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F-signature and Hilbert–Kunz multiplicity: a combined approach and comparison

Thomas Polstra and Kevin Tucker

We present a unified approach to the study of F-signature, Hilbert–Kunz multiplicity, and related limits governed by Frobenius and Cartier linear actions in positive characteristic commutative algebra. We introduce general techniques that give vastly simplified proofs of existence, semicontinuity, and positivity. Furthermore, we give an affirmative answer to a question of Watanabe and Yoshida allowing the F-signature to be viewed as the infimum of relative differences in the Hilbert–Kunz multiplicities of the cofinite ideals in a local ring.

1. Introduction

Throughout this paper, we shall assume all rings R are commutative and Noetherian with prime characteristic $p > 0$. Central to the study of such rings is the use of the Frobenius or p-th power endomorphism $F\colon R \to R$ defined by $r \mapsto r^p$ for all $r \in R$. Following the result of Kunz [1969] characterizing regularity by the flatness of Frobenius, it has long been understood that the behavior of Frobenius governs the singularities of such rings. In this article, we will be primarily concerned with two important numerical invariants that measure the failure of flatness for the iterated Frobenius: the Hilbert–Kunz multiplicity [Monsky 1983] and the F-signature [Smith and Van den Bergh 1997; Huneke and Leuschke 2002]. Our aim is to revisit a number of core results about these invariants — existence [Tucker 2012], semicontinuity [Smirnov 2016; Polstra 2015], and positivity [Hochster and Huneke 1994; Aberbach and Leuschke 2003] — and provide vastly simplified proofs, which in turn yield new and important results. In particular, we confirm the suspicion of Watanabe and Yoshida [2004, Question 1.10] allowing the F-signature to be viewed as the infimum of relative differences in the Hilbert–Kunz multiplicities of the cofinite ideals in a local ring.

For the sake of simplicity in introducing Hilbert–Kunz multiplicity and F-signature, assume that $(R, \mathfrak{m}, k)$ is a complete local domain of dimension d and $k = k^{1/p}$ is perfect. If $I \subseteq R$ is an ideal with finite colength $\ell_R(R/I) < \infty$, the Hilbert–Kunz multiplicity along I is defined by $e_{\mathrm{HK}}(I) = \lim_{e\to\infty}(1/p^{ed})\ell_R(R/I^{[p^e]})$ where $I^{[p^e]} = (F^e(I))$ is the expansion of I along the e-iterated Frobenius. Note that, unlike for Hilbert–Samuel multiplicity, the function $e \mapsto \ell_R(R/I^{[p^e]})$ is far from polynomial in p^e [Han and Monsky 1993]. The existence of Hilbert–Kunz limits was shown by Monsky [1983], and

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it has recently been shown by Brenner [2013] that there exist irrational Hilbert–Kunz multiplicities. Of particular interest is the Hilbert–Kunz multiplicity along the maximal ideal $\mathfrak{m}$ denoted by $e_{\mathrm{HK}}(R) = e_{\mathrm{HK}}(\mathfrak{m})$, where it is known that $e_{\mathrm{HK}}(R) \geq 1$ with equality if and only if R is regular [Watanabe and Yoshida 2000, Theorem 1.5]. More generally, if the Hilbert–Kunz multiplicity of R is sufficiently small, then R is Gorenstein and strongly F-regular [Blickle and Enescu 2004, Proposition 2.5; Aberbach and Enescu 2008, Corollary 3.6]. Recently, it has been shown that the Hilbert–Kunz multiplicity determines an upper semicontinuous $\mathbb{R}$-valued function on ring spectra [Smirnov 2016].

Another useful perspective comes from viewing the Hilbert–Kunz function $\ell_R(R/\mathfrak{m}^{[p^e]}) = \mu_R(R^{1/p^e})$ as measuring the minimal number of generators of the finitely generated R-module R^{1/p^e}, the ring of p^e-th roots of R inside an algebraic closure of its fraction field. As the rank $\mathrm{rk}_R(R^{1/p^e})$ of this torsion-free R-module is p^{ed}, we see immediately that R^{1/p^e} is free — and hence F^e is flat and R is regular — if and only if $\mu_R(R^{1/p^e}) = p^{ed}$. Similarly, letting $a_e = \mathrm{frk}(R^{1/p^e})$ denote the largest rank of a free summand appearing in an R-module direct sum decomposition of R^{1/p^e}, we have that R^{1/p^e} is free if and only if $a_e = p^{ed}$. The limit $s(R) = \lim_{e\to\infty}(1/p^{ed})\,\mathrm{frk}(R^{1/p^e})$ was first studied in [Smith and Van den Bergh 1997] and revisited in [Huneke and Leuschke 2002], where it was coined the F-signature and shown to exist for Gorenstein rings; existence in full generality was first shown in [Tucker 2012]. Aberbach and Leuschke [2003] have shown that the positivity of the F-signature characterizes the notion of strong F-regularity introduced by Hochster and Huneke [1994] in their celebrated study of tight closure [1990]. In [Polstra 2015] the first author showed that the F-signature determines a lower semicontinuous $\mathbb{R}$-valued function on ring spectra; an unpublished and independent proof was simultaneously found and shown to experts by the second author, and has been incorporated into this article.

At the heart of this work is an elementary argument simultaneously proving the existence of Hilbert–Kunz multiplicity and F-signature (Theorem 3.2), derived from basic properties of the functions $\mu_R(\_)$ and $\mathrm{frk}_R(\_)$. Building upon the philosophy introduced in [Tucker 2012], we further attempt to carefully track the uniform constants controlling convergence in the proof and throughout this article. To that end, we rely heavily on the uniform bounds for Hilbert–Kunz functions over ring spectra (Theorem 3.3) shown in [Polstra 2015, Theorem 4.3]. In particular, this also allows us to give vastly simplified proofs of the upper semicontinuity of Hilbert–Kunz multiplicity (Theorem 3.4) and lower semicontinuity of the F-signature (Theorem 3.8).

The F-signature has also long been known to be closely related to the relative differences $e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}(J)$ in Hilbert–Kunz multiplicities for ideals $I \subseteq J$ with finite colength [Huneke and Leuschke 2002, Theorem 15] (see Lemma 6.1). The infimum among these differences was studied independently by Watanabe and Yoshida [2004], and the connection to F-signature was made by Yao [2006]. Moreover, Watanabe and Yoshida [2004, Question 1.10] expected and formally questioned if the infimum of the relative Hilbert–Kunz differences was equal to the F-signature. We confirm their suspicions in Section 6.

Theorem A (Corollary 6.5). *If $(R, \mathfrak{m}, k)$ is an F-finite local ring, then*

$$s(R) = \inf_{\substack{I\subseteq J\subseteq R,\ \ell_R(R/I)<\infty \\ I\neq J,\ \ell_R(R/J)<\infty}} \frac{e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}(J)}{\ell_R(J/I)} = \inf_{\substack{I\subseteq R,\ \ell_R(R/I)<\infty \\ x\in R,\ (I:x)=\mathfrak{m}}} e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}((I,x)).$$

Much interest in relative differences of Hilbert–Kunz multiplicities stems from the result of Hochster and Huneke [1990, Theorem 8.17] that such differences can be used to detect instances of tight closure. Together with the characterization of strong F-regularity in terms of the positivity of the F-signature [Aberbach and Leuschke 2003], these two results are at the core of the use of asymptotic Frobenius techniques to measure singularities. Building off of previous work of the second author [Blickle et al. 2012; Schwede and Tucker 2014; 2015], we present new and highly simplified proofs of these results. In particular, the first proof of Theorem 5.1 gives a readily computable lower bound for the F-signature of a strongly F-regular ring.

Using standard reduction to characteristic $p > 0$ techniques, the singularities governed by Frobenius in positive characteristic commutative algebra have been closely related to those appearing in complex algebraic geometry. This connection has motivated several important generalized settings for tight closure. Hara and Watanabe [2002] have defined tight closure for divisor pairs (compare to [Takagi 2004]), and Hara and Yoshida [2003] have done the same for ideals with a real coefficient; both of these works build on the important results of Smith [1997], Hara [1998], and Mehta and Srinivas [1997]. Previous work of the second author [Blickle et al. 2012; 2013] has extended the notion of F-signature to these settings and beyond, incorporating the notion of a Cartier subalgebra [Blickle and Schwede 2013]. We work to extend all of our results as far as possible to these settings, including new and simplified proofs of existence (Theorem 4.7), semicontinuity (Theorems 4.9, 4.11, and 4.12), and positivity (Corollaries 5.12, 5.7(ii), and 5.8(ii)); many of these results have not appeared previously. In particular, while Hilbert–Kunz theory of divisor and ideal pairs has not been introduced, we are able to give an analogue of relative Hilbert–Kunz differences in these settings (Corollaries 5.7(i) and 5.8(i)). Once more, the F-signatures can be viewed as a minimum among the relative Hilbert–Kunz differences (Corollaries 6.9 and 6.10).

To give an overview of our methods, for each $e \in \mathbb{N}$ let $I_e^{\mathrm{HK}} = \mathfrak{m}^{[p^e]}$ and $I_e^{F\text{-sig}} = (r \in R \mid \psi(r^{1/p^e}) \in \mathfrak{m}$ for all $\psi \in \operatorname{Hom}_R(R^{1/p^e}, R))$. As k is perfect, straightforward computations show that $(1/p^{ed})\mu(R^{1/p^e}) = (1/p^{ed})\ell(R/I_e^{\mathrm{HK}})$ and $(1/p^{ed})\operatorname{frk}(R^{1/p^e}) = (1/p^{ed})\ell(R/I_e^{F\text{-sig}})$, so that the Hilbert–Kunz multiplicity and the F-signature can be understood by studying properties of sequences of ideals $\{I_e\}_{e\in\mathbb{N}}$ satisfying properties similar to those of $\{I_e^{\mathrm{HK}}\}_{e\in\mathbb{N}}$ and $\{I_e^{F\text{-sig}}\}_{e\in\mathbb{N}}$.

Theorem B (Corollary 4.5, Theorem 5.5). *Let $(R, \mathfrak{m}, k)$ be an F-finite local domain of dimension d, and consider a sequence of ideals $\{I_e\}_{e\in\mathbb{N}}$ such that $\mathfrak{m}^{[p^e]} \subseteq I_e$ for each e. Then the limit $\eta = \lim_{e\to\infty}(1/p^{ed})\ell(R/I_e)$ exists provided one of the following conditions holds.*

(i) *There exists $0 \neq c \in R$ so that $cI_e^{[p]} \subseteq I_{e+1}$ for all $e \in \mathbb{N}$.*

(ii) *There exists a nonzero $\psi \in \operatorname{Hom}_R(R^{1/p}, R)$ so that $\psi(I_{e+1}^{1/p}) \subseteq I_e$ for all $e \in \mathbb{N}$.*

Moreover, *in case* (ii), *we have $\eta > 0$ if and only if $\bigcap_{e\in\mathbb{N}} I_e = 0$.*

It is easy to check that the sequences $\{I_e^{\mathrm{HK}}\}_{e\in\mathbb{N}}$ or $\{I_e^{F\text{-sig}}\}_{e\in\mathbb{N}}$ satisfy both conditions (i) and (ii) above (with *any* choice of c and ψ). Our results on positivity come from the stated criterion for sequences of ideals satisfying (ii), while the relative Hilbert–Kunz statements largely come through careful tracking of uniform constants bounding the growth rates for sequences satisfying (i).

The article is organized as follows. Section 2 recalls a number of preliminary statements about rings in positive characteristic; in particular, we review properties of F-finite rings, which will be the primary setting of this article. We also discuss elementary properties of the maximal free rank (see Lemma 2.1) which are used in the unified proof of the existence of the Hilbert–Kunz multiplicity and the F-signature provided in Section 3; new proofs of semicontinuity follow immediately thereafter. Section 4 is the technical heart of the paper, and generalizes the methods of the previous sections for Hilbert–Kunz multiplicity and F-signature alone to the various sequences of ideals satisfying the conditions in Theorem B above.

Having dispatched with the results on existence and semicontinuity, we turn in Section 5 to a discussion of positivity statements. In particular, two simple proofs of the positivity of the F-signature for strongly F-regular rings appear at the beginning of this section. The relative Hilbert–Kunz criteria for tight closure along a divisor or ideal pair appear at the end of this section, together with a discussion of the positivity of the F-splitting ratio. Section 6 is devoted to relating the F-signature with minimal relative Hilbert–Kunz differences giving a proof of Theorem A, and we conclude in Section 7 with a discussion of a number of related open questions.

2. Preliminaries

Throughout this paper, we shall assume all rings R are commutative with a unit, Noetherian, and have prime characteristic $p > 0$. If $\mathfrak{p} \in \operatorname{Spec}(R)$, we let $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ denote the corresponding residue field. A local ring is a triple $(R, \mathfrak{m}, k)$ where $\mathfrak{m}$ is the unique maximal ideal of the ring R and $k = k(\mathfrak{m}) = R/\mathfrak{m}$.

Maximal free rank. For any ring R and R-module M, $\ell_R(M)$ denotes the length of M, and $\mu_R(M)$ denotes the minimal number of generators of M. When ambiguity is unlikely, we freely omit the subscript R from these and similar notations. We define the *maximal free rank* $\operatorname{frk}_R(M)$ of M to be the maximal rank of a free R-module quotient of M. As a surjection onto a free module necessarily admits a section, $\operatorname{frk}_R(M)$ is also the maximal rank of a free direct summand of M. Equivalently, M admits a direct sum decomposition $M = R^{\oplus \operatorname{frk}_R(M)} \oplus N$ where N has no free direct summands — a property characterized by $\phi(N) \neq R$ for all $\phi \in \operatorname{Hom}_R(N, R)$. It is immediate that $\operatorname{frk}_R(M) \leq \mu_R(M)$ with equality if and only if M is a free R-module.

When R is a domain, we will use $\operatorname{rk}_R(M)$ to denote the torsion-free rank of M. It is easily checked that

$$\operatorname{frk}_R(M) \leq \operatorname{rk}_R(M) \leq \mu_R(M) \tag{1}$$

and the second inequality is strict when M is not free; assuming M is torsion-free but not free, the first inequality is strict as well. Notice that, whereas $\mu(\_)$ is subadditive on short exact sequences over local rings, this is not the case for $\operatorname{frk}(\_)$ and motivates the following lemma.

Lemma 2.1. *Let $(R, \mathfrak{m}, k)$ be a local ring.*

(i) *If M_1 and M_2 are R-modules*, *then* $\operatorname{frk}(M_1 \oplus M_2) = \operatorname{frk}(M_1) + \operatorname{frk}(M_2)$.

(ii) *If M is an R-module with $M' \subseteq M$ a submodule and $M'' = M/M'$, then*

$$\operatorname{frk}(M'') \leq \operatorname{frk}(M) \leq \operatorname{frk}(M') + \mu(M'').$$

Proof. (i) If $M_1 = R^{\oplus \operatorname{frk}_R(M_1)} \oplus N_1$ and $M_2 = R^{\oplus \operatorname{frk}_R(M_2)} \oplus N_2$ where N_1, N_2 have no free direct summands, then $M_1 \oplus M_2 = R^{\oplus(\operatorname{frk}_R(M_1)+\operatorname{frk}_R(M_2))} \oplus (N_1 \oplus N_2)$. If $\phi \in \operatorname{Hom}_R(N_1 \oplus N_2, R)$, then $\phi(N_1), \phi(N_2) \subseteq \mathfrak{m}$ as N_1, N_2 have no free direct summands. It follows that $\phi(N_1 \oplus N_2) = \phi(N_1) + \phi(N_2) \subseteq \mathfrak{m}$ as well, showing $N_1 \oplus N_2$ has no free direct summands as desired.

(ii) For the first inequality, a surjection $M'' \to R^{\oplus \operatorname{frk}(M'')}$ induces another $M \to M/M' = M'' \to R^{\oplus \operatorname{frk}(M'')}$, showing $\operatorname{frk}(M'') \leq \operatorname{frk}(M)$. To show the final inequality, let n be the maximal rank of a mutual free direct summand of M and M'. In other words, simultaneously decompose $M = R^{\oplus n} \oplus N$ and $M' = R^{\oplus n} \oplus N'$ where $M' \subseteq M$ is given by equality on $R^{\oplus n}$ and an inclusion $N' \subseteq N$, and we have $\phi(N') \subseteq \mathfrak{m}$ for every $\phi \colon N \to R$. Taking a surjection $\Phi \colon N \to R^{\oplus \operatorname{frk}(N)}$, it follows that $\Phi(N') \subseteq \mathfrak{m}^{\oplus \operatorname{frk}(N)}$. Thus, Φ induces a surjection $M'' = N/N' \to k^{\oplus \operatorname{frk}(N)}$ and hence also $M''/\mathfrak{m}M'' \to k^{\oplus \operatorname{frk}(N)}$, which shows $\mu(M'') \geq \operatorname{frk}(N)$. This gives $\operatorname{frk}(M) = n + \operatorname{frk}(N) \leq \operatorname{frk}(M') + \mu(M'')$ as desired. □

For X any topological space, recall that a function $f \colon X \to \mathbb{R}$ is lower semicontinuous if and only if for any $\delta \in \mathbb{R}$ the set $\{x \in X \mid f(x) > \delta\}$ is open. Similarly, $f \colon X \to \mathbb{R}$ is upper semicontinuous if and only if for any $\delta \in \mathbb{R}$ the set $\{x \in X \mid f(x) < \delta\}$ is open.

Lemma 2.2. *If M is a finitely generated R-module, the function $\operatorname{Spec}(R) \to \mathbb{R}$ given by $\mathfrak{p} \mapsto \operatorname{frk}_{R_\mathfrak{p}}(M_\mathfrak{p})$ is lower semicontinuous. Similarly, the function $\operatorname{Spec}(R) \to \mathbb{R}$ given by $\mathfrak{p} \mapsto \mu_{R_\mathfrak{p}}(M_\mathfrak{p})$ is upper semicontinuous.*

Proof. For $\mathfrak{p} \in \operatorname{Spec}(R)$, let $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ denote the corresponding residue field. If $\operatorname{frk}_{R_\mathfrak{p}}(M_\mathfrak{p}) = n$, we can find a surjection $M_\mathfrak{p} \to R_\mathfrak{p}^{\oplus n}$. Without loss of generality, since M is finitely generated, we may assume this surjection is the localization of an R-module homomorphism $M \to R^{\oplus n}$. As it becomes surjective when localized at $\mathfrak{p}$, there is some $g \in R \setminus \mathfrak{p}$ so that $M_g \to R_g^{\oplus n}$ is surjective. Localizing at any $\mathfrak{q} \in \operatorname{Spec}(R)$ with $g \notin \mathfrak{q}$ yields a surjection $M_\mathfrak{q} \to R_\mathfrak{q}^{\oplus n}$, implying $\operatorname{frk}_{R_\mathfrak{q}}(M_\mathfrak{q}) \geq n = \operatorname{frk}_{R_\mathfrak{p}}(M_\mathfrak{p})$ for any $\mathfrak{q}$ in the Zariski open neighborhood $D(g) = \{\mathfrak{q} \in \operatorname{Spec}(R) \mid g \notin \mathfrak{q}\}$ of $\mathfrak{p}$ in $\operatorname{Spec}(R)$. If $\delta \in \mathbb{R}$ and $\operatorname{frk}_{R_\mathfrak{p}}(M_\mathfrak{p}) > \delta$, then so also $\operatorname{frk}_{R_\mathfrak{q}}(M_\mathfrak{q}) > \delta$ for any $\mathfrak{q} \in D(g)$. This shows the function $\operatorname{Spec}(R) \to \mathbb{R}$ given by $\mathfrak{p} \mapsto \operatorname{frk}_R(M_\mathfrak{p})$ is lower semicontinuous. That the function $\operatorname{Spec}(R) \to \mathbb{R}$ given by $\mathfrak{p} \mapsto \mu(M_\mathfrak{p})$ is upper semicontinuous proceeds in an analogous manner. □

***F*-finite rings.** The Frobenius or p-th power endomorphism $F \colon R \to R$ is defined by $r \mapsto r^p$ for all $r \in R$. Similarly, for $e \in \mathbb{N}$, we have $F^e \colon R \to R$ given by $r \mapsto r^{p^e}$. The expansion of an ideal I over F^e is denoted $I^{[p^e]} = (F^e(I))$. In case R is a domain, we let R^{1/p^e} denote the subring of p^e-th roots of R inside a fixed algebraic closure of the fraction field of R. By taking p^e-th roots, R^{1/p^e} is abstractly isomorphic to R, and the Frobenius map is identified with the ring extension $R \subseteq R^{1/p^e}$. More generally, for any R and any R-module M, we will write M^{1/p^e} for the R-module given by restriction of scalars for F^e; this is an exact functor on the category of R modules. We say that R is F-finite when R^{1/p^e} is a finitely generated R-module, which further implies that R is excellent [Kunz 1976, Theorem 2.5]. A complete local ring $(R, \mathfrak{m}, k)$ is F-finite if and only if its residue field k is F-finite. When R is F-finite and M is a finitely generated R-module, M^{1/p^e} is also a finitely generated R-module and

$$\ell_R(M^{1/p^e}) = [k^{1/p^e} : k] \cdot \ell_{R^{1/p^e}}(M^{1/p^e}) = [k^{1/p^e} : k] \cdot \ell_R(M).$$

In particular, note that

$$\mu_R(M^{1/p^e}) = \ell_R(M^{1/p^e} \otimes_R k) = \ell_R((M/\mathfrak{m}^{[p^e]}M)^{1/p^e}) = [k^{1/p^e} : k] \cdot \ell_R(M/\mathfrak{m}^{[p^e]}M). \tag{2}$$

Lemma 2.3. *Suppose $(R, \mathfrak{m}, k)$ is a complete local F-finite domain with coefficient field k and system of parameters $x_1, \dots, x_d$ chosen so that R is module finite and generically separable over the regular subring $A = k[\![x_1, \dots, x_d]\!]$. Then there exists $0 \neq c \in A$ so that $c \cdot R^{1/p^e} \subseteq R[A^{1/p^e}] = R \otimes_A A^{1/p^e}$ for all $e \in \mathbb{N}$.*

Proof. Note first that, by the Cohen–Gabber structure theorem, every complete local ring admits such a coefficient field and system of parameters; see [Kurano and Shimomoto 2015] for an elementary proof. Let $K = \mathrm{Frac}(A) \subseteq L = \mathrm{Frac}(R)$ be the corresponding extension of fraction fields. The separability assumption implies L and K^{1/p^e} are linearly disjoint over K, so that $L \otimes_K K^{1/p^e} = LK^{1/p^e}$ is reduced. Since A^{1/p^e} is a free A-module, $R \otimes_A A^{1/p^e}$ is a free R-module and injects into its localization $L \otimes_A A^{1/p^e} = L \otimes_K K^{1/p^e}$. Thus, $R \otimes_A A^{1/p^e}$ is also reduced, which gives the equality $R \otimes_A A^{1/p^e} = R[A^{1/p^e}]$. If $e_1, \dots, e_s \in R$ generate R as an A-module, it is easy to see $c = (\det(\mathrm{tr}(e_i e_j)))^2$ satisfies $c \cdot R^{1/p^e} \subseteq R[A^{1/p^e}]$ where tr is the trace map; see [Hochster and Huneke 2000, Lemma 4.3] for a proof and [Hochster and Huneke 1990, §6] for further discussion. □

The corollaries below follow immediately; see [Kunz 1976] for details.

Corollary 2.4. *If $(R, \mathfrak{m}, k)$ is an F-finite local domain of dimension d, then $\mathrm{rk}_R(R^{1/p^e}) = [k^{1/p^e} : k] \cdot p^{ed}$.*

Corollary 2.5. *If R is F-finite, for any two prime ideals $\mathfrak{p} \subseteq \mathfrak{q}$ of R it follows that*

$$[k(\mathfrak{p})^{1/p^e} : k(\mathfrak{p})] = [k(\mathfrak{q})^{1/p^e} : k(\mathfrak{q})] \cdot p^{e \dim(R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q})}.$$

If additionally R is locally equidimensional, we have

$$[k(\mathfrak{p})^{1/p^e} : k(\mathfrak{p})] \cdot p^{e \dim(R_\mathfrak{p})} = [k(\mathfrak{q})^{1/p^e} : k(\mathfrak{q})] \cdot p^{e \dim(R_\mathfrak{q})}$$

and the function $\mathrm{Spec}(R) \to \mathbb{N}$ given by $\mathfrak{p} \mapsto [k(\mathfrak{p})^{1/p^e} : k(\mathfrak{p})] \cdot p^{e \dim(R_\mathfrak{p})}$ is constant on the connected components of $\mathrm{Spec}(R)$.

Corollary 2.6. *If R is a locally equidimensional F-finite reduced ring with connected spectrum and $p^\gamma = [k(\mathfrak{p})^{1/p} : k(\mathfrak{p})] \cdot p^{\dim(R_\mathfrak{p})}$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$, there exist short exact sequences*

$$0 \longrightarrow R^{\oplus p^\gamma} \longrightarrow R^{1/p} \longrightarrow M \longrightarrow 0, \tag{3}$$

$$0 \longrightarrow R^{1/p} \longrightarrow R^{\oplus p^\gamma} \longrightarrow N \longrightarrow 0 \tag{4}$$

so that $\dim(M_\mathfrak{p}) < \dim(R_\mathfrak{p})$ and $\dim(N_\mathfrak{p}) < \dim(R_\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$.

3. F-signature and Hilbert–Kunz multiplicity revisited

The first goal of this section is to provide an elementary proof that simultaneously shows the existence of the F-signature and Hilbert–Kunz multiplicity for an F-finite local domain $(R, \mathfrak{m}, k)$. At the heart of all known proofs of existence is a basic observation on the growth rate of a Hilbert–Kunz function.

Lemma 3.1 [Monsky 1983, Lemma 1.1]. *Let $(R, \mathfrak{m}, k)$ be a local ring and M a finitely generated R-module. Then there exists a positive constant $C(M, \mathfrak{m}) \in \mathbb{R}$ so that*

$$\ell(M/\mathfrak{m}^{[p^e]}M) \leq C(M, \mathfrak{m}) \cdot p^{e \dim(M)}$$

for each $e \in \mathbb{N}$.

Proof. If $\mathfrak{m}$ is generated by t elements, then $\mathfrak{m}^{tp^e} \subseteq \mathfrak{m}^{[p^e]}$ for each $e \in \mathbb{N}$. Thus, $\ell(M/\mathfrak{m}^{[p^e]}M) \leq \ell(M/\mathfrak{m}^{tp^e}M)$, which is eventually polynomial of degree $\dim(M)$ in the entry p^e. □

Theorem 3.2 [Monsky 1983, Theorem 1.8; Tucker 2012, Theorem 4.9]. *Suppose that $(R, \mathfrak{m}, k)$ is an F-finite local domain and $\dim(R) = d$. Then the limits*

$$e_{\mathrm{HK}}(R) = \lim_{e\to\infty} \frac{\mu_R(R^{1/p^e})}{[k^{1/p^e} : k] \cdot p^{ed}}, \qquad s(R) = \lim_{e\to\infty} \frac{\mathrm{frk}_R(R^{1/p^e})}{[k^{1/p^e} : k] \cdot p^{ed}}$$

exist.

Proof. Let $p^{\gamma} = [k^{1/p} : k] \cdot p^d = \mathrm{rk}_R(R^{1/p^e})$ and $\nu(\_)$ denote either $\mu(\_)$ or $\mathrm{frk}(\_)$. Set $\eta_e = (1/p^{e\gamma})\nu_R(R^{1/p^e})$, and observe using Lemma 3.1 with (1) and (2) that the sequence $\{\eta_e\}_{e\in\mathbb{N}}$ is bounded above. Put $\eta^+ = \limsup_{e\to\infty} \eta_e$ and $\eta^- = \liminf_{e\to\infty} \eta_e$.

As in (3), fix a short exact sequence

$$0 \longrightarrow R^{\oplus p^{\gamma}} \longrightarrow R^{1/p} \longrightarrow M \longrightarrow 0$$

where $\dim(M) < d$. Applying the exact functors $(\_)^{1/p^e}$ gives the short exact sequences

$$0 \longrightarrow (R^{1/p^e})^{\oplus p^{\gamma}} \longrightarrow R^{1/p^{e+1}} \longrightarrow M^{1/p^e} \longrightarrow 0,$$

and Lemma 2.1 implies

$$\nu(R^{1/p^{e+1}}) \leq p^{\gamma} \cdot \nu(R^{1/p^e}) + \mu(M^{1/p^e})$$

for each $e \in \mathbb{N}$. By Lemma 3.1 and using (2), there exists a positive constant $C(M, \mathfrak{m}) \in \mathbb{R}$ with $\mu(M^{1/p^e}) \leq C(M, \mathfrak{m})p^{e \dim(M)} \leq C(M, \mathfrak{m})p^{e(d-1)}$. Dividing through by $p^{(e+1)d}$ and setting $C = C(M, \mathfrak{m})/p^d \in \mathbb{R}$ yields $\eta_{e+1} \leq \eta_e + C/p^e$. Iterating this inequality gives

$$\eta_{e+e'} \leq \eta_e + \frac{C}{p^e}\left(1 + \frac{1}{p} + \cdots + \frac{1}{p^{e'-1}}\right) \leq \eta_e + \frac{2C}{p^e}$$

for all $e, e' \in \mathbb{N}$. For each e, taking $\limsup_{e'\to\infty}$ then gives $\eta^+ \leq \eta_e + 2C/p^e$. Now taking $\liminf_{e\to\infty}$ gives $\eta^+ \leq \eta^-$ as desired. □

Our next goal is to give a direct proof of the upper semicontinuity of Hilbert–Kunz multiplicity. The key observation is that the constant appearing in Lemma 3.1 can be taken uniformly on ring spectra.

Theorem 3.3 [Polstra 2015, Theorem 4.3]. *For any F-finite ring R and finitely generated module M, there exists a positive constant $C(M)$ such that for all* $\mathfrak{p} \in \operatorname{Spec}(R)$

$$\ell_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}/\mathfrak{p}^{[p^e]}M_{\mathfrak{p}}) \leq C(M)p^{e\dim(M_{\mathfrak{p}})}.$$

From the proof of Theorem 3.2 above, which is valid without change for a local equidimensional reduced ring, it follows that there is a positive constant $C \in \mathbb{R}$ such that

$$e_{\mathrm{HK}}(R_{\mathfrak{p}}) \leq \frac{\mu_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}^{1/p^e})}{[k(\mathfrak{p})^{1/p^e} : k(\mathfrak{p})] \cdot p^{e\dim(R_{\mathfrak{p}})}} + \frac{2C}{p^e}, \qquad s(R_{\mathfrak{p}}) \leq \frac{\operatorname{frk}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}^{1/p^e})}{[k(\mathfrak{p})^{1/p^e} : k(\mathfrak{p})] \cdot p^{e\dim(R_{\mathfrak{p}})}} + \frac{2C}{p^e}$$

for all $\mathfrak{p} \in \operatorname{Spec}(R)$ and all $e \in \mathbb{N}$.

Theorem 3.4 [Smirnov 2016, Main result]. *For any F-finite locally equidimensional reduced ring R, the function $e_{\mathrm{HK}}\colon \operatorname{Spec}(R) \to \mathbb{R}$ given by $\mathfrak{p} \mapsto e_{\mathrm{HK}}(R_{\mathfrak{p}})$ is upper semicontinuous.*

Proof. Restricting to a connected component, we may assume without loss of generality that $\operatorname{Spec}(R)$ is connected and hence $p^{\gamma} = [k(\mathfrak{p})^{1/p} : k(\mathfrak{p})] \cdot p^{\dim(R_{\mathfrak{p}})}$ is constant for all $\mathfrak{p} \in \operatorname{Spec}(R)$ by Corollary 2.5. If $\delta \in \mathbb{R}$ and $e_{\mathrm{HK}}(R_{\mathfrak{q}}) < \delta$ for some $\mathfrak{q} \in \operatorname{Spec}(R)$, then

$$\frac{\mu_{R_{\mathfrak{q}}}(R_{\mathfrak{q}}^{1/p^e})}{[k(\mathfrak{q})^{1/p^e} : k(\mathfrak{q})] \cdot p^{e\dim(R_{\mathfrak{q}})}} < \delta - \epsilon$$

for some $0 < \epsilon \ll 1$ and some $e \in \mathbb{N}$ with $2C/p^e < \epsilon$. By Lemma 2.2, the same holds true for all $\mathfrak{p}$ in a neighborhood of $\mathfrak{q}$. This yields

$$e_{\mathrm{HK}}(R_{\mathfrak{p}}) \leq \frac{\mu_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}^{1/p^e})}{[k(\mathfrak{p})^{1/p^e} : k(\mathfrak{p})] \cdot p^{e\dim(R_{\mathfrak{p}})}} + \frac{2C}{p^e} < \delta - \epsilon + \frac{2C}{p^e} < \delta$$

as desired and completes the proof. ☐

To get a similar argument for lower semicontinuity of F-signature, we need to reverse the estimates arising in the proof of existence. To that end, we first record the following elementary lemma.

Lemma 3.5. *Let p be a prime number, $d \in \mathbb{N}$, and $\{\lambda_e\}_{e\in\mathbb{N}}$ be a sequence of real numbers so that $\{\lambda_e/p^{ed}\}_{e\in\mathbb{N}}$ is bounded.*

(i) *If there exists a positive constant $C \in \mathbb{R}$ so that $\lambda_{e+1}/p^{(e+1)d} \leq \lambda_e/p^{ed} + C/p^e$ for all $e \in \mathbb{N}$, then the limit $\lambda = \lim_{e\to\infty} \lambda_e/p^{ed}$ exists and $\lambda - \lambda_e/p^{ed} \leq 2C/p^e$ for all $e \in \mathbb{N}$.*

(ii) *If there exists a positive constant $C \in \mathbb{R}$ so that $\lambda_e/p^{ed} \leq \lambda_{e+1}/p^{(e+1)d} + C/p^e$ for all $e \in \mathbb{N}$, then the limit $\lambda = \lim_{e\to\infty} \lambda_e/p^{ed}$ exists and $\lambda_e/p^{ed} - \lambda \leq 2C/p^e$ for all $e \in \mathbb{N}$.*

(iii) *If there exists a positive constant $C \in \mathbb{R}$ so that $|\lambda_{e+1}/p^{(e+1)d} - \lambda_e/p^{ed}| \leq C/p^e$ for all $e \in \mathbb{N}$, then the limit $\lambda = \lim_{e\to\infty} \lambda_e/p^{ed}$ exists and $|\lambda_e/p^{ed} - \lambda| \leq 2C/p^e$ for all $e \in \mathbb{N}$. In particular, $\lambda_e = \lambda p^{ed} + O(p^{e(d-1)})$.*

Proof. Let $\lambda^+ = \limsup_{e\to\infty} \lambda_e/p^{ed}$ and $\lambda^- = \liminf_{e\to\infty} \lambda_e/p^{ed}$, which are finite as $\{\lambda_e/p^{ed}\}_{e\in\mathbb{N}}$ is bounded. Iterating the inequality in (i) yields

$$\frac{1}{p^{(e+e')d}}\lambda_{e+e'} \le \frac{1}{p^{ed}}\lambda_e + \frac{C}{p^e}\left(1+\frac{1}{p}+\cdots+\frac{1}{p^{e'-1}}\right) \le \frac{1}{p^{ed}}\lambda_e + \frac{2C}{p^e}$$

for all $e, e' \in \mathbb{N}$. Taking $\limsup_{e'\to\infty}$ for each fixed $e \in \mathbb{N}$ gives $\lambda^+ \le \lambda_e/p^{ed} + 2C/p^e$, and then applying $\liminf_{e\to\infty}$ gives $\lambda^+ \le \lambda^-$ so that $\lambda = \lim_{e\to\infty} \lambda_e/p^{ed}$ exists and $\lambda - \lambda_e/p^{ed} \le 2C/p^e$ for all $e \in \mathbb{N}$. Similarly, iterating the inequality in (ii) yields

$$\frac{1}{p^{ed}}\lambda_e \le \frac{1}{p^{(e+e')d}}\lambda_{e+e'} + \frac{C}{p^e}\left(1+\frac{1}{p}+\cdots+\frac{1}{p^{e'-1}}\right) \le \frac{1}{p^{(e+e')d}}\lambda_{e+e'} + \frac{2C}{p^e}$$

for all $e, e' \in \mathbb{N}$. Taking $\liminf_{e'\to\infty}$ for each fixed $e \in \mathbb{N}$ gives $\lambda_e/p^{ed} \le \lambda^- + 2C/p^e$, and then applying $\limsup_{e\to\infty}$ gives $\lambda^+ \le \lambda^-$ so that $\lambda = \lim_{e\to\infty} \lambda_e/p^{ed}$ exists and $\lambda_e/p^{ed} - \lambda \le 2C/p^e$ for all $e \in \mathbb{N}$. The final statement (iii) follows immediately from a combination of (i) and (ii). $\square$

Theorem 3.6. *If R is a locally equidimensional reduced F-finite ring of dimension d, there is a positive constant $C \in \mathbb{R}$ so that*

$$\left|\frac{\mu_{R_\mathfrak{p}}(R_\mathfrak{p}^{1/p^e})}{[k(\mathfrak{p})^{1/p^e} : k(\mathfrak{p})]\cdot p^{e\dim R_\mathfrak{p}}} - e_{\mathrm{HK}}(R_\mathfrak{p})\right| \le \frac{C}{p^e}, \qquad \left|\frac{\mathrm{frk}_{R_\mathfrak{p}}(R_\mathfrak{p}^{1/p^e})}{[k(\mathfrak{p})^{1/p^e} : k(\mathfrak{p})]\cdot p^{e\dim R_\mathfrak{p}}} - s(R_\mathfrak{p})\right| \le \frac{C}{p^e}$$

for all $e \in \mathbb{N}$ and $\mathfrak{p} \in \operatorname{Spec}(R)$.

Proof. Restricting to a connected component, we may assume without loss of generality that $\operatorname{Spec}(R)$ is connected and hence $p^\gamma = [k(\mathfrak{p})^{1/p} : k(\mathfrak{p})]\cdot p^{\dim(R_\mathfrak{p})}$ is constant for all $\mathfrak{p} \in \operatorname{Spec}(R)$ by Corollary 2.5.

The proofs of Theorems 3.2 and 3.4 above relied on a short exact sequence as in (3) of Corollary 2.6 and produced inequalities as in Lemma 3.5(i). Repeating those same arguments on a short exact sequence as in (4) of Corollary 2.6 yields inequalities as in Lemma 3.5(ii), which combine to give the desired result. $\square$

Corollary 3.7. *If $(R, \mathfrak{m}, k)$ is an F-finite local equidimensional reduced ring of dimension d,*

$$\frac{\mu_R(R^{1/p^e})}{[k : k^{p^e}]} = e_{\mathrm{HK}}(R)p^{ed} + O(p^{e(d-1)}), \qquad \frac{\mathrm{frk}_R(R^{1/p^e})}{[k : k^{p^e}]} = s(R)p^{ed} + O(p^{e(d-1)}).$$

Theorem 3.8 [Polstra 2015, Theorem 5.7]. *For any F-finite locally equidimensional reduced ring R, the function $s\colon \operatorname{Spec}(R) \to \mathbb{R}$ given by $\mathfrak{p} \mapsto s(R_\mathfrak{p})$ is lower semicontinuous.*

Proof. From Theorem 3.6, there is a $C \in \mathbb{R}$ such that

$$\frac{\mathrm{frk}_{R_\mathfrak{p}}(R_\mathfrak{p}^{1/p^e})}{[k(\mathfrak{p})^{1/p^e} : k(\mathfrak{p})]\cdot p^{e\dim(R_\mathfrak{p})}} \le s(R_\mathfrak{p}) + \frac{C}{p^e}$$

for all $\mathfrak{p} \in \operatorname{Spec}(R)$ and all $e \in \mathbb{N}$. The argument in Theorem 3.4 immediately gives the desired result. Alternatively, using the whole of Theorem 3.6, both semicontinuity statements follow from the fact that the uniform limit of upper [Kunz 1976, Corollary 3.4] or lower [Enescu and Yao 2011, Corollary 2.5] semicontinuous functions is upper or lower semicontinuous, respectively. □

4. Limits via Frobenius and Cartier linear maps

Background. Suppose that M is an R-module. Recall that an additive map $\phi \in \operatorname{Hom}_{\mathbb{Z}}(M, M)$ is said to be Frobenius linear or *p-linear* if $\phi(rm) = r^p\phi(m)$ for all $r \in R$ and $m \in M$. Similarly, we say that $\phi \in \operatorname{Hom}_{\mathbb{Z}}(M, M)$ is p^e-linear if $\phi(rm) = r^{p^e}m$ for all $r \in R$ and $m \in M$. The set of all p^e-linear maps on M is denoted $\mathscr{F}_e(M)$ and is naturally both a left and right R-module readily identified with $\operatorname{Hom}_R(M, M^{1/p^e})$. The composition of a p^{e_1}-linear map ϕ_1 and a p^{e_2}-linear map ϕ_2 yields a $p^{e_1+e_2}$-linear map $\phi_1 \circ \phi_2$, so that the ring of Frobenius linear operators $\mathscr{F}(M) = \bigoplus_{e\geq 0} \mathscr{F}_e(M)$ on M forms a noncommutative $\mathbb{N}$-graded ring with $R \to \mathscr{F}_0(M) = \operatorname{Hom}_R(M, M)$. In the case $M = R$, we have that every p^e-linear map $\phi \in \operatorname{Hom}_R(R, R^{1/p^e}) = R^{1/p^e}$ is a postmultiple of the Frobenius F^e by some $c^{1/p^e} \in R$.

Dual to p-linear maps are the Cartier linear or *p^{-1}-linear maps*; an additive map $\phi \in \operatorname{Hom}_{\mathbb{Z}}(M, M)$ is said to be p^{-1}-linear if $\phi(r^p m) = r\phi(m)$ for all $r \in R$ and $m \in M$. Similarly, we say that $\phi \in \operatorname{Hom}_{\mathbb{Z}}(M, M)$ is p^{-e}-linear if $\phi(r^{p^e}m) = rm$ for all $r \in R$ and $m \in M$. The set of all p^{-e}-linear maps on M is denoted $\mathscr{C}_e(M)$ and is naturally both a left and right R-module readily identified with $\operatorname{Hom}_R(M^{1/p^e}, M)$. The composition of a p^{-e_1}-linear map ϕ_1 and a p^{-e_2}-linear map ϕ_2 yields a $p^{-(e_1+e_2)}$-linear map $\phi_1 \circ \phi_2$, so that the ring of Cartier linear operators $\mathscr{C}(M) = \bigoplus_{e\geq 0} \mathscr{C}_e(M)$ on M forms a noncommutative $\mathbb{N}$-graded ring with $R \to \mathscr{C}_0(M) = \operatorname{Hom}_R(M, M)$.

If we have $\phi_1 \in \operatorname{Hom}_R(M^{1/p^{e_1}}, M)$ and $\phi_2 \in \operatorname{Hom}_R(M^{1/p^{e_2}}, M)$, we write $\phi_1 \cdot \phi_2$ for the composition $\phi_1 \circ (\phi_2)^{1/p^{e_1}}$, which coincides with their product when viewed as elements of $\mathscr{C}(M)$. In particular, given $\phi \in \operatorname{Hom}_R(M^{1/p}, M)$ we write $\phi^e \in \operatorname{Hom}_R(M^{1/p^e}, M)$ for the corresponding e-th iterate. If $\psi \in \operatorname{Hom}_R(M^{1/p^e}, M)$, we let $\psi(r^{1/p^e} \cdot \_)$ denote the R-linear map

$$M^{1/p^e} \xrightarrow{r^{1/p^e}} M^{1/p^e} \xrightarrow{\psi} R$$

given by premultiplying with $r^{1/p^e} \in R^{1/p^e}$.

Lemma 4.1. *Let R be an F-finite domain of dimension d with* $\operatorname{rk}_R(R^{1/p}) = p^\gamma$.

(i) *If $0 \neq c \in R$, then there exists a short exact sequence*

$$0 \longrightarrow R^{\oplus p^\gamma} \xrightarrow{\Phi} R^{1/p} \longrightarrow M \longrightarrow 0$$

so that $\dim(M) < R$ *and* $\Phi(R^{\oplus p^\gamma}) \subseteq (cR)^{1/p}$.

(ii) *Let $0 \neq \psi \in \operatorname{Hom}_R(R^{1/p}, R)$ be a nonzero map. There exists a short exact sequence*

$$0 \longrightarrow R^{1/p} \xrightarrow{\Psi} R^{\oplus p^\gamma} \longrightarrow N \longrightarrow 0$$

so that every component function of Ψ is a premultiple of ψ. In other words, there exists $r_1, \dots, r_{p^\gamma} \in R$ so that $\Psi = (\psi(r_1^{1/p} \cdot \_), \dots, \psi(r_{p^\gamma}^{1/p} \cdot \_))$.

Proof. For (i), start with any inclusion $R^{\oplus p^\gamma} \subseteq R^{1/p}$ with a torsion quotient as in (3) and postmultiply by $c^{1/p}$ on $R^{1/p}$. For (ii), note that $\mathrm{rk}_{R^{1/p}}(\mathrm{Hom}_R(R^{1/p}, R)) = 1$ so that there is some nonzero $z \in R$ with $z^{1/p} \cdot \mathrm{Hom}_R(R^{1/p}, R) \subseteq \psi \cdot R^{1/p}$. Start with an inclusion $R^{1/p} \subseteq R^{\oplus p^\gamma}$ with torsion quotient as in (4) and premultiply by $z^{1/p}$ on $R^{1/p}$ to achieve the desired sequence. □

If R is an F-finite normal domain and D is a (Weil) divisor on $X = \mathrm{Spec}(R)$, we use $R(D)$ to denote $\Gamma(X, \mathcal{O}_X(D))$. There is a well known correspondence between p^{-e}-linear maps and certain effective $\mathbb{Q}$-divisors. Fixing a canonical divisor K_R, standard duality arguments for finite extensions show that $\mathrm{Hom}_R(R^{1/p^e}, R) = (R((1-p^e)K_R))^{1/p^e}$ for each $e \in \mathbb{N}$. Thus, to each $0 \neq \phi \in \mathrm{Hom}_R(R^{1/p^e}, R)$ we can associate a divisor D_ϕ so that $D_\phi \sim_{\mathbb{Z}} (1-p^e)K_R$, and we set $\Delta_\phi = (1/(p^e-1))D_\phi$. If $\phi, \psi \in \mathrm{Hom}_R(R^{1/p^e}, R)$, then $\Delta_\phi \geq \Delta_\psi$ if and only if $\phi(\_) = \psi(r^{1/p^e} \cdot \_)$ for some $0 \neq r \in R$, in which case $\Delta_\phi = \Delta_\psi + (1/(p^e-1))\,\mathrm{div}_R(r)$. Moreover, if $\phi_1 \in \mathrm{Hom}_R(R^{1/p^{e_1}}, R)$ and $\phi_2 \in \mathrm{Hom}_R(R^{1/p^{e_2}}, R)$ with $\phi = \phi_1 \cdot \phi_2 = \phi_1 \circ (\phi_2)^{1/p^{e_1}}$, then $\Delta_\phi = (1/(p^{e_1+e_2}-1))((p^{e_2}-1)\Delta_{\phi_2} + p^{e_2}(p^{e_1}-1)\Delta_{\phi_1})$. In particular, $\Delta_\phi = \Delta_{\phi^n}$ for all $e, n \in \mathbb{N}$ and all $\phi \in \mathrm{Hom}_R(R^{1/p^e}, R)$. If Δ is an effective $\mathbb{Q}$-divisor on $X = \mathrm{Spec}(R)$ and $\phi \in \mathrm{Hom}_R(R^{1/p^e}, R)$, note that $\Delta_\phi \geq \Delta$ if and only if $(p^e-1)\Delta_\phi \geq \lceil (p^e-1)\Delta \rceil$ (as $(p^e-1)\Delta_\phi$ is integral), which is equivalent to asking that ϕ is in the image of the natural restriction mapping

$$\mathrm{Hom}_R(R(\lceil (p^e-1)\Delta \rceil)^{1/p^e}, R) \to \mathrm{Hom}_R(R^{1/p^e}, R).$$

See [Schwede and Tucker 2012, §4] or [2014, §2] for further details.

Lemma 4.2. *Let R be an F-finite domain and $0 \neq \psi \in \mathrm{Hom}_R(R^{1/p}, R)$. There exists an element $0 \neq z \in R$ so that, for any $e \in \mathbb{N}$ and any $\phi \in \mathrm{Hom}_R(R^{1/p^e}, R)$, the map $z \cdot \phi(\_) = \phi(z \cdot \_) = \phi((z^{p^e})^{1/p^e} \cdot \_) = \psi^e(r^{1/p^e} \cdot \_)$ for some $r^{1/p^e} \in R^{1/p^e}$. In other words, $z\phi$ is a premultiple of ψ^e.*

Proof. Let $\bar{R}$ be the normalization of R in its fraction field, and take $0 \neq c \in R$ inside the conductor ideal $\mathfrak{c} = \mathrm{Ann}_R(\bar{R}/R)$. One can show [Blickle and Schwede 2013, Exercise 6.14] that every p^{-e}-linear map $\phi\colon R^{1/p^e} \to R$ extends uniquely to a p^{-e}-linear map on $\bar{R}$ compatible with $\mathfrak{c}$, which we denoted here by $\bar{\phi}\colon \bar{R}^{1/p^e} \to \bar{R}$. Let $0 \neq x \in R$ be such that $\mathrm{div}_{\bar{R}}(x) \geq \Delta_{\bar{\psi}}$, and put $z = cx$. Then $\Delta_{x\bar{\phi}} = \Delta_{\bar{\phi}} + (p^e/(p^e-1))\,\mathrm{div}_{\bar{R}}(x) \geq \Delta_{\bar{\psi}} = \Delta_{\bar{\psi}^e}$ so that $\bar{\phi}(x \cdot \_) = \psi^e(\bar{r}^{1/p^e} \cdot \_)$ for some $\bar{r} \in \bar{R}$. Letting $r^{1/p^e} = c\bar{r}^{1/p^e} \in R^{1/p^e}$ and restricting back to R, this gives $\phi(z \cdot \_) = \psi^e(r^{1/p^e} \cdot \_)$ as desired. □

Recall that a Cartier subalgebra on R is a graded subring $\mathcal{D} \subseteq \mathcal{C}(R)$ of the ring of Cartier linear operators on R containing $\mathcal{C}_0(R) = \mathrm{Hom}_R(R, R) = R$. Cartier subalgebras can be seen as a natural generalization of a number of commonly studied settings in positive characteristic commutative algebra. For instance, if R is an F-finite domain and $0 \neq \mathfrak{a} \subseteq R$ is an ideal and $t \in \mathbb{R}_{\geq 0}$, the Cartier subalgebra $\mathcal{C}^{\mathfrak{a}^t} = \bigoplus_{e \geq 0} \mathcal{C}_e^{\mathfrak{a}^t}$ where

$$\begin{aligned}\mathcal{C}_e^{\mathfrak{a}^t} &= \mathfrak{a}^{\lceil t(p^e-1) \rceil / p^e} \mathrm{Hom}_R(R^{1/p^e}, R) \\ &= \{\phi(x^{1/p^e} \cdot \_) \mid x \in \mathfrak{a}^{\lceil t(p^e-1) \rceil} \text{ and } \phi \in \mathrm{Hom}_R(R^{1/p^e}, R)\}\end{aligned}$$

recovers the framework of [Hara and Yoshida 2003]. Similarly, if R is an F-finite normal domain and Δ is an effective $\mathbb{Q}$-divisor on $\operatorname{Spec}(R)$, the Cartier subalgebra $\mathscr{C}^{(R,\Delta)} = \bigoplus_{e\geq 0} \mathscr{C}_e^{(R,\Delta)}$ where

$$\begin{aligned}\mathscr{C}_e^{(R,\Delta)} &= \{\phi \in \operatorname{Hom}_R(R^{1/p^e}, R) \mid \Delta_\phi \geq \Delta\}\\ &= \operatorname{im}(\operatorname{Hom}_R(R(\lceil (p^e-1)\Delta\rceil)^{1/p^e}, R) \to \operatorname{Hom}_R(R^{1/p^e}, R))\end{aligned}$$

recovers the setting of [Hara and Watanabe 2002; Takagi 2004]. For more information on Cartier subalgebras, see [Blickle and Schwede 2013]. We will mainly be interested in the generalization of F-signature to a Cartier subalgebra $\mathscr{D}$ introduced in [Blickle et al. 2012; 2013].

Existence and semicontinuity.

Theorem 4.3. *Let $(R, \mathfrak{m}, k)$ be an F-finite local domain of dimension d, and $\{I_e\}_{e\in\mathbb{N}}$ a sequence of ideals such that $\mathfrak{m}^{[p^e]} \subseteq I_e$ for all $e \in \mathbb{N}$.*

(i) *If there exists $0 \neq c \in R$ so that $cI_e^{[p]} \subseteq I_{e+1}$ for all $e \in \mathbb{N}$, then $\eta = \lim_{e\to\infty}(1/p^{ed})\ell_R(R/I_e)$ exists. Moreover, there exists a positive constant $C(c)$ depending only on $c \in \mathbb{R}$ with $\eta - (1/p^{ed})\ell_R(R/I_e) \leq C(c)/p^e$ for all $e \in \mathbb{N}$.*

(ii) *If there exists a nonzero R-linear map $\psi \colon R^{1/p} \to R$ so that $\psi(I_{e+1}^{1/p}) \subseteq I_e$ for all $e \in \mathbb{N}$, then $\eta = \lim_{e\to\infty}(1/p^{ed})\ell_R(R/I_e)$ exists. Moreover, there exists a positive constant $C(\psi) \in \mathbb{R}$ depending only on ψ such that $(1/p^{ed})\ell_R(R/I_e) - \eta \leq C(\psi)/p^e$ for all $e \in \mathbb{N}$.*

Remark 4.4. The conditions on ideal sequences $\{I_e\}_{e\in\mathbb{N}}$ in Theorem 4.3(i)–(ii) are far more symmetric when phrased in terms of p-linear and p^{-1}-linear maps. In (i), the requirement is simply that there exists a p-linear map $0 \neq \phi \in \mathscr{F}_1(R)$ on R such that $\phi(I_e) \subseteq I_{e+1}$ for each $e \in \mathbb{N}$. Similarly, for (ii) the requirement is that there exists a p^{-1}-linear map $0 \neq \phi \in \mathscr{C}_1(R)$ such that $\phi(I_{e+1}) \subseteq I_e$ for each $e \in \mathbb{N}$.

Proof of Theorem 4.3. For (i), put $\operatorname{rk}_R(R^{1/p}) = [k^{1/p} : k] \cdot p^d = p^\gamma$ and consider a short exact sequence

$$0 \longrightarrow R^{\oplus p^\gamma} \xrightarrow{\ \Phi\ } R^{1/p} \longrightarrow M \longrightarrow 0 \tag{5}$$

as in Lemma 4.1(i) with $\dim(M) < \dim(R)$ and $\Phi(R^{\oplus p^\gamma}) \subseteq (Rc)^{1/p}$. Note that the condition $cI_e^{[p]} \subseteq I_{e+1}$ for all $e \in \mathbb{N}$ can be restated as $I_e(Rc)^{1/p} \subseteq (I_{e+1})^{1/p}$, so that $\Phi(I_e^{\oplus p^\gamma}) = \Phi(I_e(R^{\oplus p^\gamma})) \subseteq I_e(Rc)^{1/p} \subseteq (I_{e+1})^{1/p}$. In particular, Φ induces a quotient map

$$\bar{\Phi} \colon (R/I_e)^{\oplus p^\gamma} \to (R/I_{e+1})^{1/p}$$

and it follows that

$$[k^{1/p} : k]\ell_R(R/I_{e+1}) = \ell_R((R/I_{e+1})^{1/p}) \leq p^\gamma \ell_R(R/I_e) + \ell_R(\operatorname{coker}(\bar{\Phi})).$$

Since $\mathfrak{m}^{[p^{e+1}]} \subseteq I_{e+1}$ and $\operatorname{coker}(\bar{\Phi})$ is a quotient of $(R/I_{e+1})^{1/p}$, we have $\mathfrak{m}^{[p^e]} \subseteq \operatorname{Ann}_R(\operatorname{coker}(\bar{\Phi}))$. But $\operatorname{coker}(\bar{\Phi})$ is also a quotient of $\operatorname{coker}(\Phi) = M$ and thus $M/\mathfrak{m}^{[p^e]}M$, so $\ell_R(\operatorname{coker}(\bar{\Phi})) \leq \ell_R(M/\mathfrak{m}^{[p^e]}M) \leq$

$C(M, \mathfrak{m})p^{e(d-1)}$ by Lemma 3.1. Dividing through by $[k^{1/p} : k]p^{(e+1)d} = p^{\gamma+ed}$ yields

$$\frac{1}{p^{(e+1)d}}\ell_R(R/I_{e+1}) \le \frac{1}{p^{ed}}\ell_R(R/I_e) + \frac{C(M, \mathfrak{m})/p^{\gamma}}{p^e},$$

and the result now follows from Lemma 3.5 with $C(c) = 2C(M, \mathfrak{m})/p^{\gamma}$ independent of the sequence of ideals $\{I_e\}_{e\in\mathbb{N}}$ satisfying the condition in (i).

Similarly for (ii), consider a short exact sequence

$$0 \longrightarrow R^{1/p} \xrightarrow{\Psi} R^{\oplus p^{\gamma}} \longrightarrow N \longrightarrow 0$$

as in Lemma 4.1(ii) with $\dim(N) < \dim(R)$ so that every component function of Ψ is a premultiple of ψ. It follows that $\Psi((I_{e+1})^{1/p}) \subseteq I_e^{\oplus p^{\gamma}}$, so that Ψ induces a quotient map

$$\bar{\Psi} : (R/I_{e+1})^{1/p} \to (R/I_e)^{\oplus p^{\gamma}}$$

and thus

$$\begin{aligned} p^{\gamma}\ell_R(R/I_e) &\le \ell_R((R/I_{e+1})^{1/p}) + \ell_R(\operatorname{coker}(\bar{\Psi})) \\ &= [k^{1/p} : k]\ell_R(R/I_{e+1}) + \ell_R(\operatorname{coker}(\bar{\Psi})). \end{aligned}$$

Since $\mathfrak{m}^{[p^e]} \subseteq I_e$ and $\operatorname{coker}(\bar{\Psi})$ is a quotient of $(R/I_e)^{\oplus p^{\gamma}}$, we have that $\mathfrak{m}^{[p^e]} \subseteq \operatorname{Ann}_R(\operatorname{coker}(\bar{\Psi}))$. But $\operatorname{coker}(\bar{\Psi})$ is also a quotient of $\operatorname{coker}(\Psi) = N$ and thus $N/\mathfrak{m}^{[p^e]}N$, so $\ell_R(\operatorname{coker}(\bar{\Psi})) \le \ell_R(N/\mathfrak{m}^{[p^e]}N) \le C(N, \mathfrak{m})p^{e(d-1)}$ by Lemma 3.1. Dividing through by $[k^{1/p} : k]p^{(e+1)d} = p^{\gamma+ed}$ yields

$$\frac{1}{p^{ed}}\ell_R(R/I_e) \le \frac{1}{p^{(e+1)d}}\ell_R(R/I_{e+1}) + \frac{C(N, \mathfrak{m})/p^{\gamma}}{p^e},$$

and the result now follows from Lemma 3.5 with $C(\psi) = 2C(N, \mathfrak{m})/p^{\gamma}$ independent of the sequence of ideals $\{I_e\}_{e\in\mathbb{N}}$ satisfying the condition in (ii). □

Corollary 4.5. *Let $(R, \mathfrak{m}, k)$ be an F-finite local domain of dimension d, and $\{I_e\}_{e\in\mathbb{N}}$ a sequence of ideals such that $\mathfrak{m}^{[p^e]} \subseteq I_e$ for all $e \in \mathbb{N}$. Suppose there exists $0 \ne c \in R$ so that $cI_e^{[p]} \subseteq I_{e+1}$ for all $e \in \mathbb{N}$, and a nonzero R-linear map $\psi : R^{1/p} \to R$ so that $\psi(I_{e+1}^{1/p}) \subseteq I_e$ for all $e \in \mathbb{N}$. Then $\eta = \lim_{e\to\infty}(1/p^{ed})\ell_R(R/I_e)$ exists, and there is a positive constant $C \in \mathbb{R}$ such that $|(1/p^{ed})\ell_R(R/I_e) - \eta| \le C/p^e$ for all $e \in \mathbb{N}$ and all such sequence of ideals $\{I_e\}_{e\in\mathbb{N}}$. In particular, $\ell_R(R/I_e) = \eta p^{ed} + O(p^{e(d-1)})$.*

Corollary 4.5 gives yet another perspective on the existence proofs for Hilbert–Kunz multiplicity and F-signature in terms of the properties of certain sequences of ideals. If $(R, \mathfrak{m}, k)$ is a local domain and we set $I_e^{\mathrm{HK}} = \mathfrak{m}^{[p^e]}$ and $I_e^{F\text{-sig}} = (r \in R \mid \phi(r^{1/p^e}) \in \mathfrak{m} \text{ for all } \phi \in \operatorname{Hom}_R(R^{1/p^e}, R))$, it is easy to see $\mu_R(R^{1/p^e}) = \ell_R((R/I_e^{\mathrm{HK}})^{1/p^e})$ and $\operatorname{frk}_R(R^{1/p^e}) = \ell_R((R/I_e^{F\text{-sig}})^{1/p^e})$. It is shown in [Tucker 2012] that both sequences satisfy the conditions of Corollary 4.5; in fact, any choice of $0 \ne c \in R$ and $0 \ne \psi \in \operatorname{Hom}_R(R^{1/p}, R)$ will suffice.

As in [Tucker 2012], in settings such as Theorem 4.3, we have opted to consider sequences of ideals $\{I_e\}_{e\in\mathbb{N}}$ with $\mathfrak{m}^{[p^e]} \subseteq I_e$. In light of Theorem 3.3, uniformity of constants over $\operatorname{Spec}(R)$ would seem more transparent for such sequences. Nonetheless, for any $\mathfrak{m}$-primary ideal J, one could consider sequences with $J^{[p^e]} \subseteq I_e$. One has $\mathfrak{m}^{[p^{e_0}]} \subseteq J$ for some $e_0 \in \mathbb{N}$, so that $\mathfrak{m}^{[p^{e+e_0}]} \subseteq J^{[p^e]} \subseteq I_e$ for such sequences. In particular, our techniques carry over to this setting after allowing for reindexing, and it is straightforward to recover Monsky's original limit existence result.

Corollary 4.6 [Monsky 1983]. *If $(R, \mathfrak{m}, k)$ is a local ring of dimension d and $J \subseteq R$ is an $\mathfrak{m}$-primary ideal, then*

$$e_{\mathrm{HK}}(R, J) = \lim_{e\to\infty} \frac{1}{p^{ed}} \ell_R(R/J^{[p^e]})$$

exists and there is a positive constant $C \in \mathbb{R}$ such that

$$\left| e_{\mathrm{HK}}(R, J) - \frac{1}{p^{ed}} \ell_R(R/J^{[p^e]}) \right| < \frac{C}{p^e}$$

for all e.

Proof. The reindexing argument preceding the statement together with Theorem 4.3 yields the desired conclusion when R is an F-finite domain, which can be used to deduce the general case. For completeness, we include the reduction from [Monsky 1983]. First, passing to a faithfully flat unramified local extension, we may assume that R is complete with perfect residue field (and hence F-finite). Suppose $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ are the minimal primes of R with dimension d. Take $e_0 \gg 0$ so that $(\sqrt{0})^{[p^{e_0}]} = 0$, and put $N_i = \ell_{R_{\mathfrak{p}_i}}(F^{e_0}_* R_{\mathfrak{p}_i})$ for $i = 1, \dots, s$. If

$$M = \bigoplus_{i=1}^{s} ((R/\mathfrak{p}_i)^{\oplus N_i}),$$

then M and $F^{e_0}_* R$ agree upon localization to any of the $\mathfrak{p}_i$, and it follows that there exist R-module homomorphisms $M \to F^{e_0}_* R$ and $F^{e_0}_* R \to M$ with cokernels of dimension strictly less than d. Applying $R/I^{[p^e]} \otimes \_$ and using Lemma 3.1 yields a positive constant C' so that

$$|\ell_R(R/I^{[p^{e+e_0}]}) - \ell_R(M/I^{[p^e]}M)| \le C' p^{e(d-1)}$$

for all $e \in \mathbb{N}$. Dividing through by $p^{(e+e_0)d}$ and using

$$\ell_R(M/I^{[p^e]}M) = \sum_{i=1}^{s} N_i \cdot \ell_R\left(\frac{R/\mathfrak{p}_i}{I^{[p^e]} R/\mathfrak{p}_i} \right),$$

the statement for R now follows from that of the $R/\mathfrak{p}_i$, which are F-finite complete domains. □

Another useful modification of the setup of Theorem 4.3 proceeds in the following manner. Fixing $e_0 \in \mathbb{N}$, one can modify the condition on the sequences of ideals $\{I_e\}_{e\in\mathbb{N}}$ of Theorem 4.3(i) to require that $cI_e^{[p^{e_0}]} \subseteq I_{e+e_0}$ for all $e \in \mathbb{N}$. Similarly, in Theorem 4.3(ii), one can ask that there exists $0 \ne \psi \in \mathrm{Hom}_R(R^{1/p^{e_0}}, R)$ such that $\psi((I_{e+e_0})^{1/p^{e_0}}) \subseteq I_e$ for all $e \in \mathbb{N}$. In both cases, after reindexing, the same methods apply to yield analogous results.

For example, this last generalization is particularly relevant when working with an arbitrary Cartier subalgebra $\mathfrak{D}$ on a local F-finite domain $(R, \mathfrak{m}, k)$. As $\mathfrak{D}$ is closed under composition, it is easily verified that

$$\Gamma_{\mathscr{D}} = \{e \in \mathbb{N} \mid \mathscr{D}_e \neq 0\}$$

is a subsemigroup of $(\mathbb{N}, +)$, and is called the semigroup of $\mathscr{D}$. For $e \in \Gamma_{\mathscr{D}}$, one defines the e-th F-splitting number $a_e^{\mathscr{D}}$ of R along $\mathscr{D}$ to be the maximal number of copies of R with projection maps in $\mathscr{D}_e$ appearing in an R-module direct sum decomposition of R^{1/p^e}. In other words, $a_e^{\mathscr{D}}$ is the largest integer so that there is a surjection $R^{1/p^e} \to R^{\oplus a_e^{\mathscr{D}}}$ where the induced map $\operatorname{Hom}_R(R^{\oplus a_e^{\mathscr{D}}}, R) \to \operatorname{Hom}_R(R^{1/p^e}, R) = \mathscr{C}_e(R)$ has image inside of $\mathscr{D}_e$. See [Blickle et al. 2012, §3.1] for further details. Setting $I_e^{\mathscr{D}} = (r \in R \mid \phi(r^{1/p^e}) \in \mathfrak{m}$ for all $\phi \in \mathscr{D}_e)$, it is again easy to check that $a_e^{\mathscr{D}} = \ell_R((R/I_e^{\mathscr{D}})^{1/p^e})$, and the method of Theorem 4.3(ii) yields the following result.

Theorem 4.7. *Let $(R, \mathfrak{m}, k)$ be an F-finite local domain of dimension d and $\mathscr{D}$ a Cartier subalgebra on R. Then the F-signature $s(R, \mathscr{D}) = \lim_{e \to \infty, e \in \Gamma_{\mathscr{D}}} (1/([k^{1/p^e} : k] \cdot p^{ed}))a_e^{\mathscr{D}}$ of R along $\mathscr{D}$ exists, and there is a positive constant $C \in \mathbb{R}$ so that*

$$\frac{1}{[k^{1/p^e} : k] \cdot p^{ed}} a_e^{\mathscr{D}} \leq s(R, \mathscr{D}) + \frac{C}{p^e}$$

for all $e \in \Gamma_{\mathscr{D}}$.

Proof. Let $p^\gamma = \operatorname{rk}_R(R^{1/p}) = [k^{1/p} : k] \cdot p^d$, and take a set[1] of generators $e_1, \dots, e_s$ for the semigroup $\Gamma_{\mathscr{D}}$. Fixing $0 \neq \psi_i \in \mathscr{D}_{e_i}$ for each $i = 1, \dots, s$, we can find a short exact sequence of R-modules

$$0 \longrightarrow R^{1/p^{e_i}} \xrightarrow{\Psi_i} R^{\oplus p^{\gamma e_i}} \longrightarrow M_i \longrightarrow 0 \tag{6}$$

where $\dim(M_i) < d$ and every component function of Ψ_i is a premultiple of ψ_i as in Lemma 4.1(ii). Since the Cartier linear maps in $\mathscr{D}$ are closed under composition, it follows readily that $\psi_i((I_{e+e_i}^{\mathscr{D}})^{1/p^{e_i}}) \subseteq I_e^{\mathscr{D}}$ for any $e \in \Gamma_{\mathscr{D}}$. In particular, $\Psi_i((I_{e+e_i}^{\mathscr{D}})^{1/p^{e_i}}) \subseteq (I_e^{\mathscr{D}})^{\oplus p^{\gamma e_i}}$ and proceeding as in Theorem 4.3(ii), we see that

$$\frac{1}{p^{e\gamma}} a_e^{\mathscr{D}} \leq \frac{1}{p^{(e+e_i)\gamma}} a_{e+e_i}^{\mathscr{D}} + \frac{C'}{p^e}$$

for any $i = 1, \dots, s$ and $e \in \Gamma_{\mathscr{D}}$ where $C' = \max\{C(M_1, \mathfrak{m})/p^{e_1\gamma}, \dots, C(M_s, \mathfrak{m})/p^{e_s\gamma}\}$. It follows as in Lemma 3.5(ii) that $s(R, \mathscr{D}) = \lim_{e \to \infty, e \in \Gamma_{\mathscr{D}}} (1/p^{e\gamma}) a_e^{\mathscr{D}}$ exists and $s(R, \mathscr{D}) - (1/p^{e\gamma}) a_e^{\mathscr{D}} \leq 2C'/p^e$ for all $e \in \Gamma_{\mathscr{D}}$ as desired. □

Remark 4.8 (compare to [Polstra 2015, Condition (1)]). Consider a Cartier subalgebra $\mathscr{D}$ on an F-finite local domain $(R, \mathfrak{m}, k)$ with the following property: if $\phi \in \mathscr{D}_{e+1}$ for some $e \in \mathbb{N}$, then $\phi|_{R^{1/p^e}} \in \mathscr{D}_e$. It is easy to check $(I_e^{\mathscr{D}})^{[p]} \subseteq I_{e+1}^{\mathscr{D}}$ for all $e \in \mathbb{N} = \Gamma_{\mathscr{D}}$, and Theorem 4.3(i) gives that there is a positive constant $C \in \mathbb{R}$ so that

$$\left| s(R, \mathscr{D}) - \frac{1}{[k^{1/p^e} : k] \cdot p^{ed}} a_e^{\mathscr{D}} \right| \leq \frac{C}{p^e} \tag{7}$$

[1] Every subsemigroup of $\mathbb{N}$ is finitely generated [Grillet 2001, Proposition 4.1].

for all $e \in \mathbb{N}$. Arbitrary Cartier subalgebras will not satisfy such properties, and we know of no reason to expect (7) to hold in general. However, we will see below in Theorems 4.11 and 4.12 that the Cartier subalgebras constructed using ideals and divisors do satisfy (7) by showing they enjoy a perturbation of the above property.

The proof of Theorem 4.3 made use of positive constants arising from Lemma 3.1. However, in case R is not local and the p-linear map in (i) or p^{-1}-linear map in (ii) extend to R, one can make these constants spread uniformly over $\operatorname{Spec}(R)$ using Theorem 3.3 instead. In the case of Theorem 4.7, this immediately yields the following result.

Theorem 4.9. *Let R be an F-finite domain and $\mathscr{D}$ a Cartier subalgebra on R. There is a positive constant $C \in \mathbb{R}$ so that*

$$\frac{1}{[k(\mathfrak{p})^{1/p^e} : k(\mathfrak{p})] \cdot p^{e\,\mathrm{ht}(\mathfrak{p})}} a_e^{\mathscr{D}_\mathfrak{p}} \leq s(R_\mathfrak{p}, \mathscr{D}_\mathfrak{p}) + \frac{C}{p^e}$$

for all $e \in \Gamma_{\mathscr{D}}$ and all $\mathfrak{p} \in \operatorname{Spec}(R)$. Moreover, the function $\operatorname{Spec}(R) \to \mathbb{R}$ given by $\mathfrak{p} \mapsto s(R_\mathfrak{p}, \mathscr{D}_\mathfrak{p})$ is lower semicontinuous.

Note that, for the lower semicontinuity property, one can relax the above requirement that R is a domain — any F-finite ring will suffice. The F-signature function is identically zero off of the strongly F-regular locus of R, so that $s(R_\mathfrak{p}, \mathscr{D}_\mathfrak{p}) = 0$ for any $\mathfrak{p} \in \operatorname{Spec}(R)$ where $R_\mathfrak{p}$ is not a normal domain. Thus, it suffices to check lower semicontinuity on the normal locus $U \subseteq \operatorname{Spec}(R)$, which follows from the lower semicontinuity on each $\operatorname{Spec}(R_f)$ for each $f \in R$ where R_f is a normal domain.

Proof. Following along in the proof of Theorem 4.7, observe that sequences in (6) can be taken to be global and then localized to each $\mathfrak{p} \in \operatorname{Spec}(R)$. By Theorem 3.3, we may then take the positive constant $C = \max\{2C(M_1)/p^{e_1\gamma}, \dots, 2C(M_s)/p^{e_s\gamma}\}$ independent of $\mathfrak{p} \in \operatorname{Spec}(R)$. For the remainder, first note the argument of Lemma 2.2 readily adapts to show the function $\operatorname{Spec}(R) \to \mathbb{R}$ given by $\mathfrak{p} \mapsto \mathfrak{a}_e^{\mathscr{D}_\mathfrak{p}}$ is lower semicontinuous. The lower semicontinuity of the F-signatures thus follows from the argument given in Theorem 3.8. ☐

Remark 4.10. Analogous to the argument above, for an $\mathbb{R}$-valued function on $\operatorname{Spec}(R)$ governed locally by sequences of ideals as in Theorem 4.3(i), upper semicontinuity passes from the individual terms in the sequences to the limit as in Theorem 3.4. However, we are unaware of any such functions that are not directly related to Hilbert–Kunz multiplicity itself.

Theorem 4.11. *Let R be an F-finite domain, $\mathfrak{a} \subseteq R$ a nonzero ideal, $t \in \mathbb{R}_{\geq 0}$, and $\mathfrak{p} \in \operatorname{Spec}(R)$. Then the F-signature*

$$s(R_\mathfrak{p}, \mathfrak{a}_\mathfrak{p}^t) = s(R_\mathfrak{p}, \mathscr{C}^{\mathfrak{a}_p^t}) = \lim_{e\to\infty} \frac{1}{[k(\mathfrak{p})^{1/p^e} : k(\mathfrak{p})] \cdot p^{e\,\mathrm{ht}(\mathfrak{p})}} a_e^{\mathfrak{a}_\mathfrak{p}^t}$$

of $R_{\mathfrak{p}}$ *along* $\mathfrak{a}_{\mathfrak{p}}^t$ *exists and determines a lower semicontinuous* $\mathbb{R}$*-valued function on* $\operatorname{Spec}(R)$. *Moreover, there is a positive constant* $C \in \mathbb{R}$ *so that*

$$\left| s(R_{\mathfrak{p}}, \mathfrak{a}_{\mathfrak{p}}^t) - \frac{a_e^{\mathfrak{a}_{\mathfrak{p}}^t}}{[k(\mathfrak{p})^{1/p^e} : k(\mathfrak{p})] \cdot p^{e\,\operatorname{ht}(\mathfrak{p})}} \right| \le \frac{C}{p^e} \tag{8}$$

for all $e \in \mathbb{N}$, *all* $\mathfrak{p} \in \operatorname{Spec}(R)$, *and all* $t \in \mathbb{R}_{\ge 0}$. *In particular*, $a_e^{\mathfrak{a}_{\mathfrak{p}}^t}/[k(\mathfrak{p})^{1/p^e} : k(\mathfrak{p})] = s(R_{\mathfrak{p}}, \mathfrak{a}_{\mathfrak{p}}^t) p^{e\,\operatorname{ht}(\mathfrak{p})} + O(p^{e(\operatorname{ht}(\mathfrak{p})-1)})$.

Proof. The existence and semicontinuity statements follow immediately from Theorems 4.7 and 4.9 above. Let $f_1, \dots, f_s$ be a set of generators for $\mathfrak{a}$. If $t > s$, then $a_e^{\mathfrak{a}_{\mathfrak{p}}^t} = 0$ for all $e \in \mathbb{N}$ and any $\mathfrak{p} \in \operatorname{Spec}(R)$. Thus, we are free to assume $t \le s$ going forward.

For a fixed $t \in \mathbb{R}_{\ge 0}$, one inequality in (8) also follows from Theorem 4.7 above; however, it remains to show that the positive constant C can be chosen independent of $t \in \mathbb{R}_{\ge 0}$. To that end, fix a choice of $0 \ne x \in \mathfrak{a}^{s(p-1)}$ and $\phi \in \operatorname{Hom}_R(R^{1/p}, R)$. Then $\psi(\_) = \phi(x^{1/p} \cdot \_) \in \mathscr{C}_1^{\mathfrak{a}^t}$ for any $t \le s$. The desired independence follows from the observation that a short exact sequence as in Lemma 4.1(ii) can be used for (6) in the proof of Theorem 4.7 for any $t \le s$.

To show the reverse inequality, choose an element $0 \ne c \in R$ that satisfies $c\mathfrak{a}^{\lceil t(p^{e+1}-1)\rceil} \subseteq (\mathfrak{a}^{\lceil t(p^e-1)\rceil})^{[p]}$ for all $e \in \mathbb{N}$ and any $t \in \mathbb{R}_{\ge 0}$; one can check that $c = f_1^p \cdots f_s^p$ will suffice. We will show that $c(I_e^{\mathfrak{a}_{\mathfrak{p}}^t})^{[p]} \subseteq I_{e+1}^{\mathfrak{a}_{\mathfrak{p}}^t}$ for all $t \in \mathbb{R}_{\ge 0}$ and $\mathfrak{p} \in \operatorname{Spec}(R)$, after which the result follows by using a short exact sequence as in Lemma 4.1(i) for (5) in the proof of Theorem 4.3(i) for any $t \le s$ and $\mathfrak{p} \in \operatorname{Spec}(R)$ with $C(M_{\mathfrak{p}}, \mathfrak{p}R_{\mathfrak{p}}) = C(M)$ from Theorem 3.3. Supposing that $\phi \in \operatorname{Hom}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}^{1/p^{e+1}}, R_{\mathfrak{p}})$ and $y \in \mathfrak{a}_{\mathfrak{p}}^{\lceil t(p^{e+1}-1)\rceil}$, we must show $\phi((yc(I_e^{\mathfrak{a}_{\mathfrak{p}}^t})^{[p]})^{1/p^{e+1}}) \subseteq \mathfrak{p}R_{\mathfrak{p}}$. Write $cy = \sum_{i=1}^n g_i^p r_i$ where each $r_i \in R_{\mathfrak{p}}$ and $g_i \in \mathfrak{a}_{\mathfrak{p}}^{\lceil t(p^e-1)\rceil}$. Let $\phi_i \in \operatorname{Hom}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}^{1/p^e}, R_{\mathfrak{p}})$ be the map defined by $\phi_i(z^{1/p^e}) = \phi(r_i^{1/p^{e+1}} z^{1/p^e})$ for all $z \in R_{\mathfrak{p}}$. If $x \in I_e^{\mathfrak{a}_{\mathfrak{p}}^t}$, then $\phi((ycx^p)^{1/p^{e+1}}) = \sum_{i=1}^n \phi(r_i^{1/p^{e+1}}(g_i x)^{1/p^e}) = \sum_{i=1}^n \phi_i((g_i x)^{1/p^e}) \in \mathfrak{p}R_{\mathfrak{p}}$ as $\phi_i(g_i^{1/p^e} \cdot \_) \in \mathscr{C}_e^{\mathfrak{a}_{\mathfrak{p}}^t}$ for $i = 1, \dots, n$. □

Theorem 4.12. *Let* R *be an* F*-finite normal domain*, Δ *an effective* $\mathbb{Q}$*-divisor on* $\operatorname{Spec}(R)$, *and* $\mathfrak{p} \in \operatorname{Spec}(R)$. *Then the* F*-signature*

$$s(R_{\mathfrak{p}}, \Delta) = s(R_{\mathfrak{p}}, \mathscr{C}^{(R_{\mathfrak{p}},\Delta)}) = \lim_{e\to\infty} \frac{1}{[k(\mathfrak{p})^{1/p^e} : k(\mathfrak{p})] \cdot p^{e\,\operatorname{ht}(\mathfrak{p})}} a_e^{(R_{\mathfrak{p}},\Delta)}$$

of $R_{\mathfrak{p}}$ *along* Δ *exists and determines a lower semicontinuous* R*-valued function on* $\operatorname{Spec}(R)$. *Moreover, if* $\bar{\Delta}$ *is a fixed effective integral divisor on* $\operatorname{Spec}(R)$, *there exists a positive constant* $C \in \mathbb{R}$ *so that*

$$\left| s(R_{\mathfrak{p}}, \Delta) - \frac{a_e^{(R_{\mathfrak{p}},\Delta)}}{[k(\mathfrak{p})^{1/p^e} : k(\mathfrak{p})] \cdot p^{e\,\operatorname{ht}(\mathfrak{p})}} \right| \le \frac{C}{p^e} \tag{9}$$

for all $e \in \mathbb{N}$, *all* $\mathfrak{p} \in \operatorname{Spec}(R)$, *and all effective* $\mathbb{Q}$*-divisors* Δ *with* $\Delta \le \bar{\Delta}$. *In particular, we have that* $a_e^{(R_{\mathfrak{p}},\Delta)}/[k(\mathfrak{p})^{1/p^e} : k(\mathfrak{p})] = s(R_{\mathfrak{p}}, \Delta) p^{e\,\operatorname{ht}(\mathfrak{p})} + O(p^{e(d-1)})$.

Proof. The existence and semicontinuity statements follow immediately from Theorems 4.7 and 4.9 above. For a fixed Δ, one inequality in (9) also follows from Theorem 4.7 above; however, it remains to show that the positive constant C can be chosen independent of $\Delta \leq \bar{\Delta}$. To that end, fix a choice of $0 \neq \psi \in \mathscr{C}_1^{(R,\bar{\Delta})}$. Since $\mathscr{C}_1^{(R,\bar{\Delta})} \subseteq \mathscr{C}_1^{(R,\Delta)}$, the desired independence follows from the observation that a short exact sequence as in Lemma 4.1(ii) can be used for (6) in the proof of Theorem 4.7 for any $\Delta \leq \bar{\Delta}$.

For the reverse inequality, choose $0 \neq c \in R$ so that $\operatorname{div}_R(c) \geq p\bar{\Delta}$. We will show that $c(I_e^{(R_\mathfrak{p},\Delta)})^{[p]} \subseteq I_{e+1}^{(R_\mathfrak{p},\Delta)}$ for all $\Delta \leq \bar{\Delta}$, after which the result follows by using a short exact sequence as in Lemma 4.1(i) for (5) in the proof of Theorem 4.3(i) for any $\Delta \leq \bar{\Delta}$ and $\mathfrak{p} \in \operatorname{Spec}(R)$ with $C(M_\mathfrak{p}, \mathfrak{p}R_\mathfrak{p}) = C(M)$ from Theorem 3.3. Supposing $\mathfrak{p} \in \operatorname{Spec}(R)$ and $\phi \in \mathscr{C}_{e+1}^{(R_\mathfrak{p},\Delta)}$, we must show $\phi((c(I_e^{(R_\mathfrak{p},\Delta)})^{[p]})^{1/p^{e+1}}) \subseteq \mathfrak{p}R_\mathfrak{p}$. Let $\psi \in \operatorname{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}^{1/p^{e+1}}, R_\mathfrak{p})$ be the map given by $\psi(\_) = \phi(c^{1/p^{e+1}} \cdot \_)$, and $\psi_e = \psi|_{R_\mathfrak{p}^{1/p^e}} \in \operatorname{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}^{1/p^e}, R_\mathfrak{p})$ be the restriction to $R_\mathfrak{p}^{1/p^e}$. It suffices to show that $\Delta_{\psi_e} \geq \Delta$, so that for any $x \in I_e^{(R_\mathfrak{p},\Delta)}$ we have $\phi((cx^p)^{1/p^{e+1}}) = \psi_e(x^{1/p^e}) \in \mathfrak{p}R_\mathfrak{p}$.

To that end, consider the inclusions for each $e \in \mathbb{N}$

$$\begin{array}{ccc} R^{1/p^e} & \subseteq & R^{1/p^{e+1}} \\ \cap\!| & & \cap\!| \\ (R(\lceil (p^e-1)\Delta \rceil))^{1/p^e} & \subseteq & (R(p\lceil (p^e-1)\Delta \rceil))^{1/p^{e+1}} \\ & & \cap\!| \\ & & (R(\lceil (p^{e+1}-1)\Delta \rceil + \operatorname{div}_R(c)))^{1/p^{e+1}} \end{array}$$

which hold because $\lceil (p^e-1)\Delta \rceil \leq (p^e-1)\Delta + \bar{\Delta}$ implies

$$\begin{aligned} p\lceil (p^e-1)\Delta \rceil &\leq (p^{e+1}-p)\Delta + p\bar{\Delta} \\ &\leq (p^{e+1}-1)\Delta + p\bar{\Delta} \\ &\leq \lceil (p^{e+1}-1)\Delta \rceil + \operatorname{div}_R(c). \end{aligned}$$

We know that ψ is the restriction to $R_\mathfrak{p}^{1/p^{e+1}}$ of a map in

$$\operatorname{Hom}_{R_\mathfrak{p}}((R_\mathfrak{p}(\lceil (p^{e+1}-1)\Delta \rceil + \operatorname{div}_R(c)))^{1/p^{e+1}}, R_\mathfrak{p}),$$

and by localizing the above inclusions at $\mathfrak{p}$, we see that $\psi_e = \psi|_{R_\mathfrak{p}^{1/p^e}}$ can be extended to a map in $\operatorname{Hom}_{R_\mathfrak{p}}((R_\mathfrak{p}(\lceil (p^e-1)\Delta)\rceil)^{1/p^e}, R_\mathfrak{p})$. It follows immediately that $\Delta_{\psi_e} \geq \Delta$. □

As the above examples demonstrate, Theorem 4.3 can be used to show the existence of limits in a large number of important settings. Moreover, the following well known lemma allows similar techniques to be used more broadly still.

Lemma 4.13. *Let $(R, \mathfrak{m}, k)$ be an F-finite local domain of dimension d and $0 \neq c \in R$. Suppose $\{I_e\}_{e\in\mathbb{N}}$ and $\{J_e\}_{e\in\mathbb{N}}$ are two sequences of ideals in R so that*

$$\mathfrak{m}^{[p^e]} \subseteq I_e \subseteq J_e \subseteq (I_e : c)$$

for all $e \in \mathbb{N}$. Then there exists a positive constant $C \in \mathbb{R}$ so that

$$\left| \frac{1}{p^{ed}} \ell_R(R/I_e) - \frac{1}{p^{ed}} \ell_R(R/J_e) \right| = \frac{1}{p^{ed}} \ell_R(J_e/I_e) \leq \frac{C}{p^e}$$

for all $e \in \mathbb{N}$ *and so* $\lim_{e\to\infty}((1/p^{ed})\ell_R(R/I_e) - (1/p^{ed})\ell_R(R/J_e)) = 0$. *Hence*, $\lim_{e\to\infty}(1/p^{ed})\ell_R(R/J_e)$ *exists if and only if* $\lim_{e\to\infty}(1/p^{ed})\ell_R(R/I_e)$ *exists, in which case they are equal.*

Proof. Since $(I_e : c)/I_e$ is the kernel and $R/(I_e, c)$ the cokernel of multiplication by c on R/I_e, their lengths are equal. Thus,

$$\frac{1}{p^{ed}}\ell_R(J_e/I_e) \le \frac{1}{p^{ed}}\ell_R((I_e : x)/I_e) = \frac{1}{p^{ed}}\ell_R(R/(I_e, c)) \le \frac{1}{p^{ed}}\ell_R(R/(\mathfrak{m}^{[p^e]}, c)) \le \frac{C}{p^e}$$

by applying Lemma 3.1 with $M = R/(c)$. □

Once again, the constant C in the above lemma depends only on the choice of $0 \ne c \in R$ and not on any particular sequence of ideals; moreover, in case R is not local, the constant C can be chosen uniformly over $\operatorname{Spec}(R)$ using Theorem 3.3.

5. Positivity

Positivity of the F-signature. Recall that an F-finite ring R is said to be strongly F-regular if and only if, for all $x \in R$ not contained in any minimal prime (e.g., $x \ne 0$ when R is a domain), there exist some $e \in \mathbb{N}$ and $\phi \in \operatorname{Hom}_R(R^{1/p^e}, R)$ with $\phi(x^{1/p^e}) = 1$. Strongly F-regular rings are always products of Cohen–Macaulay normal domains. Moreover, a strongly F-regular local domain remains so after completion.

Theorem 5.1 [Aberbach and Leuschke 2003, Main Result]. *Suppose* $(R, \mathfrak{m}, k)$ *is a local* F*-finite domain. Then* R *is strongly* F*-regular if and only if* $s(R) > 0$.

We will give two new proofs of Theorem 5.1. The first proof is notable in that it gives a readily computable lower bound for the F-signature.

First proof of Theorem 5.1. We will assume for simplicity in the exposition that $k = k^p$ is perfect — the proof is easily adapted to arbitrary k. Suppose first that R is not strongly F-regular, so that there exists some $0 \ne x \in R$ with $\phi(x^{1/p^e}) \in \mathfrak{m}$ for all $e \in \mathbb{N}$ and all $\phi \in \operatorname{Hom}_R(R^{1/p^e}, R)$. Take a surjection $\Phi\colon R^{1/p^e} \to R^{\oplus \operatorname{frk}_R(R^{1/p^e})}$ of R-modules. It follows that $\Phi((xR)^{1/p^e}) \subseteq \mathfrak{m}^{\oplus \operatorname{frk}_R(R^{1/p^e})}$, so that Φ induces a surjection of R-modules $(R/xR)^{1/p^e} \to k^{\oplus \operatorname{frk}_R(R^{1/p^e})}$ and hence $\operatorname{frk}_R(R^{1/p^e}) \le \mu_R((R/xR)^{1/p^e}) \le C(R/xR, \mathfrak{m})p^{e(d-1)}$ using (2) and Lemma 3.1 for some positive constant $C(R/xR, \mathfrak{m}) \in \mathbb{R}$. Dividing through by p^{ed} and taking $\lim_{e\to\infty}$ then gives $s(R) = 0$. Thus, $s(R) > 0$ implies that R is strongly F-regular.

Conversely, suppose that R is strongly F-regular. The F-signature remains unchanged upon completion, so we may assume R is complete and choose a coefficient field k and system of parameters $x_1, \dots, x_d$ so that R is module finite and generically separable over the regular subring $A = k[\![x_1, \dots, x_d]\!]$. Take $0 \ne c \in A$ so that $cR^{1/p^e} \subseteq R[A^{1/p^e}] = R \otimes_A A^{1/p^e}$ as in Lemma 2.3 for all $e \in \mathbb{N}$. Since R is strongly F-regular, we can find $e_0 \in \mathbb{N}$ and $\phi \in \operatorname{Hom}_R(R^{1/p^{e_0}}, R)$ with $\phi(c^{1/p^{e_0}}) = 1$. We will show $s(R) \ge 1/p^{e_0 d} > 0$.

For any $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}^d$ and $e \in \mathbb{N}$, we write $\underline{x}^{\alpha/p^e}$ for $x_1^{\alpha_1/p^e} \cdots x_d^{\alpha_d/p^e}$. Let $\mathcal{I}_e = \{\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}^d \mid 0 \le \alpha_i < p^e \text{ for } i = 1, \dots, d\}$. The monomials $\underline{x}^{\alpha/p^e}$ for $\alpha \in \mathcal{I}_e$ are a free basis for A^{1/p^e} over A. As such, for each $\alpha \in \mathcal{I}_e$, we can find an A-linear map $\pi_\alpha \colon A^{1/p^e} \to A$ with $\pi_\alpha(\underline{x}^{\alpha/p^e}) = 1$ and $\pi_\alpha(\underline{x}^{\beta/p^e}) = 0$ for all $\alpha \ne \beta \in \mathcal{I}_e$. Applying $R \otimes_A \_$ gives an R-linear map $\tilde{\pi}_\alpha \colon R[A^{1/p^e}] \to R$ with $\tilde{\pi}_\alpha(\underline{x}^{\alpha/p^e}) = 1$ and $\tilde{\pi}_\alpha(\underline{x}^{\beta/p^e}) = 0$ for all $\alpha \ne \beta \in \mathcal{I}_e$. Multiplication by c gives an R-linear map $R^{1/p^e} \to R[A^{1/p^e}]$, and composing with $\tilde{\pi}_\alpha$ gives an R-linear map $\tilde{\pi}_\alpha(c \cdot \_) \colon R^{1/p^e} \to R$ so that $\tilde{\pi}_\alpha(c\underline{x}^{\alpha/p^e}) = c\tilde{\pi}_\alpha(\underline{x}^{\alpha/p^e}) = c$ and $\tilde{\pi}_\alpha(c\underline{x}^{\beta/p^e}) = c\tilde{\pi}_\alpha(\underline{x}^{\beta/p^e}) = 0$ for all $\alpha \ne \beta \in \mathcal{I}_e$ (since $c \in A$). Setting $\psi_\alpha(\_) = \phi \circ (\tilde{\pi}_\alpha(c \cdot \_))^{1/p^{e_0}} \colon R^{1/p^{e+e_0}} \to R$, we have $\psi_\alpha \in \operatorname{Hom}_R(R^{1/p^{e+e_0}}, R)$ with $\psi_\alpha(\underline{x}^{\alpha/p^{e+e_0}}) = 1$ and $\psi_\alpha(\underline{x}^{\beta/p^{e+e_0}}) = 0$ for all $\alpha \ne \beta \in \mathcal{I}_e$. It follows that $\Psi = \bigoplus_{\alpha \in \mathcal{I}_e} \psi_\alpha \colon R^{1/p^{e+e_0}} \to R^{\oplus p^{ed}}$ is an R-module surjection and hence $\operatorname{frk}_R(R^{1/p^{e+e_0}}) \ge p^{ed}$. Dividing through by $p^{(e+e_0)d}$ and taking $\lim_{e\to\infty}$ yields $s(R) \ge (1/p^{e_0 d}) > 0$ as desired. □

For the second proof, we will need the following lemma, which should be compared with [Hochster and Huneke 1991, Theorem 3.3]. In the next subsection, we will show how to adapt this proof to arbitrary sequences of ideals as in Theorem 4.3(ii).

Lemma 5.2. *Let $(R, \mathfrak{m}, k)$ be a complete local F-finite Cohen–Macaulay domain of dimension d. There exists $N \in \mathbb{N}$ with the following property: for any $e \in \mathbb{N}$ and all $x \in R \setminus \mathfrak{m}^{[p^e]}$, there exists a map $\phi \in \operatorname{Hom}_R(R^{1/p^e}, R)$ with $\phi(x^{1/p^e}) \notin \mathfrak{m}^N$.*

Proof. Choosing a coefficient field k and system of parameters $x_1, \dots, x_d$, we have that R is a finitely generated free module over $A = k[\![x_1, \dots, x_d]\!]$ [Bruns and Herzog 1993, Proposition 2.2.11]; let $\mathfrak{m}_A$ denote the maximal ideal of A. Fix a nonzero map $\tau \in \operatorname{Hom}_A(R, A)$. Since $\operatorname{Hom}_A(R, A)$ is a rank-one torsion-free R-module, we can find $0 \ne y \in \operatorname{Ann}_R(\operatorname{Hom}_A(R, A)/(R\tau))$; replacing by a nonzero multiple, we may further assume $0 \ne y \in A$. If $N', N \in \mathbb{N}$ are sufficiently large so that $y \notin \mathfrak{m}_A^{N'}$ and $\mathfrak{m}^N \subseteq \mathfrak{m}_A^{N'} R$, we will show N satisfies the desired property.

Fix $e \in \mathbb{N}$ and $x \in R \setminus \mathfrak{m}^{[p^e]}$, so also $x \notin \mathfrak{m}_A^{[p^e]} R$. Since R^{1/p^e} is a free A-module and $x^{1/p^e} \notin \mathfrak{m}_A R^{1/p^e}$, x^{1/p^e} can be taken as part of a free basis and there exists an A-linear map $\chi \colon R^{1/p^e} \to A$ with $\chi(x^{1/p^e}) = 1$. If $\operatorname{Tr} \colon \operatorname{Hom}_A(R, A) \to A$ is the A-linear map given by evaluation at 1, there is a commutative diagram

$$\begin{array}{ccccccccc} R^{1/p^e} & \xrightarrow{\tilde{\chi}} & \operatorname{Hom}_A(R^{1/p^e}, A) & \longrightarrow & \operatorname{Hom}_A(R, A) & \xrightarrow{\cdot y} & R\tau & \xrightarrow{\simeq} & R \\ & \searrow^{\chi} & & & \downarrow{\scriptstyle \operatorname{Tr}} & & \downarrow{\scriptstyle \operatorname{Tr}} & & \downarrow{\scriptstyle \tau} \\ & & & & A & \xrightarrow{\cdot y} & A & \xrightarrow{=} & A \end{array} \tag{10}$$

where $\tilde{\chi}$ is the R^{1/p^e}-linear map sending $1 \mapsto \chi$ and where $\operatorname{Hom}_A(R^{1/p^e}, A) \to \operatorname{Hom}_A(R, A)$ is the R-linear map given by restriction to $R \subseteq R^{1/p^e}$. Let $\phi \in \operatorname{Hom}_R(R^{1/p^e}, R)$ be the composition along the top row of this diagram, so that $y\chi = \tau \circ \phi$. We have $\tau(\phi(x^{1/p^e})) = y\chi(x^{1/p^e}) = y \notin \mathfrak{m}_A^{N'}$, and thus $\phi(x^{1/p^e}) \notin \mathfrak{m}_A^{N'} R$ as τ is A-linear. In particular, since $\mathfrak{m}^N \subseteq \mathfrak{m}_A^{N'} R$, we have $\phi(x^{1/p^e}) \notin \mathfrak{m}^N$ as desired. □

Second proof of Theorem 5.1. Suppose that R is strongly F-regular, and as such Cohen–Macaulay. As before, without loss of generality, we may assume R is complete. If $I_e^{F\text{-sig}} = (r \in R \mid \phi(r^{1/p^e}) \in \mathfrak{m}$

for all $\phi \in \mathrm{Hom}_R(R^{1/p^e}, R))$ and $\psi \in \mathrm{Hom}_R(R^{1/p^{e'}}, R)$, it is easy to check that $\psi((I_{e+e'}^{F\text{-sig}})^{1/p^{e'}}) \subseteq I_e^{F\text{-sig}}$ for all $e, e' \in \mathbb{N}$. Since R is F-split, we may conclude $I_{e+1}^{F\text{-sig}} \subseteq I_e^{F\text{-sig}}$ for all $e \in \mathbb{N}$; moreover, by the definition of strong F-regularity, we have $\bigcap_{e\in\mathbb{N}} I_e^{F\text{-sig}} = 0$. Thus, for N as in Lemma 5.2, Chevalley's lemma [1943, Lemma 7] gives $e_0 \in \mathbb{N}$ with $I_{e_0}^{F\text{-sig}} \subseteq \mathfrak{m}^N$. It follows from Lemma 5.2 that $I_{e+e_0}^{F\text{-sig}} \subseteq \mathfrak{m}^{[p^e]}$ for all $e \in \mathbb{N}$. Indeed for each $x \in R \setminus \mathfrak{m}^{[p^e]}$ there is some $\phi \in \mathrm{Hom}_R(R^{1/p^e}, R)$ with $\phi(x^{1/p^e}) \notin \mathfrak{m}^N$ and hence $\phi(x^{1/p^e}) \notin I_{e_0}^{F\text{-sig}}$; as $\phi((I_{e+e_0}^{F\text{-sig}})^{1/p^e}) \subseteq I_{e_0}^{F\text{-sig}}$ we must have $x \notin I_{e+e_0}^{F\text{-sig}}$. Thus, we compute

$$s(R) = \lim_{e\to\infty} \frac{1}{p^{(e+e_0)d}} \ell_R(R/I_{e+e_0}^{F\text{-sig}}) \geq \lim_{e\to\infty} \frac{1}{p^{(e+e_0)d}} \ell_R(R/\mathfrak{m}^{[p^e]}) = \frac{1}{p^{e_0 d}} e_{\mathrm{HK}}(R) \geq \frac{1}{p^{e_0 d}} > 0$$

as desired. □

Positivity via a Cartier linear map. In this section, we generalize the method of the second proof of Theorem 5.1 to examine the positivity of the limits appearing in Theorem 4.3(ii). The essential technique is to exploit the following generalization of Lemma 5.2, which should again be compared with [Hochster and Huneke 1991, Theorem 3.3].

Proposition 5.3. *Let $(R, \mathfrak{m}, k)$ be a complete local F-finite domain of dimension d, and suppose that $0 \neq \psi \in \mathrm{Hom}_R(R^{1/p}, R)$. There exist $N \in \mathbb{N}$ and $0 \neq c \in R$ so that $\psi^e((xR)^{1/p^e}) \nsubseteq \mathfrak{m}^N$ for any $e \in \mathbb{N}$ and all $x \in R \setminus (\mathfrak{m}^{[p^e]} : c)$.*

Proof. Choose a coefficient field k and system of parameters $x_1, \dots, x_d$ so that R is module finite over the regular subring $A = k[\![x_1, \dots, x_d]\!]$. As R is a torsion-free A-module, choose a free A-submodule $G = A^{\oplus \mathrm{rank}_A(R)} \subseteq R$ of maximal rank and take $0 \neq c \in \mathrm{Ann}_A(R/G)$. In particular, $c^{1/p^e} R^{1/p^e} \subseteq G^{1/p^e} \subseteq R^{1/p^e}$ for all $e \in \mathbb{N}$.

Fixing a nonzero map $\tau \in \mathrm{Hom}_A(R, A)$, we can find $0 \neq y \in \mathrm{Ann}_R(\mathrm{Hom}_A(R, A)/(R\tau))$ as $\mathrm{Hom}_A(R, A)$ is a rank-one torsion-free R-module. Let $0 \neq z \in R$ be as in Lemma 4.2; replacing by nonzero multiples, we may further assume both $0 \neq y \in A$ and $0 \neq z \in A$. Let $N', N \in \mathbb{N}$ be sufficiently large so that $yz \notin \mathfrak{m}_A^{N'}$ and $\mathfrak{m}^N \subseteq \mathfrak{m}_A^{N'} R$; we will show that the integer N and element $c = yz$ satisfy the desired property.

Suppose $e \in \mathbb{N}$ and $x \in R \setminus (\mathfrak{m}^{[p^e]} : c)$. Since $xc \notin \mathfrak{m}_A^{[p^e]} R \subseteq \mathfrak{m}^{[p^e]}$, we have $(xc)^{1/p^e} \notin \mathfrak{m}_A R^{1/p^e}$ and so also $(xc)^{1/p^e} \notin \mathfrak{m}_A G^{1/p^e} \subseteq \mathfrak{m}_A R^{1/p^e}$. As $G^{1/p^e} = (A^{1/p^e})^{\oplus \mathrm{rank}_A(R)}$ is a free A-module, there is a $\chi' \in \mathrm{Hom}_A(G^{1/p^e}, A)$ with $\chi'((xc)^{1/p^e}) = 1$. Let $\chi \in \mathrm{Hom}_A(R^{1/p^e}, A)$ be the composition

$$R^{1/p^e} \xrightarrow{\cdot c^{1/p^e}} G^{1/p^e} \xrightarrow{\chi'} A$$

which satisfies $\chi(x^{1/p^e}) = 1$. If $\mathrm{Tr}\colon \mathrm{Hom}_A(R, A) \to A$ is the A-linear map given by evaluation at 1, the same diagram (10) from Lemma 5.2 commutes where $\tilde{\chi}$ is the R^{1/p^e}-linear map sending $1 \mapsto \chi$ and $\mathrm{Hom}_A(R^{1/p^e}, A) \to \mathrm{Hom}_A(R, A)$ is the R-linear map given by restriction to $R \subseteq R^{1/p^e}$. Let $\phi \in \mathrm{Hom}_R(R^{1/p^e}, R)$ be the composition along the top row of this diagram, so that $y\chi = \tau \circ \phi$. By Lemma 4.2, we can find $r \in R$ so that $z\phi(\_) = \psi^e(r^{1/p^e} \cdot \_)$. We compute

$$\tau(\psi^e((rx)^{1/p^e})) = \tau(z\phi(x^{1/p^e})) = z\tau(\phi(x^{1/p^e})) = yz\chi(x^{1/p^e}) = yz \notin \mathfrak{m}_A^{N'}$$

and conclude $\psi^e((rx)^{1/p^e}) \notin \mathfrak{m}_A^{N'}R$ as τ is A-linear. In particular, since $\mathfrak{m}^N \subseteq \mathfrak{m}_A^{N'}R$, we have that $\psi^e((xR)^{1/p^e}) \not\subseteq \mathfrak{m}^N$ as desired. □

In the proof of the main result of this section, we will need to use a well known result on the stabilization of the images of an iterated p^{-1}-linear map. Rooted in the work of Hartshorne and Speiser [1977, Proposition 1.11] and later generalized by Lyubeznik [1997], the following version is due to Gabber [2004].

Theorem 5.4 [Gabber 2004, Lemma 13.1; Blickle and Böckle 2011, Proposition 2.14]. *Let M be a finitely generated module over an F-finite ring R and $0 \neq \phi \in \operatorname{Hom}_R(M^{1/p}, M)$. Then the descending chain of R-modules*

$$M \supseteq \phi(M^{1/p}) \supseteq \phi^2(M^{1/p^2}) \supseteq \cdots \supseteq \phi^e(M^{1/p^e}) \supseteq \cdots$$

stabilizes for $e \gg 0$.

Theorem 5.5. *Let $(R, \mathfrak{m}, k)$ be a complete local F-finite domain of dimension d and $\{I_e\}_{e\in\mathbb{N}}$ a sequence of ideals so that $\mathfrak{m}^{[p^e]} \subseteq I_e$ for all $e \in \mathbb{N}$. Suppose there exists a nonzero $\psi \in \operatorname{Hom}_R(R^{1/p}, R)$ so that $\psi((I_{e+1})^{1/p}) \subseteq I_e$ for all $e \in \mathbb{N}$. Then the limit $\lim_{e\to\infty}(1/p^{ed})\ell_R(R/I_e)$ is positive if and only if $\bigcap_{e\in\mathbb{N}} I_e = 0$.*

Proof. By Lemma 3.1, if $0 \neq c \in \bigcap_{e\in\mathbb{N}} I_e$, then $\ell_R(R/I_e) \leq \ell_R(R/(\mathfrak{m}^{[p^e]}, c)) \leq C(R/(c), \mathfrak{m})p^{e(d-1)}$ for all $e \in \mathbb{N}$. It follows that $\bigcap_{e\in\mathbb{N}} I_e \neq 0$ implies that $\lim_{e\to\infty}(1/p^{ed})\ell_R(R/I_e) = 0$.

For the converse, suppose $\bigcap_{e\in\mathbb{N}} I_e = 0$. If $0 \neq \sigma = \psi^e(R^{1/p^e})$ for $e \gg 0$ is the stable image of ψ as in Theorem 5.4, we also have $\bigcap_{e\in\mathbb{N}}(I_e : \sigma) = 0$ as $\sigma\big(\bigcap_{e\in\mathbb{N}}(I_e : \sigma)\big) \subseteq \bigcap_{e\in\mathbb{N}} I_e = 0$ and R is a domain. In addition, it follows from $\psi(\sigma^{1/p}) = \sigma$ that $(I_{e+1} : \sigma) \subseteq (I_e : \sigma)$ for all $e \in \mathbb{N}$ as

$$(I_{e+1} : \sigma)\sigma = (I_{e+1} : \sigma)\psi(\sigma^{1/p}) = \psi(((I_{e+1} : \sigma)^{[p]}\sigma)^{1/p}) \subseteq \psi(((I_{e+1} : \sigma)\sigma)^{1/p}) \subseteq \psi(I_{e+1}^{1/p}) \subseteq I_e.$$

Take $N \in \mathbb{N}$ and $0 \neq c \in R$ as in Proposition 5.3. By Chevalley's lemma [1943, Lemma 7], there is some $e_0 \in \mathbb{N}$ with $I_{e_0} \subseteq (I_{e_0} : \sigma) \subseteq \mathfrak{m}^N$. It follows from Proposition 5.3 that $I_{e+e_0} \subseteq (\mathfrak{m}^{[p^e]} : c)$ for all $e \in \mathbb{N}$. Indeed for each $x \in R \setminus (\mathfrak{m}^{p^e} : c)$ we have $\psi^e((xR)^{1/p^e}) \not\subseteq \mathfrak{m}^N$ and hence $\psi^e((xR)^{1/p^e}) \not\subseteq I_{e_0}$; since $\psi^e((I_{e+e_0})^{1/p^e}) \subseteq I_{e_0}$ we must have $x \notin I_{e+e_0}$. Using Lemma 4.13, we compute

$$\begin{aligned}\lim_{e\to\infty} \frac{1}{p^{(e+e_0)d}}\ell_R(R/I_{e+e_0}) &\geq \frac{1}{p^{e_0 d}} \lim_{e\to\infty} \frac{1}{p^{ed}}\ell_R(R/(\mathfrak{m}^{[p^e]} : c)) \\ &= \frac{1}{p^{e_0 d}} \lim_{e\to\infty} \frac{1}{p^{ed}}\ell_R(R/\mathfrak{m}^{[p^e]}) \\ &= \frac{1}{p^{e_0 d}} e_{\mathrm{HK}}(R) \geq \frac{1}{p^{e_0 d}} > 0.\end{aligned}$$

□

Theorem 5.5 is quite powerful and allows one to show positivity of the limits appearing in Theorem 4.3(ii) in a number of different situations. One should view this result not only as a generalization of the work done by Aberbach and Leuschke [2003], but also the work of Hochster and Huneke [1990]. If R is a domain and $I \subseteq R$, recall that $x \in R$ is in the tight closure I^* of I if there exists an element $0 \neq c \in R$

such that $cx^{p^e} \in I^{[p^e]}$ for all $e \in \mathbb{N}$. The tight closure I^* is an ideal containing I, and $I^{**} = I$. A ring is said to be weakly F-regular if all ideals I of R are tightly closed, i.e., satisfy $I^* = I$. See [Huneke 1996] or [Hochster 2007] for further details.

Corollary 5.6 [Hochster and Huneke 1990, Theorem 8.17] (length criterion for tight closure). *Let $(R, \mathfrak{m}, k)$ be a complete local F-finite domain of dimension d. Suppose that $I \subseteq J$ is an inclusion of $\mathfrak{m}$-primary ideals in R. Then $I^* = J^*$ if and only if $e_{\mathrm{HK}}(I) = e_{\mathrm{HK}}(J)$.*

Proof. Applying the criterion to each term of a composition series of J/I, we may assume $J = (I, x)$ for some $x \in R$ with $(I : x) = \mathfrak{m}$. Consider the sequence of ideals $(I^{[p^e]} : x^{p^e})$ for each $e \in \mathbb{N}$. We have that $(I^{[p^e]} : x^{p^e})^{[p]} \subseteq (I^{[p^{e+1}]} : x^{p^{e+1}})$ for all $e \geq 0$, so that in particular $\mathfrak{m}^{[p^e]} \subseteq (I^{[p^e]} : x^{p^e})$ for each $e \in \mathbb{N}$. Moreover, for any nonzero $\psi \in \mathrm{Hom}_R(R^{1/p}, R)$ it is easy to check $\psi((I^{[p^{e+1}]} : x^{p^{e+1}})^{1/p}) \subseteq (I^{[p^e]} : x^{p^e})$. Since $(I, x)^{[p^e]}/I^{[p^e]} \simeq R/(I^{[p^e]} : x^{p^e})$, we see by Theorem 5.5 that

$$e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}((I, x)) = \lim_{e\to\infty} \frac{1}{p^{ed}} \ell_R(R/(I^{[p^e]} : x^{p^e})) \tag{11}$$

is zero if and only if $\bigcap_{e\in\mathbb{N}} (I^{[p^e]} : x^{p^e}) \neq 0$, which is equivalent to $x \in I^*$ by definition. □

Recall that there are also a number of well known generalizations of tight closure and strong F-regularity [Hara and Yoshida 2003; Hara and Watanabe 2002; Takagi 2004]. If R is an F-finite local domain, $\mathfrak{a}$ is a nonzero ideal of R, and $t \in \mathbb{R}_{\geq 0}$, one can speak of the $\mathfrak{a}^t$-tight closure $I^{*\mathfrak{a}^t}$ of an ideal $I \subseteq R$. By definition, if $x \in R$, then $x \in I^{*\mathfrak{a}^t}$ if and only if there exists $0 \neq c \in R$ with $c\mathfrak{a}^{\lceil t(p^e-1)\rceil} x^{p^e} \in I^{[p^e]}$ for all $e \in \mathbb{N}$. The pair $(R, \mathfrak{a}^t)$ is strongly F-regular provided for any $0 \neq x \in R$ there exists $e \in \mathbb{N}$ and $\phi \in \mathscr{C}_e^{\mathfrak{a}^t}$ with $\phi(x^{1/p^e}) = 1$. Similarly, if R is an F-finite normal local domain and Δ is an effective $\mathbb{Q}$-divisor on $\mathrm{Spec}(R)$, we have the Δ-tight closure $I^{*\Delta}$ of an ideal $I \subseteq R$. By definition, if $x \in R$, then $x \in I^{*\Delta}$ if and only if there exists $0 \neq c \in R$ with $cx^{p^e} \in I^{[p^e]}(R(\lceil (p^e - 1)\Delta \rceil))$ for all $e \in \mathbb{N}$. The pair (R, Δ) is strongly F-regular provided for any $0 \neq x \in R$ there exists $e \in \mathbb{N}$ and $\phi \in \mathscr{C}_e^{(R,\Delta)}$ with $\phi(x^{1/p^e}) = 1$. As of yet, there does not exist a well formed theory of Hilbert–Kunz multiplicity for an ideal or divisor; nonetheless, one can use Theorem 5.5 to give analogues of Corollary 5.6 for these operations.

Corollary 5.7 (length criterion for $\mathfrak{a}^t$-tight closure). *Let $(R, \mathfrak{m}, k)$ be a complete local F-finite domain of dimension d, $\mathfrak{a}$ a nonzero ideal of R, and $t \in \mathbb{R}_{\geq 0}$.*

(i) *For any $\mathfrak{m}$-primary ideal I and $x \in R$, $\lim_{e\to\infty}(1/p^{ed})\ell_R(R/(I^{[p^e]} : \mathfrak{a}^{\lceil t(p^e-1)\rceil} x^{p^e}))$ exists; moreover, this limit equals zero if and only if $x \in I^{*\mathfrak{a}^t}$.*

(ii) *The F-signature $s(R, \mathfrak{a}^t) = \lim_{e\to\infty}(1/p^{ed})\ell_R(R/I_e^{\mathfrak{a}^t})$ of R along $\mathfrak{a}^t$ is positive if and only if $(R, \mathfrak{a}^t)$ is strongly F-regular.*

Proof. For (i), take $0 \neq \psi \in \mathscr{C}_1^{\mathfrak{a}^t} = (\mathfrak{a}^{\lceil t(p-1)\rceil})^{1/p} \cdot \mathrm{Hom}_R(R^{1/p}, R)$, and consider the sequence of ideals $(I^{[p^e]} : \mathfrak{a}^{\lceil t(p^e-1)\rceil} x^{p^e})$ for $e \in \mathbb{N}$. It is easy to check $\psi((I^{[p^{e+1}]} : \mathfrak{a}^{\lceil t(p^{e+1}-1)\rceil} x^{p^{e+1}})^{1/p}) \subseteq (I^{[p^e]} : \mathfrak{a}^{\lceil t(p^e-1)\rceil} x^{p^e})$ for all $e \in \mathbb{N}$. If $e_0 \in \mathbb{N}$ is sufficiently large that $\mathfrak{m}^{[p^{e_0}]} \subseteq I$, then we have $\mathfrak{m}^{[p^{e+e_0}]} \subseteq (I^{[p^e]} : \mathfrak{a}^{\lceil t(p^e-1)\rceil} x^{p^e})$ for all $e \in \mathbb{N}$. After shifting the index, Theorem 4.3(ii) and Theorem 5.5 apply, giving the existence of the limit and showing that it equals zero if and only if $\bigcap_{e\in\mathbb{N}}(I^{[p^e]} : \mathfrak{a}^{\lceil t(p^e-1)\rceil}x^{p^e}) \neq 0$, which is equivalent to $x \in I^{*\mathfrak{a}^t}$ by definition. To see (ii), simply note once more that $\psi((I_{e+1}^{\mathfrak{a}^t})^{1/p}) \subseteq I_e^{\mathfrak{a}^t}$ and apply Theorem 5.5. □

Corollary 5.8 (length criterion for Δ-tight closure). *Let $(R, \mathfrak{m}, k)$ be a complete local F-finite normal domain of dimension d, and Δ an effective $\mathbb{Q}$-divisor on* $\operatorname{Spec}(R)$.

(i) *For any $\mathfrak{m}$-primary ideal $I \subseteq R$ and $x \in R$,* $\lim_{e\to\infty}(1/p^{ed})\ell_R(R/(I^{[p^e]}R(\lceil(p^e-1)\Delta\rceil) : x^{p^e}))$ *exists; moreover, this limit equals zero if and only if $x \in I^{*\Delta}$.*

(ii) *The F-signature $s(R,\Delta) = \lim_{e\to\infty}(1/p^{ed})\ell_R(R/I_e^{(R,\Delta)})$ of R along Δ is positive if and only if (R,Δ) is strongly F-regular.*

Proof. Let $0 \neq \psi \in \mathscr{C}_1^{\Delta} = \operatorname{im}(\operatorname{Hom}_R((R(\lceil(p-1)\Delta\rceil))^{1/p}, R) \to \operatorname{Hom}_R(R^{1/p}, R))$. That is, identifying ψ with its unique extension to a p^{-1}-linear map on $K = \operatorname{Frac}(R)$, we have that $\psi((R(\lceil(p-1)\Delta\rceil))^{1/p}) \subseteq R$. Twisting by $\lceil(p^e-1)\Delta\rceil$ and using the fact that $p\lceil(p^e-1)\Delta\rceil + \lceil(p-1)\Delta\rceil \geq \lceil(p^{e+1}-1)\Delta\rceil$, observe $\psi((R(\lceil(p^{e+1}-1)\Delta\rceil))^{1/p}) \subseteq R(\lceil(p^e-1)\Delta\rceil)$ for all $e \in \mathbb{N}$.

For (i), consider the sequence of ideals $(I^{[p^e]}R(\lceil(p^e-1)\Delta\rceil) : x^{p^e})$ for $e \in \mathbb{N}$. It is easy to check $\psi((I^{[p^{e+1}]}R(\lceil(p^{e+1}-1)\Delta\rceil) : x^{p^{e+1}})^{1/p}) \subseteq (I^{[p^e]}R(\lceil(p^e-1)\Delta\rceil) : x^{p^e})$ for all $e \in \mathbb{N}$. If $e_0 \in \mathbb{N}$ is sufficiently large that $\mathfrak{m}^{[p^{e_0}]} \subseteq I$, then we have $\mathfrak{m}^{[p^{e+e_0}]} \subseteq (I^{[p^e]}R(\lceil(p^e-1)\Delta\rceil) : x^{p^e})$ for all $e \in \mathbb{N}$. After shifting the index, Theorems 4.3(ii) and 5.5 apply, giving the existence of the limit and showing that it equals zero if and only if $\bigcap_{e\in\mathbb{N}}(I^{[p^e]}R(\lceil(p^e-1)\Delta\rceil) : x^{p^e}) \neq 0$, which is equivalent to $x \in I^{*\Delta}$ by definition. For (ii), simply note once more that $\psi((I_{e+1}^{(R,\Delta)})^{1/p}) \subseteq I_e^{(R,\Delta)}$ and apply Theorem 5.5. □

Remark 5.9. The analogue of Theorem 5.5 for sequences of ideals satisfying Theorem 4.3(i) instead of (ii) is false. Here is an easy counterexample. Let $(R, \mathfrak{m}, k)$ be a complete regular local ring of dimension d, and consider the sequence of ideals $I_e = \mathfrak{m}^{[p^{\lfloor e/2\rfloor}]}$ for $e \in \mathbb{N}$. Then $\mathfrak{m}^{[p^e]} \subseteq I_e$ and $I_e^{[p]} \subseteq I_{e+1}$ for all $e \in \mathbb{N}$ and certainly $\bigcap_{e\in\mathbb{N}} I_e = 0$. However, we have $\lim_{e\to\infty}(1/p^{ed})\ell_R(R/I_e) = \lim_{e\to\infty} 1/p^{(e-\lfloor e/2\rfloor)d} = 0$.

To see yet another application of Theorem 5.5, suppose that $(R, \mathfrak{m}, k)$ is an F-split F-finite local ring. If $I_e^{F\text{-sig}} = (r \in R \mid \phi(r^{1/p^e}) \in \mathfrak{m}$ for all $\phi \in \operatorname{Hom}_R(R^{1/p^e}, R))$, it is straightforward to check that $\mathscr{P} = \bigcap_{e\in\mathbb{N}} I_e^{F\text{-sig}}$ is a prime ideal [Tucker 2012, Lemma 4.7], coined the F-splitting prime by Aberbach and Enescu [2005]. Furthermore, they suspected that the dimension of the F-splitting prime governed the growth rate of $\operatorname{frk}_R(R^{1/p^e})$ when R is not strongly F-regular; this observation was verified in joint work of the second author with Blickle and Schwede.

Theorem 5.10 [Blickle et al. 2012]. *Suppose $(R, \mathfrak{m}, k)$ is an F-split F-finite local ring and $\mathscr{P}$ is the F-splitting prime. If $n = \dim(R/\mathscr{P})$, the limit $r_F(R) = \lim_{e\to\infty} 1/(([k^{1/p^e} : k]\cdot p^{en}))\operatorname{frk}_R(R^{1/p^e})$ exists and is positive, called the F-splitting ratio. Moreover,* $(1/[k^{1/p^e} : k])\operatorname{frk}_R(R^{1/p^e}) = r_F(R)p^{en} + O(p^{e(n-1)})$.

Proof. Without loss of generality, we may assume R is complete. If $I_e^{F\text{-sig}} = \langle r \in R \mid \phi(r^{1/p^e}) \in \mathfrak{m}$ for all $\phi \in \operatorname{Hom}_R(R^{1/p^e}, R)\rangle$, we have $\mathfrak{m}^{[p^e]} \subseteq I_e^{F\text{-sig}}$ and $(I_e^{F\text{-sig}})^{[p]} \subseteq I_{e+1}^{F\text{-sig}}$ for all $e \in \mathbb{N}$. Fixing a surjective map $\psi \in \operatorname{Hom}_R(R^{1/p}, R)$, we have $\psi((I_{e+1}^{F\text{-sig}})^{1/p}) \subseteq I_e^{F\text{-sig}}$. In particular, this implies

$\psi(\mathcal{P}^{1/p}) \subseteq \mathcal{P}$ and so ψ induces a map $\bar{\psi} \in \mathrm{Hom}_{\bar{R}}(\bar{R}^{1/p}, \bar{R})$ where $\bar{R} = R/\mathcal{P}$. Note that $\bar{\psi}$ is still surjective and hence nonzero. Thus, passing to the sequence of ideals $\bar{I_e} = I_e^{F\text{-sig}}\bar{R}$, we have $\mathfrak{m}^{[p^e]}\bar{R} \subseteq \bar{I_e}$, $\bar{I_e}^{[p]} \subseteq \overline{I_{e+1}}$, and $\bar{\psi}((\overline{I_{e+1}})^{1/p}) \subseteq \bar{I_e}$ for all $e \in \mathbb{N}$. Moreover, $\bigcap_{e\in\mathbb{N}} \bar{I_e} = 0$ in $\bar{R}$. The result now follows immediately from Corollary 4.5 together with Theorem 5.5, using that $\mathrm{frk}_R(R^{1/p^e})/[k^{1/p^e} : k] = \ell_R(R/I_e^{F\text{-sig}}) = \ell_{\bar{R}}(\bar{R}/\bar{I_e})$ for all $e \in \mathbb{N}$. □

Remark 5.11. It is straightforward to generalize the notions of F-splitting prime and F-splitting ratio to arbitrary Cartier subalgebras; see [Blickle et al. 2012] for further details. In all cases, the argument from Theorem 5.10 applies and greatly simplifies the proofs. In particular, the methods of the proofs of Theorems 4.7 and 5.5 immediately give an alternative proof of the following result. Recall that a Cartier subalgebra $\mathcal{D}$ on a domain R is said to be strongly F-regular provided that for any $0 \neq x \in R$ there exists an $e \in \mathbb{N}$ and $\phi \in \mathcal{D}_e$ with $\phi(x^{1/p^e}) = 1$. This definition is compatible with the generalizations of strong F-regularity to ideal and divisor pairs discussed prior to Corollary 5.7 as well.

Corollary 5.12 [Blickle et al. 2012, Theorem 3.18]. *Let $(R, \mathfrak{m}, k)$ be an F-finite local domain of dimension d and $\mathcal{D}$ a Cartier subalgebra on R. Then the F-signature $s(R, \mathcal{D}) = \lim_{e\to\infty, e\in\Gamma_{\mathcal{D}}}(1/([k^{1/p^e} : k]\cdot p^{ed}))a_e^{\mathcal{D}}$ of R along $\mathcal{D}$ is positive if and only if $(R, \mathcal{D})$ is strongly F-regular.*

As in the proof of Theorem 4.7, certain reindexing arguments are required in order to deduce Corollary 5.12 from Theorem 5.5. More generally, it is straightforward to modify Theorems 4.3(ii) and 5.5 for sequences of $\mathfrak{m}$-primary ideals governed by a nonzero p^{-e_0}-linear map for some $e_0 \in \mathbb{N}$. Rather than requiring the reader to trace through the various arguments above, we provide a simpler direct proof.

Corollary 5.13. *Let $(R, \mathfrak{m}, k)$ be a complete local F-finite domain of dimension d and $e_0 \in \mathbb{N}$. Suppose $\{I_n\}_{n\in\mathbb{N}}$ is a sequence of ideals with $\mathfrak{m}^{[p^{ne_0}]} \subseteq I_n$ for all $n \in \mathbb{N}$, and there is a nonzero $\psi \in \mathrm{Hom}_R(R^{1/p^{e_0}}, R)$ so that $\psi((I_{n+1})^{1/p^{e_0}}) \subseteq I_n$ for all $n \in \mathbb{N}$. Then the limit $\lim_{n\to\infty}(1/p^{ne_0 d})\ell_R(R/I_n)$ exists, and it is positive if and only if $\bigcap_{n\in\mathbb{N}} I_n = 0$.*

Proof. Replacing with a premultiple and using Lemma 4.2, we may assume there is a nonzero $\phi \in \mathrm{Hom}_R(R^{1/p}, R)$ so that $\psi = \phi^{e_0}$. Define a new sequence of ideals $\{J_e\}_{e\in\mathbb{N}}$ by

$$J_e = \phi^r((I_{\lceil e/e_0\rceil})^{1/p^r}) + \mathfrak{m}^{[p^e]} \quad \text{where } r = \lceil e/e_0\rceil e_0 - e$$

so that $J_{ne_0} = I_n$ for all $n \in \mathbb{N}$. By construction, we have that $\phi(J_{e+1}^{1/p}) \subseteq J_e$ for all $e \in \mathbb{N}$ so that Theorem 4.3(ii) applies directly to show the limits

$$\lim_{e\to\infty} \frac{1}{p^{ed}}\ell_R(R/J_e) = \lim_{n\to\infty} \frac{1}{p^{ne_0 d}}\ell_R(R/J_{ne_0}) = \lim_{n\to\infty} \frac{1}{p^{ne_0 d}}\ell_R(R/I_n)$$

exist. For the positivity statement, if $\bigcap_{n\in\mathbb{N}} I_n = 0$, then so too $\bigcap_{e\in\mathbb{N}} J_e = 0$ as $\bigcap_{e\in\mathbb{N}} J_e \subseteq \bigcap_{n\in\mathbb{N}} I_n$. Thus, Theorem 5.5 gives that the limits above are positive. Else, if $0 \neq c \in \bigcap_{n\in\mathbb{N}} I_n$, then $\ell_R(R/I_n) \leq \ell_R(R/(\mathfrak{m}^{[p^{ne_0}]}, c)) \leq C(R/(c), \mathfrak{m})p^{ne_0(d-1)}$ for all $n \in \mathbb{N}$ by Lemma 3.1, so $\lim_{n\to\infty}(1/p^{ne_0 d})\ell_R(R/I_n)$ vanishes as desired. □

6. F-signature and minimal relative Hilbert–Kunz multiplicity

Our next aim is to realize the F-signature as the infimum of relative differences in the Hilbert–Kunz multiplicities of the cofinite ideals in a local ring (Corollary 6.5). After first bounding such differences from below by the F-signature (Lemma 6.1), we will make use of approximately Gorenstein sequences to find differences arbitrarily close to the F-signature. The crucial step and main technical result is Theorem 6.3, which uses the uniformity of the constants tracked above to swap limits between iterations of Frobenius and progression in an approximately Gorenstein sequence.

The remainder of the section is reserved for constructions of explicit sequences of relative Hilbert–Kunz differences that approach the F-signature (Corollary 6.6); we also analyze when the infimum is known to be achieved (Corollary 6.8). Generalizations to divisor and ideal pairs are also given (Corollaries 6.9 and 6.10).

Lemma 6.1. *Suppose that $(R, \mathfrak{m}, k)$ is an F-finite local ring and $I_e^{F\text{-sig}} = (r \in R \mid \phi(r^{1/p^e}) \in \mathfrak{m}$ for all $\phi \in \operatorname{Hom}_R(R^{1/p^e}, R))$ for $e \in \mathbb{N}$. Then*

$$I_e^{F\text{-sig}} \supseteq \sum_{\substack{I \subseteq J \subseteq R \\ 0 < \ell_R(J/I) < \infty}} (I^{[p^e]} : J^{[p^e]}) \supseteq \sum_{\substack{I \subseteq R,\, \ell_R(R/I) < \infty \\ x \in R,\, (I:x) = \mathfrak{m}}} (I^{[p^e]} : x^{p^e}) \tag{12}$$

and we have

$$s(R) \leq \inf_{\substack{I \subseteq J \subseteq R,\, \ell_R(R/I) < \infty \\ I \neq J,\, \ell_R(R/J) < \infty}} \frac{e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}(J)}{\ell_R(J/I)} \leq \inf_{\substack{I \subseteq R,\, \ell_R(R/I) < \infty \\ x \in R,\, (I:x) = \mathfrak{m}}} e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}((I, x)). \tag{13}$$

Proof. If $I \subsetneq J$ is a proper inclusion of ideals with $\ell_R(J/I) < \infty$ and $\phi \in \operatorname{Hom}_R(R^{1/p^e}, R)$, we have that $\phi((I^{[p^e]} : J^{[p^e]})^{1/p^e}) \subseteq (I : J) \subseteq \mathfrak{m}$. It follows that

$$\sum_{\substack{I \subseteq J \subseteq R \\ 0 < \ell_R(J/I) < \infty}} (I^{[p^e]} : J^{[p^e]}) \subseteq I_e^{F\text{-sig}}.$$

Since the sum on the right is over a smaller set of proper inclusions, (12) follows immediately.

For (13), if $I \subseteq R$ is $\mathfrak{m}$-primary and $x \in R$ with $(I : x) = \mathfrak{m}$, then $(I^{[p^e]} : x^{p^e}) \subseteq I_e^{F\text{-sig}}$ for all $e \in \mathbb{N}$ implies $e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}((I, x)) \geq s(R)$ using (11). Moreover, if $I \subsetneq J$ is a proper inclusion of $\mathfrak{m}$-primary ideals, summing up this inequality for each factor in a composition series of J/I shows $\ell_R(J/I)s(R) \leq (e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}(J))$. The inequalities in (13) now follow immediately, noting again that the infimum on the right is over a smaller set of proper inclusions. □

Recall that a local ring $(R, \mathfrak{m}, k)$ is said to be approximately Gorenstein if there exists a descending chain of irreducible ideals $\{J_t\}_{t \in \mathbb{N}}$ cofinal with powers of the maximal ideal. In particular, each R/J_t is a zero-dimensional Gorenstein local ring and has a one-dimensional socle. By [Hochster 1977, Theorem 1.6], a reduced excellent local ring is always approximately Gorenstein. It is easy to check that equality holds throughout (12) for such rings.

Lemma 6.2. *Suppose that* $(R, \mathfrak{m}, k)$ *is an approximately Gorenstein* F*-finite local ring, and* $\{J_t\}_{t\in\mathbb{N}}$ *is a descending chain of irreducible ideals cofinal with the powers of* $\mathfrak{m}$. *If* $\delta_t \in R$ *generates the socle of* R/J_t, *then for all* $e \in \mathbb{N}$ *we have* $(J_t^{[p^e]} : \delta_t^{p^e}) \subseteq (J_{t+1}^{[p^e]} : \delta_{t+1}^{p^e})$ *and*

$$\begin{aligned} I_e^{F\text{-sig}} &= \langle r \in R \mid \phi(r^{1/p^e}) \in \mathfrak{m} \text{ for all } \phi \in \operatorname{Hom}_R(R^{1/p^e}, R)\rangle \\ &= \sum_{t\in\mathbb{N}} (J_t^{[p^e]} : \delta_t^{p^e}) \\ &= \bigcup_{t\in\mathbb{N}} (J_t^{[p^e]} : \delta_t^{p^e}) \\ &= (J_{t_e}^{[p^e]} : \delta_{t_e}^{p^e}) \quad \text{for all } t_e \gg 0 \text{ sufficiently large.} \end{aligned}$$

Moreover, R *is weakly* F*-regular if and only if* $J_t^* = J_t$ *is tightly closed for all* $t \in \mathbb{N}$.

Proof. Since each R/J_t is an Artinian Gorenstein local ring, we have $\operatorname{Ann}_E(J_t) \simeq R/J_t$ where $E = E_R(k)$ is the injective hull of the residue field of R. Thus, we may view $E = E_R(k) = \varinjlim R/J_t$ as the direct limit of inclusions $R/J_t \to R/J_{t+1}$ mapping (the class of) $\delta_t \mapsto \delta_{t+1}$. In particular, after applying $\_ \otimes_R R^{1/p^e}$, one sees that $(J_t^{[p^e]} : \delta_t^{p^e}) \subseteq (J_{t+1}^{[p^e]} : \delta_{t+1}^{p^e})$ for all $t \in \mathbb{N}$. It follows immediately that

$$\sum_{t\in\mathbb{N}} (J_t^{[p^e]} : \delta_t^{p^e}) = \bigcup_{t\in\mathbb{N}} (J_t^{[p^e]} : \delta_t^{p^e}) = (J_{t_e}^{[p^e]} : \delta_{t_e}^{p^e}) \quad \text{for all } t_e \gg 0.$$

An inclusion $R \to M$ to a finitely generated R-module M determined by $1 \mapsto m$ splits if and only if $E \to E \otimes_R M$ remains injective, which is equivalent to $\delta_t m \notin J_t M$ for all $t \in \mathbb{N}$. See [Hochster 2007, p. 155] for further details. In particular, if $x \in R$, applying this splitting criterion to the map $R \to R^{1/p^e}$ with $1 \mapsto x^{1/p^e}$ gives that $x \in R \setminus I_e^{F\text{-sig}}$ if and only if $x \in R \setminus (J_t^{[p^e]} : \delta_t^{p^e})$ for all $t \in \mathbb{N}$, and so we have that $I_e^{F\text{-sig}} = (J_{t_e}^{[p^e]} : \delta_{t_e}^{p^e})$ for $t_e \gg 0$.

Lastly, suppose there is an ideal $I \subseteq R$ that is not tightly closed. Then $I = \bigcap_{n\in\mathbb{N}} (I + \mathfrak{m}^n)$ is an intersection of $\mathfrak{m}$-primary ideals. The arbitrary intersection of tightly closed ideals is tightly closed [Hochster and Huneke 1990, Proposition 4.1(b)]. Hence, we may replace I with an $\mathfrak{m}$-primary ideal which is not tightly closed and choose $x \in I^*$ with $\mathfrak{m} = (I : x)$. Since R/I injects into a direct sum of copies of E, we can find an R-module homomorphism $R/I \to E$ so that $x + I$ has nonzero image in E and hence must generate the socle $k \subseteq E$. Using that $E = \varinjlim R/J_t$, we may assume $R/I \to R/J_t$ and $x + I \mapsto \delta_t + J_t$ for some $t \in \mathbb{N}$. For each $e \in \mathbb{N}$, applying $\_ \otimes_R R^{1/p^e}$ and viewing as an R^{1/p^e}-module gives $R/I^{[p^e]} \to R/J_t^{[p^e]}$ where $x^{p^e} + I^{[p^e]} \mapsto \delta_t^{p^e} + J_t^{[p^e]}$. In particular, it follows that $(I^{[p^e]} : x^{p^e}) \subseteq (J_t^{[p^e]} : \delta_t^{p^e})$. Thus, $\bigcap_{e\in\mathbb{N}} (I^{[p^e]} : x^{p^e}) \subseteq \bigcap_{e\in\mathbb{N}} (J_t^{[p^e]} : \delta_t^{p^e})$ is not contained in any minimal prime of R; it follows that $\delta_t \in J_t^*$ and hence J_t is not tightly closed (compare to [Hochster and Huneke 1990, Proposition 8.23(f)]). ☐

In the next result, we show how to make use of the uniformity of constants from Theorem 4.3 together with approximately Gorenstein sequences in order to compare F-signature and Hilbert–Kunz multiplicity. As an application, we answer a question posed by Watanabe and Yoshida [2004, Question 1.10].

Theorem 6.3. *Let $(R, \mathfrak{m}, k)$ be a complete local F-finite domain of dimension d, and fix $0 \neq c \in R$. Suppose we are given sequences of ideals $\{I_{t,e}\}_{t,e\in\mathbb{N}}$ satisfying $\mathfrak{m}^{[p^e]} \subseteq I_{t,e}$, $c(I_{t,e}^{[p]}) \subseteq I_{t,e+1}$, and $I_{t,e} \subseteq I_{t+1,e}$ for all $t, e \in \mathbb{N}$. Then*

$$\lim_{e\to\infty}\lim_{t\to\infty}\frac{1}{p^{ed}}\ell_R(R/I_{t,e}) = \lim_{e\to\infty}\frac{1}{p^{ed}}\ell_R(R/I_e) = \lim_{t\to\infty}\lim_{e\to\infty}\frac{1}{p^{ed}}\ell_R(R/I_{t,e})$$

where $I_e = \sum_{t\in\mathbb{N}} I_{t,e}$.

Proof. For each fixed $t \in \mathbb{N}$, as $\mathfrak{m}^{[p^e]} \subseteq I_{t,e}$ and $c(I_{t,e}^{[p]}) \subseteq I_{t,e+1}$ for $e \in \mathbb{N}$, Theorem 4.3(i) guarantees the existence of $\eta_t = \lim_{e\to\infty}(1/p^{ed})\ell_R(R/I_{t,e})$ and provides a uniform positive constant $C \in \mathbb{R}$ so that

$$\eta_t \leq \frac{1}{p^{ed}}\ell_R(R/I_{t,e}) + \frac{C}{p^e} \tag{14}$$

for all $t, e \in \mathbb{N}$. The sequence $\{I_e\}_{e\in\mathbb{N}}$ inherits the properties $\mathfrak{m}^{[p^e]} \subseteq I_e$ and $c(I_e^{[p]}) \subseteq I_{e+1}$ for $e \in \mathbb{N}$; hence, $\lim_{e\to\infty}(1/p^{ed})\ell_R(R/I_e)$ exists as well. Since $I_{t,e} \subseteq I_e$ and so $\ell_R(R/I_e) \leq \ell_R(R/I_{t,e})$ for all $t, e \in \mathbb{N}$, applying $\lim_{e\to\infty}$ gives

$$\lim_{e\to\infty}\frac{1}{p^{ed}}\ell_R(R/I_e) \leq \eta_t \tag{15}$$

for all $t \in \mathbb{N}$.

Since $I_{t,e} \subseteq I_{t+1,e}$ is increasing in t for fixed e, it follows that $\eta_t \geq \eta_{t+1} \geq 0$ for all $t \in \mathbb{N}$ and hence $\lim_{t\to\infty}\eta_t$ exists. We also have $I_e = I_{t_e,e}$ for $t_e \gg 0$, so that $\lim_{e\to\infty}\lim_{t\to\infty}(1/p^{ed})\ell_R(R/I_{t,e}) = \lim_{e\to\infty}(1/p^{ed})\ell_R(R/I_e)$. Applying $\lim_{t\to\infty}$ to (14) and (15) gives

$$\lim_{e\to\infty}\frac{1}{p^{ed}}\ell_R(R/I_e) \leq \lim_{t\to\infty}\eta_t \leq \frac{1}{p^{ed}}\ell_R(R/I_e) + \frac{C}{p^e}$$

for all $e \in \mathbb{N}$. Further taking $\lim_{e\to\infty}$ gives $\lim_{t\to\infty}\eta_t = \lim_{e\to\infty}(1/p^{ed})\ell_R(R/I_e)$ and completes the proof. □

Theorem 6.4. *Let $(R, \mathfrak{m}, k)$ be an approximately Gorenstein F-finite local ring of dimension d. Suppose $\{J_t\}_{t\in\mathbb{N}}$ is a descending chain of irreducible ideals cofinal with the powers of $\mathfrak{m}$, and $\delta_t \in R$ generates the socle of R/J_t. Then $s(R) = \lim_{t\to\infty} e_{\mathrm{HK}}(J_t) - e_{\mathrm{HK}}((J_t, \delta_t))$.*

Proof. Both invariants are unchanged after completion, so we may assume R is complete. Suppose first that R is not weakly F-regular, so that $\delta_t \in J_t^*$ for some $t \in \mathbb{N}$ and $(J_t^{[p^e]} : \delta_t^{p^e}) \subseteq (J_{t+1}^{[p^e]} : \delta_{t+1}^{p^e}) \subseteq \cdots \subseteq (J_{t+t'}^{[p^e]} : \delta_{t+t'}^{p^e}) \subseteq \cdots$ for all $e, t' \in \mathbb{N}$ by Lemma 6.2. If $c \in \bigcap_{e\in\mathbb{N}}(J_t^{[p^e]} : \delta_t^{p^e}) \subseteq \bigcap_{e\in\mathbb{N}}(J_{t+t'}^{[p^e]} : \delta_{t+t'}^{p^e})$ is not in any minimal prime, then we have $0 \leq e_{\mathrm{HK}}(J_{t+t'}) - e_{\mathrm{HK}}((J_{t+t'}, \delta_{t+t'})) \leq \lim_{e\to\infty}(1/p^{ed})\ell_R(R/(c, \mathfrak{m}^{[p^e]})) = 0$ using (11) from Corollary 5.6 and applying Lemma 3.1 with $M = R/(c)$. Using Lemma 6.1, we have that $s(R) = \lim_{t'\to\infty} e_{\mathrm{HK}}(J_{t+t'}) - e_{\mathrm{HK}}((J_{t+t'}, \delta_{t+t'})) = 0$ as desired.

Thus, we assume for the remainder that R is weakly F-regular and hence a domain. Consider the sequences of ideals $I_{t,e} = (J_t^{[p^e]} : \delta_t^{p^e})$ for $t, e \in \mathbb{N}$. We check

$$\begin{aligned} \mathfrak{m}^{[p^e]} &= (J_t : \delta_t)^{[p^e]} \subseteq (J_t^{[p^e]} : \delta_t^{p^e}) = I_{t,e}, \\ I_{t,e}^{[p]} &= (J_t : \delta_t)^{[p^e]} \subseteq (J_t^{[p^e]} : \delta_t^{p^e})^{[p]} \subseteq (J_t^{[p^{e+1}]} : \delta_t^{p^{e+1}}) = I_{t,e+1}, \\ I_{t,e} &= (J_t^{[p^e]} : \delta_t^{p^e}) \subseteq (J_{t+1}^{[p^e]} : \delta_{t+1}^{p^e}) = I_{t+1,e}, \end{aligned}$$

so Theorem 6.3 applies with $c = 1$. Using Lemma 6.2 and (11) from Corollary 5.6, we conclude

$$\begin{aligned} s(R) = \lim_{e\to\infty} \frac{1}{p^{ed}} \ell_R(R/I_e^{F\text{-sig}}) \quad &= \lim_{e\to\infty} \lim_{t\to\infty} \frac{1}{p^{ed}} \ell_R(R/I_{t,e}) \\ = \lim_{t\to\infty} \lim_{e\to\infty} \frac{1}{p^{ed}} \ell_R(R/I_{t,e}) &= \lim_{t\to\infty} e_{\mathrm{HK}}(J_t) - e_{\mathrm{HK}}((J_t, \delta_t)). \end{aligned}$$ □

Theorem 6.4 provides a positive answer to [Watanabe and Yoshida 2004, Question 1.10].

Corollary 6.5. *If $(R, \mathfrak{m}, k)$ is an F-finite local ring, then*

$$s(R) = \inf_{\substack{I\subseteq J\subseteq R,\, \ell_R(R/I)<\infty \\ I\neq J,\, \ell_R(R/J)<\infty}} \frac{e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}(J)}{\ell_R(J/I)} = \inf_{\substack{I\subseteq R,\, \ell_R(R/I)<\infty \\ x\in R,\, (I:x)=\mathfrak{m}}} e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}((I, x)).$$

Proof. If R is not reduced, we can find some $0 \neq x \in R$ with $x^p = 0$. For $n \gg 0$, we have $x \notin \mathfrak{m}^n$ and can find an ideal I with $\mathfrak{m}^n \subseteq I \subseteq (\mathfrak{m}^n, x)$ where $(I : x) = \mathfrak{m}$. Since $x^{p^e} = 0$ for all $e \in \mathbb{N}$, we have $I^{[p^e]} = (I, x)^{[p^e]}$, and thus $e_{\mathrm{HK}}(I) = e_{\mathrm{HK}}((I, x))$. It follows from Lemma 6.1 that $s(R) = 0$ and equality holds throughout (13). Thus, we may assume R is reduced. By [Hochster 1977, Theorem 1.7], R is approximately Gorenstein and Theorem 6.4 implies equality holds throughout (13) as desired. □

When $(R, \mathfrak{m}, k)$ is a complete Cohen–Macaulay local F-finite domain of dimension d, one can make Theorem 6.4 more explicit still. Recall that a canonical ideal $J \subseteq R$ is an ideal such that J is isomorphic to a canonical module ω_R, which exists as R is assumed complete. A canonical ideal J is necessarily unmixed with height 1, and moreover, R/J will be Gorenstein of dimension $d - 1$ [Bruns and Herzog 1993, Proposition 3.3.18]. When R is also normal, fixing a canonical ideal is equivalent to fixing a choice of effective anticanonical divisor.

Corollary 6.6. *Suppose that $(R, \mathfrak{m}, k)$ is a complete Cohen–Macaulay local F-finite domain of dimension d, and J is a canonical ideal of R. Let $x_1 \in J$ and $x_2, \dots, x_d \in R$ be chosen so that $x_1, \dots, x_d$ give a system of parameters for R, and suppose $\delta \in R$ generates the socle of $R/(J, x_2, \dots, x_d)$. Then*

$$s(R) = \lim_{t\to\infty} e_{\mathrm{HK}}((J, x_2^t, \dots, x_d^t)) - e_{\mathrm{HK}}((J, x_2^t, \dots, x_d^t, (x_2^{t-1} \cdots x_d^{t-1}\delta))).$$

Proof. Note that $x_1^{t-1} J$ is yet another canonical ideal for any $t \in \mathbb{N}$, and $x_2^t, \dots, x_d^t$ give a system of parameters for $R/(x_1^{t-1} J)$. Thus, the sequence $J_t = (x_1^{t-1} J, x_2^t, \dots, x_d^t)$ gives a descending chain of irreducible ideals cofinal with the powers of $\mathfrak{m}$. It is easy to check that $\delta_t = x_1^{t-1} x_2^{t-1} \cdots x_d^{t-1}\delta$ generates the socle of R/J_t using that $x_1,\dots,x_d$ form a regular sequence, which further implies $(J_t^{p^e}:\delta_t^{p^e}) = ((J,x_2^t,\dots,x_d^t)^{[p^e]}:(x_2^{t-1}\cdots x_d^{t-1}\delta)^{p^e})$ for all $t,e\in\mathbb{N}$. Thus, Theorem 6.4 gives

$$\begin{aligned} s(R) &= \lim_{t\to\infty}\frac{1}{p^{ed}}\ell_R(R/(J_t^{p^e}:\delta_t^{p^e}))\\ &= \lim_{t\to\infty} e_{\mathrm{HK}}((x_1^{t-1}J,x_2^t,\dots,x_d^t)) - e_{\mathrm{HK}}((x_1^{t-1}J,x_2^t,\dots,x_d^t,(x_1^{t-1}x_2^{t-1}\cdots x_d^{t-1}\delta)))\\ &= \lim_{t\to\infty}\frac{1}{p^{ed}}\ell_R(R/((J,x_2^t,\dots,x_d^t)^{[p^e]}:(x_2^{t-1}\cdots x_d^{t-1}\delta)^{p^e}))\\ &= \lim_{t\to\infty} e_{\mathrm{HK}}((J,x_2^t,\dots,x_d^t)) - e_{\mathrm{HK}}((J,x_2^t,\dots,x_d^t,(x_2^{t-1}\cdots x_d^{t-1}\delta))) \end{aligned}$$

using the relation in (11) once more. □

One can push the above analysis further still. In the notation of the previous proof, when R is normal x_2 can be chosen so that R_{x_2} is Gorenstein and J_{x_2} is principal. This allows one to remove the exponent t on x_2 in the limit above using (i) from the subsequent lemma. Following the methods of [Aberbach 2002] (compare to [Aberbach and Enescu 2006; MacCrimmon 1996; Yao 2006]), we present a complete treatment in Corollary 6.8 below.

Lemma 6.7. *Suppose that $(R,\mathfrak{m},k)$ is a complete Cohen–Macaulay local F-finite normal domain of dimension d, and D an effective Weil divisor on* Spec(R). *Put $J=R(-D)\subseteq R$ so that $J^{(n)}=R(-nD)$ for $n\in\mathbb{N}$. Let $x_1\in J$ and $x_2,\dots,x_d\in R$ be chosen so that $x_1,\dots,x_d$ give a system of parameters for R, and fix $e\in\mathbb{N}$.*

(i) *If $x_2J\subseteq a_2R$ for some $a_2\in J$, there exists $b_2\in J$ so that $a_2,x_2+b_2,x_3,\dots,x_d$ give a system of parameters for R. Moreover, for any nonnegative integers $N_2,\dots,N_d$ with $N_2\geq 2$, we have that*

$$\begin{aligned} ((J^{(p^e)},x_2^{N_2p^e},x_3^{N_3p^e},\dots,x_d^{N_dp^e}):x_2^{(N_2-1)p^e}) &= ((J^{[p^e]},x_2^{N_2p^e},x_3^{N_3p^e},\dots,x_d^{N_dp^e}):x_2^{(N_2-1)p^e})\\ &= ((J^{[p^e]},x_2^{2p^e},x_3^{N_3p^e},\dots,x_d^{N_dp^e}):x_2^{p^e}). \end{aligned}$$

(ii) *Suppose $x_d^nJ^{(n)}\subseteq a_dR$ for some $n\in\mathbb{N}$ and $a_d\in J^{(n)}$. Then there exists $b_d\in J$ such that $a_d,x_2,\dots,x_{d-1},x_d+b_d$ give a system of parameters for R. Moreover, for any nonnegative integers $N_2,\dots,N_d$ with $N_d\geq 2$, we have that*

$$((J^{(p^e)},x_2^{N_2p^e},\dots,x_{d-1}^{N_{d-1}p^e},x_d^{N_dp^e}):x_d^{(N_d-1)p^e})\subseteq((J^{(p^e)},x_2^{N_2p^e},\dots,x_{d-1}^{N_{d-1}p^e},x_d^{2p^e}):x_1^nx_d^{p^e})$$

Proof. For (i), note first a_2 is a nonzero divisor on $R/(x_3,\dots,x_d)$ as its multiple $x_1x_2\in x_2J$ is a nonzero divisor. Hence, $\dim(R/(x_3,\dots,x_d,a_2))=d-1$. Since $\ell_R(R/(J,x_2,\dots,x_d))\leq\ell_R(R/(x_1,\dots,x_d))<\infty$, we know that (x_2,J) is not contained in any minimal prime ideal of $(x_3,\dots,x_d,a_2)$. In particular, using standard prime avoidance arguments, one can find $b_2\in J$ so that x_2+b_2 also avoids all of the minimal primes of $(x_3,\dots,x_d,a_2)$; it follows that the sequence $x_3,\dots,x_d,a_2,x_2+b_2$ is a system of parameters for R.

Since $R(-\operatorname{div}(x_2)-D)=x_2J\subseteq a_2R=R(-\operatorname{div}(a_2))$, we have that $D+\operatorname{div}(x_2)\geq\operatorname{div}(a_2)$ and hence also $p^eD+\operatorname{div}(x_2^{p^e})\geq\operatorname{div}(a_2^{p^e})$; it follows that $x_2^{p^e}J^{(p^e)}\subseteq a_2^{p^e}R$. The ideals inclusions $\supseteq$ in (i) are clear, so we need only check the reverse. Let $I = (x_3^{N_3 p^e}, \dots, x_d^{N_d p^e})$, and suppose that we have $cx_2^{(N_2-1)p^e} \in (J^{(p^e)}, x_2^{N_2 p^e}, I)$ for some $c \in R$. We can find $r_2 \in R$ with $(c - r_2 x_2^{p^e}) x_2^{(N_2-1)p^e} \in (J^{(p^e)}, I)$. Since $b_2 \in J$, we have $b^{p^e} \in J^{(p^e)}$ and so $(c - r_2 x_2^{p^e})(x_2 + b_2)^{(N_2-1)p^e} \in (J^{(p^e)}, I)$. Multiplying through by $x_2^{p^e}$ and using that $x_2^{p^e} J^{(p^e)} \subseteq a_2^{p^e}$ gives $(cx_2^{p^e} - r_2 x_2^{2p^e})(x_2 + b_2)^{(N_2-1)p^e} \in (a_2^{p^e}, I)$. Using that $a_2, x_2 + b_2, x_3, \dots, x_d$ give a system of parameters for R, we see $(cx_2^{p^e} - r_2 x_2^{2p^e}) \in (a_2^{p^e}, I) \subseteq (J^{[p^e]}, I)$ and conclude $cx_2^{p^e} \in (J^{[p^e]}, x_2^{2p^e}, I)$ as desired.

Statement (ii) proceeds similarly. We have that a_d is a nonzero divisor on $R/(x_2, \dots, x_{d-1})$ as its multiple $x_d^n x_1^n$ is a nonzero divisor, and again (x_d, J) is not contained in any minimal prime of $(a_d, x_2, \dots, x_{d-1})$. We can find $b_d \in J$ so that $x_d + b_d$ is not in any minimal prime of $(a_d, x_2, \dots, x_{d-1})$ and thus $a_d, x_2, \dots, x_{d-1}, x_d + b_d$ is a system of parameters for R. From $x_d^n J^{(n)} \subseteq a_d R$, we have $nD + \operatorname{div}(x_d^n) \geq \operatorname{div}(a_d)$ and it follows that $x_d^{p^e} J^{(p^e)} \subseteq a_d^{\lfloor p^e/n \rfloor} R$. To see the final inclusion, let $I = (x_2^{N_2 p^e}, \dots, x_{d-1}^{N_{d-1} p^e})$ and suppose $cx_d^{(N_d-1)p^e} \in (J^{(p^e)}, I, x_d^{N_d p^e})$ for some $c \in R$. We can find $r_d \in R$ so that $(c - r_d x_d^{p^e}) x_d^{(N_d-1)p^e} \in (J^{(p^e)}, I)$. As $b_d \in J$ so $b_d^{p^e} \in J^{(p^e)}$, hence also $(c - r_d x_d^{p^e})(x_d + b_d)^{(N_d-1)p^e} \in (J^{(p^e)}, I)$. Multiplying through by $x_d^{p^e}$ gives $(cx_d^{p^e} - r_d x_d^{2p^e})(x_d + b_d)^{(N_d-1)p^e} \in (a_d^{\lfloor p^e/n \rfloor}, I)$. Using that $a_d, x_2, \dots, x_{d-1}, x_d + b_d$ are a system of parameters, we see that $(cx_d^{p^e} - r_d x_d^{2p^e}) \in (a_d^{\lfloor p^e/n \rfloor}, I)$. Multiplying through by x_1^n and using that $a_d^{\lfloor p^e/n \rfloor} x_1^n \in (J^{(n)})^{\lfloor p^e/n \rfloor + 1} \subseteq J^{(p^e)}$ gives $(cx_1^n x_d^{p^e} - r_d x_1^n x_d^{2p^e}) \in (J^{(p^e)}, I)$ and in particular $cx_1^n x_d^{p^e} \in (J^{(p^e)}, I, x_d^{2p^e})$ as desired. □

Corollary 6.8. *Suppose $(R, \mathfrak{m}, k)$ is a complete Cohen–Macaulay local F-finite domain of dimension d.*

(i) *If R is Gorenstein and $x_1, \dots, x_d$ are any system of parameters for R and $\delta \in R$ generates the socle of $R/(x_1, \dots, x_d)$, then $I_e^{F\text{-sig}} = ((x_1, \dots, x_d)^{[p^e]} : \delta^{p^e})$ for all $e \in \mathbb{N}$ and $s(R) = e_{\mathrm{HK}}((x_1, \dots, x_d)) - e_{\mathrm{HK}}((x_1, \dots, x_d, \delta))$* [Huneke and Leuschke 2002].

(ii) *If the punctured spectrum $\operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$ is $\mathbb{Q}$-Gorenstein, there exists an $\mathfrak{m}$-primary ideal I and $x \in R$ with $(I : x) = \mathfrak{m}$ so that $s(R) = e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}((I, x))$ and moreover $I_e^{F\text{-sig}} \subseteq ((I^{[p^e]})^* : x^{p^e})$ for all $e \in \mathbb{N}$. In particular, if R is weakly F-regular, $I_e^{F\text{-sig}} = (I^{[p^e]} : x^{p^e})$ for all $e \in \mathbb{N}$.*

Proof. (i) Since R is Gorenstein, we may take $J = (x_1)$ to be the canonical ideal above in Corollary 6.6. For fixed $e \in \mathbb{N}$, we have that $I_e^{F\text{-sig}} = ((x_1^{tp^e}, \dots, x_d^{tp^e}) : (x_1^{(t-1)p^e} \cdots x_d^{(t-1)p^e} \delta^{p^e}))$ for all $t \gg 0$. However, since $x_1, \dots, x_d$ are a regular sequence on R, it follows that $((x_1^{tp^e}, \dots, x_d^{tp^e}) : (x_1^{(t-1)p^e} \cdots x_d^{(t-1)p^e} \delta^{p^e})) = ((x_1^{p^e}, \dots, x_d^{p^e}) : \delta^{p^e})$ for any $t \in \mathbb{N}$. The final statement follows again using the relation in (11) once more.

For (ii), we begin with the following construction. Let J be a canonical ideal of R, and choose $0 \neq x_1 \in J$. Let U_2 be the complement of the minimal primes of x_1, which are all height one and contain the set of minimal primes of J. We have that $U_2^{-1} J$ is principal (as $U_2^{-1} R$ is a semilocal Dedekind domain) and can choose $x_2 \in U_2$ and $a_2 \in J$ so that $J_{x_2} = a_2 R_{x_2}$ and $x_2 J \subseteq a_2 R$.

As $R_{\mathfrak{p}}$ is $\mathbb{Q}$-Gorenstein for all $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$, we may choose $n \in \mathbb{N}$ so that $J_{\mathfrak{p}}^{(n)}$ is principal for all $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$. Assuming $x_1, \dots, x_{i-1}$ have already been chosen, let U_i be the complement of the set of minimal primes of $x_1, \dots, x_{i-1}$. Again, $U_i^{-1} J^{(n)}$ is principal (as $U_i^{-1} R$ is a semilocal domain, every locally principal ideal is principal by [Kaplansky 1974, Theorem 60]), so we can choose $x_i \in U_i$ and $a_i \in J^{(n)}$ so that $J_{x_i}^{(n)} = a_i R_{x_i}$ and $x_i^n J^{(n)} \subseteq a_i R$. Continuing in this manner gives a system of parameters

$x_1, \dots, x_d$ of R with $x_1 \in J$, and a sequence of elements $a_3, \dots, a_d \in J^{(n)}$ so that $x_i^n J^{(n)} \subseteq a_i R$ for each $i = 3, \dots, d$. For fixed $e \in \mathbb{N}$ and $t \gg 0$, we have from Corollary 6.6 and repeated application of Lemma 6.7(ii) for $x_d, \dots, x_3$ and then Lemma 6.7(i) for x_2 that

$$\begin{aligned} I_e^{F\text{-sig}} &= ((J^{[p^e]}, x_2^{tp^e}, \dots, x_{d-1}^{tp^e}, x_d^{tp^e}) : (x_2^{(t-1)p^e} \cdots x_{d-1}^{(t-1)p^e} x_d^{(t-1)p^e} \delta^{p^e})) \\ &\subseteq ((J^{(p^e)}, x_2^{tp^e}, \dots, x_{d-1}^{tp^e}, x_d^{tp^e}) : (x_2^{(t-1)p^e} \cdots x_{d-1}^{(t-1)p^e} x_d^{(t-1)p^e} \delta^{p^e})) \\ &\subseteq ((J^{(p^e)}, x_2^{tp^e}, \dots, x_{d-1}^{tp^e}, x_d^{2p^e}) : (x_1^n x_2^{(t-1)p^e} \cdots x_{d-1}^{(t-1)p^e} x_d^{p^e} \delta^{p^e})) \\ &\ \ \vdots \\ &\subseteq ((J^{(p^e)}, x_2^{tp^e}, x_3^{2p^e}, \dots, x_d^{2p^e}) : (x_1^{(d-2)n} x_2^{(t-1)p^e} x_3^{p^e} \cdots x_d^{p^e} \delta^{p^e})) \\ &= ((J^{[p^e]}, x_2^{2p^e}, x_3^{2p^e}, \dots, x_d^{2p^e}) : (x_1^{(d-2)n} x_2^{p^e} x_3^{p^e} \cdots x_d^{p^e} \delta^{p^e})) \\ &= (((J^{[p^e]}, x_2^{2p^e}, \dots, x_d^{2p^e}) : (x_2^{p^e} \cdots x_d^{p^e} \delta^{p^e})) : x_1^{(d-2)n}), \end{aligned} \tag{16}$$

so setting $I = (J, x_2^2, \dots, x_d^2)$ and $x = x_2 \cdots x_d \delta$ gives $s(R) = e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}((I, x))$ using Lemma 4.13 (with $I_e = ((J^{[p^e]}, x_2^{2p^e}, \dots, x_d^{2p^e}) : (x_2^{p^e} \cdots x_d^{p^e} \delta^{p^e}))$, $J_e = I_e^{F\text{-sig}}$, and $c = x_1^{(d-2)n}$).

For the final statement, since $(I_e^{F\text{-sig}})^{[p^{e'}]} \subseteq I_{e+e'}^{F\text{-sig}}$ for all $e, e' \in \mathbb{N}$, it follows from (16) that

$$0 \neq x_1^{(d-2)n} \in \bigcap_{e' \in \mathbb{N}} (I^{[p^{e+e'}]} : (x^{p^e} I_e^{F\text{-sig}})^{[p^{e'}]})$$

so that $x^{p^e} I_e^{F\text{-sig}} \subseteq (I^{[p^e]})^*$ or $I_e^{F\text{-sig}} \subseteq ((I^{[p^e]})^* : x^{p^e})$ as claimed. In particular, if R is weakly F-regular, we have that $I_e^{F\text{-sig}} \subseteq ((I^{[p^e]}) : x^{p^e})$ and equality follows from Lemma 6.1. □

One can also use Theorem 6.3 to show that the length criteria in Corollaries 5.7 and 5.8 also determine the F-signatures in those settings.

Corollary 6.9. *Let $(R, \mathfrak{m}, k)$ be a complete F-finite local domain of dimension d. Let $\mathfrak{a} \subseteq R$ be a nonzero ideal and $\xi \in \mathbb{R}_{\geq 0}$. Suppose that $\{J_t\}_{t \in \mathbb{N}}$ is a descending chain of irreducible ideals cofinal with the powers of $\mathfrak{m}$, and $\delta_t \in R$ generates the socle of R/J_t. Then*

$$s(R, \mathfrak{a}^\xi) = \lim_{t \to \infty} \lim_{e \to \infty} (1/p^{ed}) \ell_R \cdot (R/(J_t^{[p^e]} : \mathfrak{a}^{\lceil \xi(p^e-1) \rceil} \delta_t^{p^e})).$$

Proof. It is straightforward to check for each $e \in \mathbb{N}$ that

$$\begin{aligned} I_e^{\mathfrak{a}^\xi} &= (x \in R \mid \phi(x^{1/p^e}) \in \mathfrak{m} \text{ for all } \phi \in \mathscr{C}_e^{\mathfrak{a}^\xi}) \\ &= (x \in R \mid (xy)^{1/p^e} R \subseteq R^{1/p^e} \text{ is not split for any } y \in \mathfrak{a}^{\lceil \xi(p^e-1) \rceil}), \end{aligned}$$

and it follows that $I_e^{\mathfrak{a}^\xi} = \sum_{t \in \mathbb{N}} I_{t,e}^{\mathfrak{a}^\xi}$ where $I_{t,e}^{\mathfrak{a}^\xi} = (J_t^{[p^e]} : \delta_t^{p^e} \mathfrak{a}^{\lceil (p^e-1)\xi \rceil})$ for $t, e \in \mathbb{N}$. Choosing an element $0 \neq c \in R$ such that $c\mathfrak{a}^{\lceil \xi(p^{e+1}-1) \rceil} \subseteq (\mathfrak{a}^{\lceil \xi(p^e-1) \rceil})^{[p]}$ for all $e \in \mathbb{N}$ as in the proof of Theorem 4.11, we have that $\mathfrak{m}^{[p^e]} \subseteq I_{t,e}^{\mathfrak{a}^\xi}$, $c((I_{t,e}^{\mathfrak{a}^\xi})^{[p]}) \subseteq I_{t,e+1}^{\mathfrak{a}^\xi}$, and $I_{t,e}^{\mathfrak{a}^\xi} \subseteq I_{t+1,e}^{\mathfrak{a}^\xi}$ for all $t, e \in \mathbb{N}$. The result now follows immediately from Theorem 6.3 as we know $s(R, \mathfrak{a}^\xi) = \lim_{e \to \infty} (1/p^{ed}) \ell_R(R/I_e^{\mathfrak{a}^\xi})$. □

Corollary 6.10. *Let $(R, \mathfrak{m}, k)$ be a complete F-finite local normal domain of dimension d. Suppose $\{J_t\}_{t\in\mathbb{N}}$ is a descending chain of irreducible ideals cofinal with the powers of $\mathfrak{m}$, and $\delta_t \in R$ generates the socle of R/J_t. If Δ is an effective $\mathbb{Q}$-divisor on* $\operatorname{Spec}(R)$, *then* $s(R,\Delta) = \lim_{t\to\infty}\lim_{e\to\infty}(1/p^{ed})\ell_R \cdot (R/(J_t^{[p^e]}R(\lceil (p^e-1)\Delta\rceil) : \delta_t^{p^e}))$.

Proof. Again it is straightforward to check for each $e \in \mathbb{N}$ that

$$\begin{aligned} I_e^{(R,\Delta)} &= (x \in R \mid \phi(x^{1/p^e}) \in \mathfrak{m} \text{ for all } \phi \in \mathscr{C}_e^{(R,\Delta)}) \\ &= (x \in R \mid (x)^{1/p^e} R \subseteq (R(\lceil (p^e-1)\Delta\rceil))^{1/p^e} \text{ is not split}), \end{aligned}$$

and it follows that $I_e^{(R,\Delta)} = \sum_{t\in\mathbb{N}} I_{t,e}^{(R,\Delta)}$ where $I_{t,e}^{(R,\Delta)} = (J_t^{[p^e]}R(\lceil (p^e-1)\Delta\rceil) : \delta_t^{p^e})$ for $t, e \in \mathbb{N}$. Choosing an element $0 \neq c \in R$ so that $\operatorname{div}_R(c) \geq p\lceil\Delta\rceil$ as in the proof of Theorem 4.12, we have that $\mathfrak{m}^{[p^e]} \subseteq I_{t,e}^{(R,\Delta)}$, $c((I_{t,e}^{(R,\Delta)})^{[p]}) \subseteq I_{t,e+1}^{(R,\Delta)}$, and $I_{t,e}^{(R,\Delta)} \subseteq I_{t+1,e}^{\mathfrak{a}^{\xi}}$ for all $t, e \in \mathbb{N}$. The result now follows immediately from Theorem 6.3 as we know $s(R,\Delta) = \lim_{e\to\infty}(1/p^{ed})\ell_R(R/I_e^{(R,\Delta)})$. □

Lastly, note that Theorem 6.3 can also be applied outside the context of descending chains of irreducible ideals, and gives yet another perspective on the original proof of the existence of the F-signature.

Corollary 6.11 [Tucker 2012, Proof of Theorem 4.9]. *Let $(R, \mathfrak{m}, k)$ be a complete F-finite local domain of dimension d and $I_e^{F\text{-sig}} = (r \in R \mid \phi(r^{1/p^e}) \in \mathfrak{m}$ for all $\phi \in \operatorname{Hom}_R(R^{1/p^e}, R))$. Then $s(R) = \lim_{t\to\infty}(1/p^{td})e_{\mathrm{HK}}(I_t^{F\text{-sig}})$.*

Proof. Let

$$I_{t,e} = \begin{cases} (I_t^{F\text{-sig}})^{[p^{e-t}]}, & t < e, \\ I_e^{F\text{-sig}}, & t \geq e. \end{cases}$$

It is easily checked that $\mathfrak{m}^{[p^e]} \subseteq I_{t,e}$, $I_{t,e}^{[p]} \subseteq I_{t,e+1}$, and $I_{t,e} \subseteq I_{t+1,e}$ for all $t, e \in \mathbb{N}$ using that $\mathfrak{m}^{[p^e]} \subseteq I_e^{F\text{-sig}}$ and $(I_e^{F\text{-sig}})^{[p]} \subseteq I_{e+1}^{F\text{-sig}}$ for all $e \in \mathbb{N}$. The result now follows immediately from Theorem 6.3, as we have

$$\begin{aligned} s(R) &= \lim_{e\to\infty}\frac{1}{p^{ed}}\ell_R(I_e^{F\text{-sig}}) = \lim_{e\to\infty}\lim_{t\to\infty}\frac{1}{p^{ed}}\ell_R(R/I_{t,e}) = \lim_{t\to\infty}\lim_{e\to\infty}\frac{1}{p^{ed}}\ell_R(R/I_{t,e}) \\ &= \lim_{t\to\infty}\lim_{e\to\infty}\frac{1}{p^{ed}}\ell_R(R/(I_t^{F\text{-sig}})^{[p^{e-t}]}) = \lim_{t\to\infty}\frac{1}{p^{td}}e_{\mathrm{HK}}(I_t^{F\text{-sig}}) \end{aligned}$$

as desired. □

7. Open questions

In this section, we collect together some information on the important questions left unanswered in this article. We have seen that Hilbert–Kunz multiplicity and the F-signature enjoy semicontinuity properties for F-finite rings; it is not difficult to extend these results to rings which are essentially of finite type over an excellent local ring. More generally, however, this raises the following question.

Question 7.1. *If R is an excellent domain that is not F-finite or essentially of finite type over an excellent local ring, do the Hilbert–Kunz multiplicity and F-signature determine semicontinuous $\mathbb{R}$-valued functions on* $\operatorname{Spec}(R)$?

In the case of F-signature, a positive answer would imply the openness of the strongly F-regular locus for such a ring. Note that this question would seem closely related to the existence of (locally and completely stable) test elements for excellent domains, which also remains unanswered.

Perhaps the most important question left open in this article regarding the relationship between Hilbert–Kunz multiplicity and F-signature is the following, which (in light of the results of the previous section) is attributed to Watanabe and Yoshida [2004].

Question 7.2. *Let $(R, \mathfrak{m}, k)$ be a complete local F-finite normal domain. Do there exist $\mathfrak{m}$-primary ideals $I \subsetneq J$ so that $e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}(J) = s(R)$?*

For any ring such that Question 7.2 has a positive answer, it that weak F-regularity is equivalent to strong F-regularity by the length criterion for tight closure [Hochster and Huneke 1994, Theorem 8.17] (Corollary 5.6 above).

We see from Corollary 6.8 that Question 7.2 is true provided R is $\mathbb{Q}$-Gorenstein on the punctured spectrum. When in addition R is weakly F-regular, the stronger condition below is satisfied as well.

Question 7.3. *Let $(R, \mathfrak{m}, k)$ be a complete local F-finite normal domain. Do there exist $\mathfrak{m}$-primary ideals $I \subsetneq J$ so that*

$$\frac{\operatorname{frk}_R(R^{1/p^e})}{[k^{1/p^e} : k]} = \frac{\ell_R(R/I^{[p^e]}) - \ell_R(R/J^{[p^e]})}{\ell_R(J/I)}$$

for all $e \in \mathbb{N}$?

The importance of Question 7.3 stems from the observation that it allows one to apply the results of Huneke, McDermott, and Monsky [Huneke et al. 2004]; for an $\mathfrak{m}$-primary ideal $I \subseteq R$, they show the existence of a constant $\alpha(I) \in \mathbb{R}$ so that $\ell(R/I^{[p^e]}) = e_{\mathrm{HK}}(I)p^{ed} + \alpha(I)p^{e(d-1)} + O(p^{e(d-2)})$. In other words, Hilbert–Kunz functions in normal local F-finite domains are polynomial in p^e to an extra degree. In particular, for any ring such that Question 7.3 has a positive answer, so also does the following question.

Question 7.4. *If $(R, \mathfrak{m}, k)$ is a complete local F-finite normal domain, when does there exist a positive constant $\alpha(R) \in \mathbb{R}$ so that $\operatorname{frk}_R(R^{1/p^e})/[k^{1/p^e} : k] = s(R)p^{ed} + \alpha(R)p^{e(d-1)} + O(p^{e(d-2)})$?*

Finally, we have tried to emphasize the applicability of our techniques to the settings of divisor and ideal pairs throughout, and the questions above are readily generalized to those settings. Moreover, it may well be the case that answers to the questions above require the use of such pairs (particularly to remove Gorenstein or $\mathbb{Q}$-Gorenstein hypotheses). In this direction, and in view of [Schwede 2011], one could imagine a positive and constructive answer to the question below could prove quite useful.

Question 7.5. *Suppose $(R, \mathfrak{m}, k)$ is an F-finite local normal domain and $X = \operatorname{Spec}(R)$. For each $\epsilon > 0$, does there exist an effective $\mathbb{Q}$-divisor Δ on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier with index prime to p and so that $s(R) - s(R, \Delta) < \epsilon$? In case R is not local, can Δ be chosen globally to have this property locally over $\operatorname{Spec}(R)$?*

References

[Aberbach 2002] I. M. Aberbach, "Some conditions for the equivalence of weak and strong F-regularity", *Comm. Algebra* **30**:4 (2002), 1635–1651. MR Zbl

[Aberbach and Enescu 2005] I. M. Aberbach and F. Enescu, "The structure of F-pure rings", *Math. Z.* **250**:4 (2005), 791–806. MR Zbl

[Aberbach and Enescu 2006] I. M. Aberbach and F. Enescu, "When does the F-signature exists?", *Ann. Fac. Sci. Toulouse Math.* (6) **15**:2 (2006), 195–201. MR Zbl

[Aberbach and Enescu 2008] I. M. Aberbach and F. Enescu, "Lower bounds for Hilbert–Kunz multiplicities in local rings of fixed dimension", *Michigan Math. J.* **57** (2008), 1–16. MR Zbl

[Aberbach and Leuschke 2003] I. M. Aberbach and G. J. Leuschke, "The F-signature and strong F-regularity", *Math. Res. Lett.* **10**:1 (2003), 51–56. MR Zbl

[Blickle and Böckle 2011] M. Blickle and G. Böckle, "Cartier modules: finiteness results", *J. Reine Angew. Math.* **661** (2011), 85–123. MR Zbl

[Blickle and Enescu 2004] M. Blickle and F. Enescu, "On rings with small Hilbert–Kunz multiplicity", *Proc. Amer. Math. Soc.* **132**:9 (2004), 2505–2509. MR Zbl

[Blickle and Schwede 2013] M. Blickle and K. Schwede, "p^{-1}-linear maps in algebra and geometry", pp. 123–205 in *Commutative algebra*, edited by I. Peeva, Springer, 2013. MR Zbl

[Blickle et al. 2012] M. Blickle, K. Schwede, and K. Tucker, "F-signature of pairs and the asymptotic behavior of Frobenius splittings", *Adv. Math.* **231**:6 (2012), 3232–3258. MR Zbl

[Blickle et al. 2013] M. Blickle, K. Schwede, and K. Tucker, "F-signature of pairs: continuity, p-fractals and minimal log discrepancies", *J. Lond. Math. Soc.* (2) **87**:3 (2013), 802–818. MR Zbl

[Brenner 2013] H. Brenner, "Irrational Hilbert–Kunz multiplicities", preprint, 2013. arXiv

[Bruns and Herzog 1993] W. Bruns and J. Herzog, *Cohen–Macaulay rings*, Cambridge Studies in Advanced Mathematics **39**, Cambridge University, 1993. MR

[Chevalley 1943] C. Chevalley, "On the theory of local rings", *Ann. of Math.* (2) **44** (1943), 690–708. MR Zbl

[Enescu and Yao 2011] F. Enescu and Y. Yao, "The lower semicontinuity of the Frobenius splitting numbers", *Math. Proc. Cambridge Philos. Soc.* **150**:1 (2011), 35–46. MR Zbl

[Gabber 2004] O. Gabber, "Notes on some t-structures", pp. 711–734 in *Geometric aspects of Dwork theory*, vol. II, edited by A. Adolphson et al., de Gruyter, 2004. MR Zbl

[Grillet 2001] P. A. Grillet, *Commutative semigroups*, Advances in Mathematics (Dordrecht) **2**, Kluwer, 2001. MR Zbl

[Han and Monsky 1993] C. Han and P. Monsky, "Some surprising Hilbert–Kunz functions", *Math. Z.* **214**:1 (1993), 119–135. MR Zbl

[Hara 1998] N. Hara, "A characterization of rational singularities in terms of injectivity of Frobenius maps", *Amer. J. Math.* **120**:5 (1998), 981–996. MR Zbl

[Hara and Watanabe 2002] N. Hara and K.-I. Watanabe, "F-regular and F-pure rings vs. log terminal and log canonical singularities", *J. Algebraic Geom.* **11**:2 (2002), 363–392. MR Zbl

[Hara and Yoshida 2003] N. Hara and K.-I. Yoshida, "A generalization of tight closure and multiplier ideals", *Trans. Amer. Math. Soc.* **355**:8 (2003), 3143–3174. MR Zbl

[Hartshorne and Speiser 1977] R. Hartshorne and R. Speiser, "Local cohomological dimension in characteristic p", *Ann. of Math.* (2) **105**:1 (1977), 45–79. MR Zbl

[Hochster 1977] M. Hochster, "Cyclic purity versus purity in excellent Noetherian rings", *Trans. Amer. Math. Soc.* **231**:2 (1977), 463–488. MR Zbl

[Hochster 2007] M. Hochster, "Foundations of tight closure theorey", lecture notes, University of Michigan, 2007, Available at http://www.math.lsa.umich.edu/~hochster/711F07/fndtc.pdf.

[Hochster and Huneke 1990] M. Hochster and C. Huneke, "Tight closure, invariant theory, and the Briançon–Skoda theorem", *J. Amer. Math. Soc.* **3**:1 (1990), 31–116. MR Zbl

[Hochster and Huneke 1991] M. Hochster and C. Huneke, "Tight closure and elements of small order in integral extensions", *J. Pure Appl. Algebra* **71**:2–3 (1991), 233–247. MR Zbl

[Hochster and Huneke 1994] M. Hochster and C. Huneke, "F-regularity, test elements, and smooth base change", *Trans. Amer. Math. Soc.* **346**:1 (1994), 1–62. MR Zbl

[Hochster and Huneke 2000] M. Hochster and C. Huneke, "Localization and test exponents for tight closure", *Michigan Math. J.* **48** (2000), 305–329. MR Zbl

[Huneke 1996] C. Huneke, *Tight closure and its applications*, CBMS Regional Conference Series in Mathematics **88**, American Mathematical Society, 1996. MR Zbl

[Huneke and Leuschke 2002] C. Huneke and G. J. Leuschke, "Two theorems about maximal Cohen–Macaulay modules", *Math. Ann.* **324**:2 (2002), 391–404. MR Zbl

[Huneke et al. 2004] C. Huneke, M. A. McDermott, and P. Monsky, "Hilbert–Kunz functions for normal rings", *Math. Res. Lett.* **11**:4 (2004), 539–546. MR Zbl

[Kaplansky 1974] I. Kaplansky, *Commutative rings*, revised ed., University of Chicago, 1974. MR Zbl

[Kunz 1969] E. Kunz, "Characterizations of regular local rings for characteristic p", *Amer. J. Math.* **91** (1969), 772–784. MR Zbl

[Kunz 1976] E. Kunz, "On Noetherian rings of characteristic p", *Amer. J. Math.* **98**:4 (1976), 999–1013. MR Zbl

[Kurano and Shimomoto 2015] K. Kurano and K. Shimomoto, "An elementary proof of Cohen–Gabber theorem in the equal characteristic $p > 0$ case", preprint, 2015. To appear in *Tohoku Math. J.* arXiv

[Lyubeznik 1997] G. Lyubeznik, "F-modules: applications to local cohomology and D-modules in characteristic $p > 0$", *J. Reine Angew. Math.* **491** (1997), 65–130. MR

[MacCrimmon 1996] B. C. MacCrimmon, *Strong F-regularity and boundedness questions in tight closure*, Ph.D. thesis, University of Michigan, 1996, Available at https://search.proquest.com/docview/304246031. MR

[Mehta and Srinivas 1997] V. B. Mehta and V. Srinivas, "A characterization of rational singularities", *Asian J. Math.* **1**:2 (1997), 249–271. MR Zbl

[Monsky 1983] P. Monsky, "The Hilbert–Kunz function", *Math. Ann.* **263**:1 (1983), 43–49. MR Zbl

[Polstra 2015] T. Polstra, "Uniform bounds in F-finite rings and lower semi-continuity of the F-signature", preprint, 2015. To appear in *T. Am. Math. Soc.* arXiv

[Schwede 2011] K. Schwede, "Test ideals in non-$\mathbb{Q}$-Gorenstein rings", *Trans. Amer. Math. Soc.* **363**:11 (2011), 5925–5941. MR Zbl

[Schwede and Tucker 2012] K. Schwede and K. Tucker, "A survey of test ideals", pp. 39–99 in *Progress in commutative algebra 2*, edited by C. Francisco et al., de Gruyter, 2012. MR Zbl

[Schwede and Tucker 2014] K. Schwede and K. Tucker, "On the behavior of test ideals under finite morphisms", *J. Algebraic Geom.* **23**:3 (2014), 399–443. MR Zbl

[Schwede and Tucker 2015] K. Schwede and K. Tucker, "Explicitly extending Frobenius splittings over finite maps", *Comm. Algebra* **43**:10 (2015), 4070–4079. MR Zbl

[Smirnov 2016] I. Smirnov, "Upper semi-continuity of the Hilbert–Kunz multiplicity", *Compos. Math.* **152**:3 (2016), 477–488. MR Zbl

[Smith 1997] K. E. Smith, "F-rational rings have rational singularities", *Amer. J. Math.* **119**:1 (1997), 159–180. MR Zbl

[Smith and Van den Bergh 1997] K. E. Smith and M. Van den Bergh, "Simplicity of rings of differential operators in prime characteristic", *Proc. London Math. Soc.* (3) **75**:1 (1997), 32–62. MR Zbl

[Takagi 2004] S. Takagi, "An interpretation of multiplier ideals via tight closure", *J. Algebraic Geom.* **13**:2 (2004), 393–415. MR Zbl

[Tucker 2012] K. Tucker, "F-signature exists", *Invent. Math.* **190**:3 (2012), 743–765. MR Zbl

[Watanabe and Yoshida 2000] K.-i. Watanabe and K.-i. Yoshida, "Hilbert–Kunz multiplicity and an inequality between multiplicity and colength", *J. Algebra* **230**:1 (2000), 295–317. MR Zbl

[Watanabe and Yoshida 2004] K.-i. Watanabe and K.-i. Yoshida, "Minimal relative Hilbert–Kunz multiplicity", *Illinois J. Math.* **48**:1 (2004), 273–294. MR Zbl

[Yao 2006] Y. Yao, "Observations on the F-signature of local rings of characteristic p", *J. Algebra* **299**:1 (2006), 198–218. MR

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