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Rationality does not specialize among terminal fourfolds

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Rationality does not specialize among terminal fourfolds

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We show that rationality does not specialize in flat projective families of complex fourfolds with terminal singularities. This answers a question of Totaro, who established the corresponding result in dimensions greater than 4.

1. Introduction

Rationality behaves subtly in families of complex algebraic varieties. In general, given a flat projective family, the locus of rational fibers forms a countable union of locally closed subsets of the base [de Fernex and Fusi 2013, Proposition 2.3]. Recently, Hassett, Pirutka, and Tschinkel [2016] produced a smooth projective family of fourfolds where none of these locally closed subsets is dense, but their union is dense (even in the Euclidean topology). In particular, rationality is neither an open nor closed condition in smooth families.

This paper concerns the question of whether the locally closed subsets parametrizing the rational fibers of a family are actually closed, i.e., whether rationality specializes.

Question 1. Given a flat projective family of complex varieties, does geometric rationality of the generic fiber imply the same of every fiber?

Without further restrictions, the answer is negative: specializations of rational varieties need not even be rationally connected, as shown by a family of smooth cubic surfaces degenerating to a cone over a smooth cubic curve. However, if the fibers of the family are required to be smooth of dimension at most 3, Timmerscheidt [1982] proved the answer is positive. In fact, as Totaro observed, it follows from the results of de Fernex and Fusi [2013] and Hacon and Mckernan [2007] that the answer remains positive if the fibers are allowed to have log terminal singularities and dimension at most 3.

In higher dimensions, however, Totaro [2016b] showed that rationality does not

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specialize among varieties with mild singularities. Namely, specialization fails in every dimension greater than 4 if terminal singularities (the mildest type of singularity arising from the minimal model program) are allowed, and in dimension 4 if canonical singularities (the second mildest type of singularity) are allowed. This left open the possibility that rationality specializes among terminal fourfolds. The purpose of this paper is to show that this fails too.

Theorem 2. *There is a flat projective family of fourfolds over a Zariski open neighborhood U of the origin $0 \in \mathbb{A}^1$ in the complex affine line such that*:

(1) *All the fibers have terminal singularities.*

(2) *The fibers over $U \setminus \{0\}$ are rational.*

(3) *The fiber over 0 is stably irrational.*

Our proof of Theorem 2 closely follows [Totaro 2016b]. There, starting from a stably irrational smooth quartic fourfold $Y \subset \mathbb{P}^5$ (known to exist by [Totaro 2016a]), Totaro constructs a family of fivefolds satisfying conditions (1)–(3) in Theorem 2 by deforming the cone over Y to rational fivefolds. More generally, starting from any smooth hypersurface $Y \subset \mathbb{P}^n$ which is Fano of index at least 2, his construction produces a family of n-folds satisfying (1) and (2), whose fiber over 0 is birational to $Y \times \mathbb{P}^1$. It is thus tempting to take $Y \subset \mathbb{P}^4$. However, then the only potential candidate for Y is a cubic threefold such that $Y \times \mathbb{P}^1$ is irrational, the existence of which is a difficult open problem.

Our idea is to instead take Y to be a quartic double solid. Then Y is a Fano threefold of index 2, and can be chosen to be stably irrational by Voisin's seminal work [2015]. Although Y is not a hypersurface in projective space, it is a hypersurface in a *weighted* projective space, which we show is enough to run Totaro's argument.

The natural question left open by this paper is whether rationality specializes among smooth varieties of dimension greater than 3.

Conventions. We work over the field of complex numbers $\mathbb{C}$. For positive integers $a_0, \dots, a_n$, we denote by $\mathbb{P}(a_0, \dots, a_n)$ the weighted projective space with weights a_i. We use superscripts to denote that a weight is repeated with multiplicity, e.g., $\mathbb{P}(1^4, 2) = \mathbb{P}(1, 1, 1, 1, 2)$. For a vector bundle $\mathcal{E}$ on a scheme S, the associated projective bundle is $\mathbb{P}(\mathcal{E}) = \mathrm{Proj}_S(\mathrm{Sym}(\mathcal{E}^\vee))$.

2. Proof of Theorem 2

Let $Y \to \mathbb{P}^3$ be a quartic double solid, i.e., a double cover of $\mathbb{P}^3$ branched along a smooth quartic surface. We regard Y as a hypersurface in the weighted projective space $\mathbb{P}(1^4, 2)$, cut out by a polynomial of the form

$$f_4(x_0, \dots, x_4) = x_4^2 - h_4(x_0, \dots, x_3),$$

where $h_4(x_0, \dots, x_3)$ is a quartic. Let $X \subset \mathbb{P}(1^4, 2, 1)$ be the cone over Y defined by the same polynomial $f_4(x_0, \dots, x_4)$ in the bigger weighted projective space $\mathbb{P}(1^4, 2, 1)$. For a stably irrational choice of Y, the variety X will form the central fiber in the promised family of fourfolds.

Lemma 3. *X is birational to $Y \times \mathbb{P}^1$, and has terminal singularities.*

Proof. This can be deduced from a general result on cones (see [Kollár 2013, §3.1]), but we give a direct argument. Let H denote the pullback of the hyperplane class on $\mathbb{P}^3$ to Y. Define

$$\pi : \widetilde{X} = \mathbb{P}(\mathcal{O}_Y(-H) \oplus \mathcal{O}_Y) \to Y.$$

There is a natural morphism $\widetilde{X} \to \mathbb{P}(1^4, 2, 1)$ given as follows. Let ζ denote the divisor corresponding to the relative $\mathcal{O}(1)$ line bundle on $\widetilde{X}$. Then

$$\pi_*(\mathcal{O}_{\widetilde{X}}(\zeta)) = \mathcal{O}_Y(H) \oplus \mathcal{O}_Y \quad \text{and} \quad \pi_*(\mathcal{O}_{\widetilde{X}}(2\zeta)) = \mathcal{O}_Y(2H) \oplus \mathcal{O}_Y(H) \oplus \mathcal{O}_Y.$$

Hence $\mathrm{H}^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(\zeta)) \cong \mathbb{C}^4 \oplus \mathbb{C}$, and $\mathrm{H}^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(2\zeta))$ has a canonical 1-dimensional subspace corresponding to the canonical section of $\mathcal{O}_Y(2H)$. This data specifies the morphism $\widetilde{X} \to \mathbb{P}(1^4, 2, 1)$. In fact, this morphism factors through $X \subset \mathbb{P}(1^4, 2, 1)$ and gives a resolution of singularities $f : \widetilde{X} \to X$ with a single exceptional divisor

$$E = \mathbb{P}(\mathcal{O}_Y) \subset \widetilde{X},$$

which is contracted to $[0, 0, 0, 0, 0, 1] \in X$. Thus the first claim of the lemma holds.

Note that X is normal with $\mathbb{Q}$-Cartier canonical divisor. We show that the discrepancy of the exceptional divisor E above is 1, so that X has terminal singularities, completing the proof. Write $K_{\widetilde{X}} = f^*(K_X) + aE$. Then by adjunction

$$K_E = (K_{\widetilde{X}} + E)|_E = (a + 1)E|_E.$$

Observe that $E \cong Y$, so $K_E = -2H$, and $E = \zeta - \pi^*H$, so $E|_E = -H$. We conclude $a = 1$. □

Next, choose a nonzero polynomial $g_3(x_0, \dots, x_4) \in \mathrm{H}^0(\mathbb{P}(1^4, 2), \mathcal{O}(3))$ of weighted degree 3. We consider the flat family $\mathcal{X} \to \mathbb{A}^1$ over the affine line whose fiber $\mathcal{X}_t \subset \mathbb{P}(1^4, 2, 1)$ over $t \in \mathbb{A}^1$ is given by

$$f_4(x_0, \dots, x_4) + t g_3(x_0, \dots, x_4) x_5 = 0.$$

Note that $X = \mathcal{X}_0$.

Lemma 4. *There is a Zariski open neighborhood U of $0 \in \mathbb{A}^1$ such that*:

(1) *$\mathcal{X}_t$ has terminal singularities for all $t \in U$.*

(2) *$\mathcal{X}_t$ is rational for $t \in U \setminus \{0\}$.*

Proof. The fiber $\mathcal{X}_0$ has terminal singularities by Lemma 3. Since this condition is Zariski open in families [Nakayama 2004, Corollary VI.5.3], there is a Zariski open neighborhood U of $0 \in \mathbb{A}^1$ such that all fibers of $\mathcal{X}_U \to U$ are terminal. Further, observe that for $t \neq 0$, projection away from the x_5-coordinate gives a birational map from $\mathcal{X}_t$ to $\mathbb{P}(1^4, 2)$. Indeed, this map is an isomorphism over the locus where $g_3(x_0, \dots x_4) \neq 0$ in $\mathbb{P}(1^4, 2)$. Hence $\mathcal{X}_t$ is rational for $t \neq 0$. □

Now we can prove Theorem 2. By [Voisin 2015, Theorem 1.1], a very general quartic double solid is stably irrational. Taking such a Y in the above construction and combining Lemmas 3 and 4, we conclude that $\mathcal{X}_U \to U$ is a family of fourfolds satisfying all of the required conditions. □

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