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The role of defect and splitting in finite generation of extensions of associated graded rings along a valuation

Steven Dale Cutkosky

Suppose that R is a 2-dimensional excellent local domain with quotient field K, K^* is a finite separable extension of K and S is a 2-dimensional local domain with quotient field K^* such that S dominates R. Suppose that ν^* is a valuation of K^* such that ν^* dominates S. Let ν be the restriction of ν^* to K. The associated graded ring $\mathrm{gr}_\nu(R)$ was introduced by Bernard Teissier. It plays an important role in local uniformization. We show that the extension $(K,\nu)\to(K^*,\nu^*)$ of valued fields is without defect if and only if there exist regular local rings R_1 and S_1 such that R_1 is a local ring of a blowup of R, S_1 is a local ring of a blowup of S, ν^* dominates S_1, S_1 dominates R_1 and the associated graded ring $\mathrm{gr}_{\nu^*}(S_1)$ is a finitely generated $\mathrm{gr}_\nu(R_1)$-algebra.

We also investigate the role of splitting of the valuation ν in K^* in finite generation of the extensions of associated graded rings along the valuation. We say that ν does not split in S if ν^* is the unique extension of ν to K^* which dominates S. We show that if R and S are regular local rings, ν^* has rational rank 1 and is not discrete and $\mathrm{gr}_{\nu^*}(S)$ is a finitely generated $\mathrm{gr}_\nu(R)$-algebra, then S is a localization of the integral closure of R in K^*, the extension $(K,\nu)\to(K^*,\nu^*)$ is without defect and ν does not split in S. We give examples showing that such a strong statement is not true when ν does not satisfy these assumptions. As a consequence, we deduce that if ν has rational rank 1 and is not discrete and if $R\to R'$ is a nontrivial sequence of quadratic transforms along ν, then $\mathrm{gr}_\nu(R')$ is not a finitely generated $\mathrm{gr}_\nu(R)$-algebra.

Suppose that K is a field. Associated to a valuation ν of K is a value group Φ_ν and a valuation ring V_ν with maximal ideal m_ν. Let R be a local domain with quotient field K. We say that ν dominates R if $R\subset V_\nu$ and $m_\nu\cap R=m_R$, where m_R is the maximal ideal of R. We have an associated semigroup $S^R(\nu)=\{\nu(f)\mid f\in R\}$,

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as well as the associated graded ring along the valuation

$$\mathrm{gr}_\nu(R) = \bigoplus_{\gamma\in\Phi_\nu} \mathcal{P}_\gamma(R)/\mathcal{P}^+_\gamma(R) = \bigoplus_{\gamma\in S^R(\nu)} \mathcal{P}_\gamma(R)/\mathcal{P}^+_\gamma(R), \tag{1}$$

which is defined in [Teissier 2003]. Here

$$\mathcal{P}_\gamma(R) = \{f \in R \mid \nu(f) \geq \gamma\} \quad \text{and} \quad \mathcal{P}^+_\gamma(R) = \{f \in R \mid \nu(f) > \gamma\}.$$

This ring plays an important role in local uniformization of singularities [Teissier 2003; 2014]. The ring $\mathrm{gr}_\nu(R)$ is a domain, but it is often not Noetherian, even when R is.

Suppose that $K \to K^*$ is a finite extension of fields and ν^* is a valuation which is an extension of ν to K^*. We have the classical indices

$$e(\nu^*/\nu) = [\Phi_{\nu^*} : \Phi_\nu] \quad \text{and} \quad f(\nu^*/\nu) = [V_{\nu^*}/m_{\nu^*} : V_\nu/m_\nu]$$

as well as the defect $\delta(\nu^*/\nu)$ of the extension. Ramification of valuations and the defect are discussed in [Zariski and Samuel 1960, Chapter VI; Endler 1972; Kuhlmann 2000; 2010]; a survey is given in [Cutkosky and Piltant 2004, Section 7.1]. By Ostrowski's lemma, if ν^* is the unique extension of ν to K^*, we have that

$$[K^* : K] = e(\nu^*/\nu) f(\nu^*/\nu) p^{\delta(\nu^*/\nu)}, \tag{2}$$

where p is the characteristic of the residue field V_ν/m_ν. From this formula, the defect can be computed using Galois theory in an arbitrary finite extension. If V_ν/m_ν has characteristic 0, then $\delta(\nu^*/\nu) = 0$ and $p^{\delta(\nu^*/\nu)} = 1$, so there is no defect. Further, if $\Phi_\nu = \mathbb{Z}$ and K^* is separable over K then there is no defect.

If K is an algebraic function field over a field k, then an algebraic local ring R of K is a local domain which is essentially of finite type over k and has K as its field of fractions. In [Cutkosky 1999], it is shown that if $K \to K^*$ is a finite extension of algebraic function fields over a field k of characteristic 0, ν^* is a valuation of K^* (which is trivial on k) with restriction ν to K and if $R \to S$ is an inclusion of algebraic regular local rings of K and K^* such that ν^* dominates S and S dominates R, then there exists a commutative diagram

$$\begin{array}{ccc} R_1 & \longrightarrow & S_1 \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array} \tag{3}$$

where the vertical arrows are products of blowups of nonsingular subschemes along the valuation ν^* (monoidal transforms) and $R_1 \to S_1$ is dominated by ν^* and is a monomial mapping; that is, there exist regular parameters $x_1, \ldots, x_n$ in R_1, regular parameters $y_1, \ldots, y_n$ in S_1, units $\delta_i \in S_1$, and a matrix $A = (a_{ij})$ of natural numbers with $\mathrm{Det}(A) \neq 0$ such that

$$x_i = \delta_i \prod_{j=1}^{n} y_j^{a_{ij}} \quad \text{for } 1 \leq i \leq n. \tag{4}$$

In [Cutkosky and Piltant 2004], it is shown that this theorem is true, giving a monomial form of the mapping (4) after appropriate blowing up (3) along the valuation, if $K \to K^*$ is a separable extension of two dimension algebraic function fields over an algebraically closed field, which has no defect. This result is generalized to the situation of this paper, namely when R is a 2-dimensional excellent local ring, in [Cutkosky 2016b]. However, it may be that such monomial forms do not exist, even after blowing up, if the extension has defect, as is shown by examples in [Cutkosky 2015].

In the case when k has characteristic 0 and for separable defectless extensions of 2-dimensional algebraic function fields in positive characteristic, it is further shown in [Cutkosky and Piltant 2004] that the expressions (3) and (4) are stable under further simple sequences of blowups along ν^* and the form of the matrix A stably reflects invariants of the valuation.

We always have an inclusion of graded domains $\mathrm{gr}_\nu(R) \to \mathrm{gr}_{\nu^*}(S)$ and the index of their quotient fields is

$$[\mathrm{QF}(\mathrm{gr}_{\nu^*}(S)) : \mathrm{QF}(\mathrm{gr}_\nu(R))] = e(\nu^*/\nu) f(\nu^*/\nu), \tag{5}$$

as shown in [Cutkosky 2016a, Proposition 3.3]. Comparing with Ostrowski's lemma (2), we see that the defect has disappeared in (5).

Even though $\mathrm{QF}(\mathrm{gr}_{\nu^*}(S))$ is finite over $\mathrm{QF}(\mathrm{gr}_\nu(R))$, it is possible for $\mathrm{gr}_{\nu^*}(S)$ to not be a finitely generated $\mathrm{gr}_\nu(R)$-algebra. Examples showing this for extensions $R \to S$ of 2-dimensional algebraic local rings over arbitrary algebraically closed fields are given in Example 9.4 of [Cutkosky and Vinh 2014].

It was shown by Ghezzi, Hà and Kashcheyeva [Ghezzi et al. 2006] for extensions of 2-dimensional algebraic function fields over an algebraically closed field k of characteristic 0, and later by Ghezzi and Kashcheyeva [2007] for defectless separable extensions of 2-dimensional algebraic functions fields over an algebraically closed field k of positive characteristic, that there exists a commutative diagram (3) such that $\mathrm{gr}_{\nu^*}(S_1)$ is a finitely generated $\mathrm{gr}_\nu(R_1)$-algebra. Further, this property is stable under further suitable sequences of blowups.

In [Cutkosky 2016a, Theorem 1.6], it is shown that for algebraic regular local rings of arbitrary dimension, if the ground field k is algebraically closed of characteristic 0 and the valuation has rank 1 and is 0-dimensional ($V_\nu/m_\nu = k$), then we can also construct a commutative diagram (3) such that $\mathrm{gr}_{\nu^*}(S_1)$ is a finitely generated $\mathrm{gr}_\nu(R_1)$-algebra, and this property is stable under further suitable sequences of blowups.

An example is given in [Cutkosky and Piltant 2004] of an inclusion $R \to S$ in a separable defect extension of 2-dimensional algebraic function fields such that $\mathrm{gr}_{\nu^*}(S_1)$ is stably not a finitely generated $\mathrm{gr}_\nu(R_1)$-algebra in diagram (3) under sequences of blowups. This raises the question of whether the existence of a finitely generated extension of associated graded rings along the valuation implies that K^* is a defectless extension of K.

We find that we must impose the condition that K^* is a separable extension of K to obtain a positive answer to this question, as there are simple examples of inseparable defect extensions such that $\mathrm{gr}_{\nu^*}(S)$ is a finitely generated $\mathrm{gr}_\nu(R)$-algebra, such as in the following example, which is Example 8.6 of [Kuhlmann 2000]. Let k be a field of characteristic $p > 0$ and $k((x))$ the field of formal power series over k with the x-adic valuation ν_x. Let $y \in k((x))$ be transcendental over $k(x)$ with $\nu_x(y) > 0$. Let $\tilde{y} = y^p$, and $K = k(x, \tilde{y}) \subset K^* = k(x, y)$. Let $\nu^* = \nu_x | K^*$ and $\nu = \nu_x | K$. Then we have equality of value groups $\Phi_\nu = \Phi_{\nu^*} = \nu(x)\mathbb{Z}$ and equality of residue fields of valuation rings $V_\nu / m_\nu = V_{\nu^*} / m_{\nu^*} = k$, so $e(\nu^*/\nu) = 1$ and $f(\nu^*/\nu) = 1$. We have that ν^* is the unique extension of ν to K^* since K^* is purely inseparable over K. By Ostrowski's lemma (2), the extension $(K, \nu) \to (K^*, \nu^*)$ is a defect extension with defect $\delta(\nu^*/\nu) = 1$. Let $R = k[x, \tilde{y}]_{(x,\tilde{y})} \to S = k[x, y]_{(x,y)}$. Then we have equality

$$\mathrm{gr}_\nu(R) = k[t] = \mathrm{gr}_{\nu^*}(S),$$

where t is the class of x.

In this paper we show that the question does have a positive answer for separable extensions in the following theorem.

Theorem 0.1. *Suppose that R is a 2-dimensional excellent local domain with quotient field K. Further suppose that K^* is a finite separable extension of K and S is a 2-dimensional local domain with quotient field K^* such that S dominates R. Suppose that ν^* is a valuation of K^* such that ν^* dominates S. Let ν be the restriction of ν^* to K. Then the extension $(K, \nu) \to (K^*, \nu^*)$ is without defect if and only if there exist regular local rings R_1 and S_1 such that R_1 is a local ring of a blowup of R, S_1 is a local ring of a blowup of S, ν^* dominates S_1, S_1 dominates R_1 and $\mathrm{gr}_{\nu^*}(S_1)$ is a finitely generated $\mathrm{gr}_\nu(R_1)$-algebra.*

We immediately obtain the following corollary for 2-dimensional algebraic function fields.

Corollary 0.2. *Suppose $K \to K^*$ is a finite separable extension of 2-dimensional algebraic function fields over a field k and ν^* is a valuation of K^* with restriction ν to K. Then the extension $(K, \nu) \to (K^*, \nu^*)$ is without defect if and only if there exist algebraic regular local rings R of K and S of K^* such that ν^* dominates S, S dominates R and $\mathrm{gr}_{\nu^*}(S)$ is a finitely generated $\mathrm{gr}_\nu(R)$-algebra.*

We see from Theorem 0.1 that the defect, which is completely lost in the extension of quotient fields of the associated graded rings along the valuation (5), can be recovered from knowledge of all extensions of associated graded rings along the valuation of regular local rings $R_1 \to S_1$ within the field extensions which dominate $R \to S$ and are dominated by the valuation.

The fact that there exists $R_1 \to S_1$ as in the conclusions of the theorem if the assumptions of the theorem hold and the extension is without defect is proven within 2-dimensional algebraic function fields over an algebraically closed field in [Ghezzi et al. 2006; Ghezzi and Kashcheyeva 2007], and in the generality of the assumptions of Theorem 0.1 in Theorems 4.3 and 4.4 of [Cutkosky 2016b]. Further, if the assumptions of the theorem hold and $\delta(\nu^*/\nu) \neq 0$, then the value group Φ_{ν^*} is not finitely generated by [Cutkosky and Piltant 2004, Theorem 7.3] in the case of algebraic function fields over an algebraically closed field. With the full generality of the hypothesis of Theorem 0.1, the defect is zero by [Endler 1972, Corollary 18.7] in the case of discrete rank 1 valuations and the defect is zero by [Cutkosky 2016b, Theorem 3.7] in the case of rational rank 2 valuations, so by Abhyankar's inequality [Abhyankar 1956, Proposition 2] or [Zariski and Samuel 1960, Appendix 2], if $\delta(\nu^*/\nu) \neq 0$, then the value group Φ_{ν^*} has rational rank 1 and is not discrete and V_{ν^*}/m_{ν^*} is algebraic over S/m_S. Thus, to prove Theorem 0.1, we have reduced to the following proposition, which we establish in this paper.

Proposition 0.3. *Suppose that R is a 2-dimensional excellent local domain with quotient field K. Further suppose that K^* is a finite separable extension of K and S is a 2-dimensional local domain with quotient field K^* such that S dominates R. Suppose that ν^* is a valuation of K^* such that ν^* dominates S. Let ν be the restriction of ν^* to K.*

Suppose that ν^* has rational rank 1 and ν^* is not discrete. Further suppose that there exist regular local rings R_1 and S_1 such that R_1 is a local ring of a blowup of R, S_1 is a local ring of a blowup of S, ν^* dominates S_1, S_1 dominates R_1 and $\mathrm{gr}_{\nu^*}(S_1)$ is a finitely generated $\mathrm{gr}_{\nu}(R_1)$-algebra. Then $\delta(\nu^*/\nu) = 0$.

Another factor in the question of finite generation of extensions of associated graded rings along a valuation is the splitting of ν in K^*. We say that ν does not split in S if ν^* is the unique extension of ν to K^* such that ν^* dominates S. After a little blowing up, we can always obtain nonsplitting, as the following lemma shows.

Lemma 0.4. *Given an extension $R \to S$ as in the hypotheses of Theorem 0.1, there exists a normal local ring R' which is a local ring of a blowup of R such that ν dominates R', and if*

$$\begin{array}{ccc} R_1 & \longrightarrow & S_1 \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

is a commutative diagram of normal local rings, where R_1 is a local ring of a blowup of R and S_1 is a local ring of a blowup of S, ν^* dominates S_1 and R_1 dominates R', then ν does not split in S_1.

Lemma 0.4 will be proven in Section 1. We have the following theorem.

Theorem 0.5. *Suppose that R is a 2-dimensional excellent regular local ring with quotient field K. Further suppose that K^* is a finite separable extension of K and S is a 2-dimensional regular local ring with quotient field K^* such that S dominates R. Suppose that ν^* is a valuation of K^* such that ν^* dominates S. Let ν be the restriction of ν^* to K. Further suppose that ν^* has rational rank 1 and ν^* is not discrete. Suppose that $\mathrm{gr}_{\nu^*}(S)$ is a finitely generated $\mathrm{gr}_\nu(R)$-algebra. Then S is a localization of the integral closure of R in K^*, $\delta(\nu^*/\nu) = 0$ and ν^* does not split in S.*

We give examples showing that the condition that ν^* has rational rank 1 and is not discrete in Theorem 0.5 are necessary. As an immediate consequence of Theorem 0.5, we obtain the following corollary.

Corollary 0.6. *Suppose that R is a 2-dimensional excellent regular local ring with quotient field K. Suppose that ν is a valuation of K such that ν dominates R. Further suppose that ν has rational rank 1 and ν is not discrete. Suppose that $R \to R'$ is a nontrivial sequence of quadratic transforms along ν. Then $gr_\nu(R')$ is not a finitely generated $gr_\nu(R)$-algebra.*

Michel Vaquié [2007] extended Mac Lane's theory of key polynomials [Mac Lane 1936] to show that if $(K, \nu) \to (K^*, \nu^*)$ is a finite extension of valued fields with $\delta(\nu^*/\nu) = 0$ and ν^* is the unique extension of ν to K^*, then ν^* can be constructed from ν by a finite sequence of augmented valuations. This suggests that a converse of Theorem 0.5 may be true.

1. Local degree and defect

We will use the following criterion to measure defect. This result is implicit in [Cutkosky and Piltant 2004] with the assumptions of Proposition 0.3.

Proposition 1.1 [Cutkosky 2016b, Proposition 3.4]. *Suppose R is a 2-dimensional excellent local domain with quotient field K. Further suppose that K^* is a finite separable extension of K and S is a 2-dimensional local domain with quotient field K^* such that S dominates R. Suppose that ν^* is a valuation of K^* such that ν^* dominates S, the residue field V_{ν^*}/m_{ν^*} of V_{ν^*} is algebraic over S/m_S and the value group Φ_{ν^*} of ν^* has rational rank 1. Let ν be the restriction of ν^* to K. There exists a local ring R' of K which is essentially of finite type over R, is dominated by ν and* *dominates R such that if we have a commutative diagram*

$$\begin{array}{ccc} V_\nu & \longrightarrow & V_{\nu^*} \\ \uparrow & & \uparrow \\ R_1 & \longrightarrow & S_1 \\ \uparrow & & \uparrow \\ R' & & \Big| \\ \uparrow & & \Big| \\ R & \longrightarrow & S \end{array} \tag{6}$$

where

- R_1 *is a regular local ring of* K *which is essentially of finite type over* R *and dominates* R,
- S_1 *is a regular local ring of* K^* *which is essentially of finite type over* S *and dominates* S, *and*
- R_1 *has a regular system of parameters* u, v *and* S_1 *has a regular system of parameters* x, y *such that there is an expression*

$$u = \gamma x^a, \qquad v = x^b f,$$

where $a > 0$, $b \geq 0$, γ *is a unit in* S, $x \nmid f$ *in* S_1 *and* f *is not a unit in* S_1,

then

$$ad[S_1/m_{S_1} : R_1/m_{R_1}] = e(\nu^*/\nu) f(\nu^*/\nu) p^{\delta(\nu^*/\nu)}, \tag{7}$$

where $d = \bar{\nu}(f \bmod x)$ *with* $\bar{\nu}$ *being the natural valuation of the DVR* S/xS.

Proof of Lemma 0.4. Let $\nu_1 = \nu^*, \nu_2, \ldots, \nu_r$ be the extensions of ν to K^*. Let T be the integral closure of V_ν in K^*. Then $T = V_{\nu_1} \cap \cdots \cap V_{\nu_r}$ is the integral closure of V_{ν^*} in K^* by [Abhyankar 1959, Propositions 2.36 and 2.38]. Let $m_i = m_{\nu_i} \cap T$ be the maximal ideals of T. By the Chinese remainder theorem, there exists $u \in T$ such that $u \in m_1$ and $u \notin m_i$ for $2 \leq i \leq r$. Let

$$u^n + a_1 u^{n-1} + \cdots + a_n = 0$$

be an equation of integral dependence of u over V_ν. Let A be the integral closure of $R[a_1, \ldots, a_n]$ in K and let $R' = A_{A \cap m_\nu}$. Let T' be the integral closure of R' in K^*. We have that $u \in T' \cap m_i$ if and only if $i = 1$. Let $S' = T'_{T' \cap m_1}$. Then ν does not split in S' and R' has the property of the conclusions of the lemma. □

2. Generating sequences

Given an additive group G with $\lambda_0, \ldots, \lambda_r \in G$, let $G(\lambda_0, \ldots, \lambda_r)$ and $S(\lambda_0, \ldots, \lambda_r)$ denote the subgroup and the semigroup, respectively, generated by $\lambda_0, \ldots, \lambda_r$.

In this section, we suppose that R is a regular local ring of dimension 2, with maximal ideal m_R and residue field R/m_R. For $f \in R$, let $\bar{f}$ or $[f]$ denote the residue of f in R/m_R.

The following theorem is Theorem 4.2 of [Cutkosky and Vinh 2014], as interpreted by [Cutkosky and Vinh 2014, Remark 4.3].

Theorem 2.1. *Suppose that ν is a valuation of the quotient field of R dominating R. Let $L = V_\nu/m_\nu$ be the residue field of the valuation ring V_ν of ν. For $f \in V_\nu$, let $[f]$ denote the class of f in L. Suppose that x, y are regular parameters in R. Then there exist $\Omega \in \mathbb{Z}_+ \cup \{\infty\}$ and $P_i(\nu, R) \in m_R$ for $i \in \mathbb{Z}_+$ with $i < \min\{\Omega+1, \infty\}$ such that $P_0(\nu, R) = x$, $P_1(\nu, R) = y$ and for $1 \le i < \Omega$, there is an expression*

$$P_{i+1}(\nu, R) = P_i(\nu, R)^{n_i(\nu,R)} + \sum_{k=1}^{\lambda_i} c_k P_0(\nu, R)^{\sigma_{i,0}(k)} P_1(\nu, R)^{\sigma_{i,1}(k)} \cdots P_i(\nu, R)^{\sigma_{i,i}(k)}, \quad (8)$$

where $n_i(\nu, R) \ge 1$, $\lambda_i \ge 1$, the c_k are nonzero units in R for $1 \le k \le \lambda_i$, $\sigma_{i,s}(k) \in \mathbb{N}$ for all s, k, and $0 \le \sigma_{i,s}(k) < n_s(\nu, R)$ for $s \ge 1$. Further,

$$n_i(\nu, R)\nu(P_i(\nu, R)) = \nu\big(P_0(\nu, R)^{\sigma_{i,0}(k)} P_1(\nu, R)^{\sigma_{i,1}(k)} \cdots P_i(\nu, R)^{\sigma_{i,i}(k)}\big) \quad (9)$$

for all k.

For all $i \in \mathbb{Z}_+$ with $i < \Omega$, the following are true:

(1) *$\nu(P_{i+1}(\nu, R)) > n_i(\nu, R)\nu(P_i(\nu, R))$.*

(2) *Suppose that $r \in \mathbb{N}$, $m \in \mathbb{Z}_+$, $j_k(l) \in \mathbb{N}$ for $1 \le l \le m$ and $0 \le j_k(l) < n_k(\nu, R)$ for $1 \le k \le r$ are such that $(j_0(l), j_1(l), \ldots, j_r(l))$ are distinct for $1 \le l \le m$, and*

$$\nu\big(P_0(\nu, R)^{j_0(l)} P_1(\nu, R)^{j_1(l)} \cdots P_r(\nu, R)^{j_r(l)}\big) = \nu\big(P_0(\nu, R)^{j_0(1)} \cdots P_r(\nu, R)^{j_r(1)}\big)$$

for $1 \le l \le m$. Then

$$1, \left[\frac{P_0(\nu, R)^{j_0(2)} P_1(\nu, R)^{j_1(2)} \cdots P_r(\nu, R)^{j_r(2)}}{P_0(\nu, R)^{j_0(1)} P_1(\nu, R)^{j_1(1)} \cdots P_r(\nu, R)^{j_r(1)}}\right], \ldots, \left[\frac{P_0(\nu, R)^{j_0(m)} P_1(\nu, R)^{j_1(m)} \cdots P_r(\nu, R)^{j_r(m)}}{P_0(\nu, R)^{j_0(1)} P_1(\nu, R)^{j_1(1)} \cdots P_r(\nu, R)^{j_r(1)}}\right]$$

are linearly independent over R/m_R.

(3) *Let*

$$\bar{n}_i(\nu, R) = \big[G\big(\nu(P_0(\nu, R)), \ldots, \nu(P_i(\nu, R))\big) : G\big(\nu(P_0(\nu, R)), \ldots, \nu(P_{i-1}(\nu, R))\big)\big].$$

Then $\bar{n}_i(\nu, R) \mid \sigma_{i,i}(k)$ for all k in (8). In particular, $n_i(\nu, R) = \bar{n}_i(\nu, R) d_i(\nu, R)$ with $d_i(\nu, R) \in \mathbb{Z}_+$.

(4) *There exists* $U_i(\nu,R)=P_0(\nu,R)^{w_0(i)}P_1(\nu,R)^{w_1(i)}\cdots P_{i-1}(\nu,R)^{w_{i-1}(i)}$ *for* $i\geq 1$ *with* $w_0(i),\dots,w_{i-1}(i)\in\mathbb{N}$ *and* $0\leq w_j(i)<n_j(\nu,R)$ *for* $1\leq j\leq i-1$ *such that* $\nu(P_i(\nu,R)^{\bar{n}_i})=\nu(U_i(\nu,R))$. *Setting*

$$\alpha_i(\nu,R)=\left[\frac{P_i(\nu,R)^{\bar{n}_i(\nu,R)}}{U_i(\nu,R)}\right],$$

we have

$$b_{i,t}=\left[\sum_{\sigma_{i,i}(k)=t\bar{n}_i(\nu,R)} c_k\frac{P_0(\nu,R)^{\sigma_{i,0}(k)}P_1(\nu,R)^{\sigma_{i,1}(k)}\cdots P_{i-1}(\nu,R)^{\sigma_{i,i-1}(k)}}{U_i(\nu,R)^{(d_i(\nu,R)-t)}}\right]\in R/m_R(\alpha_1(\nu,R),\dots,\alpha_{i-1}(\nu,R))$$

for $0\leq t\leq d_i(\nu,R)-1$, *and*

$$f_i(u)=u^{d_i(\nu,R)}+b_{i,d_i(\nu,R)-1}u^{d_i(\nu,R)-1}+\cdots+b_{i,0}$$

is the minimal polynomial of $\alpha_i(\nu,R)$ *over* $R/m_R(\alpha_1(\nu,R),\dots,\alpha_{i-1}(\nu,R))$.

The algorithm terminates with $\Omega<\infty$ *if and only if either*

$$\bar{n}_\Omega(\nu,R)=\big[G\big(\nu(P_0(\nu,R)),\dots,\nu(P_\Omega(\nu,R))\big):G\big(\nu(P_0(\nu,R)),\dots,\nu(P_{\Omega-1}(\nu,R))\big)\big]=\infty \tag{10}$$

or

$$\bar{n}_\Omega(\nu,R)<\infty\quad(\text{so that }\alpha_\Omega(\nu,R)\text{ is defined as in (4)})\quad\text{and}$$

$$d_\Omega(\nu,R)=\big[R/m_R(\alpha_1(\nu,R),\dots,\alpha_\Omega(\nu,R)):R/m_R(\alpha_1(\nu,R),\dots,\alpha_{\Omega-1}(\nu,R))\big]=\infty. \tag{11}$$

If $\bar{n}_\Omega(\nu,R)=\infty$, *set* $\alpha_\Omega(\nu,R)=1$.

Let notation be as in Theorem 2.1. The following formula is statement $B(i)$ on page 360 of [Cutkosky and Vinh 2014].

Suppose M *is a Laurent monomial in* $P_0(\nu,R),P_1(\nu,R),\dots,P_i(\nu,R)$ *and* $\nu(M)=0$. *Then there exist* $s_i\in\mathbb{Z}$ *such that*

$$M=\prod_{j=1}^{i}\left[\frac{P_j(\nu,R)^{\bar{n}_j}}{U_j(\nu,R)}\right]^{s_j}, \tag{12}$$

so that

$$[M]\in R/m_R[\alpha_1(\nu,R),\dots,\alpha_i(\nu,R)].$$

Define $\beta_i(\nu,R)=\nu(P_i(\nu,R))$ for $0\leq i$.

Since ν is a valuation of the quotient field of R, we have that

$$\Phi_\nu=\bigcup_{i=1}^{\infty}G(\beta_0(\nu,R),\beta_1(\nu,R),\dots,\beta_i(\nu,R)) \tag{13}$$

and

$$V_\nu/m_\nu = \bigcup_{i=1}^{\infty} R/m_R[\alpha_1(\nu, R), \dots, \alpha_i(\nu, R)]. \tag{14}$$

The following is [Cutkosky and Vinh 2014, Theorem 4.10].

Theorem 2.2. *Suppose that ν is a valuation dominating R. Let*

$$P_0(\nu, R) = x, \quad P_1(\nu, R) = y, \quad P_2(\nu, R), \quad \dots$$

be the sequence of elements of R constructed by Theorem 2.1. Suppose that $f \in R$ and there exists $n \in \mathbb{Z}_+$ such that $\nu(f) < n\nu(m_R)$. Then there exists an expansion

$$f = \sum_I a_I P_0(\nu, R)^{i_0} P_1(\nu, R)^{i_1} \cdots P_r(\nu, R)^{i_r} + \sum_J \varphi_J P_0(\nu, R)^{j_0} \cdots P_r(\nu, R)^{j_r} + h,$$

where $r \in \mathbb{N}$ and

- *$I = (i_0, \dots, i_r) \in \mathbb{N}^{r+1}$ with $0 \le i_k < n_k(\nu, R)$ for $1 \le k \le r$, the a_I are units in R and $\nu(P_0(\nu, R)^{i_0} P_1(\nu, R)^{i_1} \cdots P_r(\nu, R)^{i_r}) = \nu(f)$ for all I in the first sum*;
- *$J = (j_0, \dots, j_r) \in \mathbb{N}^{r+1}$, $\varphi_J \in R$ and $\nu(P_0(\nu, R)^{j_0} \cdots P_r(\nu, R)^{j_r}) > \nu(f)$ for all J in the second sum*; *and*
- *$h \in m_R^n$.*

The terms in the first sum are uniquely determined, *up to the choice of units a_I*, *whose residues in R/m_R are uniquely determined.*

Let $\sigma_0(\nu, R) = 0$ and inductively define

$$\sigma_{i+1}(\nu, R) = \min\{j > \sigma_i(\nu, R) \mid n_j(\nu, R) > 1\}. \tag{15}$$

In Theorem 2.2, we see that all of the monomials in the expansion of f are in terms of the P_{σ_i}.

We have that

$$S(\beta_0(\nu, R), \beta_1(\nu, R), \dots, \beta_{\sigma_j(\nu,R)}(\nu, R)) = S(\beta_{\sigma_0}(\nu, R), \beta_{\sigma_1(\nu,R)}(\nu, R), \dots, \beta_{\sigma_j(\nu,R)}(\nu, R))$$

for all $j \ge 0$ and

$$R/m_R[\alpha_1(\nu, R), \alpha_2(\nu, R), \dots, \alpha_{\sigma_j(\nu,R)}(\nu, R)] = R/m_R[\alpha_{\sigma_1(\nu,R)}(\nu, R), \alpha_{\sigma_2(\nu,R)}(\nu, R), \dots, \alpha_{\sigma_j(\nu,R)}(\nu, R)]$$

for all $j \ge 1$.

Suppose that R is a regular local ring of dimension 2 which is dominated by a valuation ν. The quadratic transform T_1 of R along ν is defined as follows. Let

u, v be a system of regular parameters in R. Then $R[v/u] \subset V_\nu$ if $\nu(u) \le \nu(v)$ and $R[u/v] \subset V_\nu$ if $\nu(u) \ge \nu(v)$. Let

$$T_1 = R\left[\frac{v}{u}\right]_{R[v/u]\cap m_\nu} \quad \text{or} \quad T_1 = R\left[\frac{u}{v}\right]_{R[u/v]\cap m_\nu},$$

depending on whether $\nu(u) \le \nu(v)$ or $\nu(u) > \nu(v)$. T_1 is a 2-dimensional regular local ring which is dominated by ν. Let

$$R \to T_1 \to T_2 \to \cdots \tag{16}$$

be the infinite sequence of quadratic transforms along ν, so that $V_\nu = \bigcup_{i\ge 1} T_i$ [Abhyankar 1959, Lemma 4.5] and $L = V_\nu/m_\nu = \bigcup_{i\ge 1} T_i/m_{T_i}$.

For $f \in R$ and $R \to R^*$ a sequence of quadratic transforms along ν, we define a strict transform of f in R^* to be f_1 if $f_1 \in R^*$ is a local equation of the strict transform in R^* of the subscheme $f = 0$ of R. In this way, a strict transform is only defined up to multiplication by a unit in R^*. This ambiguity will not be a difficulty in our proof. We will denote a strict transform of f in R^* by $\mathrm{st}_{R^*}(f)$.

We use the notation of Theorem 2.1 and its proof for R and the $P_i(\nu, R)$. Recall that $U_1 = U^{w_0(1)}$. Let $w = w_0(1)$. Since $\bar{n}_1(\nu, R)$ and w are relatively prime, there exist $a, b \in \mathbb{N}$ such that

$$\varepsilon := \bar{n}_1(\nu, R)b - wa = \pm 1.$$

Define elements of the quotient field of R by

$$x_1 = (x^b y^{-a})^\varepsilon, \qquad y_1 = (x^{-w} y^{\bar{n}_1(\nu,R)})^\varepsilon. \tag{17}$$

We have that

$$x = x_1^{\bar{n}_1(\nu,R)} y_1^a, \qquad y = x_1^w y_1^b. \tag{18}$$

Since $\bar{n}_1(\nu, R)\nu(y) = w\nu(x)$, it follows that

$$\bar{n}_1(\nu, R)\nu(x_1) = \nu(x) > 0 \quad \text{and} \quad \nu(y_1) = 0.$$

We further have that

$$\alpha_1(\nu, R) = [y_1]^\varepsilon \in V_\nu/m_\nu. \tag{19}$$

Let $A = R[x_1, y_1] \subset V_\nu$ and $m_A = m_\nu \cap A$.

Let $R_1 = A_{m_A}$. We have that R_1 is a regular local ring and the divisor of xy in R_1 has only one component ($x_1 = 0$). In particular, $R \to R_1$ is "free" [Cutkosky and Piltant 2004, Definition 7.5]. $R \to R_1$ factors (uniquely) as a product of quadratic transforms and the divisor of xy in R_1 has two distinct irreducible factors in all intermediate rings.

Theorem 2.3 [Cutkosky and Vinh 2014, Theorem 7.1]. *Let R be a 2-dimensional regular local ring with regular parameters x, y. Suppose that R is dominated by a valuation ν. Let $P_0(\nu, R) = x$, $P_1(\nu, R) = y$ and $\{P_i(\nu, R)\}$ be the sequence* *of elements of R constructed in Theorem 2.1. Suppose that $\Omega \geq 2$. Then there exists some smallest value i in the sequence* (16) *such that the divisor of xy in* $\mathrm{Spec}(T_i)$ *has only one component. Let $R_1 = T_i$. Then $R_1/m_{R_1} \cong R/m_R(\alpha_1(\nu, R))$, and there exists $x_1 \in R_1$ and $w \in \mathbb{Z}_+$ such that $x_1 = 0$ is a local equation of the exceptional divisor of* $\mathrm{Spec}(R_1) \to \mathrm{Spec}(R)$, *and $Q_0 = x_1$, $Q_1 = P_2/x_1^{wn_1}$ are regular parameters in R_1. We have that*

$$P_i(\nu, R_1) = \frac{P_{i+1}(\nu, R)}{P_0(\nu, R_1)^{wn_1(\nu,R)\cdots n_i(\nu,R)}}$$

for $1 \leq i < \max\{\Omega, \infty\}$ satisfy the conclusions of Theorem 2.1 for the ring R_1.

We have that

$$G(\beta_0(\nu, R_1), \ldots, \beta_i(\nu, R_1)) = G(\beta_0(\nu, R), \ldots, \beta_{i+1}(\nu, R))$$

for $i \geq 1$, so that

$$\bar{n}_i(\nu, R_1) = \bar{n}_{i+1}(\nu, R) \quad \text{for } i \geq 1$$

and

$$R_1/m_{R_1}[\alpha_1(\nu, R_1), \ldots, \alpha_i(\nu, R_1)] = R/m_R[\alpha_1(\nu, R), \ldots, \alpha_{i+1}(\nu, R)] \quad \text{for } i \geq 1,$$

giving

$$d_i(\nu, R_1) = d_{i+1}(\nu, R) \quad \text{and} \quad n_i(\nu, R_1) = n_{i+1}(\nu, R) \quad \text{for } i \geq 1.$$

Let $\sigma_0(\nu, R_1) = 0$ and inductively define

$$\sigma_{i+1}(\nu, R_1) = \min\{j > \sigma_i(1) \mid n_j(\nu, R_1) > 1\}.$$

We then have that $\sigma_0(\nu, R_1) = 0$ and for $i \geq 1$, $\sigma_i(\nu, R_1) = \sigma_{i+1}(\nu, R) - 1$ if $n_1(\nu, R) > 1$, and $\sigma_i(\nu, R_1) = \sigma_i(\nu, R) - 1$ if $n_1(\nu, R) = 1$. For all $j \geq 0$,

$$\begin{aligned} &S(\beta_0(\nu, R_1), \beta_1(\nu, R_1), \ldots, \beta_{\sigma_{j+1}(\nu,R_1)}(\nu, R_1)) \\ &\qquad = S(\beta_{\sigma_0(1)}(\nu, R_1), \beta_{\sigma_1(\nu,R_1)}, \ldots, \beta_{\sigma_j(\nu,R_1)}(\nu, R_1)). \end{aligned}$$

Iterating this construction, we produce a sequence of sequences of quadratic transforms along ν,

$$R \to R_1 \to \cdots \to R_{\sigma_1(\nu,R)}.$$

Now $x, \bar{y} = P_{\sigma_1(\nu,R)}$ are regular parameters in R. By (17) (with y replaced with $\bar{y}$) we have that $R_{\sigma_1(\nu,R)}$ has regular parameters

$$x_1 = (x^b \bar{y}^{-a})^{\varepsilon}, \quad y_1 = (x^{-\omega} \bar{y}^{\bar{n}_{\sigma_1(\nu,R)}(\nu,R)})^{\varepsilon}, \tag{20}$$

where $\omega, a, b \in \mathbb{N}$ satisfy $\varepsilon = \bar{n}_{\sigma_1(\nu,R)}(\nu, R)b - \omega a = \pm 1$. Furthermore, $R_{\sigma_1(\nu,R_1)}$

has regular parameters $x_{\sigma_1(\nu,R)}$, $y_{\sigma_1(\nu,R)}$, where

$$x=\delta x_{\sigma_1(\nu,R_1)}^{\bar{n}_{\sigma_1(\nu,R)}(\nu,R_1)} \quad \text{and} \quad y_{\sigma_1(\nu,R_1)}=\mathrm{st}_{R_{\sigma_1}(\nu,R_1)}P_{\sigma_1(\nu,R)}(\nu,R)$$

with $\delta\in R_{\sigma_1(\nu,R)}$ a unit.

For the remainder of this section, we suppose that R is a 2-dimensional regular local ring and ν is a nondiscrete rational rank 1 valuation of the quotient field of R with valuation ring V_ν, so that V_ν/m_ν is algebraic over R/m_R. Suppose that $f\in R$ and $\nu(f)=\gamma$. We denote the class of f in $\mathcal{P}_\gamma(R)/\mathcal{P}^+_\gamma(R)\subset \mathrm{gr}_\nu(R)$ by $\mathrm{in}_\nu(f)$. By Theorem 2.2, we have that $\mathrm{gr}_\nu(R)$ is generated by the initial forms of the $P_i(\nu,R)$ as an R/m_R-algebra. That is,

$$\begin{aligned}\mathrm{gr}_\nu(R)&=R/m_R\big[\mathrm{in}_\nu(P_0(\nu,R)),\mathrm{in}_\nu(P_1(\nu,R)),\dots\big]\\&=R/m_R\big[\mathrm{in}_\nu(P_{\sigma_0(\nu,R)}(\nu,R)),\mathrm{in}_\nu(P_{\sigma_1(\nu,R)}(\nu,R)),\dots\big].\end{aligned}$$

Thus the semigroup $S^R(\nu)=\{\nu(f)\mid f\in R\}$ is equal to

$$S^R(\nu)=S(\beta_0(\nu,R),\beta_1(\nu,R),\dots)=S(\beta_{\sigma_0(\nu,R)}(\nu,R),\beta_{\sigma_1(\nu,R)}(\nu,R),\dots),$$

the value group Φ_ν is equal to

$$G(\beta_0(\nu,R),\beta_1(\nu,R),\dots)$$

and the residue field of the valuation ring V_ν/m_ν is

$$R/m_R[\alpha_1(\nu,R),\alpha_2(\nu,R),\dots]=R/m_R[\alpha_{\sigma_1}(\nu,R),\alpha_{\sigma_2}(\nu,R),\dots].$$

By (1) of Theorem 2.1, every element $\beta\in S^R(\nu)$ has a unique expression

$$\beta=\sum_{i=0}^{r}a_i\beta_i(\nu,R)$$

for some r with $a_i\in\mathbb{N}$ for all i and $0\le a_i<n_i(\nu,R)$ for $1\le i$. In particular, if $a_i\neq 0$ in the expansion then $\beta_i(\nu,R)=\beta_{\sigma_j(\nu,R)}(\nu,R)$ for some j.

Lemma 2.4. *Let*

$$\begin{aligned}\sigma_i&=\sigma_i(\nu,R), & \sigma_i(1)&=\sigma_i(\nu,R_{\sigma_1}),\\ \beta_i&=\beta_i(\nu,R), & \beta_i(1)&=\beta_i(\nu,R_{\sigma_1}),\\ P_i&=P_i(\nu,R), & P_i(1)&=P_i(\nu,R_{\sigma_1}),\\ n_i&=n_i(\nu,R), & n_i(1)&=n_i(\nu,R_{\sigma_1}),\\ \bar{n}_i&=\bar{n}_i(\nu,R), & \bar{n}_i(1)&=\bar{n}_i(\nu,R_{\sigma_1}).\end{aligned}$$

Suppose that $i\in\mathbb{N}$, $r\in\mathbb{N}$ and $a_j\in\mathbb{N}$ for $j=0,\dots,r$, with $0\le a_j<n_{\sigma_j}$ for $j\ge 1$, are such that

$$\nu(P_{\sigma_0}^{a_0}\cdots P_{\sigma_r}^{a_r})>\nu(P_{\sigma_i})$$

or $r < i$ and

$$\nu(P_{\sigma_0}^{a_0}\cdots P_{\sigma_r}^{a_r}) = \nu(P_{\sigma_i}).$$

By (18) *and Theorem 2.3, we have expressions in*

$$R_{\sigma_1} = R[x_1, y_1]_{m_\nu \cap R[x_1,y_1]}$$

where x_1, y_1 are defined by (20),

$$P_{\sigma_0}^{a_0}\cdots P_{\sigma_r}^{a_r} = y_1^{aa_0+ba_1} P_{\sigma_1(1)}(1)^{a_2}\cdots P_{\sigma_{r-1}(1)}(1)^{a_r} P_{\sigma_0(1)}(1)^t$$

where $t = \bar{n}_{\sigma_1}a_0 + \omega a_1 + \omega n_{\sigma_1}a_2 + \cdots + \omega n_{\sigma_1}\cdots n_{\sigma_{r-1}}a_r$ and

$$P_{\sigma_i} = \begin{cases} y_1^a P_{\sigma_0(1)}(1)^{\bar{n}_{\sigma_1}} & \text{if } i = 0,\\ y_1^b P_{\sigma_0(1)}(1)^{\omega} & \text{if } i = 1,\\ P_{\sigma_{i-1}(1)}(1) P_{\sigma_0(1)}(1)^{\omega n_{\sigma_1}\cdots n_{\sigma_{i-1}}} & \text{if } i \geq 2.\end{cases}$$

Let

$$\lambda = \begin{cases} \bar{n}_{\sigma_1} & \text{if } i = 0,\\ \omega & \text{if } i = 1,\\ \omega n_{\sigma_1}\cdots n_{\sigma_{i-1}} & \text{if } i \geq 2.\end{cases}$$

Then $t > \lambda$, except in the case where $i = 1$, $P_{\sigma_0}^{a_0}\cdots P_{\sigma_r}^{a_r} = P_{\sigma_0}$ and $\bar{n}_{\sigma_1} = \omega = 1$. In this case we have $\lambda = t$.

Proof. First suppose that $i \geq 2$ and $r \geq i$. Then

$$t - \lambda = (\bar{n}_{\sigma_1}a_0 + \omega a_1 + \omega n_{\sigma_1}a_2 + \cdots + \omega n_{\sigma_1}\cdots n_{\sigma_{r-1}}a_r) - \omega n_{\sigma_1}\cdots n_{\sigma_{i-1}} > 0.$$

Now suppose that $i \geq 2$ and $r < i$. We have that

$$(\bar{n}_{\sigma_1}a_0 + \omega a_1 + \cdots + \omega n_{\sigma_1}\cdots n_{\sigma_{r-1}}a_r - \omega n_{\sigma_1}\cdots n_{\sigma_{i-1}})\beta_{\sigma_0(1)}(1)$$
$$\geq \beta_{\sigma_{i-1}(1)}(1) - a_2\beta_{\sigma_1(1)}(1) - \cdots - a_r\beta_{\sigma_{r-1}(1)}(1) > 0,$$

since $n_{\sigma_j(1)}(1) = n_{\sigma_{j+1}}$ for all j, and so $n_{\sigma_{j+1}}\beta_{\sigma_j(1)}(1) < \beta_{\sigma_{j+1}(1)}(1)$ for all j.

Now suppose that $i = 1$. As in the proof for the case $i \geq 2$, we have that $t - \lambda > 0$ if $r \geq 1$, so suppose that $i = 1$ and $r = 0$. Then $\bar{n}_{\sigma_1}\beta_{\sigma_1} = \omega\beta_{\sigma_0}$. From our assumption $a_0\nu(P_0) \geq \nu(P_1)$, we obtain $t - \lambda = \bar{n}_{\sigma_1}a_0 - \omega \geq 0$ with equality if and only if $a_0 = \omega = \bar{n}_{\sigma_1} = 1$ since $\gcd(\omega, \bar{n}_{\sigma_1}) = 1$.

Now suppose $i = 0$. As in the previous cases, we have $t - \lambda > 0$ if $r > 1$ and $t - \lambda > 0$ if $r = 1$ except possibly if $P_0^{a_0}\cdots P_r^{a_r} = P_1^{a_1}$. We then have that $\nu(P_{\sigma_1}^{a_1}) > \nu(P_{\sigma_0})$, and so

$$a_1\frac{\beta_{\sigma_1}}{\beta_{\sigma_0}} > 1.$$

Since

$$\frac{\beta_{\sigma_1}}{\beta_{\sigma_0}} = \frac{\omega}{\bar{n}_{\sigma_1}},$$

we have that $t - \lambda = \omega a_1 - \bar{n}_{\sigma_1} > 0$. □

Lemma 2.5. *Let notation be the same as in Lemma 2.4. Suppose that $f \in R$, with $\nu(f) = \nu(P_{\sigma_i})$ for some $i \geq 0$, and that f has an expression of the form of Theorem 2.2,*

$$f = cP_{\sigma_i} + \sum_{j=1}^{s} c_i P_{\sigma_0}^{a_0(j)} P_{\sigma_1}^{a_1(j)} \cdots P_{\sigma_r}^{a_r(j)} + h,$$

where

- $s, r \in \mathbb{N}$;
- c, c_j *are units in* R;
- $0 \leq a_k(j) < n_k$ *for* $1 \leq k \leq r$ *and* $1 \leq j \leq s$;
- $\nu(f) = \nu(P_{\sigma_i}) \leq \nu\big(P_{\sigma_0}^{a_0(j)} P_{\sigma_1}^{a_1(j)} \cdots P_{\sigma_r}^{a_r(j)}\big)$ *for* $1 \leq j \leq s$;
- $a_k(j) = 0$ *for* $k \geq i$ *if* $\nu(f) = \nu\big(P_{\sigma_0}^{a_p(j)} \cdots P_{\sigma_r}^{a_r(j)}\big)$; *and*
- $h \in m_R^n$ *with* $n > \nu(f)$.

Then $\mathrm{st}_{R_{\sigma_1}}(f)$ *is a unit in* R_{σ_1} *if* $i = 0$ *or* 1, *and if* $i > 1$, *there exists a unit* $\bar{c}$ *in* R_{σ_1} *and* $\Omega \in R_{\sigma_1}$ *such that*

$$\mathrm{st}_{R_{\sigma_1}}(f) = \bar{c} P_{\sigma_{i-1}(1)}(1) + x_1 \Omega$$

with $\nu(\mathrm{st}_{R_{\sigma_1}}(f)) = \nu(P_{\sigma_{i-1}(1)}(1))$ *and* $\nu(P_{\sigma_{i-1}(1)}(1)) \leq \nu(x_1 \Omega)$.

Proof. Let

$$\lambda = \begin{cases} \bar{n}_1 & \text{if } i = 0, \\ \omega & \text{if } i = 1, \\ \omega n_{\sigma_1} \cdots n_{\sigma_{r-1}} & \text{if } i \geq 2. \end{cases}$$

Then

$$f = cH_i + \sum_{j=1}^{s} c_j (y_1)^{aa_0(j)+ba_1(j)} P_{\sigma_0(1)}(1)^{t_j} P_{\sigma_1}(1)^{a_2(j)} \cdots P_{\sigma_{r-1}(1)}(1)^{a_r(j)} + P_{\sigma_0(1)}(1)^t h'$$

with

$$H_i = \begin{cases} (y_1)^a P_{\sigma_0(1)}(1)^{\bar{n}_1} & \text{if } i = 0, \\ (y_1)^b P_{\sigma_0(1)}(1)^{\omega} & \text{if } i = 1, \\ P_{\sigma_0(1)}(1)^{\omega n_1 \cdots n_{i-1}} P_{\sigma_{i-1}(1)}(1) & \text{if } i \geq 2, \end{cases}$$

$h' \in R_{\sigma_1}$ and

$$t_j = \bar{n}_1 a_0(j) + \omega a_1(j) + \omega n_{\sigma_1} a_2(j) + \cdots + \omega n_{\sigma_1} \cdots n_{\sigma_{r-1}} a_r(j)$$

for $1 \leq j \leq s$ and $t > \lambda$. By Lemma 2.4, if $i \geq 2$ or $i = 0$, we have that $t_j > \lambda$ for all j. Thus $f = P_{\sigma_0(1)}(1)^{\lambda} \bar{f}$, where

$$\bar{f} = cG_i + \sum_{j=1}^{s} c_j P_{\sigma_0(1)}(1)^{t_j - \lambda} P_{\sigma_1(1)}(1)^{a_2(j)} \cdots P_{\sigma_{r-1}(1)}(1)^{a_r(j)} + P_{\sigma_0(1)}(1)^{t-\lambda} h'$$

is a strict transform $\bar{f} = \mathrm{st}_{R_1}(f)$ of f in R_1, with

$$G_i=\begin{cases}(y_1)^a & \text{if } i=0,\\ (y_1)^b & \text{if } i=1,\\ P_{\sigma_{i-1}(1)}(1) & \text{if } i\geq 2.\end{cases}$$

If $i=1$, then by Lemma 2.4, $t_j>\lambda$ for all j, except possibly for a single term (which we can assume is t_1) which is P_{σ_0}, and we have that $\omega=\bar{n}_{\sigma_1}=1$. In this case $t_1=\lambda$. Then

$$\left[\frac{P_{\sigma_1}}{P_{\sigma_0}}\right]=\alpha_{\sigma_1}(\nu,R)\in V_\nu/m_\nu,$$

which has degree $d_{\sigma_1}(\nu,R)=n_{\sigma_1}>1$ over R/m_R. By (18), $x=x_1$, $y=x_1y_1$ and

$$f=x_1[c+c_1y_1+x_1\Omega]$$

with $\Omega\in R_{\sigma_1}$. We have that $c+c_1y_1$ is a unit in R_{σ_1} since

$$[y_1]=\left[\frac{P_{\sigma_0}}{P_{\sigma_1}}\right]\notin R/m_R. \qquad \square$$

3. Finite generation implies no defect

Suppose that R is a 2-dimensional excellent regular local ring and S is a 2-dimensional regular local ring such that S dominates R. Let K be the quotient field of R and K^* the quotient field of S. Suppose that $K\to K^*$ is a finite separable field extension. Suppose that ν^* is a nondiscrete rational rank 1 valuation of K^* such that V_{ν^*}/m_{ν^*} is algebraic over S/m_S and that ν^* dominates S. Then we have a natural graded inclusion $\mathrm{gr}_\nu(R)\to\mathrm{gr}_{\nu^*}(S)$, so that for $f\in R$, we have that $\mathrm{in}_\nu(f)=\mathrm{in}_{\nu^*}(f)$. Let $\nu=\nu^*|K$. Let $L=V_{\nu^*}/m_{\nu^*}$. Suppose that $\mathrm{gr}_{\nu^*}(S)$ is a finitely generated $\mathrm{gr}_\nu(R)$-algebra.

Let x, y be regular parameters in R, with associated generating sequence to ν, $P_0=P_0(\nu,R)=x$, $P_1=P_1(\nu,R)=y$, $P_2=P_2(\nu,R),\ldots$ in R as constructed in Theorem 2.1, with $U_i=U_i(\nu,R)$, $\beta_i=\beta_i(\nu,R)=\nu(P_i)$, $\gamma_i=\alpha_i(\nu,R)$, $m_i=m_i(\nu,R)$, $\bar{m}_i=\bar{m}_i(\nu,R)$, $d_i=d_i(\nu,R)$ and $\sigma_i=\sigma_i(\nu,R)$ defined as in Section 2.

Similarly, let u, v be regular parameters in S, with associated generating sequence to ν^*, $Q_0=P_0(\nu^*,S)=u$, $Q_1=P_1(\nu^*,S)=v$, $Q_2=P_2(\nu^*,S),\ldots$ in S as constructed in Theorem 2.1, with $V_i=U_i(\nu^*,S)$, $\gamma_i=\beta_i(\nu^*,S)=\nu^*(Q_i)$, $\delta_i=\alpha_i(\nu^*,S)$, $n_i=n_i(\nu^*,S)$, $\bar{n}_i=\bar{n}_i(\nu^*,S)$, $e_i=\alpha_i(\nu^*,S)$ and $\tau_i=\sigma_i(\nu^*,S)$ defined as in Section 2.

With our assumption that $\mathrm{gr}_{\nu^*}(S)$ is a finitely generated $\mathrm{gr}_\nu(R)$-algebra, we have that for all sufficiently large l,

$$\mathrm{gr}_{\nu^*}(S)=\mathrm{gr}_\nu(R)[\mathrm{in}_{\nu^*}Q_{\tau_0},\ldots,\mathrm{in}_{\nu^*}Q_{\tau_l}]. \tag{21}$$

Proposition 3.1. *With our assumption that* $\mathrm{gr}_{\nu^*}(S)$ *is a finitely generated* $\mathrm{gr}_\nu(R)$*-algebra, there exist integers* $s>1$ *and* $r>1$ *such that for all* $j\geq 0$,

$$\beta_{\sigma_{r+j}}=\gamma_{\tau_{s+j}},\quad \overline{m}_{\sigma_{r+j}}=\bar{n}_{\tau_{s+j}},\quad d_{\sigma_{r+j}}=e_{\tau_{s+j}},\quad m_{\sigma_{r+j}}=n_{\tau_{s+j}},$$
$$G(\beta_{\sigma_0},\dots,\beta_{\sigma_{r+j}})\subset G(\gamma_{\tau_0},\dots,\gamma_{\tau_{s+j}}),$$
$$[G(\gamma_{\tau_0},\dots,\gamma_{\tau_{s+j}}):G(\beta_{\sigma_0},\dots,\beta_{\sigma_{r+j}})]=e(\nu^*/\nu),$$
$$R/m_R[\delta_{\sigma_1},\dots,\delta_{\sigma_{r+j}}]\subset S/m_S[\varepsilon_{\tau_1},\dots,\varepsilon_{\tau_{s+j}}]$$

and

$$[S/m_S[\varepsilon_{\tau_1},\dots,\varepsilon_{\tau_{s+j}}]:R/m_R[\delta_{\sigma_1},\dots,\delta_{\sigma_{r+j}}]]=f(\nu^*/\nu).$$

Proof. Let l be as in (21). For $s\geq l$, define the subalgebra A_{τ_s} of $\mathrm{gr}_{\nu^*}(S)$ by

$$A_{\tau_s}=S/m_S[\mathrm{in}_{\nu^*}Q_{\tau_0},\dots,\mathrm{in}_{\nu^*}Q_{\tau_s}].$$

For $s\geq l$, let

$$r_s=\max\{j\mid \mathrm{in}_{\nu^*}P_{\sigma_j}\in A_{\tau_s}\},$$
$$\lambda_s=[G(\gamma_{\tau_0},\dots,\gamma_{\tau_s}):G(\beta_{\sigma_0},\dots,\beta_{\sigma_{r_s}})],$$

and

$$\chi_s=[S/m_S[\varepsilon_{\tau_0},\dots,\varepsilon_{\tau_s}]:R/m_R[\delta_{\sigma_0},\dots,\delta_{\sigma_{r_s}}]].$$

To simplify notation, we write $r=r_s$.

We now show that $\beta_{\sigma_{r+1}}=\gamma_{\tau_{s+1}}$. Suppose that $\beta_{\sigma_{r+1}}>\gamma_{\tau_{s+1}}$. We have that

$$\mathrm{in}_{\nu^*}Q_{\tau_{s+1}}\in\mathrm{gr}_\nu(R)[\mathrm{in}_{\nu^*}Q_{\tau_0},\dots,\mathrm{in}_{\nu^*}Q_{\tau_s}].$$

Since

$$\beta_{\sigma_{r+1}}<\beta_{\sigma_{r+2}}<\cdots,$$

we then have that $\mathrm{in}_{\nu^*}Q_{\tau_{s+1}}\in A_{\tau_s}$, which is impossible. Thus $\beta_{\sigma_{r+1}}\leq\gamma_{\tau_{s+1}}$. If $\beta_{\sigma_{r+1}}<\gamma_{\tau_{s+1}}$, then since

$$\gamma_{\tau_{s+1}}<\gamma_{\tau_{s+2}}<\cdots\quad\text{and}\quad \mathrm{in}_{\nu^*}P_{\sigma_{r+1}}\in\mathrm{gr}_{\nu^*}(S),$$

we have that $\mathrm{in}_{\nu^*}P_{\sigma_{r+1}}\in A_{\tau_s}$, which is impossible. Thus $\beta_{\sigma_{r+1}}=\gamma_{\tau_{s+1}}$.

We now establish that either we have a reduction $\lambda_{s+1}<\lambda_s$ or

$$\lambda_{s+1}=\lambda_s,\qquad \beta_{\sigma_{r+1}}=\gamma_{\tau_{s+1}}\quad\text{and}\quad \overline{m}_{\sigma_{r+1}}=\bar{n}_{\tau_{s+1}}. \tag{22}$$

Let ω be a generator of the group $G(\gamma_{\tau_1},\dots,\gamma_{\tau_s})$, so that $G(\gamma_{\tau_1},\dots,\gamma_{\tau_s})=\mathbb{Z}\omega$. We have that

$$G(\gamma_{\tau_0},\dots,\gamma_{\tau_{s+1}})=\frac{1}{\bar{n}_{\tau_{s+1}}}\mathbb{Z}\omega$$

and

$$G(\beta_{\sigma_0},\dots,\beta_{\sigma_{r+1}})=\frac{1}{\overline{m}_{\sigma_{r+1}}}\mathbb{Z}(\lambda_s\omega).$$

There exists a positive integer f with $\gcd(f, \bar{n}_{\tau_{s+1}}) = 1$ such that

$$\gamma_{\tau_{s+1}} = \frac{f}{\bar{n}_{\tau_{s+1}}}\omega,$$

and a positive integer g with $\gcd(g, \overline{m}_{\sigma_{r+1}}) = 1$ such that

$$\beta_{\sigma_{r+1}} = \frac{g}{\overline{m}_{\sigma_{r+1}}}\lambda_s\omega.$$

Since $\beta_{\sigma_{r+1}} = \gamma_{\tau_{s+1}}$, we have

$$g\lambda_s\bar{n}_{\tau_{s+1}} = f\overline{m}_{\sigma_{r+1}}.$$

Thus $\bar{n}_{\tau_{s+1}}$ divides $\overline{m}_{\sigma_{r+1}}$ and $\overline{m}_{\sigma_{r+1}}$ divides $\lambda_s\bar{n}_{\tau_{s+1}}$, so that

$$a = \frac{\overline{m}_{\sigma_{r+1}}}{\bar{n}_{\tau_{s+1}}}$$

is a positive integer, and defining

$$\bar{\lambda} = \frac{\lambda_s}{a},$$

$\bar{\lambda}$ is a positive integer with

$$\frac{\lambda_s}{\overline{m}_{\sigma_{r+1}}} = \frac{\bar{\lambda}}{\bar{n}_{\tau_{s+1}}}$$

and

$$\bar{\lambda} = [G(\gamma_{\tau_0}, \ldots, \gamma_{\tau_{s+1}}) : G(\beta_{\sigma_0}, \ldots, \beta_{\sigma_{r+1}})].$$

Since $\lambda_{s+1} \leq \bar{\lambda}$, either $\lambda_{s+1} < \lambda_s$ or $\lambda_{s+1} = \lambda_s$ and $\overline{m}_{\sigma_{r+1}} = \bar{n}_{\tau_{s+1}}$.

We now suppose that s is sufficiently large that (22) holds. Since

$$\mathrm{in}_{\nu^*} Q_{\tau_{s+1}} \in \mathrm{gr}_{\nu^*}(S) = \mathrm{gr}_{\nu}(R)[\mathrm{in}_{\nu^*} Q_{\tau_0}, \ldots, \mathrm{in}_{\nu^*} Q_{\tau_s}],$$

if $\bar{n}_{\tau_{s+1}} > 1$ we have an expression

$$\mathrm{in}_{\nu^*} P_{\sigma_{r+1}} = \mathrm{in}_{\nu^*}(\alpha)\mathrm{in}_{\nu^*} Q_{\tau_{s+1}} \tag{23}$$

in $\mathcal{P}_{\gamma_{\tau_{s+1}}}(S)/\mathcal{P}^+_{\gamma_{\tau_{s+1}}}(S)$ with α a unit in S, and if $\bar{n}_{\tau_{s+1}} = 1$, since $\mathrm{in}_{\nu^*} P_{\sigma_{r+1}} \notin A_{\tau_s}$ we have an expression

$$\mathrm{in}_{\nu^*} P_{\sigma_{r+1}} = \mathrm{in}_{\nu^*}(\alpha)\mathrm{in}_{\nu^*} Q_{\tau_{s+1}} + \sum \mathrm{in}_{\nu^*}(\alpha_J)(\mathrm{in}_{\nu^*} Q_{\tau_0})^{j_0} \cdots (\mathrm{in}_{\nu^*} Q_{\tau_s})^{j_s} \tag{24}$$

in $\mathcal{P}_{\gamma_{\tau_{s+1}}}(S)/\mathcal{P}^+_{\gamma_{\tau_{s+1}}}(S)$ with α a unit in S, where the sum is taken over certain $J = (j_0, \ldots, j_s) \in \mathbb{N}^{s+1}$ such that the α_J are units in S, and the terms $\mathrm{in}_{\nu^*} Q_{\tau_{s+1}}$ and $(\mathrm{in}_{\nu^*} Q_{\tau_0})^{j_0} \cdots (\mathrm{in}_{\nu^*} Q_{\tau_s})^{j_s}$ are all linearly independent over S/m_S.

The monomial $U_{\sigma_{r+1}}$ in $P_{\sigma_0}, \ldots, P_{\sigma_r}$ and the monomial $V_{\tau_{s+1}}$ in $Q_{\tau_0}, \ldots, Q_{\tau_s}$ both have the value $\bar{n}_{\tau_{s+1}}\gamma_{\tau_{s+1}} = \overline{m}_{\sigma_{r+1}}\beta_{\sigma_{r+1}}$, and satisfy

$$\varepsilon_{\tau_{s+1}} = \left[\frac{Q_{\tau_{s+1}}^{\bar{n}_{\tau_{s+1}}}}{V_{\tau_{s+1}}}\right]$$

and

$$\delta_{\sigma_{r+1}} = \left[\frac{P_{\sigma_{r+1}}^{\bar{n}_{\tau_{s+1}}}}{U_{\sigma_{r+1}}}\right].$$

Since $U_{\sigma_{r+1}}, V_{\tau_{s+1}} \in A_{\tau_s}$ and by (12) and Theorem 2.1(2), we have that

$$\left[\frac{V_{\tau_{s+1}}}{U_{\sigma_{r+1}}}\right] \in S/m_S[\varepsilon_{\tau_1}, \ldots, \varepsilon_{\tau_s}].$$

If $\bar{n}_{\tau_{s+1}} > 1$, then by (23), we have

$$\left[\frac{P_{\sigma_{r+1}}^{\bar{n}_{\tau_{s+1}}}}{U_{\sigma_{r+1}}}\right] = \left[\frac{V_{\tau_{s+1}}}{U_{\sigma_{r+1}}}\right]\left([\alpha]^{\bar{n}_{\tau_{s+1}}}\left[\frac{Q_{\tau_{s+1}}^{\bar{n}_{\tau_{s+1}}}}{V_{\tau_{s+1}}}\right]\right)$$

in $L = V_{\nu^*}/m_{\nu^*}$, and if $\bar{n}_{\tau_{s+1}} = 1$, then by (24) we have

$$\left[\frac{P_{\sigma_{r+1}}}{U_{\sigma_{r+1}}}\right] = \left[\frac{V_{\tau_{s+1}}}{U_{\sigma_{r+1}}}\right]\left([\alpha]\left[\frac{Q_{\tau_{s+1}}}{V_{\tau_{s+1}}}\right] + \sum [\alpha_J]\left[\frac{Q_{\tau_0}^{j_0} \cdots Q_{\tau_s}^{j_s}}{V_{\tau_{s+1}}}\right]\right).$$

Thus by (12),

$$S/m_S[\varepsilon_{\tau_1}, \ldots, \varepsilon_{\tau_s}][\varepsilon_{\tau_{s+1}}] = S/m_S[\varepsilon_{\tau_1}, \ldots, \varepsilon_{\tau_s}][\delta_{\sigma_{r+1}}]. \tag{25}$$

We have a commutative diagram

$$\begin{array}{ccc} S/m_S[\varepsilon_{\tau_1}, \ldots, \varepsilon_{\tau_s}] & \longrightarrow & S/m_S[\varepsilon_{\tau_1}, \ldots, \varepsilon_{\tau_s}, \varepsilon_{\tau_{s+1}}] = S/m_S[\varepsilon_{\tau_1}, \ldots, \varepsilon_{\tau_s}][\delta_{\sigma_{r+1}}] \\ \uparrow & & \uparrow \\ R/m_R[\delta_{\sigma_1}, \ldots, \delta_{\sigma_r}] & \longrightarrow & R/m_R[\delta_{\sigma_1}, \ldots, \delta_{\sigma_r}][\delta_{\sigma_{r+1}}] \end{array}$$

Let

$$\bar{\chi} = [S/m_S[\varepsilon_{\tau_1}, \ldots, \varepsilon_{\tau_s}, \varepsilon_{\tau_{s+1}}] : R/m_R[\delta_{\sigma_1}, \ldots, \delta_{\sigma_r}, \delta_{\sigma_{r+1}}]].$$

Since

$$S/m_S[\varepsilon_{\tau_1}, \ldots, \varepsilon_{\tau_s}, \varepsilon_{\tau_{s+1}}] = S/m_S[\varepsilon_{\tau_1}, \ldots, \varepsilon_{\tau_s}][\delta_{\sigma_{r+1}}],$$

we have that $e_{\tau_{s+1}} \mid d_{\sigma_{r+1}}$. Further,

$$\frac{d_{\sigma_{r+1}}}{e_{\tau_{s+1}}}\bar{\chi} = \chi_s,$$

whence $\bar{\chi} \leq \chi_s$. Thus $\chi_{s+1} \leq \chi_s$ and if $\chi_{s+1} = \chi_s$, then $d_{\sigma_{r+1}} = e_{\tau_{s+1}}$ and $r_{s+1} = r_s + 1$, since $P_{\sigma_{r+2}} \in A_{\tau_{s+1}}$ implies $\lambda_{s+1} < \lambda_s$ or $\chi_{s+1} < \chi_s$.

We may thus choose s sufficiently large that there exists an integer $r > 1$ such that for all $j \geq 0$,

$$\beta_{\sigma_{r+j}} = \gamma_{\tau_{s+j}}, \qquad \bar{m}_{\sigma_{r+j}} = \bar{n}_{\tau_{s+j}}, \qquad d_{\sigma_{r+j}} = e_{\tau_{s+j}}, \qquad m_{\sigma_{r+j}} = n_{\tau_{s+j}},$$
$$G(\beta_{\sigma_0}, \dots, \beta_{\sigma_{r+j}}) \subset G(\gamma_{\tau_0}, \dots, \gamma_{\tau_{s+j}}),$$

there is a constant λ (which does not depend on j) such that

$$[G(\gamma_{\tau_0}, \dots, \gamma_{\tau_{s+j}}) : G(\beta_{\sigma_0}, \dots, \beta_{\sigma_{r+j}})] = \lambda,$$
$$R/m_R[\delta_{\sigma_1}, \dots, \delta_{\sigma_{r+j}}] \subset S/m_S[\varepsilon_{\tau_1}, \dots, \varepsilon_{\tau_{s+j}}],$$

and there is a constant χ (which does not depend on j) such that

$$[S/m_S[\varepsilon_{\tau_1}, \dots, \varepsilon_{\tau_{s+j}}] : R/m_R[\delta_{\sigma_1}, \dots, \delta_{\sigma_{r+j}}]] = \chi.$$

Then

$$\Phi_{\nu^*} = \bigcup_{j \geq 1} \frac{1}{\bar{n}_{\tau_{s+1}} \cdots \bar{n}_{\tau_{s+j}}} \mathbb{Z}\omega,$$

where $G(\gamma_{\tau_0}, \dots, \gamma_{\tau_s}) = \mathbb{Z}\omega$, and

$$\Phi_\nu = \bigcup_{j \geq 1} \frac{1}{\bar{m}_{\sigma_{r+1}} \cdots \bar{m}_{\sigma_{r+j}}} \lambda\mathbb{Z}\omega = \bigcup_{j \geq 1} \frac{1}{\bar{n}_{\tau_{s+1}} \cdots \bar{n}_{\tau_{s+j}}} \lambda\mathbb{Z}\omega,$$

so that

$$\lambda = [\Phi_{\nu^*} : \Phi_\nu] = e(\nu^*/\nu).$$

For $i \geq 0$, let $K_i = R/m_R[\delta_{\sigma_1}, \dots, \delta_{\sigma_{r+i}}]$ and $M_i = S/m_S[\varepsilon_{\tau_1}, \dots, \varepsilon_{\tau_{s+i}}]$. We have that $M_{i+1} = M_i[\delta_{\sigma_{r+i+1}}]$ for $i \geq 0$ and $\chi = [M_i : K_i]$ for all i. Further,

$$\bigcup_{i=0}^{\infty} M_i = V_{\nu^*}/m_{\nu^*} \quad \text{and} \quad \bigcup_{i=0}^{\infty} K_i = V_\nu/m_\nu.$$

Thus if $g_1, \dots, g_\lambda \in M_0$ form a basis of M_0 as a K_0-vector space, then $g_1, \dots, g_\lambda$ form a basis of M_i as a K_i-vector space for all $i \geq 0$. Thus

$$\chi = [V_{\nu^*}/m_{\nu^*} : V_\nu/m_\nu] = f(\nu^*/\nu). \qquad \square$$

Let r and s be as in the conclusions of Proposition 3.1. There exists τ_t with $t \geq s$ such that we have a commutative diagram of inclusions of regular local rings

$$\begin{array}{ccc} R_{\sigma_r} & \longrightarrow & S_{\tau_t} \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

(with the notation introduced in Section 2). After possibly increasing s and r, we may assume that $R' \subset R_{\sigma_r}$, where R' is the local ring of the conclusions of Proposition 1.1. Recall that R has regular parameters $x = P_0$, $y = P_1$ and S has regular parameters $u = Q_0$, $v = Q_1$; and that R_{σ_r} has regular parameters x_{σ_r}, y_{σ_r} such that

$$x = \delta x_{\sigma_r}^{\overline{m}_{\sigma_1}\cdots\overline{m}_{\sigma_r}}, \qquad y_{\sigma_r} = \mathrm{st}_{R_{\sigma_r}} P_{\sigma_{r+1}},$$

where δ is a unit in R_{σ_r} and S_{τ_t} has regular parameters u_{τ_t}, v_{τ_t} such that

$$u = \varepsilon u_{\tau_t}^{\bar{n}_{\tau_1}\cdots\bar{n}_{\tau_t}}, \qquad v_{\tau_t} = \mathrm{st}_{S_{\tau_t}} Q_{\tau_{t+1}},$$

where ε is a unit in S_{τ_t}. We may choose $t \gg 0$ so that we have an expression

$$x_{\sigma_r} = \varphi u_{\tau_t}^{\lambda} \tag{26}$$

for some positive integer λ, where φ is a unit in S_{τ_t}, since $\bigcup_{t=0}^{\infty} S_{\tau_t} = V_{\nu^*}$.

We have expressions $P_i = \psi_i x_{\sigma_r}^{c_i}$ in R_{σ_r}, where the ψ_i are units in R_{σ_r} for $i \leq \sigma_r$, so that $P_i = \psi_i^* u_{\tau_t}^{c_i\lambda}$ in S_{τ_t}, where the ψ_i^* are units in S_{τ_t} for $i \leq \sigma_r$, by (26).

Lemma 3.2. *For $j \geq 1$ we have*

$$\mathrm{st}_{R_{\sigma_r}}(P_{\sigma_{r+j}}) = u_{\tau_t}^{\lambda_j}\, \mathrm{st}_{S_{\tau_t}}(P_{\sigma_{r+j}})$$

for some $\lambda_j \in \mathbb{N}$, where we regard $P_{\sigma_{r+j}}$ as an element of R on the left-hand side of the equation and regard $P_{\sigma_{r+j}}$ as an element of S on the right-hand side.

Proof. Using (26), we have

$$P_{\sigma_{r+j}} = \mathrm{st}_{R_{\sigma_r}}(P_{\sigma_{r+j}}) x_{\sigma_r}^{f_j} = \mathrm{st}_{R_{\sigma_r}}(P_{\sigma_{r+j}}) u_{\tau_t}^{\lambda f_j} \varphi^{f_j},$$

where $f_j \in \mathbb{N}$. Viewing $P_{\sigma_{r+j}}$ as an element of S, we have that

$$P_{\sigma_{r+j}} = \mathrm{st}_{S_{\tau_t}}(P_{\sigma_{r+j}}) u_{\tau_t}^{g_j}$$

for some $g_j \in \mathbb{N}$. Since $u_{\tau_t} \nmid \mathrm{st}_{S_{\tau_t}}(P_{\sigma_{r+j}})$, we have that $f_j\lambda \leq g_j$ and therefore $\lambda_j = g_j - f_j\lambda \geq 0$. □

By induction on the sequence of quadratic transforms above R and S from Lemma 2.5, and since $\nu^*(P_{\sigma_{r+j}}) = \beta_{\sigma_{r+j}} = \gamma_{\tau_{s+j}}$ by Proposition 3.1, we have by (23) and (24) an expression

$$\mathrm{st}_{S_{\tau_t}}(P_{\sigma_{r+j}}) = c\, \mathrm{st}_{S_{\tau_t}}(Q_{\tau_{s+j}}) + u_{\tau_t}\Omega \tag{27}$$

with $c \in S_{\tau_t}$ a unit, $\Omega \in S_{\tau_t}$ and $\nu^*(u_{\tau_t}\Omega) \geq \nu^*(\mathrm{st}_{S_{\tau_t}}(Q_{\tau_{s+j}}))$ if $s + j > t$; and

$$S_{\tau_t}(P_{\sigma_{r+j}}) \text{ is a unit in } S_{\tau_t} \tag{28}$$

if $s + j \leq t$. Thus $P_{\sigma_{r+j}} = u_{\tau_t}^{d_j}\bar{\varphi}_j$ in S_{τ_t}, where d_j is a positive integer and $\bar{\varphi}_j$ is a unit in S_{τ_t} if $s + j \leq t$.

Suppose $s < t$. Then

$$y_{\sigma_r} = \mathrm{st}_{R_{\sigma_r}}(P_{\sigma_{r+1}}) = \tilde{\varphi} u_{\tau_t}^{h},$$

where $\tilde{\varphi}$ is a unit in S_{τ_t} and h is a positive integer. As shown in (20) of Section 2,

$$R_{\sigma_{r+1}} = R_{\sigma_r}[\bar{x}_1, \bar{y}_1]_{m_\nu \cap R_{\sigma_r}[\bar{x}_1, \bar{y}_1]},$$

where

$$\bar{x}_1 = (x_{\sigma_r}^b y_{\sigma_r}^{-a})^\varepsilon, \qquad \bar{y}_1 = (x_{\sigma_r}^{-\omega} y_{\sigma_r}^{m_{\sigma_r}})^\varepsilon,$$

with $\varepsilon = m_{\sigma_r} b - \omega a = \pm 1$, $\nu(\bar{x}_1) > 0$ and $\nu(\bar{y}_1) = 0$. Substituting

$$x_{\sigma_r} = \varphi u_{\tau_t}^\lambda \quad \text{and} \quad y_{\sigma_1} = \tilde{\varphi} u_{\tau_t}^h,$$

we see that $R_{\sigma_{r+1}}$ is dominated by S_{τ_t}. We thus have a factorization

$$R_{\sigma_r} \to R_{\sigma_{r+1}} \to S_{\tau_t}$$

with $x_{\sigma_{r+1}} = \bar{x}_1 = \hat{\varphi} u_{\tau_t}^{\lambda'}$, where $\hat{\varphi}$ is a unit in S_{τ_t} and λ' is a positive integer. We may thus replace s with $s+1$, r with $r+1$ and R_{σ_r} with $R_{\sigma_{r+1}}$.

Iterating this argument, we may assume that $s = t$ (with $r = r_s$), so that by Lemma 3.2, (28) and (27),

$$y_{\sigma_r} = \mathrm{st}_{R_{\sigma_r}}(P_{\sigma_{r+1}}) = u_{\tau_s}^\mu \mathrm{st}_{S_{\tau_s}}(P_{\sigma_{r+1}}),$$

where

$$\mathrm{st}_{S_{\tau_s}}(P_{\sigma_{r+1}}) = c\, \mathrm{st}_{S_{\tau_s}}(Q_{\tau_{s+1}}) + u_{\tau_s}\Omega$$

with c a unit in S_{τ_s} and $\Omega \in S_{\tau_s}$. Thus by (26), we have an expression

$$x_{\sigma_r} = \varphi u_{\tau_s}^\lambda, \qquad y_{\sigma_r} = \bar{\varepsilon} u_{\tau_s}^\alpha (v_{\tau_s} + u_{\tau_s}\Omega),$$

where λ is a positive integer, $\alpha \in \mathbb{N}$, φ and $\bar{\varepsilon}$ are units in S_{τ_s} and $\Omega \in S_{\tau_s}$.

We have that

$$\nu^*(x_{\sigma_r}) = \lambda \nu^*(u_{\tau_s}),$$
$$\nu(x_{\sigma_r})\mathbb{Z} = G(\nu(x_{\sigma_r})) = G(\beta_{\sigma_0}, \ldots, \beta_{\sigma_r}),$$
$$\nu^*(u_{\tau_s})\mathbb{Z} = G(\nu^*(u_{\tau_s})) = G(\gamma_{\tau_0}, \ldots, \gamma_{\tau_s}).$$

Thus

$$\lambda = [G(\gamma_{\tau_0}, \ldots, \gamma_{\tau_s}) : G(\beta_{\sigma_0}, \ldots, \beta_{\sigma_r})] = e(\nu^*/\nu)$$

by Proposition 3.1.

By Theorem 2.3, we have that

$$R_{\sigma_r}/m_{R_{\sigma_r}} = R/m_R[\delta_{\sigma_1}, \ldots, \delta_{\sigma_r}] \quad \text{and} \quad S_{\tau_s}/m_{S_{\tau_s}} = S/m_S[\varepsilon_{\tau_1}, \ldots, \varepsilon_{\tau_s}].$$

Thus

$$[S_{\tau_s}/m_{S_{\tau_s}} : R_{\sigma_r}/m_{R_{\sigma_r}}] = f(\nu^*/\nu)$$

by Proposition 3.1.

Since the ring R' of Proposition 1.1 is contained in R_{σ_r} by our construction, we have by Proposition 1.1 that $(K, \nu) \to (K^*, \nu^*)$ is without defect, completing the proofs of Proposition 0.3 and Theorem 0.1.

4. Nonsplitting and finite generation

In this section, we maintain the following assumptions. Suppose that R is a 2-dimensional excellent local domain with quotient field K. Further suppose that K^* is a finite separable extension of K and S is a 2-dimensional local domain with quotient field K^* such that S dominates R. Suppose that ν^* is a valuation of K^* such that ν^* dominates S. Let ν be the restriction of ν^* to K.

Suppose that ν^* has rational rank 1 and ν^* is not discrete. Then V_{ν^*}/m_{ν^*} is algebraic over S/m_S, by Abhyankar's inequality [1956, Proposition 2].

Lemma 4.1. *Let assumptions be as above. Then the associated graded ring* $\mathrm{gr}_{\nu^*}(S)$ *is an integral extension of* $\mathrm{gr}_{\nu}(R)$.

Proof. It suffices to show that $\mathrm{in}_{\nu^*}(f)$ is integral over $\mathrm{gr}_{\nu}(R)$ whenever $f\in S$. Suppose that $f\in S$. There exists $n_1>0$ such that $n_1\nu^*(f)\in\Phi_\nu$. Let $x\in m_R$ and $\omega=\nu(x)$. Then there exists a positive integer b and natural number a such that $bn_1\nu^*(f)=a\omega$, so

$$\nu^*\left(\frac{f^{bn_1}}{x^a}\right)=0.$$

Let

$$\xi=\left[\frac{f^{bn_1}}{x^a}\right]\in V_{\nu^*}/m_{\nu^*},$$

and let $g(t)=t^r+\bar{a}_{r-1}t^{r-1}+\cdots+\bar{a}_0$ with $\bar{a}_i\in R/m_R$ be the minimal polynomial of ξ over R/m_R. Let a_i be lifts of the $\bar{a}_i$ to R. Then

$$\nu^*(f^{b_1n_1r}+a_{r-1}x^af^{bn_1(r-1)}+\cdots+a_0x^{ar})\\ >\nu^*(f^{bn_1r})=\nu^*(a_{r-1}x^af^{bn_1(r-1)})=\cdots=\nu^*(a_0x^{ar}).$$

Thus,

$$\mathrm{in}_{\nu^*}(f)^{b_1n_1r}+\mathrm{in}_{\nu}(a_{r-1}x^a)\mathrm{in}_{\nu^*}(f)^{bn_1(r-1)}+\cdots+\mathrm{in}_{\nu}(a_0x^{ar})=0$$

in $\mathrm{gr}_{\nu^*}(S)$, so $\mathrm{in}_{\nu^*}(f)$ is integral over $\mathrm{gr}_{\nu^*}(R)$. □

We now establish Theorem 0.5. Recall (as defined after Proposition 0.3) that ν^* does not split in S if ν^* is the unique extension of ν to K^* which dominates S.

Theorem 0.5. *Let assumptions be as above and suppose that R and S are regular local rings. Suppose that* $\mathrm{gr}_{\nu^*}(S)$ *is a finitely generated* $\mathrm{gr}_{\nu}(R)$*-algebra. Then S is a localization of the integral closure of R in* K^*, $\delta(\nu^*/\nu)=0$ *and* ν^* *does not split in S.*

Proof. Let s and r be as in the conclusions of Proposition 3.1. We first show that $P_{\sigma_{r+j}}$ is irreducible in $\hat{S}$ for all $j>0$. There exists a unique extension of ν^* to the quotient field of $\hat{S}$ which dominates $\hat{S}$ [Spivakovsky 1990; Cutkosky and Vinh 2014; Herrera Govantes et al. 2014]. The extension is immediate since ν^* is not discrete; that is, there is no increase in value group or residue field for the extended valuation. It has the property that if $f \in \hat{S}$ and $\{f_i\}$ is a Cauchy sequence in $\hat{S}$ which converges to f, then $\nu^*(f) = \nu^*(f_i)$ for all $i \gg 0$.

Suppose that $P_{\sigma_{r+j}}$ is not irreducible in $\hat{S}$ for some $j > 0$. We derive a contradiction. With this assumption, $P_{\sigma_{r+j}} = fg$ with $f, g \in m_{\hat{S}}$. Let $\{f_i\}$ be a Cauchy sequence in S which converges to f and let $\{g_i\}$ be a Cauchy sequence in S which converges to g. For i sufficiently large, $f - f_i, g - g_i \in m_{\hat{S}}^n$, where n is so large that $n\nu^*(m_{\hat{S}}) = n\nu^*(m_S) > \nu(P_{\sigma_{r+j}})$. Thus $P_{\sigma_{r+j}} = f_i g_i + h$ with $h \in m_{\hat{S}}^n \cap S = m_S^n$, and so $\mathrm{in}_{\nu^*}(P_{\sigma_{r+j}}) = \mathrm{in}_{\nu^*}(f_i)\mathrm{in}_{\nu^*}(g_i)$. Now

$$\nu^*(f_i), \nu^*(g_i) < \nu(P_{\sigma_{r+j}}) = \beta_{\sigma_{r+j}} = \gamma_{\tau_{s+j}} = \nu^*(Q_{\tau_{s+j}}),$$

so that

$$\mathrm{in}_{\nu^*}(f_i), \mathrm{in}_{\nu^*}(g_i) \in S/m_S[\mathrm{in}_{\nu^*}(Q_{\tau_0}), \ldots, \mathrm{in}_{\nu^*}(Q_{\tau_{s+j-1}})],$$

which implies

$$\mathrm{in}_{\nu^*}(P_{\sigma_{r+j}}) \in S/m_S[\mathrm{in}_{\nu^*}(Q_{\tau_0}), \ldots, \mathrm{in}_{\nu^*}(Q_{\tau_{s+j-1}})].$$

But then (24) implies

$$\mathrm{in}_{\nu^*}(Q_{\tau_{s+j}}) \in S/m_S[\mathrm{in}_{\nu^*}(Q_{\tau_0}), \ldots, \mathrm{in}_{\nu^*}(Q_{\tau_{s+j-1}})],$$

which is impossible. Thus $P_{\sigma_{r+j}}$ is irreducible in $\hat{S}$ for all $j > 0$.

If S is not a localization of the integral closure of R in K^*, then by Zariski's main theorem (Theorem 1 of Chapter 4 in [Raynaud 1970]), $m_R S = fN$, where $f \in m_S$ and N is an m_S-primary ideal. Thus f divides P_i in S for all i, which is impossible since we have shown that $P_{\sigma_{r+j}}$ is analytically irreducible in S for all $j > 0$; we cannot have $P_{\sigma_{r+j}} = a_j f$ where a_j is a unit in S for $j > 0$ since $\nu(P_{\sigma_{r+j}}) = \nu^*(Q_{\tau_{s+j}})$ by Proposition 3.1.

Now suppose that ν^* is not the unique extension of ν to K^* which dominates S. Recall that V_ν is the union of all quadratic transforms above R along ν and V_{ν^*} is the union of all quadratic transforms above S along ν^* [Abhyankar 1959, Lemma 4.5].

Then for all $i \gg 0$, we have a commutative diagram

$$\begin{array}{ccc} R_{\sigma_i} & \longrightarrow & T_i \\ \uparrow & & \uparrow \\ R & \longrightarrow & T \end{array}$$

where T is the integral closure of R in K^*, T_i is the integral closure of R_{σ_i} in K^*, $S = T_{\mathfrak{p}}$ for some maximal ideal $\mathfrak{p}$ in T which lies over m_R, and there exist $r \geq 2$ prime ideals $\mathfrak{p}_1(i), \ldots, \mathfrak{p}_r(i)$ in T_i which lie over $m_{R_{\sigma_i}}$ and whose intersection with T is $\mathfrak{p}$. We may assume that $\mathfrak{p}_1(i)$ is the center of ν^*.

There exists an m_R-primary ideal I_i in R with blowup $\gamma : X_{\sigma_i} \to \mathrm{Spec}(R)$, where X_{σ_i} is regular and R_{σ_i} is a local ring of X_{σ_i}. Let Z_{σ_i} be the integral closure of X_{σ_i} in K^*. Let $Y_{\sigma_i} = Z_{\sigma_i} \times_{\mathrm{Spec}(T)} \mathrm{Spec}(S)$. We have a commutative diagram of morphisms

$$\begin{array}{ccc} Y_{\sigma_i} & \xrightarrow{\beta} & X_{\sigma_i} \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ \mathrm{Spec}(S) & \xrightarrow{\alpha} & \mathrm{Spec}(R) \end{array}$$

The morphism δ is projective by [EGA II 1961, Proposition II.5.5.5 and Corollary II.6.1.11] and it is birational, so since Y_{σ_i} and $\mathrm{Spec}(S)$ are integral, it is a blowup of an ideal J_i in S [EGA III$_1$ 1961, Proposition III.2.3.5], which we can take to be m_S-primary since S is a regular local ring and hence factorial. Define curves $C = \mathrm{Spec}(R/(P_{\sigma_i}))$ and $C' = \alpha^{-1}(C) = \mathrm{Spec}(S/(P_{\sigma_i}))$. Denote the Zariski closure of a set W by $\overline{W}$. The strict transform C^* of C' in Y_{σ_i} is the Zariski closure

$$\begin{aligned} C^* &= \overline{\delta^{-1}(C' \setminus m_S)} = \overline{\delta^{-1}\alpha^{-1}(C \setminus m_R)} = \overline{\beta^{-1}\gamma^{-1}(C \setminus m_R)} \\ &= \beta^{-1}(\overline{\gamma^{-1}(C \setminus m_R)}) \qquad \text{(since } \beta \text{ is quasifinite)} \\ &= \beta^{-1}(\widetilde{C}), \end{aligned} \tag{29}$$

where $\widetilde{C}$ is the strict transform of C in X_{σ_i}. We have $Z_{\sigma_i} \times_{X_{\sigma_i}} \mathrm{Spec}(R_{\sigma_i}) \cong \mathrm{Spec}(T_i)$, so

$$Y_{\sigma_i} \times_{X_{\sigma_i}} \mathrm{Spec}(R_{\sigma_i}) \cong \mathrm{Spec}(T_i \otimes_T S).$$

Let x_{σ_i} be a local equation in R_{σ_i} of the exceptional divisor of $\mathrm{Spec}(R_{\sigma_i}) \to \mathrm{Spec}(R)$ and let $y_{\sigma_i} = \mathrm{st}_{R_{\sigma_i}}(P_{\sigma_i})$. Then $x_{\sigma_i}, y_{\sigma_i}$ are regular parameters in R_{σ_i}. We have that

$$\sqrt{m_{R_{\sigma_i}}(T_i \otimes_T S)} = \bigcap_{j=1}^{r} \mathfrak{p}_j(i)(T_i \otimes_T S).$$

The blowup of $J_i(S/(P_{\sigma_i}))$ in C' is $\bar{\delta} : C^* \to C'$, where $\bar{\delta}$ is the restriction of δ to C^* [Hartshorne 1977, Corollary II.7.15]. Since y_{σ_i} is a local equation of $\widetilde{C}$ in R_{σ_i}, we have by (29) that

$$\mathfrak{p}_1(i), \ldots, \mathfrak{p}_r(i) \in \bar{\delta}^{-1}(m_S) \subset C^*.$$

Since $\bar{\delta}$ is proper and C' is a curve, $C^* = \mathrm{Spec}(A)$ for some excellent 1-dimensional domain A such that the inclusion $S/(P_{\sigma_i}) \to A$ is finite [Milne 1980, Corollary I.1.10]. Let $B = A \otimes_{S/(P_{\sigma_i})} \hat{S}/(P_{\sigma_i})$. Then

$$C^* \times_{\mathrm{Spec}(S/(P_{\sigma_i}))} \mathrm{Spec}(\hat{S}/(P_{\sigma_i})) = \mathrm{Spec}(B) \to \mathrm{Spec}(\hat{S}/(P_{\sigma_i}))$$

is the blowup of $J_i(\hat{S}/(P_{\sigma_i}))$ in $\hat{S}/(P_{\sigma_i})$. The extension $\hat{S}/(P_{\sigma_i}) \to B$ is finite since $S/(P_{\sigma_i}) \to A$ is finite.

Now assume that $S/(P_{\sigma_i})$ is analytically irreducible. Then B has only one minimal prime since the blowup $\mathrm{Spec}(B) \to \mathrm{Spec}(\hat{S}/(P_{\sigma_i}))$ is birational.

Since a complete local ring is Henselian, B is a local ring [Milne 1980, Theorem I.4.2 on page 32], a contradiction to our assumption that $r > 1$. □

As a consequence of the above theorem, we now obtain Corollary 0.6.

Corollary 0.6. *Let assumptions be as above and suppose that R is a regular local ring. Suppose that $R \to R'$ is a nontrivial sequence of quadratic transforms along ν. Then $gr_\nu(R')$ is not a finitely generated $gr_\nu(R)$-algebra.*

Proof. The integral closure of R in its quotient field is R, which is not equal to R' since $m_R R'$ is a principal ideal. Thus $\mathrm{gr}_\nu(R')$ is not a finitely generated $\mathrm{gr}_\nu(R)$-algebra by Theorem 0.5. □

The conclusions of Theorem 0.5 do not hold if we remove the assumption that ν^* is not discrete, when V_ν/m_ν is finite over R/m_R. We give a simple example. Let k be an algebraically closed field of characteristic not equal to 2 and let $p(u)$ be a transcendental series in the power series ring $k[[u]]$ such that $p(0) = 1$. Then $f = v - up(u)$ is irreducible in the power series ring $k[[u, v]]$ and $k[[u, v]]/(f)$ is a discrete valuation ring with regular parameter u. Let ν be the natural valuation of this ring. Let $R = k[u, v]_{(u,v)}$ and $S = k[x, y]_{(x,y)}$. Define a k-algebra homomorphism $R \to S$ by $u \mapsto x^2$ and $v \mapsto y^2$. The series $f(x^2, y^2)$ factors as

$$f = \left(y - x\sqrt{p(x^2)}\right)\left(y + x\sqrt{p(x^2)}\right)$$

in $k[[x, y]]$. Let $f_1 = y - x\sqrt{p(x^2)}$ and $f_2 = y + x\sqrt{p(x^2)}$. The rings $k[[x, y]]/(f_i)$ are discrete valuation rings with regular parameter x. Let ν_1 and ν_2 be the natural valuations of these ring.

Let ν be the valuation of the quotient field of R which dominates R and is defined by the natural inclusion $R \to k[[u, v]]/(f)$, and let ν_i for $i = 1, 2$ be the valuations of the quotient field of S which dominate S and are defined by the respective natural inclusions $S \to k[[x, y]]/(f_i)$. Then ν_1 and ν_2 are distinct extensions of ν to the quotient field of S which dominate S. However, we have that $\mathrm{gr}_\nu(R) = k[\mathrm{in}_\nu(u)]$ and $\mathrm{gr}_{\nu_i}(S) = k[\mathrm{in}_{\nu^*}(x)]$ with $\mathrm{in}_{\nu^*}(x)^2 = \mathrm{in}_\nu(u)$. Thus $\mathrm{gr}_{\nu_i}(S)$ is a finite $\mathrm{gr}_\nu(R)$-algebra.

We now give an example where ν^* has rational rank 2 and ν splits in S but $\mathrm{gr}_{\nu^*}(S)$ is a finitely generated $\mathrm{gr}_\nu(R)$-algebra. Suppose that k is an algebraically closed field of characteristic not equal to 2. Let $R = k[x, y]_{(x,y)}$ and $S = k[u, v]_{(u,v)}$. The substitutions $u = x^2$ and $v = y^2$ make S into a finite separable extension of R. Define a valuation ν_1 of the quotient field K^* of S by $\nu_1(x) = 1$ and $\nu_1(y - x) = \pi + 1$, and define a valuation ν_2 of the quotient field K^* by $\nu_2(x) = 1$ and $\nu_2(y + x) = \pi + 1$. Since $u = x^2$ and $v - u = (y - x)(y + x)$, we have that $\nu_1(u) = \nu_2(u) = 2$ and $\nu_1(v - u) = \nu_2(v - u) = \pi + 2$. Let ν be the common restriction of ν_1 and ν_2 to the quotient field K of R. Then ν splits in S. However, $\mathrm{gr}_{\nu_1}(S)$ is a finitely generated $\mathrm{gr}_\nu(R)$-algebra since $\mathrm{gr}_{\nu_1}(S)=k[\mathrm{in}_{\nu_1}(x), \mathrm{in}_{\nu_1}(y-x)]$ is a finitely generated k-algebra. Note that $\mathrm{gr}_\nu(R)=k[\mathrm{in}_\nu(u), \mathrm{in}_\nu(v-u)]$ with $\mathrm{in}_{\nu_1}(x)^2=\mathrm{in}_\nu(u)$ and $\mathrm{in}_\nu(v-u)=2\mathrm{in}_{\nu_1}(y-x)\mathrm{in}_{\nu_1}(x)$.

References

[Abhyankar 1956] S. Abhyankar, "On the valuations centered in a local domain", *Amer. J. Math.* **78** (1956), 321–348. MR Zbl

[Abhyankar 1959] S. Abhyankar, *Ramification theoretic methods in algebraic geometry*, Annals of Math. Studies **43**, Princeton University Press, 1959. MR Zbl

[Cutkosky 1999] S. D. Cutkosky, *Local monomialization and factorization of morphisms*, Astérisque **260**, Société Mathématique de France, Paris, 1999. MR Zbl

[Cutkosky 2015] S. D. Cutkosky, "Counterexamples to local monomialization in positive characteristic", *Math. Ann.* **362**:1-2 (2015), 321–334. MR Zbl

[Cutkosky 2016a] S. D. Cutkosky, "A generalisation of the Abhyankar Jung theorem to associated graded rings of valuations", *Math. Proc. Cambridge Philos. Soc.* **160**:2 (2016), 233–255. MR

[Cutkosky 2016b] S. D. Cutkosky, "Ramification of valuations and local rings in positive characteristic", *Comm. Algebra* **44**:7 (2016), 2828–2866. MR Zbl

[Cutkosky and Piltant 2004] S. D. Cutkosky and O. Piltant, "Ramification of valuations", *Adv. Math.* **183**:1 (2004), 1–79. MR Zbl

[Cutkosky and Vinh 2014] S. D. Cutkosky and P. A. Vinh, "Valuation semigroups of two-dimensional local rings", *Proc. Lond. Math. Soc.* (3) **108**:2 (2014), 350–384. MR Zbl

[EGA II 1961] A. Grothendieck, "Eléments de géométrie algébrique, II: Étude globale élémentaire de quelques classes de morphismes", *Inst. Hautes Études Sci. Publ. Math.* **8** (1961), 5–222. MR Zbl

[EGA III_1 1961] A. Grothendieck, "Eléments de géométrie algébrique, III: Étude cohomologique des faisceaux cohérents, I", *Inst. Hautes Études Sci. Publ. Math.* **11** (1961), 5–167. MR Zbl

[Endler 1972] O. Endler, *Valuation theory*, Springer, 1972. MR Zbl

[Ghezzi and Kashcheyeva 2007] L. Ghezzi and O. Kashcheyeva, "Toroidalization of generating sequences in dimension two function fields of positive characteristic", *J. Pure Appl. Algebra* **209**:3 (2007), 631–649. MR Zbl

[Ghezzi et al. 2006] L. Ghezzi, H. T. Hà, and O. Kashcheyeva, "Toroidalization of generating sequences in dimension two function fields", *J. Algebra* **301**:2 (2006), 838–866. MR Zbl

[Hartshorne 1977] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Math. **52**, Springer, 1977. MR Zbl

[Herrera Govantes et al. 2014] F. J. Herrera Govantes, M. A. Olalla Acosta, M. Spivakovsky, and B. Teissier, "Extending valuations to formal completions", pp. 252–265 in *Valuation theory in interaction*, edited by A. Campillo et al., European Mathematical Society, Zürich, 2014. MR Zbl

[Kuhlmann 2000] F.-V. Kuhlmann, "Valuation theoretic and model theoretic aspects of local uniformization", pp. 381–456 in *Resolution of singularities* (Obergurgl, 1997), edited by H. Hauser et al., Progr. Math. **181**, Birkhäuser, Basel, 2000. MR Zbl

[Kuhlmann 2010] F.-V. Kuhlmann, "A classification of Artin–Schreier defect extensions and characterizations of defectless fields", *Illinois J. Math.* **54**:2 (2010), 397–448. MR Zbl

[Mac Lane 1936] S. Mac Lane, "A construction for absolute values in polynomial rings", *Trans. Amer. Math. Soc.* **40**:3 (1936), 363–395. MR Zbl

[Milne 1980] J. S. Milne, *Étale cohomology*, Princeton Mathematical Series **33**, Princeton University Press, 1980. MR Zbl

[Raynaud 1970] M. Raynaud, *Anneaux locaux henséliens*, Lecture Notes in Math. **169**, Springer, 1970. MR Zbl

[Spivakovsky 1990] M. Spivakovsky, "Valuations in function fields of surfaces", *Amer. J. Math.* **112**:1 (1990), 107–156. MR Zbl

[Teissier 2003] B. Teissier, "Valuations, deformations, and toric geometry", pp. 361–459 in *Valuation theory and its applications* (Saskatoon, SK, 1999), vol. 2, edited by F.-V. Kuhlmann et al., Fields Inst. Commun. **33**, American Mathematical Society, Providence, RI, 2003. MR Zbl

[Teissier 2014] B. Teissier, "Overweight deformations of affine toric varieties and local uniformization", pp. 474–565 in *Valuation theory in interaction*, edited by A. Campillo et al., European Mathematical Society, Zürich, 2014. MR Zbl

[Vaquié 2007] M. Vaquié, "Famille admissible de valuations et défaut d'une extension", *J. Algebra* **311**:2 (2007), 859–876. MR Zbl

[Zariski and Samuel 1960] O. Zariski and P. Samuel, *Commutative algebra*, vol. 2, Van Nostrand, Princeton, 1960. MR Zbl

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