Algebra & Number Theory

Volume 11

2017

No. 7

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ALGEBRA AND NUMBER THEORY 11:7 (2017)
dx.doi.org/10.2140/ant.2017.11.1627

Quantitative equidistribution of Galois orbits of small points in the N-dimensional torus

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We present a quantitative version of Bilu's theorem on the limit distribution of Galois orbits of sequences of points of small height in the N-dimensional algebraic torus. Our result gives, for a given point, an explicit bound for the discrepancy between its Galois orbit and the uniform distribution on the compact subtorus, in terms of the height and the generalized degree of the point.

1. Introduction

One of the first results concerning the distribution of Galois orbits of points of small height in algebraic varieties is due to Bilu [1997]. It establishes that the Galois orbits of strict sequences of points of small Weil height in an algebraic torus tend to the uniform distribution around the unit polycircle.

Let us introduce some notation before giving the precise formulation of this result. Fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ together with an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. By $\mathbb{C}^\times$ and $\overline{\mathbb{Q}}^\times$ we denote the multiplicative groups of $\mathbb{C}$ and $\overline{\mathbb{Q}}$, respectively. Let $N \geq 1$; the *Galois orbit* of a point in $(\overline{\mathbb{Q}}^\times)^N$ is its orbit under the action of the *absolute Galois group*, $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

For a finite set $T \subset (\mathbb{C}^\times)^N$, the discrete probability measure on $(\mathbb{C}^\times)^N$ associated to it is given by

$$\mu_T = \frac{1}{\#T} \sum_{\alpha \in T} \delta_\alpha,$$

where $\#T$ denotes the cardinality of T and δ_α the Dirac delta measure on $(\mathbb{C}^\times)^N$ supported on α. The *unit polycircle* $(S^1)^N$ is the set of points $(z_1, \dots, z_n) \in \mathbb{C}^N$ such that $|z_1| = \cdots = |z_N| = 1$. It is a compact subgroup of $(\mathbb{C}^\times)^N$. We denote by $\lambda_{(S^1)^N}$ the Haar probability measure of $(S^1)^N$, considered as a measure on $(\mathbb{C}^\times)^N$.

D'Andrea and Narváez-Clauss were partially supported by the MICINN research project MTM2010-20279-C02-01 and the MINECO research project MTM2013-40775-P. Sombra was partially supported by the MINECO research projects MTM2012-38122-C03-02 and MTM2015-65361-P.
MSC2010: primary 11G50; secondary 11K38, 43A25.
Keywords: height of points, algebraic torus, equidistribution of Galois orbits.

A sequence $(\mu_k)_{k\geq 1}$ of probability measures on $(\mathbb{C}^\times)^N$ *converges weakly* to a probability measure μ on $(\mathbb{C}^\times)^N$ if, for every compactly supported continuous function $F:(\mathbb{C}^\times)^N \to \mathbb{R}$,

$$\lim_{k\to\infty} \int_{(\mathbb{C}^\times)^N} F\, d\mu_k = \int_{(\mathbb{C}^\times)^N} F\, d\mu.$$

Let $\xi \in \overline{\mathbb{Q}}^\times$ and $f_\xi \in \mathbb{Z}[x]$ be the minimal polynomial of ξ over the integers. The *Weil height* of ξ is defined as

$$\mathrm{h}(\xi) = \frac{m(f_\xi)}{\deg(\xi)},$$

where $m(f_\xi)$ is the *(logarithmic) Mahler measure* of f_ξ, given by

$$m(f_\xi) = \frac{1}{2\pi}\int_0^{2\pi} \log|f_\xi(e^{i\theta})|\, d\theta,$$

and $\deg(\xi) = [\mathbb{Q}(\xi):\mathbb{Q}]$ is the degree of the point ξ.

This notion of height extends to $(\overline{\mathbb{Q}}^\times)^N$ as follows:

$$\mathrm{h}(\boldsymbol{\xi}) = \mathrm{h}(\xi_1) + \cdots + \mathrm{h}(\xi_N), \quad \text{for every } \boldsymbol{\xi} = (\xi_1,\dots,\xi_N) \in (\overline{\mathbb{Q}}^\times)^N. \tag{1-1}$$

A sequence $(\boldsymbol{\xi}_k)_{k\geq 1}$ in $(\overline{\mathbb{Q}}^\times)^N$ is *strict* if, for every proper algebraic subgroup $Y \subset (\overline{\mathbb{Q}}^\times)^N$, the cardinality of the set $\{k : \boldsymbol{\xi}_k \in Y\}$ is finite.

Theorem 1.1 [Bilu 1997, Theorem 1.1]. *Let $(\boldsymbol{\xi}_k)_{k\geq 1}$ be a strict sequence in $(\overline{\mathbb{Q}}^\times)^N$ such that $\lim_{k\to\infty} \mathrm{h}(\boldsymbol{\xi}_k) = 0$. Then*

$$\lim_{k\to\infty} \mu_{S_k} = \lambda_{(S^1)^N},$$

where μ_{S_k} is the discrete probability measure associated to the Galois orbit S_k of $\boldsymbol{\xi}_k$.

This result was inspired by a previous work of Szpiro, Ullmo and Zhang [Szpiro et al. 1997] on the equidistribution of points of small Néron–Tate height in abelian varieties. It was originally motivated by Bogomolov's conjecture, solved in [Ullmo 1998; Zhang 1998]. The results of Szpiro, Ullmo and Zhang and of Bilu were largely generalized to other heights and places [Rumely 1999; Baker and Hsia 2005; Favre and Rivera-Letelier 2006; Baker and Rumely 2006; Chambert-Loir 2006; Yuan 2008; Gubler 2008; Berman and Boucksom 2010; Chen 2011; Burgos Gil et al. 2015]. In particular, these results established the equidistribution of Galois orbits of sequences of small points for all places of $\mathbb{Q}$ and heights associated to algebraic dynamical systems. Moreover, this equidistribution phenomenon holds for the bigger set of test functions with logarithmic singularities along divisors with minimal height; see [Chambert-Loir and Thuillier 2009].

As a general fact, these equidistribution theorems are formulated in a qualitative way, in the sense that no information is provided on the rate of convergence towards the equidistribution. An exception is [Favre and Rivera-Letelier 2006], where a bound for this rate of convergence is given for a large class of heights of points in the projective line and all places of $\mathbb{Q}$. Independently, Petsche [2005] gave a quantitative version of Bilu's result for the case of dimension one.

In this paper, we present a quantitative version of Theorem 1.1 for the general N-dimensional case. In particular, we provide a bound for the integral of a suitable test function with respect to the signed measure defined by the difference of the discrete probability measure associated to the Galois orbit of a point in $(\overline{\mathbb{Q}}^\times)^N$ and the measure $\lambda_{(S^1)^N}$. This bound is given in terms of the height of the point, a higher dimensional generalization of the notion of the degree of an algebraic number, and a constant depending only on the test function.

To state our main result properly, let us introduce further definitions and notations. For every $\boldsymbol{n} = (n_1, \dots, n_N) \in \mathbb{Z}^N$, consider the monomial map

$$\begin{aligned} \chi^{\boldsymbol{n}} : (\overline{\mathbb{Q}}^\times)^N &\to \overline{\mathbb{Q}}^\times \\ \boldsymbol{z} = (z_1, \dots, z_N) &\mapsto \chi^{\boldsymbol{n}}(\boldsymbol{z}) = z_1^{n_1} \cdots z_N^{n_N}. \end{aligned}$$

We define the *generalized degree* of a point $\boldsymbol{\xi} \in (\overline{\mathbb{Q}}^\times)^N$ by

$$\mathscr{D}(\boldsymbol{\xi}) = \min_{\boldsymbol{n} \neq \boldsymbol{0}} \bigl\{ \|\boldsymbol{n}\|_1 \deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) \bigr\}, \tag{1-2}$$

where $\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))$ is the degree of the point $\chi^{\boldsymbol{n}}(\boldsymbol{\xi}) \in \overline{\mathbb{Q}}^\times$ and $\|\cdot\|_1$ is the 1-norm on $\mathbb{C}^N$. For a particular choice of $\boldsymbol{\xi}$, the generalized degree can be computed with a finite number of operations; see Remark 2.7.

Let us write $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ and identify $\mathbb{T}^N \times \mathbb{R}^N$ with $(\mathbb{C}^\times)^N$ via the logarithmic-polar coordinate change of variables

$$(\boldsymbol{\theta}, \boldsymbol{u}) = ((\theta_1, \dots, \theta_N), (u_1, \dots, u_N)) \mapsto (e^{2\pi i \theta_1 + u_1}, \dots, e^{2\pi i \theta_N + u_N}).$$

On $\mathbb{T}^N \times \mathbb{R}^N \simeq (\mathbb{C}^\times)^N$, consider the translation invariant distance, defined as

$$\mathrm{d}((\boldsymbol{\theta}, \boldsymbol{u}), (\boldsymbol{\theta}', \boldsymbol{u}')) = \left(\sum_{l=1}^{N} \mathrm{d}_{\mathrm{ang}}(\theta_l, \theta_l')^2 + |u_l - u_l'|^2 \right)^{\frac{1}{2}},$$

where $\mathrm{d}_{\mathrm{ang}}(\theta_l, \theta_l')$ is the Euclidean distance in S^1 between $e^{2\pi i \theta_l}$ and $e^{2\pi i \theta_l'}$, divided by 2π.

A function $F : (\mathbb{C}^\times)^N \to \mathbb{R}$ belongs to the set of test functions $\mathcal{F}$ if it satisfies:

(i) F is a Lipschitz function with respect to the distance d;

(ii) the restriction $F_0 = F|_{(S^1)^N}$ is in $\mathscr{C}^{N+1}((S^1)^N, \mathbb{R})$.

The set $\mathcal{F}$ contains all compactly supported functions in $\mathscr{C}^{N+1}((\mathbb{C}^\times)^N, \mathbb{R})$.

The following is the main result of this paper.

Theorem 1.2. *There is a constant $C \leq 64$ such that, for every $\boldsymbol{\xi} \in (\overline{\mathbb{Q}}^\times)^N$ with $\mathrm{h}(\boldsymbol{\xi}) \leq 1$ and every $F \in \mathcal{F}$,*

$$\left| \int_{(\mathbb{C}^\times)^N} F \, d\mu_S - \int_{(\mathbb{C}^\times)^N} F \, d\lambda_{(S^1)^N} \right| \leq c(F) \left(4\,\mathrm{h}(\boldsymbol{\xi}) + C \frac{\log(\mathscr{D}(\boldsymbol{\xi})+1)}{\mathscr{D}(\boldsymbol{\xi})} \right)^{\frac{1}{2}},$$

where S is the Galois orbit of $\boldsymbol{\xi}$, μ_S the discrete probability measure associated to it and $c(F)$ a positive constant depending only on F.

For every test function $F \in \mathcal{F}$, the function F_0, its Fourier transform $\widehat{F}_0$, all the first order partial derivatives of F_0, and their corresponding Fourier transforms are integrable with respect to a Haar measure (Theorem A.1). In logarithmic-polar coordinates $F_0(\boldsymbol{\theta}) = F(\boldsymbol{\theta}, \mathbf{0})$. Then, as shown in the proof of Theorem 1.2, the constant $c(F)$ can be bounded by

$$c(F) \leq 2\,\mathrm{Lip}(F) + 16 \sum_{l=1}^{N} \left\| \widehat{\frac{\partial F_0}{\partial \theta_l}} \right\|_{\mathrm{L}^1},$$

where $\mathrm{Lip}(F)$ is the Lipschitz constant of F with respect to the distance d of $(\mathbb{C}^\times)^N$ and where $\|\cdot\|_{\mathrm{L}^1}$ stands for the L^1-norm of a function on the locally compact abelian group $\mathbb{Z}^N$ with respect to the standard Haar measure.

Our main theorem is a quantitative version of Bilu's result. Indeed, if we consider a strict sequence $(\boldsymbol{\xi}_k)_{k\geq 1}$ in $(\overline{\mathbb{Q}}^\times)^N$ such that $\mathrm{h}(\boldsymbol{\xi}_k) \to 0$ as $k \to \infty$, we necessarily have that $\mathscr{D}(\boldsymbol{\xi}_k) \to \infty$ as $k \to \infty$ (Lemma 2.8). Hence, for every function $F \in \mathcal{F}$, Theorem 1.2 implies that

$$\lim_{k\to\infty} \int_{(\mathbb{C}^\times)^N} F \, d\mu_{S_k} = \int_{(\mathbb{C}^\times)^N} F \, d\lambda_{(S^1)^N},$$

where μ_{S_k} is the discrete probability measure associated to the Galois orbit S_k of $\boldsymbol{\xi}_k$. Since $\mathcal{F}$ contains a dense subset of the set of compactly supported continuous functions on $(\mathbb{C}^\times)^N$, we deduce Theorem 1.1.

The rate of convergence in Theorem 1.2 has the expected exponent $\frac{1}{2}$, as in [Favre and Rivera-Letelier 2006], see also Theorem 3.1. On the other hand, one could ask if, for the general N-dimensional case, the constant $c(F)$ might be bounded by the Lipschitz constant of the test function, as in their paper.

The idea of the proof of our result is to reduce the problem, via monomial maps, to the one-dimensional situation as it was done in [Bilu 1997; D'Andrea et al. 2014]. In this setting, we apply Favre and Rivera-Letelier's result (Theorem 3.1). Then, we lift the obtained quantitative control to the N-dimensional torus by applying the Fourier inversion formula and a study of the Fourier–Stieltjes transform of the discrete probability measure associated to the orbit of the point.

This paper is structured as follows. Section 2 contains preliminary theory and general results on Fourier analysis, measures on the Riemann sphere, Galois invariant sets, and the generalized degree. In Section 3, we give the proof of Theorem 1.2, which is divided in several propositions and lemmas. At the end of the paper there are two appendices, the first one studies the set of test functions $\mathcal{F}$, and the second the Lipschitz constant of an auxiliary function used in Section 3.

2. Preliminaries

2.1. ***Fourier analysis.*** In this section we review basic concepts of Fourier analysis on $\mathbb{T}^N$. We refer the reader to [Rudin 1962] for the proof of the stated results.

Let $p \geq 1$. Given a function $H : \mathbb{T}^N \to \mathbb{C}$, its L^p*-norm* is defined by

$$\|H\|_{\mathrm{L}^p} = \left(\int_{\mathbb{T}^N} |H(\boldsymbol{\theta})|^p \, d\boldsymbol{\theta} \right)^{\frac{1}{p}} \in \mathbb{R}_{\geq 0} \cup \{+\infty\}.$$

We say that $H \in \mathrm{L}^p(\mathbb{T}^N)$ if this norm is finite. In particular, the function H is *Haar-integrable* if it lies in $\mathrm{L}^1(\mathbb{T}^N)$. Similarly, for a function $G : \mathbb{Z}^N \to \mathbb{C}$, its L^p*-norm* is defined by

$$\|G\|_{\mathrm{L}^p} = \left(\sum_{\boldsymbol{n} \in \mathbb{Z}^N} |G(\boldsymbol{n})|^p \right)^{\frac{1}{p}} \in \mathbb{R}_{\geq 0} \cup \{+\infty\},$$

and we say that $G \in \mathrm{L}^p(\mathbb{Z}^N)$ if this norm is finite. Also, G is *Haar-integrable* if it lies in $\mathrm{L}^1(\mathbb{Z}^N)$.

Let $H : \mathbb{T}^N \to \mathbb{C}$ be Haar-integrable. Its Fourier transform is the function $\widehat{H} : \mathbb{Z}^N \to \mathbb{C}$, defined as

$$\widehat{H}(\boldsymbol{n}) = \int_{\mathbb{T}^N} H(\boldsymbol{\theta}) e^{-2\pi i \boldsymbol{n} \cdot \boldsymbol{\theta}} \, d\boldsymbol{\theta},$$

where

$$\boldsymbol{n} \cdot \boldsymbol{\theta} = (n_1, \dots, n_N) \cdot (\theta_1, \dots, \theta_N) = n_1\theta_1 + \cdots + n_N\theta_N.$$

If $\widehat{H}$ is also Haar-integrable, the *Fourier inversion formula* states that

$$H(\boldsymbol{\theta}) = \sum_{\boldsymbol{n} \in \mathbb{Z}^N} \widehat{H}(\boldsymbol{n}) e^{2\pi i \boldsymbol{n} \cdot \boldsymbol{\theta}}.$$

For $H \in (\mathrm{L}^1 \cap \mathrm{L}^2)(\mathbb{T}^N)$, *Plancherel's theorem* states that $\widehat{H} \in \mathrm{L}^2(\mathbb{Z}^N)$ and

$$\|\widehat{H}\|_{\mathrm{L}^2} = \|H\|_{\mathrm{L}^2}.$$

For every finite and regular positive measure λ on $\mathbb{T}^N$, its *Fourier–Stieltjes transform* is the function $\widehat{\lambda} : \mathbb{Z}^N \to \mathbb{C}$ given by

$$\widehat{\lambda}(\boldsymbol{n}) = \int_{\mathbb{T}^N} e^{-2\pi i \boldsymbol{n}\cdot\boldsymbol{\theta}}\, d\lambda(\boldsymbol{\theta}).$$

We now establish some auxiliary results that will be useful for the proof of Theorem 1.2.

Lemma 2.1. *Let $H : \mathbb{T}^N \to \mathbb{C}$ be a Haar-integrable function such that its Fourier transform $\widehat{H}$ is also Haar-integrable. For any finite regular measure λ on $\mathbb{T}^N$ we have that H is integrable with respect to λ and $\widehat{H}\widehat{\lambda}$ is Haar-integrable. Moreover,*

$$\int_{\mathbb{T}^N} H\, d\lambda = \sum_{\boldsymbol{n}\in\mathbb{Z}^N} \widehat{H}(\boldsymbol{n})\overline{\widehat{\lambda}(\boldsymbol{n})}.$$

Proof. Let λ be a finite regular measure on $\mathbb{T}^N$. Its Fourier–Stieltjes transform is the function $\widehat{\lambda} : \mathbb{Z}^N \to \mathbb{C}$ given by

$$\widehat{\lambda}(\boldsymbol{n}) = \int_{\mathbb{T}^N} e^{-2\pi i \boldsymbol{n}\cdot\boldsymbol{\theta}}\, d\lambda(\boldsymbol{\theta}).$$

Since both H and $\widehat{H}$ are Haar-integrable, we apply the Fourier inversion formula that, together with Fubini's theorem, leads to

$$\begin{aligned}\int_{\mathbb{T}^N} H d\lambda &= \int_{\mathbb{T}^N} \Big(\sum_{\boldsymbol{n}\in\mathbb{Z}^N} \widehat{H}(\boldsymbol{n}) e^{2\pi i \boldsymbol{n}\cdot\boldsymbol{\theta}} \Big) d\lambda(\boldsymbol{\theta}) \\ &= \sum_{\boldsymbol{n}\in\mathbb{Z}^N} \widehat{H}(\boldsymbol{n}) \Big(\int_{\mathbb{T}^N} e^{2\pi i \boldsymbol{n}\cdot\boldsymbol{\theta}}\, d\lambda(\boldsymbol{\theta}) \Big) \\ &= \sum_{\boldsymbol{n}\in\mathbb{Z}^N} \widehat{H}(\boldsymbol{n})\overline{\widehat{\lambda}(\boldsymbol{n})},\end{aligned}$$

this equality containing the fact that H is integrable with respect to λ and that $\widehat{H}\widehat{\lambda}$ is Haar-integrable. ☐

Lemma 2.2. *Let $H : \mathbb{T}^N \to \mathbb{C}$ be a Haar-integrable function such that $\widehat{H}$ is also Haar-integrable, and let λ be a finite regular measure on $\mathbb{T}^N$. Then*

$$\int_{\mathbb{T}^N} H\, d\lambda - \int_{\mathbb{T}^N} H\, d\lambda_{(S^1)^N} = \widehat{H}(\boldsymbol{0})\big(\overline{\widehat{\lambda}(\boldsymbol{0})} - 1\big) + \sum_{\boldsymbol{n}\neq\boldsymbol{0}} \widehat{H}(\boldsymbol{n})\overline{\widehat{\lambda}(\boldsymbol{n})}.$$

Proof. Since $\lambda_{(S^1)^N}$ is the Haar probability measure of $\mathbb{T}^N$, for any $\boldsymbol{n} \in \mathbb{Z}^N$,

$$\widehat{\lambda}_{(S^1)^N}(\boldsymbol{n}) = \int_{\mathbb{T}^N} e^{-2\pi i \boldsymbol{n}\cdot\boldsymbol{\theta}}\, d\boldsymbol{\theta} = \begin{cases} 1 & \text{if } \boldsymbol{n} = \boldsymbol{0}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, by Lemma 2.1 we obtain

$$\int_{(\mathbb{C}^\times)^N} H\, d\lambda_{(S^1)^N} = \sum_{\boldsymbol{n}\in\mathbb{Z}^N} \widehat{H}(\boldsymbol{n})\overline{\widehat{\lambda}_{(S^1)^N}(\boldsymbol{n})} = \widehat{H}(\boldsymbol{0}).$$

Then we have

$$\int_{\mathbb{T}^N} H d\lambda - \int_{\mathbb{T}^N} H d\lambda_{(S^1)^N} = \Big(\sum_{\boldsymbol{n}\in\mathbb{Z}^N} \widehat{H}(\boldsymbol{n})\overline{\widehat{\lambda}(\boldsymbol{n})}\Big) - \widehat{H}(\boldsymbol{0}) = \widehat{H}(\boldsymbol{0})\big(\overline{\widehat{\lambda}(\boldsymbol{0})} - 1\big) + \sum_{\boldsymbol{n}\neq\boldsymbol{0}} \widehat{H}(\boldsymbol{n})\overline{\widehat{\lambda}(\boldsymbol{n})}. \quad \square$$

2.2. ***Galois invariant sets.*** In this section we work with Galois invariant sets and study their height. For further details on basic Galois theory we refer to [Lang 2002], and on heights of points to [Bombieri and Gubler 2006].

Let $\xi \in \overline{\mathbb{Q}}^\times$ and $f_\xi \in \mathbb{Z}[x]$ be the minimal polynomial of ξ over the integers. Recall that the Weil height of ξ is defined as

$$\mathrm{h}(\xi) = \frac{m(f_\xi)}{\deg(\xi)},$$

where $m(f_\xi)$ is the Mahler measure of f_ξ, given by

$$m(f_\xi) = \frac{1}{2\pi}\int_0^{2\pi} \log|f_\xi(e^{i\theta})|\, d\theta,$$

and $\deg(\xi) = [\mathbb{Q}(\xi) : \mathbb{Q}]$ is the degree of the point ξ. This notion of height coincides with that in [Bombieri and Gubler 2006, §1.5], which is defined using local decompositions.

Let $T \subset (\overline{\mathbb{Q}}^\times)^N$ be a finite Galois-invariant set, its height is defined as

$$\mathrm{h}(T) = \sum_{\boldsymbol{\alpha}\in T} \mathrm{h}(\boldsymbol{\alpha}),$$

where $\mathrm{h}(\alpha)$ is the height of $\alpha \in (\overline{\mathbb{Q}}^\times)^N$ as in (1-1). In particular, since the height of two Galois conjugate points coincide, if $T \subset (\overline{\mathbb{Q}}^\times)^N$ is a Galois orbit of cardinality D, then

$$\mathrm{h}(T) = D\,\mathrm{h}(\boldsymbol{\alpha}),$$

for any $\boldsymbol{\alpha} \in T$.

Lemma 2.3. *Let $\boldsymbol{\xi} = (\xi_1, \dots, \xi_N)$ in $(\overline{\mathbb{Q}}^\times)^N$, S its Galois orbit, and set $D = \#S$. Then*

(1) $D = [\mathbb{Q}(\xi_1, \dots, \xi_N) : \mathbb{Q}]$,

(2) $\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))$ *divides D for every $\boldsymbol{n} \in \mathbb{Z}^N$.*

Proof. If a group G acts on a finite set S transitively, then for any $x \in S$ the index of the stabilizer G_x is equal to $\#S$, because the cosets G/G_x stay in a natural one-to-one correspondence with the element of S. Applying this to $G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, $x = \boldsymbol{\xi}$, and S the Galois orbit of $\boldsymbol{\xi}$, we find $[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\boldsymbol{\xi}))] = D$, which by Galois theory implies that $[\mathbb{Q}(\boldsymbol{\xi}) : \mathbb{Q}] = D$, proving the first statement. The second statement is immediate because $\chi^{\boldsymbol{n}}(\boldsymbol{\xi}) \in \mathbb{Q}(\boldsymbol{\xi})$. □

Lemma 2.4. *Let* $\xi \in \overline{\mathbb{Q}}^{\times}$, $d = \deg(\xi)$, *and S its Galois orbit. Then*

$$\frac{1}{d}\sum_{\alpha\in S} \big|\log|\alpha|\big| \leq 2\,\mathrm{h}(\xi).$$

Proof. We have

$$\begin{aligned}\frac{1}{d}\sum_{\alpha\in S} \big|\log|\alpha|\big| &= \frac{1}{d}\sum_{\alpha\in S} \max\{-\log|\alpha|, \log|\alpha|\}\\ &= \frac{1}{d}\sum_{\alpha\in S} \log\max\Big\{\frac{1}{|\alpha|}, |\alpha|\Big\}\\ &= \frac{1}{d}\sum_{\alpha\in S} \big(\log\max\{1, |\alpha|^2\} - \log|\alpha|\big).\end{aligned}$$

Let $P_\xi(x) = a_d x^d + \cdots + a_0 \in \mathbb{Z}[x]$ be the minimal polynomial of ξ over $\mathbb{Z}$. Since S is the Galois orbit of ξ,

$$P_\xi(x) = a_d \prod_{\alpha\in S}(x-\alpha) \quad \text{and} \quad a_0 = (-1)^d a_d \prod_{\alpha\in S}\alpha.$$

Since $|a_0|$ is a nonzero positive integer, we obtain

$$\begin{aligned}\frac{1}{d}\sum_{\alpha\in S}(\log\max\{1,|\alpha|^2\} - \log|\alpha|) &= \frac{1}{d}\sum_{\alpha\in S}\log\max\{1,|\alpha|^2\} + \log\frac{|a_d|}{|a_0|}\\ &\leq \frac{1}{d}\sum_{\alpha\in S}\log\max\{1,|\alpha|^2\} + \log|a_d|\\ &\leq 2\Big(\frac{1}{d}\sum_{\alpha\in S}\log\max\{1,|\alpha|\} + \log|a_d|\Big)\\ &= 2\,\mathrm{h}(\xi),\end{aligned}$$

where the last equality is given by Jensen's formula for the Mahler measure [Bombieri and Gubler 2006, Proposition 1.6.5]. □

Lemma 2.5. *Let $\boldsymbol{\xi}_1 \in (\overline{\mathbb{Q}}^{\times})^N$ and consider its Galois orbit $\{\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_D\}$, where $\boldsymbol{\xi}_j = (\xi_{j,1}, \dots, \xi_{j,N})$ for every $j = 1, \dots, D$. Then*

$$\frac{1}{D}\sum_{l=1}^{N}\sum_{j=1}^{D}\bigl|\log|\xi_{j,l}|\bigr| \le 2\,\mathrm{h}(\boldsymbol{\xi}_1).$$

Proof. For every $l = 1, \dots, N$, the elements $\xi_{j,l}$ and $\xi_{k,l}$ are conjugates. Denote by S_l the Galois orbit of $\xi_{1,l}$. By Lemma 2.3, we have that $\#S_l = \deg(\xi_{1,l})$ divides D. That is, there is a positive integer k_l such that $D = \deg(\xi_{1,l})k_l$, where k_l is exactly the number of times each element of the orbit is repeated in $\{\xi_{1,l}, \dots, \xi_{D,l}\}$. Thus

$$\begin{aligned}\frac{1}{D}\sum_{l=1}^{N}\sum_{j=1}^{D}\bigl|\log|\xi_{j,l}|\bigr| &= \sum_{l=1}^{N}\frac{1}{k_l\deg(\xi_{1,l})}\sum_{j=1}^{D}\bigl|\log|\xi_{j,l}|\bigr| \\ &= \sum_{l=1}^{N}\frac{1}{\deg(\xi_{1,l})}\sum_{\alpha\in S_l}\bigl|\log|\alpha|\bigr| \\ &\le \sum_{l=1}^{N} 2\,\mathrm{h}(\xi_{1,l}) = 2\,\mathrm{h}(\boldsymbol{\xi}_1),\end{aligned}$$

where the inequality follows from Lemma 2.4. □

Lemma 2.6. *Let $S \subset \overline{\mathbb{Q}}^{\times}$ be a Galois-invariant set of cardinality D. For every $0 < \delta < 1$,*

$$\#S_\delta < 2\Bigl(\log\frac{1}{\delta}\Bigr)^{-1}\mathrm{h}(S),$$

where $S_\delta = \Bigl\{\alpha \in S : \bigl|\log|\alpha|\bigr| > \log\frac{1}{\delta}\Bigr\}$.

Proof. Write S as a finite disjoint union of Galois orbits

$$S = S_1 \sqcup \cdots \sqcup S_m.$$

By definition, for any $\alpha \in S_\delta$,

$$1 < \Bigl(\log\frac{1}{\delta}\Bigr)^{-1}\bigl|\log|\alpha|\bigr|.$$

Hence,

$$\begin{aligned}\#S_\delta &< \sum_{\alpha\in S_\delta}\Bigl(\log\frac{1}{\delta}\Bigr)^{-1}\bigl|\log|\alpha|\bigr| \le \Bigl(\log\frac{1}{\delta}\Bigr)^{-1}\sum_{\alpha\in S}\bigl|\log|\alpha|\bigr| \\ &= \Bigl(\log\frac{1}{\delta}\Bigr)^{-1}\sum_{l=1}^{m}\sum_{\alpha\in S_l}\bigl|\log|\alpha|\bigr| \le \Bigl(\log\frac{1}{\delta}\Bigr)^{-1}\sum_{l=1}^{m}2\,\mathrm{h}(S_l) \\ &= 2\Bigl(\log\frac{1}{\delta}\Bigr)^{-1}\mathrm{h}(S),\end{aligned}$$

where the last inequality holds by Lemma 2.4. □

2.3. *The generalized degree.* We now study the notion of the generalized degree of a point in the algebraic torus defined in (1-2). First of all, let us see that in dimension one, it coincides with the notion of the degree of the algebraic number. Let $\xi \in \overline{\mathbb{Q}}^{\times}$; then

$$\mathscr{D}(\xi) = \min_{n \neq 0}\{|n| \deg(\xi^n)\}.$$

For every nonzero integer n, let $Q_n(x)$ be the minimal polynomial of $\xi^{|n|}$ over $\mathbb{Z}$, which is of degree $\deg(\xi^{|n|}) = \deg(\xi^n)$. By setting $R_n(x) = Q_n(x^{|n|}) \in \mathbb{Z}[x]$ we obtain that $R_n(\xi) = 0$ and this implies that

$$\deg(\xi) \leq \deg(R_n(x)) = |n| \deg(\xi^n).$$

Hence, $\mathscr{D}(\xi) = \deg(\xi)$.

Remark 2.7. For $N \geq 1$ and every $\boldsymbol{\xi} = (\xi_1, \dots, \xi_N)$ in $(\overline{\mathbb{Q}}^{\times})^N$,

$$\mathscr{D}(\boldsymbol{\xi}) \leq \min\{\deg(\xi_1), \dots, \deg(\xi_N)\}.$$

This holds since

$$\{\deg(\xi_1), \dots, \deg(\xi_N)\} \subset \{\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) : \boldsymbol{n} \neq \boldsymbol{0}\}.$$

Thus, for a particular choice of $\boldsymbol{\xi}$, the generalized degree can be computed after a finite number of steps by considering all $\boldsymbol{n} \neq \boldsymbol{0}$ such that

$$\|\boldsymbol{n}\|_1 \leq \min\{\deg(\xi_1), \dots, \deg(\xi_N)\}.$$

For $N = 1$, a strict sequence $(\xi_k)_{k \geq 1}$ in $\overline{\mathbb{Q}}^{\times}$ such that $\lim_{k\to\infty} \mathrm{h}(\xi_k) = 0$ shows that $\lim_{k\to\infty} \deg(\xi_k) = \infty$. Indeed, to the contrary suppose there is some $c > 0$ such that $\deg(\xi_k) \leq c$ for every $k \geq 0$. By Northcott's theorem [Bombieri and Gubler 2006, Theorem 1.6.8], there are only finitely many elements with bounded degree and height. Hence, there is some $\alpha \in \overline{\mathbb{Q}}^{\times}$ such that $\xi_k = \alpha$ for infinitely many values of k. Since $\mathrm{h}(\xi_k)$ tends to 0 as k goes to infinity, by Kronecker's theorem [op. cit., Theorem 1.5.9] we necessarily have $\mathrm{h}(\alpha) = 0$, which implies that α is a root of unity. In particular, there is an infinite subsequence of $(\xi_k)_{k \geq 1}$ contained in a proper algebraic subgroup of $\overline{\mathbb{Q}}^{\times}$ which is not possible by the assumption that the sequence is strict.

The following lemma is a generalization of this fact to higher dimensions.

Lemma 2.8. *Let $(\boldsymbol{\xi}_k)_{k \geq 1}$ be a strict sequence in $(\overline{\mathbb{Q}}^{\times})^N$ such that*

$$\lim_{k\to\infty} \mathrm{h}(\boldsymbol{\xi}_k) = 0.$$

Then

$$\lim_{k\to\infty} \mathscr{D}(\boldsymbol{\xi}_k) = \infty.$$

Proof. Since the sequence $(\boldsymbol{\xi}_k)_{k\geq 0}$ is strict, the sequence $(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}_k))_{k\geq 0}$ is a strict sequence in $\overline{\mathbb{Q}}^{\times}$ for every $\boldsymbol{n}\neq \boldsymbol{0}$.

Write $\boldsymbol{\xi}_k = (\xi_{k,1},\dots,\xi_{k,N})$ and let $\boldsymbol{n}=(n_1,\dots,n_N)\neq\boldsymbol{0}$. Then

$$\begin{aligned}\mathrm{h}(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}_k)) &= \mathrm{h}(\xi_{k,1}^{n_1}\cdots\xi_{k,N}^{n_N})\\ &\leq \mathrm{h}(\xi_{k,1}^{n_1})+\ldots+\mathrm{h}(\xi_{k,N}^{n_N})\\ &= |n_1|\,\mathrm{h}(\xi_{k,1})+\cdots+|n_N|\,\mathrm{h}(\xi_{k,N})\\ &\leq \|\boldsymbol{n}\|_1\,\mathrm{h}(\boldsymbol{\xi}_k)\xrightarrow{k\to\infty} 0,\end{aligned}$$

where the first inequality follows from [Bombieri and Gubler 2006, §1.5.14].

Thus, as we just saw, for every $\boldsymbol{n}\neq\boldsymbol{0}$

$$\lim_{k\to\infty}\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}_k))=\infty.$$

Finally, by Remark 2.7, for every $k\geq 0$ there is $\boldsymbol{n}_k\neq\boldsymbol{0}$ with bounded 1-norm such that $\mathscr{D}(\boldsymbol{\xi}_k)=\|\boldsymbol{n}_k\|_1\deg(\chi^{\boldsymbol{n}_k}(\boldsymbol{\xi}_k))$. Hence

$$\lim_{k\to\infty}\mathscr{D}(\boldsymbol{\xi}_k)=\infty,$$

completing the proof. □

3. Proof of the main result

In this section we give the proof of Theorem 1.2. As we mentioned in the introduction, we do so by using Fourier analysis techniques and reducing the problem, via projections, to the one-dimensional case, where the result follows from [Favre and Rivera-Letelier 2006, Corollary 1.4].

Before stating this result, we give the definition of the spherical distance on the Riemann sphere. Let us identify the projective complex line with the unit sphere S^2 of $\mathbb{R}^3$. Let $S^2\setminus\{(0,0,1)\}\to\mathbb{C}$ be the stereographic projection, where we identify the equator of S^2 with the set $\{z\in\mathbb{C}:|z|=1\}$. Composing it with the standard inclusion $\mathbb{C}\hookrightarrow\mathbb{P}^1(\mathbb{C})$ gives a map $S^2\setminus\{(0,0,1)\}\to\mathbb{P}^1(\mathbb{C})\setminus\{(0:1)\}$, that we extend to a homeomorphism $\rho:S^2\to\mathbb{P}^1(\mathbb{C})$ by setting $\rho(0,0,1)=(0:1)$. The spherical distance $\mathrm{d}_{\mathrm{sph}}$ on $\mathbb{P}^1(\mathbb{C})$ is given by the length of the arc on S^2 under this identification and extended to $\mathbb{P}^1(\mathbb{C})^N$ for $\boldsymbol{p}=(p_1,\dots,p_N)$ and $\boldsymbol{p}'=(p_1',\dots,p_N')$ as

$$\mathrm{d}_{\mathrm{sph}}(\boldsymbol{p},\boldsymbol{p}')=\left(\sum_{l=1}^{N}\mathrm{d}_{\mathrm{sph}}(p_l,p_l')^2\right)^{\frac{1}{2}}.$$

A function $f:\mathbb{P}^1(\mathbb{C})^N\to\mathbb{C}$ is a *Lipschitz function* with respect to the distance

d_{sph} if there is a constant $K \geq 0$ such that

$$|f(\boldsymbol{p}) - f(\boldsymbol{p}')| \leq K\, d_{sph}(\boldsymbol{p}, \boldsymbol{p}') \quad \text{for every } \boldsymbol{p}, \boldsymbol{p}' \in \mathbb{P}^1(\mathbb{C})^N. \tag{3-1}$$

If f is a Lipschitz function with respect to the spherical distance, then its *Lipschitz constant* $\mathrm{Lip}_{sph}(f)$ is the smallest $K \geq 0$ such that (3-1) holds.

We now state the result of Favre and Rivera-Letelier together with the explicit constants computed in the Ph.D. thesis of Narváez-Clauss [2016, Theorem II].

Theorem 3.1. *There is a positive constant $C_0 \leq 15$ such that for every $\mathscr{C}^1$-function $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{R}$ and every $\xi \in \overline{\mathbb{Q}}^\times$*

$$\left|\int_{\mathbb{P}^1(\mathbb{C})} f\, d\mu_S - \int_{\mathbb{P}^1(\mathbb{C})} f\, d\lambda_{S^1}\right| \\ \leq \mathrm{Lip}_{sph}(f)\left(\frac{\pi}{\deg(\xi)} + \left(4\,\mathrm{h}(\xi) + C_0\frac{\log(\deg(\xi)+1)}{\deg(\xi)}\right)^{\frac{1}{2}}\right),$$

where S is the Galois orbit of ξ, μ_S is the discrete probability measure associated to it, and Lip_{sph} stands for the Lipschitz constant with respect to the spherical distance on the Riemann sphere.

In particular, if $\mathrm{h}(\xi) \leq 1$, then

$$\left|\int_{\mathbb{P}^1(\mathbb{C})} f\, d\mu_S - \int_{\mathbb{P}^1(\mathbb{C})} f\, d\lambda_{S^1}\right| \leq \mathrm{Lip}_{sph}(f)\left(4\,\mathrm{h}(\xi) + C\frac{\log(\deg(\xi)+1)}{\deg(\xi)}\right)^{\frac{1}{2}},$$

for $C \leq 64$.

The proof of this result relies on the interpretation of the height of a point in terms of the potential theory over the complex projective line. Given $\xi \in \overline{\mathbb{Q}}^\times$, it can be shown that the mutual energy of the signed measure $\mu_S - \lambda_{S^1}$ is bounded above by twice the height of the point. Since this signed measure is not regular enough, Favre and Rivera-Letelier consider a regularization such that it has vanishing total mass and its trace measure has continuous potential. This allows them to apply a Cauchy–Schwartz type inequality to the integral of the function with respect to the regularized measure. Together with the study of the integral of the function with respect to the difference of the measure and its regularization, this leads to their result. The constant in [Narváez-Clauss 2016] is made explicit by considering a specific regularization of the measure, which is done by convolution with a specific *mollifier*.

Consider the projection

$$\pi : \mathbb{T}^N \times \mathbb{R}^N \to \mathbb{T}^N, \\ (\boldsymbol{\theta}, \boldsymbol{u}) \mapsto \boldsymbol{\theta}.$$

Under the natural identifications

$$\begin{aligned}(\mathbb{C}^\times)^N &\to \mathbb{T}^N \times \mathbb{R}^N,\\ (z_1,\dots,z_N) &\mapsto \left(\left(\frac{\arg(z_1)}{2\pi},\dots,\frac{\arg(z_N)}{2\pi}\right), (\log|z_1|,\dots,\log|z_N|)\right)\end{aligned}$$

and

$$\begin{aligned}(S^1)^N &\to \mathbb{T}^N,\\ (z_1,\dots,z_N) &\mapsto \left(\frac{\arg(z_1)}{2\pi},\dots,\frac{\arg(z_N)}{2\pi}\right),\end{aligned}$$

the map π can be rewritten as

$$\begin{aligned}(\mathbb{C}^\times)^N &\to (S^1)^N,\\ (z_1,\dots,z_N) &\mapsto \left(\frac{z_1}{|z_1|},\dots,\frac{z_N}{|z_N|}\right).\end{aligned}$$

Let $\boldsymbol{\xi} \in (\mathbb{Q}^\times)^N$, S its Galois orbit, and μ_S the discrete probability measure associated to it. If $F : \mathbb{T}^N \times \mathbb{R}^N \to \mathbb{C}$ is integrable with respect to the measure $\lambda_{(S^1)^N}$, then

$$\begin{aligned}&\left|\int_{\mathbb{T}^N\times\mathbb{R}^N} F\,d\mu_S - \int_{\mathbb{T}^N\times\mathbb{R}^N} F\,d\lambda_{(S^1)^N}\right| \\ &\quad\le \left|\int_{\mathbb{T}^N\times\mathbb{R}^N} F\,d\mu_S - \int_{\mathbb{T}^N} F_0\,d\nu_S\right| + \left|\int_{\mathbb{T}^N} F_0\,d\nu_S - \int_{\mathbb{T}^N\times\mathbb{R}^N} F\,d\lambda_{(S^1)^N}\right|, \quad (3\text{-}2)\end{aligned}$$

where $F_0 : \mathbb{T}^N \to \mathbb{R}$ is defined by $F_0(\boldsymbol{\theta}) = F(\boldsymbol{\theta},\mathbf{0})$, and the measure ν_S is the pushforward of the measure μ_S, which is given by

$$\nu_S = \pi_*\mu_S = \frac{1}{\#S}\sum_{\boldsymbol{\alpha}\in S} \delta_{\boldsymbol{\alpha}/|\boldsymbol{\alpha}|}. \quad (3\text{-}3)$$

Using (3-2), we are able to divide the proof of the main result into two parts. The following proposition corresponds to the first one.

Proposition 3.2. *Let $\boldsymbol{\xi} \in (\overline{\mathbb{Q}}^\times)^N$ and S its Galois orbit. Let $F : (\mathbb{C}^\times)^N \to \mathbb{R}$ be a Lipschitz function with respect to the distance* d *and such that it is integrable with respect to $\lambda_{(S^1)^N}$. Then*

$$\left|\int_{\mathbb{T}^N\times\mathbb{R}^N} F\,d\mu_S - \int_{\mathbb{T}^N} F_0\,d\nu_S\right| \le 2\,\mathrm{Lip}(F)\,\mathrm{h}(\boldsymbol{\xi}),$$

where $F_0(\boldsymbol{\theta}) = F(\boldsymbol{\theta},\mathbf{0})$ and $\mathrm{Lip}(F)$ *is the Lipschitz constant of F.*

Proof. With the above notation, we have

$$\begin{aligned}\left|\int_{\mathbb{T}^N\times\mathbb{R}^N} F\,d\mu_S - \int_{\mathbb{T}^N} F_0\,d\nu_S\right| \\ &= \left|\int_{\mathbb{T}^N\times\mathbb{R}^N} F\,d\mu_S - \int_{\mathbb{T}^N\times\mathbb{R}^N}(F\circ\pi)\,d\mu_S\right| \\ &= \left|\int_{(\mathbb{C}^\times)^N}\left(F(z_1,\dots,z_N) - F\left(\frac{z_1}{|z_1|},\dots,\frac{z_N}{|z_N|}\right)\right) d\mu_S(z_1,\dots,z_N)\right| \\ &\le \frac{1}{\#S}\sum_{(\alpha_1,\dots,\alpha_N)\in S}\left|F(\alpha_1,\dots,\alpha_N) - F\left(\frac{\alpha_1}{|\alpha_1|},\dots,\frac{\alpha_N}{|\alpha_N|}\right)\right| \\ &\le \frac{1}{\#S}\,\mathrm{Lip}(F)\sum_{(\alpha_1,\dots,\alpha_N)\in S}\mathrm{d}\left((\alpha_1,\dots,\alpha_N),\left(\frac{\alpha_1}{|\alpha_1|},\dots,\frac{\alpha_N}{|\alpha_N|}\right)\right),\end{aligned}$$

where the last inequality is given by the fact that F is a Lipschitz function with respect to the distance d of $(\mathbb{C}^\times)^N$. By the definition of this distance,

$$\mathrm{d}\left((\alpha_1,\dots,\alpha_N),\left(\frac{\alpha_1}{|\alpha_1|},\dots,\frac{\alpha_N}{|\alpha_N|}\right)\right) = \left(\sum_{l=1}^N \big|\log|\alpha_l|\big|^2\right)^{\frac{1}{2}} \le \sum_{l=1}^N\big|\log|\alpha_l|\big|.$$

Hence, by Lemma 2.5, we conclude

$$\begin{aligned}\left|\int_{\mathbb{T}^N\times\mathbb{R}^N} F\,d\mu_S - \int_{\mathbb{T}^N} F_0\,d\nu_S\right| &\le \frac{1}{\#S}\,\mathrm{Lip}(F)\sum_{(\alpha_1,\dots,\alpha_N)\in S}\sum_{l=1}^N\big|\log|\alpha_l|\big| \\ &\le 2\,\mathrm{Lip}(F)\,\mathrm{h}(\boldsymbol{\xi}).\end{aligned}$$ □

Let us study now the second summand in (3-2). First of all we observe that, since the measure $\lambda_{(S^1)^N}$ is supported on $\mathbb{T}^N\times\{\mathbf{0}\}$, we can reduce the problem to the compact torus $(S^1)^N$. Indeed, with the notation as in (3-2), we have

$$\left|\int_{\mathbb{T}^N} F_0\,d\nu_S - \int_{\mathbb{T}^N\times\mathbb{R}^N} F\,d\lambda_{(S^1)^N}\right| = \left|\int_{\mathbb{T}^N} F_0\,d\nu_S - \int_{\mathbb{T}^N} F_0\,d\lambda_{(S^1)^N}\right|,$$

where ν_S is given by (3-3).

If $F_0:\mathbb{T}^N\to\mathbb{R}$ is Haar-integrable and such that its Fourier transform $\widehat{F_0}$ is also Haar-integrable, by Lemma 2.2,

$$\int_{\mathbb{T}^N} F_0\,d\nu_S - \int_{\mathbb{T}^N} F_0\,d\lambda_{(S^1)^N} = \widehat{F_0}(\mathbf{0})\big(\overline{\widehat{\nu}_S(\mathbf{0})} - 1\big) + \sum_{\boldsymbol{n}\neq\mathbf{0}}\widehat{F_0}(\boldsymbol{n})\overline{\widehat{\nu}_S(\boldsymbol{n})},$$

where the Fourier–Stieltjes transform of ν_S is

$$\widehat{\nu}_S(\boldsymbol{n}) = \int_{\mathbb{T}^N} e^{-2\pi i \boldsymbol{n}\cdot\boldsymbol{\theta}}\, d\nu_S(\boldsymbol{\theta}) = \frac{1}{\# S} \sum_{(\alpha_1,\dots,\alpha_N)\in S} e^{-i\boldsymbol{n}\cdot(\arg(\alpha_1),\dots,\arg(\alpha_N))}, \tag{3-4}$$

for every $\boldsymbol{n} \in \mathbb{Z}^N$. In particular, $\widehat{\nu}_S(\boldsymbol{0}) = 1$.

We obtain the following lemma.

Lemma 3.3. *Let $F_0 : \mathbb{T}^N \to \mathbb{R}$ be Haar-integrable and such that its Fourier transform is also Haar-integrable. With the notation as above,*

$$\int_{\mathbb{T}^N} F_0\, d\nu_S - \int_{\mathbb{T}^N} F_0\, d\lambda_{(S^1)^N} = \sum_{\boldsymbol{n}\neq\boldsymbol{0}} \widehat{F_0}(\boldsymbol{n})\overline{\widehat{\nu}_S(\boldsymbol{n})}.$$

We now study the Fourier–Stieltjes transform of the measure $\nu_S = \pi_*\mu_S$.

Proposition 3.4. *There is a constant $C \le 64$ such that, for every $\boldsymbol{n} \neq \boldsymbol{0}$ and every $0 < \delta < 1$, if $\mathrm{h}(\boldsymbol{\xi}) \le 1$,*

$$|\widehat{\nu}_S(\boldsymbol{n})| \le \frac{-2}{\log\delta}\|\boldsymbol{n}\|_1\,\mathrm{h}(\boldsymbol{\xi}) + \frac{4\sqrt{2}(\delta^2+9)}{\delta^3}\|\boldsymbol{n}\|_1\left(4\,\mathrm{h}(\boldsymbol{\xi}) + C\,\frac{\log(\mathscr{D}(\boldsymbol{\xi})+1)}{\mathscr{D}(\boldsymbol{\xi})}\right)^{\frac{1}{2}}.$$

Proof. Let $\boldsymbol{n} \neq \boldsymbol{0}$ and let $S_{\boldsymbol{n}}$ be the Galois orbit of $\chi^{\boldsymbol{n}}(\boldsymbol{\xi})$. By Lemma 2.3, there is an integer $l_{\boldsymbol{n}}$ such that $\# S = l_{\boldsymbol{n}} \# S_{\boldsymbol{n}}$ and we know that every element $\alpha \in S_{\boldsymbol{n}}$ is repeated $l_{\boldsymbol{n}}$ times in $\{\chi^{\boldsymbol{n}}(\boldsymbol{\alpha}) : \boldsymbol{\alpha} \in S\}$. Hence, by (3-4), we obtain

$$\overline{\widehat{\nu}_S(\boldsymbol{n})} = \frac{1}{\# S} \sum_{(\alpha_1,\dots,\alpha_N)\in S} e^{i\boldsymbol{n}\cdot(\arg(\alpha_1),\dots,\arg(\alpha_N))} = \frac{1}{\# S}\sum_{\boldsymbol{\alpha}\in S}\frac{\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})}{|\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})|} = \frac{1}{\# S_{\boldsymbol{n}}}\sum_{\alpha\in S_{\boldsymbol{n}}}\frac{\alpha}{|\alpha|}.$$

For $0 < \delta < 1$, consider the function $f_\delta : \mathbb{P}^1(\mathbb{C}) \to \mathbb{C}$ given by

$$f_\delta(0:1) = 0 \quad \text{and} \quad f_\delta(1:z) = \rho_\delta(|z|)\frac{z}{|z|} \quad \text{for any } z \in \mathbb{C},$$

where the function $\rho_\delta : \mathbb{R} \to [0,1]$ is given by

$$\rho_\delta(r) = \begin{cases} 0 & \text{if } r < \delta/2, \\ (5\delta - 4r)(\delta - 2r)^2/\delta^3 & \text{if } \delta/2 \le r \le \delta, \\ 1 & \text{if } \delta < r < 1/\delta, \\ (-2+\delta r)^2(-1+2\delta r) & \text{if } 1/\delta \le r \le 2/\delta, \\ 0 & \text{if } r > 2/\delta. \end{cases}$$

In Lemma B.1, we prove that f_δ is a $\mathscr{C}^1$-function such that, if we write $f_\delta = u_\delta + i v_\delta$,

$$\mathrm{Lip}_{\mathrm{sph}}(u_\delta), \mathrm{Lip}_{\mathrm{sph}}(v_\delta) \le \frac{2\sqrt{2}(\delta^2+9)}{\delta^3},$$

where $\mathrm{Lip}_{\mathrm{sph}}$ stands for the Lipschitz constant with respect to the spherical distance on the Riemann sphere.

For every $\boldsymbol{n} \neq \boldsymbol{0}$,

$$
\begin{aligned}
\Bigl| \overline{\widehat{\nu}_S(\boldsymbol{n})} - \frac{1}{\#S} \sum_{\boldsymbol{\alpha}\in S}^{D} f_\delta(1:\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})) \Bigr|
&= \Bigl| \frac{1}{\#S} \sum_{\boldsymbol{\alpha}\in S} \frac{\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})}{|\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})|} - \frac{1}{\#S} \sum_{\boldsymbol{\alpha}\in S} \rho_\delta(|\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})|) \frac{\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})}{|\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})|} \Bigr| \\
&= \Bigl| \frac{1}{\#S} \sum_{\boldsymbol{\alpha}\in S} \frac{\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})}{|\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})|} \bigl(1-\rho_\delta(|\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})|)\bigr) \Bigr| \\
&\leq \frac{1}{\#S} \sum_{\boldsymbol{\alpha}\in S} \bigl|1-\rho_\delta(|\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})|)\bigr|. \qquad (3\text{-}5)
\end{aligned}
$$

Let us define, for every $\boldsymbol{n} \neq \boldsymbol{0}$ and $0<\delta<1$, the set

$$J_{\boldsymbol{n},\delta} = \Bigl\{ \boldsymbol{\alpha} \in S : \delta \leq |\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})| \leq \frac{1}{\delta} \Bigr\}.$$

If $\boldsymbol{\alpha} \in J_{\boldsymbol{n},\delta}$, then $\rho_\delta(|\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})|)=1$, and otherwise $0 \leq \rho_\delta(|\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})|) < 1$. Hence,

$$\frac{1}{\#S} \sum_{\boldsymbol{\alpha}\in S} \bigl|1-\rho_\delta(|\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})|)\bigr| = \frac{1}{\#S} \sum_{\boldsymbol{\alpha}\notin J_{\boldsymbol{n},\delta}} 1-\rho_\delta(|\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})|) \leq \frac{1}{\#S} \sum_{\boldsymbol{\alpha}\notin J_{\boldsymbol{n},\delta}} 1. \qquad (3\text{-}6)$$

Set

$$S_{\boldsymbol{n},\delta} = \Bigl\{ \alpha \in S_{\boldsymbol{n}} : \bigl|\log|\boldsymbol{\alpha}|\bigr| > \log \frac{1}{\delta} \Bigr\},$$

then we obtain

$$\frac{1}{\#S} \sum_{\boldsymbol{\alpha}\notin J_{\boldsymbol{n},\delta}} 1 = \frac{1}{\#S_{\boldsymbol{n}}} \sum_{\alpha\in S_{\boldsymbol{n},\delta}} 1 \leq 2\Bigl(\log\frac{1}{\delta}\Bigr)^{-1} \mathrm{h}(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})), \qquad (3\text{-}7)$$

where the last inequality is given by Lemma 2.6.

As we saw in the proof of Lemma 2.8, for $\boldsymbol{n} \neq \boldsymbol{0}$ we have

$$\mathrm{h}(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) \leq \|\boldsymbol{n}\|_1 \, \mathrm{h}(\boldsymbol{\xi}).$$

Thus, putting this together with (3-5), (3-6) and (3-7) we deduce that

$$\Bigl| \overline{\widehat{\nu}_S(\boldsymbol{n})} - \frac{1}{\#S} \sum_{\boldsymbol{\alpha}\in S} f_\delta(1:\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})) \Bigr| \leq 2\Bigl(\log\frac{1}{\delta}\Bigr)^{-1} \|\boldsymbol{n}\|_1 \, \mathrm{h}(\boldsymbol{\xi}). \qquad (3\text{-}8)$$

On the other hand,

$$\frac{1}{\#S} \sum_{\boldsymbol{\alpha}\in S} f_\delta(1:\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})) = \frac{1}{l_{\boldsymbol{n}}\#S_{\boldsymbol{n}}} \sum_{\alpha\in S_{\boldsymbol{n}}} l_{\boldsymbol{n}} f_\delta(1:\alpha) = \int_{\mathbb{P}^1(\mathbb{C})} f_\delta \, d\mu_{S_{\boldsymbol{n}}}, \qquad (3\text{-}9)$$

where $\mu_{S_{\boldsymbol{n}}}$ is the discrete probability measure on $\mathbb{P}^1(\mathbb{C})$ associated to the Galois orbit $S_{\boldsymbol{n}}$ of $\chi^{\boldsymbol{n}}(\boldsymbol{\xi})$.

Since λ_{S^1} is the measure on $\mathbb{P}^1(\mathbb{C})$ supported on the unit circle, where it coincides with the Haar probability measure and, by definition, $f_\delta(1:z)=z$ if $|z|=1$,

$$\int_{\mathbb{P}^1(\mathbb{C})} f_\delta \, d\lambda_{S^1} = \int_{\mathbb{C}^\times} z \, d\lambda_{S^1}(z) = 0.$$

By Theorem 3.1, we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{P}^1(\mathbb{C})} f_\delta \, d\mu_{S_{\boldsymbol{n}}}\right| \\
&\quad = \left|\int_{\mathbb{P}^1(\mathbb{C})} f_\delta \, d\mu_{S_{\boldsymbol{n}}} - \int_{\mathbb{P}^1(\mathbb{C})} f_\delta \, d\lambda_{S^1}\right| \\
&\quad \le \left|\int_{\mathbb{P}^1(\mathbb{C})} u_\delta \, d\mu_{S_{\boldsymbol{n}}} - \int_{\mathbb{P}^1(\mathbb{C})} u_\delta \, d\lambda_{S^1}\right| \\
&\qquad\quad + \left|\int_{\mathbb{P}^1(\mathbb{C})} v_\delta \, d\mu_{S_{\boldsymbol{n}}} - \int_{\mathbb{P}^1(\mathbb{C})} v_\delta \, d\lambda_{S^1}\right| \\
&\quad \le (\mathrm{Lip}_{\mathrm{sph}}(u_\delta) + \mathrm{Lip}_{\mathrm{sph}}(v_\delta)) \\
&\qquad\quad \times \left(\frac{\pi}{\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))} + \left(4\,\mathrm{h}(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) + C_0 \frac{\log(\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))+1)}{\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))}\right)^{\frac12}\right) \\
&\quad \le \frac{4\sqrt{2}(\delta^2+9)}{\delta^3} \\
&\qquad\quad \times \left(\frac{\pi}{\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))} + \left(4\,\mathrm{h}(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) + C_0 \frac{\log(\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))+1)}{\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))}\right)^{\frac12}\right),
\end{aligned} \tag{3-10}$$

where $C_0 \le 15$.

Since $\mathrm{h}(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) \le \|\boldsymbol{n}\|_1 \,\mathrm{h}(\boldsymbol{\xi})$,

$$\begin{aligned}
&\left(4\,\mathrm{h}(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) + C_0 \frac{\log(\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))+1)}{\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))}\right)^{\frac12} \\
&\qquad \le \left(4\|\boldsymbol{n}\|_1 \,\mathrm{h}(\boldsymbol{\xi}) + C_0 \frac{\|\boldsymbol{n}\|_1 \log(\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))+1)}{\|\boldsymbol{n}\|_1 \deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))}\right)^{\frac12} \\
&\qquad \le \sqrt{\|\boldsymbol{n}\|_1}\left(4\,\mathrm{h}(\boldsymbol{\xi}) + C_0 \frac{\log(\|\boldsymbol{n}\|_1 \deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))+1)}{\|\boldsymbol{n}\|_1 \deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))}\right)^{\frac12} \\
&\qquad \le \|\boldsymbol{n}\|_1\left(4\,\mathrm{h}(\boldsymbol{\xi}) + C_0 \frac{\log(\|\boldsymbol{n}\|_1 \deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))+1)}{\|\boldsymbol{n}\|_1 \deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))}\right)^{\frac12}.
\end{aligned}$$

Hence, this together with (3-9) and (3-10) gives

$$\begin{aligned}\left|\int_{\mathbb{P}^1(\mathbb{C})} f_\delta \, d\mu_{S_{\boldsymbol{n}}}\right| &\\ &\le \frac{4\sqrt{2}(\delta^2+9)}{\delta^3}\|\boldsymbol{n}\|_1\left(\frac{\pi}{\|\boldsymbol{n}\|_1 \deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))}\right.\\ &\qquad\left.+\left(4\,\mathrm{h}(\boldsymbol{\xi})+C_0\frac{\log(\|\boldsymbol{n}\|_1\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))+1)}{\|\boldsymbol{n}\|_1\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))}\right)^{\frac{1}{2}}\right)\\ &\le \frac{4\sqrt{2}(\delta^2+9)}{\delta^3}\|\boldsymbol{n}\|_1\left(4\,\mathrm{h}(\boldsymbol{\xi})+C\frac{\log(\|\boldsymbol{n}\|_1\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))+1)}{\|\boldsymbol{n}\|_1\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))}\right)^{\frac{1}{2}}, \qquad (3\text{-}11)\end{aligned}$$

with $C \le 64$.

The function $\log(x+1)/x$ is monotonically decreasing for $x \ge 1$. We deduce that, for every $\boldsymbol{n} \ne \boldsymbol{0}$,

$$\frac{\log(\|\boldsymbol{n}\|_1\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))+1)}{\|\boldsymbol{n}\|_1\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))} \le \frac{\log(\mathscr{D}(\boldsymbol{\xi})+1)}{\mathscr{D}(\boldsymbol{\xi})}.$$

Together with (3-11), this implies that, for every $\boldsymbol{n} \ne \boldsymbol{0}$,

$$\left|\int_{\mathbb{P}^1(\mathbb{C})} f_\delta \, d\mu_{S_{\boldsymbol{n}}}\right| \le \left(4\,\mathrm{h}(\boldsymbol{\xi})+C\frac{\log(\mathscr{D}(\boldsymbol{\xi})+1)}{\mathscr{D}(\boldsymbol{\xi})}\right)^{\frac{1}{2}}.$$

By using this inequality and (3-5) we deduce that

$$\begin{aligned}\left|\widehat{\nu_S}(\boldsymbol{n})\right| &\le \left|\widehat{\nu_S}(\boldsymbol{n})-\frac{1}{\#S}\sum_{\boldsymbol{\alpha}\in S} f_\delta(1:\chi^{\boldsymbol{n}}(\boldsymbol{\alpha}))\right|+\left|\frac{1}{\#S}\sum_{\boldsymbol{\alpha}\in S} f_\delta(1:\chi^{\boldsymbol{n}}(\boldsymbol{\alpha}))\right|\\ &\le \frac{-2}{\log\delta}\|\boldsymbol{n}\|_1\,\mathrm{h}(\boldsymbol{\xi})+\frac{4\sqrt{2}(\delta^2+9)}{\delta^3}\|\boldsymbol{n}\|_1\left(4\,\mathrm{h}(\boldsymbol{\xi})+C\frac{\log(\mathscr{D}(\boldsymbol{\xi})+1)}{\mathscr{D}(\boldsymbol{\xi})}\right)^{\frac{1}{2}},\end{aligned}$$

proving the proposition. □

The following proposition bounds the second summand in the inequality (3-2).

Proposition 3.5. *There is a constant $C \le 64$ such that, for every $\boldsymbol{\xi} \in (\overline{\mathbb{Q}}^\times)^N$ with $\mathrm{h}(\boldsymbol{\xi}) \le 1$, every $0 < \delta < 1$ and every $F \in \mathcal{F}$,*

$$\begin{aligned}\left|\int_{\mathbb{T}^N} F_0\, d\nu_S - \int_{\mathbb{T}^N\times\mathbb{R}^N} F\, d\lambda_{(S^1)^N}\right| &\\ &\le \frac{1}{2\pi}\left(\frac{-2}{\log\delta}+\frac{4\sqrt{2}(\delta^2+9)}{\delta^3}\right)\left(4\,\mathrm{h}(\boldsymbol{\xi})+C\frac{\log(\mathscr{D}(\boldsymbol{\xi})+1)}{\mathscr{D}(\boldsymbol{\xi})}\right)^{\frac{1}{2}}\sum_{l=1}^{N}\left\|\widehat{\frac{\partial F_0}{\partial\theta_l}}\right\|_{\mathrm{L}^1},\end{aligned}$$

where S is the Galois orbit of $\boldsymbol{\xi}$, μ_S the discrete probability measure associated to it, $\mathscr{D}(\boldsymbol{\xi})$ the generalized degree of $\boldsymbol{\xi}$, and $F_0(\boldsymbol{\theta}) = F(\boldsymbol{\theta},\boldsymbol{0})$.

Proof. In Appendix A we prove that, given $F \in \mathcal{F}$, the function F_0 is Haar-integrable as well as its Fourier transform $\widehat{F}_0$. Thus, by Lemma 3.3 and Proposition 3.4,

$$\begin{aligned}
&\left|\int_{\mathbb{T}^N} F_0\, d\nu_S - \int_{\mathbb{T}^N\times\mathbb{R}^N} F\, d\lambda_{(S^1)^N}\right| \\
&\quad = \left|\int_{\mathbb{T}^N} F_0\, d\nu_S - \int_{\mathbb{T}^N} F_0\, d\lambda_{(S^1)^N}\right| \\
&\quad = \left|\sum_{\boldsymbol{n}\neq \boldsymbol{0}} \widehat{F}_0(\boldsymbol{n})\overline{\widehat{\nu}_S(\boldsymbol{n})}\right| \le \sum_{\boldsymbol{n}\neq \boldsymbol{0}} |\widehat{F}_0(\boldsymbol{n})||\widehat{\nu}_S(\boldsymbol{n})| \\
&\quad \le \sum_{\boldsymbol{n}\neq \boldsymbol{0}} |\widehat{F}_0(\boldsymbol{n})|\Bigg(\frac{-2}{\log\delta}\|\boldsymbol{n}\|_1\, \mathrm{h}(\boldsymbol{\xi}) \\
&\qquad\qquad + \frac{4\sqrt{2}(\delta^2+9)}{\delta^3}\|\boldsymbol{n}\|_1\left(4\,\mathrm{h}(\boldsymbol{\xi}) + C\,\frac{\log(\mathscr{D}(\boldsymbol{\xi})+1)}{\mathscr{D}(\boldsymbol{\xi})}\right)^{\frac{1}{2}}\Bigg) \\
&\quad \le \left(\frac{-2}{\log\delta} + \frac{4\sqrt{2}(\delta^2+9)}{\delta^3}\right)\left(4\,\mathrm{h}(\boldsymbol{\xi}) + C\,\frac{\log(\mathscr{D}(\boldsymbol{\xi})+1)}{\mathscr{D}(\boldsymbol{\xi})}\right)^{\frac{1}{2}} \sum_{\boldsymbol{n}\neq \boldsymbol{0}} |\widehat{F}_0(\boldsymbol{n})|\|\boldsymbol{n}\|_1,
\end{aligned}$$

where the last inequality is given by the fact that $\mathrm{h}(\boldsymbol{\xi}) \le 1$.

By Lemma A.2, for every $l = 1, \dots, N$,

$$\widehat{\frac{\partial F_0}{\partial \theta_l}}(\boldsymbol{n}) = 2\pi i\, n_l\, \widehat{F}_0(\boldsymbol{n}).$$

Hence, we obtain

$$\begin{aligned}
\sum_{\boldsymbol{n}\neq \boldsymbol{0}} |\widehat{F}_0(\boldsymbol{n})|\|\boldsymbol{n}\|_1 &= \frac{1}{2\pi}\sum_{l=1}^{N}\sum_{\boldsymbol{n}\neq \boldsymbol{0}} |\widehat{F}_0(\boldsymbol{n})|\cdot|2\pi n_l| \\
&= \frac{1}{2\pi}\sum_{l=1}^{N}\sum_{\boldsymbol{n}\in\mathbb{Z}^N}\left|\widehat{\frac{\partial F_0}{\partial\theta_l}}(\boldsymbol{n})\right| \\
&= \frac{1}{2\pi}\sum_{l=1}^{N}\left\|\widehat{\frac{\partial F_0}{\partial\theta_l}}\right\|_{\mathrm{L}^1}.
\end{aligned}$$

Finally, we conclude:

$$\begin{aligned}
&\left|\int_{\mathbb{T}^N} F_0\, d\pi_*\mu_S - \int_{\mathbb{T}^N\times\mathbb{R}^N} F\, d\lambda_{(S^1)^N}\right| \\
&\quad \le \left(\frac{-2}{\log\delta} + \frac{4\sqrt{2}(\delta^2+9)}{\delta^3}\right)\left(4\,\mathrm{h}(\boldsymbol{\xi}) + C\,\frac{\log(\mathscr{D}(\boldsymbol{\xi})+1)}{\mathscr{D}(\boldsymbol{\xi})}\right)^{\frac{1}{2}} \frac{1}{2\pi}\sum_{l=1}^{N}\left\|\widehat{\frac{\partial F_0}{\partial\theta_l}}\right\|_{\mathrm{L}^1} \qquad \square
\end{aligned}$$

Proof of Theorem 1.2. Let $F \in \mathcal{F}$ and set $F_0(\boldsymbol{\theta}) = F(\boldsymbol{\theta}, \mathbf{0})$. By Theorem A.1, the function F_0 and its Fourier transform $\widehat{F_0}$ are Haar-integrable and thus, as shown in (3-2),

$$\left|\int_{\mathbb{T}^N\times\mathbb{R}^N} F\, d\mu_S - \int_{\mathbb{T}^N\times\mathbb{R}^N} F\, d\lambda_{(S^1)^N}\right| \\ \leq \left|\int_{\mathbb{T}^N\times\mathbb{R}^N} F\, d\mu_S - \int_{\mathbb{T}^N} F_0\, d\nu_S\right| + \left|\int_{\mathbb{T}^N} F_0\, d\nu_S - \int_{\mathbb{T}^N\times\mathbb{R}^N} F\, d\lambda_{(S^1)^N}\right|.$$

Since the function F is Lipschitz with respect to the distance d, by Propositions 3.2 and 3.5, there is a constant $C \leq 64$ such that

$$\left|\int_{\mathbb{T}^N\times\mathbb{R}^N} F\, d\mu_S - \int_{\mathbb{T}^N\times\mathbb{R}^N} F\, d\lambda_{(S^1)^N}\right| \\ \leq 2\,\mathrm{Lip}(F)\,\mathrm{h}(\boldsymbol{\xi}) \\ + \frac{1}{2\pi}\left(\frac{-2}{\log\delta} + \frac{4\sqrt{2}(\delta^2+9)}{\delta^3}\right)\left(4\,\mathrm{h}(\boldsymbol{\xi}) + C\,\frac{\log(\mathscr{D}(\boldsymbol{\xi})+1)}{\mathscr{D}(\boldsymbol{\xi})}\right)^{\frac{1}{2}} \sum_{l=1}^{N}\left\|\widehat{\frac{\partial F_0}{\partial\theta_l}}\right\|_{\mathrm{L}^1}.$$

By Theorem A.1, the Fourier transforms of the first order partial derivatives of F_0 are Haar-integrable and so this bound is finite.

We search numerically for the minimum of the function

$$\frac{-2}{\log\delta} + \frac{4\sqrt{2}(\delta^2+9)}{\delta^3},$$

for $0 < \delta < 1$, and we obtain the value 94.9591, attained at $\delta \approx 0.9071$. Hence, since $\mathrm{h}(\boldsymbol{\xi}) \leq 1$ and $94.9591/2\pi < 16$,

$$\left|\int_{\mathbb{T}^N\times\mathbb{R}^N} F\, d\mu_S - \int_{\mathbb{T}^N\times\mathbb{R}^N} F\, d\lambda_{(S^1)^N}\right| \\ \leq 2\,\mathrm{Lip}(F)\,\mathrm{h}(\boldsymbol{\xi}) + 16\left(4\,\mathrm{h}(\boldsymbol{\xi}) + C\,\frac{\log(\mathscr{D}(\boldsymbol{\xi})+1)}{\mathscr{D}(\boldsymbol{\xi})}\right)^{\frac{1}{2}} \sum_{l=1}^{N}\left\|\widehat{\frac{\partial F_0}{\partial\theta_l}}\right\|_{\mathrm{L}^1} \\ \leq \left(2\,\mathrm{Lip}(F) + 16\sum_{l=1}^{N}\left\|\widehat{\frac{\partial F_0}{\partial\theta_l}}\right\|_{\mathrm{L}^1}\right)\left(4\,\mathrm{h}(\boldsymbol{\xi}) + C\,\frac{\log(\mathscr{D}(\boldsymbol{\xi})+1)}{\mathscr{D}(\boldsymbol{\xi})}\right)^{\frac{1}{2}}.$$

□

Remark 3.6. The functions in $\mathcal{F}$ are functions with logarithmic singularities along toric divisors in a toric compactification of $(\overline{\mathbb{Q}}^\times)^N$. The qualitative equidistribution with respect to this set of test functions is given by [Chambert-Loir and Thuillier 2009, Théorème 1.2].

Appendix A: The set of test functions

In this appendix, we show that the test functions in $\mathcal{F}$, when restricted to the unit polycircle $(S^1)^N$, are Haar-integrable as well as their Fourier transforms. We also prove the Haar-integrability of all their first order partial derivatives and their corresponding Fourier transforms.

Recall the definition of the test functions. The set $\mathcal{F}$ is given by all real-valued functions F satisfying

(i) F is Lipschitz with respect to the distance d on $(\mathbb{C}^\times)^N$,

(ii) $F_0(\boldsymbol{\theta}) = F(\boldsymbol{\theta}, \mathbf{0})$ is in $\mathscr{C}^{N+1}(\mathbb{T}^N, \mathbb{R})$.

The main theorem of this section is this:

Theorem A.1. *For any $F \in \mathcal{F}$, the function $F_0(\boldsymbol{\theta}) = F(\boldsymbol{\theta}, \mathbf{0})$ has these properties*:

(i) *F_0 is Haar-integrable.*

(ii) *$\widehat{F}_0$ is Haar-integrable.*

(iii) *$\partial F_0/\partial\theta_l$ is Haar-integrable for every $l = 1, \ldots, N$.*

(iv) *$\widehat{\partial F_0/\partial\theta_l}$ is Haar-integrable for every $l = 1, \ldots, N$.*

Before proving this result, let us consider a technical lemma. For every function $H : \mathbb{T}^N \to \mathbb{R}$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N) \in \{0, 1\}^N$, we will use the notation

$$\frac{\partial^{|\boldsymbol{\alpha}|} H}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}}}(\boldsymbol{\theta}) = \frac{\partial^{\alpha_1+\cdots+\alpha_N} H}{\partial\theta_1^{\alpha_1}\cdots\theta_N^{\alpha_N}}(\boldsymbol{\theta}),$$

whenever it makes sense.

Lemma A.2. *Let $H : \mathbb{T}^N \to \mathbb{R}$ of class $\mathscr{C}^{N+1}$ be such that*

$$\frac{\partial^{|\boldsymbol{\alpha}|} H}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}}} \in \mathrm{L}^1(\mathbb{T}^N) \quad \textit{and} \quad \frac{\partial^{|\boldsymbol{\alpha}|+1} H}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}}\theta_l} \in \mathrm{L}^1(\mathbb{T}^N),$$

for $\boldsymbol{\alpha} \in \{0, 1\}^N$ and $l = 1, \ldots, N$. Then,

$$\widehat{\frac{\partial^{|\boldsymbol{\alpha}|} H}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}}}}(\boldsymbol{n}) = \prod_{k=1}^{N}(2\pi i n_k)^{\alpha_k}\,\widehat{H}(\boldsymbol{n}) \quad \textit{and} \quad \widehat{\frac{\partial^{|\boldsymbol{\alpha}|+1} H}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}}\theta_l}} = (2\pi i n_l)\prod_{k=1}^{N}(2\pi i n_k)^{\alpha_k}\,\widehat{H}(\boldsymbol{n}).$$

Proof. This lemma is proved by recursively applying the following calculation:

$$
\begin{aligned}
\widehat{\frac{\partial F_0}{\partial \theta_1}}(\boldsymbol{n}) &= \int_{\mathbb{T}^N} \frac{\partial F_0}{\partial \theta_1}(\boldsymbol{\theta}) e^{-2\pi i \boldsymbol{n}\cdot\boldsymbol{\theta}} \, d\boldsymbol{\theta} \\
&= \int_{\mathbb{T}^{N-1}} \left(\int_{\mathbb{T}} \frac{\partial F_0}{\partial \theta_1}(\boldsymbol{\theta}) e^{-2\pi i n_1 \theta_1} \, d\theta_1 \right) e^{-2\pi i \sum_{j\neq 1} n_j \theta_j} \, d\theta_2 \cdots d\theta_N \\
&= \int_{\mathbb{T}^{N-1}} \left(2\pi i n_1 \int_{\mathbb{T}} F_0(\boldsymbol{\theta}) e^{-2\pi i n_1 \theta_1} \, d\theta_1 \right) e^{-2\pi i \sum_{j\neq 1} n_j \theta_j} \, d\theta_2 \cdots d\theta_N \\
&= (2\pi i n_1) \int_{\mathbb{T}^N} F_0(\boldsymbol{\theta}) e^{-2\pi i \boldsymbol{n}\cdot\boldsymbol{\theta}} \, d\boldsymbol{\theta} \\
&= (2\pi i n_1) \widehat{F}_0(\boldsymbol{n}). \qquad \square
\end{aligned}
$$

Proof of Theorem A.1. By definition, for every $F \in \mathcal{F}$, the function F_0 is of class $N+1$. This is, all its partial derivatives up to order $N+1$ are continuous and, since they are defined on a compact space, they are bounded. Hence, for every $\boldsymbol{\alpha} \in \{0,1\}^N$ and every $l = 1, \dots, N$,

$$
\frac{\partial^{|\boldsymbol{\alpha}|} F_0}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}}}, \frac{\partial^{|\boldsymbol{\alpha}|+1} F_0}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}} \theta_l} \in (\mathrm{L}^1 \cap \mathrm{L}^2)(\mathbb{T}^N).
$$

In particular, we obtain parts (i) and (iii).

Let us prove (ii), we have to see that

$$
\sum_{\boldsymbol{n} \in \mathbb{Z}^N} |\widehat{F}_0(\boldsymbol{n})| < \infty.
$$

To do so, we will divide the sum over all $\boldsymbol{n} \in \mathbb{Z}^N$ in several subsets. Let $\alpha \in \{0,1\}$ and set

$$
W(\alpha) = \begin{cases} \mathbf{0} & \text{if } \alpha = 0, \\ \mathbb{Z}^N \setminus \{\mathbf{0}\} & \text{if } \alpha = 1. \end{cases}
$$

For $\boldsymbol{\alpha} \in \{0,1\}^N$, set also

$$
\boldsymbol{W}(\boldsymbol{\alpha}) = W(\alpha_1) \times \cdots \times W(\alpha_N).
$$

Hence,

$$
\sum_{\boldsymbol{n} \in \mathbb{Z}^N} |\widehat{F}_0(\boldsymbol{n})| = \sum_{\boldsymbol{\alpha} \in \{0,1\}^N} \sum_{\boldsymbol{n} \in \boldsymbol{W}(\boldsymbol{\alpha})} |\widehat{F}_0(\boldsymbol{n})|.
$$

For $\boldsymbol{\alpha} \in \{0,1\}^N$,

$$
\sum_{\boldsymbol{n} \in \boldsymbol{W}(\boldsymbol{\alpha})} |\widehat{F}_0(\boldsymbol{n})| = \sum_{\boldsymbol{n} \in \boldsymbol{W}(\boldsymbol{\alpha})} \prod_{k : \alpha_k \neq 0} (2\pi n_k)^{-1} \left| \widehat{\frac{\partial^{|\boldsymbol{\alpha}|} F_0}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}}}}(\boldsymbol{n}) \right|.
$$

We saw that $\partial^{|\boldsymbol{\alpha}|} F_0 / \partial \boldsymbol{\theta}^{\boldsymbol{\alpha}} \in (\mathrm{L}^1 \cap \mathrm{L}^2)(\mathbb{T}^N)$ and so, by Plancherel's theorem,

$$
\left\| \widehat{\frac{\partial^{|\boldsymbol{\alpha}|} F_0}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}}}} \right\|_{\mathrm{L}^2(\mathbb{Z}^N)} = \left\| \frac{\partial^{|\boldsymbol{\alpha}|} F_0}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}}} \right\|_{\mathrm{L}^2(\mathbb{T}^N)}.
$$

Using the Cauchy–Schwartz inequality, we obtain

$$\begin{aligned}\Bigg(\sum_{\boldsymbol{\alpha}\in\{0,1\}^N}\sum_{\boldsymbol{n}\in\boldsymbol{W}(\boldsymbol{\alpha})}|\widehat{F}_0(\boldsymbol{n})|\Bigg)^2 &\\ &= \Bigg(\sum_{\boldsymbol{\alpha}\in\{0,1\}^N}\sum_{\boldsymbol{n}\in\boldsymbol{W}(\boldsymbol{\alpha})}\prod_{k:\alpha_k\neq 0}(2\pi n_k)^{-1}\bigg|\widehat{\frac{\partial^{|\boldsymbol{\alpha}|}F_0}{\partial\boldsymbol{\theta}^{\boldsymbol{\alpha}}}}(\boldsymbol{n})\bigg|\Bigg)^2\\ &\le \Bigg(\sum_{\boldsymbol{\alpha}\in\{0,1\}^N}\sum_{\boldsymbol{n}\in\boldsymbol{W}(\boldsymbol{\alpha})}\prod_{k:\alpha_k\neq 0}\frac{1}{4\pi^2 n_k^2}\Bigg)\Bigg(\sum_{\boldsymbol{\alpha}\in\{0,1\}^N}\sum_{\boldsymbol{n}\in\boldsymbol{W}(\boldsymbol{\alpha})}\bigg|\widehat{\frac{\partial^{|\boldsymbol{\alpha}|}F_0}{\partial\boldsymbol{\theta}^{\boldsymbol{\alpha}}}}(\boldsymbol{n})\bigg|^2\Bigg)<\infty.\end{aligned}$$

Part (iv) of the theorem is proved by applying the same argument to the function $\partial F_0/\partial\theta_l$ for every $l=1,\dots,N$. □

Appendix B: Bounds for the Lipschitz constant of the function f_δ

In this appendix, we give a bound for the Lipschitz constant with respect to the spherical distance of the function $f_\delta:\mathbb{P}^1(\mathbb{C})\to\mathbb{C}$ defined by

$$f_\delta(0:1)=0 \quad\text{and}\quad f_\delta(1:z)=\rho_\delta(|z|)\frac{z}{|z|}\quad\text{for any } z\in\mathbb{C},$$

where $\rho_\delta:\mathbb{R}\to[0,1]$, with $0<\delta<1$, is given by

$$\rho_\delta(r)=\begin{cases}0 & \text{if } r<\delta/2,\\ (5\delta-4r)(\delta-2r)^2/\delta^3 & \text{if } \delta/2\le r\le\delta,\\ 1 & \text{if } \delta<r<1/\delta,\\ (-2+\delta r)^2(-1+2\delta r) & \text{if } 1/\delta\le r\le 2/\delta,\\ 0 & \text{if } r>2/\delta.\end{cases}$$

First we prove that $f_\delta\in\mathscr{C}^1(\mathbb{P}^1(\mathbb{C}),\mathbb{C})$. Afterwards, we will study the Lipschitz constant of its real and imaginary parts. Let us define the usual charts in $\mathbb{P}^1(\mathbb{C})$,

$$U_0:=\{(z_0:z_1)\in\mathbb{P}^1(\mathbb{C}):z_0\neq 0\}\quad\text{and}\quad U_1:=\{(z_0:z_1)\in\mathbb{P}^1(\mathbb{C}):z_1\neq 0\}.$$

It is easy to see that the function f_δ is compactly supported on $U_0\cap U_1$. In fact, we have that

$$\operatorname{supp}(f_\delta)=\Big\{(1:z):\frac{\delta}{2}\le|z|\le\frac{2}{\delta}\Big\}.$$

For this reason, to prove that f_δ is in $\mathscr{C}^1(\mathbb{P}^1(\mathbb{C}),\mathbb{C})$, it is enough to prove that the function $\rho_\delta(|z|)z/|z|$ is of class $\mathscr{C}^1$ in a neighborhood of the set $\{z:\delta/2\le|z|\le 2/\delta\}$.

The piecewise-defined function ρ_δ is continuous, as well as its derivative, which is given by

$$\rho'_\delta(r) = \begin{cases} -24(\delta-2r)(\delta-r)/\delta^3 & \text{if } \delta/2 \le r \le \delta, \\ 6\delta(-2+\delta r)(-1+\delta r) & \text{if } 1/\delta \le r \le 2/\delta, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, since $|z|$ and $z/|z|$ are smooth on $\mathbb{C}^\times$, we conclude that $\rho_\delta(|z|)z/|z|$ is of class $\mathscr{C}^1$.

Lemma B.1. *Let f_δ be defined as above, and set $f_\delta = u_\delta + i v_\delta$. Then,*

$$\mathrm{Lip}_{\mathrm{sph}}(u_\delta), \mathrm{Lip}_{\mathrm{sph}}(v_\delta) \le 2\sqrt{2}\,\frac{\delta^2+9}{\delta^3}.$$

The spherical distance $\mathrm{d}_{\mathrm{sph}}$ on $\mathbb{P}^1(\mathbb{C})$ can be computed by

$$\mathrm{d}_{\mathrm{sph}}(p,p') := 2\arccos \frac{\left|z_0\overline{z'_0} + z_1\overline{z'_1}\right|}{\sqrt{|z_0|^2+|z_1|^2}\sqrt{|z'_0|^2+|z'_1|^2}},$$

for $p=(z_0:z_1)$ and $p'=(z'_0:z'_1)$ in $\mathbb{P}^1(\mathbb{C})$.

To simplify the computations, we will work with an equivalent distance, the *chordal distance* d_{ch} on $\mathbb{P}^1(\mathbb{C})$, which is given by the length of the chord joining two points of S^2. For $p=(z_0:z_1)$ and $p'=(z'_0:z'_1)$ in $\mathbb{P}^1(\mathbb{C})$, we have

$$\mathrm{d}_{\mathrm{ch}}(p,p') := \frac{2|z_0z'_1 - z_1z'_0|}{\sqrt{|z_0|^2+|z_1|^2}\sqrt{|z'_0|^2+|z'_1|^2}}.$$

These distances can be compared as follows:

Lemma B.2. *For every $p, p' \in \mathbb{P}^1(\mathbb{C})$,*

$$\frac{2}{\pi}\,\mathrm{d}_{\mathrm{sph}}(p,p') \le \mathrm{d}_{\mathrm{ch}}(p,p') \le \mathrm{d}_{\mathrm{sph}}(p,p').$$

Proof. We work on the sphere using the stereographic projection. Since the chordal distance d_{ch} between two points in the sphere is the length of the chord joining them and the spherical distance $\mathrm{d}_{\mathrm{sph}}$ is the angle between the vectors both points define, we have

$$\mathrm{d}_{\mathrm{ch}}(p,p') = 2\sin\frac{\mathrm{d}_{\mathrm{sph}}(p,p')}{2}, \quad \text{for every } p,p' \in \mathbb{P}^1(\mathbb{C}).$$

For any pair of points, we have $\mathrm{d}_{\mathrm{sph}}(p,p') \le \pi$ so we deduce

$$\mathrm{d}_{\mathrm{ch}}(p,p') \le \mathrm{d}_{\mathrm{sph}}(p,p').$$

Now, let $\beta > 0$ be such that $\beta\,\mathrm{d}_{\mathrm{sph}}(p,p') \le \mathrm{d}_{\mathrm{ch}}(p,p')$ for all $p,p' \in \mathbb{P}^1(\mathbb{C})$. This is equivalent to $\beta x \le 2\sin\frac{x}{2}$ for every $0 \le x \le \pi$. By the convexity of the function $2\sin\frac{x}{2}$, we deduce that the optimal value is $\beta = \frac{2}{\pi}$. □

Proof of Lemma B.1. Let us compute now a bound for the Lipschitz constants, with respect to the spherical distance, of the u_δ and v_δ. To do so, we choose coordinates (x, y) in $\mathbb{R}^2 \cong \mathbb{C}$. Let

$$\tilde{u}_\delta(x, y) := u_\delta(1 : x + iy) = \frac{\rho_\delta(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}} x,$$

and

$$\tilde{v}_\delta(x, y) := v_\delta(1 : x + iy) = \frac{\rho_\delta(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}} y.$$

Since the computations are symmetric for both the real and imaginary parts of f_δ, it is enough to study the Lipschitz constant of one of them. To simplify these computations, we will study the Lipschitz constant with respect to the chordal distance in the Riemann sphere and conclude by applying the comparison between the chordal and spherical distances.

First of all, recall that the chordal distance restricted to the open subset $U_0 \subset \mathbb{P}^1(\mathbb{C})$ is given by

$$\mathrm{d}_{\mathrm{ch}}\big((1 : x_0 + iy_0), (1 : x_1 + iy_1)\big) = \frac{2\|(x_0, y_0) - (x_1, y_1)\|}{\sqrt{1 + m(x_0, y_0)^2}\sqrt{1 + m(x_1, y_1)^2}},$$

where $\|\cdot\|$ denotes the Euclidean metric on $\mathbb{R}^2$ and $m(x, y) = \sqrt{x^2 + y^2}$. Now, since the function u_δ is supported on U_0,

$$\begin{aligned}\sup_{z_0, z_1 \in \mathbb{C}} & \frac{|u_\delta(1 : z_0) - u_\delta(1 : z_1)|}{\mathrm{d}_{\mathrm{ch}}((1 : z_0), (1 : z_1))} \\ &= \sup_{\substack{(x_0, y_0) \in \mathbb{R}^2 \\ (x_1, y_1) \in \mathbb{R}^2}} \frac{|\tilde{u}_\delta(x_0, y_0) - \tilde{u}_\delta(x_1, y_1)|}{\|(x_0, y_0) - (x_1, y_1)\|} \frac{\sqrt{1 + m(x_0, y_0)^2}\sqrt{1 + m(x_1, y_1)^2}}{2}.\end{aligned}$$

We consider different cases.

Case 1. If $(x_0, y_0), (x_1, y_1) \notin D(0, 2/\delta)$, we trivially obtain

$$\frac{|\tilde{u}_\delta(x_0, y_0) - \tilde{u}_\delta(x_1, y_1)|}{\mathrm{d}_{\mathrm{ch}}((1 : x_0 + iy_0), (1 : x_1 + iy_1))} = 0.$$

Case 2. Suppose $(x_0, y_0), (x_1, y_1) \in D(0, 2/\delta)$. For $t \in [0, 1]$, consider the function

$$g(t) = \tilde{u}_\delta((1 - t)(x_0, y_0) + t(x_1, y_1)).$$

By the mean value theorem, there is some $c \in (0, 1)$ such that $g(1) - g(0) = g'(c)$. Applying the chain rule, we obtain

$$\tilde{u}_\delta(x_1, y_1) - \tilde{u}_\delta(x_0, y_0) = \nabla\tilde{u}_\delta((1 - c)(x_0, y_0) + c(x_1, y_1)) \cdot (x_1 - x_0, y_1 - y_0).$$

Hence, we deduce

$$\frac{|\tilde{u}_\delta(x_0,y_0)-\tilde{u}_\delta(x_1,y_1)|}{\|(x_0,y_0)-(x_1,y_1)\|}\le \sup_{(x,y)\in D(0,\frac{2}{\delta})}\|\nabla\tilde{u}_\delta(x,y)\|. \tag{B-1}$$

Let us study the gradient of $\tilde{u}_\delta$. For every $(x,y)\in\mathbb{R}^2$,

$$\frac{\partial\tilde{u}_\delta}{\partial x}(x,y)=\left(\frac{x}{m(x,y)}\right)^2\rho'_\delta(m(x,y))+\left(\frac{y}{m(x,y)}\right)^2\frac{\rho_\delta(m(x,y))}{m(x,y)}$$

and

$$\frac{\partial\tilde{u}_\delta}{\partial y}(x,y)=\frac{xy}{m(x,y)^2}\left(\rho'_\delta(m(x,y))-\frac{\rho_\delta(m(x,y))}{m(x,y)}\right).$$

Without loss of generality, we restrict ourselves to the situation where (x,y) satisfies $\delta/2\le m(x,y)\le 2/\delta$, since otherwise both partial derivatives would vanish. It can be easily shown that $|\rho'_\delta(r)|\le 3/\delta$ for every $r\ge 0$. This, together with the fact that $0\le\rho_\delta\le 1,\ x\le m(x,y),\ y\le m(x,y)$, and $m(x,y)\ge\delta/2$, leads to

$$\left|\frac{\partial\tilde{u}_\delta}{\partial x}(x,y)\right|,\left|\frac{\partial\tilde{u}_\delta}{\partial y}(x,y)\right|\le\frac{4}{\delta}.$$

We then conclude that $\|\nabla\tilde{u}_\delta(x,y)\|\le 4\sqrt{2}/\delta$. for any $(x,y)\in\mathbb{R}^2$.

On the other hand, given $(x_0,y_0),(x_1,y_1)\in D(0,2/\delta)$ we have that

$$\frac{\sqrt{1+m(x_0,y_0)^2}\sqrt{1+m(x_1,y_1)^2}}{2}\le\frac{\delta^2+4}{2\delta^2}.$$

Therefore, we obtain

$$\frac{|\tilde{u}_\delta(x_0,y_0)-\tilde{u}_\delta(x_1,y_1)|}{\mathrm{d_{ch}}((1:x_0+iy_0),(1:x_1+iy_1))}\le 2\sqrt{2}\,\frac{\delta^2+4}{\delta^3}.$$

Case 3. Suppose now that $(x_0,y_0)\in D(0,2/\delta)$ and $(x_1,y_1)\in D(0,3/\delta)\setminus D(0,2/\delta)$. As we did in the previous case, we can deduce that

$$\frac{|\tilde{u}_\delta(x_0,y_0)-\tilde{u}_\delta(x_1,y_1)|}{\|(x_0,y_0)-(x_1,y_1)\|}\le\frac{4\sqrt{2}}{\delta}\quad\text{and}\quad\frac{\sqrt{1+m(x_0,y_0)^2}\sqrt{1+m(x_1,y_1)^2}}{2}\le\frac{\delta^2+9}{2\delta^2}.$$

and

$$\frac{\sqrt{1+m(x_0,y_0)^2}\sqrt{1+m(x_1,y_1)^2}}{2}\le\frac{\delta^2+9}{2\delta^2}.$$

Hence,

$$\frac{|\tilde{u}_\delta(x_0,y_0)-\tilde{u}_\delta(x_1,y_1)|}{\mathrm{d_{ch}}((1:x_0+iy_0),(1:x_1+iy_1))}\le 2\sqrt{2}\,\frac{\delta^2+9}{\delta^3}.$$

Case 4. Finally suppose that $(x_0, y_0) \in D(0, 2/\delta)$ and $(x_1, y_1) \notin D(0, 3/\delta)$. In this situation, we have $\tilde{u}_\delta(x_1, y_1) = 0$ and

$$|\tilde{u}_\delta(x_1, y_1)| = |\rho_\delta(m(x_0, y_0))| \frac{|x_0|}{m(x_0, y_0)} \leq 1.$$

Since

$$\mathrm{d}_{\mathrm{ch}}((1 : x_0 + iy_0), (1 : x_1 + iy_1)) \geq \mathrm{d}_{\mathrm{ch}}\Big(\Big(1 : \frac{2}{\delta}\Big), \Big(1 : \frac{3}{\delta}\Big)\Big) = \frac{2\delta}{\sqrt{(\delta^2+9)(\delta^2+4)}},$$

we conclude

$$\frac{|\tilde{u}_\delta(x_0, y_0) - \tilde{u}_\delta(x_1, y_1)|}{\mathrm{d}_{\mathrm{ch}}((1 : x_0 + iy_0), (1 : x_1 + iy_1))} \leq \frac{2\delta}{\sqrt{(\delta^2+9)(\delta^2+4)}} 2\delta.$$

Having studied all these cases, we deduce that

$$\sup_{\substack{(x_0,y_0)\in\mathbb{R}^2\\(x_1,y_1)\in\mathbb{R}^2}} \frac{|\tilde{u}_\delta(x_0, y_0) - \tilde{u}_\delta(x_1, y_1)|}{\mathrm{d}_{\mathrm{ch}}((1 : x_0 + iy_0), (1 : x_1 + iy_1))} \leq 2\sqrt{2}\,\frac{\delta^2+9}{\delta^3}.$$

As we mentioned above, we were looking for a bound of the Lipschitz constant of u_δ with respect to the spherical distance. By Lemma B.2, we know that $\mathrm{d}_{\mathrm{sph}}(p, p') \geq \mathrm{d}_{\mathrm{ch}}(p, p')$ for any pair of points $p, p' \in \mathbb{P}^1(\mathbb{C})$ and we obtain

$$\begin{aligned}
\mathrm{Lip}_{\mathrm{sph}}(u_\delta) &= \sup_{p,p'\in\mathbb{P}^1(\mathbb{C})} \frac{|u_\delta(p) - u_\delta(p')|}{\mathrm{d}_{\mathrm{sph}}(p, p')} \\
&\leq \sup_{\substack{(x_0,y_0)\in\mathbb{R}^2\\(x_1,y_1)\in\mathbb{R}^2}} \frac{|\tilde{u}_\delta(x_0, y_0) - \tilde{u}_\delta(x_1, y_1)|}{\mathrm{d}_{\mathrm{ch}}((1 : x_0 + iy_0), (1 : x_1 + iy_1))} \\
&\leq 2\sqrt{2}\,\frac{\delta^2+9}{\delta^3}.
\end{aligned}$$

Analogously, we deduce that $\mathrm{Lip}_{\mathrm{sph}}(v_\delta) \leq 2\sqrt{2}\,\frac{\delta^2+9}{\delta^3}$. □

Acknowledgements

We thank Joaquim Ortega, Juan Rivera-Letelier and the anonymous referee for useful comments and suggestions. The results of this paper are part of the Ph.D. thesis of Narváez-Clauss [2016].

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Communicated by Jonathan Pila
Received 2016-10-02 Revised 2017-04-06 Accepted 2017-05-23

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Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

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Algebra & Number Theory

Volume 11 No. 7 2017