_r}^{a_r}) = \nu(P_{\sigma_i}).$$

By (18) *and Theorem 2.3, we have expressions in*

$$R_{\sigma_1} = R[x_1, y_1]_{m_\nu \cap R[x_1,y_1]}$$

where x_1, y_1 are defined by (20),

$$P_{\sigma_0}^{a_0}\cdots P_{\sigma_r}^{a_r} = y_1^{aa_0+ba_1} P_{\sigma_1(1)}(1)^{a_2}\cdots P_{\sigma_{r-1}(1)}(1)^{a_r} P_{\sigma_0(1)}(1)^t$$

where $t = \bar{n}_{\sigma_1}a_0 + \omega a_1 + \omega n_{\sigma_1}a_2 + \cdots + \omega n_{\sigma_1}\cdots n_{\sigma_{r-1}}a_r$ and

$$P_{\sigma_i} = \begin{cases} y_1^a P_{\sigma_0(1)}(1)^{\bar{n}_{\sigma_1}} & \text{if } i = 0,\\ y_1^b P_{\sigma_0(1)}(1)^{\omega} & \text{if } i = 1,\\ P_{\sigma_{i-1}(1)}(1) P_{\sigma_0(1)}(1)^{\omega n_{\sigma_1}\cdots n_{\sigma_{i-1}}} & \text{if } i \geq 2.\end{cases}$$

Let

$$\lambda = \begin{cases} \bar{n}_{\sigma_1} & \text{if } i = 0,\\ \omega & \text{if } i = 1,\\ \omega n_{\sigma_1}\cdots n_{\sigma_{i-1}} & \text{if } i \geq 2.\end{cases}$$

Then $t > \lambda$, except in the case where $i = 1$, $P_{\sigma_0}^{a_0}\cdots P_{\sigma_r}^{a_r} = P_{\sigma_0}$ and $\bar{n}_{\sigma_1} = \omega = 1$. In this case we have $\lambda = t$.

Proof. First suppose that $i \geq 2$ and $r \geq i$. Then

$$t - \lambda = (\bar{n}_{\sigma_1}a_0 + \omega a_1 + \omega n_{\sigma_1}a_2 + \cdots + \omega n_{\sigma_1}\cdots n_{\sigma_{r-1}}a_r) - \omega n_{\sigma_1}\cdots n_{\sigma_{i-1}} > 0.$$

Now suppose that $i \geq 2$ and $r < i$. We have that

$$\begin{aligned}(\bar{n}_{\sigma_1}a_0 + \omega a_1 + \cdots + \omega n_{\sigma_1}\cdots n_{\sigma_{r-1}}a_r - \omega n_{\sigma_1}\cdots n_{\sigma_{i-1}})\beta_{\sigma_0(1)}(1)&\\ \geq \beta_{\sigma_{i-1}(1)}(1) - a_2\beta_{\sigma_1(1)}(1) - \cdots - a_r\beta_{\sigma_{r-1}(1)}(1) &> 0,\end{aligned}$$

since $n_{\sigma_j(1)}(1) = n_{\sigma_{j+1}}$ for all j, and so $n_{\sigma_{j+1}}\beta_{\sigma_j(1)}(1) < \beta_{\sigma_{j+1}(1)}(1)$ for all j.

Now suppose that $i = 1$. As in the proof for the case $i \geq 2$, we have that $t - \lambda > 0$ if $r \geq 1$, so suppose that $i = 1$ and $r = 0$. Then $\bar{n}_{\sigma_1}\beta_{\sigma_1} = \omega\beta_{\sigma_0}$. From our assumption $a_0\nu(P_0) \geq \nu(P_1)$, we obtain $t - \lambda = \bar{n}_{\sigma_1}a_0 - \omega \geq 0$ with equality if and only if $a_0 = \omega = \bar{n}_{\sigma_1} = 1$ since $\gcd(\omega, \bar{n}_{\sigma_1}) = 1$.

Now suppose $i = 0$. As in the previous cases, we have $t - \lambda > 0$ if $r > 1$ and $t - \lambda > 0$ if $r = 1$ except possibly if $P_0^{a_0}\cdots P_r^{a_r} = P_1^{a_1}$. We then have that $\nu(P_{\sigma_1}^{a_1}) > \nu(P_{\sigma_0})$, and so

$$a_1\frac{\beta_{\sigma_1}}{\beta_{\sigma_0}} > 1.$$

Since

$$\frac{\beta_{\sigma_1}}{\beta_{\sigma_0}} = \frac{\omega}{\bar{n}_{\sigma_1}},$$

we have that $t - \lambda = \omega a_1 - \bar{n}_{\sigma_1} > 0$. ☐

Lemma 2.5. *Let notation be the same as in Lemma 2.4. Suppose that $f \in R$, with $\nu(f) = \nu(P_{\sigma_i})$ for some $i \geq 0$, and that f has an expression of the form of Theorem 2.2,*

$$f = cP_{\sigma_i} + \sum_{j=1}^{s} c_i P_{\sigma_0}^{a_0(j)} P_{\sigma_1}^{a_1(j)} \cdots P_{\sigma_r}^{a_r(j)} + h,$$

where

- $s, r \in \mathbb{N}$;
- c, c_j *are units in* R;
- $0 \leq a_k(j) < n_k$ *for* $1 \leq k \leq r$ *and* $1 \leq j \leq s$;
- $\nu(f) = \nu(P_{\sigma_i}) \leq \nu\big(P_{\sigma_0}^{a_0(j)} P_{\sigma_1}^{a_1(j)} \cdots P_{\sigma_r}^{a_r(j)}\big)$ *for* $1 \leq j \leq s$;
- $a_k(j) = 0$ *for* $k \geq i$ *if* $\nu(f) = \nu\big(P_{\sigma_0}^{a_p(j)} \cdots P_{\sigma_r}^{a_r(j)}\big)$; *and*
- $h \in m_R^n$ *with* $n > \nu(f)$.

Then $\mathrm{st}_{R_{\sigma_1}}(f)$ *is a unit in* R_{σ_1} *if* $i = 0$ *or* 1, *and if* $i > 1$, *there exists a unit* $\bar{c}$ *in* R_{σ_1} *and* $\Omega \in R_{\sigma_1}$ *such that*

$$\mathrm{st}_{R_{\sigma_1}}(f) = \bar{c} P_{\sigma_{i-1}(1)}(1) + x_1 \Omega$$

with $\nu(\mathrm{st}_{R_{\sigma_1}}(f)) = \nu(P_{\sigma_{i-1}(1)}(1))$ *and* $\nu(P_{\sigma_{i-1}(1)}(1)) \leq \nu(x_1 \Omega)$.

Proof. Let

$$\lambda = \begin{cases} \bar{n}_1 & \text{if } i = 0, \\ \omega & \text{if } i = 1, \\ \omega n_{\sigma_1} \cdots n_{\sigma_{r-1}} & \text{if } i \geq 2. \end{cases}$$

Then

$$f = cH_i + \sum_{j=1}^{s} c_j (y_1)^{aa_0(j)+ba_1(j)} P_{\sigma_0(1)}(1)^{t_j} P_{\sigma_1}(1)^{a_2(j)} \cdots P_{\sigma_{r-1}(1)}(1)^{a_r(j)} + P_{\sigma_0(1)}(1)^t h'$$

with

$$H_i = \begin{cases} (y_1)^a P_{\sigma_0(1)}(1)^{\bar{n}_1} & \text{if } i = 0, \\ (y_1)^b P_{\sigma_0(1)}(1)^{\omega} & \text{if } i = 1, \\ P_{\sigma_0(1)}(1)^{\omega n_1 \cdots n_{i-1}} P_{\sigma_{i-1}(1)}(1) & \text{if } i \geq 2, \end{cases}$$

$h' \in R_{\sigma_1}$ and

$$t_j = \bar{n}_1 a_0(j) + \omega a_1(j) + \omega n_{\sigma_1} a_2(j) + \cdots + \omega n_{\sigma_1} \cdots n_{\sigma_{r-1}} a_r(j)$$

for $1 \leq j \leq s$ and $t > \lambda$. By Lemma 2.4, if $i \geq 2$ or $i = 0$, we have that $t_j > \lambda$ for all j. Thus $f = P_{\sigma_0(1)}(1)^{\lambda} \bar{f}$, where

$$\bar{f} = cG_i + \sum_{j=1}^{s} c_j P_{\sigma_0(1)}(1)^{t_j - \lambda} P_{\sigma_1(1)}(1)^{a_2(j)} \cdots P_{\sigma_{r-1}(1)}(1)^{a_r(j)} + P_{\sigma_0(1)}(1)^{t-\lambda} h'$$

is a strict transform $\bar{f} = \mathrm{st}_{R_1}(f)$ of f in R_1, with

$$G_i = \begin{cases} (y_1)^a & \text{if } i = 0, \\ (y_1)^b & \text{if } i = 1, \\ P_{\sigma_{i-1}(1)}(1) & \text{if } i \geq 2. \end{cases}$$

If $i = 1$, then by Lemma 2.4, $t_j > \lambda$ for all j, except possibly for a single term (which we can assume is t_1) which is P_{σ_0}, and we have that $\omega = \bar{n}_{\sigma_1} = 1$. In this case $t_1 = \lambda$. Then

$$\left[\frac{P_{\sigma_1}}{P_{\sigma_0}}\right] = \alpha_{\sigma_1}(\nu, R) \in V_\nu/m_\nu,$$

which has degree $d_{\sigma_1}(\nu, R) = n_{\sigma_1} > 1$ over R/m_R. By (18), $x = x_1$, $y = x_1 y_1$ and

$$f = x_1[c + c_1 y_1 + x_1 \Omega]$$

with $\Omega \in R_{\sigma_1}$. We have that $c + c_1 y_1$ is a unit in R_{σ_1} since

$$[y_1] = \left[\frac{P_{\sigma_0}}{P_{\sigma_1}}\right] \notin R/m_R. \qquad \square$$

3. Finite generation implies no defect

Suppose that R is a 2-dimensional excellent regular local ring and S is a 2-dimensional regular local ring such that S dominates R. Let K be the quotient field of R and K^* the quotient field of S. Suppose that $K \to K^*$ is a finite separable field extension. Suppose that ν^* is a nondiscrete rational rank 1 valuation of K^* such that V_{ν^*}/m_{ν^*} is algebraic over S/m_S and that ν^* dominates S. Then we have a natural graded inclusion $\mathrm{gr}_\nu(R) \to \mathrm{gr}_{\nu^*}(S)$, so that for $f \in R$, we have that $\mathrm{in}_\nu(f) = \mathrm{in}_{\nu^*}(f)$. Let $\nu = \nu^*|K$. Let $L = V_{\nu^*}/m_{\nu^*}$. Suppose that $\mathrm{gr}_{\nu^*}(S)$ is a finitely generated $\mathrm{gr}_\nu(R)$-algebra.

Let x, y be regular parameters in R, with associated generating sequence to ν, $P_0 = P_0(\nu, R) = x$, $P_1 = P_1(\nu, R) = y$, $P_2 = P_2(\nu, R), \ldots$ in R as constructed in Theorem 2.1, with $U_i = U_i(\nu, R)$, $\beta_i = \beta_i(\nu, R) = \nu(P_i)$, $\gamma_i = \alpha_i(\nu, R)$, $m_i = m_i(\nu, R)$, $\bar{m}_i = \bar{m}_i(\nu, R)$, $d_i = d_i(\nu, R)$ and $\sigma_i = \sigma_i(\nu, R)$ defined as in Section 2.

Similarly, let u, v be regular parameters in S, with associated generating sequence to ν^*, $Q_0 = P_0(\nu^*, S) = u$, $Q_1 = P_1(\nu^*, S) = v$, $Q_2 = P_2(\nu^*, S), \ldots$ in S as constructed in Theorem 2.1, with $V_i = U_i(\nu^*, S)$, $\gamma_i = \beta_i(\nu^*, S) = \nu^*(Q_i)$, $\delta_i = \alpha_i(\nu^*, S)$, $n_i = n_i(\nu^*, S)$, $\bar{n}_i = \bar{n}_i(\nu^*, S)$, $e_i = \alpha_i(\nu^*, S)$ and $\tau_i = \sigma_i(\nu^*, S)$ defined as in Section 2.

With our assumption that $\mathrm{gr}_{\nu^*}(S)$ is a finitely generated $\mathrm{gr}_\nu(R)$-algebra, we have that for all sufficiently large l,

$$\mathrm{gr}_{\nu^*}(S) = \mathrm{gr}_\nu(R)[\mathrm{in}_{\nu^*} Q_{\tau_0}, \ldots, \mathrm{in}_{\nu^*} Q_{\tau_l}]. \tag{21}$$

Proposition 3.1. *With our assumption that* $\mathrm{gr}_{\nu^*}(S)$ *is a finitely generated* $\mathrm{gr}_{\nu}(R)$*-algebra, there exist integers* $s > 1$ *and* $r > 1$ *such that for all* $j \geq 0$,

$$\beta_{\sigma_{r+j}} = \gamma_{\tau_{s+j}}, \quad \bar{m}_{\sigma_{r+j}} = \bar{n}_{\tau_{s+j}}, \quad d_{\sigma_{r+j}} = e_{\tau_{s+j}}, \quad m_{\sigma_{r+j}} = n_{\tau_{s+j}},$$
$$G(\beta_{\sigma_0}, \ldots, \beta_{\sigma_{r+j}}) \subset G(\gamma_{\tau_0}, \ldots, \gamma_{\tau_{s+j}}),$$
$$[G(\gamma_{\tau_0}, \ldots, \gamma_{\tau_{s+j}}) : G(\beta_{\sigma_0}, \ldots, \beta_{\sigma_{r+j}})] = e(\nu^*/\nu),$$
$$R/m_R[\delta_{\sigma_1}, \ldots, \delta_{\sigma_{r+j}}] \subset S/m_S[\varepsilon_{\tau_1}, \ldots, \varepsilon_{\tau_{s+j}}]$$

and

$$[S/m_S[\varepsilon_{\tau_1}, \ldots, \varepsilon_{\tau_{s+j}}] : R/m_R[\delta_{\sigma_1}, \ldots, \delta_{\sigma_{r+j}}]] = f(\nu^*/\nu).$$

Proof. Let l be as in (21). For $s \geq l$, define the subalgebra A_{τ_s} of $\mathrm{gr}_{\nu^*}(S)$ by

$$A_{\tau_s} = S/m_S[\mathrm{in}_{\nu^*} Q_{\tau_0}, \ldots, \mathrm{in}_{\nu^*} Q_{\tau_s}].$$

For $s \geq l$, let

$$r_s = \max\{j \mid \mathrm{in}_{\nu^*} P_{\sigma_j} \in A_{\tau_s}\},$$
$$\lambda_s = [G(\gamma_{\tau_0}, \ldots, \gamma_{\tau_s}) : G(\beta_{\sigma_0}, \ldots, \beta_{\sigma_{r_s}})],$$

and

$$\chi_s = [S/m_S[\varepsilon_{\tau_0}, \ldots, \varepsilon_{\tau_s}] : R/m_R[\delta_{\sigma_0}, \ldots, \delta_{\sigma_{r_s}}]].$$

To simplify notation, we write $r = r_s$.

We now show that $\beta_{\sigma_{r+1}} = \gamma_{\tau_{s+1}}$. Suppose that $\beta_{\sigma_{r+1}} > \gamma_{\tau_{s+1}}$. We have that

$$\mathrm{in}_{\nu^*} Q_{\tau_{s+1}} \in \mathrm{gr}_{\nu}(R)[\mathrm{in}_{\nu^*} Q_{\tau_0}, \ldots, \mathrm{in}_{\nu^*} Q_{\tau_s}].$$

Since

$$\beta_{\sigma_{r+1}} < \beta_{\sigma_{r+2}} < \cdots,$$

we then have that $\mathrm{in}_{\nu^*} Q_{\tau_{s+1}} \in A_{\tau_s}$, which is impossible. Thus $\beta_{\sigma_{r+1}} \leq \gamma_{\tau_{s+1}}$. If $\beta_{\sigma_{r+1}} < \gamma_{\tau_{s+1}}$, then since

$$\gamma_{\tau_{s+1}} < \gamma_{\tau_{s+2}} < \cdots \quad \text{and} \quad \mathrm{in}_{\nu^*} P_{\sigma_{r+1}} \in \mathrm{gr}_{\nu^*}(S),$$

we have that $\mathrm{in}_{\nu^*} P_{\sigma_{r+1}} \in A_{\tau_s}$, which is impossible. Thus $\beta_{\sigma_{r+1}} = \gamma_{\tau_{s+1}}$.

We now establish that either we have a reduction $\lambda_{s+1} < \lambda_s$ or

$$\lambda_{s+1} = \lambda_s, \qquad \beta_{\sigma_{r+1}} = \gamma_{\tau_{s+1}} \quad \text{and} \quad \bar{m}_{\sigma_{r+1}} = \bar{n}_{\tau_{s+1}}. \tag{22}$$

Let ω be a generator of the group $G(\gamma_{\tau_1}, \ldots, \gamma_{\tau_s})$, so that $G(\gamma_{\tau_1}, \ldots, \gamma_{\tau_s}) = \mathbb{Z}\omega$. We have that

$$G(\gamma_{\tau_0}, \ldots, \gamma_{\tau_{s+1}}) = \frac{1}{\bar{n}_{\tau_{s+1}}} \mathbb{Z}\omega$$

and

$$G(\beta_{\sigma_0}, \ldots, \beta_{\sigma_{r+1}}) = \frac{1}{\bar{m}_{\sigma_{r+1}}} \mathbb{Z}(\lambda_s \omega).$$

There exists a positive integer f with $\gcd(f, \bar{n}_{\tau_{s+1}}) = 1$ such that

$$\gamma_{\tau_{s+1}} = \frac{f}{\bar{n}_{\tau_{s+1}}}\omega,$$

and a positive integer g with $\gcd(g, \overline{m}_{\sigma_{r+1}}) = 1$ such that

$$\beta_{\sigma_{r+1}} = \frac{g}{\overline{m}_{\sigma_{r+1}}}\lambda_s\omega.$$

Since $\beta_{\sigma_{r+1}} = \gamma_{\tau_{s+1}}$, we have

$$g\lambda_s\bar{n}_{\tau_{s+1}} = f\overline{m}_{\sigma_{r+1}}.$$

Thus $\bar{n}_{\tau_{s+1}}$ divides $\overline{m}_{\sigma_{r+1}}$ and $\overline{m}_{\sigma_{r+1}}$ divides $\lambda_s\bar{n}_{\tau_{s+1}}$, so that

$$a = \frac{\overline{m}_{\sigma_{r+1}}}{\bar{n}_{\tau_{s+1}}}$$

is a positive integer, and defining

$$\bar{\lambda} = \frac{\lambda_s}{a},$$

$\bar{\lambda}$ is a positive integer with

$$\frac{\lambda_s}{\overline{m}_{\sigma_{r+1}}} = \frac{\bar{\lambda}}{\bar{n}_{\tau_{s+1}}}$$

and

$$\bar{\lambda} = [G(\gamma_{\tau_0}, \ldots, \gamma_{\tau_{s+1}}) : G(\beta_{\sigma_0}, \ldots, \beta_{\sigma_{r+1}})].$$

Since $\lambda_{s+1} \le \bar{\lambda}$, either $\lambda_{s+1} < \lambda_s$ or $\lambda_{s+1} = \lambda_s$ and $\overline{m}_{\sigma_{r+1}} = \bar{n}_{\tau_{s+1}}$.

We now suppose that s is sufficiently large that (22) holds. Since

$$\mathrm{in}_{\nu^*} Q_{\tau_{s+1}} \in \mathrm{gr}_{\nu^*}(S) = \mathrm{gr}_{\nu}(R)[\mathrm{in}_{\nu^*} Q_{\tau_0}, \ldots, \mathrm{in}_{\nu^*} Q_{\tau_s}],$$

if $\bar{n}_{\tau_{s+1}} > 1$ we have an expression

$$\mathrm{in}_{\nu^*} P_{\sigma_{r+1}} = \mathrm{in}_{\nu^*}(\alpha)\mathrm{in}_{\nu^*} Q_{\tau_{s+1}} \tag{23}$$

in $\mathcal{P}_{\gamma_{\tau_{s+1}}}(S)/\mathcal{P}^+_{\gamma_{\tau_{s+1}}}(S)$ with α a unit in S, and if $\bar{n}_{\tau_{s+1}} = 1$, since $\mathrm{in}_{\nu^*} P_{\sigma_{r+1}} \notin A_{\tau_s}$ we have an expression

$$\mathrm{in}_{\nu^*} P_{\sigma_{r+1}} = \mathrm{in}_{\nu^*}(\alpha)\mathrm{in}_{\nu^*} Q_{\tau_{s+1}} + \sum \mathrm{in}_{\nu^*}(\alpha_J)(\mathrm{in}_{\nu^*} Q_{\tau_0})^{j_0} \cdots (\mathrm{in}_{\nu^*} Q_{\tau_s})^{j_s} \tag{24}$$

in $\mathcal{P}_{\gamma_{\tau_{s+1}}}(S)/\mathcal{P}^+_{\gamma_{\tau_{s+1}}}(S)$ with α a unit in S, where the sum is taken over certain $J = (j_0, \ldots, j_s) \in \mathbb{N}^{s+1}$ such that the α_J are units in S, and the terms $\mathrm{in}_{\nu^*} Q_{\tau_{s+1}}$ and $(\mathrm{in}_{\nu^*} Q_{\tau_0})^{j_0} \cdots (\mathrm{in}_{\nu^*} Q_{\tau_s})^{j_s}$ are all linearly independent over S/m_S.

The monomial $U_{\sigma_{r+1}}$ in $P_{\sigma_0}, \ldots, P_{\sigma_r}$ and the monomial $V_{\tau_{s+1}}$ in $Q_{\tau_0}, \ldots, Q_{\tau_s}$ both have the value $\bar{n}_{\tau_{s+1}} \gamma_{\tau_{s+1}} = \overline{m}_{\sigma_{r+1}} \beta_{\sigma_{r+1}}$, and satisfy

$$\varepsilon_{\tau_{s+1}} = \left[\frac{Q_{\tau_{s+1}}^{\bar{n}_{\tau_{s+1}}}}{V_{\tau_{s+1}}} \right]$$

and

$$\delta_{\sigma_{r+1}} = \left[\frac{P_{\sigma_{r+1}}^{\bar{n}_{\tau_{s+1}}}}{U_{\sigma_{r+1}}} \right].$$

Since $U_{\sigma_{r+1}}, V_{\tau_{s+1}} \in A_{\tau_s}$ and by (12) and Theorem 2.1(2), we have that

$$\left[\frac{V_{\tau_{s+1}}}{U_{\sigma_{r+1}}} \right] \in S/m_S[\varepsilon_{\tau_1}, \ldots, \varepsilon_{\tau_s}].$$

If $\bar{n}_{\tau_{s+1}} > 1$, then by (23), we have

$$\left[\frac{P_{\sigma_{r+1}}^{\bar{n}_{\tau_{s+1}}}}{U_{\sigma_{r+1}}} \right] = \left[\frac{V_{\tau_{s+1}}}{U_{\sigma_{r+1}}} \right] \left([\alpha]^{\bar{n}_{\tau_{s+1}}} \left[\frac{Q_{\tau_{s+1}}^{\bar{n}_{\tau_{s+1}}}}{V_{\tau_{s+1}}} \right] \right)$$

in $L = V_{\nu^*}/m_{\nu^*}$, and if $\bar{n}_{\tau_{s+1}} = 1$, then by (24) we have

$$\left[\frac{P_{\sigma_{r+1}}}{U_{\sigma_{r+1}}} \right] = \left[\frac{V_{\tau_{s+1}}}{U_{\sigma_{r+1}}} \right] \left([\alpha] \left[\frac{Q_{\tau_{s+1}}}{V_{\tau_{s+1}}} \right] + \sum [\alpha_J] \left[\frac{Q_{\tau_0}^{j_0} \cdots Q_{\tau_s}^{j_s}}{V_{\tau_{s+1}}} \right] \right).$$

Thus by (12),

$$S/m_S[\varepsilon_{\tau_1}, \ldots, \varepsilon_{\tau_s}][\varepsilon_{\tau_{s+1}}] = S/m_S[\varepsilon_{\tau_1}, \ldots, \varepsilon_{\tau_s}][\delta_{\sigma_{r+1}}]. \tag{25}$$

We have a commutative diagram

$$\begin{array}{ccc}
S/m_S[\varepsilon_{\tau_1}, \ldots, \varepsilon_{\tau_s}] & \longrightarrow & S/m_S[\varepsilon_{\tau_1}, \ldots, \varepsilon_{\tau_s}, \varepsilon_{\tau_{s+1}}] = S/m_S[\varepsilon_{\tau_1}, \ldots, \varepsilon_{\tau_s}][\delta_{\sigma_{r+1}}] \\
\uparrow & & \uparrow \\
R/m_R[\delta_{\sigma_1}, \ldots, \delta_{\sigma_r}] & \longrightarrow & R/m_R[\delta_{\sigma_1}, \ldots, \delta_{\sigma_r}][\delta_{\sigma_{r+1}}]
\end{array}$$

Let

$$\bar{\chi} = [S/m_S[\varepsilon_{\tau_1}, \ldots, \varepsilon_{\tau_s}, \varepsilon_{\tau_{s+1}}] : R/m_R[\delta_{\sigma_1}, \ldots, \delta_{\sigma_r}, \delta_{\sigma_{r+1}}]].$$

Since

$$S/m_S[\varepsilon_{\tau_1}, \ldots, \varepsilon_{\tau_s}, \varepsilon_{\tau_{s+1}}] = S/m_S[\varepsilon_{\tau_1}, \ldots, \varepsilon_{\tau_s}][\delta_{\sigma_{r+1}}],$$

we have that $e_{\tau_{s+1}} \mid d_{\sigma_{r+1}}$. Further,

$$\frac{d_{\sigma_{r+1}}}{e_{\tau_{s+1}}} \bar{\chi} = \chi_s,$$

whence $\bar{\chi} \leq \chi_s$. Thus $\chi_{s+1} \leq \chi_s$ and if $\chi_{s+1} = \chi_s$, then $d_{\sigma_{r+1}} = e_{\tau_{s+1}}$ and $r_{s+1} = r_s + 1$, since $P_{\sigma_{r+2}} \in A_{\tau_{s+1}}$ implies $\lambda_{s+1} < \lambda_s$ or $\chi_{s+1} < \chi_s$.

We may thus choose s sufficiently large that there exists an integer $r > 1$ such that for all $j \geq 0$,

$$\beta_{\sigma_{r+j}} = \gamma_{\tau_{s+j}}, \qquad \bar{m}_{\sigma_{r+j}} = \bar{n}_{\tau_{s+j}}, \qquad d_{\sigma_{r+j}} = e_{\tau_{s+j}}, \qquad m_{\sigma_{r+j}} = n_{\tau_{s+j}},$$
$$G(\beta_{\sigma_0}, \ldots, \beta_{\sigma_{r+j}}) \subset G(\gamma_{\tau_0}, \ldots, \gamma_{\tau_{s+j}}),$$

there is a constant λ (which does not depend on j) such that

$$[G(\gamma_{\tau_0}, \ldots, \gamma_{\tau_{s+j}}) : G(\beta_{\sigma_0}, \ldots, \beta_{\sigma_{r+j}})] = \lambda,$$
$$R/m_R[\delta_{\sigma_1}, \ldots, \delta_{\sigma_{r+j}}] \subset S/m_S[\varepsilon_{\tau_1}, \ldots, \varepsilon_{\tau_{s+j}}],$$

and there is a constant χ (which does not depend on j) such that

$$[S/m_S[\varepsilon_{\tau_1}, \ldots, \varepsilon_{\tau_{s+j}}] : R/m_R[\delta_{\sigma_1}, \ldots, \delta_{\sigma_{r+j}}]] = \chi.$$

Then

$$\Phi_{\nu^*} = \bigcup_{j \geq 1} \frac{1}{\bar{n}_{\tau_{s+1}} \cdots \bar{n}_{\tau_{s+j}}} \mathbb{Z}\omega,$$

where $G(\gamma_{\tau_0}, \ldots, \gamma_{\tau_s}) = \mathbb{Z}\omega$, and

$$\Phi_\nu = \bigcup_{j \geq 1} \frac{1}{\bar{m}_{\sigma_{r+1}} \cdots \bar{m}_{\sigma_{r+j}}} \lambda\mathbb{Z}\omega = \bigcup_{j \geq 1} \frac{1}{\bar{n}_{\tau_{s+1}} \cdots \bar{n}_{\tau_{s+j}}} \lambda\mathbb{Z}\omega,$$

so that

$$\lambda = [\Phi_{\nu^*} : \Phi_\nu] = e(\nu^*/\nu).$$

For $i \geq 0$, let $K_i = R/m_R[\delta_{\sigma_1}, \ldots, \delta_{\sigma_{r+i}}]$ and $M_i = S/m_S[\varepsilon_{\tau_1}, \ldots, \varepsilon_{\tau_{s+i}}]$. We have that $M_{i+1} = M_i[\delta_{\sigma_{r+i+1}}]$ for $i \geq 0$ and $\chi = [M_i : K_i]$ for all i. Further,

$$\bigcup_{i=0}^{\infty} M_i = V_{\nu^*}/m_{\nu^*} \quad \text{and} \quad \bigcup_{i=0}^{\infty} K_i = V_\nu/m_\nu.$$

Thus if $g_1, \ldots, g_\lambda \in M_0$ form a basis of M_0 as a K_0-vector space, then $g_1, \ldots, g_\lambda$ form a basis of M_i as a K_i-vector space for all $i \geq 0$. Thus

$$\chi = [V_{\nu^*}/m_{\nu^*} : V_\nu/m_\nu] = f(\nu^*/\nu). \qquad \square$$

Let r and s be as in the conclusions of Proposition 3.1. There exists τ_t with $t \geq s$ such that we have a commutative diagram of inclusions of regular local rings

$$\begin{array}{ccc} R_{\sigma_r} & \longrightarrow & S_{\tau_t} \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

(with the notation introduced in Section 2). After possibly increasing s and r, we may assume that $R' \subset R_{\sigma_r}$, where R' is the local ring of the conclusions of Proposition 1.1. Recall that R has regular parameters $x = P_0$, $y = P_1$ and S has regular parameters $u = Q_0$, $v = Q_1$; and that R_{σ_r} has regular parameters x_{σ_r}, y_{σ_r} such that

$$x = \delta x_{\sigma_r}^{\overline{m}_{\sigma_1}\cdots\overline{m}_{\sigma_r}}, \qquad y_{\sigma_r} = \mathrm{st}_{R_{\sigma_r}} P_{\sigma_{r+1}},$$

where δ is a unit in R_{σ_r} and S_{τ_t} has regular parameters u_{τ_t}, v_{τ_t} such that

$$u = \varepsilon u_{\tau_t}^{\bar{n}_{\tau_1}\cdots\bar{n}_{\tau_t}}, \qquad v_{\tau_t} = \mathrm{st}_{S_{\tau_t}} Q_{\tau_{t+1}},$$

where ε is a unit in S_{τ_t}. We may choose $t \gg 0$ so that we have an expression

$$x_{\sigma_r} = \varphi u_{\tau_t}^{\lambda} \tag{26}$$

for some positive integer λ, where φ is a unit in S_{τ_t}, since $\bigcup_{t=0}^{\infty} S_{\tau_t} = V_{\nu^*}$.

We have expressions $P_i = \psi_i x_{\sigma_r}^{c_i}$ in R_{σ_r}, where the ψ_i are units in R_{σ_r} for $i \le \sigma_r$, so that $P_i = \psi_i^* u_{\tau_t}^{c_i\lambda}$ in S_{τ_t}, where the ψ_i^* are units in S_{τ_t} for $i \le \sigma_r$, by (26).

Lemma 3.2. *For $j \ge 1$ we have*

$$\mathrm{st}_{R_{\sigma_r}}(P_{\sigma_{r+j}}) = u_{\tau_t}^{\lambda_j}\, \mathrm{st}_{S_{\tau_t}}(P_{\sigma_{r+j}})$$

for some $\lambda_j \in \mathbb{N}$, where we regard $P_{\sigma_{r+j}}$ as an element of R on the left-hand side of the equation and regard $P_{\sigma_{r+j}}$ as an element of S on the right-hand side.

Proof. Using (26), we have

$$P_{\sigma_{r+j}} = \mathrm{st}_{R_{\sigma_r}}(P_{\sigma_{r+j}}) x_{\sigma_r}^{f_j} = \mathrm{st}_{R_{\sigma_r}}(P_{\sigma_{r+j}}) u_{\tau_t}^{\lambda f_j} \varphi^{f_j},$$

where $f_j \in \mathbb{N}$. Viewing $P_{\sigma_{r+j}}$ as an element of S, we have that

$$P_{\sigma_{r+j}} = \mathrm{st}_{S_{\tau_t}}(P_{\sigma_{r+j}}) u_{\tau_t}^{g_j}$$

for some $g_j \in \mathbb{N}$. Since $u_{\tau_t} \nmid \mathrm{st}_{S_{\tau_t}}(P_{\sigma_{r+j}})$, we have that $f_j\lambda \le g_j$ and therefore $\lambda_j = g_j - f_j\lambda \ge 0$. □

By induction on the sequence of quadratic transforms above R and S from Lemma 2.5, and since $\nu^*(P_{\sigma_{r+j}}) = \beta_{\sigma_{r+j}} = \gamma_{\tau_{s+j}}$ by Proposition 3.1, we have by (23) and (24) an expression

$$\mathrm{st}_{S_{\tau_t}}(P_{\sigma_{r+j}}) = c\, \mathrm{st}_{S_{\tau_t}}(Q_{\tau_{s+j}}) + u_{\tau_t}\Omega \tag{27}$$

with $c \in S_{\tau_t}$ a unit, $\Omega \in S_{\tau_t}$ and $\nu^*(u_{\tau_t}\Omega) \ge \nu^*(\mathrm{st}_{S_{\tau_t}}(Q_{\tau_{s+j}}))$ if $s + j > t$; and

$$S_{\tau_t}(P_{\sigma_{r+j}}) \text{ is a unit in } S_{\tau_t} \tag{28}$$

if $s + j \le t$. Thus $P_{\sigma_{r+j}} = u_{\tau_t}^{d_j} \bar{\varphi}_j$ in S_{τ_t}, where d_j is a positive integer and $\bar{\varphi}_j$ is a unit in S_{τ_t} if $s + j \le t$.

Suppose $s < t$. Then

$$y_{\sigma_r} = \mathrm{st}_{R_{\sigma_r}}(P_{\sigma_{r+1}}) = \tilde{\varphi} u_{\tau_t}^{h},$$

where $\tilde{\varphi}$ is a unit in S_{τ_t} and h is a positive integer. As shown in (20) of Section 2,

$$R_{\sigma_{r+1}} = R_{\sigma_r}[\bar{x}_1, \bar{y}_1]_{m_\nu \cap R_{\sigma_r}[\bar{x}_1, \bar{y}_1]},$$

where

$$\bar{x}_1 = (x_{\sigma_r}^b y_{\sigma_r}^{-a})^\varepsilon, \qquad \bar{y}_1 = (x_{\sigma_r}^{-\omega} y_{\sigma_r}^{m_{\sigma_r}})^\varepsilon,$$

with $\varepsilon = m_{\sigma_r} b - \omega a = \pm 1$, $\nu(\bar{x}_1) > 0$ and $\nu(\bar{y}_1) = 0$. Substituting

$$x_{\sigma_r} = \varphi u_{\tau_t}^\lambda \quad \text{and} \quad y_{\sigma_1} = \tilde{\varphi} u_{\tau_t}^h,$$

we see that $R_{\sigma_{r+1}}$ is dominated by S_{τ_t}. We thus have a factorization

$$R_{\sigma_r} \to R_{\sigma_{r+1}} \to S_{\tau_t}$$

with $x_{\sigma_{r+1}} = \bar{x}_1 = \hat{\varphi} u_{\tau_t}^{\lambda'}$, where $\hat{\varphi}$ is a unit in S_{τ_t} and λ' is a positive integer. We may thus replace s with $s+1$, r with $r+1$ and R_{σ_r} with $R_{\sigma_{r+1}}$.

Iterating this argument, we may assume that $s = t$ (with $r = r_s$), so that by Lemma 3.2, (28) and (27),

$$y_{\sigma_r} = \mathrm{st}_{R_{\sigma_r}}(P_{\sigma_{r+1}}) = u_{\tau_s}^\mu \mathrm{st}_{S_{\tau_s}}(P_{\sigma_{r+1}}),$$

where

$$\mathrm{st}_{S_{\tau_s}}(P_{\sigma_{r+1}}) = c\, \mathrm{st}_{S_{\tau_s}}(Q_{\tau_{s+1}}) + u_{\tau_s} \Omega$$

with c a unit in S_{τ_s} and $\Omega \in S_{\tau_s}$. Thus by (26), we have an expression

$$x_{\sigma_r} = \varphi u_{\tau_s}^\lambda, \qquad y_{\sigma_r} = \bar{\varepsilon} u_{\tau_s}^\alpha (v_{\tau_s} + u_{\tau_s} \Omega),$$

where λ is a positive integer, $\alpha \in \mathbb{N}$, φ and $\bar{\varepsilon}$ are units in S_{τ_s} and $\Omega \in S_{\tau_s}$.

We have that

$$\nu^*(x_{\sigma_r}) = \lambda \nu^*(u_{\tau_s}),$$
$$\nu(x_{\sigma_r})\mathbb{Z} = G(\nu(x_{\sigma_r})) = G(\beta_{\sigma_0}, \ldots, \beta_{\sigma_r}),$$
$$\nu^*(u_{\tau_s})\mathbb{Z} = G(\nu^*(u_{\tau_s})) = G(\gamma_{\tau_0}, \ldots, \gamma_{\tau_s}).$$

Thus

$$\lambda = [G(\gamma_{\tau_0}, \ldots, \gamma_{\tau_s}) : G(\beta_{\sigma_0}, \ldots, \beta_{\sigma_r})] = e(\nu^*/\nu)$$

by Proposition 3.1.

By Theorem 2.3, we have that

$$R_{\sigma_r}/m_{R_{\sigma_r}} = R/m_R[\delta_{\sigma_1}, \ldots, \delta_{\sigma_r}] \quad \text{and} \quad S_{\tau_s}/m_{S_{\tau_s}} = S/m_S[\varepsilon_{\tau_1}, \ldots, \varepsilon_{\tau_s}].$$

Thus

$$[S_{\tau_s}/m_{S_{\tau_s}} : R_{\sigma_r}/m_{R_{\sigma_r}}] = f(\nu^*/\nu)$$

by Proposition 3.1.

Since the ring R' of Proposition 1.1 is contained in R_{σ_r} by our construction, we have by Proposition 1.1 that $(K, \nu) \to (K^*, \nu^*)$ is without defect, completing the proofs of Proposition 0.3 and Theorem 0.1.

4. Nonsplitting and finite generation

In this section, we maintain the following assumptions. Suppose that R is a 2-dimensional excellent local domain with quotient field K. Further suppose that K^* is a finite separable extension of K and S is a 2-dimensional local domain with quotient field K^* such that S dominates R. Suppose that ν^* is a valuation of K^* such that ν^* dominates S. Let ν be the restriction of ν^* to K.

Suppose that ν^* has rational rank 1 and ν^* is not discrete. Then V_{ν^*}/m_{ν^*} is algebraic over S/m_S, by Abhyankar's inequality [1956, Proposition 2].

Lemma 4.1. *Let assumptions be as above. Then the associated graded ring* $\mathrm{gr}_{\nu^*}(S)$ *is an integral extension of* $\mathrm{gr}_{\nu}(R)$.

Proof. It suffices to show that $\mathrm{in}_{\nu^*}(f)$ is integral over $\mathrm{gr}_{\nu}(R)$ whenever $f \in S$. Suppose that $f \in S$. There exists $n_1 > 0$ such that $n_1\nu^*(f) \in \Phi_\nu$. Let $x \in m_R$ and $\omega = \nu(x)$. Then there exists a positive integer b and natural number a such that $bn_1\nu^*(f) = a\omega$, so

$$\nu^*\left(\frac{f^{bn_1}}{x^a}\right) = 0.$$

Let

$$\xi = \left[\frac{f^{bn_1}}{x^a}\right] \in V_{\nu^*}/m_{\nu^*},$$

and let $g(t) = t^r + \bar{a}_{r-1}t^{r-1} + \cdots + \bar{a}_0$ with $\bar{a}_i \in R/m_R$ be the minimal polynomial of ξ over R/m_R. Let a_i be lifts of the $\bar{a}_i$ to R. Then

$$\begin{aligned}\nu^*(f^{b_1n_1r} + a_{r-1}x^a f^{bn_1(r-1)} + \cdots + a_0x^{ar})& \\ > \nu^*(f^{bn_1r}) = \nu^*(a_{r-1}&x^a f^{bn_1(r-1)}) = \cdots = \nu^*(a_0x^{ar}).\end{aligned}$$

Thus,

$$\mathrm{in}_{\nu^*}(f)^{b_1n_1r} + \mathrm{in}_\nu(a_{r-1}x^a)\mathrm{in}_{\nu^*}(f)^{bn_1(r-1)} + \cdots + \mathrm{in}_\nu(a_0x^{ar}) = 0$$

in $\mathrm{gr}_{\nu^*}(S)$, so $\mathrm{in}_{\nu^*}(f)$ is integral over $\mathrm{gr}_{\nu^*}(R)$. □

We now establish Theorem 0.5. Recall (as defined after Proposition 0.3) that ν^* does not split in S if ν^* is the unique extension of ν to K^* which dominates S.

Theorem 0.5. *Let assumptions be as above and suppose that R and S are regular local rings. Suppose that* $\mathrm{gr}_{\nu^*}(S)$ *is a finitely generated* $\mathrm{gr}_\nu(R)$*-algebra. Then S is a localization of the integral closure of R in K^*, $\delta(\nu^*/\nu) = 0$ and ν^* does not split in S.*

Proof. Let s and r be as in the conclusions of Proposition 3.1. We first show that $P_{\sigma_{r+j}}$ is irreducible in $\hat{S}$ for all $j > 0$. There exists a unique extension of ν^* to the quotient field of $\hat{S}$ which dominates $\hat{S}$ [Spivakovsky 1990; Cutkosky and Vinh 2014; Herrera Govantes et al. 2014]. The extension is immediate since ν^* is not discrete; that is, there is no increase in value group or residue field for the extended valuation. It has the property that if $f \in \hat{S}$ and $\{f_i\}$ is a Cauchy sequence in $\hat{S}$ which converges to f, then $\nu^*(f) = \nu^*(f_i)$ for all $i \gg 0$.

Suppose that $P_{\sigma_{r+j}}$ is not irreducible in $\hat{S}$ for some $j > 0$. We derive a contradiction. With this assumption, $P_{\sigma_{r+j}} = fg$ with $f, g \in m_{\hat{S}}$. Let $\{f_i\}$ be a Cauchy sequence in S which converges to f and let $\{g_i\}$ be a Cauchy sequence in S which converges to g. For i sufficiently large, $f - f_i, g - g_i \in m_{\hat{S}}^n$, where n is so large that $n\nu^*(m_{\hat{S}}) = n\nu^*(m_S) > \nu(P_{\sigma_{r+j}})$. Thus $P_{\sigma_{r+j}} = f_i g_i + h$ with $h \in m_{\hat{S}}^n \cap S = m_S^n$, and so $\mathrm{in}_{\nu^*}(P_{\sigma_{r+j}}) = \mathrm{in}_{\nu^*}(f_i)\mathrm{in}_{\nu^*}(g_i)$. Now

$$\nu^*(f_i), \nu^*(g_i) < \nu(P_{\sigma_{r+j}}) = \beta_{\sigma_{r+j}} = \gamma_{\tau_{s+j}} = \nu^*(Q_{\tau_{s+j}}),$$

so that

$$\mathrm{in}_{\nu^*}(f_i), \mathrm{in}_{\nu^*}(g_i) \in S/m_S[\mathrm{in}_{\nu^*}(Q_{\tau_0}), \ldots, \mathrm{in}_{\nu^*}(Q_{\tau_{s+j-1}})],$$

which implies

$$\mathrm{in}_{\nu^*}(P_{\sigma_{r+j}}) \in S/m_S[\mathrm{in}_{\nu^*}(Q_{\tau_0}), \ldots, \mathrm{in}_{\nu^*}(Q_{\tau_{s+j-1}})].$$

But then (24) implies

$$\mathrm{in}_{\nu^*}(Q_{\tau_{s+j}}) \in S/m_S[\mathrm{in}_{\nu^*}(Q_{\tau_0}), \ldots, \mathrm{in}_{\nu^*}(Q_{\tau_{s+j-1}})],$$

which is impossible. Thus $P_{\sigma_{r+j}}$ is irreducible in $\hat{S}$ for all $j > 0$.

If S is not a localization of the integral closure of R in K^*, then by Zariski's main theorem (Theorem 1 of Chapter 4 in [Raynaud 1970]), $m_R S = fN$, where $f \in m_S$ and N is an m_S-primary ideal. Thus f divides P_i in S for all i, which is impossible since we have shown that $P_{\sigma_{r+j}}$ is analytically irreducible in S for all $j > 0$; we cannot have $P_{\sigma_{r+j}} = a_j f$ where a_j is a unit in S for $j > 0$ since $\nu(P_{\sigma_{r+j}}) = \nu^*(Q_{\tau_{s+j}})$ by Proposition 3.1.

Now suppose that ν^* is not the unique extension of ν to K^* which dominates S. Recall that V_ν is the union of all quadratic transforms above R along ν and V_{ν^*} is the union of all quadratic transforms above S along ν^* [Abhyankar 1959, Lemma 4.5].

Then for all $i \gg 0$, we have a commutative diagram

$$\begin{array}{ccc} R_{\sigma_i} & \longrightarrow & T_i \\ \uparrow & & \uparrow \\ R & \longrightarrow & T \end{array}$$

where T is the integral closure of R in K^*, T_i is the integral closure of R_{σ_i} in K^*, $S = T_{\mathfrak{p}}$ for some maximal ideal $\mathfrak{p}$ in T which lies over m_R, and there exist $r \geq 2$ prime ideals $\mathfrak{p}_1(i), \ldots, \mathfrak{p}_r(i)$ in T_i which lie over $m_{R_{\sigma_i}}$ and whose intersection with T is $\mathfrak{p}$. We may assume that $\mathfrak{p}_1(i)$ is the center of ν^*.

There exists an m_R-primary ideal I_i in R with blowup $\gamma : X_{\sigma_i} \to \mathrm{Spec}(R)$, where X_{σ_i} is regular and R_{σ_i} is a local ring of X_{σ_i}. Let Z_{σ_i} be the integral closure of X_{σ_i} in K^*. Let $Y_{\sigma_i} = Z_{\sigma_i} \times_{\mathrm{Spec}(T)} \mathrm{Spec}(S)$. We have a commutative diagram of morphisms

$$\begin{tikzcd} Y_{\sigma_i} \arrow[r, "\beta"] \arrow[d, "\delta"'] & X_{\sigma_i} \arrow[d, "\gamma"] \\ \mathrm{Spec}(S) \arrow[r, "\alpha"] & \mathrm{Spec}(R) \end{tikzcd}$$

The morphism δ is projective by [EGA II 1961, Proposition II.5.5.5 and Corollary II.6.1.11] and it is birational, so since Y_{σ_i} and $\mathrm{Spec}(S)$ are integral, it is a blowup of an ideal J_i in S [EGA III$_1$ 1961, Proposition III.2.3.5], which we can take to be m_S-primary since S is a regular local ring and hence factorial. Define curves $C = \mathrm{Spec}(R/(P_{\sigma_i}))$ and $C' = \alpha^{-1}(C) = \mathrm{Spec}(S/(P_{\sigma_i}))$. Denote the Zariski closure of a set W by $\overline{W}$. The strict transform C^* of C' in Y_{σ_i} is the Zariski closure

$$\begin{aligned} C^* &= \overline{\delta^{-1}(C' \setminus m_S)} = \overline{\delta^{-1}\alpha^{-1}(C \setminus m_R)} = \overline{\beta^{-1}\gamma^{-1}(C \setminus m_R)} \\ &= \beta^{-1}(\overline{\gamma^{-1}(C \setminus m_R)}) \qquad \text{(since } \beta \text{ is quasifinite)} \\ &= \beta^{-1}(\widetilde{C}), \end{aligned} \tag{29}$$

where $\widetilde{C}$ is the strict transform of C in X_{σ_i}. We have $Z_{\sigma_i} \times_{X_{\sigma_i}} \mathrm{Spec}(R_{\sigma_i}) \cong \mathrm{Spec}(T_i)$, so

$$Y_{\sigma_i} \times_{X_{\sigma_i}} \mathrm{Spec}(R_{\sigma_i}) \cong \mathrm{Spec}(T_i \otimes_T S).$$

Let x_{σ_i} be a local equation in R_{σ_i} of the exceptional divisor of $\mathrm{Spec}(R_{\sigma_i}) \to \mathrm{Spec}(R)$ and let $y_{\sigma_i} = \mathrm{st}_{R_{\sigma_i}}(P_{\sigma_i})$. Then $x_{\sigma_i}, y_{\sigma_i}$ are regular parameters in R_{σ_i}. We have that

$$\sqrt{m_{R_{\sigma_i}}(T_i \otimes_T S)} = \bigcap_{j=1}^{r} \mathfrak{p}_j(i)(T_i \otimes_T S).$$

The blowup of $J_i(S/(P_{\sigma_i}))$ in C' is $\bar{\delta} : C^* \to C'$, where $\bar{\delta}$ is the restriction of δ to C^* [Hartshorne 1977, Corollary II.7.15]. Since y_{σ_i} is a local equation of $\widetilde{C}$ in R_{σ_i}, we have by (29) that

$$\mathfrak{p}_1(i), \dots, \mathfrak{p}_r(i) \in \bar{\delta}^{-1}(m_S) \subset C^*.$$

Since $\bar{\delta}$ is proper and C' is a curve, $C^* = \mathrm{Spec}(A)$ for some excellent 1-dimensional domain A such that the inclusion $S/(P_{\sigma_i}) \to A$ is finite [Milne 1980, Corollary I.1.10]. Let $B = A \otimes_{S/(P_{\sigma_i})} \hat{S}/(P_{\sigma_i})$. Then

$$C^* \times_{\mathrm{Spec}(S/(P_{\sigma_i}))} \mathrm{Spec}(\hat{S}/(P_{\sigma_i})) = \mathrm{Spec}(B) \to \mathrm{Spec}(\hat{S}/(P_{\sigma_i}))$$

is the blowup of $J_i(\hat{S}/(P_{\sigma_i}))$ in $\hat{S}/(P_{\sigma_i})$. The extension $\hat{S}/(P_{\sigma_i}) \to B$ is finite since $S/(P_{\sigma_i}) \to A$ is finite.

Now assume that $S/(P_{\sigma_i})$ is analytically irreducible. Then B has only one minimal prime since the blowup $\mathrm{Spec}(B) \to \mathrm{Spec}(\hat{S}/(P_{\sigma_i}))$ is birational.

Since a complete local ring is Henselian, B is a local ring [Milne 1980, Theorem I.4.2 on page 32], a contradiction to our assumption that $r > 1$. □

As a consequence of the above theorem, we now obtain Corollary 0.6.

Corollary 0.6. *Let assumptions be as above and suppose that R is a regular local ring. Suppose that $R \to R'$ is a nontrivial sequence of quadratic transforms along ν. Then $gr_\nu(R')$ is not a finitely generated $gr_\nu(R)$-algebra.*

Proof. The integral closure of R in its quotient field is R, which is not equal to R' since $m_R R'$ is a principal ideal. Thus $\mathrm{gr}_\nu(R')$ is not a finitely generated $\mathrm{gr}_\nu(R)$-algebra by Theorem 0.5. □

The conclusions of Theorem 0.5 do not hold if we remove the assumption that ν^* is not discrete, when V_ν/m_ν is finite over R/m_R. We give a simple example. Let k be an algebraically closed field of characteristic not equal to 2 and let $p(u)$ be a transcendental series in the power series ring $k[[u]]$ such that $p(0) = 1$. Then $f = v - up(u)$ is irreducible in the power series ring $k[[u, v]]$ and $k[[u, v]]/(f)$ is a discrete valuation ring with regular parameter u. Let ν be the natural valuation of this ring. Let $R = k[u, v]_{(u,v)}$ and $S = k[x, y]_{(x,y)}$. Define a k-algebra homomorphism $R \to S$ by $u \mapsto x^2$ and $v \mapsto y^2$. The series $f(x^2, y^2)$ factors as

$$f = \left(y - x\sqrt{p(x^2)}\right)\left(y + x\sqrt{p(x^2)}\right)$$

in $k[[x, y]]$. Let $f_1 = y - x\sqrt{p(x^2)}$ and $f_2 = y + x\sqrt{p(x^2)}$. The rings $k[[x, y]]/(f_i)$ are discrete valuation rings with regular parameter x. Let ν_1 and ν_2 be the natural valuations of these ring.

Let ν be the valuation of the quotient field of R which dominates R and is defined by the natural inclusion $R \to k[[u, v]]/(f)$, and let ν_i for $i = 1, 2$ be the valuations of the quotient field of S which dominate S and are defined by the respective natural inclusions $S \to k[[x, y]]/(f_i)$. Then ν_1 and ν_2 are distinct extensions of ν to the quotient field of S which dominate S. However, we have that $\mathrm{gr}_\nu(R) = k[\mathrm{in}_\nu(u)]$ and $\mathrm{gr}_{\nu_i}(S) = k[\mathrm{in}_{\nu^*}(x)]$ with $\mathrm{in}_{\nu^*}(x)^2 = \mathrm{in}_\nu(u)$. Thus $\mathrm{gr}_{\nu_i}(S)$ is a finite $\mathrm{gr}_\nu(R)$-algebra.

We now give an example where ν^* has rational rank 2 and ν splits in S but $\mathrm{gr}_{\nu^*}(S)$ is a finitely generated $\mathrm{gr}_\nu(R)$-algebra. Suppose that k is an algebraically closed field of characteristic not equal to 2. Let $R = k[x, y]_{(x,y)}$ and $S = k[u, v]_{(u,v)}$. The substitutions $u = x^2$ and $v = y^2$ make S into a finite separable extension of R. Define a valuation ν_1 of the quotient field K^* of S by $\nu_1(x) = 1$ and $\nu_1(y - x) = \pi + 1$, and define a valuation ν_2 of the quotient field K^* by $\nu_2(x) = 1$ and $\nu_2(y + x) = \pi + 1$. Since $u = x^2$ and $v - u = (y - x)(y + x)$, we have that $\nu_1(u) = \nu_2(u) = 2$ and $\nu_1(v - u) = \nu_2(v - u) = \pi + 2$. Let ν be the common restriction of ν_1 and ν_2 to the quotient field K of R. Then ν splits in S. However, $\mathrm{gr}_{\nu_1}(S)$ is a finitely generated $\mathrm{gr}_\nu(R)$-algebra since $\mathrm{gr}_{\nu_1}(S) = k[\mathrm{in}_{\nu_1}(x), \mathrm{in}_{\nu_1}(y-x)]$ is a finitely generated k-algebra. Note that $\mathrm{gr}_\nu(R) = k[\mathrm{in}_\nu(u), \mathrm{in}_\nu(v-u)]$ with $\mathrm{in}_{\nu_1}(x)^2 = \mathrm{in}_\nu(u)$ and $\mathrm{in}_\nu(v-u) = 2\mathrm{in}_{\nu_1}(y-x)\mathrm{in}_{\nu_1}(x)$.

References

[Abhyankar 1956] S. Abhyankar, "On the valuations centered in a local domain", *Amer. J. Math.* **78** (1956), 321–348. MR Zbl

[Abhyankar 1959] S. Abhyankar, *Ramification theoretic methods in algebraic geometry*, Annals of Math. Studies **43**, Princeton University Press, 1959. MR Zbl

[Cutkosky 1999] S. D. Cutkosky, *Local monomialization and factorization of morphisms*, Astérisque **260**, Société Mathématique de France, Paris, 1999. MR Zbl

[Cutkosky 2015] S. D. Cutkosky, "Counterexamples to local monomialization in positive characteristic", *Math. Ann.* **362**:1-2 (2015), 321–334. MR Zbl

[Cutkosky 2016a] S. D. Cutkosky, "A generalisation of the Abhyankar Jung theorem to associated graded rings of valuations", *Math. Proc. Cambridge Philos. Soc.* **160**:2 (2016), 233–255. MR

[Cutkosky 2016b] S. D. Cutkosky, "Ramification of valuations and local rings in positive characteristic", *Comm. Algebra* **44**:7 (2016), 2828–2866. MR Zbl

[Cutkosky and Piltant 2004] S. D. Cutkosky and O. Piltant, "Ramification of valuations", *Adv. Math.* **183**:1 (2004), 1–79. MR Zbl

[Cutkosky and Vinh 2014] S. D. Cutkosky and P. A. Vinh, "Valuation semigroups of two-dimensional local rings", *Proc. Lond. Math. Soc.* (3) **108**:2 (2014), 350–384. MR Zbl

[EGA II 1961] A. Grothendieck, "Eléments de géométrie algébrique, II: Étude globale élémentaire de quelques classes de morphismes", *Inst. Hautes Études Sci. Publ. Math.* **8** (1961), 5–222. MR Zbl

[EGA III$_1$ 1961] A. Grothendieck, "Eléments de géométrie algébrique, III: Étude cohomologique des faisceaux cohérents, I", *Inst. Hautes Études Sci. Publ. Math.* **11** (1961), 5–167. MR Zbl

[Endler 1972] O. Endler, *Valuation theory*, Springer, 1972. MR Zbl

[Ghezzi and Kashcheyeva 2007] L. Ghezzi and O. Kashcheyeva, "Toroidalization of generating sequences in dimension two function fields of positive characteristic", *J. Pure Appl. Algebra* **209**:3 (2007), 631–649. MR Zbl

[Ghezzi et al. 2006] L. Ghezzi, H. T. Hà, and O. Kashcheyeva, "Toroidalization of generating sequences in dimension two function fields", *J. Algebra* **301**:2 (2006), 838–866. MR Zbl

[Hartshorne 1977] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Math. **52**, Springer, 1977. MR Zbl

[Herrera Govantes et al. 2014] F. J. Herrera Govantes, M. A. Olalla Acosta, M. Spivakovsky, and B. Teissier, "Extending valuations to formal completions", pp. 252–265 in *Valuation theory in interaction*, edited by A. Campillo et al., European Mathematical Society, Zürich, 2014. MR Zbl

[Kuhlmann 2000] F.-V. Kuhlmann, "Valuation theoretic and model theoretic aspects of local uniformization", pp. 381–456 in *Resolution of singularities* (Obergurgl, 1997), edited by H. Hauser et al., Progr. Math. **181**, Birkhäuser, Basel, 2000. MR Zbl

[Kuhlmann 2010] F.-V. Kuhlmann, "A classification of Artin–Schreier defect extensions and characterizations of defectless fields", *Illinois J. Math.* **54**:2 (2010), 397–448. MR Zbl

[Mac Lane 1936] S. Mac Lane, "A construction for absolute values in polynomial rings", *Trans. Amer. Math. Soc.* **40**:3 (1936), 363–395. MR Zbl

[Milne 1980] J. S. Milne, *Étale cohomology*, Princeton Mathematical Series **33**, Princeton University Press, 1980. MR Zbl

[Raynaud 1970] M. Raynaud, *Anneaux locaux henséliens*, Lecture Notes in Math. **169**, Springer, 1970. MR Zbl

[Spivakovsky 1990] M. Spivakovsky, "Valuations in function fields of surfaces", *Amer. J. Math.* **112**:1 (1990), 107–156. MR Zbl

[Teissier 2003] B. Teissier, "Valuations, deformations, and toric geometry", pp. 361–459 in *Valuation theory and its applications* (Saskatoon, SK, 1999), vol. 2, edited by F.-V. Kuhlmann et al., Fields Inst. Commun. **33**, American Mathematical Society, Providence, RI, 2003. MR Zbl

[Teissier 2014] B. Teissier, "Overweight deformations of affine toric varieties and local uniformization", pp. 474–565 in *Valuation theory in interaction*, edited by A. Campillo et al., European Mathematical Society, Zürich, 2014. MR Zbl

[Vaquié 2007] M. Vaquié, "Famille admissible de valuations et défaut d'une extension", *J. Algebra* **311**:2 (2007), 859–876. MR Zbl

[Zariski and Samuel 1960] O. Zariski and P. Samuel, *Commutative algebra*, vol. 2, Van Nostrand, Princeton, 1960. MR Zbl

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