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A nonarchimedean Ax–Lindemann theorem

Antoine Chambert-Loir and François Loeser

À Daniel Bertrand, en témoignage d'amitié

Motivated by the André–Oort conjecture, Pila has proved an analogue of the Ax–Lindemann theorem for the uniformization of classical modular curves. In this paper, we establish a similar theorem in nonarchimedean geometry. Precisely, we give a geometric description of subvarieties of a product of hyperbolic Mumford curves such that the irreducible components of their inverse image by the Schottky uniformization are algebraic, in some sense. Our proof uses a p-adic analogue of the Pila–Wilkie theorem due to Cluckers, Comte and Loeser, and requires that the relevant Schottky groups have algebraic entries.

1. Introduction

1.1. The classical Lindemann–Weierstrass theorem states that if algebraic numbers $\alpha_1, \dots, \alpha_n$ are $\mathbf{Q}$-linearly independent, then their exponentials $\exp(\alpha_1), \dots, \exp(\alpha_n)$ are algebraically independent over $\mathbf{Q}$. More generally, if $\alpha_1, \dots, \alpha_n$ are any $\mathbf{Q}$-linearly independent complex numbers, no longer assumed to be algebraic, Schanuel's conjecture predicts that the field $\mathbf{Q}(\alpha_1, \dots, \alpha_n, \exp(\alpha_1), \dots, \exp(\alpha_n))$ has transcendence degree at least n over $\mathbf{Q}$. Ax [1971] established power series and differential field versions of Schanuel's conjecture. In particular, the part of Ax's results corresponding to the Lindemann–Weierstrass theorem can be recast into geometrical terms as follows:

Theorem 1.2 (exponential Ax–Lindemann). *Let* $\exp : \mathbf{C}^n \to (\mathbf{C}^\times)^n$ *be the morphism* $(z_1, \dots, z_n) \mapsto (\exp(z_1), \dots, \exp(z_n))$. *Let V be an irreducible algebraic subvariety of* $(\mathbf{C}^\times)^n$ *and let W be an irreducible component of a maximal algebraic subvariety of* $\exp^{-1}(V)$. *Then W is geodesic*, *that is*, *W is defined by a finite family of equations of the form* $\sum_{i=1}^n a_i z_i = b$ *with* $a_1, \dots, a_n \in \mathbf{Q}$ *and* $b \in \mathbf{C}$.

In a breakthrough paper, Pila [2011] succeeded in providing an unconditional proof of the André–Oort conjecture for products of modular curves. One of his

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main ingredients was to prove a hyperbolic version of the above Ax–Lindemann theorem, which we now state in a simplified version.

Let $\boldsymbol{h}$ denote the complex upper half-plane and $j : \boldsymbol{h} \to \mathbf{C}$ the elliptic modular function. By an algebraic subvariety of $\boldsymbol{h}^n$, we mean the trace in $\boldsymbol{h}^n$ of an algebraic subvariety of $\mathbf{C}^n$. An algebraic subvariety of $\boldsymbol{h}^n$ is said to be geodesic if it can be defined by equations of the form $z_i = c_i$ and $z_k = g_{k\ell} z_\ell$, with $c_i \in \mathbf{C}$ and $g_{k\ell} \in \mathrm{GL}(2, \mathbf{Q})^+$.

Theorem 1.3 (hyperbolic Ax–Lindemann). *Let $j : \boldsymbol{h}^n \to \mathbf{C}^n$ be the morphism $(z_1, \ldots, z_n) \mapsto (j(z_1), \ldots, j(z_n))$. Let V be an irreducible algebraic subvariety of $\mathbf{C}^n$ and let W be an irreducible component of a maximal algebraic subvariety of $j^{-1}(V)$. Then W is geodesic.*

Pila's method to prove this Ax–Lindemann theorem is quite different from the differential approach of Ax. It follows a strategy initiated by Pila and Zannier [2008] in their new proof of the Manin–Mumford conjecture for abelian varieties; that approach makes crucial use of the bound on the number of rational points of bounded height in the transcendental part of sets definable in an o-minimal structure obtained in [Pila and Wilkie 2006]. Recently, still using the Pila and Zannier strategy, Klingler, Ullmo and Yafaev [Klingler et al. 2016] have succeeded in proving a very general form of the hyperbolic Ax–Lindemann theorem valid for any arithmetic variety; see also [Ullmo and Yafaev 2014] for the compact case.

1.4. In the recent paper [Cluckers et al. 2015], Cluckers, Comte and Loeser established a nonarchimedean analogue of the Pila–Wilkie theorem of [Pila and Wilkie 2006] in its block version of [Pila 2009]. The purpose of this paper is to use this result to prove a version of Ax–Lindemann for products of algebraic curves admitting a nonarchimedean uniformization and whose corresponding Schottky group is “arithmetic” and has rank at least 2 (Theorem 2.7). In particular, this theorem applies for products of Shimura curves admitting a p-adic uniformization à la Čerednik–Drinfel'd (see Section 3).

The basic strategy we use is strongly inspired by that of [Pila 2011] (see also [Pila 2015]), though some new ideas are required in order to adapt it to the nonarchimedean setting. Similarly as in Pila's approach one starts by working on some neighborhood of the boundary of our space (which, instead of a product of Poincaré upper half-planes, is a product of open subsets of the Berkovich projective line). Analytic continuation and monodromy arguments are replaced by more algebraic ones and explicit matrix computations by group theory considerations. We also take advantage of the fact that Schottky groups are free and of the geometric description of their fundamental domains. Compared with Pila's proof, where parabolic elements are used in a crucial way, one main difficulty of the nonarchimedean situation lies in the fact that all nontrivial elements of a Schottky groups are hyperbolic.

To conclude, let us note that there are cases where p-adic analogues of theorems in transcendental number theory seem to require other methods than those used to prove their complex counterparts. For instance, it is still an open problem to prove a p-adic analogue, for values of the p-adic exponential function, of the classical Lindemann–Weierstrass theorem.

Since his first works (see, for example, [Bertrand 1976]), Daniel Bertrand has shown deep insight into p-adic transcendental number theory, and disseminated his vision within the mathematical community. We are pleased to dedicate this paper to him.

2. Statement of the theorem

2.1. ***Nonarchimedean analytic spaces.*** Given a complete nonarchimedean valued field F, we consider in this paper F-analytic spaces in the sense of Berkovich [1990; 1993]. However, the statements, and essentially the proofs, can be carried on mutatis mutandis in the rigid analytic setting. In this context, there is a notion of irreducible component; see [Ducros 2009], or [Conrad 1999] for the rigid analytic version.

If V is an algebraic variety over F, we denote by V^{an} the corresponding F-analytic space. There is a canonical topological embedding of $V(F)$ in V^{an}, and its image is closed if F is locally compact.

If F' is a complete nonarchimedean extension of F, we denote by $X_{F'}$ the F'-analytic space deduced from an F-analytic space X by base change to F'.

2.2. ***Schottky groups.*** Let p be a prime number; we denote by $\mathbf{C}_p$ the completion of an algebraic closure of $\mathbf{Q}_p$ and let F be a finite extension of $\mathbf{Q}_p$ contained in $\mathbf{C}_p$. The group $\mathrm{PGL}(2, F)$ acts by homographies on the F-analytic projective line $\mathbf{P}_1^{\text{an}}$. In the next paragraphs, we recall from [Gerritzen and van der Put 1980] a few definitions concerning Schottky groups in $\mathrm{PGL}(2, F)$, their limit sets and the associated uniformizations of algebraic curves.

One says that a discrete subgroup Γ of $\mathrm{PGL}(2, F)$ is a *Schottky group* if it is finitely generated, and if no element ($\neq$ id) has finite order [Gerritzen and van der Put 1980, I, (1.6)]. If Γ is a Schottky group, then Γ is free; moreover, any discrete finitely generated subgroup of $\mathrm{PGL}(2, F)$ possesses a normal subgroup of finite index which is a Schottky group [Gerritzen and van der Put 1980, I, (3.1)].

We say that Γ is *arithmetic* if its elements can be represented by matrices whose coefficients lie in a number field. In this case, it follows from the hypothesis that Γ is finitely generated that there exists a number field $K \subset F$ such that $\Gamma \subset \mathrm{PGL}(2, K)$.

2.3. ***Limit sets.*** Let Γ be a Schottky subgroup of $\mathrm{PGL}(2, F)$. Its *limit set* is the set $\mathscr{L}_\Gamma$ of all points in $\mathbf{P}_1(\mathbf{C}_p)$ of the form $\lim_n(\gamma_n \cdot x)$, where (γ_n) is a sequence of distinct elements of Γ and $x \in \mathbf{P}_1(\mathbf{C}_p)$ [Gerritzen and van der Put 1980, I, (1.3)].

By [Gerritzen and van der Put 1980, I, (1.6)], the limit set $\mathscr{L}_\Gamma$ is a compact subset of $\mathbf{P}_1(F)$. If the rank of Γ is at least 2, then $\mathscr{L}_\Gamma$ is a perfect (that is, closed and without isolated points) subset of $\mathbf{P}_1(F)$; see [Gerritzen and van der Put 1980, I, (1.6.3) and (1.7.2)].

Let $\Omega_\Gamma = (\mathbf{P}_1)^{\mathrm{an}} - \mathscr{L}_\Gamma$; it is a Γ-invariant open set of $\mathbf{P}_1^{\mathrm{an}}$. By Lemma 5.4 below, it is geometrically irreducible.

2.4. ***Quotients.*** Let us assume that Γ is a Schottky group and let g be its rank. From the explicit description of the action of the group Γ given by [Gerritzen and van der Put 1980, I.4] and recalled in Section 6.5 below (see also [Berkovich 1990, p. 86]), it follows that the group Γ acts freely on Ω_Γ, and the quotient space Ω_Γ/Γ admits a unique structure of an F-analytic space such that the projection $p_\Gamma : \Omega_\Gamma \to \Omega_\Gamma/\Gamma$ is both a topological covering and a local isomorphism. Moreover, Ω_Γ/Γ is the F-analytic space associated with a smooth, geometrically connected, projective F-curve X_Γ of genus g [Gerritzen and van der Put 1980, III, (2.2); Berkovich 1990, Theorem 4.4.1, p. 86], canonically determined by the GAGA theorem in this context, [Berkovich 1990, Theorem 3.4.12, p. 68].

2.5. Let us now consider a finite family $(\Gamma_i)_{1\leq i\leq n}$ of Schottky subgroups of $\mathrm{PGL}(2, F)$ of rank ≥ 2. Let us set $\Omega = \prod_{i=1}^n \Omega_{\Gamma_i}$ and $X = \prod_{i=1}^n X_{\Gamma_i}$, and let $p : \Omega \to X^{\mathrm{an}}$ be the morphism deduced from the morphisms $p_{\Gamma_i} : \Omega_{\Gamma_i} \to X_{\Gamma_i}^{\mathrm{an}}$.

2.6. ***Flat subvarieties.*** Let K be a complete extension of F and let W be a closed analytic subspace of Ω_K.

The following terminology is borrowed from the analogous notions in the differential geometry of hermitian symmetric domains.

We say that W is *irreducible algebraic* if there exists a K-algebraic subvariety Y of $(\mathbf{P}_1^n)_K$ such that W is an irreducible component of the analytic space $\Omega_K \cap Y^{\mathrm{an}}$. In this case, one can take for Y the Zariski closure of W in $(\mathbf{P}_1^n)_K$; it is irreducible and satisfies $\dim(Y) = \dim(W)$; see [Ducros 2009, Proposition 4.22].

We say that W is *flat* if it can be defined by equations of the following form:

(1) $z_i = c$ for some $i \in \{1, \dots, n\}$ and $c \in \Omega_{\Gamma_i}(K)$;

(2) $z_j = g \cdot z_i$ for some pair (i, j) of distinct elements of $\{1, \dots, n\}$ and some $g \in \mathrm{PGL}(2, F)$.

Assume that W is flat and let Y be the subvariety of $(\mathbf{P}_1^n)_K$ defined by equations of this form which define W on Ω_K. There exists a subset I of $\{1, \dots, n\}$ such that the projection $q_I : \mathbf{P}_1^n \to \mathbf{P}_1^I$ given by the coordinates in I induces an isomorphism of Y to $(\mathbf{P}_1^I)_K$. This implies that q_I induces an isomorphism from W to $\prod_{i\in I} \Omega_{i,K}$. In particular, W is irreducible, even geometrically irreducible, and hence is irreducible algebraic. Conversely, we observe that if W is geometrically irreducible and if there exists a complete extension L of K such that W_L is flat, then W is flat.

We say that W is *geodesic* if, moreover, the elements g in (2) can be taken such that $g\Gamma_i g^{-1}$ and Γ_j are commensurable (i.e., their intersection has finite index in both of them).

Here is the main result of this paper.

Theorem 2.7 (nonarchimedean Ax–Lindemann theorem). *Let F be a finite extension of $\mathbf{Q}_p$ and let $(\Gamma_i)_{1\leq i\leq n}$ be a finite family of* arithmetic *Schottky subgroups of* $\mathrm{PGL}(2, F)$ *of ranks ≥ 2. As above, let us set $\Omega = \prod_{i=1}^n \Omega_{\Gamma_i}$ and $X = \prod_{i=1}^n X_{\Gamma_i}$, and let $p : \Omega \to X^{\mathrm{an}}$ be the morphism deduced from the morphisms $p_{\Gamma_i} : \Omega_{\Gamma_i} \to X_{\Gamma_i}^{\mathrm{an}}$.*

Let V be an irreducible algebraic subvariety of X and let W be an irreducible algebraic subvariety of Ω, maximal among those contained in $p^{-1}(V^{\mathrm{an}})$. Then every irreducible component of $W_{\mathbf{C}_p}$ is flat.

The proof of this theorem is given in Section 8; it follows the strategy of Pila–Zannier. In the archimedean setting, this strategy relies crucially on a theorem of Pila–Wilkie about rational points on definable sets; we recall in Section 4 the nonarchimedean analogue of this theorem [Cluckers et al. 2015] which is used here. It is at this point that we need the assumption that the group Γ be arithmetic. This restriction is inherent to Pila–Zannier's strategy and we do not know whether it can be bypassed.

In Section 6, we recall a few more facts on p-adic Schottky groups and p-adic uniformization, essentially borrowed from [Gerritzen and van der Put 1980].

In a final section, we prove a characterization (Theorem 9.2) of geodesic subvarieties of Ω as the geometrically irreducible algebraic subvarieties whose projection to X is algebraic ("bialgebraic subvarieties"), in analogy with what happens in the context of Ax's theorem or of Shimura varieties.

3. The example of Shimura curves

We begin by recalling the definition of Shimura curves and their p-adic uniformization. The literature is unfortunately rather scattered; we refer to [Boutot and Carayol 1992] for more detail, as well as to [Clark 2003, Chapter 0].

3.1. ***Complex Shimura curves.*** Let B be a quaternion division algebra with center $\mathbf{Q}$; we assume that it is indefinite, namely $B \otimes_{\mathbf{Q}} \mathbf{R} \simeq \mathrm{M}_2(\mathbf{R})$. Let then $\mathscr{O}_B$ be a maximal order of B, that is a maximal subring of B which is isomorphic to $\mathbf{Z}^4$ as a $\mathbf{Z}$-module. Let H be the algebraic group of units of $\mathscr{O}_B$, modulo center, considered as a $\mathbf{Z}$-group scheme. For every field R containing $\mathbf{Q}$, one has $H(R) = (B \otimes_{\mathbf{Q}} R)^\times / Z((B \otimes_{\mathbf{Q}} R)^\times)$; in particular, the group $H(\mathbf{R})$ is isomorphic to $\mathrm{PGL}(2, \mathbf{R})$, and we fix such an isomorphism. Then the group $H(\mathbf{R})$ acts by homographies on the double Poincaré upper half-plane

$$\boldsymbol{h}^{\pm} = \mathbf{C} - \mathbf{R}.$$

Let also Δ be a congruence subgroup of $H(\mathbf{Z})$; recall that this means that there exists an integer $N \geq 1$ such that Δ contains the kernel of the canonical morphism $H(\mathbf{Z}) \to H(\mathbf{Z}/N\mathbf{Z})$. We assume that Δ has been chosen small enough so that the stabilizer of every point of $\boldsymbol{h}^{\pm}$ is trivial. The quotient $\boldsymbol{h}^{\pm}/\Delta$ has a natural structure of a compact Riemann surface and the projection $p : \boldsymbol{h}^{\pm} \to \boldsymbol{h}^{\pm}/\Delta$ is an étale covering.

This curve parameterizes triples (V, ι, ν), where V is a complex two-dimensional abelian variety, $\iota : \mathscr{O}_B \to \mathrm{End}(V)$ is a faithful action of $\mathscr{O}_B$ on V and ν is a level structure "of type Δ" on V. When Δ is the kernel of $H(\mathbf{Z})$ to $H(\mathbf{Z}/N\mathbf{Z})$, for some integer $N \geq 1$, such a level structure corresponds to an equivariant isomorphism of V_N, the subgroup of N-torsion of V, with $\mathscr{O}_B/N\mathscr{O}_B$.

By [Shimura 1961], it admits a canonical structure of an algebraic curve S which can be defined over a number field E in $\mathbf{C}$.

3.2. ***The p-adic uniformization of Shimura curves.*** Let p be a prime number at which B ramifies, which means that $B \otimes_{\mathbf{Q}} \mathbf{Q}_p$ is a division algebra. Let also F be the completion of the field E at a place dividing p; we denote by $\mathbf{C}_p$ the p-adic completion of an algebraic closure of F. We still denote by S the F-curve deduced from an E-model of the complex curve S.

Let $\Omega = (\mathbf{P}_1)^{\mathrm{an}}_F - \mathbf{P}_1(\mathbf{Q}_p)$ be the extension of scalars to F of Drinfel'd's upper half-plane. According to the theorem of Čerednik and Drinfel'd [Čerednik 1976; Drinfel'd 1976] (see also [Boutot and Carayol 1992] for a detailed exposition), and up to replacing F by a finite unramified extension, the F-analytic curve S^{an} admits a "p-adic uniformization" which takes the form of a surjective analytic morphism

$$j : \Omega \to S^{\mathrm{an}},$$

identifying S^{an} with the quotient of Ω by the action of a subgroup Γ of $\mathrm{PGL}(2, \mathbf{Q}_p)$. Up to replacing Δ by a smaller congruence subgroup, which replaces S by a finite (possibly ramified) covering, we may also assume that Γ is a p-adic Schottky subgroup acting freely on Ω, and that j is topologically étale. Then the morphism $j : \Omega \to S^{\mathrm{an}}$ is the universal cover of S^{an}.

Let us describe this subgroup. Let A be the quaternion division algebra over $\mathbf{Q}$ with the same invariants as B, except for those invariants at p and ∞ which are switched. In particular, $A \otimes_{\mathbf{Q}} \mathbf{R}$ is Hamilton's quaternion algebra, while $A \otimes_{\mathbf{Q}} \mathbf{Q}_p \simeq \mathrm{M}_2(\mathbf{Q}_p)$. Let G be the algebraic group of units of A, modulo center; in particular, $G(\mathbf{Q}_p) \simeq \mathrm{PGL}(2, \mathbf{Q}_p)$. As explained in [Boutot and Carayol 1992], the discrete subgroup Γ is the intersection of $G(\mathbf{Q})$ with a compact open subgroup of $G(\mathbf{A}_{\mathrm{f}})$, the adelic group associated with G where the place at ∞ is omitted.

Lemma 3.3. *The group Γ is conjugate to an arithmetic Schottky subgroup in* $\mathrm{PGL}(2, \mathbf{Q}_p)$, *its rank is at least* 2, *and its limit set is equal to* $\mathbf{P}_1(\mathbf{Q}_p)$.

Proof. The group Γ is a discrete subgroup of $\mathrm{PGL}(2, \mathbf{Q}_p)$, so its limit set $\mathscr{L}_\Gamma$ is a Γ-invariant subset of $\mathbf{P}_1(\mathbf{Q}_p)$. In other words, the Drinfeld upper half-plane $\Omega = \mathbf{P}_1^{\mathrm{an}} - \mathbf{P}_1(\mathbf{Q}_p)$ is an open subset of $\Omega_\Gamma = \mathbf{P}_1^{\mathrm{an}} - \mathscr{L}_\Gamma$. By the theory of Mumford curves and Schottky groups (see [Gerritzen and van der Put 1980]), the analytic curve $(\mathbf{P}_1^{\mathrm{an}} - \mathscr{L}_\Gamma)/\Gamma$ is algebraic, and admits the analytic curve $S^{\mathrm{an}} = \Omega/\Gamma$ as an open subset. According to the Čerednik–Drinfel'd theorem, the curve S^{an} is projective. This implies that $\Omega = \mathbf{P}_1^{\mathrm{an}} - \mathscr{L}_\Gamma$, and hence $\mathscr{L}_\Gamma = \mathbf{P}_1(\mathbf{Q}_p)$.

After base change to $\mathbf{Q}_p$, the algebraic $\mathbf{Q}$-group G becomes isomorphic to $\mathrm{PGL}(2)_{\mathbf{Q}_p}$. Consequently, there exists a finite algebraic extension K of $\mathbf{Q}$, contained in $\mathbf{Q}_p$, such that $G_K \simeq \mathrm{PGL}(2)_K$. By such an isomorphism, $G(\mathbf{Q})$ is mapped into $\mathrm{PGL}(2, K)$; this implies that the group Γ is conjugate to an arithmetic group.

Since Γ is a Schottky group, it is free. Since it is nonabelian, its rank is at least 2. □

By this lemma, the following result is a special case of our main theorem (Theorem 2.7).

Theorem 3.4. *Let F be a finite extension of $\mathbf{Q}_p$, let $\Omega = (\mathbf{P}_1)_F^{\mathrm{an}} - \mathbf{P}_1(\mathbf{Q}_p)$ and let $j : \Omega^n \to S^{\mathrm{an}}$ be the Čerednik–Drinfel'd uniformization of a product of Shimura curves. Let V be an irreducible algebraic subvariety of S and let $W \subset \Omega^n$ be a maximal irreducible algebraic subvariety of $j^{-1}(V^{\mathrm{an}})$. Then every irreducible component of $W_{\mathbf{C}_p}$ is flat.*

3.5. By the same arguments, one can show that Theorem 2.7 also applies to the uniformizations of Shimura curves associated with quaternion division algebras over totally real fields, as considered by Čerednik [1976] and Boutot and Zink [1995].

3.6. As suggested by J. Pila and explained to us by Y. André, Theorem 3.4 can also be deduced from its complex analogue, which is a particular case of [Ullmo and Yafaev 2014]. The crucial ingredient is a deep theorem of André [2003, III, 4.7.4] stating that the p-adic uniformization and the complex uniformization of Shimura curves satisfy the *same* nonlinear differential equation. His proof relies on a delicate description of the Gauss–Manin equation in terms of convergent crystals and on the tempered fundamental group introduced by him. From that point on, one can apply Seidenberg's embedding theorem [1958] in differential algebra to prove that both the complex and nonarchimedean Ax–Lindemann theorems are equivalent to a single statement in differential algebra, in the original spirit of [Ax 1971].

4. Definability — a p-adic Pila–Wilkie theorem

4.1. There are two distinct notions of p-adic analytic geometry: one is "naïve", and the other rigid analytic. (Regarding rigid analytic geometry, we work in the framework defined by Berkovich.) These two notions give rise to three classes of sets, and we use them all in this paper. Let F be a finite extension of $\mathbf{Q}_p$.

a) *Semialgebraic* and *subanalytic* subsets of $\mathbf{Q}_p^n$ are defined by Denef and van den Dries [1988]; see also [Cluckers et al. 2015, p. 26].

   Replacing $\mathbf{Q}_p$ by a finite extension F, this leads to an analogous notion of F-semialgebraic, or F-subanalytic, subset of F^n. Considering affine charts, one then defines F-semialgebraic or F-subanalytic subsets of $V(F)$, for every (quasiprojective, say) algebraic variety V defined over F.

   On the other hand, the Weil restriction functor assigns to V an algebraic variety W defined over $\mathbf{Q}_p$ together with a canonical identification $V(F) \to W(\mathbf{Q}_p)$; we say that a subset of $V(F)$ is $\mathbf{Q}_p$-semialgebraic or $\mathbf{Q}_p$-subanalytic if its image in $W(\mathbf{Q}_p)$ is $\mathbf{Q}_p$-semialgebraic or $\mathbf{Q}_p$-subanalytic, respectively. Observe that F-semialgebraic subsets of $V(F)$ are $\mathbf{Q}_p$-semialgebraic, and that F-subanalytic subsets of $V(F)$ are $\mathbf{Q}_p$-subanalytic.

   Recall that an F-subanalytic subset S is said to be smooth of dimension d at a point x if it possesses a neighborhood U which is isomorphic to the unit ball of F^d; then S is smooth of dimension d at every point of U.

b) Lipshitz [1993] defined a notion of *rigid subanalytic subset* of $\mathbf{C}_p^n$. We use in this paper the variant [Lipshitz and Robinson 2000a, Definition 2.1.1] where the coefficients of all polynomials and power series involved belong to F; we call them *rigid F-subanalytic.* The notion extends to subsets of $V(\mathbf{C}_p)$, where V is an algebraic variety defined over F.

These classes of sets are stable under boolean operations and projections [Lipshitz and Robinson 2000b, Corollary 4.3], admit cell decompositions [Cluckers et al. 2006, Theorem 7.4], a natural notion of dimension (in fact, they are b-minimal in the sense of [Cluckers and Loeser 2007]), as well as a natural notion of smoothness.

Lemma 4.2. *Let F be a finite extension of $\mathbf{Q}_p$ contained in $\mathbf{C}_p$ and let V be an algebraic variety over F. Let Z be a rigid F-subanalytic subset of $V(\mathbf{C}_p)$. Then $Z(F) = Z \cap V(F)$ is an F-subanalytic subset of $V(F)$.*

Proof. We may assume that $V = \mathbf{A}^n$. Then Z can be defined by a quantifier-free formula of the above-mentioned variant of Lipshitz's analytic language, and our claim follows from the very definition of this language. □

4.3. A *block* in $\mathbf{Q}_p^n$ is either empty, or a singleton, or a smooth subanalytic subset of pure dimension $d > 0$ which is contained in a smooth semialgebraic subset of dimension d.

A *family of blocks* in $\mathbf{Q}_p^n \times \mathbf{Q}_p^s$ is a subanalytic subset W such that there exists an integer $t \geq 0$ and a semialgebraic set $Z \subset \mathbf{Q}_p^n \times \mathbf{Q}_p^t$ such that for every $\sigma \in \mathbf{Q}_p^s$, there exists $\tau \in \mathbf{Q}_p^t$ such that the fibers W_σ and Z_τ are smooth of the same dimension, and $W_\sigma \subset Z_\tau$. (In particular, the sets W_σ, for $\sigma \in \mathbf{Q}_p^s$, are blocks in $\mathbf{Q}_p^n$.)

Let F be a finite extension of $\mathbf{Q}_p$. Considering Weil restriction, we deduce from these notions the definition of a block in F^n, or of a family of blocks in $F^n \times \mathbf{Q}_p^t$.

4.4. Let H be the standard height function on $\overline{\mathbf{Q}}$; for $x \in \mathbf{Q}$, written as a fraction a/b in lowest terms, one has $H(x) = \max(|a|, |b|)$. We also write H for the height function on $\overline{\mathbf{Q}}^n$ defined by $H(x_1, \dots, x_n) = \max_i(H(x_i))$. Viewing $\mathrm{GL}(d, \overline{\mathbf{Q}})$ as a subspace of $\overline{\mathbf{Q}}^{d^2}$, it defines a height function on $\mathrm{GL}(d, \overline{\mathbf{Q}})$. There exists a strictly positive real number c such that $H(gg') \le cH(g)H(g')$ for every $g, g' \in \mathrm{GL}(d, \overline{\mathbf{Q}})$, and $H(g^{-1}) \ll H(g)^c$ for every $g \in \mathrm{GL}(d, \overline{\mathbf{Q}})$. When $d = 2$ and $g \in \mathrm{SL}(2, \overline{\mathbf{Q}})$, one even has $H(g^{-1}) = H(g)$.

Consider $g \in \mathrm{GL}(d, \overline{\mathbf{Q}})$. If g is diagonal, then $H(g^n) = H(g)^n$ for every $n \in \mathbf{Z}$. More generally, if g is *semisimple*, then we have upper and lower bounds $H(g)^n \ll H(g^n) \ll H(g)^n$ for every $n \in \mathbf{Z}$.

By abuse of language, if G is a linear algebraic $\overline{\mathbf{Q}}$-group, we implicitly choose an embedding in some linear group, which furnishes a height function H on $G(\overline{\mathbf{Q}})$.

The actual choice of this height function depends on the chosen embedding, but any other height function H' is equivalent, in the sense that there is a strictly positive real number c such that $H(x)^{1/c} \ll H'(x) \ll H(x)^c$ for every $x \in G(\overline{\mathbf{Q}})$.

4.5. Let Z be a subset of F^n and let K be a finite extension of $\mathbf{Q}$ contained in F. We write $Z(K) = Z \cap K^n$ (K-rational points of Z). For every real number T, we define $Z(K; T) = \{x \in Z(K) : H(x) \le T\}$; for every integer D, we also define $Z(D; T)$ to be the set of points $x \in Z(F)$ such that $[\mathbf{Q}(x_i) : \mathbf{Q}] \le D$ for every $i \in \{1, \dots, n\}$ and $H(x) \le T$. These are finite sets.

We say that Z has *many K-rational points* if there exist strictly positive real numbers c, α such that

$$\mathrm{Card}(Z(K; T)) \ge cT^\alpha$$

for all T large enough. This notion only depends on the equivalence class of the height.

4.6. In [Cluckers et al. 2015], Cluckers, Comte and Loeser established a p-adic analogue of a theorem of Pila and Wilkie [2006] concerning the rational points of a definable set. We will use the following variant of [Cluckers et al. 2015, Theorem 4.2.3].

Theorem 4.7. *Let F be a finite extension of $\mathbf{Q}_p$ and let K be a finite extension of $\mathbf{Q}$ contained in F. Let $Z \subset F^n$ be a $\mathbf{Q}_p$-subanalytic subset. Let $\varepsilon > 0$. There exist $s \in \mathbf{N}$, $c \in \mathbf{R}$ and a family of blocks $W \subset Z \times \mathbf{Q}_p^s$ satisfying the following property: for every $T > 1$, there exists a subset $S_T \subset \mathbf{Q}_p^s$ of cardinality $< cT^\varepsilon$ such that $Z(K; T) \subset \bigcup_{\sigma \in S_T} W_\sigma$.*

Proof. Let $d = [F : \mathbf{Q}_p]$. By Krasner's lemma, there exists an algebraic number $e \in F$ of degree d such that $F = \mathbf{Q}_p(e)$. Then the basis $(1, e, \dots, e^{d-1})$ defines a $\mathbf{Q}_p$-linear bijection $\psi : \mathbf{Q}_p^d \xrightarrow{\sim} F$, $(x_1, \dots, x_d) \mapsto \sum x_i e^{i-1}$. Let $\varphi : F \simeq \mathbf{Q}_p^d$ be its inverse.

By construction, if K is a number field contained in F and $x \in K^d$, then $\psi(x) \in K(e)$; in particular, $[\mathbf{Q}(\psi(x)) : \mathbf{Q}] \le d[\mathbf{Q}(x) : \mathbf{Q}]$. Conversely, if $x \in K$, then the coordinates of $\varphi(x)$ in $\mathbf{Q}_p^d$ belong to the Galois closure $K(e)'$ of the compositum $K \cdot \mathbf{Q}(e)$, hence are algebraic numbers of degrees $\le D = [K(e)' : \mathbf{Q}]$. In other words, φ and ψ induce bijections at the level of algebraic points. Since these maps are linear, there exists a positive real number $a > 0$ such that $a^{-1}H(x) \le H(\varphi(x)) \le aH(x)$ for every $x \in K$.

We deduce from φ a $\mathbf{Q}_p$-linear isomorphism $\varphi : F^n \to \mathbf{Q}_p^{nd}$. In particular, $Z' = \varphi(Z)$ is a subanalytic subset of $\mathbf{Q}_p^{nd}$. The morphism φ maps algebraic points of given degree to algebraic points of uniformly bounded degree, and there exists a positive real number $a > 0$ such that $a^{-1}H(x) \le H(\varphi(x)) \le aH(x)$ for every $x \in Z(K)$.

The definition of a family of blocks that we have adopted here is slightly stronger than the one used in Theorem 4.2.3 of [Cluckers et al. 2015]. However, all proofs go over without any modification, so that there exists a family of blocks $W' \subset Z' \times \mathbf{Q}_p^s$ such that for any $T > 1$, there exists a subset $S_T \subset \mathbf{Q}_p^s$ of cardinality $< cT^\varepsilon$ such that $Z'(D; T) \subset \bigcup_{\sigma \in S_T} W'_\sigma$. Let $\psi : F^n \times \mathbf{Q}_p^s \to \mathbf{Q}_p^{nd} \times \mathbf{Q}_p^s$ be the map $(x, y) \mapsto (\varphi(x), y)$ and let $W = \psi^{-1}(W') \subset F^n \times \mathbf{Q}_p^s$. By definition, W is a family of blocks in Z. Moreover, for any $T > 1$, one has

$$Z(F; T) \subset \psi^{-1}(Z'(D; aT)) \subset \bigcup_{\sigma \in S_{aT}} \varphi^{-1}(W'_\sigma) = \bigcup_{\sigma \in S_{aT}} W_\sigma.$$

Since $\mathrm{Card}(S_{aT}) \le ca^\varepsilon T^\varepsilon$, the family of blocks W satisfies the requirements of the theorem. □

5. Zariski closures and analytic functions

5.1. Let F be a complete nonarchimedean valued field. Let V be an F-scheme of finite type. One says that a subset K of V^{an} is *sparse* if there exist a set T and a subset Z of $V^{\mathrm{an}} \times T$ such that for every $t \in T$, $Z_t = \{x \in V^{\mathrm{an}} : (x, t) \in Z\}$ is a Zariski-closed subset of V^{an} with empty interior, and $K = \bigcup_{t \in T} Z_t$.

Lemma 5.2. *A sparse set has empty interior.*

Proof. Let us say that a point $x \in V^{\mathrm{an}}$ is maximally Abhyankar if the rational rank of the value group of $\mathscr{H}(x)$ is equal to $\dim_x(V^{\mathrm{an}})$. If V is irreducible, then maximally Abhyankar points are dense in V^{an}; moreover, each of them is Zariski dense. Let K be a sparse set in V^{an}; write $K = \bigcup_t Z_t$ as above. Let us argue by contradiction and let U be a nonempty subset of V^{an} contained in K. By what precedes, there exists a maximally Abhyankar point $x \in U$. Let $t \in T$ be such that $x \in Z_t$. Then Z_t contains the Zariski closure of x in V^{an}, so that Z_t contains an irreducible component of V^{an}, contradicting the definition of a sparse set. □

Lemma 5.3. *Let F' be an algebraically closed complete extension of F and $q : V^{\mathrm{an}}_{F'} \to V^{\mathrm{an}}$ the base change morphism. Let K be a closed sparse subset of V^{an} and let $K' = q^{-1}(K)$. Then K' is sparse.*

Proof. Indeed, if $K = \bigcup_{t \in T} Z_t^{\mathrm{an}}$ is a description of the sparse set K, then the equality $K' = \bigcup_{t \in T} (Z_t)^{\mathrm{an}}_{F'}$ shows that K' is sparse as well. □

Lemma 5.4. *Let us assume that K is sparse, and let $C \subset V$ be a geometrically irreducible curve such that $C^{\mathrm{an}} \not\subset K$. Then $C^{\mathrm{an}} - K$ is connected.*

Proof. Using Lemma 5.3, we reduce to the case where F is algebraically closed; moreover, we may assume that C is reduced. Let $K = \bigcup_{t \in T} Z_t^{\mathrm{an}}$ be a description of K as above. Up to adding the singular locus of C to K, we may assume that C is smooth. By assumption, for every $t \in T$, $C \not\subset Z_t^{\mathrm{an}}$; consequently, $Z_t^{\mathrm{an}} \cap C^{\mathrm{an}}$ consists of rigid points of C^{an}, and hence $K \cap C^{\mathrm{an}}$ consists of rigid points of C^{an}. In the topological description of smooth geometrically irreducible analytic curves as real graphs [Berkovich 1990, Chapter 4], their rigid points are endpoints, so $C^{\mathrm{an}} - (K \cap C^{\mathrm{an}})$ is connected as well. □

Proposition 5.5. *Let F be a complete nonarchimedean valued field. Let V be an F-scheme of finite type which is geometrically connected (resp. geometrically irreducible) and let K be a closed sparse subset of V^{an}. Then $V^{\mathrm{an}} - K$ is a geometrically connected (resp. geometrically irreducible) analytic space.*

The particular case $K = \varnothing$ implies the "GAGA"-type consequence that if V is geometrically connected (or geometrically irreducible), then so is V^{an}.

Proof. Using Lemma 5.3, we reduce to the case where F is algebraically closed. By assumption, V is connected. Let us prove that $V^{\mathrm{an}} - K$ is connected. Let $x, y \in V^{\mathrm{an}} - K$. Let F' an algebraically closed complete valued field containing both $\mathscr{H}(x)$ and $\mathscr{H}(y)$, and view x, y as elements of $V(F')$. Let $q : V^{\mathrm{an}}_{F'} \to V^{\mathrm{an}}$ be the base change morphism and let $K' = q^{-1}(K)$; by Lemma 5.3, this is a sparse subset of $V^{\mathrm{an}}_{F'}$. By [Mumford 1970, p. 56], there exists an irreducible curve $C \subset V_{F'}$ which passes through x and y. Then C^{an} is connected. One has $C \not\subset K'$, by definition of K'; it follows from Lemma 5.4 that $C^{\mathrm{an}} - (K' \cap C^{\mathrm{an}})$ is connected. Consequently, x and y belong to the same component of $V^{\mathrm{an}}_{F'} - K'$, and hence their images in $V^{\mathrm{an}} - K$ belong to the same connected component. This proves that $V^{\mathrm{an}} - K$ is connected.

Let us now assume that V is geometrically irreducible. The normalization morphism $p : W \to V$ is finite, and W is geometrically connected. Since $p^{-1}(K)$ is a sparse subset of W^{an}, it follows from the first part of the lemma that $W^{\mathrm{an}} - p^{-1}(K)$ is geometrically connected. Since W^{an} is the normalization of V^{an} [Ducros 2016, Lemma 2.7.15], then $W^{\mathrm{an}} - p^{-1}(K) = p^{-1}(V^{\mathrm{an}} - K)$ is the normalization of $V^{\mathrm{an}} - K$. By Theorem 5.17 of [Ducros 2009], this implies that $V^{\mathrm{an}} - K$ is geometrically irreducible. □

Corollary 5.6. *Let F be a complete valued field, let V be an F-scheme of finite type and let K be a closed sparse subset of V^{an}. The set of irreducible components of $V^{\mathrm{an}} - K$ is finite. If V is equidimensional, then each of them has dimension* $\dim(V)$.

Proof. We may assume that V is irreducible. Let $\Omega = V^{\mathrm{an}} - K$. Let E be the completion of an algebraic closure of F. By Proposition 5.5, $\Omega_E \cap Z^{\mathrm{an}}$ is irreducible for every irreducible component Z of V_E, and the family of these intersections is the family of irreducible components of Ω_E. The finiteness statement then follows from [Ducros 2009, Lemme 4.25], while the one about dimension follows from [Ducros 2009, Proposition 4.22]. □

Corollary 5.7. *Let F be a complete valued field, let V be an irreducible F-scheme of finite type and let K be a closed sparse subset of V^{an}. Let W be an irreducible component of $V^{\mathrm{an}} - K$. If W is geometrically irreducible, then V is geometrically irreducible as well, one has $W = V^{\mathrm{an}} - K$ and W is topologically dense in V^{an}.*

Proof. Let E be a complete algebraically closed extension of F, and let $V_1, \dots, V_n$ be the irreducible components of V_E. Let L be the preimage of K in V_E; it is a closed sparse subset of V_E^{an} (Lemma 5.3). Consequently, $L_j = V_j^{\mathrm{an}} \cap L$ is a closed sparse subset of V_j^{an}, for every j. By Proposition 5.5, $W_j = V_j^{\mathrm{an}} - L_j$ is geometrically irreducible. The automorphism group $\mathrm{Aut}(E/F)$ acts transitively on the set $\{V_1, \dots, V_n\}$ of irreducible components of V_E, hence on the set $\{W_1, \dots, W_n\}$ of irreducible components of $V_E^{\mathrm{an}} - L$. Since V_E is geometrically irreducible, there exists an index j such that $W_E = W_j$; then $\mathrm{Aut}(E/F)$ fixes W_j, so that $n = 1$ and $j = 1$. This proves that V is geometrically irreducible. By Proposition 5.5, one has $W = V^{\mathrm{an}} - K$. By Lemma 5.2, W is topologically dense in V^{an}. □

Proposition 5.8. *Let F be a finite extension of $\mathbf{Q}_p$. Let A be an affine scheme of finite type over F and let $\Omega \subset A^{\mathrm{an}}$ be the complement of a closed sparse subset. Let X be a closed analytic subspace of Ω. Let V be a $\mathbf{Q}_p$-semialgebraic subset of $A(F)$, contained in $X(F)$, and let W be its Zariski closure in A. Then $W^{\mathrm{an}} \cap \Omega \subset X$.*

Proof. This proof is inspired by that of [Pila and Tsimerman 2013, Lemma 4.1].

We argue by noetherian induction on W, assuming that if W' is the Zariski closure of a $\mathbf{Q}_p$-semialgebraic subset V' of $A(F)$ contained in $X(F)$, and if $W' \subsetneq W$, then $(W')^{\mathrm{an}} \cap \Omega \subset X$.

First assume that W is not irreducible. Then any irreducible component W' of W is the Zariski closure in A of $V \cap W'(F)$, a $\mathbf{Q}_p$-semialgebraic subset of $A(F)$; by induction, $(W')^{\mathrm{an}} \cap \Omega \subset X$, so that $W^{\mathrm{an}} \cap \Omega \subset X$.

We may thus assume that W is irreducible; since its subset $W(F)$ of F-rational points contains V, it is Zariski-dense in W, so that W is geometrically irreducible.

Let $K = A^{\mathrm{an}} - \Omega$. By assumption, K is closed and sparse. Let $K = \bigcup S_t^{\mathrm{an}}$ be a presentation of K, where for every t, S_t is a Zariski-closed subset with empty interior of A. Since W is irreducible and not contained in S_t, $W \cap S_t$ is a strict Zariski-closed subset of W. Consequently, $W^{\mathrm{an}} \cap K$ is a sparse subset of W^{an}. By Proposition 5.5, $W^{\mathrm{an}} \cap \Omega$ is thus a geometrically irreducible analytic space.

Let R be the Weil restriction functor from F to $\mathbf{Q}_p$. By definition, $A(F)$ is identified with $\mathrm{R}(A)(\mathbf{Q}_p)$ and we write $\mathrm{R}(V)$ for the image of V inside $\mathrm{R}(A)(\mathbf{Q}_p)$. Let then Z be the Zariski closure of $\mathrm{R}(V)$ inside $\mathrm{R}(A)$.

Let Z' be an irreducible component of Z. Then $Z' \cap \mathrm{R}(V)$ is a semialgebraic subset of $\mathrm{R}(A)$, of the form $\mathrm{R}(V')$, for a unique $\mathbf{Q}_p$-semialgebraic subset V' of V. When Z' varies, the corresponding subsets V' cover V; we may thus choose Z' such that V' is Zariski dense in W. Replacing V by V', we may assume that Z is irreducible; then it is geometrically irreducible, because its set of $\mathbf{Q}_p$-points is Zariski dense.

Since V is $\mathbf{Q}_p$-semialgebraic, the subset $\mathrm{R}(V)$ of $\mathrm{R}(A)(\mathbf{Q}_p)$ is semialgebraic; hence, the dimension of Z coincides with the dimension of V as a $\mathbf{Q}_p$-semialgebraic subset of $A(F)$. Consequently, $\dim_{\mathrm{Zar}}(Z) = \dim(Z(\mathbf{Q}_p)) = \dim(\mathrm{R}(V))$.

Since W is a Zariski closed subset of A containing V, the subscheme $\mathrm{R}(W)$ is Zariski closed in $\mathrm{R}(A)$ and contains $\mathrm{R}(V)$, so that $Z \subset \mathrm{R}(W)$. By Weil restriction, the inclusion $Z \to \mathrm{R}(W)$ corresponds to a morphism $g : Z_F \to W$. Let $x \in A(F)$ and let $\tilde{x} \in \mathrm{R}(A)(\mathbf{Q}_p)$ be the corresponding point; if $x \in V$, then $\tilde{x} \in \mathrm{R}(V) \subset Z(\mathbf{Q}_p)$, and hence $\tilde{x} \in Z_F(F)$. By the definition of the Weil restriction functor, one has $g(\tilde{x}) = x$. In particular, the image of $Z_F(F)$ under g contains V. Hence, g is dominant, by definition of W.

The morphism g induces an analytic morphism $g^{\mathrm{an}} : Z_F^{\mathrm{an}} \to W^{\mathrm{an}} \subset A^{\mathrm{an}}$. The inverse image of $W^{\mathrm{an}} \cap \Omega$ is the complement of a closed sparse subset of Z_F^{an}; since Z_F^{an} is geometrically irreducible, Corollary 5.6 implies that $(g^{\mathrm{an}})^{-1}(W^{\mathrm{an}} \cap \Omega)$ is geometrically irreducible, of dimension $\dim(Z_F^{\mathrm{an}})$. Let $Y = (g^{\mathrm{an}})^{-1}(W^{\mathrm{an}} \cap X)$; it is a Zariski closed analytic subset of $(g^{\mathrm{an}})^{-1}(W^{\mathrm{an}} \cap \Omega)$.

Let us admit for a moment that $\dim(Y) = \dim(Z_F)$ and let us conclude that $W^{\mathrm{an}} \cap \Omega \subset X$. Since $\dim(Z_F^{\mathrm{an}}) = \dim(Z_F) = \dim((g^{\mathrm{an}})^{-1}(W^{\mathrm{an}} \cap \Omega))$, we see that

$$Y = (g^{\mathrm{an}})^{-1}(W^{\mathrm{an}} \cap X) = (g^{\mathrm{an}})^{-1}(W^{\mathrm{an}} \cap \Omega).$$

The morphism $g : Z_F \to W$ being dominant, its image contains a nonempty open subset W' of W. Since W is geometrically irreducible, $(W')^{\mathrm{an}}$ is dense in W^{an};

in particular, the image of g^{an} meets any nonempty open subset of W^{an}. Since $(g^{\mathrm{an}})^{-1}(W^{\mathrm{an}} \cap (\Omega - X))$ is empty, by the preceding equality, this implies that $W^{\mathrm{an}} \cap (\Omega - X)$ is empty; hence, $W^{\mathrm{an}} \cap \Omega = W^{\mathrm{an}} \cap X$.

It remains to prove the equality $\dim(Y) = \dim(Z_F)$.

Let us consider a semialgebraic cell decomposition of $\mathrm{R}(A)(\mathbf{Q}_p)$ which is adapted to $\mathrm{R}(V)$, $Z(\mathbf{Q}_p)$, $Z_{\mathrm{sing}}(\mathbf{Q}_p)$, and to their singular loci: a finite partition of $\mathrm{R}(A)(\mathbf{Q}_p)$ into "open cells" such that these $\mathbf{Q}_p$-semialgebraic subsets are unions of cells; see [Denef 1986] and also [Cluckers and Loeser 2007].

Let $\widetilde{C}$ be a cell of dimension $\dim(\mathrm{R}(V))$ which is contained in $\mathrm{R}(V)$. Since

$$\dim(Z_{\mathrm{sing}}(\mathbf{Q}_p)) \leq \dim(Z_{\mathrm{sing}}) < \dim(Z) = \dim(\mathrm{R}(V)),$$

the cell $\widetilde{C}$ is disjoint from $Z_{\mathrm{sing}}(\mathbf{Q}_p)$. By definition of a cellular decomposition, $\widetilde{C}$ is open in $\mathrm{R}(V)$ and in $(Z - Z_{\mathrm{sing}})(\mathbf{Q}_p)$.

Let C be the subset of V corresponding to $\widetilde{C}$. Since the identification of C with $\widetilde{C}$ provided by the Weil restriction functor is a homeomorphism which respects the singular loci, C is an open subset of V.

Let x be a point of C and let $\tilde{x}$ be the corresponding point of $\widetilde{C}$. By what precedes, $\mathrm{R}(V)$, $Z(\mathbf{Q}_p)$ and Z are smooth at $\tilde{x}$, so that $\mathrm{T}_{\tilde{x}}(\mathrm{R}(V)) = \mathrm{T}_{\tilde{x}}(Z(\mathbf{Q}_p)) = \mathrm{T}_{\tilde{x}}(Z)$. In particular, these three $\mathbf{Q}_p$-vector spaces have the same dimension, equal to $\dim(T_x(V)) = \dim(V)$.

Since $g(\tilde{x}) = x \in X$, one has $\tilde{x} \in Y$; more generally, $\widetilde{C} \subset Y$. The tangent space $\mathrm{T}_{\tilde{x}}(Y)$ of Y at $\tilde{x}$ is an F-vector subspace of $\mathrm{T}_{\tilde{x}}(Z_F) = (\mathrm{T}_{\tilde{x}}(Z))_F$ which contains $\mathrm{T}_{\tilde{x}}(\widetilde{C}) = \mathrm{T}_{\tilde{x}}(Z)$. Consequently, $\mathrm{T}_{\tilde{x}}(Y) = \mathrm{T}_{\tilde{x}}(Z_F)$. This implies that the analytic space Y has dimension $\dim(Z_F)$, and concludes the proof. □

6. Complements on p-adic Schottky groups and uniformization

Let F be a finite extension of $\mathbf{Q}_p$. Unless specified otherwise, analytic spaces are F-analytic spaces.

6.1. Let $a \in F$ and $r \in \mathbf{R}_{>0}$; as usual, we let $B(a, r)$ and $E(a, r)$ be the subsets of $(\mathbf{A}^1)^{\mathrm{an}}$ of points x such that $|T(x) - a| < r$ and $|T(x) - a| \leq r$, respectively. The subspace $B(a, r)$ is called a *bounded open disk*; we say that $E(a, r)$ is the corresponding *bounded closed disk*. If B is a bounded open disk, we write B^+ for the corresponding bounded closed disk. We say that such a disk is strict if its *radius* r belongs to $|F^\times|^{\mathbf{Q}}$.

To these disks, we also add the unbounded open disks $\mathbf{P}_1^{\mathrm{an}} - E(a, r)$ and the unbounded closed disks $\mathbf{P}_1^{\mathrm{an}} - B(a, r)$. An unbounded disk is said to be strict if its complementary disk is strict.

The image by an homography $\gamma \in \mathrm{PGL}(2, F)$ of an open (resp. closed, strict) disk is again an open (resp. closed, strict) disk.

6.2. We endow $\mathbf{P}_1(\mathbf{C}_p)$ with the distance given by

$$\delta(x, y) = \frac{|x - y|}{\max(1, |x|)\max(1, |y|)}$$

for $x, y \in \mathbf{C}_p$ — it is invariant under the action of $\mathrm{PGL}(2, \mathscr{O}_{\mathbf{C}_p})$. Moreover, an elementary calculation shows that every element $g \in \mathrm{PGL}(2, \mathbf{C}_p)$ is Lipschitz for this distance; see also Theorem 1.1.1 of [Rumely 1989].

6.3. Let Γ be a Schottky group in $\mathrm{PGL}(2, F)$, $\mathscr{L}_\Gamma \subset \mathbf{P}_1(F)$ its limit set and $\Omega_\Gamma = \mathbf{P}_1^{\mathrm{an}} - \mathscr{L}_\Gamma$. For any rigid point $x \in \Omega_\Gamma$, let $\delta_\Gamma(x)$ be the δ-distance of x to $\mathscr{L}_\Gamma$.

For every $\gamma \in \mathrm{PGL}(2, F)$, there exists a real number $c \geq 1$ such that $c^{-1}\delta_\Gamma(z) \leq \delta_\Gamma(\gamma \cdot z) \leq c\delta_\Gamma(z)$ for every rigid point $z \in \Omega_\Gamma$.

Lemma 6.4. *Let $\mathfrak{G}$ be a compact subset of Ω_Γ. There exists a strictly positive real number c such that $\delta_\Gamma(x) \geq c$ for every rigid point $x \in \mathfrak{G}$.*

Proof. Arguing by contradiction, we assume that there exists a sequence (x_n) of rigid points of $\mathfrak{G}$ such that $\delta_\Gamma(x_n) \to 0$. For every n, let $\xi_n \in \mathscr{L}_\Gamma$ such that $\delta_\Gamma(x_n) = \delta(x_n, \xi_n)$; it exists since $\mathscr{L}_\Gamma$ is compact. Extracting a subsequence if necessary, we assume that the sequence (ξ_n) converges to a point ξ of $\mathscr{L}_\Gamma$. Then $\delta(x_n, \xi) \to 0$. This implies that the sequence (x_n) converges to ξ in the Berkovich space $\mathbf{P}_1^{\mathrm{an}}$. Since $\mathfrak{G}$ is compact, one has $\xi \in \mathfrak{G}$, a contradiction. □

6.5. Let Γ be a Schottky subgroup of $\mathrm{PGL}(2, F)$. Let us assume that the point at infinity ∞ does not belong to its limit set $\mathscr{L}_\Gamma$. Then, by [Gerritzen and van der Put 1980, I, (4.3)], the group Γ admits a basis $(\gamma_1, \dots, \gamma_g)$ and a *good fundamental domain* $\mathfrak{F}_\Gamma$ with respect to this basis, in the following sense:

(1) There exists a finite family $(B_1, \dots, B_g, C_1, \dots, C_g)$ of strict bounded open disks in $\mathbf{P}_1^{\mathrm{an}}$ such that $\mathfrak{F}_\Gamma = \mathbf{P}_1^{\mathrm{an}} - \big(\bigcup B_i \cup \bigcup C_i\big)$.

(2) The corresponding bounded closed disks $B_1^+, \dots, B_g^+, C_1^+, \dots, C_g^+$ are pairwise disjoint.

Let then $\mathfrak{F}_\Gamma^\circ = \mathbf{P}_1^{\mathrm{an}} - \big(\bigcup B_i^+ \cup \bigcup C_i^+\big)$.

(3) The elements $\gamma_1, \dots, \gamma_g$ satisfy $\gamma_i(\mathbf{P}_1^{\mathrm{an}} - B_i) = C_i^+$ and $\gamma_i(\mathbf{P}_1^{\mathrm{an}} - B_i^+) = C_i$ for every $i \in \{1, \dots, g\}$.

With this notation, let $W = \mathbf{P}_1^{\mathrm{an}} - \bigcup B_i$; this is an affinoid domain of $\mathbf{P}_1^{\mathrm{an}}$ containing $\mathfrak{F}$, stable under each γ_i. Indeed, one has $W \subset \mathbf{P}_1^{\mathrm{an}} - B_i$. Hence, $\gamma_i W \subset \gamma_i(\mathbf{P}_1^{\mathrm{an}} - B_i) = C_i^+$, and hence the claim since C_j^+ is disjoint from each B_i.

Moreover, the following properties are satisfied:

(4) One has $\bigcup_{\gamma \in \Gamma} \gamma \cdot \mathfrak{F}_\Gamma = \mathbf{P}_1 - \mathscr{L}_\Gamma$.

(5) For $\gamma \in \Gamma$, one has $\mathfrak{F}_\Gamma \cap \gamma \cdot \mathfrak{F}_\Gamma \neq \varnothing$ if and only if $\gamma \in \{\mathrm{id}, \gamma_1^{\pm 1}, \dots, \gamma_g^{\pm 1}\}$.

(6) For every $\gamma \in \Gamma - \{\mathrm{id}\}$, one has $\mathfrak{F}_\Gamma^\circ \cap \gamma \cdot \mathfrak{F}_\Gamma = \varnothing$.

In this context, we identify an element γ of Γ with a reduced word in the letters $\{\gamma_1^\pm, \dots, \gamma_g^\pm\}$ and denote its length by $\ell_\Gamma(\gamma)$.

For every $\gamma \in \Gamma - \{\mathrm{id}\}$, [Gerritzen and van der Put 1980, I, §4, p. 29] define a bounded open disk $B(\gamma)$, equal either to $\gamma \cdot (\mathbf{P}_1^{\mathrm{an}} - B_i^+)$ or to $\gamma \cdot (\mathbf{P}_1^{\mathrm{an}} - C_i^+)$, according to whether the last letter of the reduced word representing γ is γ_i or γ_i^{-1}; in any case, one has $\gamma \cdot \infty \in B(\gamma)$. Moreover, they prove:

(7) $B(\gamma') \subset B(\gamma)$ if and only if γ is an initial subword of γ'.

(8) For every integer n, one has

$$\mathbf{P}_1^{\mathrm{an}} - \bigcup_{\ell_\Gamma(\gamma) < n} \gamma \cdot \mathfrak{F} = \bigcup_{\ell_\Gamma(\gamma) = n} B(\gamma).$$

(9) There exists a real number $c > 1$ such that for every γ, the radius of the disk $B(\gamma)$ is $\ll c^{-\ell_\Gamma(\gamma)}$.

(10) The intersection of every decreasing sequence of open disks $(B(\gamma_n))$, where $\ell_\Gamma(\gamma_n) = n$, is reduced to a limit point of Γ, and every limit point can be obtained in this way.

Proposition 6.6. *Let Γ be a Schottky group in* $\mathrm{PGL}(2, F)$ *and let $\mathfrak{G}$ be a compact analytic domain of Ω_Γ. There exist positive real numbers a, b such that for every $\gamma \in \Gamma$ and every rigid point $x \in \gamma \cdot \mathfrak{G}$, one has*

$$\ell_\Gamma(\gamma) \le a - b \log(\delta_\Gamma(x)).$$

Proof. To prove this proposition, we may extend the scalars to a finite extension of F and henceforth assume that the limit set $\mathscr{L}_\Gamma$ is not equal to $\mathbf{P}_1(F)$. Placing a point of $\mathbf{P}_1(F) - \mathscr{L}_\Gamma$ at infinity, Section 6.5 furnishes a basis $(\gamma_1, \dots, \gamma_g)$ and a good fundamental domain with respect to this basis of the form $\mathfrak{F} = \mathbf{P}_1^{\mathrm{an}} - \big(\bigcup_{i=1}^g B_i \cup \bigcup_{i=1}^g C_i\big)$. Let b and $c > 1$ be positive real numbers such that the diameter of $B(\gamma)$ is bounded by $bc^{-\ell_\Gamma(\gamma)}$, for every $\gamma \in \Gamma - \{\mathrm{id}\}$.

Let $x \in \Omega_\Gamma$ and let $\gamma \in \Gamma$ be such that $x \in \gamma \cdot \mathfrak{F}$. Let $\xi \in \mathscr{L}_\Gamma(x)$ be such that $\delta_\Gamma(x) = \delta(x, \xi)$. As the disk $B(\gamma)$ contains both x and ξ, one has $\delta_\Gamma(x) \le bc^{-\ell_\Gamma(\gamma)}$, that is,

$$\ell_\Gamma(\gamma) \le \frac{1}{\log(c)}(-\log(\delta_\Gamma(x)) + \log(b)),$$

since $\log(c) > 0$. This proves the proposition in the particular case where $\mathfrak{G} = \mathfrak{F}$.

Let us now prove the general case. Let a be a real number such that $\delta_\gamma(x) \ge a > 0$ for every rigid point of $\mathfrak{G}$ (Lemma 6.4). The preceding inequality shows that there exists a finite subset S of Γ such that $\mathfrak{G}$ meets $\gamma \cdot \mathfrak{F}$ if and only if $\gamma \in S$. It then follows from property (8) that $\mathfrak{G}$ is contained in the finite union $\bigcup_{s \in S} s \cdot \mathfrak{F}$. To conclude the proof, we observe that if $x \in \gamma \cdot \mathfrak{G}$, then there exists $s \in S$ such that

$x \in \gamma s \cdot \mathfrak{F}$. The proposition then follows from the particular case already treated and from the inequality $\ell_\Gamma(\gamma) \leq \ell_\Gamma(\gamma s) + \ell_\Gamma(s)$. □

Corollary 6.7. *Let $\mathfrak{G}$ and $\mathfrak{G}'$ be compact analytic domains of Ω_Γ. The set of $\gamma \in \Gamma$ such that $\gamma \cdot \mathfrak{G} \cap \mathfrak{G}' \neq \varnothing$ is finite.*

Proof. Let S be this set. For $\gamma \in S$, the intersection $\gamma \cdot \mathfrak{G} \cap \mathfrak{G}'$ is a nonempty affinoid domain of $\mathbf{P}_1^{\mathrm{an}}$; hence, it contains a rigid point x_γ. With a and b as in the statement of Proposition 6.6, one has $\ell_\Gamma(\gamma) \leq a - b\log(\delta_\Gamma(x_\gamma))$. Since $x_\gamma \in \mathfrak{G}'$, $\delta_\Gamma(x_\gamma)$ is bounded from below by Lemma 6.4. This shows that $\ell_\Gamma(\gamma)$ is bounded above when γ runs over S. □

Proposition 6.8. *Let Γ be a Schottky group in* $\mathrm{PGL}(2, F)$ *and let g be its rank. Let $\xi \in \mathscr{L}_\Gamma$ and let U be an open neighborhood of ξ in $\mathbf{P}_1^{\mathrm{an}}$.*

There exist an open neighborhood U' of ξ, contained in U, a basis $\gamma_1, \dots, \gamma_g$ of Γ, an affinoid domain $\mathfrak{F} \subset \Omega_\Gamma$ such that the following properties hold:

(1) *One has $\mathfrak{F} \subset U'$.*

(2) *For every i, one has $\gamma_i(U') \subset U'$.*

(3) *One has $\bigcup_{\gamma\in\Gamma} \gamma\mathfrak{F} = \Omega_\Gamma$.*

Such an affinoid domain will be called a *fundamental set.*

Proof. We first treat the case where $\mathscr{L}_\Gamma \neq \mathbf{P}_1(F)$. Placing a point of $\mathbf{P}_1(F) - \mathscr{L}_\Gamma$ at infinity, Section 6.5 furnishes a basis $(\gamma_1, \dots, \gamma_g)$ and a good fundamental domain $\mathfrak{F}$ with respect to this basis of the form $\mathfrak{F} = \mathbf{P}_1^{\mathrm{an}} - \left(\bigcup_{i=1}^g B_i \cup \bigcup_{i=1}^g C_i\right)$.

By (10), for every integer $n \geq 1$, there is an element $\gamma \in \Gamma$ of length n such that $\xi \in B(\gamma)$; if n is large enough, one has $B(\gamma)^+ \subset U$, because the diameter of $B(\gamma)^+$ tends to 0 when $n = \ell_\Gamma(\gamma)$ tends to ∞. Since $\gamma \cdot \mathfrak{F} \subset B(\gamma)^+$, this implies that $\gamma \cdot \mathfrak{F} \subset U$.

Up to changing the basis $(\gamma_1, \dots, \gamma_g)$ into $(\gamma_1^{-1}, \dots, \gamma_g^{-1})$, and exchanging B_i and C_i for every i, we may assume that the last letter of γ is γ_s, for some $s \in \{1, \dots, g\}$. Set $W = \mathbf{P}_1^{\mathrm{an}} - \bigcup_{i=1}^g B_i$; recall that W is an affinoid domain of $\mathbf{P}_1^{\mathrm{an}}$ containing $\mathfrak{F}$ and stable under $\gamma_1, \dots, \gamma_g$. By definition, one has

$$B(\gamma)^+ = \gamma \cdot (\mathbf{P}_1^{\mathrm{an}} - B_s) \supset \gamma \cdot W,$$

since $W \subset \mathbf{P}_1^{\mathrm{an}} - B_s$.

Let us now set $\mathfrak{F}' = \gamma \cdot \mathfrak{F}$, $W' = \gamma \cdot W$ and $\gamma_i' = \gamma\gamma_i\gamma^{-1}$ for $i \in \{1, \dots, g\}$. By construction, $\mathfrak{F}'$ and W' are affinoid domains of $\mathbf{P}_1^{\mathrm{an}}$ such that $\mathfrak{F}' \subset W' \subset B(\gamma)^+ \subset U$, the translates of $\mathfrak{F}'$ under Γ cover Ω_Γ, and W' is stable under the basis $(\gamma_1', \dots, \gamma_g')$ of Γ.

This almost proves (1–3), except that W' is affinoid and not open. To conclude the construction, one sets U' to be the interior of W' and redoes the construction starting from U' instead of U. The second paragraph of the proof shows that there exists $\gamma' \in \Gamma$ such that $\gamma' \cdot \mathfrak{F}'$ is contained in U'. The affinoid $\gamma' \cdot \mathfrak{F}'$, the open subset U' and the basis $(\gamma'_1, \dots, \gamma'_g)$ satisfy the requirements of the proposition.

Let us now treat the case where $\mathscr{L}_\Gamma = \mathbf{P}_1(F)$. Let F' be a finite extension of F of degree > 1. The preceding construction can be applied starting with a point of $\mathbf{P}_1(F') \smallsetminus \mathscr{L}_\Gamma$ and furnishes an open neighborhood V' of ξ in $(\mathbf{P}_1^{\mathrm{an}})_{F'}$, contained in $U_{F'}$, a basis $(\gamma_1, \dots, \gamma_g)$ of Γ and an affinoid domain $\mathfrak{F}'$ of $\Omega_{\Gamma,F'}$ satisfying properties (1–3). The images U' of V' and $\mathfrak{F}$ of $\mathfrak{F}'$ by the projection $(\mathbf{P}_1^{\mathrm{an}})_{F'} \to \mathbf{P}_1^{\mathrm{an}}$ satisfy the required properties. □

Lemma 6.9. *Let Γ be an arithmetic Schottky group in* $\mathrm{PGL}(2, F)$ *and let H be a height function on* $\mathrm{PGL}(2, \overline{\mathbf{Q}})$. *There exists a positive real number c such that $H(\gamma) \le c^{\ell_\Gamma(\gamma)+1}$ for every $\gamma \in \Gamma$.*

Proof. Let $(\gamma_1, \dots, \gamma_g)$ be a basis of Γ as above. Let c_1 be a positive real number such that $H(hh') \le c_1 H(h)H(h')$ for every $h, h' \in \mathrm{PGL}(2, \overline{\mathbf{Q}})$. Let $c = c_1 \sup(H(\mathrm{id}), H(\gamma_1), \dots, H(\gamma_g))$. One proves by induction on $\ell_\Gamma(\gamma)$ that

$$c_1 H(\gamma) \le \sup(c_1 H(\gamma_1^{\pm}), \dots, c_1 H(\gamma_g^{\pm}))^{\ell_\Gamma(\gamma)} c_1 H(\mathrm{id}) \le c_1 c^{\ell_\Gamma(\gamma)+1}$$

for every $\gamma \in \Gamma$, as was to be shown. □

Lemma 6.10. *Let Γ be a Schottky subgroup of* $\mathrm{PGL}(2, F)$ *and let Δ be a subset of $\mathbf{P}_1(\overline{F})$ of cardinality* 2. *Let K be a number field contained in F. The stabilizer of Δ inside Γ does not have many K-rational points.*

Proof. Let S be this stabilizer; we may assume that $S \neq \{\mathrm{id}\}$. Let $g \in S \smallsetminus \{\mathrm{id}\}$. Then g is hyperbolic (see [Gerritzen and van der Put 1980, p. 7, line 2]), and hence has exactly two rational fixed points in $\mathbf{P}_1(F)$. Up to a change of projective coordinates, we may thus assume that $\Delta = \{0, \infty\}$. Then every element h of S is of the form $z \mapsto \lambda(h)z$, for some unique element $\lambda(h) \in K^\times$; moreover, unless $h = \mathrm{id}$, any such h is hyperbolic and thus is represented by a matrix having two eigenvalues with distinct absolute values, so that $|\lambda(h)| \neq 1$. Let us choose $h \in S \smallsetminus \{\mathrm{id}\}$ such that $|\lambda(h)|$ is > 1 and minimal. By euclidean division, one has $S = \langle h \rangle$.

Then $S \cap \mathrm{PGL}(2, K)$ is generated by an element of the form h^a for some $a \in \mathbf{Z}$. Since h^a is semisimple, we have $H(h^a)^n \ll H(h^{an}) \ll H(h^a)^n$, for every $n \in \mathbf{Z}$ (see Section 4.4). This shows that $S \cap \mathrm{PGL}(2, K)$ does not have many rational points. □

In Section 8, we will need the following lemma.

Lemma 6.11. *Let r be a positive real number, $f \in \mathbf{C}_p[\![z]\!]$ a power series which converges on the closed disk $E(0, r)$, and L_1 and L_2 closed subsets of $\mathbf{C}_p$ such that $f^{-1}(L_2) \subset L_1$. For every $x \in \mathbf{C}_p$, let $\delta(x; L_1)$ and $\delta(x; L_2)$ be the distances of x to L_1 and L_2, respectively. Then there exist real numbers $m \ge 0$, $c > 0$ and s such that $0 < s < r$ and such that $\delta(f(x); L_2) \ge c\delta(x; L_1)^m$ for every $x \in E(0, s)$.*

Proof. Write $f = \sum c_n z^n$. We may assume that there exists $a \in \mathbf{C}_p^\times$ such that $r = |a|$; composing f with homographies which map $E(0, r)$ to $E(0, 1)$ and $f(E(0, r))$ into the disk $E(0, 1)$, we assume that $r = 1$ and that $|c_n| \le 1$ for all n. (Recall from Section 6.2 that homographies are Lipschitz for the distance δ.)

Let us first treat the case where $f(0) \notin L_2$. Then there exists a real number $s > 0$ such that $E(f(0), s) \cap L_2 = \varnothing$. For every $x \in E(0, 1)$ such that $|x| < s$, one has $|f(x) - f(0)| < s$; hence, $\delta(f(x); L_2) > s$. It suffices to set $m = 0$ and $c = s$.

We now assume that $f(0) \in L_2$, and hence $0 \in L_1$. Let $m = \operatorname{ord}_0(f - f(0))$. Since $f'(z) = \sum_{n \ge m} n c_n z^{n-1}$, there exists a real number s such that $0 < s \le 1$ and such that $|f'(z)| = |m c_m| |z|^{m-1}$ provided $|z| \le s$. Moreover, $|f^{(n)}(z)/n!| \le 1$ for every $n \ge 0$ and any $z \in E(0, 1)$. Considering the Taylor expansion

$$f(y) = \sum_{n \ge 0} \frac{1}{n!} f^{(n)}(x)(y - x)^n,$$

we then see that there exists a real number s' such that

$$f(E(x, u)) = E(f(x), |f'(x)| u)$$

for every real number u such that $0 < u \le s'$ and $x \in E(0, 1)$ such that $0 < |x| \le s$. If $u < \delta(x; L_1)$, then $E(x, u) \cap L_1 = \varnothing$; hence, $E(f(x), |f'(x)|u) \cap L_2 = \varnothing$. Consequently, $\delta(f(x); L_2) \ge |f'(x)|\, \delta(x; L_1)$. Since $0 \in L_1$, one has $|x| \ge \delta(x; L_1)$. Consequently,

$$\delta(f(x); L_2) \ge |m c_m| |x|^{m-1} \delta(x; L_1) \ge |m c_m|\, \delta(x; L_1)^m.$$

This concludes the proof. □

7. Automorphisms of curves

The following result is already present in [Pila 2013]. For the clarity of exposition, we isolate it as a lemma.

Lemma 7.1. *Let k be an algebraically closed field of characteristic zero, B a smooth connected projective k-curve and $f : B \to \mathbf{P}_1$ a nonconstant morphism. Let $R_f \subset B$ be the ramification locus of f (the set of points of B at which f is not étale) and let $\Delta_f = f(R_f)$ be its discriminant locus.*

Assume that there exist automorphisms $g \in \operatorname{Aut}(\mathbf{P}_1)$ and $h \in \operatorname{Aut}(B)$ such that $f \circ h = g \circ f$, and that g has infinite order. Then B is isomorphic to $\mathbf{P}_1$, and one of the following cases holds:

- *The morphism f is an isomorphism (and $\Delta_f = \varnothing$).*
- *One has $\operatorname{Card}(R_f) = 2$ and $g(\Delta_f) = \Delta_f$.*

Proof. By construction, f induces a finite étale covering of $\mathbf{P}_1 - \Delta_f$.

Let $b \in R_f$. One has $df(b) = 0$; hence, $d(f \circ h)(b) = d(g \circ f)(b) = 0$. Since h is an automorphism of B, this implies that $df(h(b)) = 0$; hence, $h(b) \in R_f$. We thus have $h(R_f) \subset R_f$; hence, $h(R_f) = R_f$, because h is an isomorphism. Consequently, $g(\Delta_f) = \Delta_f$, so that some power of g fixes Δ_f pointwise. Since the identity is the only homography that fixes 3 points and g has infinite order, this implies that $\operatorname{Card}(\Delta_f) \leq 2$.

If $\operatorname{Card}(\Delta_f) \leq 1$, then $\mathbf{P}_1 - \Delta_f$ is simply connected. Hence, f is an isomorphism (and $\Delta_f = \varnothing$).

Otherwise, one has $\operatorname{Card}(\Delta_f) = 2$. Let $n = \deg(f)$. Up to a change of projective coordinates in $\mathbf{P}_1$, we may assume that $\Delta_f = \{0, \infty\}$. Then g is a homothety, because it leaves Δ_f invariant and has infinite order (otherwise, it would be of the form $g(z) = a/z$). Since all finite étale coverings of $\mathbf{P}_1 - \Delta_f$ are of Kummer type (equivalently, $\pi_1(\mathbf{P}_1 - \Delta_f) = \mathbf{Z}$), one has $B \simeq \mathbf{P}_1$ and the morphism f is conjugate to the morphism $z \mapsto z^n$ from $\mathbf{P}_1$ to itself.

We then remark that h is a homography of infinite order. Indeed, if $h^e = \mathrm{id}_B$, then $f = g^e \circ f$. Hence, $g^e = \mathrm{id}$ since f is surjective. Hence $e = 0$, since g has infinite order. As above, the formula $h(R_f) = R_f$ then implies that $\operatorname{Card}(R_f) \leq 2$. On the other hand, $\operatorname{Card}(R_f) \geq \operatorname{Card}(\Delta_f) = 2$. Hence, $\operatorname{Card}(R_f) = 2$. □

Proposition 7.2. *Let k be a field of characteristic zero. Let B be an integral k-curve in $\mathbf{P}_1^n$ possessing a smooth k-rational point. Let Γ_B be the stabilizer of B in $(\operatorname{Aut}(\mathbf{P}_1))^n$ and let $\Gamma_1 \subset \operatorname{Aut}(\mathbf{P}_1)$ be its image under the first projection. Assume that Γ_1 contains an element of infinite order. Then one of the following cases holds*:

(1) *The morphism $p_1|_B$ is constant.*

(2) *The morphism $p_1|_B$ is an isomorphism and the components of its inverse are either constant or homographies.*

(3) *There is a subset of $\mathbf{P}_1(\bar{k})$ of cardinality 2 which is invariant under every element of Γ_1.*

Proof. Assume that $p_1|_B$ is not constant. Let $\nu : B' \to B$ be the normalization of B and let $p_1' = p_1 \circ \nu : B' \to \mathbf{P}_1$. Let $g = (g_1, \dots, g_n)$ be an element of Γ_B. There exists a unique automorphism h of B' that lifts g, so $p_1' \circ h = g_1 \circ p_1'$. Since the curve B has smooth rational points, the curve B' is geometrically integral. Choosing g such that g_1 has infinite order, the preceding lemma implies that $\operatorname{Card}(R_{p_1'}) \in \{0, 2\}$.

Let us first assume that $\operatorname{Card}(R_{p_1'}) = 2$. Then $\operatorname{Card}(\Delta_{p_1'}) = 2$ as well. Moreover, the relation $p_1' \circ h = g_1 \circ p_1'$ implies that $g_1(\Delta_{p_1'}) \subset \Delta_{p_1'}$, so that case (3) holds.

Let us now assume that $\operatorname{Card}(R_{p_1'}) = 0$ and fix g such that g_1 has infinite order. By the preceding lemma, p_1' is an isomorphism; this implies that $p_1|_B$ is an isomorphism as well. Let f be its inverse and let $f_1, \dots, f_n$ be its components. Assume that case (2) does not hold, that is, for some j, the rational map f_j is neither constant, nor a homography; its ramification locus R_j is nonempty. Since g_1 has infinite order, the relation $g_j \circ f_j = f_j \circ g_1$ implies that g_j has infinite order as well. By the preceding lemma, one has $\operatorname{Card}(R_j) = 2$. Let then $g' = (g'_1, \dots, g'_n)$ be any element of Γ_B. The relation $g'_j \circ f_j = f_j \circ g'_1$ implies that $g'_1(R_j) \subset R_j$, so that case (3) holds. □

8. Proof of Theorem 2.7

We will reduce the proof of Theorem 2.7 to the following variant:

Proposition 8.1. *Let F be a finite extension of $\mathbf{Q}_p$ and let $(\Gamma_i)_{1\leq i\leq n}$ be a finite family of* arithmetic *Schottky subgroups of* $\mathrm{PGL}(2, F)$ *of ranks* ≥ 2. *As above, let us set $\Omega = \prod_{i=1}^n \Omega_{\Gamma_i}$ and $X = \prod_{i=1}^n X_{\Gamma_i}$, and let $p : \Omega \to X^{\mathrm{an}}$ be the morphism deduced from the morphisms $p_{\Gamma_i} : \Omega_{\Gamma_i} \to X^{\mathrm{an}}_{\Gamma_i}$.*

Let V be an irreducible algebraic subvariety of X and let W be an irreducible algebraic subvariety of Ω, maximal among those contained in $p^{-1}(V^{\mathrm{an}})$. If W is geometrically irreducible, then it is flat.

Lemma 8.2. *Proposition 8.1 implies Theorem 2.7.*

Proof. Let Y be the Zariski closure of W in $\mathbf{P}_1^n$; by assumption, W is an irreducible component of $Y^{\mathrm{an}} \cap \Omega$. Let W_0 be an irreducible component of $W_{\mathbf{C}_p}$. By [Ducros 2009, Théorème 7.16(v)], there exists a finite extension F' of F, contained in $\mathbf{C}_p$, and an irreducible component W' of $W_{F'}$ such that $W_0 = W'_{\mathbf{C}_p}$. Then W' is geometrically irreducible, as well as its Zariski closure Y'. By Proposition 5.5, $\Omega \cap Y'$ is geometrically irreducible. The inclusion $W' \subset \Omega \cap Y'$ and the inequality $\dim(W') = \dim(W_0) = \dim(W) = \dim(Y) \geq \dim(Y')$ imply that $W' = \Omega \cap Y'$. In particular, W' is irreducible algebraic and is contained in $p^{-1}(V^{\mathrm{an}}_{F'})$. Let us show that it is maximal. Let $W'_1 \subset \Omega_{F'}$ be an irreducible algebraic subvariety contained in $p^{-1}(V^{\mathrm{an}}_{F'})$ such that $W' \subsetneq W'_1$, and let $Y'_1 \subset (\mathbf{P}_1^n)_{F'}$ be the Zariski closure of W'_1. The image Y_1 of Y'_1 in $(\mathbf{P}_1^n)_F$ is Zariski closed, because F' is a finite extension of F, and $Y'_1 \subset (Y_1)_{F'}$. Moreover, $Y \subset Y_1$. There exists a unique irreducible component W_1 of $\Omega \cap Y_1$ that contains W, and W'_1 is an irreducible component of $W_{1,F'}$. Necessarily, W_1 is contained in $p^{-1}(V^{\mathrm{an}})$, because $W'_1 \subset p^{-1}(V^{\mathrm{an}}_{F'})$; this contradicts the maximality of W.

Applying Proposition 8.1 to W', we conclude that W' is flat. Consequently, $W_0 = W'_{\mathbf{C}_p}$ is flat, as was to be shown. □

8.3. To prove Proposition 8.1, we argue by induction and assume that it holds if there are less that n factors. Let W be an irreducible algebraic subvariety of Ω, maximal among those contained in $p^{-1}(V^{\mathrm{an}})$ and geometrically irreducible. Let Y be an irreducible subvariety of $\mathbf{P}_1^n$ such that W is an irreducible component of $Y^{\mathrm{an}} \cap \Omega$. By Corollary 5.7, Y is geometrically irreducible, $W = Y^{\mathrm{an}} \cap \Omega$ and W is topologically dense in Y.

The proof that W is flat requires intermediate steps and will be concluded in Proposition 8.11.

A crucial step will consist in proving that the stabilizer of W inside Γ has many points of bounded heights (Proposition 8.10). To that aim, we define in Section 8.7 an F-subanalytic subset R of $\mathrm{PGL}(2, F)^n$. The definition, close to that of a similar set in [Pila 2011; 2015], guarantees the following important property (Lemma 8.8): if B is a small enough subset of R then, for every $g \in B$, the translate $(g \cdot Y^{\mathrm{an}}) \cap \Omega$ is contained in $p^{-1}(V^{\mathrm{an}})$, and is independent of g. At this point, the maximality of W is invoked.

The existence of such blocks is established by applying the p-adic Pila–Wilkie theorem of [Cluckers et al. 2015]. We thus prove that R has many rational points (Lemma 8.9); these points are constructed using the action of the Schottky groups in a neighborhood of a boundary point ξ, applying material recalled in Section 6. The construction of such a point ξ, performed in Lemma 8.5, is actually the starting point of the proof.

The actual statement of Proposition 8.10 furnishes elements in Γ of a precise form. Using Proposition 7.2, we will finally conclude the proof of Proposition 8.1.

8.4. By assumption, $W = Y^{\mathrm{an}} \cap \Omega$; consequently, the j-th projection $q_j : (\mathbf{P}_1)^n \to \mathbf{P}_1$ is constant on Y if and only if it is constant on W, if and only if the j-th projection from X to X_j is constant on V, and in this case, its image is an F-rational point of $\mathbf{P}_1$, because W is geometrically irreducible. Deleting these constant factors, we thus assume that there does not exist $j \in \{1, \dots, n\}$ such that the j-th projection $q_j : (\mathbf{P}_1)^n \to \mathbf{P}_1$ is constant on Y. Consequently, $q_j|_Y : Y \to \mathbf{P}_1$ is surjective for every j; in particular, Y^{an} meets $q_j^{-1}(\mathscr{L}_{\Gamma_j})$.

Let $m = \dim(Y)$; by what precedes, we have $m > 0$, and $Y^{\mathrm{an}} \not\subset \Omega$.

Lemma 8.5. *Up to reordering the coordinates, there exists a smooth rigid point $\xi \in Y^{\mathrm{an}}$ and a connected open neighborhood U of ξ in $(\mathbf{P}_1^n)^{\mathrm{an}}$ such that the following properties hold*:

(1) *The first component $q_1(\xi)$ of ξ belongs to the limit set $\mathscr{L}_{\Gamma_1}$ of Γ_1.*

(2) *Letting $J = \{1, \dots, m\}$, the projection $q_J : \mathbf{P}_1^n \to \mathbf{P}_1^J$ induces a finite étale morphism from $U \cap Y^{\mathrm{an}}$ to its image in $(\mathbf{P}_1^J)^{\mathrm{an}}$.*

(3) *For every $j \in \{1, \dots, n\}$ and every point $y \in U \cap Y^{\mathrm{an}}$ such that $q_j(y) \in \mathscr{L}_{\Gamma_j}$, one has $q_1(y) \in \mathscr{L}_{\Gamma_1}$.*

Proof. For every subset V of Y^{an}, let us define a relation $\preceq_V$ on $\{1, \dots, n\}$ as follows: $i \preceq_V j$ if and only if, for every $y \in V$ such that $q_i(y) \in \mathscr{L}_{\Gamma_i}$, one has $q_j(y) \in \mathscr{L}_{\Gamma_j}$. This is a preordering relation. If $U \subset V \subset Y^{\mathrm{an}}$ and $i \preceq_V j$, then $i \preceq_U j$.

We define a decreasing sequence $(V_0, V_1, \dots, V_n)$ of nonempty open subsets of Y^{an} and a sequence $(j_0, j_1, \dots, j_n)$ of elements of $\{1, \dots, n\}$, such that for every k, $q_{j_k}(V_k)$ meets $\mathscr{L}_{\Gamma_{j_k}}$ and $1, \dots, k \preceq_{V_k} j_k$.

We start with $V_0 = Y^{\mathrm{an}}$. We have reduced to the case where $q_j(Y^{\mathrm{an}}) = \mathbf{P}_1$ for every j. In particular, $q_j(Y^{\mathrm{an}})$ meets $\mathscr{L}_{\Gamma_j}$. We may take $j_0 = 1$.

Let $k \geq 0$ be such that $V_0, V_1, \dots, V_k$ and $j_0, j_1, \dots, j_k$ are defined. If $k+1 \preceq_{V_k} j_k$, we set $V_{k+1} = V_k$ and $j_{k+1} = j_k$. Otherwise, one has $k+1 \not\preceq_{V_k} j_k$. Hence, there exists $y \in V_k$ such that $q_{k+1}(y) \in \mathscr{L}_{\Gamma_{k+1}}$ and $q_{j_k}(y) \notin \mathscr{L}_{\Gamma_{j_k}}$. Let $V_{k+1} = V_k \cap (q_{j_k})^{-1}(\Omega_{\Gamma_{j_k}})$; this is an open neighborhood of y in V_k such that $q_{j_{k+1}}(V_{k+1})$ meets $\mathscr{L}_{\Gamma_{j_{k+1}}}$. By construction, no element z of V_{k+1} satisfies $q_{j_k}(z) \in \mathscr{L}_{\Gamma_{j_k}}$, so that $j_k \preceq_{V_{k+1}} k+1$. We then set $j_{k+1} = k+1$.

Let $V = V_n$ and $i = j_n$, and let $y \in V$ be such that $q_i(y) \in \mathscr{L}_{\Gamma_i}$. Let Z be the dense open subscheme of Y consisting of smooth points at which dq_i does not vanish. Then Z^{an} is open and dense in Y^{an}, and $V \cap Z^{\mathrm{an}}$ is open and dense in V; hence, $q_i(V \cap Z^{\mathrm{an}})$ is dense in $q_i(V)$. Since $\mathscr{L}_{\Gamma_i}$ has no isolated points, we may assume that $y \in Z^{\mathrm{an}}$. Rigid points are dense in $q_i^{-1}(q_i(y)) \cap V \cap Z^{\mathrm{an}}$; there exists a rigid point ξ in $(q_i)^{-1}(q_i(y)) \cap V \cap Z^{\mathrm{an}}$. Since $q_i(y)$ is a rigid point, the point ξ is a rigid point of $V \cap Z^{\mathrm{an}}$ (and not only of its fiber of q_i). Moreover, $q_i(\xi) = q_i(y) \in \mathscr{L}_{\Gamma_i}$.

Since dq_i does not vanish at ξ, there exists a subset J of $\{1, \dots, n\}$ containing i such that the projection q_J from V to $(\mathbf{P}_1^J)^{\mathrm{an}}$ is finite étale at ξ. One has $\mathrm{Card}(J) = \dim(V) = m$. Consequently, there exists an open neighborhood U of ξ in $(\mathbf{P}_1^n)^{\mathrm{an}}$ such that q_J induces a finite étale morphism from $U \cap Y^{\mathrm{an}}$ to its image in $(\mathbf{P}_1^J)^{\mathrm{an}}$.

Reordering the coordinates, we may assume that $i = 1$ and $J = \{1, \dots, m\}$, hence the lemma. □

8.6. Choose ξ, $J = \{1, \dots, m\}$ and U as in the previous lemma; we may even assume that U is of the form $U_1 \times \cdots \times U_n$, where, for each i, U_i is an open neighborhood of $q_i(\xi)$ in $\mathbf{P}_1^{\mathrm{an}}$.

Let F' be a finite extension of F such that $\xi \in Y(F')$. Since W is geometrically irreducible, $W_{F'}$ is an irreducible algebraic subvariety of Ω. It is also maximal. Note that the flatness of $W_{F'}$ implies the flatness of W. Replacing F by F', we thus may assume that $\xi \in Y(F)$; then q_J induces a local isomorphism at ξ.

Let $\varphi = (\varphi_1, \dots, \varphi_n) : O \to Y^{\mathrm{an}} \cap U$ be an analytic section of $q_J|_{Y^{\mathrm{an}} \cap U}$, defined on an open neighborhood O of $q_J(\xi)$; we may assume that $O = U_1 \times \cdots \times U_m$.

By condition (3) of Lemma 8.5, $q_1(\varphi_j^{-1}(\mathscr{L}_{\Gamma_j})) \subset \mathscr{L}_{\Gamma_1}$ for every $j \in \{1, \dots, n\}$.

8.7. Let G be the $\mathbf{Q}$-algebraic group $\mathrm{PGL}(2)^n$, and let G_0 be the algebraic subgroup of G defined by

$$(g_1, \dots, g_n) \in G_0 \quad \Leftrightarrow \quad g_2 = \cdots = g_m = 1. \tag{8.7.1}$$

We denote by $q_1, \dots, q_n$ the projections of G to $\mathrm{PGL}(2)$. For every compact analytic domain $\mathfrak{F}$ of Ω, we define a subset $R_{\mathfrak{F}}$ of $G_0(F)$ by

$$g \in R_{\mathfrak{F}} \quad \Leftrightarrow \quad \dim(g \cdot Y^{\mathrm{an}} \cap \mathfrak{F} \cap p^{-1}(V^{\mathrm{an}})) = m. \tag{8.7.2}$$

Lemma 8.8. *Let $\mathfrak{F}$ be an affinoid domain of Ω.*

(1) *The set $R_{\mathfrak{F}}$ is an F-subanalytic subset of $G_0(F)$.*

(2) *For every $g \in R_{\mathfrak{F}}$, one has $(g \cdot Y^{\mathrm{an}}) \cap \Omega \subset p^{-1}(V^{\mathrm{an}})$.*

(3) *Let $M \subset R_{\mathfrak{F}}$ be a subset whose Zariski closure is irreducible; for every $g, h \in M$, one has $g \cdot Y = h \cdot Y$.*

Proof. (1) The sets V and Y are algebraic over F; hence, $V(\mathbf{C}_p)$ and $Y(\mathbf{C}_p)$ are rigid F-subanalytic. Since $\mathfrak{F}$ is affinoid, the morphism $p|_{\mathfrak{F}}$ defines a rigid F-subanalytic map from $\mathfrak{F}(\mathbf{C}_p)$ to $V(\mathbf{C}_p)$, so that $(\mathfrak{F} \cap p^{-1}(V^{\mathrm{an}}))(\mathbf{C}_p)$ is a rigid F-subanalytic set. Consequently, taking $\mathbf{C}_p$-points, $(g \cdot Y^{\mathrm{an}} \cap \mathfrak{F} \cap p^{-1}(V^{\mathrm{an}}))_g$ furnishes a rigid F-subanalytic family of rigid F-subanalytic subsets of $\Omega(\mathbf{C}_p)$, parameterized by $G_0(\mathbf{C}_p)$. By b-minimality, the set of points $g \in G_0(\mathbf{C}_p)$ such that $\dim(g \cdot Y^{\mathrm{an}} \cap \mathfrak{F} \cap p^{-1}(V^{\mathrm{an}})) = m$ is a rigid F-subanalytic subset of $G_0(\mathbf{C}_p)$. It then follows from Lemma 4.2 that $R_{\mathfrak{F}}$ is an F-subanalytic subset of $G_0(F)$.

(2) Let $g \in R_{\mathfrak{F}}$ and let us prove that $(g \cdot Y^{\mathrm{an}}) \cap \Omega \subset p^{-1}(V^{\mathrm{an}})$. Since $g \cdot Y^{\mathrm{an}}$ is irreducible and $g \cdot Y^{\mathrm{an}} \cap \mathfrak{F}$ has dimension $m = \dim(g \cdot Y^{\mathrm{an}})$, this intersection is Zariski dense in $g \cdot Y^{\mathrm{an}}$. Moreover, there exists a finite extension F' of F such that $g \cdot Y^{\mathrm{an}}_{F'} \cap \mathfrak{F}(F')$ is Zariski dense in $Y_{F'}$ (it suffices that $g \cdot Y^{\mathrm{an}} \cap \mathfrak{F}$ admits a smooth F'-point), so that the Zariski closure of $g \cdot Y^{\mathrm{an}} \cap \mathfrak{F}(F')$ in $(\mathbf{P}^n_1)_{F'}$ is equal to $g \cdot Y_{F'}$. Moreover, $g \cdot Y(F') \cap \mathfrak{F}(F')$ is F'-semialgebraic. Hence, Proposition 5.8 implies that $g \cdot Y^{\mathrm{an}}_{F'} \cap \Omega_{F'} \subset p^{-1}_{F'}(V^{\mathrm{an}}_{F'})$. Since p is defined over F and $g \in G(F)$, this implies that $(g \cdot Y^{\mathrm{an}}) \cap \Omega \subset p^{-1}(V^{\mathrm{an}})$.

(3) As a subset, $(M \cdot Y^{\mathrm{an}}) \cap \Omega$ is contained in $p^{-1}(V^{\mathrm{an}})$. By Proposition 5.8, its Zariski closure Y' satisfies $(Y')^{\mathrm{an}} \cap \Omega \subset p^{-1}(V^{\mathrm{an}})$ as well. Since Y and the Zariski closure of M are geometrically irreducible, Y' is geometrically irreducible.

Let $g \in M$; then $Y^{\mathrm{an}} \subset g^{-1} M \cdot Y^{\mathrm{an}} \subset g^{-1} \cdot (Y')^{\mathrm{an}}$, and hence $W \subset g^{-1} \cdot (Y')^{\mathrm{an}} \cap \Omega$. By maximality of W, one has $W = g^{-1} \cdot (Y')^{\mathrm{an}} \cap \Omega$. This implies $g \cdot Y = Y'$. Thus $g \cdot Y = h \cdot Y$ for every $g, h \in M$. □

We return to the context of Section 8.6. In particular, ξ is a point of $Y(F)$ such that $q_1(\xi) \in \mathscr{L}_{\Gamma_1}$, and the restriction to Y of the projection to the first m coordinates is étale at ξ, with a local analytic section φ defined on $U_1 \times \cdots \times U_m$.

Lemma 8.9. *There exist a real number $c > 0$, fundamental sets $\mathfrak{F}_i \subset \Omega_{\Gamma_i}$ and a subset Υ of $R_{\mathfrak{F}} \cap \Gamma$, where $\mathfrak{F} = \prod \mathfrak{F}_i$, such that the following hold:*

(1) *For all T large enough, one has* $\operatorname{Card}(\Upsilon_T) \geq T^c$, *where Υ_T denotes the set of all $\gamma \in \Upsilon$ such that $H(\gamma) \leq T$.*

(2) *The projection* q_1 *is injective on* Υ.

(3) *For all* $j \in \{1, \dots, n\}$ *such that* $q_j(\xi) \notin \mathscr{L}_{\Gamma_j}$, *one has* $\operatorname{Card}(q_j(\Upsilon)) = 1$.

Recall that there exists a number field K contained in F such that $\Gamma \subset \mathrm{PGL}(2, K)^n$, and H is induced by a fixed height function on $\mathrm{PGL}(2, \overline{\mathbf{Q}})^n$. In particular, Lemma 8.9 implies that the subset $R_{\mathfrak{F}}$ of $\mathrm{PGL}(2, F)^n$ has many K-rational points, in the sense of Section 4.5.

Proof. Let q be the genus of X_{Γ_1}; by Proposition 6.8, there exists a basis $\alpha_1, \dots, \alpha_q$ of Γ_1, an open neighborhood U_1' of $q_1(\xi)$ which is contained in U_1 and stable under the action of $\alpha_1, \dots, \alpha_q$, and a fundamental set $\mathfrak{F}_1$ for Γ_1 contained in U_1'. For simplicity of notation, we now assume that $U_1 = U_1'$.

We have introduced in Section 8.6 a local analytic section

$$\varphi = (\varphi_1, \dots, \varphi_n) : U_1 \times \dots \times U_m \to Y^{\mathrm{an}} \cap U_1 \times \dots \times U_n$$

of the projection $q_J : Y \to \mathbf{P}_1^J$, where $J = \{1, \dots, m\}$. Let $j \in \{1, \dots, n\}$ be such that $q_j(\xi) \notin \mathscr{L}_{\Gamma_j}$. Then $q_j(\xi)$ has a compact analytic neighborhood U_j' contained in Ω_{Γ_j}. Shrinking $U_1, \dots, U_m$ if necessary, we assume that the image of φ_j is contained in U_j' for every such j.

Let $a' = (a_1, \dots, a_n) \in W$ be a rigid point that belongs to the image of φ and such that $a_1 \in \mathfrak{F}_1$. Let $a = (a_1, \dots, a_m)$; we have $a' = \varphi(a)$. For $j \in \{2, \dots, n\}$, we also choose a fundamental set $\mathfrak{F}_j$ that contains a_j.

We claim that we can complete any element $\gamma_1 \in F_1$ which is a positive word γ_1 in $\alpha_1, \dots, \alpha_q$ to an element $\gamma \in \Gamma$ such that $\gamma^{-1} \in R_{\mathfrak{F}}$ and $H(\gamma) \ll c^{\ell_{\Gamma_1}(\gamma_1)}$, for some real number c.

Let us now prove the asserted claim. For any positive word γ_1 in $\alpha_1, \dots, \alpha_q$, one has $\gamma_1 \cdot a_1 \in U_1$; in particular, we can consider the point $a(\gamma_1) = (\gamma_1 \cdot a_1, a_2, \dots, a_m)$ of $U_1 \times \dots \times U_m$ and its image $\varphi(a(\gamma_1))$ under the section φ.

By Section 6.3, there exists a real number $c_1 \geq 1$ such that $\delta(\alpha_j \cdot a_1; \mathscr{L}_{\Gamma_1}) \geq c_1^{-1} \delta(a_1; \mathscr{L}_{\Gamma_1})$, uniformly in a_1. By induction on the length $\ell_{\Gamma_1}(\gamma_1)$ of the positive word γ_1, this implies the inequality

$$\delta(\gamma_1 \cdot a_1; \mathscr{L}_{\Gamma_1}) \geq c_1^{-\ell_{\Gamma_1}(\gamma_1)}. \tag{8.9.1}$$

We first set $\gamma_2 = \dots = \gamma_m = 1$.

Let $j > m$. Let $\psi_j : U_1 \to U_j$ be the analytic map with $\psi_j(x) = \varphi_j(x, a_2, \dots, a_m)$. By construction (Lemma 8.5), if $\psi_j(x) = \varphi_j(x, a_2, \dots, a_m) \in \mathscr{L}_{\Gamma_j}$, one has $x = q_1(x, a_2, \dots, a_m) \in \mathscr{L}_{\Gamma_1}$. In other words, one has $\psi_j^{-1}(\mathscr{L}_{\Gamma_j}) \subset \mathscr{L}_{\Gamma_1}$. Applying Lemma 6.11 to ψ_j, we obtain an inequality of the form

$$\delta(\varphi_j(x, a_2, \dots, a_m); \mathscr{L}_{\Gamma_j}) \gg \delta(x; \mathscr{L}_{\Gamma_1})^k,$$

for some integer $k \geq 0$ and all $x \in U_1$. In particular,

$$\delta(\varphi_j(a(\gamma_1)); \mathscr{L}_{\Gamma_j}) \gg \delta(\gamma_1 \cdot a_1; \mathscr{L}_{\Gamma_1})^k. \tag{8.9.2}$$

By Proposition 6.8, there exists $\gamma_j \in \Gamma_j$ such that $\varphi_j(a(\gamma_1)) \in \gamma_j \cdot \mathfrak{F}_j$. By Proposition 6.6 and Lemma 6.9, one has

$$H(\gamma_j) \ll \delta(\varphi_j(a(\gamma_1)); \mathscr{L}_{\Gamma_j})^{-\kappa}, \tag{8.9.3}$$

where κ is a positive real number, independent of γ_1. By equations (8.9.1), (8.9.2) and (8.9.3), we thus have

$$H(\gamma_j) \ll \delta(\gamma_1 \cdot a_1; \mathscr{L}_{\Gamma_j})^{-k\kappa} \ll c_1^{\ell_{\Gamma_1}(\gamma_1)k\kappa}. \tag{8.9.4}$$

Let $c = c_1^{k\kappa}$.

Let $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Gamma$. By what precedes, $H(\gamma) \ll c^{\ell_{\Gamma_1}(\gamma_1)}$. Moreover, $\varphi_j(a(\gamma_1)) \in \gamma_j \cdot \mathfrak{F}_j$ for every j; this follows from the fact that $a_j \in \mathfrak{F}_j$ if $j \leq m$, and from the construction of γ_j if $j > m$.

Let us prove $\gamma^{-1} \in R_{\mathfrak{F}}$. One has $W \subset p^{-1}(V^{\mathrm{an}})$ by assumption; since $\gamma \in \Gamma$, this implies $\gamma^{-1} \cdot W \subset p^{-1}(V^{\mathrm{an}})$. Consequently,

$$\gamma^{-1} \cdot Y^{\mathrm{an}} \cap \mathfrak{F} \cap p^{-1}(V^{\mathrm{an}}) \supset \gamma^{-1} \cdot W \cap \mathfrak{F} \cap p^{-1}(V^{\mathrm{an}}) = \gamma^{-1} \cdot W \cap \mathfrak{F}.$$

The analytic morphism

$$U_1 \times \cdots \times U_m \to W, \qquad (x_1, \ldots, x_m) \mapsto \varphi(\gamma_1 \cdot x_1, x_2, \ldots, x_m)$$

is an immersion and maps the point $a = (a_1, \ldots, a_m)$ to the point $\varphi(a(\gamma_1)) \in \gamma \cdot \mathfrak{F}$. Since a is a rigid point, this morphism maps a neighborhood of a into $\gamma \cdot \mathfrak{F}$, so that $\dim(W \cap \gamma \cdot \mathfrak{F}) \geq m$. This proves $\gamma^{-1} \in R_{\mathfrak{F}}$.

Applying Lemma 6.9 to estimate $H(\gamma_1)$, we thus have shown the existence of a positive real number c such that for every positive word γ_1 in $\alpha_1, \ldots, \alpha_q$, there exists an element $\gamma = (\gamma_1, \ldots, \gamma_n)$ completing γ_1 such that $H(\gamma) \ll c^{\ell_{\Gamma_1}(\gamma_1)}$ and $\gamma^{-1} \in R_{\mathfrak{F}} \cap \Gamma$.

Let Υ' be the set of all such elements γ^{-1}, where γ_1 ranges over positive words in $\alpha_1, \ldots, \alpha_q$. It is a subset of $R_{\mathfrak{F}} \cap \Gamma$. By construction, the projection q_1 is injective on Υ'. Moreover, since the number of positive words of length ℓ in $\alpha_1, \ldots, \alpha_q$ is q^ℓ, the cardinality of Υ'_T is bounded from below by $q^{\log(T)/\log(c)} = T^{\log(q)/\log(c)}$, and the exponent of T is strictly positive, since $q \geq 2$. Finally, let j be such that $q_j(\xi) \notin \mathscr{L}_{\Gamma_j}$. By construction, $\varphi_j(a(\gamma_1)) \in \gamma_j \mathfrak{F}_j$; hence $\gamma_j \mathfrak{F}_j$ meets U'_j. By Corollary 6.7, the set S_j of such elements γ_j in Γ_j is finite. It follows that there is a subset Υ of Υ' that satisfies the conclusion of the proposition. □

Proposition 8.10. *Let G'_0 be the subgroup of G_0 consisting of elements (g_j) such that $g_j = \mathrm{id}$ if $q_j(\xi) \notin \mathscr{L}_{\Gamma_j}$. Both the stabilizer of W inside $G'_0 \cap \Gamma$ and its image in Γ_1 under the first projection have many rational points.*

Proof. Let c, Υ, $\mathfrak{F}_i$, $\mathfrak{F} = \prod \mathfrak{F}_i$ and $R = R_{\mathfrak{F}}$ be as given by Lemma 8.9; let $T_0 > 1$ be such that $\operatorname{Card}(\Upsilon_T) \geq T^c$ for $T \geq T_0$.

Let K be a number field contained in F such that all groups Γ_j are contained in $\mathrm{PGL}(2, K)$; the points of $R \cap \Gamma$ are K-rational points. Recall that for every real number T, we denote by $R(K; T)$ the set of K-rational points of R of height $\leq T$. One has $\Upsilon_T = \Upsilon \cap R(K; T)$.

Since R is F-subanalytic (Lemma 8.8), it is also $\mathbf{Q}_p$-subanalytic and we may apply the p-adic Pila–Wilkie theorem of [Cluckers et al. 2015], as stated in Theorem 4.7. Thus let $s \in \mathbf{N}$, $d \in \mathbf{R}$, $\varepsilon > 0$ and $B \subset R \times \mathbf{Q}_p^s$ be a family of blocks such that for every $T > 1$, there exists a subset $\Sigma_T \in \mathbf{Q}_p^s$ of cardinality $< dT^\varepsilon$ such that $R(K; T) \subset \bigcup_{\sigma \in \Sigma_T} B_\sigma$. Let also $t \in \mathbf{N}$ and $Z \subset G_0(F) \times \mathbf{Q}_p^t$ be a semialgebraic subset such that for every $\sigma \in \mathbf{Q}_p^s$, there exists $\tau \in \mathbf{Q}_p^t$ such that $B_\sigma \subset Z_\tau$ and $\dim(B_\sigma) = \dim(Z_\tau)$. Let finally r be an upper bound for the number of irreducible components of the Zariski closure of the sets Z_τ, for $\tau \in \mathbf{Q}_p^t$.

Let $T > T_0$. Since $\Upsilon_T \subset R(K; T)$, by the pigeonhole principle, there exists $\sigma \in \Sigma_T$ such that

$$\operatorname{Card}(\Upsilon_T \cap B_\sigma) \geq \frac{\operatorname{Card}(\Upsilon_T)}{\operatorname{Card}(\Sigma_T)} \geq \frac{1}{d} T^{c-\varepsilon}.$$

Moreover, the Zariski closure of B_σ in $\mathrm{PGL}(2)_F^n$ has at most r irreducible components. Consequently, we may choose such an irreducible component $\overline{M}$ whose trace M on B_σ satisfies

$$\operatorname{Card}(\Upsilon_T \cap M) \geq \frac{1}{dr} T^{c-\varepsilon}.$$

(Observe that $\overline{M}$ is indeed the Zariski closure of M.)

Let $g \in \Upsilon_T \cap M$. Since the Zariski closure of M is irreducible and $M \subset R_{\mathfrak{F}}$, it follows from Lemma 8.8 that the stabilizer of W inside $G_0 \cap \Gamma$ contains $g^{-1}M$; hence $g^{-1}(\Upsilon_T \cap M)$. By construction, the image of $g^{-1}(\Upsilon_T \cap M)$ under the projection of index j is $\{\mathrm{id}\}$ if $q_j(\xi) \notin \mathscr{L}_{\Gamma_j}$. This shows in particular that the stabilizer of W inside $G_0' \cap \Gamma$ contains $g^{-1}(\Upsilon_T \cap M)$. This set contains $\geq T^{c-\varepsilon}/dr$ points, and their heights are $\ll T^2$; the same holds for its image by the first projection, since this projection is injective on $g^{-1}(\Upsilon \cap M)$.

We thus have shown that the stabilizer of W inside $G_0' \cap \Gamma$ has many rational points, as well as its image under the first projection, concluding the proof. ☐

Proposition 8.11. *The subvariety W is flat.*

Proof. We have constructed in Section 8.6 an analytic map $\varphi : U_1 \times \cdots \times U_m \to Y$, which is a local section of the projection to the m first coordinates.

Let $a \in \prod_{i=2}^m (\Omega_{\Gamma_i} \cap U_i)$; let us denote by W_a the fiber of W over a under the projection to $\prod_{i=2}^m \mathbf{P}_1^{\mathrm{an}}$, and Y_a similarly. When a varies, the number of irreducible components of Y_a is uniformly bounded.

Let $\psi_a : (U_1)_{\mathscr{H}(a)} \to Y_a^{\mathrm{an}}$ be the analytic morphism deduced from φ. We claim that the components of ψ_a are either constant or homographies.

Let $g \in G_0 \cap \Gamma$ be an element such that $g \cdot W = W$, $g_1 \neq \mathrm{id}$ and $g_j = \mathrm{id}$ if $q_j(\xi) \notin \mathscr{L}_{\Gamma_j}$ (Proposition 8.10). Since $g \cdot W = W$, one has $g \cdot Y = Y$. Hence $g \cdot W_a = W_a$ and $g \cdot Y_a = Y_a$. The element g induces a commutative diagram

$$\begin{tikzcd} Y_a \arrow[r, "g"] \arrow[d] & Y_a \arrow[d] \\ (\mathbf{P}_1)_{\mathscr{H}(a)} \arrow[r, "g_1"] \arrow[u, dashed, bend left, "\psi_a"] & (\mathbf{P}_1)_{\mathscr{H}(a)} \arrow[u, dashed, bend right, "\psi_a"'] \end{tikzcd}$$

where the section ψ_a is analytic and defined over the open subset $(U_1)_{\mathscr{H}(a)}$ of $(\mathbf{P}_1)^{\mathrm{an}}_{\mathscr{H}(a)}$. Let Y'_a be the irreducible component of Y_a that contains $\psi_a(\xi_1)$; it is geometrically irreducible. Recall that g_1 has infinite order; replacing g_1 and g by some fixed power, we may thus assume that $g \cdot Y'_a = Y'_a$.

By Proposition 7.2, either $Y'_a \to (\mathbf{P}_1)_{\mathscr{H}(a)}$ is an isomorphism and the components of its inverse are constant or homographies, or there exists a subset Δ of $\mathbf{P}_1(\overline{\mathscr{H}(a)})$ such that $\mathrm{Card}(\Delta) = 2$ and $g_1(\Delta) = \Delta$ for every element $g = (g_1, \dots, g_n) \in G'_0 \cap \Gamma$ such that $g \cdot W = W$ and $g \cdot Y'_a = Y'_a$. Let us assume that we are in the latter case. Using that $\Gamma_1 \subset \mathrm{PGL}(2, F)$, we see that $\Delta \subset \mathbf{P}_1(\bar{F})$. By Lemma 6.10, the projection to Γ_1 of the stabilizer of W inside $G'_0 \cap \Gamma$ has few rational points, contradicting Proposition 8.10.

We thus have shown that the components of the analytic map ψ_a are either constant or given by homographies.

Let $j \in \{m+1, \dots, n\}$.

First assume that $q_j(\xi) \in \Omega_{\Gamma_j}$. Then $g_j = \mathrm{id}$, whence the relation $\psi_{a,j} = \psi_{a,j} \circ g_1$. Since $g_1 \neq \mathrm{id}$, this implies that $\psi_{a,j}$ is constant, i.e., φ_j does not depend on the coordinate x_1. Since U is reduced, the morphism φ_j is deduced by pull-back of an analytic map $\theta_j : \prod_{i=2}^m U_i \to \mathbf{P}_1^{\mathrm{an}}$.

Let us then assume that $q_j(\xi) \in \mathscr{L}_{\Gamma_j}$. Since the j-th component of φ takes the value $q_j(\xi)$, the section $\psi_{a,j}$ cannot be constant. It is thus a homography $\tau_{j,a}$.

A priori, one has $\tau_{j,a} \in \mathrm{PGL}(2, \mathscr{H}(a))$ for every a. However, by condition (3) of Lemma 8.5, one has $\varphi_j^{-1}(\mathscr{L}_{\Gamma_j}) \subset \mathscr{L}_{\Gamma_1}$. The limit sets $\mathscr{L}_{\Gamma_1}$ and $\mathscr{L}_{\Gamma_j}$ are contained in $\mathbf{P}_1(F)$ and have no isolated points, so that $\tau_{j,a}^{-1}$ maps an infinite subset of $\mathbf{P}_1(F)$ into $\mathbf{P}_1(F)$; this implies that $\tau_{j,a} \in \mathrm{PGL}(2, F)$.

Observe that for $x \in U_1 \cap \mathbf{P}_1(F)$, one has $\tau_{j,a} \cdot x = \psi_{a,j}(x) = \varphi(x, a)$. In particular, the assignment $a \mapsto \tau_{j,a}$ is induced by an analytic morphism. Since it takes its values in $\mathrm{PGL}(2, F)$, it is constant.

Let J' and J'' be the set of all $j \in \{m+1, \dots, n\}$ such that $q_j(\xi)$ belongs to $\mathscr{L}_{\Gamma_j}$ and Ω_{Γ_j}, respectively. Let $\Omega' = \Omega_{\Gamma_1} \times \prod_{j \in J'} \Omega_{\Gamma_j}$ and $\Omega'' = \prod_{i=2}^m \Omega_{\Gamma_i} \times \prod_{j \in J''} \Omega_{\Gamma_j}$; similarly, write $X' = X_1 \times \prod_{j \in J'} X_j$ and $X'' = \prod_{i=2}^m X_i \times \prod_{j \in J''} X_j$, and decompose the projection $p:\Omega\to X$ as (p', p''), where $p':\Omega'\to X'$ and $p'':\Omega''\to X''$ are the natural projections.

Let Z' be the graph in $\big(\mathbf{P}_1\times\prod_{j\in J'}\mathbf{P}_1\big)^{\mathrm{an}}$ of $(\tau_j)_{j\in J'}$ and Z'' the graph in $\big(\prod_{i=2}^m\mathbf{P}_1\times\prod_{j\in J''}\mathbf{P}_1\big)^{\mathrm{an}}$ of $(\theta_j)_{j\in J''}$. Let Y' and Y'' be the Zariski closure of Z' and Z'', let W' and W'' be their traces in Ω' and Ω'', and let V' and V'' be the Zariski closures of $p'(Z')$ and $p''(Z'')$. It is clear that $Y'=Z'$ is the curve in $\mathbf{P}_1\times\prod_{j\in J'}\mathbf{P}_1$ (with coordinates x_1 and x_j for $j\in J'$) given by the equations $x_j=\tau_j(x_1)$, and W' is its trace on Ω'. In particular, W' is flat.

By construction, $Z'\times Z''$ is a subspace of Y^{an} which meets W in a Zariski dense subset of itself; hence $Y=Y'\times Y''$ and $W=\Omega\cap Y^{\mathrm{an}}=W'\times W''$. Moreover, $p(W)=p'(W')\times p''(W'')\subset V$; hence $V'\times V''\subset V$. Consequently, W'' is a maximal algebraic irreducible subset of $(p'')^{-1}((V'')^{\mathrm{an}})$. By induction, W'' is flat.

Consequently, $W=W'\times W''$ is flat, as was to be shown. □

9. A characterization of geodesic subvarieties

9.1. Let F be a finite extension of $\mathbf{Q}_p$ and let $(\Gamma_i)_{1\le i\le n}$ be a finite family of arithmetic Schottky subgroups of ranks ≥ 2 in $\mathrm{PGL}(2,F)$ Let us set $\Omega=\prod_{i=1}^n\Omega_{\Gamma_i}$, $X=\prod_{i=1}^n X_{\Gamma_i}$, and let $p:\Omega\to X^{\mathrm{an}}$ be the morphism deduced from the morphisms $p_{\Gamma_i}:\Omega_{\Gamma_i}\to X_{\Gamma_i}^{\mathrm{an}}$.

Theorem 9.2. *Let W be a Zariski closed subvariety of Ω, geometrically irreducible. Then the following properties are equivalent*:

(i) *The variety W is geodesic.*

(ii) *Its projection $p(W)$ is algebraic.*

(iii) *The dimension of the Zariski closure of $p(W)$ in X is equal to* $\dim(W)$.

Proof. Let us assume that W is geodesic and show that $p(W)$ is algebraic.

We may assume that no projection p_{Γ_i} is constant on W. Define a relation $\sim$ on $\{1,\dots,n\}$ given by $i\sim j$ if there exists $g\in\mathrm{PGL}(2,F)$ (necessarily unique) such that $g\Gamma_i g^{-1}$ and Γ_j are commensurable and $z_j=g\cdot z_i$ for every $z\in W$. This is an equivalence relation. Fix an element j in each equivalence class; for i such that $i\sim j$, we may replace Γ_i by its conjugate $g\Gamma_i g^{-1}$ and assume that $z_j=z_i$ on W. This shows that W and Ω decompose as a product indexed by the set of equivalence classes of the following particular situation: all the subgroups Γ_i are commensurable, and W is the diagonal of Ω. It thus suffices to treat this particular case.

Let $\Gamma_0=\bigcap_i\Gamma_i$ and X_0 be the algebraic curve associated with $\Omega_{\Gamma_0}/\Gamma_0$. Then, for every i, the morphism $f_i:W\to X_i^{\mathrm{an}}$ deduced from $f=p|_W$ factors as the composition of the uniformization $p_0:\Omega_{\Gamma_0}\to X_0^{\mathrm{an}}$ and of a finite morphism $X_0^{\mathrm{an}}\to X_i^{\mathrm{an}}$. By GAGA [Berkovich 1990, Corollary 3.5.2; Poineau 2010, Appendix], a finite analytic morphism of algebraic curves is algebraic; consequently, there exists a finite morphism $q_i : X_0 \to X_i$ such that $f_i = q_i^{\mathrm{an}} \circ p_0$. Then $p(W)$ is the image of X_0 by the finite morphism $q = (q_1, \dots, q_n) : X_0 \to X$, hence is algebraic. This shows that (i) implies (ii). Since it is clear that (ii) implies (iii), it remains to prove that (iii) implies (i).

Let us assume now that the dimension of the Zariski closure V of $p(W)$ in X is equal to the dimension of W. By construction, W is a maximal irreducible algebraic subvariety of $p^{-1}(V^{\mathrm{an}})$. By Proposition 8.1, W is flat. A similar analysis as in the proof of the first implication shows that there is a partition of the indices $\{1, \dots, n\}$ under which W decomposes as a product of flat curves and points. Since it suffices to prove that each of these curves is geodesic, we may assume that W is a flat curve of the form

$$W = \{(z, g_2 \cdot z, \dots, g_n \cdot z)\} \cap \Omega,$$

where $g_2, \dots, g_n \in \mathrm{PGL}(2, F)$.

First assume that $n = 2$. Let then $g \in \mathrm{PGL}(2, F)$ be such that $W = \{(z, g \cdot z)\} \cap \Omega$ and let us prove that Γ_2 and $g\Gamma_1 g^{-1}$ are commensurable, a property which is equivalent to the finiteness of both orbit sets $\Gamma_2 \backslash \Gamma_2 g \Gamma_1$ and $\Gamma_1 \backslash \Gamma_1 g^{-1} \Gamma_2$.

Let us argue by contradiction and assume that $\Gamma_2 \backslash \Gamma_2 g \Gamma_1$ is infinite. (The other finiteness is analogous, or follows by symmetry.) Fix a rigid point $z \in \Omega_{\Gamma_1}$. Let $A \subset \Gamma_1$ be a set such that gA is a set of representatives of $\Gamma_2 \backslash \Gamma_2 g \Gamma_1$; by assumption, A is infinite. Since $\Gamma \backslash W \subset V^{\mathrm{an}}$, the algebraic variety V contains the infinite set of points $p(a \cdot z, g \cdot az) = (p_1(z), p_2(ga \cdot z))$, for $a \in A$; hence it contains its Zariski closure $\{p_1(z)\} \times X_2$. Since this holds for every $z \in W$, we deduce that V contains $X_1 \times X_2$, contradicting the assumption that $\dim(W) = 1$.

Let us now return to the general case. To prove that W is geodesic, it suffices to establish that the subgroups Γ_j and $g_j \Gamma_1 g_j^{-1}$ are commensurable for every $j \in \{2, \dots, n\}$. Up to renumbering the indices, it suffices to treat the case $j = 2$. Let $\Omega' = \Omega_{\Gamma_1} \times \Omega_{\Gamma_2}$, let $p' : \Omega' \to X' = X_1 \times X_2$ be the uniformization map, and denote by π the projections from Ω to Ω' and from X to X'. Let $W' = \pi(W)$ and $V' = \pi(V)$. By Chevalley's theorem, V' is an algebraic curve in X'. Obviously, W' is a flat curve contained in $(p')^{-1}((V')^{\mathrm{an}})$, and hence is a maximal irreducible algebraic subset of $(p')^{-1}((V')^{\mathrm{an}}) \cap \Omega'$. By the case $n = 2$, the Schottky groups Γ_2 and $g_2 \Gamma_1 g_2^{-1}$ are commensurable, as was to be shown. This concludes the proof of Theorem 9.2. □

Corollary 9.3. *Let V be an irreducible* curve *in X. Then every irreducible algebraic subvariety of $\Omega_{\mathbf{C}_p}$ which is maximal among those contained in $p^{-1}(V^{\mathrm{an}}_{\mathbf{C}_p})$ is geodesic.*

Proof. Let W_0 be an irreducible algebraic subvariety of $\Omega_{\mathbf{C}_p}$, maximal among those contained in $p^{-1}(V^{\mathrm{an}}_{\mathbf{C}_p})$; let us prove that W_0 is geodesic. We may assume that $\dim(W_0) > 0$. Since p is surjective and has discrete fibers, one has $\dim(p^{-1}(V^{\mathrm{an}}_{\mathbf{C}_p})) = \dim(V^{\mathrm{an}}_{\mathbf{C}_p})$, hence $\dim(W_0) = 1$, so that W_0 is an irreducible component of $p^{-1}(V^{\mathrm{an}})_{\mathbf{C}_p}$. By Theorem 7.16 of [Ducros 2009], there exists a finite extension E of F and an irreducible component W of $p^{-1}(V^{\mathrm{an}})_E$ such that $W_0 = W_{\mathbf{C}_p}$.

By Theorem 9.2, W is geodesic. Consequently, W_0 is geodesic. □

Remark 9.4. This corollary suggests that the main results of the paper extend to maximal algebraic irreducible subvarieties of $p^{-1}(V^{\mathrm{an}})_{\mathbf{C}_p}$, without assuming that they are defined over a finite extension of F.

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