Algebra & Number Theory

Volume 12
2018
No. 1

The mean value of symmetric square L-functions

Olga Balkanova and Dmitry Frolenkov

msp

ALGEBRA AND NUMBER THEORY 12:1 (2018)
dx.doi.org/10.2140/ant.2018.12.35

The mean value of symmetric square L-functions

Olga Balkanova and Dmitry Frolenkov

We study the first moment of symmetric-square L-functions at the critical point in the weight aspect. Asymptotics with the best known error term $O(k^{-1/2})$ were obtained independently by Fomenko in 2003 and by Sun in 2013. We prove that there is an extra main term of size $k^{-1/2}$ in the asymptotic formula and show that the remainder term decays exponentially in k. The twisted first moment was evaluated asymptotically by Ng with the error bounded by $lk^{-1/2+\epsilon}$. We improve the error bound to $l^{5/6+\epsilon}k^{-1/2+\epsilon}$ unconditionally and to $l^{1/2+\epsilon}k^{-1/2}$ under the Lindelöf hypothesis for quadratic Dirichlet L-functions.

1. Introduction

Asymptotic behavior of high moments of L-functions within different families can be predicted using random matrix theory [Conrey et al. 2005] or multiple Dirichlet series [Diaconu et al. 2003]. However, obtaining asymptotic formulas with sharp error bounds is a hard problem even in the case of small moments.

One of the most challenging families is symmetric square L-functions in weight aspect. Gelbart and Jacquet [1978] proved that these are L-function attached to GL(3) cusp forms.

Despite numerous efforts, even an upper bound for the second moment of symmetric square L-functions remains an open problem. See [Khan 2010, Conjecture 1.2].

Balkanova's research was supported by Academy of Finland project no. 293876. Frolenkov's research was supported by the Russian Science Foundation under grant [14-11-00335] and performed at the Khabarovsk Division of the Institute for Applied Mathematics, Far Eastern Branch, Russian Academy of Sciences.
MSC2010: primary 11F12; secondary 33C05, 34E05, 34E20.
Keywords: symmetric square L-functions, weight aspect, Gauss hypergeometric function, Liouville–Green method, WKB approximation.

The first moment has been studied intensively during the last decades. See [Fomenko 2003; Khan 2007; Kohnen and Sengupta 2002; Lau 2002; Ng 2016; 2017; Sun 2013]. Nevertheless, even the best known asymptotic error estimates do not appear to be sharp.

The present paper aims to optimize error bounds in existing asymptotic formulas. With this goal, we prove an exact formula for the twisted first moment of symmetric square L-functions, and apply the Liouville–Green method (also called WKB approximation) to estimate remainder terms. This technique, originating from the theory of approximation of second-order differential equations, is quite unusual for analytic number theory, yet very effective. See, for example, [Balkanova and Frolenkov 2016; Zavorotny 1989].

2. Main results

Let $S_{2k}(1)$ denote the space of holomorphic cusp forms of weight $2k \geq 2$ with respect to the full modular group. Denote by H_{2k} the normalized Hecke basis for $S_{2k}(1)$. Every $f \in H_{2k}$ has a Fourier expansion of the form

$$f(z) = \sum_{n\geq 1} \lambda_f(n) n^{k-1/2} \exp(2\pi i n z), \tag{2-1}$$

$$\lambda_f(1) = 1. \tag{2-2}$$

For $\Re s > 1$ the associated symmetric square L-function is given by

$$L(\operatorname{sym}^2 f, s) = \zeta(2s) \sum_{n=1}^{\infty} \frac{\lambda_f(n^2)}{n^s}. \tag{2-3}$$

Let $\Gamma(s)$ be the Gamma function and define

$$L_\infty(s) := \pi^{-3s/2} \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{s-1}{2} + k\right) \Gamma\left(\frac{s}{2} + k\right). \tag{2-4}$$

Shimura [1975] showed that the completed L-function

$$\Lambda(\operatorname{sym}^2 f, s) := L_\infty(s) L(\operatorname{sym}^2 f, s)$$

is entire and satisfies the functional equation

$$\Lambda(\operatorname{sym}^2 f, s) = \Lambda(\operatorname{sym}^2 f, 1-s). \tag{2-5}$$

Consider

$$M_1(l, s) := \sum_{f\in H_{2k}}^{h} \lambda_f(l^2) L(\operatorname{sym}^2 f, s). \tag{2-6}$$

The superscript h in the formula above indicates that the expression in the sum is multiplied by the harmonic weight $\Gamma(2k-1)/((4\pi)^{2k-1}\langle f, f\rangle_1)$, where $\langle f, f\rangle_1$ is the Petersson inner product on the space of level 1 holomorphic modular forms.

Denote by γ the Euler constant and by $\psi(s)$ the logarithmic derivative of the Gamma function. Let ${}_2F_1(a,b,c;x)$ be the Gauss hypergeometric function and

$$\Phi_k(x):=\frac{\Gamma\big(k-\frac{1}{4}\big)\Gamma\big(\frac{3}{4}-k\big)}{\Gamma\big(\frac{1}{2}\big)}\,{}_2F_1\big(k-\tfrac{1}{4},\tfrac{3}{4}-k,\tfrac{1}{2};x\big), \tag{2-7}$$

$$\Psi_k(x):=x^k\frac{\Gamma\big(k-\frac{1}{4}\big)\Gamma\big(k+\frac{1}{4}\big)}{\Gamma(2k)}\,{}_2F_1\big(k-\tfrac{1}{4},k+\tfrac{1}{4},2k;x\big). \tag{2-8}$$

We prove the following exact formula for the twisted first moment.

Theorem 2.1. *For any $l\geq 1$ one has*

$$\begin{aligned}M_1\Big(l,\frac{1}{2}\Big)=\frac{1}{2\sqrt{l}}\Big(-2\log l-3\log 2\pi+\frac{\pi}{2}+3\gamma+\psi\Big(k-\frac{1}{4}\Big)+\psi\Big(k+\frac{1}{4}\Big)\Big)\\+\frac{\sqrt{2\pi}(-1)^k}{2\sqrt{l}}\frac{\Gamma\big(k-\frac{1}{4}\big)}{\Gamma\big(k+\frac{1}{4}\big)}\mathscr{L}_{-4l^2}\Big(\frac{1}{2}\Big)+\frac{1}{\sqrt{l}}\sum_{1\leq n<2l}\mathscr{L}_{n^2-4l^2}\Big(\frac{1}{2}\Big)\Phi_k\Big(\frac{n^2}{4l^2}\Big)\\+\frac{1}{l\sqrt{2}}\sum_{n>2l}\mathscr{L}_{n^2-4l^2}\Big(\frac{1}{2}\Big)\sqrt{n}\,\Psi_k\Big(\frac{4l^2}{n^2}\Big),\end{aligned} \tag{2-9}$$

where

$$\mathscr{L}_n(s)=\frac{\zeta(2s)}{\zeta(s)}\sum_{q=1}^{\infty}\frac{1}{q^s}\Big(\sum_{\substack{1\leq r\leq 2q\\ r^2\equiv n \pmod{4q}}}1\Big). \tag{2-10}$$

A similar formula, where the last two summands are expressed in terms of the Legendre function of the first kind, was established by a different method by Zagier [1977, Theorem 1]. Zagier's formula was applied by Kohnen and Sengupta [2002] to prove an upper bound for $M_1\big(1,\frac{1}{2}\big)$, by Fomenko [2003] to obtain an asymptotic formula for $M_1\big(1,\frac{1}{2}\big)$, and by Luo [2012] to estimate the second moment of $L\big(\mathrm{sym}^2 f,\frac{1}{2}\big)$ over short intervals.

The proof of Theorem 2.1 is quite simple and makes use of Petersson's trace formula and the functional equation for the Lerch zeta function.

When $l=1$, exact formula (2-9) allows one to isolate the second main term of size $k^{-1/2}$ in the asymptotic formula so that the remainder term decays exponentially.

Corollary 2.2. *For some $c>0$ one has*

$$\begin{aligned}M_1\Big(1,\frac{1}{2}\Big)=\frac{1}{2}\Big(\frac{\pi}{2}-3\log 2\pi+3\gamma+\psi\Big(k-\frac{1}{4}\Big)+\psi\Big(k+\frac{1}{4}\Big)\Big)+\frac{\sqrt{2\pi}(-1)^k}{2}\frac{\Gamma\big(k-\frac{1}{4}\big)}{\Gamma\big(k+\frac{1}{4}\big)}L\Big(\frac{1}{2},\chi_{-4}\Big)\\+\Phi_k\Big(\frac{1}{4}\Big)L\Big(\frac{1}{2},\chi_{-3}\Big)+O\Big(\frac{1}{\sqrt{k}}\exp(-ck)\Big),\end{aligned} \tag{2-11}$$

where $L\big(\frac{1}{2},\chi_D\big)$ is a Dirichlet L-function for the primitive quadratic character of conductor D and

$$\Phi_k\Big(\frac{1}{4}\Big)=-\frac{2^{3/2}\sqrt{\pi}}{3^{1/4}}\frac{1}{\sqrt{2k-1}}\sin\frac{\pi(2k-1)}{3}\Big(1+O\Big(\frac{1}{k}\Big)\Big). \tag{2-12}$$

Remark. After posting the first version of this paper to the arXiv, the authors have been informed by Shenhui Liu that he has independently obtained an asymptotic formula similar to (2-11) by using an approximate functional equation. See [Liu 2017].

Corollary 2.2 improves the series of previously known results with the following error bounds:

- $k^{-0.008}$ [Lau 2002]
- $k^{-1/20}$ [Khan 2007]
- $k^{-1/2}$ [Fomenko 2003; Sun 2013].

Corollary 2.3. *For any* $\epsilon > 0,\ l > 1$, *one has*

$$M_1\Big(l, \frac{1}{2}\Big) = \frac{1}{2\sqrt{l}}\Big(-2\log l - 3\log 2\pi + \frac{\pi}{2} + 4\gamma + \psi(1) + \psi\Big(k-\frac{1}{4}\Big) + \psi\Big(k+\frac{1}{4}\Big)\Big) + O\Big(\frac{l^{5/6+\epsilon}}{\sqrt{k}}\Big). \tag{2-13}$$

Assuming the Lindelöf hypothesis for quadratic Dirichlet L-functions, the error term above can be replaced by $O(l^{1/2+\epsilon}k^{-1/2})$.

This improves the error bound $lk^{-1/2+\epsilon}$ proved by Ng (see [Ng 2016, Theorem 2.1.1] and [Ng 2017]).

3. Notation and tools

Let $e(x) = \exp(2\pi i x)$. For $v \in \mathbb{C}$ let

$$\tau_v(n) = \sum_{n_1 n_2 = n} \Big(\frac{n_1}{n_2}\Big)^v. \tag{3-1}$$

The classical Kloosterman sum is defined by

$$S(n, m; c) = \sum_{\substack{a \pmod c \\ (a,c)=1}} e\Big(\frac{an + a^* m}{c}\Big), \quad aa^* \equiv 1 \pmod c.$$

Lemma 3.1 (Weil's bound [1948]). *One has*

$$|S(m, n; c)| \le \tau_0(c)\sqrt{(m, n, c)}\sqrt{c}. \tag{3-2}$$

Let $J_v(x)$ be the Bessel function of the first kind.

Lemma 3.2 (Petersson's trace formula [1932]). *For* $2k \ge 12$ *and integral* $l, n \ge 1$, *one has*

$$\sum_{f \in H_{2k}}^{h} \lambda_f(l)\lambda_f(n) = \delta_{l,n} + 2\pi i^{2k} \sum_{c=1}^{\infty} \frac{S(l, n; c)}{c} J_{2k-1}\Big(\frac{4\pi\sqrt{ln}}{c}\Big). \tag{3-3}$$

The Lerch zeta function

$$\zeta(\alpha, \beta, s) = \sum_{n+\alpha>0} \frac{e(n\beta)}{(n+\alpha)^s} \tag{3-4}$$

was introduced by Lipschitz [1857] and was named after Lerch, who proved in 1887 the functional equation. The special case $\zeta(\alpha, 0, s)$ is called the Hurwitz zeta function.

Lemma 3.3 [Lerch 1887]. *One has*

$$\zeta(\alpha, 0, s) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}}\Big(-ie\Big(\frac{s}{4}\Big)\zeta(0, \alpha, 1-s) + ie\Big(-\frac{s}{4}\Big)\zeta(0, -\alpha, 1-s)\Big). \tag{3-5}$$

4. Some properties of $\mathscr{L}_n(s)$

The main references for this section are [Bykovskiĭ 1994; Soundararajan and Young 2013; Zagier 1977]. Function (2-10) can be written as follows

$$\mathscr{L}_n(s) = \frac{\zeta(2s)}{\zeta(s)}\sum_{q=1}^{\infty}\frac{\rho_q(n)}{q^s} = \sum_{q=1}^{\infty}\frac{\lambda_q(n)}{q^s}, \tag{4-1}$$

where

$$\rho_q(n) := \#\{x \pmod{2q} : x^2 \equiv n \pmod{4q}\}, \tag{4-2}$$

$$\lambda_q(n) := \sum_{q_1^2 q_2 q_3 = q} \mu(q_2)\rho_{q_3}(n). \tag{4-3}$$

For a fixed n, both $\rho_q(n)$ and $\lambda_q(n)$ are multiplicative functions of q. Furthermore, for $n \equiv 2, 3 \pmod 4$ the function $\rho_q(n)$ is identically zero. Therefore, $\mathscr{L}_n(s)$ does not vanish only for $n \equiv 0, 1 \pmod 4$. If $n = 0$ then

$$\mathscr{L}_n(s) = \zeta(2s-1). \tag{4-4}$$

Otherwise, for $n = Dl^2$ with D fundamental discriminant we have

$$\mathscr{L}_n(s) = l^{1/2-s}T_l^{(D)}(s)L(s, \chi_D), \tag{4-5}$$

where $L(s, \chi_D)$ is a Dirichlet L-function for primitive quadratic character χ_D and

$$T_l^{(D)}(s) = \sum_{l_1 l_2 = l}\chi_D(l_1)\frac{\mu(l_1)}{\sqrt{l_1}}\tau_{s-1/2}(l_2). \tag{4-6}$$

The completed L-function

$$\mathscr{L}_n^*(s) = \Big(\frac{\pi}{|n|}\Big)^{-s/2}\Gamma\Big(\frac{s}{2} + \frac{1}{4} - \frac{\operatorname{sgn} n}{4}\Big)\mathscr{L}_n(s) \tag{4-7}$$

satisfies the functional equation

$$\mathscr{L}_n^*(s) = \mathscr{L}_n^*(1-s). \tag{4-8}$$

Lemma 4.1. *One has*

$$\sum_{q=1}^{\infty}\frac{1}{q^{1+s}}\sum_{c \pmod q} S(l^2, c^2; q)e\Big(\frac{nc}{q}\Big) = \frac{1}{\zeta(2s)}\mathscr{L}_{n^2-4l^2}(s). \tag{4-9}$$

Proof. Consider

$$S := \sum_{c \pmod q} S(l^2, c^2; q) e\Big(n\frac{c}{q}\Big) = \sum_{c \pmod q} \sum_{\substack{a \pmod q\\ (a,q)=1}} e\Big(\frac{ac^2+a^*l^2+nc}{q}\Big),$$

where $aa^* \equiv 1 \pmod q$. Making the change of variables $c = c_1 a^*$, we have

$$\begin{aligned} S &= \sum_{\substack{a \pmod q\\ (a,q)=1}} \sum_{c_1 \pmod q} e\Big(\frac{c_1^2 a^* + l^2 a^* + nc_1 a^*}{q}\Big) \\ &= \sum_{c_1 \pmod q} S(0, c_1^2 + l^2 + nc_1; q) = \sum_{c \pmod q} \sum_{\substack{bd=q\\ b \mid c^2+l^2+nc}} \mu(d) b \\ &= \sum_{bd=q} \mu(d) b \sum_{\substack{c \pmod q\\ c^2+l^2+nc \equiv 0 \pmod b}} 1 = \sum_{bd=q} \mu(d) b \sum_{\substack{c \pmod b\\ c^2+l^2+nc \equiv 0 \pmod b}} \frac{q}{b}. \end{aligned}$$

The condition $c^2 + l^2 + nc \equiv 0 \pmod b$ is equivalent to $(2c+n)^2 + 4l^2 - n^2 \equiv 0 \pmod{4b}$. Hence

$$S = q \sum_{bd=q} \mu(d) \sum_{\substack{c \pmod{2b}\\ c^2 \equiv n^2-4l^2 \pmod{4b}}} 1 = q \sum_{bd=q} \mu(d) \rho_b(n^2 - 4l^2).$$

Consequently,

$$\begin{aligned} \sum_{q=1}^{\infty} \frac{1}{q^{1+s}} \sum_{c \pmod q} S(l^2, c^2; q) e\Big(\frac{nc}{q}\Big) &= \sum_{q=1}^{\infty} \frac{1}{q^s} \sum_{bd=q} \mu(d) \rho_b(n^2 - 4l^2) \\ &= \sum_{b=1}^{\infty} \frac{\rho_b(n^2-4l^2)}{b^s} \sum_{q=1}^{\infty} \frac{\mu(q)}{q^s} \\ &= \frac{1}{\zeta(s)} \sum_{q=1}^{\infty} \frac{\rho_q(n^2-4l^2)}{q^s} = \frac{\mathscr{L}_{n^2-4l^2}(s)}{\zeta(2s)}. \end{aligned}$$

□

Lemma 4.2. *Assume that $d \neq 0$. For any $\epsilon > 0$, one has*

$$\mathscr{L}_d\big(\tfrac{1}{2}\big) \ll d^{1/6+\epsilon}. \tag{4-10}$$

If the Lindelöf hypothesis for Dirichlet L-functions is true, then

$$\mathscr{L}_d\big(\tfrac{1}{2}\big) \ll d^{\epsilon}. \tag{4-11}$$

Proof. For $d = Dl^2$, where D is the fundamental discriminant, one has

$$\mathscr{L}_d\big(\tfrac{1}{2}\big) = T_l^{(D)}\big(\tfrac{1}{2}\big) L\big(\tfrac{1}{2}, \chi_D\big)$$

by equation (4-5). It follows from equality (4-6) that

$$T_l^{(D)}\Big(\frac{1}{2}\Big) \ll \sum_{l_1 l_2 = l} \frac{\sigma(l_2)}{\sqrt{l_1}} \ll \sum_{l_1 | l} \frac{(l/l_1)^{\epsilon}}{\sqrt{l_1}} \ll l^{\epsilon}.$$

By [Conrey and Iwaniec 2000, Corollary 1.5] for any $\epsilon > 0$ one has

$$L\big(\tfrac{1}{2}, \chi_D\big) \ll D^{1/6+\epsilon}.$$

This implies the required bounds for the function $\mathscr{L}_d\big(\frac{1}{2}\big)$. □

5. Exact formula

Lemma 5.1. *For $\Re s > \frac{3}{2}$ one has*

$$M_1(l,s) = \frac{\zeta(2s)}{l^s} + \frac{(2\pi)^s i^{2k}}{2l^{1-s}} \frac{\Gamma(k-s/2)}{\Gamma(k+s/2)} \mathscr{L}_{-4l^2}(s) + (2\pi)^s i^{2k} \sum_{n=1}^{\infty} \frac{1}{n^{1-s}} \mathscr{L}_{n^2-4l^2}(s) I\Big(\frac{n}{l}\Big), \tag{5-1}$$

where

$$I(x) := \frac{1}{2\pi i} \int_{(\Delta)} \frac{\Gamma\big(k-\frac{1}{2}+\frac{1}{2}w\big)}{\Gamma\big(k+\frac{1}{2}-\frac{1}{2}w\big)} \Gamma(1-s-w) \sin\Big(\pi \frac{s+w}{2}\Big) x^w \, dw \tag{5-2}$$

with $1-2k < \Delta < 1-\Re s$.

Proof. By the Petersson trace formula

$$\begin{aligned} M_1(l,s) &= \zeta(2s) \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{f \in H_{2k}}^{h} \lambda_f(l^2) \lambda_f(n^2) \\ &= \frac{\zeta(2s)}{l^s} + 2\pi i^{2k} \zeta(2s) \sum_{q=1}^{\infty} \frac{1}{q} \sum_{n=1}^{\infty} \frac{S(l^2,n^2;q)}{n^s} J_{2k-1}\Big(4\pi \frac{ln}{q}\Big). \end{aligned}$$

The change of order of summation above is justified by the absolute convergence for $\Re s > \frac{3}{2}$, which follows from the standard estimates

$$S(l^2,n^2;q) \ll q^{1/2+\epsilon} (l^2,n^2,q)^{1/2} \quad \text{for any } \epsilon > 0$$

and

$$J_{2k-1}\Big(4\pi \frac{ln}{q}\Big) \ll \begin{cases} \Big(\dfrac{ln}{q}\Big)^{2k-1}, & q > ln \\ \Big(\dfrac{ln}{q}\Big)^{-1/2}, & q < ln. \end{cases}$$

Next, we use the Mellin–Barnes representation for the Bessel function

$$M_1(l,s) = \frac{\zeta(2s)}{l^s} + 2\pi i^{2k} \zeta(2s) \sum_{q=1}^{\infty} \frac{1}{q} \times \frac{1}{4\pi i} \int_{(\Delta)} \frac{\Gamma\big(k-\frac{1}{2}+\frac{1}{2}w\big)}{\Gamma\big(k+\frac{1}{2}-\frac{1}{2}w\big)} \sum_{n=1}^{\infty} \frac{S(l^2,n^2;q)}{n^{w+s}} \Big(\frac{q}{2\pi l}\Big)^w dw,$$

where $1-2k<\Delta<0$. To guarantee the absolute convergence of the integral over w and the sums over q, n we require that

$$\max(1-2k,\, 1-\Re s)<\Delta<-\tfrac{1}{2},$$

which is true for $\Re s>\frac{3}{2}$. Consider

$$\sum_{n=1}^{\infty}\frac{S(l^2,n^2;q)}{n^{w+s}}=\sum_{c \pmod q}\sum_{n\equiv c \pmod q}\frac{S(l^2,c^2;q)}{n^{w+s}}=\sum_{c \pmod q}S(l^2,c^2;q)\sum_{n=1}^{\infty}\frac{1}{(c+nq)^{w+s}}$$
$$=\sum_{c \pmod q}\frac{S(l^2,c^2;q)}{q^{w+s}}\zeta\Big(\frac{c}{q},0,w+s\Big).$$

Note that for all c, the Lerch zeta function has a simple pole at $w=1-s$ with residue one. The next step is to apply functional equation (3-5) for the Lerch zeta function which is only possible when $\Re(s+w)<0$. Accordingly, we move the w-contour to the left up to $\Delta_1:=-s-\epsilon$, crossing a simple pole at $w=1-s$. Therefore,

$$M_1(l,s)=\frac{\zeta(2s)}{l^s}+2\pi i^{2k}\zeta(2s)\frac{\Gamma\big(k-\frac{1}{2}s\big)}{\Gamma\big(k+\frac{1}{2}s\big)}\sum_{q=1}^{\infty}\sum_{c \pmod q}\frac{S(l^2,c^2;q)}{2q^2}\Big(\frac{q}{2\pi l}\Big)^{1-s}+2\pi i^{2k}\zeta(2s)\sum_{q=1}^{\infty}\frac{1}{q}\frac{1}{4\pi i}$$
$$\times\int_{(\Delta_1)}\frac{\Gamma\big(k-\frac{1}{2}+\frac{1}{2}w\big)}{\Gamma\big(k+\frac{1}{2}-\frac{1}{2}w\big)}\Big(\frac{q}{2\pi l}\Big)^{w}\sum_{c \pmod q}\frac{S(l^2,c^2;q)}{q^{s+w}}\zeta\Big(\frac{c}{q},0;s+w\Big)\,dw.$$

Using the functional equation (3-5), we obtain

$$\sum_{c \pmod q}S(l^2,c^2;q)\zeta\Big(\frac{c}{q},0;s+w\Big)$$
$$=2(2\pi)^{s+w-1}\Gamma(1-s-w)\sin\Big(\pi\frac{s+w}{2}\Big)\sum_{c \pmod q}S(l^2,c^2;q)\zeta\Big(0,\frac{c}{q};1-s-w\Big).$$

Substituting this into $M_1(l,s)$ and opening the Lerch zeta function, one has

$$M_1(l,s)=\frac{\zeta(2s)}{l^s}+\frac{(2\pi)^s i^{2k}}{2l^{1-s}}\zeta(2s)\frac{\Gamma\big(k-\frac{1}{2}s\big)}{\Gamma\big(k+\frac{1}{2}s\big)}\sum_{q=1}^{\infty}\sum_{c \pmod q}\frac{S(l^2,c^2;q)}{q^{1+s}}$$
$$+(2\pi)^s i^{2k}\zeta(2s)\sum_{n=1}^{\infty}\frac{1}{n^{1-s}}\sum_{q=1}^{\infty}\frac{1}{q^{1+s}}\sum_{c \pmod q}S(l^2,c^2;q)e\Big(\frac{nc}{q}\Big)I\Big(\frac{n}{l}\Big),$$

where $I(x)$ is defined by equation (5-2). Finally, computing the sums over c and q using formula (4-9), we prove the lemma. □

Lemma 5.2. *If $x\geq 2$, then*

$$I(x)=\frac{2^{2k}(-1)^k}{2^s\sqrt{\pi}}\cos\Big(\frac{\pi s}{2}\Big)x^{1-2k}\frac{\Gamma\big(k-\frac{1}{2}s\big)\Gamma\big(k+\frac{1}{2}-\frac{1}{2}s\big)}{\Gamma(2k)}\times{}_2F_1\Big(k-\frac{s}{2},k+\frac{1}{2}-\frac{s}{2},2k;\frac{4}{x^2}\Big). \quad (5\text{-}3)$$

Proof. Moving the contour of integration in (5-2) to the left, we cross simple poles at $w = 1-2k-2j$, $j = 0, 1, 2, \ldots$. Therefore,

$$I(x) = 2(-1)^k \cos\Bigl(\frac{\pi s}{2}\Bigr) x^{1-2k} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{\Gamma(2k-s+2j)}{\Gamma(2k+j)} x^{-2j}.$$

By the duplication formula we have

$$\Gamma\bigl(2\bigl(k+j-\tfrac{1}{2}s\bigr)\bigr) = \frac{2^{2k-1-s+2j}}{\sqrt{\pi}} \Gamma\bigl(k+j-\tfrac{1}{2}s\bigr)\Gamma\bigl(k+j+\tfrac{1}{2}1-\tfrac{1}{2}s\bigr).$$

This yields

$$\begin{aligned} I(x) &= \frac{2(-1)^k}{\sqrt{\pi}} 2^{2k-1-s} \cos\Bigl(\frac{\pi s}{2}\Bigr) x^{1-2k} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{\Gamma\bigl(k-\frac{1}{2}s+j\bigr)}{\Gamma(2k+j)} \Gamma\Bigl(k+\frac{1}{2}-\frac{s}{2}+j\Bigr)\Bigl(\frac{4}{x^2}\Bigr)^j \\ &= \frac{2^{2k}(-1)^k}{2^s\sqrt{\pi}} \cos\Bigl(\frac{\pi s}{2}\Bigr) x^{1-2k} \frac{\Gamma\bigl(k-\frac{1}{2}s\bigr)\Gamma\bigl(k+\frac{1}{2}-\frac{1}{2}s\bigr)}{\Gamma(2k)} {}_2F_1\Bigl(k-\frac{s}{2}, k+\frac{1}{2}-\frac{s}{2}, 2k; \frac{4}{x^2}\Bigr). \end{aligned}$$

□

Lemma 5.3. *One has*

$$I(2) = \frac{2(-1)^k}{2^s\sqrt{\pi}} \cos\bigl(\tfrac{1}{2}\pi s\bigr) \frac{\Gamma\bigl(k-\frac{1}{2}s\bigr)\Gamma\bigl(k+\frac{1}{2}-\frac{1}{2}s\bigr)}{\Gamma\bigl(k+\frac{1}{2}s\bigr)\Gamma\bigl(k-\frac{1}{2}+\frac{1}{2}s\bigr)} \Gamma\bigl(s-\tfrac{1}{2}\bigr). \tag{5-4}$$

Proof. Letting $x = 2$ in (5-3) and applying [Olver et al. 2010, Equation 15.4.20], we find

$${}_2F_1\bigl(k-\tfrac{1}{2}s, k+\tfrac{1}{2}-\tfrac{1}{2}s, 2k; 1\bigr) = \frac{\Gamma(2k)\Gamma\bigl(s-\frac{1}{2}\bigr)}{\Gamma\bigl(k+\frac{1}{2}s\bigr)\Gamma\bigl(k-\frac{1}{2}+\frac{1}{2}s\bigr)}.$$

The assertion follows. □

Lemma 5.4. *If $x < 2$, then*

$$I(x) = \frac{(-1)^k}{\sqrt{\pi}} \sin\bigl(\tfrac{1}{2}\pi s\bigr) x^{1-s} \frac{\Gamma\bigl(k-\frac{1}{2}s\bigr)\Gamma\bigl(1-k-\frac{1}{2}s\bigr)}{\Gamma\bigl(\frac{1}{2}\bigr)} {}_2F_1\bigl(k-\tfrac{1}{2}s, 1-k-\tfrac{1}{2}s, \tfrac{1}{2}; \tfrac{1}{4}x^2\bigr). \tag{5-5}$$

Proof. Moving the contour of integration in (5-2) to the right we cross simple poles at $w = 1-s+j$, $j = 0, 1, 2, \ldots$. Accordingly,

$$I(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{\Gamma\bigl(k-\frac{1}{2}s+\frac{1}{2}j\bigr)}{\Gamma\bigl(k+\frac{1}{2}s-\frac{1}{2}j\bigr)} \sin\Bigl(\pi\frac{1+j}{2}\Bigr) x^{1-s+j}.$$

Note that

$$\sin\Bigl(\pi\frac{1+j}{2}\Bigr) = \cos\Bigl(\frac{\pi j}{2}\Bigr) = \begin{cases} 0, & j \text{ is odd}, \\ (-1)^m, & j = 2m. \end{cases}$$

Thus

$$I(x) = \sum_{m=0}^{\infty} \frac{1}{(2m)!} \frac{\Gamma\bigl(k-\frac{1}{2}s+m\bigr)}{\Gamma\bigl(k+\frac{1}{2}s-m\bigr)} (-1)^m x^{1-s+2m}.$$

In order to express $I(x)$ in terms of the Gauss hypergeometric function we apply the duplication formula, obtaining

$$(2m)! = \Gamma\big(2\big(m+\tfrac{1}{2}\big)\big) = \frac{1}{\sqrt{\pi}} 2^{2m}\Gamma\big(m+\tfrac{1}{2}\big)\Gamma(m+1).$$

Furthermore, by Euler's reflection formula

$$\Gamma\big(k+\tfrac{1}{2}s-m\big) = \frac{\pi}{(-1)^{k-m}\sin\big(\tfrac{1}{2}\pi s\big)\Gamma\big(1-k-\tfrac{1}{2}s+m\big)}.$$

Finally,

$$\begin{aligned} I(x) &= \frac{(-1)^k}{\sqrt{\pi}}\sin\big(\tfrac{1}{2}\pi s\big)x^{1-s}\sum_{m=0}^{\infty}\frac{1}{m!}\frac{\Gamma\big(k-\tfrac{1}{2}s+m\big)}{\Gamma\big(m+\tfrac{1}{2}\big)}\Gamma\big(1-k-\tfrac{1}{2}s+m\big)\Big(\frac{x^2}{4}\Big)^m \\ &= \frac{(-1)^k}{\sqrt{\pi}}\sin\big(\tfrac{1}{2}\pi s\big)x^{1-s}\frac{\Gamma\big(k-\tfrac{1}{2}s\big)\Gamma\big(1-k-\tfrac{1}{2}s\big)}{\Gamma\big(\tfrac{1}{2}\big)}{}_2F_1\big(k-\tfrac{1}{2}s,\,1-k-\tfrac{1}{2}s,\,\tfrac{1}{2};\,\tfrac{1}{4}x^2\big). \qquad \square \end{aligned}$$

Next, we substitute equations (5-3), (5-4), (5-5) into expression (5-1), proving the exact formula for the shifted first moment.

Theorem 5.5. *For any $l \geq 1$ and $2-2k < \Re s < 2k-1$, one has*

$$\begin{aligned} M_1(l,s) &= \frac{\zeta(2s)}{l^s} + \frac{(2\pi)^s i^{2k}}{2l^{1-s}}\frac{\Gamma\big(k-\tfrac{1}{2}s\big)}{\Gamma\big(k+\tfrac{1}{2}s\big)}\mathscr{L}_{-4l^2}(s) \\ &\quad + \frac{(2\pi)^s}{\sqrt{\pi}}\frac{\zeta(2s-1)}{l^{1-s}}\cos\big(\tfrac{1}{2}\pi s\big)\frac{\Gamma\big(k-\tfrac{1}{2}s\big)\Gamma\big(k+\tfrac{1}{2}-\tfrac{1}{2}s\big)}{\Gamma\big(k+\tfrac{1}{2}s\big)\Gamma\big(k-\tfrac{1}{2}+\tfrac{1}{2}s\big)}\Gamma\big(s-\tfrac{1}{2}\big) \\ &\quad + \frac{(2\pi)^s\sin\big(\tfrac{1}{2}\pi s\big)}{\sqrt{\pi}l^{1-s}}\sum_{1\leq n<2l}\mathscr{L}_{n^2-4l^2}(s)\frac{\Gamma\big(k-\tfrac{1}{2}s\big)\Gamma\big(1-k-\tfrac{1}{2}s\big)}{\Gamma\big(\tfrac{1}{2}\big)} \\ &\qquad\qquad \times {}_2F_1\Big(k-\frac{s}{2},\,1-k-\frac{s}{2},\,\frac{1}{2};\,\Big(\frac{n}{2l}\Big)^2\Big) \\ &\quad + \frac{2^{2k}\pi^s\cos\big(\tfrac{1}{2}\pi s\big)}{\sqrt{\pi}}\sum_{n>2l}\mathscr{L}_{n^2-4l^2}(s)\frac{\Gamma\big(k-\tfrac{1}{2}s\big)\Gamma\big(k+\tfrac{1}{2}-\tfrac{1}{2}s\big)}{\Gamma(2k)} \\ &\qquad\qquad \times \frac{1}{n^{1-s}}\Big(\frac{n}{l}\Big)^{1-2k}{}_2F_1\Big(k-\frac{s}{2},\,k+\frac{1}{2}-\frac{s}{2},\,2k;\,\Big(\frac{2l}{n}\Big)^2\Big). \end{aligned} \tag{5-6}$$

Note that in equation (5-6) only the first and the third summands have poles at $s=\frac{1}{2}$. Computing the limit as $s \to \frac{1}{2}$ we find that these poles cancel each other. This allows us to prove the exact formula for the first moment of symmetric square L-functions at the critical point given by Theorem 2.1.

6. Liouville–Green approximation of special functions

The aim of this section is to approximate the functions $\Phi_k(x)$ and $\Psi_k(x)$ appearing in exact formula (2-9). Both of these special functions can be expressed in terms of the Gauss hypergeometric function, whose asymptotic approximation has been a subject of intensive research during the last century. The works of Watson [1948], Jones [2001] and Farid Khwaja and Olde Daalhuis [2014; 2017] are the most relevant in our case. However, application of these results to approximation of $\Phi_k(x)$ and $\Psi_k(x)$ is not straightforward and involves certain technical difficulties due to either the region of validity of asymptotic expansion or the lack of asymptotic error estimates.

Therefore, we choose to follow the approach of [Olver 1974] and [Boyd and Dunster 1986]. Accordingly, we apply the Liouville–Green approximation directly to the functions $\Phi_k(x)$ and $\Psi_k(x)$. It turns out that these special functions have similar behavior to the ones occurring in the exact formula for the second moment of cusp form L-functions. See [Balkanova and Frolenkov 2016, Theorem 4.2].

Properties of Φ_k. Consider the function $\Phi_k(x)$ for $0 < x < 1$.

Lemma 6.1. *One has*

$$\Phi_k(x) = -\pi\bigl({}_2F_1\bigl(2k-\tfrac{1}{2}, \tfrac{3}{2}-2k, 1; \tfrac{1}{2}(1-\sqrt{x})\bigr) + {}_2F_1\bigl(2k-\tfrac{1}{2}, \tfrac{3}{2}-2k, 1; \tfrac{1}{2}(1+\sqrt{x})\bigr)\bigr). \tag{6-1}$$

Proof. Applying the quadratic transformation given by [Olver et al. 2010, Equation 15.8.27] we obtain

$$\Phi_k(x) = \Gamma\bigl(k-\tfrac{1}{4}\bigr)\Gamma\bigl(\tfrac{3}{4}-k\bigr)\Gamma\bigl(k+\tfrac{1}{4}\bigr)\Gamma\bigl(\tfrac{5}{4}-k\bigr)\frac{1}{2\Gamma^2\bigl(\tfrac{1}{2}\bigr)} \times \bigl({}_2F_1\bigl(2k-\tfrac{1}{2}, \tfrac{3}{2}-2k, 1; \tfrac{1}{2}(1-\sqrt{x})\bigr) + {}_2F_1\bigl(2k-\tfrac{1}{2}, \tfrac{3}{2}-2k, 1; \tfrac{1}{2}(1+\sqrt{x})\bigr)\bigr).$$

Note that $\Gamma\bigl(\tfrac{1}{2}\bigr) = \sqrt{\pi}$. Euler's reflection formula yields

$$\Gamma\bigl(k-\tfrac{1}{4}\bigr)\Gamma\bigl(k+\tfrac{1}{4}\bigr)\Gamma\bigl(\tfrac{3}{4}-k\bigr)\Gamma\bigl(\tfrac{5}{4}-k\bigr) = -2\pi^2.$$

The assertion follows. □

Making the change of variables

$$m := 2k - \tfrac{1}{2}, \quad k \in \mathbb{N}, \tag{6-2}$$

$$y := \tfrac{1}{2}(1-\sqrt{x}), \quad 0 < y < \tfrac{1}{2}, \tag{6-3}$$

one has

$$\Phi_k(x) = -\pi({}_2F_1(m, 1-m, 1; y) + {}_2F_1(m, 1-m, 1; 1-y)). \tag{6-4}$$

At the point $y = 0$, the function ${}_2F_1(m, 1-m, 1; y)$ is recessive and the function ${}_2F_1(m, 1-m, 1; 1-y)$ is dominant. Therefore, further transformations are required to apply the Liouville–Green method to the second function. In particular, we show that ${}_2F_1(m, 1-m, 1; 1-y)$ has a similar shape to $\phi_k(x)$ studied in [Balkanova and Frolenkov 2016]. Note that the parameter m is now half-integral.

Lemma 6.2. *Let $m := 2k - \tfrac{1}{2}$, $k \in \mathbb{N}$. Then*

$${}_2F_1(m, 1-m, 1; 1-y) = (-\log y + 2\psi(1) - 2\psi(m)) \times {}_2F_1(m, 1-m, 1; y) + \frac{1}{\pi}\Bigl(\frac{\partial}{\partial a} + \frac{\partial}{\partial b} + 2\frac{\partial}{\partial c}\Bigr){}_2F_1(a, b, c; y)\Big|_{\substack{a=m \\ b=1-m \\ c=1}}. \tag{6-5}$$

Proof. By [Beitman and Erdelyi 1953, Equation 33, p. 107] we have

$$
{}_2F_1(m+u,1-m+u,1;1-y) = {}_2F_1(m+u,1-m+u,1+2u;y)\frac{\Gamma(1)\Gamma(-2u)}{\Gamma(1-m-u)\Gamma(m-u)} + {}_2F_1(m-u,1-m-u,1-2u;y)\frac{\Gamma(1)\Gamma(2u)}{\Gamma(1-m+u)\Gamma(m+u)}y^{-2u}.
$$

Computing the limit as $u\to 0$, we prove the assertion. □

Lemma 6.3. *For $m = 2k-\frac{1}{2}$, $k\in\mathbb{N}$ one has*

$$
{}_2F_1\bigl(m,1-m,1;\tfrac{1}{2}\bigr) = -\frac{(-1)^k}{\sqrt{2\pi}}\frac{\Gamma\bigl(k-\frac{1}{4}\bigr)}{\Gamma\bigl(k+\frac{1}{4}\bigr)}, \tag{6-6}
$$

$$
\frac{d}{dx}\bigl({}_2F_1(m,1-m,1;x)\bigr)\big|_{x=1/2} = \frac{4(-1)^k}{\sqrt{2\pi}}\frac{\Gamma\bigl(k+\frac{1}{4}\bigr)}{\Gamma\bigl(k-\frac{1}{4}\bigr)}. \tag{6-7}
$$

Proof. On the one hand, by equation (6-1) we have

$$
{}_2F_1\bigl(m,1-m,1;\tfrac{1}{2}\bigr) = -\tfrac{1}{2\pi}\Phi_k(0).
$$

On the other hand, equation (2-7) yields

$$
\Phi_k(0) = \frac{\Gamma\bigl(k-\frac{1}{4}\bigr)\Gamma\bigl(\frac{3}{4}-k\bigr)}{\Gamma\bigl(\frac{1}{2}\bigr)} = (-1)^k\sqrt{2\pi}\frac{\Gamma\bigl(k-\frac{1}{4}\bigr)}{\Gamma\bigl(k+\frac{1}{4}\bigr)}.
$$

The last two equalities imply (6-6).

As a consequence of [Olver et al. 2010, Equation 15.8.25] we obtain

$$
{}_2F_1(m,1-m,1;x) = \frac{\Gamma\bigl(\frac{1}{2}\bigr)}{\Gamma\bigl(\frac{1}{2}m+\frac{1}{2}\bigr)\Gamma\bigl(1-\frac{1}{2}m\bigr)}\times {}_2F_1\bigl(\tfrac{1}{2}m,\tfrac{1}{2}-\tfrac{1}{2}m,\tfrac{1}{2};(1-2x)^2\bigr) + (1-2x)\frac{\Gamma\bigl(-\frac{1}{2}\bigr)}{\Gamma\bigl(\frac{1}{2}m\bigr)\Gamma\bigl(\frac{1}{2}-\frac{1}{2}m\bigr)}{}_2F_1\bigl(\tfrac{1}{2}m+\tfrac{1}{2},1-\tfrac{1}{2}m,\tfrac{3}{2};(1-2x)^2\bigr).
$$

Then equality (6-7) follows by differentiating the last expression in x and setting $x=\frac{1}{2}$. □

Lemma 6.4. *The functions*

$$
{}_2F_1(m,1-m,1;y) \quad \textit{and} \quad {}_2F_1(m,1-m,1;1-y)
$$

are solutions of differential equation

$$
y(1-y)F''(y)+(1-2y)F'(y)+m(m-1)F(y)=0. \tag{6-8}
$$

Proof. This follows from the differential equation for hypergeometric functions. □

Approximation of Φ_k. In order to find a Liouville–Green approximation for $\Phi_k(y)$, we use formula (6-4) and study separately each of the hypergeometric functions ${}_2F_1(m, 1-m, 1; y)$ and ${}_2F_1(m, 1-m, 1; 1-y)$. As shown in Lemma 6.4, these functions are solutions of differential equation (6-8) that was already approximated in [Balkanova and Frolenkov 2016, Section 5.2]. So our problem reduces to computation of the Liouville–Green constants C_Y and C_J in the approximation of ${}_2F_1(m, 1-m, 1; 1-y)$.

For the reader's convenience, we briefly recall the required results of [Balkanova and Frolenkov 2016, Section 5.2]. It follows from Lemma 6.4 that the functions

$$G_1(y) := {}_2F_1(m, 1-m, 1; y)\sqrt{y(1-y)}, \tag{6-9}$$

$$G_2(y) := {}_2F_1(m, 1-m, 1; 1-y)\sqrt{y(1-y)} \tag{6-10}$$

are solutions of the differential equation

$$G''(y) = (u^2 f(y) + g(y))G(y), \tag{6-11}$$

where

$$u := 2k-1, \quad f(y) := -\frac{1}{y(1-y)}, \tag{6-12}$$

$$g(y) := -\frac{1}{4y^2(1-y)^2} + \frac{1}{4y(1-y)}. \tag{6-13}$$

Making the change of variables

$$Z(y) := \frac{G(y)}{\alpha(y)}, \quad \alpha(y) := \frac{(y-y^2)^{1/4}}{2(\arcsin\sqrt{y})^{1/2}}, \tag{6-14}$$

$$\xi := 4\arcsin^2\sqrt{y}, \tag{6-15}$$

we transform equation (6-11) into the following shape

$$\frac{d^2Z}{d\xi^2} + \left[\frac{u^2}{4\xi} + \frac{1}{4\xi^2} + \frac{\psi(\xi)}{\xi}\right]Z = 0, \tag{6-16}$$

with

$$\psi(\xi) := \frac{1}{16\sin^2\sqrt{\xi}} - \frac{1}{16\xi}. \tag{6-17}$$

Removing the summand with $\psi(\xi)/\xi$ in equation (6-16), we have

$$\frac{d^2Z}{d\xi^2} + \left[\frac{u^2}{4\xi} + \frac{1}{4\xi^2}\right]Z = 0. \tag{6-18}$$

The solutions of (6-18) are defined by

$$Z_C = \sqrt{\xi}\,C_0(u\sqrt{\xi}), \tag{6-19}$$

where C_i is either J or Y Bessel function of index i.

Then according to [Olver 1974, Chapter 12] solutions of original differential equation (6-16) can be found in the form

$$Z_C(\xi)=\sqrt{\xi}C_0(u\sqrt{\xi})\sum_{n=0}^{\infty}\frac{A(n;\xi)}{u^{2n}}-\frac{\xi}{u}C_1(u\sqrt{\xi})\sum_{n=0}^{\infty}\frac{B(n;\xi)}{u^{2n}}. \tag{6-20}$$

In order to determine coefficients $A(n;\xi)$, $B(n;\xi)$ we use differential equations (see [Gradshteyn and Ryzhik 2007, Equation 8.491(3)]) for functions

$$W(\xi):=\sqrt{\xi}C_0(u\sqrt{\xi}),\qquad V(\xi):=\xi C_1(u\sqrt{\xi}) \tag{6-21}$$

and substitute (6-20) in equation (6-16). This yields

$$W(\xi)\sum_{n=0}^{\infty}\frac{C_n(\xi)}{u^{2n}}-V(\xi)\sum_{n=0}^{\infty}\frac{D_n(\xi)}{u^{2n-1}}=0, \tag{6-22}$$

where

$$C_n(\xi):=A''(n;\xi)+\frac{1}{\xi}A'(n;\xi)-\frac{\psi(\xi)}{\xi}A(n;\xi)-B'(n;\xi)-\frac{B(n;\xi)}{2\xi},$$
$$D_n(\xi):=B''(n-1;\xi)+\frac{1}{\xi}B'(n-1;\xi)-\frac{\psi(\xi)}{\xi}B(n-1;\xi)+\frac{1}{\xi}A'(n;\xi).$$

Letting $C_n(\xi)=D_n(\xi)=0$, we find the required recurrence relations

$$A(n;\xi)=-\xi B'(n-1;\xi)+\int_0^{\xi}\psi(x)B(n-1;x)\,dx+\lambda_n, \tag{6-23}$$

$$\sqrt{\xi}B(n;\xi)=\int_0^{\xi}\frac{1}{\sqrt{x}}\big(xA''(n;x)+A'(n;x)-\psi(x)A(n;x)\big)\,dx \tag{6-24}$$

for some real constants of integration λ_n.

Assume that $A(0;\xi)=1$. Then

$$B(0;\xi)=-\frac{1}{8\sqrt{\xi}}\left(\cot\sqrt{\xi}-\frac{1}{\sqrt{\xi}}\right), \tag{6-25}$$

$$A(1;\xi)=\frac{1}{8}\left(\frac{1}{\xi}-\frac{\cot\sqrt{\xi}}{2\sqrt{\xi}}-\frac{1}{2\sin^2\sqrt{\xi}}\right)-\frac{1}{128}\left(\cot\sqrt{\xi}-\frac{1}{\sqrt{\xi}}\right)^2+\lambda_1. \tag{6-26}$$

Furthermore, solutions (6-20) can be approximated by finite series using [Olver 1974, Theorem 4.1, p. 444] or [Boyd and Dunster 1986, Theorem 1].

Theorem 6.5. *Let $\xi_2=\frac{1}{4}\pi^2$. For each value of u and each nonnegative integer N, equation (6-16) has solutions $Z_Y(\xi)$, $Z_J(\xi)$ which are infinitely differentiable in ξ on interval $(0,\xi_2)$, and are given by*

$$Z_Y(\xi)=\sqrt{\xi}Y_0(u\sqrt{\xi})\sum_{n=0}^{N}\frac{A_Y(n;\xi)}{u^{2n}}-\frac{\xi}{u}Y_1(u\sqrt{\xi})\sum_{n=0}^{N-1}\frac{B_Y(n;\xi)}{u^{2n}}+\epsilon_{2N+1,1}(u,\xi), \tag{6-27}$$

$$Z_J(\xi)=\sqrt{\xi}J_0(u\sqrt{\xi})\sum_{n=0}^{N}\frac{A_J(n;\xi)}{u^{2n}}-\frac{\xi}{u}J_1(u\sqrt{\xi})\sum_{n=0}^{N-1}\frac{B_J(n;\xi)}{u^{2n}}+\epsilon_{2N+1,2}(u,\xi), \tag{6-28}$$

where

$$\epsilon_{2N+1,1}(u,\xi) \ll \frac{\sqrt{\xi}\,|Y_0(u\sqrt{\xi})|}{u^{2N+1}}\sqrt{\xi_2-\xi}, \tag{6-29}$$

$$\epsilon_{2N+1,2}(u,\xi) \ll \frac{\sqrt{\xi}\,|J_0(u\sqrt{\xi})|}{u^{2N+1}} \min(\sqrt{\xi},1) \tag{6-30}$$

and coefficients $(A_Y(n;\xi), B_Y(n;\xi))$, $(A_J(n;\xi), B_J(n;\xi))$ *are defined by* (6-23)–(6-24).

The functions $\xi^{1/4}(\sin\sqrt{\xi})^{1/2}G_1(\sin^2\sqrt{\xi}/2)$ and $Z_J(\xi)$ are recessive solutions of equation (6-16) as $\xi \to 0$. Therefore, there is c_0 such that

$$\xi^{1/4}(\sin\sqrt{\xi})^{1/2}G_1(\sin^2\sqrt{\xi}/2) = c_0 Z_J(\xi). \tag{6-31}$$

The value of the constant c_0 is determined by computing the limit of the left- and right-hand sides of equation (6-31) as $\xi \to 0$. On the one hand,

$$\lim_{\xi\to 0} {}_2F_1(k, 1-k, 1; \sin^2\sqrt{\xi}/2) = 1. \tag{6-32}$$

On the other hand,

$$Z_J(\xi) = \sqrt{\xi}\sum_{n=0}^{N}\frac{A_J(n;\xi)}{u^{2n}} + O(\xi) \quad \text{as } \xi \to 0. \tag{6-33}$$

Choosing $A_J(n;\xi)$ such that $A_J(0;0)=1$ and $A_J(n;0)=0$ for $n \geq 1$ we find that $c_0 = 1$.

To sum up, we have proved the following lemma:

Lemma 6.6. *Let* $\xi_2 = \frac{1}{4}\pi^2$. *For* $\xi \in (0,\xi_2)$, *one has*

$$\xi^{1/4}(\sin\sqrt{\xi})^{1/2}G_1\left(\sin^2\frac{\sqrt{\xi}}{2}\right) = Z_J(\xi), \tag{6-34}$$

where $Z_J(\xi)$ *is given by* (6-28).

This concludes the summary of results of [Balkanova and Frolenkov 2016, Section 5.2]. Our final goal is to compute $C_Y = C_Y(u)$ and $C_J = C_J(u)$ such that

$$\xi^{1/4}(\sin\sqrt{\xi})^{1/2}G_2\big(\sin^2\tfrac{1}{2}\sqrt{\xi}\big) = C_Y Z_Y(\xi) + C_J Z_J(\xi). \tag{6-35}$$

Note that there exist c_1, c_2 such that

$$\begin{aligned} Z_Y(\xi) &= \big(c_1 G_1\big(\sin^2\tfrac{1}{2}\sqrt{\xi}\big) + c_2 G_2\big(\sin^2\tfrac{1}{2}\sqrt{\xi}\big)\big) \times \xi^{1/4}(\sin\sqrt{\xi})^{1/2} \\ &= c_1 Z_J(\xi) + \xi^{1/4}\big(\sin\sqrt{\xi}\big)^{1/2} c_2 G_2\big(\sin^2\tfrac{1}{2}\sqrt{\xi}\big). \end{aligned} \tag{6-36}$$

The last two equalities imply

$$C_Y = \frac{1}{c_2}, \quad C_J = -\frac{c_1}{c_2}. \tag{6-37}$$

Lemma 6.7. *One has*

$$2c_1=(-1)^{k+1}\sqrt{2\pi}\frac{\Gamma\left(k+\frac{1}{4}\right)}{\Gamma\left(k-\frac{1}{4}\right)}\frac{Z_Y(\xi_2)}{\xi_2^{1/4}}+(-1)^k\sqrt{2\pi}\frac{\Gamma\left(k-\frac{1}{4}\right)}{\Gamma\left(k+\frac{1}{4}\right)}\xi_2^{1/4}\left(Z_Y'(\xi_2)-\frac{Z_Y(\xi_2)}{4\xi_2}\right), \tag{6-38}$$

$$2c_2=(-1)^{k+1}\sqrt{2\pi}\frac{\Gamma\left(k+\frac{1}{4}\right)}{\Gamma\left(k-\frac{1}{4}\right)}\frac{Z_Y(\xi_2)}{\xi_2^{1/4}}-(-1)^k\sqrt{2\pi}\frac{\Gamma\left(k-\frac{1}{4}\right)}{\Gamma\left(k+\frac{1}{4}\right)}\xi_2^{1/4}\left(Z_Y'(\xi_2)-\frac{Z_Y(\xi_2)}{4\xi_2}\right). \tag{6-39}$$

Proof. To determine coefficients c_1, c_2 we consider the pair of equations

$$Z_Y(\xi_2)=\xi_2^{1/4}\left(c_1G_1\left(\tfrac{1}{2}\right)+c_2G_2\left(\tfrac{1}{2}\right)\right),\quad Z_Y'(\xi_2)=\frac{Z_Y(\xi_2)}{4\xi_2}+\frac{1}{4\xi_2^{1/4}}\left(c_1G_1'\left(\tfrac{1}{2}\right)+c_2G_2'\left(\tfrac{1}{2}\right)\right).$$

Note that

$$G_1\left(\tfrac{1}{2}\right)=G_2\left(\tfrac{1}{2}\right)\quad\text{and}\quad G_1'\left(\tfrac{1}{2}\right)=-G_2'\left(\tfrac{1}{2}\right).$$

Therefore,

$$(c_1+c_2)G_1\left(\tfrac{1}{2}\right)=\frac{Z_Y(\xi_2)}{\xi_2^{1/4}},\quad (c_1-c_2)G_1'\left(\tfrac{1}{2}\right)=4\xi_2^{1/4}\left(Z_Y'(\xi_2)-\frac{Z_Y(\xi_2)}{4\xi_2}\right).$$

The assertion follows by Lemma 6.3. □

Lemma 6.8. *For $\xi_2=\frac{1}{4}\pi^2$, one has*

$$Z_Y(\xi_2)=\frac{(-1)^{k+1}}{\sqrt{2u}}\left(1+\frac{1}{u^2}\left(\lambda_1-\frac{1}{16}\right)+O\left(\frac{1}{u^3}\right)\right), \tag{6-40}$$

$$Z_Y'(\xi_2)=\frac{(-1)^{k+1}}{\sqrt{2u}}\left(\frac{u}{\pi}+\frac{1}{\pi^2}+\frac{2\lambda_1+\frac{1}{8}}{2\pi u}+O\left(\frac{1}{u^2}\right)\right). \tag{6-41}$$

Proof. Applying [Boyd and Dunster 1986, Theorem 1] with $N=1$ we obtain

$$\epsilon_{3,1}(u;\xi_2)=0\quad\text{and}\quad \frac{\partial}{\partial\xi}\epsilon_{3,1}(u;\xi)\Big|_{\xi=\xi_2}=0.$$

Therefore,

$$Z_Y(\xi_2)=\sqrt{\xi_2}Y_0(u\sqrt{\xi_2})\left(1+\frac{A_Y(1,\xi_2)}{u^2}\right)-\xi_2Y_1(u\sqrt{\xi_2})\frac{B_Y(0,\xi_2)}{u}$$

and

$$\begin{aligned}Z_Y'(\xi_2)=\sqrt{\xi_2}Y_0(u\sqrt{\xi_2})&\left(\frac{1}{2\xi_2}\left[1+\frac{A_Y(1;\xi_2)}{u^2}\right]+\frac{A_Y'(1;\xi_2)}{u^2}-\frac{1}{2}B_Y(0;\xi_2)\right)\\&-\xi_2Y_1(u\sqrt{\xi_2})\left(\frac{u}{2\xi_2}\left[1+\frac{A_Y(1;\xi_2)}{u^2}\right]+\frac{1}{2\xi_2u}B_Y(0;\xi_2)+\frac{1}{u}B_Y'(0;\xi_2)\right).\end{aligned}$$

By means of the Hankel asymptotic expansion (see [Olver et al. 2010, Equation 10.17.1, 10.17.4] and [Gradshteyn and Ryzhik 2007, Equation 8.451(1,7,8)]) we evaluate

$$\sqrt{\xi_2}Y_0(u\sqrt{\xi_2}) = \frac{\pi}{2}Y_0\Big((2k-1)\frac{\pi}{2}\Big) = \frac{(-1)^{k+1}}{\sqrt{2u}}\Big(\sum_{j=0}^{\infty}(-1)^j\frac{a_{2j}(0)}{(\pi u/2)^{2j}} + \sum_{j=0}^{\infty}(-1)^j\frac{a_{2j+1}(0)}{(\pi u/2)^{2j+1}}\Big),$$

$$\xi_2 Y_1(u\sqrt{\xi_2}) = \frac{\pi}{2}\frac{(-1)^k}{\sqrt{2u}}\Big(\sum_{j=0}^{\infty}(-1)^j\frac{a_{2j}(1)}{(\pi u/2)^{2j}} - \sum_{j=0}^{\infty}(-1)^j\frac{a_{2j+1}(1)}{(\pi u/2)^{2j+1}}\Big),$$

where

$$a_j(v) = \frac{\Gamma\big(v+j+\frac{1}{2}\big)}{2^j j!\,\Gamma\big(v-j+\frac{1}{2}\big)}.$$

This yields

$$Z_Y(\xi_2) = \frac{(-1)^{k+1}}{\sqrt{2u}}\Big(1 + \frac{a_1(0)}{\pi u/2} - \frac{a_2(0)}{(\pi u/2)^2} + O(u^{-3})\Big)\Big(1 + \frac{A_Y(1;\xi_2)}{u^2}\Big) - \frac{\pi}{2}\frac{(-1)^k}{\sqrt{2u}}\Big(1 - \frac{a_1(1)}{\pi u/2} - \frac{a_2(1)}{(\pi u/2)^2} + O(u^{-3})\Big)\frac{B_Y(0,\xi_2)}{u}.$$

Simplifying the expression above, one has

$$Z_Y(\xi_2) = \frac{(-1)^{k+1}}{\sqrt{2u}}\Big(1 + \frac{1}{u}\Big(\frac{2}{\pi}a_1(0) + \frac{\pi}{2}B_Y(0,\xi_2)\Big) + \frac{1}{u^2}\Big(A_Y(1,\xi_2) - \frac{4}{\pi^2}a_2(0) - a_1(1)B_Y(0,\xi_2)\Big)\Big) + O(u^{-7/2}).$$

Using formulas (6-25) and (6-26), we find

$$a_1(0) = -\frac{1}{8},\quad \frac{2}{\pi}a_1(0) + \frac{\pi}{2}B_Y(0,\xi_2) = 0,\quad A_Y(1,\xi_2) - \frac{4}{\pi^2}a_2(0) - a_1(1)B_Y(0,\xi_2) = \lambda_1 - \frac{1}{16}.$$

This gives equation (6-40).

Similarly, we obtain

$$Z_Y'(\xi_2) = \sqrt{\xi_2}Y_0(u\sqrt{\xi_2})\Big[\frac{7}{4\pi^2} + \frac{1}{u^2}\Big(\frac{1}{\pi^2}\Big(2\lambda_1 - \frac{1}{32}\Big) - \frac{15}{8\pi^4}\Big)\Big] - \xi_2 Y_1(u\sqrt{\xi_2})\Big[u\frac{2}{\pi^2} + \frac{1}{u}\Big(\frac{1}{\pi^2}\Big(2\lambda_1 + \frac{1}{8}\Big) - \frac{1}{16\pi^4}\Big)\Big].$$

Hankel's expansion for Bessel functions yields the following asymptotics:

$$Z_Y'(\xi_2) = \frac{(-1)^{k+1}}{\sqrt{2u}}\Big[u\frac{1}{\pi} + \Big(\frac{7}{4\pi^2} - \frac{2}{\pi^2}a_1(1)\Big) + \frac{1}{u}\Big(\frac{7}{2\pi^3}a_1(0) + \frac{\pi}{2}\Big(\frac{2\lambda_1 + \frac{1}{8}}{\pi^2} - \frac{1}{16\pi^4}\Big) - \frac{4}{\pi^3}a_2(1)\Big) + \frac{1}{u^2}\Big(-\frac{7}{\pi^4}a_2(0) - a_1(1)\Big(\frac{2\lambda_1 + \frac{1}{8}}{\pi^2} - \frac{1}{16\pi^4}\Big) + \frac{8}{\pi^4}a_3(1)\Big) + O(u^{-3})\Big].$$

Finally, substituting

$$a_1(0) = -\tfrac{1}{8},\quad a_1(1) = \tfrac{3}{8},\quad a_2(1) = -\tfrac{15}{128},$$

we prove equation (6-41). □

Corollary 6.9. *One has*

$$C_Y = 1 + O\Big(\frac{1}{k}\Big),\quad C_J = O\Big(\frac{1}{k^2}\Big). \tag{6-42}$$

Proof. By Lemma 6.8

$$Z_Y'(\xi_2) - \frac{Z_Y(\xi_2)}{4\xi_2} = \frac{(-1)^{k+1}}{\sqrt{2u}}\left(\frac{u}{\pi} + \frac{2\lambda_1 + \frac{1}{8}}{2\pi u} + O(u^{-2})\right).$$

It follows from [Olver et al. 2010, Equation 5.11.13] that

$$\frac{\Gamma\left(k+\frac{1}{4}\right)}{\Gamma\left(k-\frac{1}{4}\right)} = k^{1/2} - \tfrac{1}{4}k^{-1/2} + O(k^{-3/2}) \quad \text{and} \quad \frac{\Gamma\left(k-\frac{1}{4}\right)}{\Gamma\left(k+\frac{1}{4}\right)} = k^{-1/2} + O(k^{-3/2}).$$

Then Lemma 6.7 gives

$$c_1 = O(k^{-2}), \quad c_2 = 1 + O(k^{-1}).$$

Equations (6-37) yield the assertion. □

As a consequence of equation (6-35), Lemma 6.6, Theorem 6.5 and Corollary 6.9, we obtain the main result.

Theorem 6.10. *Let $u = 2k-1$, $\xi_2 = \frac{1}{4}\pi^2$. Then for $\xi \in (0, \xi_2)$ one has*

$$\Phi_k(\cos^2\sqrt{\xi}) = \frac{-\pi}{\xi^{1/4}(\sin\sqrt{\xi})^{1/2}}[Z_J(\xi) + C_Y Z_Y(\xi) + C_J Z_J(\xi)], \tag{6-43}$$

where Z_Y, Z_J are given by (6-27), (6-28) and

$$C_Y = 1 + O\left(\frac{1}{k}\right), \quad C_J = O\left(\frac{1}{k^2}\right). \tag{6-44}$$

Corollary 6.11. *We have*

$$\Phi_k\left(\frac{1}{4}\right) = -\frac{2^{3/2}\sqrt{\pi}}{3^{1/4}}\frac{1}{\sqrt{2k-1}}\sin\frac{\pi(2k-1)}{3}\left(1 + O\left(\frac{1}{k}\right)\right).$$

Approximation of Ψ_k. Next, we find a Liouville–Green approximation for the function Ψ_k. With this goal, we follow the arguments of [Balkanova and Frolenkov 2016, Section 5.3] with minor changes. In particular, the differential equation for $\Psi_k(x)$ is slightly different, and, therefore, one must recompute various functions and constants appearing in the Liouville–Green approximation. We provide all details here to make the presentation self-contained.

Consider the function

$$y(x) := \sqrt{1-x}\,\Psi_k(x). \tag{6-45}$$

Let $u := k - \frac{1}{2}$ and

$$f(x) := \frac{1}{x^2(1-x)}, \quad g(x) := -\frac{1}{4x^2(1-x)^2} + \frac{3}{16x(1-x)}. \tag{6-46}$$

Lemma 6.12. *The function $y = y(x)$ is a solution of equation*

$$y''(x) - (u^2 f(x) + g(x))y(x) = 0. \tag{6-47}$$

Proof. Using the differential equation for the hypergeometric function, we find that $y = y(x)$ satisfies the following differential equation

$$y'' + \left(\frac{1-(2k-1)^2}{4x^2} + \frac{1}{4(1-x)^2} + \frac{\frac{5}{4}-(2k-1)^2}{4x(1-x)}\right)y = 0.$$

The assertion follows. □

Making the change

$$Z(x) := \frac{y(x)}{\alpha(x)}, \quad \alpha(x) := \frac{(x^2-x^3)^{1/4}}{2(\operatorname{artanh}\sqrt{1-x})^{1/2}} \tag{6-48}$$

and the substitution

$$\xi := 4\operatorname{artanh}^2\sqrt{1-x}, \tag{6-49}$$

we transform equation (6-47) to the type

$$\frac{d^2Z}{d\xi^2} + \left(-\frac{u^2}{4\xi} + \frac{1}{4\xi^2} - \frac{\psi(\xi)}{\xi}\right)Z = 0, \tag{6-50}$$

where

$$\psi(\xi) = \frac{1}{16}\left(\frac{1}{\xi} - \frac{1}{4\sinh^2\sqrt{\xi}/2}\right) \tag{6-51}$$

is an analytic function as $\xi \to 0$.

In order to find a Liouville–Green approximation to equation (6-50), we remove the term with $\psi(\xi)/\xi$ in (6-50). The resulting equation

$$Z'' + \left(-\frac{u^2}{4\xi} + \frac{1}{4\xi}\right)Z = 0 \tag{6-52}$$

has I and K Bessel functions as solutions (see [Olver et al. 2010, Equation 10.13.2]), namely

$$Z_L = \sqrt{\xi}L_0(u\sqrt{\xi}), \tag{6-53}$$

where

$$L_v(x) := \begin{cases} I_v(x) \\ e^{\pi i v}K_v(x). \end{cases} \tag{6-54}$$

This suggests that solutions of the original differential equation (6-50) can be written in the form (see [Olver 1974, Equation 2.09, Chapter 12])

$$Z_L = \sqrt{\xi}L_0(u\sqrt{\xi})\sum_{n=0}^{\infty}\frac{A(n;\xi)}{u^{2n}} + \frac{\xi}{u}L_1(u\sqrt{\xi})\sum_{n=0}^{\infty}\frac{B(n;\xi)}{u^{2n}}. \tag{6-55}$$

Lemma 6.13. *The coefficients $A(n;\xi)$ and $B(n;\xi)$ are given by*

$$\sqrt{\xi}B(n;\xi) = -\sqrt{\xi}A'(n;\xi) + \int_0^{\xi}\left(\psi(x)A(n;x) - \frac{1}{2}A'(n;x)\right)\frac{dx}{\sqrt{x}}, \tag{6-56}$$

$$A(n;\xi) = -\xi B'(n-1;\xi) + \int_0^{\xi}\psi(x)B(n-1;x)\,dx + \lambda_n \tag{6-57}$$

for some real constants of integration λ_n.

Proof. By [Olver et al. 2010, Equations 10.13.2, 10.13.5, 10.36, 10.29.2, 10.29.3] the functions

$$W(\xi) := \sqrt{\xi} L_0(u\sqrt{\xi}), \quad V(\xi) := \xi L_1(u\sqrt{\xi})$$

satisfy the relations

$$W'' + \left(-\frac{u^2}{4\xi} + \frac{1}{4\xi^2}\right)W = 0, \quad V'' - \frac{1}{\xi}V' + \left(-\frac{u^2}{4\xi} + \frac{3}{4\xi^2}\right)V = 0, \quad V' = \frac{1}{2\xi}V + \frac{u}{2}W, \quad W' = \frac{1}{2\xi}W + \frac{u}{2\xi}V.$$

Using this and substituting solution (6-55) into equation (6-50), we obtain that

$$W(\xi)\sum_{n=0}^{\infty}\frac{C(n;\xi)}{u^{2n}} + V(\xi)\sum_{n=0}^{\infty}\frac{D(n;\xi)}{u^{2n+1}} = 0,$$

where

$$C(n;\xi) = A''(n;\xi) + \frac{A'(n;\xi)}{\xi} - \frac{\psi(\xi)}{\xi}A(n;\xi) + B'(n;\xi) + \frac{B(n;\xi)}{2\xi},$$
$$D(n;\xi) = B''(n-1;\xi) + \frac{B'(n-1;\xi)}{\xi} - \frac{\psi(\xi)}{\xi}B(n-1;\xi) + \frac{A'(n;\xi)}{\xi}.$$

Setting $C(n;\xi) = D(n;\xi) = 0$ we find the required recurrence relations. □

Let $A(0;\xi) = 1$. Then

$$B(0;\xi) = \frac{1}{16}\left(\frac{\coth\sqrt{\xi}}{\sqrt{\xi}} - \frac{2}{\xi}\right), \tag{6-58}$$

$$A(1;\xi) = -\frac{1}{32}\left(\frac{4}{\xi} - \frac{\coth\sqrt{\xi/4}}{\sqrt{\xi}} - \frac{1}{2\sinh^2\sqrt{\xi/4}}\right) + \frac{1}{512}\left(\coth\sqrt{\xi/4} - \frac{2}{\sqrt{\xi}}\right)^2 + \lambda_1. \tag{6-59}$$

Note that

$$\lim_{\xi\to\infty}\sqrt{\xi}B(0;\xi) = \tfrac{1}{16}, \quad \lim_{\xi\to\infty}A(1;\xi) = \tfrac{1}{512} + \lambda_1. \tag{6-60}$$

The variation of the function is given by

$$V_{a,b}(f(x)) := \int_a^b |(f(x))'|\,dx. \tag{6-61}$$

Lemma 6.14. *The function $V_{\xi,\infty}(\sqrt{x}B(1;x))$ is bounded. For $n > 1$, the function $V_{\xi,\infty}(\sqrt{x}B(n;x))$ converges.*

Proof. As a consequence of recurrence relation (6-56), we find

$$(\sqrt{x}B(1;x))' = O(x^{-1/2}) \quad \text{as } x\to 0 \quad \text{and} \quad (\sqrt{x}B(1;x))' = O(x^{-2}) \quad \text{as } x\to\infty.$$

Thus $V_{\xi,\infty}(\sqrt{x}B(1;x))$ is bounded. Note that

$$\psi^{(s)}(\xi) = O\left(\frac{1}{|\xi|^{s+1}}\right).$$

The convergence of variation for $n > 1$ then follows by [Olver 1974, Exercise 4.2, p. 445]. □

Using Lemma 6.14, we can truncate the infinite summation in (6-55) up to N summands with a negligible error term. The value of N determines the quality of approximation: the error is smaller for larger N.

Lemma 6.15. *For each value of u and each nonnegative integer N, equation (6-50) has a solution $Z_K(\xi)$ which is infinitely differentiable in ξ on interval $(0, \infty)$ and is given by*

$$Z_K(\xi) = \sqrt{\xi} K_0(u\sqrt{\xi}) \sum_{n=0}^{N} \frac{A_K(n;\xi)}{u^{2n}} - \frac{\xi}{u} K_1(u\sqrt{\xi}) \sum_{n=0}^{N-1} \frac{B_K(n;\xi)}{u^{2n}} + \epsilon_{2N+1,3}(u,\xi), \tag{6-62}$$

where

$$|\epsilon_{2N+1,3}(u,\xi)| \le \frac{\sqrt{\xi} K_0(u\sqrt{\xi})}{u^{2N+1}} \times V_{\xi,\infty}(\sqrt{\xi} B_K(N;\xi)) \exp\Bigl(\frac{1}{u} V_{\xi,\infty}(\sqrt{\xi} B_K(0;\xi))\Bigr). \tag{6-63}$$

In particular, for $N = 1$

$$\epsilon_{3,3}(u,\xi) \ll \frac{\sqrt{\xi} K_0(u\sqrt{\xi})}{u^3} \min\Bigl(\sqrt{\xi}, \frac{1}{\xi}\Bigr). \tag{6-64}$$

As $\xi \to \infty$, the differential equation (6-50) has two recessive solutions, namely

$$Z(\xi) = \Psi_k\biggl(\frac{1}{\cosh^2 \sqrt{\xi}/2}\biggr)(\xi \sinh^2 \sqrt{\xi})^{1/4} \tag{6-65}$$

and $Z_K(\xi)$ given by (6-62). Thus there is $C_K = C_K(u)$ such that

$$\Psi_k\biggl(\frac{1}{\cosh^2 \sqrt{\xi}/2}\biggr)(\xi \sinh^2 \sqrt{\xi})^{1/4} = C_K Z_K(\xi). \tag{6-66}$$

The last step is to compute $C_K = C_K(u)$.

Lemma 6.16. *One has*

$$C_K = 2 + O(k^{-1}). \tag{6-67}$$

Proof. To determine C_K, we compute the limit of the left- and right-hand sides of equation (6-66) as $\xi \to \infty$. This implies

$$C_K = \frac{\Gamma\bigl(k-\frac14\bigr)\Gamma\bigl(k+\frac14\bigr)}{\Gamma(2k)} \frac{2^{2k}\sqrt{u}}{\sqrt{\pi}} \biggl(\sum_{n=0}^{N} \frac{a_n}{u^{2n}} - \sum_{n=0}^{N-1} \frac{b_n}{u^{2n+1}}\biggr)^{-1},$$

where

$$a_n = \lim_{\xi\to\infty} A(n;\xi), \quad b_n = \lim_{\xi\to\infty} B(n;\xi)\sqrt{\xi}.$$

According to (6-60), we know that $a_0 = 1,\ a_1 = \frac{1}{512} + \lambda_1,\ b_0 = \frac{1}{16}$. Furthermore,

$$\frac{\Gamma\bigl(k-\frac14\bigr)\Gamma\bigl(k+\frac14\bigr)}{\Gamma(2k)} = \frac{\Gamma^2(k)}{\Gamma(2k)}\Bigl(1+O\Bigl(\frac1k\Bigr)\Bigr) = \frac{2\sqrt{\pi}}{\sqrt{k}2^{2k}}\Bigl(1+O\Bigl(\frac1k\Bigr)\Bigr).$$

The assertion follows. □

Finally, we obtain the main theorem.

Theorem 6.17. *For $\xi \in (0, \infty)$ the following equality holds*

$$\Psi_k\left(\frac{1}{\cosh^2\sqrt{\xi}/2}\right)(\xi\sinh^2\sqrt{\xi})^{1/4} = C_K Z_K(\xi), \tag{6-68}$$

where $Z_K(\xi)$ is defined by (6-62) and $C_K = 2 + O(k^{-1})$.

7. Asymptotic formula

Corollaries 2.2 and 2.3 are derived from Theorem 2.1 by estimating the last two summands in exact formula (2-9), as we now show.

Lemma 7.1. *For any $\epsilon > 0$,*

$$E_1(k,l) := \frac{1}{\sqrt{l}}\sum_{1\le n<2l}\mathscr{L}_{n^2-4l^2}\left(\frac{1}{2}\right)\Phi_k\left(\frac{n^2}{4l^2}\right) \ll \frac{l^{5/6+\epsilon}}{\sqrt{k}}. \tag{7-1}$$

Proof. Using the subconvexity bound (4-10) we obtain

$$E_1(k,l) \ll \frac{l^\epsilon}{\sqrt{l}}\sum_{1\le n<2l}(2l-n)^{1/6}(2l+n)^{1/6}\left|\Phi_k\left(\frac{n^2}{4l^2}\right)\right| \ll \frac{l^{1/6+\epsilon}}{\sqrt{l}}\sum_{1\le n<2l}n^{1/6}\left|\Phi_k\left(\left(1-\frac{n}{2l}\right)^2\right)\right|$$

Let $\xi = 4\left(\arcsin\sqrt{n/4l}\right)^2$, $u = 2k-1$. Then by Theorem 6.10 one has

$$\Phi_k\left(\left(1-\frac{n}{2l}\right)^2\right) \ll \frac{\sqrt{\xi}Y_0(u\sqrt{\xi})}{\left(\arcsin\sqrt{n/4l}\right)^{1/2}(n/l)^{1/4}} \ll \frac{\left(\arcsin\sqrt{n/4l}\right)^{1/2}}{(n/l)^{1/4}}Y_0(u\sqrt{\xi}).$$

If $l \ll k^2$, one has $u\sqrt{\xi} \gg 1$. Then the estimate for the Bessel function

$$Y_0(u\sqrt{\xi}) \ll \frac{1}{u^{1/2}\xi^{1/4}}$$

yields

$$\Phi_k\left(\left(1-\frac{n}{2l}\right)^2\right) \ll \frac{1}{k^{1/2}(n/l)^{1/4}}.$$

Consequently,

$$E_1(k,l) \ll \frac{l^{1/6+\epsilon}}{\sqrt{l}}\sum_{n<2l}\frac{n^{1/6}l^{1/4}}{k^{1/2}n^{1/4}} \ll \frac{l^{5/6+\epsilon}}{\sqrt{k}}.$$ □

Corollary 7.2. *If the Lindelöf hypothesis for Dirichlet L-functions is true, then*

$$E_1(k,l) \ll \frac{l^{1/2+\epsilon}}{\sqrt{k}} \quad \textit{for any } \epsilon > 0. \tag{7-2}$$

Lemma 7.3. *For some $c > 0$,*

$$E_2(k,l) := \frac{1}{l\sqrt{2}} \sum_{n>2l} \mathscr{L}_{n^2-4l^2}\Big(\frac{1}{2}\Big)\sqrt{n}\,\Psi_k\Big(\frac{4l^2}{n^2}\Big) \ll \frac{l^{-1/12}}{\sqrt{k}} \exp\Big(-\frac{ck}{\sqrt{l}}\Big). \tag{7-3}$$

Proof. It follows from the subconvexity bound (4-10) that

$$\begin{aligned} E_2(k,l) &\ll \frac{1}{l} \sum_{n>2l} (n^2-4l^2)^{1/6}\sqrt{n}\Big|\Psi_k\Big(\frac{4l^2}{n^2}\Big)\Big| \\ &\ll \frac{1}{l}\int_{2l+1}^{\infty} x^{1/2}(x^2-4l^2)^{1/6}\Big|\Psi_k\Big(\frac{4l^2}{x^2}\Big)\Big|\,dx + l^{-1/3}\Big|\Psi_k\Big(\frac{4l^2}{(2l+1)^2}\Big)\Big|. \end{aligned}$$

Next, we make the change of variables $x = 2l\cosh\big(\frac{1}{2}\sqrt{\xi}\big)$ and estimate $E_2(k,l)$ using Theorem 6.17 with $N = 0$. Consider the first summand

$$\begin{aligned} E_{2,1}(k,l) := \frac{1}{l}\int_{2l+1}^{\infty} x^{1/2}(x^2-4l^2)^{1/6}\Big|\Psi_k\Big(\frac{4l^2}{x^2}\Big)\Big|\,dx &\ll l^{5/6}\int_{\xi_0}^{\infty}\Big(\sinh\frac{\sqrt{\xi}}{2}\Big)^{5/6}\frac{|Z_k(\xi)|}{\xi^{3/4}}\,d\xi \\ &\ll l^{5/6}\int_{\xi_0}^{\infty}\Big(\sinh\frac{\sqrt{\xi}}{2}\Big)^{5/6}\frac{|K_0(u\sqrt{\xi})|}{\xi^{1/4}}\,d\xi, \end{aligned}$$

where $u = k - \frac{1}{2}$ and the limit of integration ξ_0 is defined by

$$\cosh\frac{\sqrt{\xi_0}}{2} = 1 + \frac{1}{2l}.$$

Making the change of variables $\sqrt{\xi} = t$, one has

$$t_0 = 4\operatorname{arcsinh}\frac{1}{\sqrt{4l}}.$$

Since $t_0 \gg 1/\sqrt{l}$ and $ut \ge ut_0 \gg k/\sqrt{l} \gg 1$, we estimate the Bessel function as follows

$$K_0(ut) \ll \frac{\exp(-ut)}{\sqrt{ut}}.$$

Finally,

$$\begin{aligned} E_{2,1}(k,l) &\ll l^{5/6}\int_{t_0}^{\infty}\Big(\sinh\frac{t}{2}\Big)^{5/6}\frac{|K_0(ut)|}{\sqrt{t}}\,t\,dt \ll \frac{l^{5/6}}{\sqrt{k}}\int_{t_0}^{\infty}\Big(\sinh\frac{t}{2}\Big)^{5/6} e^{-ut}\,dt \\ &\ll \frac{l^{5/6}}{\sqrt{k}}\Big(\int_{t_0}^{1} t^{5/6}\exp(-ut)\,dt + \int_1^{\infty}\exp(-ut+5t/12)\,dt\Big) \ll \frac{l^{5/12}}{u\sqrt{k}}\exp(-ut_0). \end{aligned}$$

The second summand can be estimated similarly:

$$\begin{aligned} E_{2,2}(k,l) := l^{-1/3}\Big|\Psi_k\Big(\frac{4l^2}{(2l+1)^2}\Big)\Big| &\ll l^{-1/3}\frac{C_K|Z_k(\xi)|}{\xi^{1/4}(\sinh\sqrt{\xi})^{1/2}} \\ &\ll l^{-1/3}l^{1/2}|Z_K(\xi)| \ll l^{1/6}\sqrt{\xi}|K_0(u\sqrt{\xi})| \ll \frac{\exp(-ut_0)}{k^{1/2}l^{1/12}}. \end{aligned}$$

To sum up,

$$E_2(k,l) \ll E_{2,1}(k,l) + E_{2,2}(k,l) \ll \frac{1}{l^{1/12}\sqrt{k}} \exp\left(\frac{-ck}{\sqrt{l}}\right)$$

for some $c > 0$. □

Acknowledgments

We thank the referee for the helpful comments and questions, which have improved the manuscript.

References

[Balkanova and Frolenkov 2016] O. Balkanova and D. Frolenkov, "Moments of L-functions and the Liouville–Green method", 2016. arXiv

[Beitman and Erdelyi 1953] H. Beitman and A. Erdelyi, *Higher transcendental functions*, McGraw-Hill, New York, 1953.

[Boyd and Dunster 1986] W. G. C. Boyd and T. M. Dunster, "Uniform asymptotic solutions of a class of second-order linear differential equations having a turning point and a regular singularity, with an application to Legendre functions", *SIAM J. Math. Anal.* **17**:2 (1986), 422–450. MR Zbl

[Bykovskiĭ 1994] V. A. Bykovskiĭ, "Density theorems and the mean value of arithmetic functions on short intervals", *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov.* (*POMI*) **212**:Anal. Teor. Chisel i Teor. Funktsiĭ. 12 (1994), 56–70, 196. MR

[Conrey and Iwaniec 2000] J. B. Conrey and H. Iwaniec, "The cubic moment of central values of automorphic L-functions", *Ann. of Math.* (2) **151**:3 (2000), 1175–1216. MR Zbl

[Conrey et al. 2005] J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein, and N. C. Snaith, "Integral moments of L-functions", *Proc. London Math. Soc.* (3) **91**:1 (2005), 33–104. MR Zbl

[Diaconu et al. 2003] A. Diaconu, D. Goldfeld, and J. Hoffstein, "Multiple Dirichlet series and moments of zeta and L-functions", *Compositio Math.* **139**:3 (2003), 297–360. MR Zbl

[Farid Khwaja and Olde Daalhuis 2014] S. Farid Khwaja and A. B. Olde Daalhuis, "Uniform asymptotic expansions for hypergeometric functions with large parameters IV", *Anal. Appl.* **12**:6 (2014), 667–710. MR Zbl

[Fomenko 2003] O. M. Fomenko, "The behavior of automorphic L-functions at the points $s = 1$ and $s = \frac{1}{2}$", *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov.* (*POMI*) **302**:Anal. Teor. Chisel i Teor. Funkts. 19 (2003), 149–167, 201–202. In Russian; translated in *Journal of Mathematical Sciences* **129**:3 (2005), 3898–3909. MR

[Gelbart and Jacquet 1978] S. Gelbart and H. Jacquet, "A relation between automorphic representations of GL(2) and GL(3)", *Ann. Sci. École Norm. Sup.* (4) **11**:4 (1978), 471–542. MR

[Gradshteyn and Ryzhik 2007] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, 7th ed., Elsevier/Academic Press, Amsterdam, 2007. MR Zbl

[Jones 2001] D. S. Jones, "Asymptotics of the hypergeometric function", *Math. Methods Appl. Sci.* **24**:6 (2001), 369–389. MR Zbl

[Khan 2007] R. Khan, "The first moment of the symmetric-square L-function", *J. Number Theory* **124**:2 (2007), 259–266. MR Zbl

[Khan 2010] R. Khan, "Non-vanishing of the symmetric square L-function at the central point", *Proc. Lond. Math. Soc.* (3) **100**:3 (2010), 736–762. MR Zbl

[Khwaja and Daalhuis 2017] S. F. Khwaja and A. B. O. Daalhuis, "Computation of the coefficients appearing in the uniform asymptotic expansions of integrals", *Stud. Appl. Math.* **139**:4 (2017), 551–567. Zbl

[Kohnen and Sengupta 2002] W. Kohnen and J. Sengupta, "On the average of central values of symmetric square L-functions in weight aspect", *Nagoya Math. J.* **167** (2002), 95–100. MR Zbl

[Lau 2002] Y.-K. Lau, "Non-vanishing of symmetric square L-functions", *Proc. Amer. Math. Soc.* **130**:11 (2002), 3133–3139. MR Zbl

[Lerch 1887] M. Lerch, "Note sur la fonction $\mathfrak{K}(w, x, s) = \sum_{k=0}^{\infty} \frac{e^{2k\pi ix}}{(w+k)^s}$", *Acta Math.* **11**:1-4 (1887), 19–24. MR Zbl

[Lipschitz 1857] R. Lipschitz, "Untersuchung einer aus vier Elementen gebildeten Reihe", *J. Reine Angew. Math.* **54** (1857), 313–328. MR

[Liu 2017] S. Liu, "The first moment of central values of symmetric square L-functions in the weight aspect", *Ramanujan J* (online publication August 2017).

[Luo 2012] W. Luo, "Central values of the symmetric square L-functions", *Proc. Amer. Math. Soc.* **140**:10 (2012), 3313–3322. MR Zbl

[Ng 2016] M.-H. Ng, *Moments of automorphic L-functions*, Ph.D. thesis, University of Hong Kong, 2016.

[Ng 2017] M.-H. Ng, "The first moment of central values of symmetric square L-functions of cusp forms", *Acta Arith.* **177**:3 (2017), 277–291. MR Zbl

[Olver 1974] F. W. J. Olver, *Asymptotics and special functions*, Computer Science and Applied Mathematics, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1974. MR Zbl

[Olver et al. 2010] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (editors), *NIST handbook of mathematical functions*, Cambridge University Press, Cambridge, 2010. MR Zbl

[Petersson 1932] H. Petersson, "Über die Entwicklungskoeffizienten der automorphen Formen", *Acta Math.* **58**:1 (1932), 169–215. MR Zbl

[Shimura 1975] G. Shimura, "On the holomorphy of certain Dirichlet series", *Proc. London Math. Soc.* (3) **31**:1 (1975), 79–98. MR Zbl

[Soundararajan and Young 2013] K. Soundararajan and M. P. Young, "The prime geodesic theorem", *J. Reine Angew. Math.* **676** (2013), 105–120. MR Zbl

[Sun 2013] Q. Sun, "On the first moment of the symmetric-square L-function", *Proc. Amer. Math. Soc.* **141**:2 (2013), 369–375. MR Zbl

[Weil 1948] A. Weil, "On some exponential sums", *Proc. Nat. Acad. Sci. U. S. A.* **34** (1948), 204–207. MR Zbl

[Zagier 1977] D. Zagier, "Modular forms whose Fourier coefficients involve zeta-functions of quadratic fields", pp. 105–169 in *Modular functions of one variable, VI (Proc. Second Internat. Conf.)* (Univ. Bonn, Bonn, 1976), edited by J.-P. Serre and D. B. Zagier, Lecture Notes in Math. **627**, Springer, Berlin, Heidelberg, 1977. MR Zbl

[Zavorotny 1989] N. Zavorotny, "Automorphic functions and number theory, part I, II", *Akad. Nauk SSSR, Dal'nevostochn. Otdel., Vladivostok* **254** (1989), 69–124a. In Russian.

Communicated by Philippe Michel
Received 2016-10-20 Revised 2017-06-30 Accepted 2017-11-15

olgabalkanova@gmail.com *Department of Mathematics and Statistics, University of Turku, Turku, Finland*

frolenkov@mi.ras.ru *National Research University Higher School of Economics, Steklov Mathematical Institute of Russian Academy of Sciences, Moscow, Russia*

Algebra & Number Theory

msp.org/ant

See inside back cover or msp.org/ant for submission instructions.

The subscription price for 2018 is US $340/year for the electronic version, and $535/year (+$55, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY
mathematical sciences publishers
nonprofit scientific publishing
http://msp.org/

Algebra & Number Theory

Volume 12 No. 1 2018