Algebra & Number Theory

Volume 11
2017
No. 9

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ALGEBRA AND NUMBER THEORY 11:9 (2017)
dx.doi.org/10.2140/ant.2017.11.2091

Elementary equivalence versus isomorphism, II

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In this note we give sentences ϑ_K in the language of fields which describe the isomorphy type of K among finitely generated fields, provided the Kronecker dimension $\dim(K)$ satisfies $\dim(K) < 3$. This extends results by Rumely (1980) concerning global fields; see also Scanlon (2008).

1. Introduction

We begin by recalling Rumely's result [1980] showing that for every global field k there exists a sentence $\vartheta_k^{\mathrm{Ru}}$ which *characterizes the isomorphy type of k among global fields*, i.e., if l is any *global field,* then $\vartheta_k^{\mathrm{Ru}}$ holds in l if and only if $l \cong k$ as fields.

It is one of the *main open questions* in the first-order theory of finitely generated fields whether a fact similar to Rumely's result mentioned above holds for all finitely generated fields K. We notice that the question above is related to, but much stronger than, the still open *elementary equivalence versus isomorphism problem,* which asks whether the isomorphism type of every finitely generated field K is encoded in the whole first-order theory $\mathfrak{Th}(K)$ of K; see, e.g., [Pop 2003] for details and literature about this, as well as [Scanlon 2008].[1]

In the present note we show that the answer to the above question is positive for finitely generated fields K having Kronecker dimension $\dim(K) < 3$, which are precisely the finite fields, the global fields, and the function fields of (algebraic) curves over global fields.

Supported by the John Templeton Foundation Grant ID 13394 and the NSF grant DMS-1265290.
MSC2010: primary 11G30, 14H25; secondary 03C62, 11G99, 12F20, 12G10, 12L12, 13F30.
Keywords: elementary equivalence versus isomorphism, first-order definability, finitely generated fields, Milnor K-groups, Galois étale cohomology, Kato's higher local-global principles.

[1]To the best of my knowledge, Bjorn Poonen was among the first to ask whether Rumely's result [1980] might hold in higher dimensions. It was claimed in [Scanlon 2008] that the answer to this question is positive for all finitely generated fields. Unfortunately, the proof has a gap (see *Erratum,* J. Amer. Math. Soc. **24**:3 (2011), p. 917). Nevertheless, Scanlon's work appears to reduce the problem to "definability of valuations".

Main Result. *For every finitely generated field K having* $\dim(K) < 3$, *there exists a first-order sentence in the language of fields ϑ_K such that for finitely generated fields L, the sentence ϑ_K holds in L if and only if $L \cong K$ as fields.*

The more precise form of the result is as follows: Recall that by one of the main results in [Pop 2002], for every $d \geq 0$, there exists a sentence φ_d (in the language of fields) such that for all finitely generated fields K, φ_d holds in K if and only if $\dim(K) = d$. In particular, if K is any finitely generated field, then φ_0 holds in K if and only if K is a finite field, φ_1 holds in K if and only if K is a global field, and finally φ_2 holds in K if and only if $\dim(K) = 2$.

In particular, given a global field K, consider the sentence ϑ_K given by $\varphi_1 \wedge \vartheta_K^{\mathrm{Ru}}$. Then if ϑ_K holds in a finitely generated field L, one has the following: First, $\dim(L) = 1$, because φ_1 holds in L, and hence L is a global field. Second, $L \cong K$ because $\vartheta_K^{\mathrm{Ru}}$ holds in the global field L.

In the case $\dim(K) = 2$, let $k_0 = K^{\mathrm{abs}}$ be the constant subfield of K, i.e., the set of elements of K which are algebraic over the prime field of K. Then k_0 is finite if and only if $\mathrm{char}(K) > 0$, and if so, K is the function field of a projective smooth geometrically integral surface over k_0. Letting (t_0, t_1) be a separable transcendence basis of K, there exists $t_2 \in K$ such that $K = k_0(t_0, t_1, t_2)$, with t_0, t_1, t_2 satisfying an absolutely irreducible polynomial $f(T_0, T_1, T_2)$ over k_0. And if $\mathrm{char}(K) = 0$, then K is the function field of a projective smooth k_0-curve, and for every nonconstant $t_1 \in K$ there exists $t_2 \in K$ such that $K = k_0(t_1, t_2)$, with t_1, t_2 satisfying an irreducible polynomial $f(T_1, T_2) \in k_0[T_1, T_2]$. The precise result proven will be the following; see Section 5 for proofs.

Theorem 1.1. *Let K be a finitely generated field. The following hold*:

(1) *For every finite field k_0 and absolutely irreducible polynomial $f = f(T_0, T_1, T_2)$ over k_0, there exists a formula $\psi_{k_0,f}(\mathfrak{t}_0, \mathfrak{t}_1, \mathfrak{t}_2)$ with free variables* $\mathfrak{t}_0, \mathfrak{t}_1, \mathfrak{t}_2$ *such that the following are equivalent*:

   (i) *The sentence ϑ_K defined by $\exists\, \mathfrak{t}_0, \mathfrak{t}_1, \mathfrak{t}_2\ \psi_{k_0,f}(\mathfrak{t}_0, \mathfrak{t}_1, \mathfrak{t}_2)$ holds in K.*

   (ii) *There exist $t_0, t_1, t_2 \in K$ such that $K = k_0(t_0, t_1, t_2)$ and $f(t_0, t_1, t_2) = 0$.*

      (∗) *In particular*, *suppose that ϑ_K holds in K. Then for all finitely generated fields L, ϑ_K holds in L if and only if $L \cong K$ as abstract fields.*

(2) *For every number field k_0 and absolutely irreducible polynomial $f = f(T_1, T_2)$ over k_0, there exists a formula $\psi_{k_0,f}(\mathfrak{t}_1, \mathfrak{t}_2)$ with free variables* $\mathfrak{t}_1, \mathfrak{t}_2$ *such that the following are equivalent*:

   (i) *The sentence ϑ_K defined by $\exists\, \mathfrak{t}_1, \mathfrak{t}_2\ \psi_{k_0,f}(\mathfrak{t}_1, \mathfrak{t}_2)$ holds in K.*

   (ii) *There exist $t_1, t_2 \in K$ such that $K = k_0(t_1, t_2)$ and $f(t_1, t_2) = 0$.*

      (∗) *In particular*, *suppose that ϑ_K holds in K. Then for all finitely generated fields L, ϑ_K holds in L if and only if $L \cong K$ as abstract fields.*

The result above is based on and uses in an essential way, among other things, previous results by Rumely, Poonen, and Pop. First, the above-mentioned sentences φ_d single out the finite fields, the global fields, and the fields of curves over global fields among all finitely generated fields K. Second, Poonen [2007] showed that there exists a predicate, i.e., formula $\psi^{\text{abs}}(\mathfrak{x})$ with one free variable $\mathfrak{x}$ such that for all finitely generated fields K, one has $k_0 := K^{\text{abs}} = \{x \in K \mid \psi^{\text{abs}}(x) \text{ is true in } K\}$. Further, techniques developed in [Poonen 2007] (using [Pop 2002] as well) give formulas $\psi_r(\mathfrak{x}_1, \dots, \mathfrak{x}_r, \mathfrak{x}_{r+1})$ with $r+1$ free variables such that for $x_1, \dots, x_{r+1} \in K$, one has that $\psi_r(x_1, \dots, x_r, x_{r+1})$ holds in K if and only if $x_1, \dots, x_r$ are algebraically independent over k_0, but $x_1, \dots, x_r, x_{r+1}$ are not. Hence, for $x_1, \dots, x_r \in K$ algebraically independent over k_0, the relative algebraic closure of $k_0(x_1, \dots, x_r)$ in K is given by $L := \{x_{r+1} \in K \mid \psi_r(x_1, \dots, x_r, x_{r+1}) \text{ holds in } K\}$. Finally, Poonen [2007] showed that there exists a sentence ψ_0 which holds in a finitely generated field K if and only if $\operatorname{char}(K) = 0$.

Hence, in the case $\dim(K) = 2$, one has the following: First, $k_0 = K^{\text{abs}}$ is finite if and only if $\operatorname{char}(K) > 0$ if and only if ψ_0 does not hold in K. If so, K is the function field of a projective smooth surface over k_0. Therefore, there exist separable transcendence bases t_0, t_1 of $K\,|\,k_0$ satisfying that the relative algebraic closure $k \subset K$ of $k_0(t_0)$ in K is a global function field, and furthermore that $K\,|\,k$ is the function field of a (projective smooth) geometrically integral k-curve X. Thus $K = k(t_1, t_2)$ for a properly chosen t_2. Second, if K has characteristic zero, then $k := k_0 = K^{\text{abs}}$ is a number field, and K is the function field of a projective smooth geometrically integral k-curve X. So for $t_1 \in K \setminus k$, and properly chosen $t_2 \in K$, one has $K = k(t_1, t_2)$. Hence, one can deduce Theorem 1.1 above from the following theorem; see Section 5 for detailed proofs.

Theorem 1.2. *The k-valuations of function fields $K = k(X)$ of projective smooth geometrically integral k-curves X over global fields k are* uniformly first-order definable. *In particular, there exist formulas* $\deg_N(\mathfrak{t})$, $\psi^R(\mathfrak{t},\mathfrak{t}')$, $\psi^0(\mathfrak{t},\mathfrak{t}')$, *with free variables* $\mathfrak{t}$, $\mathfrak{t}'$, *such that for every $K\,|\,k$ as above and $t \in K \setminus k$, the following hold*:

(a) $\deg_N(t)$ *is true in K if and only if t has degree N as a function of $K\,|\,k$, i.e.,* $[K : k(t)] = N$.

(b) $R := \{t' \in K \mid \psi^R(t, t')$ *is true in* $K\}$ *is the integral closure of* $k[t]$ *in* K.

(c) $k[t] = \{t' \in K \mid \psi^0(t, t')$ *is true in* $K\}$.

We mention that the formulas $\deg_N(\mathfrak{t})$, $\psi^R(\mathfrak{t},\mathfrak{t}')$, $\psi^0(\mathfrak{t},\mathfrak{t}')$ are quite explicit; see Section 5. In particular, so is Theorem 5.3, which is slightly more general than Theorem 1.2 above. The main technical tool in the proof is one of Kato's higher Hasse local-global principles (LGPs) for H^3; see Theorem 2.1 below. If similar LGPs would be available in higher dimensions, it would be possible to extend the methods of this paper to higher dimensions.

2. Reviewing well known facts

2A. ***The Hasse–Brauer–Noether local-global principle.*** We recall briefly the famous Hasse–Brauer–Noether LGP for the Brauer group of a global field k. Let $\mathbb{P}(k)$ be the set of nontrivial places of k. For $v \in \mathbb{P}(k)$, we denote by k_v the completion of k with respect to v. Then k_v is a locally compact (nondiscrete) field, and the Brauer group $\mathrm{Br}(k_v)$ of k_v admits a canonical embedding $\mathrm{inv}_v : \mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z}$, called the *invariant* (*isomorphism*), satisfying the following:

(a) If $k_v = \mathbb{C}$, then $\mathrm{Br}(k_v) = 0$ and inv_v is the trivial map.

(b) If $k_v = \mathbb{R}$, then $\mathrm{inv}_v : \mathrm{Br}(k_v) \to \frac{1}{2}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$ is an isomorphism.

(c) In the remaining cases, $\mathrm{inv}_v : \mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z}$ is an isomorphism.

The Hasse–Brauer–Noether LGP asserts that the canonical sequence

$$0 \to \mathrm{Br}(k) \to \bigoplus_v \mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z} \to 0$$

is exact. Here, the first map is the direct sum of all the canonical restriction maps $\mathrm{Br}(k) \to \mathrm{Br}(k_v)$; thus implicitly, for every division algebra D over k there exist only finitely many v such that $D \otimes_k k_v$ is not a matrix algebra. And the second map is the sum of the invariant morphisms.

Moreover, if ${}_n(\ )$ denotes the n-torsion, then identifying the n-torsion in $\mathbb{Q}/\mathbb{Z}$ canonically with $\mathbb{Z}/n$, the above exact sequence gives rise canonically to an exact sequence

$$0 \to {}_n\mathrm{Br}(k) \to \bigoplus_v {}_n\mathrm{Br}(k_v) \to \mathbb{Z}/n \to 0.$$

2B. ***Hasse higher LGPs*** **(*after Kato*).** It is a fundamental observation by Kato [1986] that the above local-global principle has higher dimensional variants as follows: First, following [Kato 1986], for every positive integer n, say $n = mp^r$ with p the characteristic, and an integer twist i, one sets $\mathbb{Z}/n(0) = \mathbb{Z}/n$, and defines in general $\mathbb{Z}/n(i) := \mu_m^{\otimes i} \oplus W_r\,\Omega^i_{\log}[-i]$, where $W_r\,\Omega_{\log}$ is the logarithmic part of the de Rham–Witt complex on the étale site; see [Illusie 1979] for details. In this notation, for every (finitely generated) field K one has

$$\mathrm{H}^1(K, \mathbb{Z}/n) = \mathrm{Hom}_{\mathrm{cont}}(G_K, \mathbb{Z}/n), \qquad \mathrm{H}^2(K, \mathbb{Z}/n(1)) = {}_n\mathrm{Br}(K),$$

where G_K is the absolute Galois group of K. Hence, the cohomology groups $\mathrm{H}^{i+1}(K, \mathbb{Z}/n(i))$ have a particular arithmetical significance for $i = 0, 1$. Further, in this notation, the Hasse–Brauer–Noether LGP is a local-global principle for the cohomology group $\mathrm{H}^2(K, \mathbb{Z}/n(1))$, and note that global fields have Kronecker dimension $d = \dim(K) = 1$.

This led Kato to the fundamental idea that for finitely generated fields K of Kronecker dimension d there should exist similar LGPs for $\mathrm{H}^{d+1}(K, \mathbb{Z}/n(d))$. And

Kato [1986] proved that such higher dimension LGPs do indeed hold for $d = 2$, i.e., for $\mathrm{H}^3(K, \mathbb{Z}/n(2))$, where K is the function field of an integral curve over some global field, or equivalently the function field of an integral two dimensional scheme of finite type.

We describe below one of Kato's local-global principles for $\mathrm{H}^3(K, \mathbb{Z}/n(2))$, and will use that LGP in the cases $n = 2$, $\mathrm{char}(K) \neq 2$ as well as $n = 3$, $\mathrm{char}(K) = 2$. The situation is as follows. Let k be a global field, and $K\,|\,k$ be the function field of a complete smooth geometrically integral k-curve X. Let S be the arithmetical complete normal curve with function field $\kappa(S) = k$; hence $S = \mathrm{Spec}\,\mathcal{O}_k$ if k is a number field, and S is the unique projective smooth curve with function field k if k is a global field of positive characteristic. Then by Abhyankar's regularization theorems of surfaces [1965], $X \to \mathrm{Spec}\,k$ is the generic fiber of a proper morphism $\mathcal{X} \to S$ of regular schemes (and having further properties, e.g., having NCD on $\mathcal{X}$ as reduced fibers, etc.). For $i \geq 0$, we denote by $\mathcal{X}_i \subset \mathcal{X}$ the points of dimension i in $\mathcal{X}$. Then for $x \in \mathcal{X}$ one has:

(a) $x \in \mathcal{X}_0 \Leftrightarrow \mathcal{O}_x$ is a two dimensional local ring $\Leftrightarrow \kappa(x)$ is a finite field.

(b) $x \in \mathcal{X}_1 \Leftrightarrow \mathcal{O}_x$ is a discrete valuation ring $\Leftrightarrow \kappa(x)$ is a global field.

For $s \in S_0$ we denote by v_s the canonical valuation of $\mathcal{O}_s$ and by k_s the completion of k at s. For $x_1 \in \mathcal{X}_1$ we denote by v_{x_1} the canonical valuation of $\mathcal{O}_{x_1}$, and by K_{x_1} the completion of K at x_1. Notice that $x_1 \mapsto s$ under $\mathcal{X} \to S$ if and only if $\mathcal{O}_s \prec \mathcal{O}_{x_1}$, that is, the local ring $\mathcal{O}_s$ is dominated by the local ring $\mathcal{O}_{x_1}$ under $k \hookrightarrow K$.

Next let L be an arbitrary field, and recall the canonical isomorphism (generalizing the classical Kummer theory isomorphism) $h^1 : L^\times/n \to \mathrm{H}^1(L, \mathbb{Z}/n(1))$.[2] As explained in [Kato 1986, §1], the isomorphism h^1 gives rise canonically for all $q \neq 0$ to morphisms[3]

$$h^q : K_q^{\mathrm{M}}(L)/n \to \mathrm{H}^q(L, \mathbb{Z}/n(q)),$$
$$\{a_1, \dots, a_q\}/n \mapsto h^1(a_1) \cup \cdots \cup h^1(a_q) =: a_1 \cup \cdots \cup a_q.$$

Let $\mathfrak{v}$ be a discrete valuation of L. Then for every uniformizing parameter $\pi \in L$ at $\mathfrak{v}$, one defines the *boundary homomorphism*

$$\partial_{\mathfrak{v}} : \mathrm{H}^{q+1}(L, \mathbb{Z}/n(q+1)) \to \mathrm{H}^q(\lambda, \mathbb{Z}/n(q))$$

by $\pi \cup a_1 \cdots \cup a_q \mapsto a_1 \cup \cdots \cup a_q$ and $a_0 \cup a_1 \cdots \cup a_q \mapsto 0$, provided all $a_0, a_1, \dots, a_q$ are $\mathfrak{v}$-units. We notice that in general, this homomorphism depends on the uniformizing parameter π. Further, if the Galois action on $\mathbb{Z}/n(1)$ is trivial, then

[2] Recall that for every abelian group A, we denote $A/n := A/(nA)$.

[3] By the (now proven) Milnor–Bloch–Kato conjecture, h^q are isomorphisms. Nevertheless, that fact in its full generality is not needed here, because one could work as well with the subgroup generated by symbols $\mathrm{H}^q_\cup(L, \mathbb{Z}/n(q)) \subseteq \mathrm{H}^q(L, \mathbb{Z}/n(q))$.

all Galois modules $\mathbb{Z}/n(q)$ are actually isomorphic to $\mathbb{Z}/n$, and $\partial_{\mathfrak{v}}$ gives rise to morphisms

$$\partial_{\mathfrak{v}} : \mathrm{H}^{q+2}(L, \mathbb{Z}/n(q+1)) \to \mathrm{H}^{q+1}(\lambda, \mathbb{Z}/n(q)).$$

We will use two instances of these homomorphisms for $q \le 2$ and L a finitely generated field of Kronecker dimension equal to q containing μ_{2n} (thus having no orderings).[4] First, let k be a global field, $K\,|\,k$ be the function field of a complete smooth k-curve X, and $\mathcal{X} \to S$, etc., be as introduced above. For $x_1 \in \mathcal{X}_1$, let $\mathfrak{v} := v_{x_1}$ be the corresponding discrete valuation of K. The boundary homomorphisms we will consider are

$$\partial_{x_1} : \mathrm{H}^3(K, \mathbb{Z}/n(2)) \to \mathrm{H}^2(\kappa(x_1), \mathbb{Z}/n(1)).$$

For later use we notice that for $f, g, h \in K$ such that g, h are v_{x_1}-units, one has

$$\partial_{x_1}(f \cup g \cup h) = v_{x_1}(f) \cdot \bar{g} \cup \bar{h} \quad \text{in } \mathrm{H}^2(\kappa(x_1), \mathbb{Z}/n(1)),$$

where $u \mapsto \bar{u}$ is the residue map $\mathcal{O}_{x_1} \to \kappa(x_1)$. In particular, if the v_{x_1}-values of $f, g, h \in K$ are all divisible by n (for instance, if f, g, h are all v_{x_1}-units), then $\partial_{x_1}(f \cup g \cup h) = 0$.

For $q = 1$, $L = \kappa(x_1)$ is the residue field of K at some $x_1 \in \mathcal{X}_1$ and $\mathfrak{v}$ is some finite place $\mathfrak{p}_0$ of $\kappa(x_1)$. Thus the boundary homomorphisms we will consider are

$$\partial_{\mathfrak{p}_0} : \mathrm{H}^2(\kappa(x_1), \mathbb{Z}/n(1)) \to \mathrm{H}^1(\kappa(\mathfrak{p}_0), \mathbb{Z}/n(0)) = \mathbb{Z}/n.$$

Notice that $\partial_{\mathfrak{p}_0}$ is nothing but the local component of the Hasse–Brauer–Noether LGP for the global field $\kappa(x_1)$ defined by the place $\mathfrak{p}_0$.

Following [Kato 1986, §1], for all $x_1 \in \mathcal{X}_1$, $x_0 \in \mathcal{X}_0$, one defines *boundary homomorphisms*

$$\partial_{x_1 x_0} : \mathrm{H}^2(\kappa(x_1), \mathbb{Z}/n(1)) \to \mathrm{H}^1(\kappa(x_0), \mathbb{Z}/n)$$

as follows. First, if $x_0 \notin \overline{\{x_1\}}$, set $\partial_{x_1 x_0} = 0$. Second, if $x_0 \in \overline{\{x_1\}}$, proceed as follows: Recall that $\mathcal{O}_{x_0}$ is a two dimensional regular local ring, and set $\mathcal{X}_{x_0} := \operatorname{Spec} \mathcal{O}_{x_0} \hookrightarrow \mathcal{X}$. Then $x_0 \in \overline{\{x_1\}}$ if and only if $x_1 \in \mathcal{X}_{x_0}$. If so, then $\mathcal{O}_{x_1}$ is some localization of $\mathcal{O}_{x_0}$ and the image $\overline{\mathcal{O}}_{x_0} \subset \kappa(x_1)$ of $\mathcal{O}_{x_0}$ under the projection $\mathcal{O}_{x_1} \to \kappa(x_1)$ is a local Noetherian ring of Krull dimension one. Thus its integral closure $\widetilde{\mathcal{O}}_{x_0}$ in $\kappa(x_1)$ is a Dedekind domain with finitely many maximal ideals $\mathfrak{p}_i$, and thus a principal ideal domain. Further, every completion $\kappa(x_1)_{\mathfrak{p}_i}$ is a localization of the global field $\kappa(x_1)$ and the residue fields $\kappa(\mathfrak{p}_i)\,|\,\kappa(x_0)$ are finite fields. Kato defined $\partial_{x_1 x_0}$ as follows, where the last map is the sum of the corestriction maps:

$$\partial_{x_1 x_0} : \mathrm{H}^2(\kappa(x_1), \mathbb{Z}/n(1)) \to \bigoplus_{\mathfrak{p}_i} \mathrm{H}^1(\kappa(\mathfrak{p}_i), \mathbb{Z}/n) \to \mathrm{H}^1(\kappa(x_0), \mathbb{Z}/n).$$

[4] Recall that in these cases, the equality $\mathrm{H}^3_{\cup} = \mathrm{H}^3$ has been known for a while already.

Finally, one of the local-global principles Kato gives — which is essential for the methods of this paper — is the following; see [Kato 1986, p. 145, Corollary].

Theorem 2.1. *With the above notation, suppose that K has no orderings, e.g., $\mu_{2n} \subset K$. Then via the obvious direct sums of the above boundary homomorphisms one gets a long exact sequence of the following form, where the last map is given by the sum*:

$$0 \to \mathrm{H}^3(K, \mathbb{Z}/n(2)) \to \bigoplus_{x_1 \in \mathcal{X}_1} \mathrm{H}^2(\kappa(x_1), \mathbb{Z}/n(1)) \to \bigoplus_{x_0 \in \mathcal{X}_0} \mathrm{H}^1(\kappa(x_0), \mathbb{Z}/n) \to \mathbb{Z}/n \to 0.$$

In particular, recalling that $\mathcal{X}_{x_0}$ is the set of all the $x_1 \in \mathcal{X}_1$ such that $x_0 \in \overline{\{x_1\}}$, the map $\mathrm{H}^3(K, \mathbb{Z}/n(2)) \to \bigoplus_{x_1 \in \mathcal{X}_{x_0}} \mathrm{H}^2(\kappa(x_1), \mathbb{Z}/n(1)) \to \mathrm{H}^1(\kappa(x_0), \mathbb{Z}/n)$ is trivial for each $x_0 \in \mathcal{X}_0$.

2C. ***An arithmetical application/interpretation.*** In the following discussion, suppose that the Galois action on $\mathbb{Z}/n(1)$ is trivial, so that $\mathbb{Z}/n(q)$ are isomorphic to $\mathbb{Z}/n$ as Galois modules.

(1) Recall ${}_n\mathrm{Br}(L) = H^2(L, \mathbb{Z}/n(2))$ and $H^3(L, \mathbb{Z}/n(3))$ are generated by symbols $a \cup b$ and $a \cup b \cup c$, respectively, with $a, b, c \in L^\times$.

(2) Now suppose that n is a prime number. For $a, b \in L^\times$ consider the field extension $L_a \,|\, L$ defined by $h^1(a) \in \mathrm{H}^1(L, \mathbb{Z}/n(1))$, the norm map $N_a : L_a^\times \to L^\times$, and the cyclic algebra $A_{a,b}$ with $[A_{a,b}] = a \cup b \in \mathrm{H}^2(L, \mathbb{Z}/n(1))$. Then $a \cup b \in \mathrm{H}^2(L, \mathbb{Z}/n(1))$ is trivial if and only if $b \in N_a(L_a^\times)$. Furthermore, if $A_{a,b}$ is a division algebra, let $N_{a,b} : A_{a,b}^\times \to L^\times$ be the reduced norm of $A_{a,b}$. Then by [Merkurjev and Suslin 1982], $N_{a,b}$ represents $c \in L^\times$ if and only if $a \cup b \cup c \in H^3(L, \mathbb{Z}/n(3))$ is trivial.

Therefore, since the conditions $b \in \mathrm{im}(N_a)$ and/or $c \in \mathrm{im}(N_{a,b})$ are first-order expressible, we conclude that $a \cup b$ and/or $a \cup b \cup c$ being (non)trivial are first-order expressible. Hence, the following hold:

(∗) *The subsets $\Sigma_2 \subset L^\times \times L^\times$ and $\Sigma_3 \subset L^\times \times L^\times \times L^\times$ defined by*

$$\Sigma_2 := \{(a, b) \mid a \cup b \textit{ is nontrivial}\},$$
$$\Sigma_3 := \{(a, b, c) \mid a \cup b \cup c \textit{ is nontrivial}\}$$

are first-order definable subsets.

(3) Let $K \,|\, k$ be a function field in one variable over a global field k as above. Let $\tilde{k} \,|\, k$ be a finite extension with $\mu_n \subset \tilde{k}$, and $\widetilde{K} := K\tilde{k}$. Then $\widetilde{K} \,|\, \tilde{k}$ is the function field of the complete smooth geometrically integral $\tilde{k}$-curve $\widetilde{X} := X \times_k \tilde{k}$. As in the case of $K \,|\, k$, we consider proper regular models $\widetilde{\mathcal{X}} \to \widetilde{S}$ of $\widetilde{X} \to \operatorname{Spec} \tilde{k}$, and the sets $\widetilde{\mathcal{X}}_i \subset \widetilde{\mathcal{X}}$ for $i = 0, 1, 2$. In particular, for every $\tilde{x}_1 \in \widetilde{\mathcal{X}}_1$, the local rings $\mathcal{O}_{\tilde{x}_1}$ are discrete valuation rings of $\widetilde{K}$, and we denote by $\widetilde{K}_{\tilde{x}_1}$ the

corresponding completions of $\widetilde{K}$. Then if $A_{a,b}$ is a division algebra as in (2) above, the following holds:

$N_{a,b}$ represents $c \in \widetilde{K}^{\times}$ over $\widetilde{K}$ if and only if $N_{a,b}$ represents c over $\widetilde{K}_{\tilde{x}_1}$ for all $\tilde{x}_1 \in \widetilde{\mathcal{X}}_1$.

3. Consequences of Kato's local-global principles

3A. *General facts.* Let K be a finitely generated field of Kronecker dimension $\dim(K) = 2$, and $k_0 = K^{\mathrm{abs}}$ be its absolute subfield. If $\mathrm{char}(K) = 0$, then $k := k_0$ is a number field, and $S = \mathrm{Spec}\,\mathcal{O}_k$ is the "canonical global curve" with function field k. Further, K is the function field of a projective regular S-surface $\mathcal{X} \to S$, having as a generic fiber a smooth projective geometrically integral k-curve X. If $\mathrm{char}(K) = p > 0$, there exist (many) global function subfields $k \subset K$ of K with $k = \bar{k} \cap K$ such that letting S be the unique projective smooth k_0-curve, there exist projective smooth S-surfaces $\mathcal{X} \to S$ having as generic fiber a projective smooth k-curve X.

In the above notation, we denote by $S_i \subset S$, $X_i \subset X$, $\mathcal{X}_i \subset \mathcal{X}$ the points of dimension i in the corresponding schemes. In particular, one has the following:

- $S_0 \subset S$, $\mathcal{X}_0 \subset \mathcal{X}$, $X_0 \subset X$ are the closed points in the corresponding schemes.
- $S = S_0 \cup \{\eta\}$ and $X = X_0 \cup \{\eta_X\}$, where $\eta \in S$, $\eta_X \in X$ are the generic points.
- $\mathcal{X} = \mathcal{X}_0 \cup \mathcal{X}_1 \cup \{\eta_{\mathcal{X}}\}$, and $\eta_{\mathcal{X}} = \eta_X$, $X_0 \subset \mathcal{X}_1$ under the canonical inclusion $X \hookrightarrow \mathcal{X}_1$.

Notation/Remarks 3.1. Let n be a fixed prime number such that the group of roots of unity μ_{2n} of order $2n$ is contained in K. We notice/define the following:

(1) The local rings $\mathcal{O}_s$ at the closed points $s \in S_0$ of S are exactly the valuation rings of the nonarchimedean places of k. Further, for $x \in \mathcal{X}$ one has $x \mapsto s$ if and only if the corresponding local rings dominate each other: $\mathcal{O}_s \prec \mathcal{O}_x$ under $k \hookrightarrow K$.

(2) For $x_1 \in \mathcal{X}_1$, let $C_{x_1} = \overline{\{x_1\}} \subset \mathcal{X}$ be the schematic closure of x_1 in $\mathcal{X}$. Then C_{x_1} is an *arithmetic curve* on $\mathcal{X}$ with generic point $x_1 \in \mathcal{X}_1$. For $s \in S$, let $\mathcal{X}_s \to \kappa(s)$ be the fiber of $\mathcal{X} \to S$ at s. Then $\mathcal{X}_s$ is a projective (maybe nonreduced) one dimensional $\kappa(v)$-scheme of finite type. In the above notation, one has the following:

(a) $x_1 \mapsto \eta \in S$ if and only if $x_1 \in X_0$ if and only if $C_{x_1} \to S$ is finite dominant. If so, C_{x_1} is called a *horizontal curve* on $\mathcal{X}$, and we denote

$$\mathcal{X}_{1,\eta} := \{x_1 \in \mathcal{X}_1 \mid C_{x_1} \text{ is a horizontal curve}\}.$$

(b) $x_1 \mapsto s \in S_0$ if and only if C_{x_1} is a reduced irreducible component of $\mathcal{X}_s \to \kappa(s)$. If so, C_{x_1} is called a *vertical curve* on $\mathcal{X}$, and we denote

$$\mathcal{X}_{1,0} := \{x_1 \in \mathcal{X}_1 \mid C_{x_1} \text{ is a vertical curve}\}.$$

(3) One obviously has $\mathcal{X}_1 = \mathcal{X}_{1,\eta} \cup \mathcal{X}_{1,0}$, and the map $\mathcal{X}_{1,0} \to S_0$ has finite fibers.

(4) Since the generic fiber $X \to k$ of $\mathcal{X} \to S$ is a projective smooth geometrically integral k-curve, there exists a (unique) nonempty maximal open subset $U = U_{\mathcal{X}}$ of S such that $\mathcal{X}_U := \mathcal{X} \times_S U \to U$ is a family of projective smooth curves with geometrically integral fibers. For $s \in U_0 := U \cap S_0$, letting $x_s \in \mathcal{X}_s \subset \mathcal{X}$ be the generic point, one has:

(a) x_s is the unique preimage of s in $\mathcal{X}_1$, so $x_s \in \mathcal{X}_{1,0}$ and $C_{x_s} = \mathcal{X}_s$.

(b) $\kappa(s)$ is relatively algebraically closed in $\kappa(x_s)$.

(5) For $f \in K^\times$, let $|\mathrm{div}(f)| := \{P \in X_0 \mid v_P(f) \neq 0\} \subset X_0$ be the support of the divisor (f) of f viewed as a function on $X \to k$. Then $C_P = \overline{\{P\}}$ with $P \in |\mathrm{div}(f)|$ are distinct horizontal curves and $\bigcup_{P\in|\mathrm{div}(f)|} C_P$ is the closure of $|\mathrm{div}(f)|$ in $\mathcal{X}$. Therefore, there exists a unique maximal open subset $U_f = U_{\mathcal{X}f} \subset U$ satisfying:

(a) $\bigcup_{P\in|\mathrm{div}(f)|} C_P \to S$ is étale above U_f. Hence, $C_P \cap \mathcal{X}_s \cap C_{P'} = \varnothing$ for $P' \neq P$.

(b) For $s \in U_f$ and its unique preimage $x_s \in \mathcal{X}_{1,0}$ under $\mathcal{X}_{1,0} \to S$, the following hold:
   - n is invertible in $\kappa(s)$.
   - f is a v_{x_s}-unit, and its residue $\bar{f} \in \kappa(x_s)$ nonconstant: $\bar{f} \in \kappa(x_s) \setminus \kappa(s)$.

(6) Finally, we notice that $U_f \subseteq U_0$ has the following permanence property: Let $\tilde{k}|k$ be a finite extension, and set $\widetilde{K} := K\tilde{k}$. Let $\widetilde{S} \to S$ be the normalization of S in $k \hookrightarrow \tilde{k}$, and $\widetilde{\mathcal{X}} \to \widetilde{S}$ be a minimal proper regular model of $\widetilde{K}|\tilde{k}$ which dominates $\mathcal{X} \to S$. In particular, the generic fiber $\widetilde{X} \to \tilde{k}$ of $\widetilde{\mathcal{X}} \to \widetilde{S}$ is the normalization $\widetilde{X} \to X$ of X in $K \hookrightarrow \widetilde{K}$. Let $U_{\widetilde{\mathcal{X}}} \subset \widetilde{S}$ be the maximal open subset such that $\widetilde{\mathcal{X}}_{U_{\widetilde{\mathcal{X}}}} := \widetilde{\mathcal{X}} \times_{\widetilde{S}} U_{\widetilde{\mathcal{X}}} \to U_{\widetilde{\mathcal{X}}}$ is smooth and has reduced geometrically integral fibers, and define the subsets $U_{\widetilde{\mathcal{X}}f} \subseteq U_{\widetilde{\mathcal{X}}}$ for the model $\widetilde{\mathcal{X}} \to \widetilde{S}$ of $\widetilde{K}|\tilde{k}$ and $f \in \widetilde{K}$ in the way the subsets $U_{\mathcal{X}f} \subseteq U_{\mathcal{X}}$ of S were defined above for the model $\mathcal{X} \to S$ of $K|k$ and $f \in K$. Then one has:

Lemma 3.2. *In the above notation, let $\widetilde{U}_{\mathcal{X}f} \subseteq \widetilde{U}_{\mathcal{X}}$ be the preimages of $U_{\mathcal{X}f} \subseteq U_{\mathcal{X}}$ under the map $\widetilde{S} \to S$. Then $\widetilde{\mathcal{X}} \times_{\widetilde{S}} \widetilde{U}_{\mathcal{X}} \to \widetilde{U}_{\mathcal{X}}$ is smooth, whence $\widetilde{U}_{\mathcal{X}} \subseteq U_{\widetilde{\mathcal{X}}}$ and $\widetilde{\mathcal{X}} \times_{\widetilde{S}} \widetilde{U}_{\mathcal{X}} = \mathcal{X} \times_S \widetilde{U}_{\mathcal{X}}$. Further, $\widetilde{U}_{\mathcal{X}f} \subseteq U_{\widetilde{\mathcal{X}}f}$, and the morphism $\widetilde{\mathcal{X}} \to \mathcal{X}$ is finite above $\mathcal{X} \times_S U_{\mathcal{X}}$.*

Proof. For the first inclusion, let $\mathcal{X}^{\mathrm{n}} \to \widetilde{S}$ denote the normalization of $\mathcal{X} \to S$ in the field extension $K \hookrightarrow \widetilde{K}$. Then $\widetilde{\mathcal{X}}$ being regular, it is also normal. Thus $\widetilde{\mathcal{X}} \to \widetilde{S}$ dominates $\mathcal{X}^{\mathrm{n}} \to \widetilde{S}$. Moreover, since the base change $\mathcal{X} \times_S \widetilde{S} \to \widetilde{S}$ is dominant and finite over $\mathcal{X} \to S$, it follows that $\mathcal{X}^{\mathrm{n}} \to \widetilde{S}$ dominates $\mathcal{X} \times_S \widetilde{S} \to \widetilde{S}$. Thus finally $\widetilde{\mathcal{X}} \to \widetilde{S}$ dominates $\mathcal{X} \times_S \widetilde{S} \to \widetilde{S}$. To simplify notation, set $U := U_{\mathcal{X}}$ and $\widetilde{U} := \widetilde{U}_{\mathcal{X}}$. Since $\mathcal{X}_U := \mathcal{X} \times_S U \to U$ is smooth and has reduced geometrically integral fibers, so is the base change $\mathcal{X}_U \times_S \widetilde{U} \to \widetilde{U}$, and in particular $\mathcal{X}_U \times_S \widetilde{U}$ is regular. Hence, by the minimality of $\widetilde{\mathcal{X}} \to \widetilde{S}$, one has $\widetilde{\mathcal{X}} \times_{\widetilde{S}} \widetilde{U} = \mathcal{X}^{\mathrm{n}} \times_{\widetilde{S}} \widetilde{U} = \mathcal{X}_U \times_S \widetilde{U}$. Therefore, $\widetilde{\mathcal{X}}_{\widetilde{U}} := \widetilde{\mathcal{X}} \times_{\widetilde{S}} \widetilde{U} \to \mathcal{X}_U$ is finite because $\mathcal{X}^{\mathrm{n}} \to \mathcal{X}$ is. Since $\widetilde{\mathcal{X}}_{\widetilde{U}} = \mathcal{X} \times_S \widetilde{U} \to \widetilde{U}$ is also smooth, one has $\widetilde{U}_{\mathcal{X}} \subseteq U_{\widetilde{\mathcal{X}}}$ by the maximality of the latter. The other inclusion follows immediately from the fact that being étale is preserved under base change, and the fact that $\widetilde{\mathcal{X}} \times_{\widetilde{S}} \widetilde{U}_{\mathcal{X}} = \mathcal{X} \times_S \widetilde{U}_{\mathcal{X}}$. □

3B. ***A local-global principle for*** $\mathbf{H}^3(\mathcal{X}, f)$**.** We work in the context and the notation of the previous subsection. Let $\mathcal{X}_{1f} \subset \mathcal{X}_1$ be the preimage of U_f under $\mathcal{X}_1 \to S$. We notice that $\mathcal{X}_1 \setminus \mathcal{X}_{1f}$ is the finite closed subset of $\mathcal{X}_{1,0}$ consisting of all $x_1 \in \mathcal{X}_1$ which map into the (finite) closed set $S_0 \setminus U_f$.

Notation. Let $\mathrm{H}^3(\mathcal{X}, f) \subset \mathrm{H}^3(K, \mathbb{Z}/n(2))$ denote the set of all the symbols $f \cup a \cup b$ with $a, b \in k^\times$ which are nontrivial over some completion K_{x_1} with $x_1 \in \mathcal{X}_{1f}$.

Lemma 3.3. *Let $D_f \subseteq |\mathrm{div}(f)|$ be the set of all P such that $v_P(f)$ is not divisible by n in $v_P(K)$. Suppose that K has no orderings. Then for every $f \cup a \cup b \in \mathrm{H}^3(\mathcal{X}, f)$ there exists $P \in D_f$ such that $f \cup a \cup b$ is nontrivial over K_P.*

Proof. Let $z_1 \in \mathcal{X}_1$ be a given point. Then by the concrete description of the boundary homomorphism $\hat{\partial}_{z_1}$ as given before Theorem 2.1, one has that if $v_{z_1}(f)$, $v_{z_1}(a)$, $v_{z_1}(b)$ are all divisible by n, then $f \cup a \cup b$ is trivial over the completion K_{z_1} at v_{z_1}. In particular, if $z_1 \in \mathcal{X}_{1,\eta}$, then $a, b \in k^\times$ are v_{z_1}-units. Thus $v_{z_1}(a) = 0 = v_{z_1}(b)$ are divisible by n. Hence, if $v_{z_1}(f)$ is divisible by n, then $f \cup a \cup b$ is trivial over K_{z_1}.

Returning to the proof of the lemma, let $f \cup a \cup b \in \mathrm{H}^3(\mathcal{X}, f)$ be a given element, and let $x_1 \in \mathcal{X}_{1f} \subset \mathcal{X}_1$ be such that $f \cup a \cup b$ is nontrivial over the completion K_{x_1}.

Case 1. $x_1 \in \mathcal{X}_{1,\eta} = X_0$. Then v_{x_1} is trivial on k, so $v_{x_1}(a) = 0 = v_{x_1}(b)$. Hence, since $f \cup a \cup b \in \mathrm{H}^3(\mathcal{X}, f)$ is nontrivial over the completion K_{x_1}, it follows by the discussion above that $v_{x_1}(f)$ is not divisible by n. Thus $P := x_1 \in D_f$, and we are done.

Case 2. $x_1 \in \mathcal{X}_{1,0}$. Let $s \in U_f$ be the image of x_1 under $\mathcal{X}_{1f} \to U_f \subset S$. Then by the definition of $\mathcal{X}_{1f}$, one has that $x_s := x_1$ is the unique preimage of s in $\mathcal{X}_1$, and the following hold:

- $\mathcal{X}_s$ is a projective smooth geometrically integral $\kappa(s)$-curve, and $C_{x_s} = \mathcal{X}_s$.
- For all $P \neq P'$ in $|\mathrm{div}(f)|$, if $x_0 \in \mathcal{X}_s \cap C_P$, then $x_0 \notin \mathcal{X}_s \cap C_{P'}$.
- n is invertible in $\kappa(s)$.
- f is a v_{x_s}-unit, and its residue $\bar{f} \in \kappa(x_s)$ is nonconstant, i.e., $\bar{f} \in \kappa(x_s) \setminus \kappa(s)$.

From this we reason as follows. Let $x_0 \in \mathcal{X}_0$ be a closed point with $x_0 \mapsto s \in U_f$, and let $z_1 \in \mathcal{X}_1$ satisfy that $x_0 \in C_{z_1}$ and not all $v_{z_1}(a)$, $v_{z_1}(b)$, $v_{z_1}(f)$ are zero. Then we have:

(a) If $z_1 \in \mathcal{X}_{1,0}$, i.e., z_1 maps to some $s' \in S_0$, then C_{z_1} is a vertical curve; and since $C_{z_1} \ni x_0 \mapsto s$, we must have $C_{z_1} = \mathcal{X}_s$, so that $z_1 = x_1$, etc.

(b) If $P := z_1 \in \mathcal{X}_{1,\eta} = X_0$, then a, b are v_P-units, and therefore $v_P(f) \neq 0$. Thus $P \in |\mathrm{div}(f)|$, and $x_0 \in \mathcal{X}_s \cap C_P$. Moreover, P is the unique point in $|\mathrm{div}(f)|$ with the property $x_0 \in C_P \cap \mathcal{X}_s$.

From this we can conclude the following. Let $x_0 \in \mathcal{X}$ be a closed point above the point $s \in U_f$. Then there exist at most two points $z_1 \in \mathcal{X}_1$, each satisfying that $x_0 \in C_{z_1}$ and at least one of the values $v_{z_1}(a)$, $v_{z_1}(b)$, $v_{z_1}(f)$ is nonzero. Further, the two (potential) points are

- the given point $x_s = x_1 \in \mathcal{X}_{1,0}$ — note that $v_{x_s}(f) = 0$, $\bar{f} \in \kappa(x_s)$ is nonconstant, etc.;
- the unique $P_0 \in |\mathrm{div}(f)|$ such that $x_0 \in C_{P_0} \cap \mathcal{X}_s$ — note that a, b are v_{P_0}-units.

Therefore, the image of $f \cup a \cup b \in \mathrm{H}^3(K, \mathbb{Z}/n(2))$ in $\bigoplus_{x_1 \in \mathcal{X}_{x_0}} \mathrm{H}^2(\kappa(x_1), \mathbb{Z}/n(1))$ under the homomorphism of Theorem 2.1 actually lies in

$$\mathrm{H}^2(\kappa(P_0), \mathbb{Z}/n(1)) \oplus \mathrm{H}^2(\kappa(x_s), \mathbb{Z}/n(1)).$$

Let us compute $\partial_{x_s}(f \cup a \cup b)$. First, since $s \in U_f$, we have $f \in \mathcal{O}_{x_s}^\times$ and $\bar{f} \in \kappa(x_s)$ is nonconstant. Second, every uniformizing parameter $\pi \in k$ at s is also a uniformizing parameter π at x_s, because $\mathcal{X}_s$ is reduced by the fact that $s \in U_f$. For such a π, set $a = \pi^q a'$ and $b = \pi^r b'$ with $a', b' \in \mathcal{O}_s^\times$. Then setting $c = a'^r / b'^q \in \mathcal{O}_s^\times$, it follows by the definition of ∂_{x_s} that we have $0 \neq \partial_{x_s}(f \cup a \cup b) = \bar{f} \cup \bar{c} \in \mathrm{H}^2(\kappa(x_s), \mathbb{Z}/n(1))$.

Next, recall that since $s \in U_f$, the special fiber $\mathcal{X}_s \to \kappa(s)$ is a complete smooth geometrically integral model of the global function field $\kappa(x_s) | \kappa(s)$. Since $\bar{f} \cup \bar{c}$ is nontrivial in $\mathrm{H}^2(\kappa(x_s), \mathbb{Z}/n(1))$, by the Hasse–Brauer–Noether LGP, there exists a closed point $x_0 \in \mathcal{X}_{s,0} \subset \mathcal{X}_0$ such that $\bar{f} \cup \bar{c}$ is nontrivial over the completion $\kappa(x_s)_{x_0}$. Equivalently, the boundary homomorphism

$$\partial_{x_0} : \mathrm{H}^2(\kappa(x_s), \mathbb{Z}/n(1)) \to \mathrm{H}^1(\kappa(x_0), \mathbb{Z}/n)$$

maps $\bar{f} \cup \bar{\gamma}$ to some nontrivial element in $\mathrm{H}^1(\kappa(x_0), \mathbb{Z}/n)$.

Let $\mathcal{O}_{x_0}$ be the local ring of $x_0 \in \mathcal{X}$ viewed as a closed point of $\mathcal{X}$, and let $\overline{\mathcal{O}}_{x_0} \subset \kappa(x_s)$ be the image of $\mathcal{O}_{x_0}$ under the canonical projection $\mathcal{O}_{x_s} \to \kappa(x_s)$. Then by scheme-theoretical nonsense, it follows that $\overline{\mathcal{O}}_{x_0}$ is the local ring of the point $x_0 \in \mathcal{X}_{s,0}$ viewed as a closed point of $\mathcal{X}_s$. Hence, since the latter is a smooth curve over $\kappa(s)$, and thus regular, it follows that $\overline{\mathcal{O}}_{x_0} = \mathcal{O}_{\mathcal{X}_s, x_0}$ is regular. Therefore, by the definition of ∂_{x_s, x_0} as described before Theorem 2.1, it follows that $\partial_{x_s x_0}(\bar{f} \cup \bar{c}) = \partial_{x_0}(\bar{f} \cup \bar{c})$. Hence we conclude that $\partial_{x_s x_0}(\bar{f} \cup \bar{c}) \in \mathrm{H}^1(\kappa(x_0), \mathbb{Z}/n)$ is nontrivial. Viewing x_0 as a closed point of $\mathcal{X}$, we conclude that the image of $f \cup a \cup b$ under $\mathrm{H}^3(K, \mathbb{Z}/n(2)) \to \mathrm{H}^2(\kappa(x_s), \mathbb{Z}/n(1)) \to \mathrm{H}^1(\kappa(x_0), \mathbb{Z}/n)$ is nontrivial.

On the other hand, the image of $f \cup a \cup b$ in $\bigoplus_{x_1 \in \mathcal{X}_{x_0}} \mathrm{H}^2(\kappa(x_1), \mathbb{Z}/n(1))$ lies in $\mathrm{H}^2(\kappa(P_0), \mathbb{Z}/n(1)) \oplus \mathrm{H}^2(\kappa(x_s), \mathbb{Z}/n(1))$, by the discussion above.

For a contradiction, suppose that the image of $f \cup a \cup b$ in $\mathrm{H}^2(\kappa(P_0), \mathbb{Z}/n(1))$ is trivial. Then the image of $f \cup a \cup b$ in $\bigoplus_{x_1 \in \mathcal{X}_{x_0}} \mathrm{H}^2(\kappa(x_1), \mathbb{Z}/n(1))$ lies in $\mathrm{H}^2(\kappa(x_s), \mathbb{Z}/n(1))$, and this image is $\bar{f} \cup \bar{c}$. Thus, the image of $f \cup a \cup b$ under the canonical map

$$\mathrm{H}^3(K, \mathbb{Z}/n(2)) \to \bigoplus_{x_1 \in \mathcal{X}_{x_0}} \mathrm{H}^2(\kappa(x_1), \mathbb{Z}/n(1)) \to \mathrm{H}^1(\kappa(x_0), \mathbb{Z}/n)$$

is nontrivial. This contradicts Theorem 2.1.

Therefore, the image of $f \cup a \cup b$ in $\mathrm{H}^2(\kappa(P_0), \mathbb{Z}/n(1))$ must be nontrivial. Since that image is $v_{P_0}(f) \cdot a \cup b$, we conclude, first, that $v_{P_0}(f)$ is not divisible by n, so that $P_0 \in D_f$, and second, that $f \cup a \cup b$ is nontrivial over K_{P_0}. □

3C. ***The Chebotarev density theorem and the size of*** $\mathbf{H^3(\mathcal{X}, f)}$. Let $\lambda | k$ be a finite extension with $\mu_{2n} \subset \lambda$. Consider $\alpha \in k$ which is not an n-th power in λ, or equivalently, $\tilde{\lambda} := \lambda[\sqrt[n]{\alpha}\,]$ is a cyclic extension of degree n of λ. Let further $\hat{\lambda} | k$ be some finite Galois extension of k containing $\tilde{\lambda}$, and let $\hat{T} \to \widetilde{T} \to T \to S$ be the normalizations of S in the field extensions $k \hookrightarrow \lambda \hookrightarrow \tilde{\lambda} \hookrightarrow \hat{\lambda}$. For a generator $\sigma \in \mathrm{Gal}(\tilde{\lambda} | \lambda)$, consider a preimage $\tau \in \mathrm{Gal}(\hat{\lambda} | \lambda) \subseteq \mathrm{Gal}(\hat{\lambda} | \lambda)$. Let $\hat{T}_\alpha \to S_\alpha$ be the sets of all the points $\hat{z} \mapsto s$ such that α is a v_s-unit and τ is the Frobenius $\kappa(\hat{z}) | \kappa(s)$. Notice that by the Chebotarev density theorem, S_α has a positive Dirichlet density, and that for $\hat{z} \in \hat{T}_\alpha$ and its image $s \in S_\alpha$ one has $\kappa(\hat{z}) = \kappa(s)[\hat{\gamma}]$ with $\hat{\gamma}^m = \bar{\alpha}$ and m the order of τ.

Finally, let $\tilde{z}_\alpha \to z_\alpha$ be the images of $\hat{z}_\alpha$ in $\widetilde{T} \to T$. Then for $\widetilde{S}_\alpha \ni \tilde{z} \mapsto z \in T_\alpha$, one has that $\tilde{z} | z$ is unramified and has σ as Frobenius automorphism: $\kappa(z) = \kappa(s)$ and $\kappa(\tilde{z}) = \kappa(s)[\gamma]$, where $\gamma^n = \bar{\alpha}$. Thus we have showed the following:

Fact 3.4. *Let $\lambda | k$ be a finite extension of global fields, $\mu_{2n} \subset \lambda$, and $\alpha \in k$ not an n-th power in λ. Let $T \to S$ be the normalization of S in $k \hookrightarrow \lambda$. There exist subsets $S_\alpha \subset S$ of positive Dirichlet density and $T_\alpha \subset T$ mapping onto S_α such that for all $T_\alpha \ni z \mapsto s \in S_\alpha$ one has $k_s = \lambda_z$, and α and n are v_s-units, and α is not an n-th power in $k_s = \lambda_z$.*

Notation/Remarks 3.5. For $\alpha, \delta \in k^\times$ and $V_\delta := \{s \in S \mid v_s(\delta) = 0\} \subset S$ open and nonempty, and the subgroup $\mathrm{H}_\delta = \{\beta \in k^\times \mid v_s(\beta - 1) > 2v_s(2n) \text{ if } s \notin V_\delta\} \subset k^\times$, consider/define

(1) $\mathrm{H}_{\delta\alpha} := \alpha \cup \mathrm{H}_\delta \subset \mathrm{H}^2(k, \mathbb{Z}/n(1))$,

(2) $\mathrm{H}_{f\delta\alpha} := f \cup \alpha \cup \mathrm{H}_\delta = f \cup \mathrm{H}_{\delta\alpha} \subset \mathrm{H}^3(K, \mathbb{Z}/n(2))$.

Notice that by Hensel's lemma we have that if $\beta \in \mathrm{H}_\delta$ then β is an n-th power in k_s for all $s \notin V_\delta$.

Lemma 3.6. *Suppose that $V_\delta \subseteq U_f$ and that $\mathrm{H}_{f\delta\alpha} \neq 0$. Then every nonzero $f \cup \alpha \cup \beta \in \mathrm{H}_{f\delta\alpha}$ lies in $\mathrm{H}^3(\mathcal{X}, f)$, and for every such $f \cup \alpha \cup \beta$ the following hold*:

(1) *There exists $P \in X$ such that $f \cup \alpha \cup \beta$ is nontrivial over K_P. Hence, $P \in D_f$ and α is not an n-th power in K_P nor in the residue field $\kappa(P)$.*

(2) $\mathrm{H}_{f\delta^*\alpha}$ *is nontrivial over K_P for all P as in (1) above and all $\delta^* \in k^\times$.*

Proof. We first prove that every nonzero $f \cup \alpha \cup \beta \in \mathrm{H}_{f\delta\alpha}$ actually lies in $\mathrm{H}^3(\mathcal{X}, f)$. Indeed, by Theorem 2.1, there exists some $x_1 \in \mathcal{X}_1$ such that $f \cup \alpha \cup \beta$ is nontrivial over the completion K_{x_1}. Let $x_1 \mapsto s \in S$ be the image of x_1 in S. We claim that $s \in V_\delta$. By contradiction suppose that $s \notin V_\delta$. Then by Notation/Remarks 3.5, β is an n-th power in k_s and in particular, $\alpha \cup \beta$ is trivial over k_s. Further, since $x_1 \mapsto s$, we have $k_s \subseteq K_{x_1}$. Hence $f \cup \alpha \cup \beta$ is trivial over $\widetilde{K}_{\tilde{x}_1}$, a contradiction! Thus finally $s \in V_\delta$, and since $V_\delta \subseteq U_f$ we have $s \in U_f$.

By (1), since $f \cup \alpha \cup \beta \in \mathrm{H}^3(\mathcal{X}, f)$, by Lemma 3.3 it follows that there exists $P \in D_f$ such that $f \cup \alpha \cup \beta$ is nontrivial over K_P. In particular, α is not an n-th power in K_P, etc.

For (2), by the discussion above, α is not an n-th power in $\lambda := \kappa(P)$. In the notation from Fact 3.4, for some fixed $s^* \in V_{\delta^*} \cap S_\alpha$, let β^* be a uniformizing parameter at s^* such that β^* is an n-th power in k_s for all $s \notin V_{\delta^*}$. Then $\alpha \cup \beta^*$ satisfies first, $\beta^* \in \mathrm{H}_{\delta^*}$, so $\alpha \cup \beta^* \in \mathrm{H}_{\delta^*\alpha}$ and $f \cup \alpha \cup \beta^* \in \mathrm{H}_{f\delta^*\alpha}$. Second, $\alpha \cup \beta^*$ is trivial over all k_s with $s \notin V_{\delta^*}$, because β^* is an n-th power in k_s. On the other hand, $\alpha \cup \beta^*$ is not trivial over λ_z for $z \mapsto s^*$ because β^* is a uniformizing parameter at s^* and at all $z \mapsto s^*$. Hence $\alpha \cup \beta^*$ is not trivial over $\kappa(P) \subset \kappa(P)_z$. But then, since $\hat{\partial}_P : \mathrm{H}^3(K_P, \mathbb{Z}/n(2)) \to \mathrm{H}^2(\kappa(P), \mathbb{Z}/n(1))$ is an isomorphism and $\hat{\partial}_P(f \cup \alpha \cup \beta^*) = v_P(f) \cdot \alpha \cup \beta^* \neq 0$, we get that $f \cup \alpha \cup \beta^*$ is nontrivial over K_P. □

4. Detecting the *k*-valuations of *K* | *k*

In this section we work in the context/notation of the previous sections: $n \neq \mathrm{char}(K)$ is a prime number and $\mu_{2n} \subset K$. So if $n = 2$, then $\mu_4 \subset K$, and if $n \neq 2$, then $\mu_n \subset K$.

4A. *The sets $\mathcal{U}_\bullet$.*

Notation/Remarks 4.1. In the usual context we have the following:

(1) For $u \in K$ and $\alpha, c \in k$, set $u_{\alpha,c} = 1 - c(1-u) + \alpha c^n (1-u)^n$, and further define $u_c := u_{0,c} = 1 + c(u-1)$ and $u_\alpha := u_{\alpha,1} = u + \alpha(1-u)^n$.

(2) For $u \in K^\times$ and $c \in k^\times$ we set $K_{u,\alpha,c} := K[\sqrt[n]{u_c}, \sqrt[n]{u_{\alpha,c}}\,]$, and notice that $K_{u,\alpha,c} \,|\, K$ is a $\mathbb{Z}/n$-elementary abelian extension of degree 1, n, or n^2.

(3) In Notation/Remarks 3.5, suppose that $\mathrm{H}_{f\delta\alpha} \neq 0$. Thus $\mathrm{H}_{f\delta^*\alpha} \neq 0$ for all $\delta^* \in k^\times$. We set $\mathcal{U}_{f\alpha} := \{u \in K^\times \mid \mathrm{H}_{f\delta^*\alpha}$ is nontrivial over $K_{u,\alpha,c}$ for all $c, \delta^* \in k^\times\}$.

(4) For u, α, c as above, set $D_{u,\alpha,c} := \{P \in D_f \mid u_c, u_{\alpha,c} \in \mathcal{O}_P^\times\}$.

(5) Finally, let $D_{f\alpha} := \{P \in D_f \mid \alpha$ is not an n-th power in $\kappa(P)\}$. Note that by Lemma 3.6, if $\mathrm{H}_{f\delta\alpha}$ is nontrivial then $D_{f\alpha}$ is nonempty.

Lemma 4.2. *Let $Y \to X$, $Q \mapsto P$, be the normalization of X in $K \hookrightarrow L := K_{u,\alpha,c}$ and suppose that $\sqrt[n]{\alpha} \notin L_Q$. Then $u_c, u_{\alpha,c} \in \mathcal{O}_P^\times$ and hence $P \in D_{u,\alpha,c}$.*

Proof. Let us analyze what happens if either u_c or $u_{\alpha,c}$ is not a v_P-unit. We first claim that u is v_P-integral. Indeed, by contradiction, suppose that $v_P(u) < 0$. Then $v_P(1/u) > 0$, and $u_{\alpha,c} = \eta\alpha(-cu)^n$, where η is a principal v_P-unit. But then η is a principal v_Q-unit too, and hence η is an n-th power in L_Q. Conclude that $\sqrt[n]{\alpha} \in L_Q$: contradiction! Thus finally u must be v_P-integral. Further, if u is a principal v_P-unit, then so are $u_c, u_{\alpha,c}$, and thus $u_c, u_{\alpha,c} \in \mathcal{O}_P^\times$. Hence, it is left to analyze what happens if $v_P(u) \geq 0$ and u is not a principal v_P-unit. First we remark that $1-u$ is a v_P-unit, and hence so is $\alpha c^n(1-u)^n$. Second, both u_c and $u_{\alpha,c}$ are v_P-integral. Therefore, since $u_{\alpha,c} = u_c + \alpha c^n(1-u)^n$, it follows that at least one of the elements u_c and $u_{\alpha,c}$ is a v_P-unit. By contradiction, suppose that either u_c or $u_{\alpha,c}$ is not a v_P-unit. Then either $v_P(u_c) = 0$ and $v_P(u_{\alpha,c}) > 0$, or vice versa.

Case 1. $v_P(u_c) = 0$ and $v_P(u_{\alpha,c}) > 0$.

Then $\alpha = -u_c(1-u_{\alpha,c}/u_c)/c^n(1-u)^n$. Since $\mu_{2n} \subset K$, it follows that -1 is an n-th power in K, and since $1-u_{\alpha,c}/u_c$ is a principal v_P-unit, it is an n-th power in L_Q. Hence all the factors on the right-hand side are n-th powers in L_Q. Thus $\sqrt[n]{\alpha} \in L_Q$: contradiction!

Case 2. $v_P(u_{\alpha,c}) = 0$ and $v_P(u_c) > 0$.

Then $\alpha = u_{\alpha,c}(1-u_c/u_{\alpha,c})/c^n(1-u)^n$ with $1-u_c/u_{\alpha,c}$ a principal v_P-unit. But then all the factors on the right-hand side are n-th powers in $K_P \subset L_Q$. Hence $\sqrt[n]{\alpha} \in K_P \subseteq L_Q$: contradiction!

We thus conclude that $u_c, u_{\alpha,c} \in \mathcal{O}_P^\times$, as claimed. □

Lemma 4.3. *Suppose that $V_\delta \subseteq U_f$ and $\mathrm{H}_{f\delta\alpha}$ is nontrivial. Then the following hold*:

(1) *If $u \in \mathcal{U}_{f\alpha}$ then $u_c \in \mathcal{U}_{f\alpha}$ for all $c \in k$. And if $c \neq 0$ and $u_c \in \mathcal{U}_{f\alpha}$, then $u \in \mathcal{U}_{f\alpha}$.*

(2) $1 + \bigcup_{P \in D_{f\alpha}} \mathfrak{m}_P \subseteq \mathcal{U}_{f\alpha}$.

(3) *For every $u \in \mathcal{U}_{f\alpha}$ and each resulting $u_c, u_{\alpha,c}$ the following hold*:
   - (a) *There exists $P \in D_{f\alpha}$ with $u_c, u_{\alpha,c} \in \mathcal{O}_P^\times$ and $\mathrm{H}_{f\delta^*\alpha}$ nontrivial over $K_P K_{u,\alpha,c}$ for all δ^*.*
   - (b) *There exists δ^* such that if $\mathrm{H}_{f\delta^*\alpha}$ is nontrivial over $K_P K_{u,\alpha,c}$, then $u_c, u_{\alpha,c} \in \mathcal{O}_P^\times$ and $P \in D_{f\alpha}$.*

Proof. (1): For all $a, c, c' \in k$, $(u_c)_{a,c'} = 1 - cc'(1-u) + a(cc')^n(1-u)^n = u_{a,cc'}$, and therefore $(u_c)_{c'} = u_{cc'}$ and $(u_c)_{\alpha,c'} = u_{\alpha,cc'}$. Hence $\{(u_c)_{c'}, (u_c)_{\alpha,c'}\} = \{u_{cc'}, u_{\alpha,cc'}\}$. Now suppose that $u \in \mathcal{U}_{f\alpha}$. Then by the definition of $\mathcal{U}_{f\alpha}$ it follows that $\mathrm{H}_{f\delta^*\alpha}$ is nontrivial over $K_{u,\alpha,c''}$ for all $c'' \in k$ and all $\delta^* \in k^\times$. In particular, setting $c'' := cc'$, it follows that $\mathrm{H}_{f\delta^*\alpha}$ is nontrivial over $K_{u_c,\alpha,c'}$, etc. The converse is clear, because given c' and u_c, by the discussion above one has that $\{u_{c'}, u_{\alpha,c'}\} = \{(u_c)_{c'/c}, (u_c)_{\alpha,c'/c}\}$, etc.

For the proof of assertions (2) and (3) we first set up notation as follows: For $u \in K^\times$ and $c \in k$, set as usual $L := K_{u,\alpha,c}$, and further, $l := L \cap \bar{k}$. Let $T \to S$ be the normalization of S in $k \hookrightarrow l$, and $\mathcal{Y} \to T$ be the minimal proper regular model of $L\,|\,l$ which dominates $\mathcal{X} \to S$. In particular, the generic fiber $Y \to l$ of $\mathcal{Y} \to T$ is the normalization $Y \to X$ of X in the field extension $K \hookrightarrow L$.

(2): Let $u \in 1 + \mathfrak{m}_P$ be a principal unit at some $P \in D_{f\alpha}$. Since $P \in D_{f\alpha}$, we have by definition that α is not an n-th power in $\kappa(P)$ nor in K_P, and $\mathrm{H}_{f\delta^*\alpha}$ is nontrivial over K_P for all $\delta^* \in k^\times$ by Lemma 3.6. On the other hand, since u is a principal v_P-unit, $u_c, u_{\alpha,c}$ are principal v_P-units too (by mere definitions). Therefore, P is totally split in the field extension $L\,|\,K$, and thus for every $Q \mapsto P$ one has $L_Q = K_P$. Hence, $\mathrm{H}_{f\delta^*\alpha}$ is nontrivial over $L_Q = L_P$ (because it was nontrivial over K_P). But then $\mathrm{H}_{f\delta^*\alpha}$ is nontrivial over $L \subset L_Q$ too.

(3): For the proper regular model $\mathcal{Y} \to T$ of $L\,|\,l$ and $f \in L$, we define the open nonempty subsets $U_{\mathcal{Y}f} \subseteq U_{\mathcal{Y}}$ of T, as we defined the sets $U_f \subseteq U_{\mathcal{X}}$ of S for the proper regular model $\mathcal{X} \to S$ of $K\,|\,k$ and $f \in K$ at Notation/Remarks 3.1(5). For both assertions (a) and (b), we consider δ^* which satisfy $V_{\delta^*} \subseteq U_f$, and the preimage of V_{δ^*} under $T \to S$ is contained in $U_{\mathcal{Y}f}$. For such a $\delta^* \in k^\times$ let $f \cup \alpha \cup \beta^* \in \mathrm{H}_{f\delta^*\alpha}$ be nontrivial over L.

Claim. The image of $f \cup \alpha \cup \beta^*$ in $\mathrm{H}^3(L, \mathbb{Z}/n(2))$ lies in $\mathrm{H}^3(\mathcal{Y}, f)$.

Indeed, since $f \cup \alpha \cup \beta^*$ is nontrivial over L, by Theorem 2.1, there exists some $y_1 \in \mathcal{Y}_1$ such that $f \cup \alpha \cup \beta^*$ is nontrivial over L_{y_1}. Let $y_1 \mapsto z \mapsto s^*$ be the images of y_1 in $T \to S$. We claim that $s \in V_{\delta^*}$, and thus $z \in U_{\mathcal{Y}f}$ by the definition of δ^*. Indeed, by contradiction, suppose that $s^* \notin V_{\delta^*}$. Then reasoning as in the proof of Lemma 3.6, taking into account that β^* is an n-th power in k_s for $s \notin V_{\delta^*}$, we conclude that $\alpha \cup \beta^*$ is trivial over k_{s^*} because β^* is in k_{s^*}. Hence $f \cup \alpha \cup \beta^*$ is trivial over L_{y_1}, because $k_{s^*} \subset L_{y_1}$. Contradiction! The claim is proved.

(a): For $u \in \mathcal{U}_{f\alpha}$ and $f \cup \alpha \cup \beta^* \in \mathrm{H}_{f\delta^*\alpha}$, which is nontrivial over L, by Lemma 3.3 applied to $f \cup \alpha \cup \beta^* \in \mathrm{H}^3(\mathcal{Y}, f)$, one found that there exists some $Q \in Y$ such that $v_Q(f)$ is not divisible by n in $v_Q(L)$, and α is not an n-th power in L_Q, nor in $\kappa(Q)$. Further, $\mathrm{H}_{f\delta^*\alpha}$ is nontrivial over $L_Q = K_P K_{u,\alpha,c}$ for all $\delta^* \in k^\times$. Let $Q \mapsto P \in X$ be the image of Q in X, and consider the canonical embeddings $K_P \hookrightarrow L_Q$, $\kappa(P) \hookrightarrow \kappa(Q)$, and recall that $v_Q = e(Q\,|\,P)\, v_P$, where $e(Q\,|\,P)$ is the ramification index of $v_Q\,|\,v_P$. Hence the following hold:

- Since $v_Q(f) \notin n \cdot v_Q(L)$, one has that $v_P(f) \notin n \cdot v_P(K)$. Therefore, $P \in D_f$.
- Since $\sqrt[n]{\alpha} \notin \kappa(Q)$, one has that $\sqrt[n]{\alpha} \notin \kappa(P)$. Therefore, $P \in D_{f\alpha}$.

Hence $\mathrm{H}_{f\delta^*\alpha}$ is nontrivial over $K_P K_{u,\alpha,c}$, and $P \in D_{f\alpha}$ and $u_c, u_{\alpha,c} \in \mathcal{O}_P^\times$ by Lemma 4.2.

(b): Clear from the discussion above. $\square$

4B. ***The k-rings*** $\mathfrak{R}_\bullet$ ***and*** $\boldsymbol{R_\bullet}$**.** In the above notation and context, we introduce the ring stabilizer $\mathfrak{R}_{f\alpha}$ of $\mathcal{U}_{f\alpha}$, which will play an essential role in describing k-valuations of $K\,|\,k$.

Notation/Remarks 4.4. (1) Let $A, +, \cdot$ be a commutative ring with 1_A, and if $k \subset A$ is a subfield in A having identity equal to 1_A, we consider A as a k-algebra. It seems that the following are well known facts to (a nonempty set of) experts:

> *Let $X \subset A$ satisfy $X = -X$ and $0_A \in X$, and set $X_0 := \{x \in A \mid x + X \subseteq X\}$. Then $X_0 \subseteq X$, and $R_X := \{a \in X_0 \mid a \cdot X_0 \subseteq X_0\}$ is a subring of R, which contains 1_A if and only if $1_A \in X_0$. Moreover, if A is a k-algebra, and X is stable under multiplication with k, then R_X is a k vector subspace.*

(2) Given a commutative ring $A, +, \cdot$ with 1_A as above, let $*$ and $\circ$ be the transport of the usual addition and multiplication, respectively, on the underlying set A via $a \mapsto a + 1_A$. Hence $a * b = a + b - 1_R$ and $a \circ b = ab - a - b + 2$, and A endowed with $*, \circ$ is an isomorphic copy of $A, +, \cdot$ which we denote $\mathfrak{A}$.

If $X \subset A$ is a nonempty subset as in (1) above, we let $\mathfrak{R}_X \subset \mathfrak{A}$, or simply $\mathfrak{R}$ if no confusion is possible, be the corresponding subring of $\mathfrak{A}$.

(3) In the context and notation of the previous subsection, recall the nonempty set $X := \mathcal{U}_{f\alpha}$ of K. We notice that in Lemma 4.3(1) one has $u_c \in \mathcal{U}_{f\alpha}$ if (and only if) $u \in \mathcal{U}_{f\alpha}$ (provided $c \neq 0$). On the other hand, $c \circ u = u \circ c = (c-1)(u-1)+1 = u_{c-1}$. Hence for $u \in K$, $c \in k$, one has $c \circ u \in \mathcal{U}_{f\alpha}$ if (and only if) $u \in \mathcal{U}_{f\alpha}$ (provided $c \neq 1$).

In particular, $X := \mathcal{U}_{f\alpha}$ is closed with respect to multiplication $\circ$ by elements of k, and therefore, X is symmetric with respect to the addition $*$.

(4) For $X := \mathcal{U}_{f\alpha}$, we denote by $\mathfrak{R}_{f\alpha} := \mathfrak{R}_{\mathcal{U}_{f\alpha}}$ the corresponding subring of $K, *, \circ$ (the latter being an isomorphic copy of the field $K, +, \cdot$ as mentioned above).

Hence $R_{f\alpha} := \mathfrak{R}_{f\alpha} - 1$ is a subring of the field $K, +, \cdot$ with the usual addition and multiplication.

Lemma 4.5. *The ring $\mathfrak{R}_{f\alpha}, *$ is a $k, *, \circ$ vector space. Thus $R_{f\alpha}$ is a k-subspace of $K, +$.*

Proof. Clear by the discussion at (1) and (3) above. $\square$

Lemma 4.6. *In the above notation, let $X := \mathcal{U}_{f\alpha}$. Then one has $X_0 \subseteq \bigcap_{P \in D_{f\alpha}} \mathcal{O}_P^\times$.*

Proof. Indeed, by contradiction, suppose that there exists $u_0 \in X_0$ such that $v_{P_0}(u_0) \neq 0$ for some $P_0 \in D_{f\alpha}$. Then using the weak approximation lemma, we can choose $t \in K$ such that $v_{P_0}(t-1) > 0$, i.e., t is a principal v_{P_0}-unit, and $v_{P'}(u_0+t-1)<0$ for all $P'\neq P_0$, $P'\in D_{f\alpha}$. Then $v_{P''}(u_0+t-1)\neq 0$ for all $P''\in D_{f\alpha}$. On the other hand, since u is a v_{P_0} principal unit, it follows by Lemma 4.3(2) that $t\in\mathcal{U}_{f\alpha}=X$. Hence, since $u_0\in X_0$, we must have $t*u_0\in\mathcal{U}_{f\alpha}$ (by the definition of X_0), i.e, $u:=t*u_0=u_0+t-1\in\mathcal{U}_{f\alpha}$. But then by Lemma 4.3(3a), it follows that for every c, there must exist some $P\in D_{f\alpha}$ such that $u_c, u_{\alpha,c}\in\mathcal{O}_P^\times$. Hence, for $c=1$, one gets $u_0+t-1=u=u_c\in\mathcal{O}_P^\times$ for some $P\in D_{f\alpha}$, contradicting the fact that $v_{P''}(u_0+t-1)\neq 0$ for all $P''\in D_{f\alpha}$. $\square$

Key Lemma 4.7. *One has* $\mathfrak{R}_{f\alpha}=1+\bigcap_{P\in D_{f\alpha}}\mathfrak{m}_P$, *and therefore*, $R_{f\alpha}=\bigcap_{P\in D_{f\alpha}}\mathfrak{m}_P$.

Proof. Let $X=\mathcal{U}_{f\alpha}$ and $X_0\subset X$ as in Notation/Remarks 4.4.

For the inclusion "$\subseteq$", consider the partition $D_{f\alpha}=D^2\cup D^1\cup D^0$, where

- $P\in D^2$ if and only if $v_P(\mathfrak{R}_{f\alpha})\neq 0$,
- $P\in D^1$ if and only if $v_P(\mathfrak{R}_{f\alpha})=0$ and $\mathfrak{R}_{f\alpha}$ is not contained in $1+\mathfrak{m}_P$,
- $P\in D^0$ if and only if $\mathfrak{R}_{f\alpha}\subseteq 1+\mathfrak{m}_P$.

Clearly, in order to show that $\mathfrak{R}_{f\alpha}\subseteq 1+\bigcap_{P\in D_{f\alpha}}\mathfrak{m}_P$, we have to show that D^2 and D^1 are empty. By contradiction, suppose that at least one of the sets D^2, D^1 is nonempty.

Case 1. *D^2 is nonempty.*

Let $P\in D^2$ and $t\in\mathfrak{R}_{f\alpha}$ be such that $v_P(t)\neq 0$. Using the weak approximation lemma, choose any principal v_P-unit u' such that $v_{P'}(t+u'-1)>0$ for all $P'\neq P$ from $D_{f\alpha}$. Since $u'\in 1+\mathfrak{m}_P$, it follows by Lemma 4.3 that $u'\in\mathcal{U}_{f\alpha}$. Since $t\in\mathfrak{R}_{f\alpha}\subseteq X_0$, and $u'\in X=\mathcal{U}_{f\alpha}$, we get (by the definition of X_0) that $t*u'\in\mathcal{U}_{f\alpha}$. On the other hand, one has $u:=t*u'=t+u'-1$, and therefore $v_{P''}(u)\neq 0$ for all $P''\in D_{f\alpha}$, thus contradicting Lemma 4.3(3).

Case 2. *D^2 is empty, and D^1 nonempty.*

For $P\in D^1$ we have $\mathfrak{R}_{f\alpha}\subset\mathcal{O}_P^\times$ and $\mathfrak{R}_{f\alpha}$ not contained in $1+\mathfrak{m}_P$. In particular, the image $\overline{R}_{f\alpha}$ of $R_{f\alpha}$ under the residue map $\mathcal{O}_P\to\kappa(P)$ is a nontrivial k-subring of $\kappa(P)$. Since $\kappa(P)$ is a finite field extension of k, it follows that $\overline{R}_{f\alpha}$ is a k-subfield of $\kappa(P)$. Hence there exists $t\in\mathfrak{R}_{f\alpha}$ whose image in $\kappa(P)$ is the given element $\alpha\in k$. In order to conclude, using the weak approximation lemma, choose $u'\in 1+\mathfrak{m}_P\subseteq\mathcal{U}_{f\alpha}$ such that $v_P(t+u'-1-\alpha)>0$ for all $P'\neq P$ from $D_{f\alpha}$. Then reasoning as above, it follows that $t*u'\in\mathcal{U}_{f\alpha}=X$. On the other hand, as above, $u:=t*u'=u+u'-1$ has the property that $v_{P''}(u-\alpha)>0$ for all $P''\in D_{f\alpha}$. Therefore, u is a $v_{P''}$-unit with residue $\bar{u}=\alpha$ in $\kappa(P'')$ for all $P''\in D_{f\alpha}$. Recalling that $u=t*u'\in\mathcal{U}_{f\alpha}$, for $c=1$ one has $\{u_c, u_{\alpha,c}\}=\{u, u+\alpha(1-u)^n\}$, and $\alpha=\bar{u}\in\kappa(P)$ for all $P\in D_{f\alpha}$. Thus α is an n-th power in $K_P K_{u,\alpha,c}$ for all $P\in D_{f\alpha}$, contradicting Lemma 4.3(3).

For the converse inclusion "$\supseteq$", we have to show that for every $u_1 = 1 + \tilde{u} \in 1 + \bigcap_{P \in D_{f\alpha}} \mathfrak{m}_P$ with $\tilde{u} \in \bigcap_{P \in D_{f\alpha}} \mathfrak{m}_P$, the following hold:

(a) $u_1 \in X_0$, or equivalently, $u_1 * u \in X$ for all $u \in X$.

(b) $u_1 \circ X_0 \subseteq X_0$, or equivalently, $u_1 \circ u_0 * u \in X$ for all $u_0 \in X_0$, $u \in X$.

For (a), we show that $u_1 * u \in X$ for all $u \in X$. Indeed, $t := u_1 * u = u_1 + u - 1 = u + \tilde{u}$. Since $u \in X$, by Lemma 4.3(3), there exists $P \in D_{f\alpha}$ such that $u_c, u_{\alpha,c} \subset \mathcal{O}_P^\times$, and for all δ^* one has that $\mathrm{H}_{f\delta^*\alpha}$ is nontrivial over $K_P K_{u,\alpha,c}$. We now claim that $K_P K_{u,\alpha,c} = K_P K_{t,\alpha,c}$. Indeed, since $t = u + \tilde{u}$, $v_P(\tilde{u}) > 0$, and $u_c, u_{\alpha,c} \in \mathcal{O}_P^\times$, it follows that $t_c, t_{\alpha,c} \in \mathcal{O}_P$ and $\bar{t}_c = \bar{u}_c$ and $\bar{t}_{\alpha,c} = \bar{u}_{\alpha,c}$ in $\kappa(P)^\times$. Hence, by Hensel's lemma it follows that $\sqrt[n]{u_c}$, $\sqrt[n]{u_{\alpha,c}}$ and $\sqrt[n]{t_c}$, $\sqrt[n]{t_{\alpha,c}}$ generate the same extension of K_P. Therefore, if $f \cup \alpha \cup \beta^* \in \mathrm{H}_{f\delta^*\alpha}$ is nontrivial over $L_Q = K_P K_{u,\alpha,c}$, it is nontrivial over $K_P K_{u,\alpha,c} = K_P K_{t,\alpha,c}$, and thus also over $K_{t,\alpha,c} \subset K_P K_{t,\alpha,c}$, etc.

For (b), we show that $u_0 * u \in X$ for all $u \in X$ implies that $(u_1 \circ u_0) * u \in X$ for all $u \in X$. First, recall that by Notation/Remarks 4.4(4), it follows that $u_0 \in \mathcal{O}_P^\times$ for all $P \in D_{f\alpha}$. Hence one gets

$$t := u_1 \circ u_0 * u = ((u_1 - 1)(u_0 - 1) + 1) + u - 1 = \tilde{u}(u_0 - 1) + u = u + \tilde{u}',$$

where $\tilde{u}' = \tilde{u}(u_0 - 1) \in \bigcap_{P \in D_{f\alpha}} \mathfrak{m}_P$ since $u_0 \in \bigcap_{P \in D_{f\alpha}} \mathcal{O}_P^\times$ and $\tilde{u} \in \bigcap_{P \in D_{f\alpha}} \mathfrak{m}_P$. On the other hand, $u \in \mathcal{U}_{f\alpha}$ and $P \in D_{f\alpha}$ are such that $\mathrm{H}_{f\delta^*\alpha}$ is nontrivial over $K_P K_{u,\alpha,c}$. Hence $K_P K_{u,\alpha,c} = K_P K_{t,\alpha,c}$, by the fact that $\bar{u} = \bar{t}$ in $\kappa(P)^\times$ (and Hensel's lemma). Finally it follows that $t \in \mathcal{U}_{f\alpha} = X$, as claimed.

This concludes the proof of Key Lemma 4.7. $\square$

5. Proof of Theorems 1.1 and 1.2

5A. ***Defining the k-valuation rings.*** In the notation and hypotheses of the previous sections, let $K\,|\,k$ be a smooth fibration of a finitely generated field K with $\dim(K) = 2$, and X the complete smooth k-curve with $K = k(X)$. By Riemann–Roch, if $P \in X$ is a closed point and $m \gg 0$, there exist functions $f \in K$ such that $(f)_\infty = mP$, and letting m be prime to n, we have $P \in D_f$. Further, setting $\lambda := \kappa(P)$, there exist "many" $\alpha \in k^\times$ such that α is not an n-th power in $\kappa(P)$. Hence there exists α such that $D_{f\alpha}$ is nonempty. Thus by Key Lemma 4.7, it follows that $\mathfrak{R}_{f\alpha} = 1 + \bigcap_{P \in D_{f\alpha}} \mathfrak{m}_P$.

For f and α as above, we set $g := f + 1$, and notice that $(g)_\infty = mP$, etc. We repeat the constructions above and we get $\mathfrak{R}_{g\alpha} = 1 + \bigcap_{Q \in D_{g\alpha}} \mathfrak{m}_Q$.

Since $|\mathrm{div}(f)| \cap |\mathrm{div}(g)| = \{P\}$, by the weak approximation lemma one has

$$\Big(1 + \bigcap_{P \in D_{f\alpha}} \mathfrak{m}_P\Big) \cdot \Big(1 + \bigcap_{Q \in D_{g\alpha}} \mathfrak{m}_Q\Big) = 1 + \mathfrak{m}_P.$$

Therefore, one can recover $\mathcal{O}_P, \mathfrak{m}_P$ from $\mathfrak{R}_{f\alpha}$ and $\mathfrak{R}_{g\alpha}$ as follows:

$$\mathfrak{m}_P = \mathfrak{R}_{f\alpha} \cdot \mathfrak{R}_{g\alpha} - 1 \quad \text{and} \quad \mathcal{O}_P = \{t \in K \mid t\,\mathfrak{m}_P \subseteq \mathfrak{m}_P\}.$$

We thus have a *first-order recipe* to define all the k-valuation rings of $K\,|\,k$:

Recipe 5.1. Let $K = k(X)$ with X a complete smooth k-curve X. Suppose that a predicate $\psi(\mathfrak{x})$ is given which defines k inside K, i.e., $k = \{x \in K \mid \psi(x) \text{ holds in } K\}$. Let $n \neq \operatorname{char}(K)$ be a prime number, and notice that describing the k-valuation rings of $K\,|\,k$ is equivalent to describing the $k[\mu_{2n}]$-valuation rings of $K[\mu_{2n}]$. Supposing that $\mu_{2n} \subset K$, consider the following steps:

(1) For every $\delta \in k^\times$ let $\mathrm{H}_\delta = \{\beta \in k^\times \mid v_s(\beta - 1) > 2v_s(2n) \text{ if } s \notin V_\delta\} \subset k^\times$.
Note that H_δ is a definable subset of K, provided a predicate $\psi(\mathfrak{x})$ is given which defines k inside K. Indeed, given the global field k, the valuation rings $\mathcal{O}_s, \mathfrak{m}_s$ of k are definable inside k by Rumely's recipe [1980] mentioned in Section 1. Thus, one has

$$\beta \in \mathrm{H}_\delta \quad \text{if and only if} \quad \forall \mathcal{O}_s, \mathfrak{m}_s\, (\delta \notin \mathcal{O}_s \Rightarrow \beta - 1 \in 4n^2 \cdot \mathfrak{m}_s).$$

(2) For every $f \in K$ and $\alpha \in k^\times$, set $\mathrm{H}_{f\delta\alpha} := f \cup \alpha \cup \mathrm{H}_\delta \subset \mathrm{H}^3(K, \mathbb{Z}/n(2))$.

(3) Let $\mathcal{U}_{f\alpha} := \{u \in K^\times \mid \mathrm{H}_{f\delta\alpha} \text{ is nontrivial over } K_{u,\alpha,c} \text{ for all } c \in k,\ \delta \in k^\times\}$, where $K_{u,\alpha,c} := K[\sqrt[n]{u_c},\ \sqrt[n]{u_{\alpha,c}}\,]$ and $u_c := 1 - c(1-u)$, $u_{\alpha,c} =: u_c + \alpha c^n (1-u)^n$.
Note that the fact that $\mathrm{H}_{f\delta\alpha}$ is nontrivial over $K_{u,\alpha,c}$ is first-order expressible as follows (see Section 2C): $f \cup \alpha$ is nontrivial and there exists $\beta \in \mathrm{H}_\delta$ such that the reduced norm $N_{f,\alpha}$ of the division algebra $A_{f,\alpha}$ does not represent β over $K_{u,\alpha,c}$.

(4) Let $\mathfrak{R}_{f\alpha} := \{u \in \mathcal{U}_{f\alpha} \mid u + \mathcal{U}_{f\alpha} - 1 \subseteq \mathcal{U}_{f\alpha},\ (u-1)(\mathcal{U}_{f\alpha} - 1) + 1 \subseteq \mathcal{U}_{f\alpha}\}$.

(5) Repeat the process above for $g := f + 1$.

(6) Set $\mathfrak{m} := \mathfrak{R}_{f\alpha} \cdot \mathfrak{R}_{g\alpha} - 1$ and $\mathcal{O} := \{t \in K \mid t\,\mathfrak{m} \subseteq \mathfrak{m}\}$.

Conclusion 5.2. The k-valuation rings $\mathcal{O}_P, \mathfrak{m}_P$ of $K\,|\,k$ are among the definable sets $\mathcal{O}, \mathfrak{m}$. Precisely, for every $\mathcal{O}_P, \mathfrak{m}_P$ there exist f and α such that $\mathcal{O} = \mathcal{O}_P$, and $1/f \in \mathfrak{m} = \mathfrak{m}_P$.

5B. ***Concluding the proof of Theorem 1.2.*** First, the above Recipe 5.1 is a uniform first-order description of the k-valuation rings of function fields of complete smooth k-curves.

In order to give the formula $\deg_N(\mathrm{t})$, we first recall that k is a Hilbertian field. Hence, for every nonconstant function $t \in K$ there exist (infinitely many) specializations $t \mapsto a \in k$ such that the point $P \in X$ with $\bar{t} = a$ in $\kappa(P)$ is unique. Hence $[\kappa(P) : k] = [K : k(t)]$ is the degree of t. Thus, one possibility for the formula

$\mathrm{def}_N(\mathfrak{t})$ would be

$$(\forall \mathcal{O}, \mathfrak{m} : \mathfrak{t} \in k + \mathfrak{m} \Rightarrow [\mathcal{O}/\mathfrak{m} : k] \le N) \,\&\, (\exists \mathcal{O}, \mathfrak{m} : \mathfrak{t} \in k + \mathfrak{m} \,\&\, [\mathcal{O}/\mathfrak{m} : k] = N).$$

For the formula $\psi^R(\mathfrak{t}, \mathfrak{t}')$, let $R \subset K$ be the integral closure of $k[t]$ in K. Then for any k-valuation ring $\mathcal{O}$, one has $t \in \mathcal{O}$ if and only if $k[t] \subset \mathcal{O}$ if and only if $R \subset \mathcal{O}$. Further, R is actually the intersection of all the $\mathcal{O}$ which contain $k[t]$, or equivalently, which contain t. Thus the formula $\psi^R(\mathfrak{t}, \mathfrak{t}')$ could be

$$\forall \mathcal{O}, \mathfrak{m}\, (\mathfrak{t} \in \mathcal{O} \Rightarrow \mathfrak{t}' \in \mathcal{O}).$$

Finally, for the formula $\psi^0(\mathfrak{t}, \mathfrak{t}')$, let $R \subset K$ be the integral closure of $k[t]$ in K. Recall that $t' \in K$ lies in $k[t]$ if and only if $t' \in R$ and for all $\mathcal{O}, \mathfrak{m}$ one has that if the residue $\bar{t} \in \mathcal{O}/\mathfrak{m}$ lies in $k \subset \mathcal{O}/\mathfrak{m}$, then the residue $\bar{t}' \in \mathcal{O}/\mathfrak{m}$ lies in $k \subset \mathcal{O}/\mathfrak{m}$. (Indeed, this follows again from the fact that k is Hilbertian.) Thus one possibility for the formula $\psi^0(\mathfrak{t}, \mathfrak{t}')$ could be

$$\forall \mathcal{O}, \mathfrak{m}\, ((\mathfrak{t} \in \mathcal{O} \Rightarrow \mathfrak{t}' \in \mathcal{O}) \,\&\, (\mathfrak{t} \in \mathfrak{m} + k \Rightarrow \mathfrak{t}' \in \mathfrak{m} + k)).$$

This completes the proof of Theorem 1.2.

5C. ***Proof of Theorem 1.1.*** In order to prove Theorem 1.1, we recall that by the main results of [Pop 2002] combined with the description of the absolute constants in [Poonen 2007], one has the following:

(1) For every finitely generated field K over a number field k with $d := \mathrm{tr.deg}(K\,|\,k)$, there exists a formula with d free variables $\varphi^0_K(\mathfrak{t}_1, \dots, \mathfrak{t}_d)$ such that the sentence
$$\exists t_1, \dots, t_d\, \varphi^0_K(t_1, \dots, t_d)$$
is true in K. Moreover, if L is any other finitely generated field such that $\varphi^0_K(u_1, \dots, u_d)$ is true in L for some choice of $u_1, \dots, u_d \in L$, then the map $(t_1, \dots, t_d) \mapsto (u_1, \dots, u_d)$ extends to an embedding of fields $K \hookrightarrow L$.

(2) Now suppose that $d = 1$. Then using the above $\mathrm{deg}_N(\mathfrak{t})$ we have proved the following theorem.

Theorem 5.3. *Let ϑ_K be the sentence $\exists \mathfrak{t}\, (\varphi^0_K(\mathfrak{t}) \,\&\, \mathrm{deg}_N(\mathfrak{t}))$. Then the following hold*:

(1) *The sentence ϑ_K is true in K if and only if there exists some $t \in K$ such that $\varphi^0_K(t)$ is true in K, and t has degree N in K.*

(2) *Suppose that ϑ_K is true in K. Then for every finitely generated field L, the sentence ϑ_K is true in L if and only if K and L are isomorphic as fields.*

Acknowledgements

I would like to thank the participants at several activities, e.g., AWS 2003, AIM Workshop 2004, INI Cambridge 2005, HIM Bonn 2009, ALANT III in 2014, for the debates on the topic and suggestions concerning this problem. Special thanks are due to Bjorn Poonen and Jakob Stix for discussing technical aspects of the proofs. The author would also like to thank the University of Heidelberg and University of Bonn for the excellent working conditions during his visit in 2015–16 as a *Humboldt Preisträger.*

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Communicated by Bjorn Poonen
Received 2016-04-23 Revised 2017-05-11 Accepted 2017-08-03

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ANT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

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Algebra & Number Theory

Volume 11 No. 9 2017