**Graded Steinberg algebras
and their representations**

Pere Ara, Roozbeh Hazrat, Huanhuan Li and Aidan Sims

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We study the category of left unital graded modules over the Steinberg algebra of a graded ample Hausdorff groupoid. In the first part of the paper, we show that this category is isomorphic to the category of unital left modules over the Steinberg algebra of the skew-product groupoid arising from the grading. To do this, we show that the Steinberg algebra of the skew product is graded isomorphic to a natural generalisation of the Cohen–Montgomery smash product of the Steinberg algebra of the underlying groupoid with the grading group. In the second part of the paper, we study the minimal (that is, irreducible) representations in the category of graded modules of a Steinberg algebra, and establish a connection between the annihilator ideals of these minimal representations, and effectiveness of the groupoid.

Specialising our results, we produce a representation of the monoid of graded finitely generated projective modules over a Leavitt path algebra. We deduce that the lattice of order-ideals in the K_0-group of the Leavitt path algebra is isomorphic to the lattice of graded ideals of the algebra. We also investigate the graded monoid for Kumjian–Pask algebras of row-finite k-graphs with no sources. We prove that these algebras are graded von Neumann regular rings, and record some structural consequences of this.

1. Introduction

There has long been a trend of "algebraisation" of concepts from operator theory into algebra. This trend seems to have started with von Neumann and Kaplansky and their students Berberian and Rickart to see what properties in operator algebra theory arise naturally from discrete underlying structures [Kaplansky 1968]. As Berberian [1972] puts it, "if all the functional analysis is stripped away ... what remains should stand firmly as a substantial piece of algebra, completely accessible through algebraic avenues".

In the last decade, Leavitt path algebras [Abrams and Aranda Pino 2005; Ara et al. 2007] were introduced as an algebraisation of graph C^*-algebras [Kumjian et al. 1997; Raeburn 2005] and in particular Cuntz–Krieger algebras. Later, Kumjian–Pask algebras [Aranda Pino et al. 2013] arose as an algebraisation of higher-rank graph C^*-algebras [Kumjian and Pask 2000]. Quite recently Steinberg algebras were introduced in [Steinberg 2010; Clark et al. 2014] as an algebraisation of the groupoid C^*-algebras first studied by Renault [1980]. Groupoid C^*-algebras include all graph C^*-algebras and higher-rank graph C^*-algebras, and Steinberg algebras include Leavitt and Kumjian–Pask algebras as

well as inverse semigroup algebras. More generally, groupoid C^*-algebras provide a model for inverse-semigroup C^*-algebras, and the corresponding inverse-semigroup algebras are the Steinberg algebras of the corresponding groupoids. All of these classes of algebras have been attracting significant attention, with particular interest in whether K-theoretic data can be used to classify various classes of Leavitt path algebras, inspired by the Kirchberg–Phillips classification theorem for C^*-algebras [Phillips 2000].

In this note we study graded representations of Steinberg algebras. For a Γ-graded groupoid $\mathcal{G}$, (i.e., a groupoid $\mathcal{G}$ with a cocycle map $c : \mathcal{G} \to \Gamma$) Renault [1980, Theorem 5.7] proved that if Γ is a discrete abelian group with Pontryagin dual $\widehat{\Gamma}$, then the C^*-algebra $C^*(\mathcal{G} \times_c \Gamma)$ of the skew-product groupoid is isomorphic to a crossed-product C^*-algebra $C^*(\mathcal{G}) \times \widehat{\Gamma}$. Kumjian and Pask [1999] used Renault's results to show that if there is a free action of a group Γ on a graph E, then the crossed product of graph C^*-algebra by the induced action is strongly Morita equivalent to $C^*(E/\Gamma)$, where E/Γ is the quotient graph.

Parallelling Renault's work, we first consider the Steinberg algebras of skew-product groupoids (for arbitrary discrete groups Γ). We extend Cohen and Montgomery's definition of the smash product of a graded ring by the grading group (introduced and studied in their seminal paper [Cohen and Montgomery 1984]) to the setting of nonunital rings. We then prove that the Steinberg algebra of the skew-product groupoid is isomorphic to the corresponding smash product. This allows us to relate the category of graded modules of the algebra to the category of modules of its smash product. Specialising to Leavitt path algebras, the smash product by the integers arising from the canonical grading yields an ultramatricial algebra. This allows us to give a presentation of the monoid of graded finitely generated projective modules for Leavitt path algebras of arbitrary graphs. In particular, we prove that this monoid is cancellative. The group completion of this monoid is called the graded Grothendieck group, K_0^{gr}, which is a crucial invariant in study of Leavitt path algebras. It is conjectured [Hazrat 2016, §3.9] that the graded Grothendieck group is a complete invariant for Leavitt path algebras. We study the lattice of order ideals of K_0^{gr} and establish a lattice isomorphism between order ideals of K_0^{gr} and graded ideals of Leavitt path algebras.

We then apply the smash product to Kumjian–Pask algebras $\mathrm{KP}_K(\Lambda)$. Unlike Leavitt path algebras, Kumjian–Pask algebras of arbitrary higher rank graphs are poorly understood, so we restrict our attention to row finite k-graphs with no sources. We show that the smash product of $\mathrm{KP}_K(\Lambda)$ by $\mathbb{Z}^k$ is also an ultramatricial algebra. This allows us to show that $\mathrm{KP}_K(\Lambda)$ is a graded von Neumann regular ring and, as in the case of Leavitt path algebras, its graded monoid is cancellative. Several very interesting properties of Kumjian–Pask algebras follow as a consequence of general results for graded von Neumann regular rings.

We then proceed with a systematic study of the irreducible representations of Steinberg algebras. Chen [2015] used infinite paths in a graph E to construct an irreducible representation of the Leavitt path algebra E. These representations were further explored in a series of papers [Abrams et al. 2015; Ara and Rangaswamy 2014; 2015; Hazrat and Rangaswamy 2016; Rangaswamy 2016]. The infinite path representations of Kumjian–Pask algebras were also defined in [Aranda Pino et al. 2013]. In the setting of a groupoid $\mathcal{G}$, the infinite path space becomes the unit space of the groupoid. For any invariant subset W of the unit space, the free module RW with basis W is a representation of the Steinberg algebra $A_R(\mathcal{G})$

[Brown et al. 2014]. These representations were used to construct nontrivial ideals of the Steinberg algebra, and ultimately to characterise simplicity.

For the Γ-graded groupoid $\mathcal{G}$, we introduce what we call Γ-aperiodic invariant subsets of the unit space of the groupoid $\mathcal{G}$. We obtain graded (irreducible) representations of the Steinberg algebra via these Γ-aperiodic invariant subsets. We then describe the annihilator ideals of these graded representations and establish a connection between these annihilator ideals and effectiveness of the groupoid. Specialising to the case of Leavitt and Kumjian–Pask algebras we obtain new results about representations of these algebras.

The paper is organised as follows. In Section 2, we recall the background we need on graded ring theory, and then introduce the smash product $A \# \Gamma$ of an arbitrary Γ-graded ring A, possibly without unit. We establish an isomorphism of categories between the category of unital left $A \# \Gamma$-modules and the category of unital left Γ-graded A-modules. This theory is used in Section 3, where we consider the Steinberg algebra associated to a Γ-graded ample groupoid $\mathcal{G}$. We prove that the Steinberg algebra of the skew-product of $\mathcal{G} \times_c \Gamma$ is graded isomorphic to the smash product of $A_R(\mathcal{G})$ with the group Γ.

In Section 4 we collect the facts we need to study the monoid of graded rings with graded local units. In Section 5 and Section 6, we apply the isomorphism of categories in Section 2 and the graded isomorphism of Steinberg algebras (Theorem 3.4) on the setting of Leavitt path algebras and Kumjian–Pask algebras. Although Kumjian–Pask algebras are a generalisation of Leavitt path algebras, we treat these classes separately as we are able to study Leavitt path algebras associated to any arbitrary graph, whereas for Kumjian–Pask algebras we consider only row-finite k-graphs with no sources, as the general case is much more complicated [Raeburn et al. 2004; Sims 2006]. We describe the monoids of graded finitely generated projective modules over Leavitt path algebras and Kumjian–Pask algebras, and obtain a new description of their lattices of graded ideals. In Section 7, we turn our attention to the irreducible representations of Steinberg algebras. We consider what we call Γ-aperiodic invariant subset of the groupoid $\mathcal{G}$ and construct graded simple $A_R(\mathcal{G})$-modules. This covers, as a special case, previous work done in the setting of Leavitt path algebras, and gives new results in the setting of Kumjian–Pask algebras. We describe the annihilator ideals of the graded modules over a Steinberg algebra and prove that these ideals reflect the effectiveness of the groupoid.

2. Graded rings and smash products

2A. *Graded rings.* Let Γ be a group with identity ε. A ring A (possibly without unit) is called a Γ-*graded ring* if $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$ such that each A_γ is an additive subgroup of A and $A_\gamma A_\delta \subseteq A_{\gamma\delta}$ for all $\gamma, \delta \in \Gamma$. The group A_γ is called the γ-*homogeneous component* of A. When it is clear from context that a ring A is graded by group Γ, we simply say that A is a *graded ring*. If A is an algebra over a ring R, then A is called a *graded algebra* if A is a graded ring and A_γ is a R-submodule for any $\gamma \in \Gamma$. A Γ-graded ring $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$ is called *strongly graded* if $A_\gamma A_\delta = A_{\gamma\delta}$ for all γ, δ in Γ.

The elements of $\bigcup_{\gamma \in \Gamma} A_\gamma$ in a graded ring A are called *homogeneous elements* of A. The nonzero elements of A_γ are called *homogeneous of degree* γ and we write $\deg(a) = \gamma$ for $a \in A_\gamma \setminus \{0\}$. The set

$\Gamma_A = \{\gamma \in \Gamma \mid A_\gamma \neq 0\}$ is called the *support* of A. We say that a Γ-graded ring A is *trivially graded* if the support of A is the trivial group $\{\varepsilon\}$ — that is, $A_\varepsilon = A$, so $A_\gamma = 0$ for $\gamma \in \Gamma \setminus \{\varepsilon\}$. Any ring admits a trivial grading by any group. If A is a Γ-graded ring and $s \in A$, then we write $s_\alpha, \alpha \in \Gamma$ for the unique elements $s_\alpha \in A_\alpha$ such that $s = \sum_{\alpha \in \Gamma} s_\alpha$. Note that $\{\alpha \in \Gamma \mid s_\alpha \neq 0\}$ is finite for every $s \in A$.

We say a Γ-graded ring A has *graded local units* if for any finite set of homogeneous elements $\{x_1, \ldots, x_n\} \subseteq A$, there exists a homogeneous idempotent $e \in A$ such that $\{x_1, \ldots, x_n\} \subseteq eAe$. Equivalently, A has graded local units, if A_ε has local units and $A_\varepsilon A_\gamma = A_\gamma A_\varepsilon = A_\gamma$ for every $\gamma \in \Gamma$.

Let M be a left A-module. We say M is unital if $AM = M$ and it is Γ-*graded* if there is a decomposition $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$ such that $A_\alpha M_\gamma \subseteq M_{\alpha\gamma}$ for all $\alpha, \gamma \in \Gamma$. We denote by A-Mod the category of unital left A-modules and by A-Gr the category of Γ-graded unital left A-modules with morphisms the A-module homomorphisms that preserve grading.

For a graded left A-module M, we define the α-*shifted* graded left A-module $M(\alpha)$ as

$$M(\alpha) = \bigoplus_{\gamma \in \Gamma} M(\alpha)_\gamma, \tag{2-1}$$

where $M(\alpha)_\gamma = M_{\gamma\alpha}$. That is, as an ungraded module, $M(\alpha)$ is a copy of M, but the grading is shifted by α. For $\alpha \in \Gamma$, the *shift functor*

$$\mathcal{T}_\alpha : A\text{-Gr} \to A\text{-Gr}, \quad M \mapsto M(\alpha)$$

is an isomorphism with the property $\mathcal{T}_\alpha \mathcal{T}_\beta = \mathcal{T}_{\alpha\beta}$ for $\alpha, \beta \in \Gamma$.

2B. *Smash products.* Let A be a Γ-graded unital R-algebra where Γ is a finite group. In an influential paper, Cohen and Montgomery [1984] introduced the smash product associated to A, denoted by $A \# R[\Gamma]^*$. They proved two main theorems, duality for actions and coactions, which related the smash product to the ring A. In turn, these theorems relate the graded structure of A to nongraded properties of A. The construction has been extended to the case of infinite groups (see for example [Beattie 1988; Liu and Van Oystaeyen 1988; Năstăsescu and Van Oystaeyen 2004, §7]). We need to adopt the construction of smash products for algebras with local units as the main algebras we will be concerned with are Steinberg algebras which are not necessarily unital but have local units. The main theorem of Section 3 shows that the Steinberg algebra of the skew-product of a groupoid by a group can be represented using the smash product construction (Theorem 3.4).

We start with a general definition of smash product for any ring.

Definition 2.1. For a Γ-graded ring A (possibly without unit), the *smash product* ring $A \# \Gamma$ is defined as the set of all formal sums $\sum_{\gamma \in \Gamma} r^{(\gamma)} p_\gamma$, where $r^{(\gamma)} \in A$ and p_γ are symbols. Addition is defined component-wise and multiplication is defined by linear extension of the rule $(rp_\alpha)(sp_\beta) = rs_{\alpha\beta^{-1}} p_\beta$, where $r, s \in A$ and $\alpha, \beta \in \Gamma$.

It is routine to check that $A \# \Gamma$ is a ring. We emphasise that the symbols p_γ do not belong to $A \# \Gamma$; however if the ring A has unit, then we regard the p_γ as elements of $A \# \Gamma$ by identifying $1_A p_\gamma$ with p_γ.

Each p_γ is then an idempotent element of $A \# \Gamma$. In this case $A \# \Gamma$ coincides with the ring $A \# \Gamma^*$ of [Beattie 1988]. If Γ is finite, then $A \# \Gamma$ is the same as the smash product $A \# k[\Gamma]^*$ of [Cohen and Montgomery 1984]. Note that $A \# \Gamma$ is always a Γ-graded ring with

$$(A \# \Gamma)_\gamma = \sum_{\alpha \in \Gamma} A_\gamma \, p_\alpha. \tag{2-2}$$

Next we define a shift functor on $A \# \Gamma$-Mod. This functor will coincide with the shift functor on A-Gr (see Proposition 2.5). This does not seem to be exploited in the literature and will be crucial in our study of K-theory of Leavitt path algebras (Section 5C).

For each $\alpha \in \Gamma$, there is an algebra automorphism

$$\mathcal{S}^\alpha : A \# \Gamma \to A \# \Gamma,$$

such that $\mathcal{S}^\alpha(sp_\beta) = sp_{\beta\alpha}$, for $sp_\beta \in A \# \Gamma$ with $s \in A$ and $\beta \in \Gamma$. We sometimes call $\mathcal{S}^\alpha$ the *shift map* associated to α. For $M \in A \# \Gamma$-Mod and $\alpha \in \Gamma$, we obtain a shifted $A \# \Gamma$-module $\mathcal{S}_*^\alpha M$ obtained by setting $\mathcal{S}_*^\alpha M := M$ as a group, and defining the left action by $a \cdot_{\mathcal{S}_*^\alpha M} m := \mathcal{S}^\alpha(a) \cdot_M m$. For $\alpha \in \Gamma$, the *shift functor*

$$\tilde{\mathcal{S}}^\alpha : A \# \Gamma\text{-Mod} \to A \# \Gamma\text{-Mod}, \quad M \mapsto \mathcal{S}_*^\alpha M,$$

is an isomorphism satisfying $\tilde{\mathcal{S}}_\alpha \tilde{\mathcal{S}}_\beta = \tilde{\mathcal{S}}_{\alpha\beta}$ for $\alpha, \beta \in \Gamma$.

If A is a unital ring then $A \# \Gamma$ has local units [Beattie 1988, Proposition 2.3]. We extend this to rings with graded local units.

Lemma 2.2. *Let A be a Γ-graded ring with graded local units. Then the ring $A \# \Gamma$ has graded local units.*

Proof. Take a finite subset $X = \{x_1, x_2, \ldots, x_n\} \subseteq A \# \Gamma$ such that all x_i are homogeneous elements. Since homogeneous elements of $A \# \Gamma$ are sums of elements of the form rp_α for $r \in A$ a homogeneous element and $\alpha \in \Gamma$, we may assume that $x_i = r_i \, p_{\alpha_i}$, $1 \leq i \leq n$, where $r_i \in A$ are homogeneous of degree γ_i and $\alpha_i \in \Gamma$. Since A has graded local units, there exists a homogeneous idempotent $e \in A$ such that $er_i = r_i e = r_i$ for all i. Consider the finite set

$$Y = \big\{ \gamma \in \Gamma \mid \gamma = \alpha_i \ \text{ or } \ \gamma = \gamma_i \alpha_i \quad \text{for } 1 \leq i \leq n \big\},$$

and let $w = \sum_{\gamma \in Y} ep_\gamma$. Since the idempotent $e \in A$ is homogeneous, w is a homogeneous element of $A \# \Gamma$. It is easy now to check that $w^2 = w$ and $wx_i = x_i = x_i w$ for all i. □

As we will see in Sections 5 and 6, smash products of Leavitt path algebras or of Kumjian–Pask algebras are ultramatricial algebras, which are very well-behaved. This allows us to obtain results about the path algebras via their smash product. For example, ultramatricial algebras are von Neumann regular rings. The following lemma allows us to exploit this property (see Theorems 6.4 and 6.5). Recall that a graded ring is called *graded von Neumann regular* if for any homogeneous element a, there is an element b such that $aba = a$.

Lemma 2.3. *Let A be a Γ-graded ring (possibly without unit). Then $A \# \Gamma$ is graded von Neumann regular if and only if A is graded von Neumann regular.*

Proof. Suppose $A \# \Gamma$ is graded regular and $a \in A_\gamma$, for some $\gamma \in G$. Since $ap_e \in (A \# \Gamma)_\gamma$ (see (2-2)), there is an element $\sum_{\alpha \in \Gamma} b_{\gamma_\alpha} p_\alpha \in (A \# \Gamma)_{\gamma^{-1}}$ with $\deg(b_{\gamma_\alpha}) = \gamma^{-1}$, $\alpha \in \Gamma$, such that

$$ap_e \left(\sum_{\alpha \in \Gamma} b_{\gamma_\alpha} p_\alpha \right) ap_e = ap_e.$$

This identity reduces to $ab_{\gamma_\gamma} ap_e = ap_e$. Thus $ab_{\gamma_\gamma} a = a$. This shows that A is graded regular.

Conversely, suppose A is graded regular and $x := \sum_{\alpha \in \Gamma} a_{\gamma_\alpha} p_\alpha \in (A \# \Gamma)_\gamma$. By (2-2) we have $\deg(a_{\gamma_\alpha}) = \gamma$, $\alpha \in \Gamma$. Then there are $b_{\gamma_\alpha^{-1}} \in A_{\gamma^{-1}}$ such that $a_{\gamma_\alpha} b_{\gamma_\alpha^{-1}} a_{\gamma_\alpha} = a_{\gamma_\alpha}$, for $\alpha \in \Gamma$. Consider the element $y := \sum_{\alpha \in \Gamma} b_{\gamma_\alpha^{-1}} p_{\gamma \alpha} \in (A \# \Gamma)_{\gamma^{-1}}$. One can then check that $xyx = x$. Thus $A \# \Gamma$ is graded regular. $\quad\square$

2C. *An isomorphism of module categories.* In this section we first prove that, for a Γ-graded ring A with graded local units, there is an isomorphism between the categories $A \# \Gamma$-Mod and A-Gr (Proposition 2.5). This is a generalisation of [Cohen and Montgomery 1984, Theorem 2.2; Beattie 1988, Theorem 2.6]. We check that the isomorphism respects the shifting in these categories. This in turn translates the shifting of modules in the category of graded modules to an action of the group on the category of modules for the smash-product. Since graded Steinberg algebras have graded local units, using this result and Theorem 3.4, we obtain a shift preserving isomorphism

$$A_R(\mathcal{G} \times_c \Gamma)\text{-Mod} \cong A_R(\mathcal{G})\text{-Gr}.$$

In Section 5 we will use this in the setting of Leavitt path algebras to establish an isomorphism between the category of graded modules of $L_R(E)$ and the category of modules of $L_R(\bar{E})$, where $\bar{E}$ is the covering graph of E (Section 5B). This yields a presentation of the monoid of graded finitely generated projective modules of a Leavitt path algebra.

We start with the following fact, which extends [Beattie 1988, Corollary 2.4] to rings with local units.

Lemma 2.4. *Let A be a Γ-graded ring with a set of graded local units E. A left $A \# \Gamma$-module M is unital if and only if for every finite subset F of M, there exists $w = \sum_{i=1}^{n} up_{\gamma_i}$ with $\gamma_i \in \Gamma$, and $u \in E$ such that $wx = x$ for all $x \in F$.*

Proof. Suppose that M is unital. Then each $m \in F$ may be written as $m = \sum_{n \in G_m} y_n n$ for some finite $G_m \subseteq M$ and choice of scalars $\{y_n \mid n \in G_m\} \subseteq A \# \Gamma$. Let $T := \bigcup_{m \in F} G_m$. By Lemma 2.2, there exists a finite set Y of Γ such that $w = \sum_{\gamma \in Y} up_\gamma$ satisfies $wy = y$ for all $y \in T$. So $wm = m$ for all $m \in F$.

Conversely, for $m \in M$, take $F = \{m\}$. Then there exists w such that $m = wm \in (A \# \Gamma)M$; that is, $(A \# \Gamma)M = M$. $\quad\square$

Proposition 2.5. *Let A be a Γ-graded ring with graded local units. Then there is an isomorphism of categories $\psi : A\text{-Gr} \to A \# \Gamma\text{-Mod}$ such that the following diagram commutes for every $\alpha \in \Gamma$.*

$$A\text{-Gr} \xrightarrow{\;\psi\;} A \# \Gamma\text{-Mod}$$

$$\left\downarrow \mathcal{T}_\alpha \qquad\qquad \left\downarrow \tilde{\mathcal{S}}^\alpha \right.\right.$$ (2-3)

$$A\text{-Gr} \xrightarrow{\;\psi\;} A \# \Gamma\text{-Mod}$$

Proof. We first define a functor $\phi : A \# \Gamma\text{-Mod} \to A\text{-Gr}$ as follows. Fix a set E of graded local units for A. Let M be a unital left $A \# \Gamma$-module. We view M as a Γ-graded left A-module M' as follows. For each $\gamma \in \Gamma$, define

$$M'_\gamma := \sum_{u \in E} u p_\gamma M.$$

We first show that for $\alpha \in \Gamma$, we have $M'_\alpha \cap \sum_{\gamma \in \Gamma, \gamma \neq \alpha} M'_\gamma = \{0\}$. Suppose this is not the case, so there exist finite index sets F and $\{F'_\gamma \mid \gamma \in \Gamma\}$ (only finitely many nonempty), elements $\{u_i \mid i \in F\}$ and $\{v_{\gamma,j} \mid \gamma \in \Gamma \text{ and } j \in F'_\gamma\}$ in E, and elements $\{m_i \mid i \in F\}$ and $\{n_{\gamma,j} \mid \gamma \in \Gamma \text{ and } j \in F'_\gamma\}$ such that

$$x = \sum_{i \in F} u_i p_\alpha m_i = \sum_{\gamma \in \Gamma, \gamma \neq \alpha} \sum_{j \in F'_\gamma} v_{\gamma,j} p_\gamma n_{\gamma,j},$$

Fix $e \in E$ such that $e u_i = u_i = u_i e$ for all $i \in F$. Using that the u_i are homogeneous elements of trivial degree at the second equality, we have

$$e p_\alpha x = \sum_{i \in F} (e p_\alpha u_i p_\alpha) m_i = \sum_{i \in F} e u_i p_\alpha m_i = x.$$

We also have

$$e p_\alpha x = \sum_{\gamma \in \Gamma \setminus \{\alpha\}} \sum_{j \in F'_\gamma} e p_\alpha v_{\gamma,j} p_\gamma n_{\gamma,j} = 0.$$

Hence $x = 0$.

For $r \in A_\gamma$ and $m \in M'_\alpha$, define $rm := r p_\alpha m$. This determines a left A-action on M'_α. For $u \in E$ satisfying $ur = r = ru$, we have

$$u p_{\gamma\alpha} rm = (u p_{\gamma\alpha} r p_\alpha) m = u r p_\alpha m = rm.$$

Hence $rm \in M'_{\gamma\alpha}$. One can easily check the associativity of the A-action. Using Lemma 2.4 we see that $M = M'$ as sets. We claim that M' is a unital A-module. For $m \in M'_\gamma$, we write $m = \sum_{u \in E'} u p_\gamma m_u$, where $E' \subseteq E$ is a finite set and $m_u \in M$. Since u is a homogeneous idempotent,

$$u(u p_\gamma m_u) = u p_\gamma (u p_\gamma m_u) = u p_\gamma m_u.$$

Thus $u(u p_\gamma m_u) = u p_\gamma m_u \in AM'$ implies that $m \in AM'$ showing that $M' = AM'$. We can therefore define

$$\phi : \mathrm{Obj}(A \# \Gamma\text{-Mod}) \to \mathrm{Obj}(A\text{-Gr}),$$

by $\phi(M) = M'$.

To define ϕ on morphisms, fix a morphism f in $A \# \Gamma$-Mod. For $m = \sum_{\gamma \in \Gamma} m_\gamma \in M'$ such that $m_\gamma = \sum_{u \in F_\gamma} u p_\gamma m_u$ with F_γ a finite subset of E, we define $f' : M' \to N'$ by

$$f'(m_\gamma) = f\left(\sum_{u \in F_\gamma} u p_\gamma m_u\right) = \sum_{u \in F_\gamma} u p_\gamma f(m_u) = f(m)_\gamma. \tag{2-4}$$

To see that f' is an A-module homomorphism, fix $m \in M'_\gamma$ and $r \in A$. Since $f(m) \in M'_\gamma$, we have

$$f'(rm) = f(r p_\gamma m) = r p_\gamma f(m) = r f'(m).$$

The definition (2-4) shows that it preserves the gradings. That is, f' is a Γ-graded A-module homomorphism. So we can define ϕ on morphisms by $\phi(f) = f'$. It is routine to check that ϕ is a functor.

Next we define a functor $\psi : A\text{-Gr} \to A \# \Gamma$-Mod as follows. Let $N = \bigoplus_{\gamma \in \Gamma} N_\gamma$ be a Γ-graded unital left A-module. Let N'' be a copy of N as a group. Fix $n \in N$, and write $n = \sum_{\gamma \in \Gamma} n_\gamma$. Fix $r \in A$ and $\alpha \in \Gamma$, and define

$$(r p_\alpha) n = r n_\alpha.$$

It is straightforward to check that this determines an associative left $A \# \Gamma$-action on N''. We claim that N'' is a unital $A \# \Gamma$-module. To see this, fix $n \in N''$. Since $AN = N$, we can express $n = \sum_{i=1}^{l} r_i n_i$, with the n_i homogeneous in N and the $r_i \in A$, and we can then write each r_i as $r_i = \sum_{\beta \in \Gamma} r_{i,\beta}$ as a sum of homogeneous elements $r_{i,\beta} \in A_\beta$. For any $\gamma \in \Gamma$,

$$n_\gamma = \sum_{i,\beta} r_{i,\beta}(n_i)_{\beta^{-1}\gamma} = \sum_{i,\beta}^{l} (r_{i,\beta} p_{\beta^{-1}\gamma}) n_i \in (A \# \Gamma) N''.$$

So we can define

$$\psi : \text{Obj}(A\text{-Gr}) \to \text{Obj}(A \# \Gamma\text{-Mod}),$$

by $\psi(N) = N''$. Since $\psi(N) = N''$ is just a copy of N as a module, we can define ψ on morphisms simply as the identity map; that is, if $f : M \to N$ is a homomorphism of graded A-modules, then for $m \in M$ we write m'' for the same element regarded as an element of M'', and we have $\psi(f)(m'') = f(m)''$. Again, it is straightforward to check that ψ is a functor.

To prove that $\psi \circ \phi = \text{Id}_{A\#\Gamma\text{-Mod}}$ and $\phi \circ \psi = \text{Id}_{A\text{-Gr}}$, it suffices to show that $(M')'' = M$ for $M \in A\#\Gamma$-Mod and $(N'')' = N$ for $N \in A$-Gr; but this is straightforward from the definitions.

To prove the commutativity of the diagram in (2-3), it suffices to show that the $A \# \Gamma$-actions on $(\psi \circ \mathcal{T}_\alpha)(N) = N(\alpha)''$ and $(\tilde{\mathcal{S}}^\alpha \circ \psi)(N) = N''(\alpha)$ coincide for any $N \in A$-Gr. Take any $n \in N$ and $s p_\beta \in A \# \Gamma$ with $s \in A$ and $\beta \in \Gamma$. For $n \in N''(\alpha)$ and a typical spanning element $s p_\beta$ of $A \# \Gamma$, we have $(s p_\beta) n = (s p_{\beta\alpha}) n = s n_{\beta\alpha}$. On the other hand, for the same n regarded as an element of N'', and the same $s p_\beta \in A\#\Gamma$, we have $(s p_\beta) n = s n'_\beta = s n_{\beta\alpha}$. Since $N(\alpha)_\beta = N_{\beta\alpha}$ by definition, this completes the proof. $\square$

3. The Steinberg algebra of the skew-product

In this section, we consider the skew-product of an ample groupoid $\mathcal{G}$ carrying a grading by a discrete group Γ. We prove that the Steinberg algebra of the skew-product is graded isomorphic to the smash product by Γ of the Steinberg algebra associated to $\mathcal{G}$. This result will be used in Section 5 to study the category of graded modules over Leavitt path algebras and give a representation of the graded finitely generated projective modules.

3A. *Graded groupoids.* A groupoid is a small category in which every morphism is invertible. It can also be viewed as a generalisation of a group which has partial binary operation. Let $\mathcal{G}$ be a groupoid. If $x \in \mathcal{G}$, $d(x) = x^{-1}x$ is the *domain* of x and $r(x) = xx^{-1}$ is its *range*. The pair (x, y) is composable if and only if $r(y) = d(x)$. The set $\mathcal{G}^{(0)} := d(\mathcal{G}) = r(\mathcal{G})$ is called the *unit space* of $\mathcal{G}$. Elements of $\mathcal{G}^{(0)}$ are units in the sense that $xd(x) = x$ and $r(x)x = x$ for all $x \in \mathcal{G}$. For $U, V \in \mathcal{G}$, we define

$$UV = \big\{ \alpha\beta \mid \alpha \in U, \ \beta \in V \text{ and } r(\beta) = d(\alpha) \big\}.$$

A topological groupoid is a groupoid endowed with a topology under which the inverse map is continuous, and such that composition is continuous with respect to the relative topology on $\mathcal{G}^{(2)} :=$ $\{(\beta, \gamma) \in \mathcal{G} \times \mathcal{G} \mid d(\beta) = r(\gamma)\}$ inherited from $\mathcal{G} \times \mathcal{G}$. An *étale* groupoid is a topological groupoid $\mathcal{G}$ such that the domain map d is a local homeomorphism. In this case, the range map r is also a local homeomorphism. An *open bisection* of $\mathcal{G}$ is an open subset $U \subseteq \mathcal{G}$ such that $d|_U$ and $r|_U$ are homeomorphisms onto an open subset of $\mathcal{G}^{(0)}$. We say that an étale groupoid $\mathcal{G}$ is *ample* if there is a basis consisting of compact open bisections for its topology.

Let Γ be a discrete group and $\mathcal{G}$ a topological groupoid. A Γ-grading of $\mathcal{G}$ is a continuous function $c : \mathcal{G} \to \Gamma$ such that $c(\alpha)c(\beta) = c(\alpha\beta)$ for all $(\alpha, \beta) \in \mathcal{G}^{(2)}$; such a function c is called a *cocycle* on $\mathcal{G}$. In this paper, we shall also refer to c as the *degree map* on $\mathcal{G}$. Observe that $\mathcal{G}$ decomposes as a topological disjoint union $\bigsqcup_{\gamma \in \Gamma} c^{-1}(\gamma)$ of subsets satisfying $c^{-1}(\beta)c^{-1}(\gamma) \subseteq c^{-1}(\beta\gamma)$. We say that $\mathcal{G}$ is *strongly graded* if $c^{-1}(\beta)c^{-1}(\gamma) = c^{-1}(\beta\gamma)$ for all β, γ. For $\gamma \in \Gamma$, we say that $X \subseteq \mathcal{G}$ is γ-graded if $X \subseteq c^{-1}(\gamma)$. We always have $\mathcal{G}^{(0)} \subseteq c^{-1}(\varepsilon)$, so $\mathcal{G}^{(0)}$ is ε-graded. We write $B^{\mathrm{co}}_\gamma(\mathcal{G})$ for the collection of all γ-graded compact open bisections of $\mathcal{G}$ and

$$B^{\mathrm{co}}_*(\mathcal{G}) = \bigcup_{\gamma \in \Gamma} B^{\mathrm{co}}_\gamma(\mathcal{G}).$$

Throughout this note we only consider Γ-graded ample Hausdorff groupoids.

3B. *Steinberg algebras.* Steinberg algebras were introduced in [Steinberg 2010] in the context of discrete inverse semigroup algebras and independently in [Clark et al. 2014] as a model for Leavitt path algebras. We recall the notion of the Steinberg algebra as a universal algebra generated by certain compact open subsets of an ample Hausdorff groupoid.

Definition 3.1. Let $\mathcal{G}$ be a Γ-graded ample Hausdorff groupoid and $B^{\mathrm{co}}_*(\mathcal{G}) = \bigcup_{\gamma \in \Gamma} B^{\mathrm{co}}_\gamma(\mathcal{G})$ the collection of all graded compact open bisections. Given a commutative ring R with identity, the *Steinberg R-algebra*

associated to $\mathcal{G}$, denoted $A_R(\mathcal{G})$, is the algebra generated by the set $\{t_B \mid B \in B_*^{\mathrm{co}}(\mathcal{G})\}$ with coefficients in R, subject to

(R1) $t_\varnothing = 0$;

(R2) $t_B t_D = t_{BD}$ for all $B, D \in B_*^{\mathrm{co}}(\mathcal{G})$; and

(R3) $t_B + t_D = t_{B \cup D}$, whenever B and D are disjoint elements of $B_\gamma^{\mathrm{co}}(\mathcal{G})$ for some $\gamma \in \Gamma$ such that $B \cup D$ is a bisection.

Every element $f \in A_R(\mathcal{G})$ can be expressed as $f = \sum_{U \in F} a_U t_U$, where F is a finite subset of elements of $B_*^{\mathrm{co}}(\mathcal{G})$. It was proved in [Clark and Edie-Michell 2015, Proposition 2.3] (see also [Clark et al. 2014, Theorem 3.10]) that the Steinberg algebra defined above is isomorphic to the following construction:

$$A_R(\mathcal{G}) = \mathrm{span}\{1_U \mid U \text{ is a compact open bisection of } \mathcal{G}\},$$

where $1_U : \mathcal{G} \to R$ denotes the characteristic function on U. Equivalently, if we give R the discrete topology, then continuous functions from $\mathcal{G}$ to R are exactly locally constant functions from $\mathcal{G}$ to R, and so $A_R(\mathcal{G}) = C_c(\mathcal{G}, R)$, the space of compactly supported continuous functions from $\mathcal{G}$ to R. Addition is point-wise and multiplication is given by convolution

$$(f * g)(\gamma) = \sum_{\{\alpha\beta = \gamma\}} f(\alpha) g(\beta).$$

It is useful to note that

$$1_U * 1_V = 1_{UV}$$

for compact open bisections U and V (see [Steinberg 2010, Proposition 4.5(3)]) and the isomorphism between the two constructions is given by $t_U \mapsto 1_U$ on the generators. By [Clark and Edie-Michell 2015, Lemma 2.2; Clark et al. 2014, Lemma 3.5], every element $f \in A_R(\mathcal{G})$ can be expressed as

$$f = \sum_{U \in F} a_U 1_U, \tag{3-1}$$

where F is a finite subset of mutually disjoint elements of $B_*^{\mathrm{co}}(\mathcal{G})$.

Recall from [Clark and Sims 2015, Lemma 3.1] that if $c : \mathcal{G} \to \Gamma$ is a cocycle into a discrete group Γ, then the Steinberg algebra $A_R(\mathcal{G})$ is a Γ-graded algebra with homogeneous components

$$A_R(\mathcal{G})_\gamma = \{f \in A_R(\mathcal{G}) \mid \mathrm{supp}(f) \subseteq c^{-1}(\gamma)\}.$$

The family of all idempotent elements of $A_R(\mathcal{G}^{(0)})$ is a set of local units for $A_R(\mathcal{G})$ ([Clark et al. 2017, Lemma 2.6]). Here, $A_R(\mathcal{G}^{(0)}) \subseteq A_R(\mathcal{G})$ is a subalgebra. Since $\mathcal{G}^{(0)} \subseteq c^{-1}(\varepsilon)$ is trivially graded, $A_R(\mathcal{G})$ is a graded algebra with graded local units. Note that any ample Hausdorff groupoid admits the trivial cocycle from $\mathcal{G}$ to the trivial group $\{\varepsilon\}$, which gives rise to a trivial grading on $A_R(\mathcal{G})$.

3C. *Skew-products.* Let $\mathcal{G}$ be an ample Hausdorff groupoid, Γ a discrete group, and $c : \mathcal{G} \to \Gamma$ a cocycle. Then $\mathcal{G}$ admits a basis $\mathcal{B}$ of compact open bisections. Replacing $\mathcal{B}$ with $\mathcal{B}' = \{U \cap c^{-1}(\gamma) \mid U \in \mathcal{B}, \gamma \in \Gamma\}$, we obtain a basis of compact open homogeneous bisections.

To a Γ-graded groupoid $\mathcal{G}$ one can associate a groupoid called the skew-product of $\mathcal{G}$ by Γ. The aim of this section is to relate the Steinberg algebra of the skew-product groupoid to the Steinberg algebra of $\mathcal{G}$. We recall the notion of skew-product of a groupoid (see [Renault 1980, Definition 1.6]).

Definition 3.2. Let $\mathcal{G}$ be an ample Hausdorff groupoid, Γ a discrete group and $c : \mathcal{G} \to \Gamma$ a cocycle. The *skew-product* of $\mathcal{G}$ by Γ is the groupoid $\mathcal{G} \times_c \Gamma$ such that (x, α) and (y, β) are composable if x and y are composable and $\alpha = c(y)\beta$. The composition is then given by $(x, c(y)\beta)(y, \beta) = (xy, \beta)$ with the inverse $(x, \alpha)^{-1} = (x^{-1}, c(x)\alpha)$.

Note that our convention for the composition of the skew-product here is slightly different from that in [Renault 1980, Definition 1.6]. The two determine isomorphic groupoids, but when we establish the isomorphism of Theorem 3.4, the composition formula given here will be more obviously compatible with the multiplication in the smash product.

Lemma 3.3. *Let $\mathcal{G}$ be a Γ-graded ample groupoid. Then the skew-product $\mathcal{G} \times_c \Gamma$ is a Γ-graded ample groupoid under the product topology on $\mathcal{G} \times \Gamma$ and with degree map $\tilde{c}(x, \gamma) := c(x)$.*

Proof. We can directly check that under the product topology on $\mathcal{G} \times \Gamma$, the inverse and composition of the skew-product $\mathcal{G} \times_c \Gamma$ are continuous making it a topological groupoid. Since the domain map $d : \mathcal{G} \to \mathcal{G}^{(0)}$ is a local homeomorphism, the domain map (also denoted d) from $\mathcal{G} \times_c \Gamma$ to $\mathcal{G}^{(0)} \times \Gamma$ is $d \times \mathrm{id}_\Gamma$ so restricts to a homeomorphism on $U \times \Gamma$ for any set U on which d is a homeomorphism. So $d : \mathcal{G} \times_c \Gamma \to (\mathcal{G} \times_c \Gamma)^{(0)}$ is a local homeomorphism. Since the inverse map is clearly a homeomorphism, it follows that the range map is also a local homeomorphism.

If $\mathcal{B}$ is a basis of compact open bisections for $\mathcal{G}$, then $\{B \times \{\gamma\} \mid B \in \mathcal{B} \text{ and } \gamma \in \Gamma\}$ is a basis of compact open bisections for the topology on $\mathcal{G} \times_c \Gamma$. Since composition on $\mathcal{G} \times_c \Gamma$ agrees with composition in $\mathcal{G}$ in the first coordinate, it is clear that $\tilde{c}$ is a cocycle. $\square$

The Steinberg algebra $A_R(\mathcal{G} \times_c \Gamma)$ associated to $\mathcal{G} \times_c \Gamma$ is a Γ-graded algebra, with homogeneous components

$$A_R(\mathcal{G} \times_c \Gamma)_\gamma = \big\{ f \in A_R(\mathcal{G} \times_c \Gamma) \mid \mathrm{supp}(f) \subseteq c^{-1}(\gamma) \times \Gamma \big\},$$

for $\gamma \in \Gamma$.

We are in a position to state the main result of this section.

Theorem 3.4. *Let $\mathcal{G}$ be a Γ-graded ample, Hausdorff groupoid and R a unital commutative ring. Then there is an isomorphism of Γ-graded algebras $A_R(\mathcal{G} \times_c \Gamma) \cong A_R(\mathcal{G}) \# \Gamma$, assigning $1_{U \times \{\alpha\}}$ to $1_U p_\alpha$ for each compact open bisection U of $\mathcal{G}$ and $\alpha \in \Gamma$.*

Proof. We first define a representation $\{t_U \mid U \in B^{co}_*(\mathcal{G} \times_c \Gamma)\}$ in the algebra $A_R(\mathcal{G}) \# \Gamma$ (see Definition 3.1). If U is a graded compact open bisection of $\mathcal{G} \times_c \Gamma$, say $U \subseteq \tilde{c}^{-1}(\alpha)$, then for each $\gamma \in \Gamma$, the set $U \cap \mathcal{G} \times \{\gamma\}$ is a compact open bisection. Since these are mutually disjoint and U is compact, there are

finitely many (distinct) $\gamma_1, \ldots, \gamma_l \in \Gamma$ such that $U = \bigsqcup_{i=1}^l U \cap \mathcal{G} \times \{\gamma_i\}$. Each $U \cap \mathcal{G} \times \{\gamma_i\}$ has the form $U_i \times \{\gamma_i\}$ where $U_i \subseteq U$ is compact open. The U_i have mutually disjoint sources because the domain map on $\mathcal{G} \times_c \Gamma$ is just $d \times \mathrm{id}$, and U is a bisection. So each $U_i \in B_\alpha^{\mathrm{co}}(\mathcal{G})$, and $U = \bigsqcup_{i=1}^l U_i \times \{\gamma_i\}$. Using this decomposition, we define

$$t_U = \sum_{i=1}^l 1_{U_i} p_{\gamma_i}.$$

We show that these elements t_U satisfy (R1)–(R3). Certainly if $U = \varnothing$, then $t_U = 0$, giving (R1). For (R2), take $V \in B_\beta^{\mathrm{co}}(\mathcal{G} \times_c \Gamma)$, and decompose $V = \bigcup_{j=1}^m V_j \times \{\gamma_j'\}$ as above. Then

$$t_U t_V = \sum_{i=1}^l 1_{U_i} p_{\gamma_i} \sum_{j=1}^m 1_{V_j} p_{\gamma_j'} = \sum_{i=1}^l \sum_{j=1}^m 1_{U_i} p_{\gamma_i} 1_{V_j} p_{\gamma_j'} = \sum_{j=1}^m \sum_{i=1}^l 1_{U_i} p_{\gamma_i} 1_{V_j} p_{\gamma_j'}$$

$$= \sum_{j=1}^m \sum_{\{1 \le i \le l \,|\, \gamma_i (\gamma_j')^{-1} = \beta\}} 1_{U_i V_j} p_{\gamma_j'}. \tag{3-2}$$

On the other hand, by the composition of the skew-product $\mathcal{G} \times_c \Gamma$, we have

$$UV = \bigcup_{i=1}^l \bigcup_{j=1}^m U_i \times \{\gamma_i\} \cdot V_j \times \{\gamma_j'\} = \bigcup_{j=1}^m \bigcup_{i=1}^l U_i \times \{\gamma_i\} \cdot V_j \times \{\gamma_j'\}$$

$$= \bigcup_{j=1}^m \bigcup_{\{1 \le i \le l \,|\, \gamma_i = \beta \gamma_j'\}} U_i V_j \times \{\gamma_j'\}.$$

For each $1 \le j \le m$, there exists at most one $1 \le i \le l$ such that $\gamma_i = \beta \gamma_j'$ and $U_i V_j \in B_{\alpha\beta}^{\mathrm{co}}(\mathcal{G})$. It follows that $t_{UV} = \sum_{j=1}^m \sum_{\{1 \le i \le l \,|\, \gamma_i (\gamma_j')^{-1} = \beta\}} 1_{U_i V_j} p_{\gamma_j'}$. Comparing this with (3-2), we obtain $t_U t_V = t_{UV}$.

To check (R3), suppose that U and V are disjoint elements of $B_\omega^{\mathrm{co}}(\mathcal{G} \times_c \Gamma)$ for some $\omega \in \Gamma$ such that $U \cup V$ is a bisection of $\mathcal{G} \times_c \Gamma$. Write them as $U = \bigcup_{i=1}^l U_i \times \{\gamma_i\}$ and $V = \bigcup_{j=1}^m V_i \times \{\gamma_j'\}$ as above. We have

$$t_U + t_V = \sum_{i=1}^l 1_{U_i} p_{\gamma_i} + \sum_{j=1}^m 1_{V_j} p_{\gamma_j'}.$$

On the other hand

$$U \cup V = \left(\bigcup_{i=1}^l U_i \times \{\gamma_i\} \right) \cup \left(\bigcup_{j=1}^m V_i \times \{\gamma_j'\} \right).$$

If $\gamma_i = \gamma_j'$, then $U_i \times \{\gamma_i\} \cup V_j \times \{\gamma_j'\} = (U_i \cup V_j) \times \{\gamma_i\}$. Since U and V are disjoint and $U \cup V$ is a bisection, we deduce that $r(U_i) \cap r(V_j) = \varnothing = d(U_i) \cap d(V_j)$ so that $U_i \cup V_j$ is a bisection. So

$$t_{U_i \times \{\gamma_i\} \cup V_j \times \{\gamma_j'\}} = t_{U_i \cup V_j \times \{\gamma_i\}} = 1_{U_i \cup V_j} p_{\gamma_i} = 1_{U_i} p_{\gamma_i} + 1_{V_j} p_{\gamma_i} = 1_{U_i} p_{\gamma_i} + 1_{V_j} p_{\gamma_j'}.$$

This shows that after combining pairs where $\gamma_i = \gamma_j'$ as above, we obtain $t_U + t_V = t_{U \cup V}$.

By the universality of Steinberg algebras, we have an R-homomorphism,

$$\phi : A_R(\mathcal{G} \times_c \Gamma) \to A_R(\mathcal{G}) \# \Gamma$$

such that $\phi(1_{U \times \{\alpha\}}) = 1_U p_\alpha$ for each compact open bisection U of $\mathcal{G}$ and $\alpha \in \Gamma$. From the definition of ϕ, it is evident that ϕ preserves the grading. Hence, ϕ is a homomorphism of Γ-graded algebras.

Next we prove that ϕ is an isomorphism. For any element $a p_\gamma \in A_R(\mathcal{G}) \# \Gamma$ with $a \in A_R(\mathcal{G})$ and $\gamma \in \Gamma$, there is a finite index set T, elements $\{r_i \mid i \in T\}$ of R, and compact open bisections $K_i \in B_*^{\mathrm{co}}(\mathcal{G})$ such that

$$a p_\gamma = \sum_{i \in T} r_i 1_{K_i} p_\gamma = \sum_{i \in T} r_i \phi(1_{K_i \times \{\gamma\}}) \in \operatorname{Im} \phi.$$

So ϕ is surjective. It remains to prove that ϕ is injective. Take an element $x \in A_R(\mathcal{G} \times_c \Gamma)$ such that $\phi(x) = 0$. Since ϕ is graded, we can assume that x is homogeneous, say $x \in A_R(\mathcal{G} \times_c \Gamma)_\gamma$. By (3-1), there is a finite set F, mutually disjoint $B_i \in B_\gamma^{\mathrm{co}}(\mathcal{G} \times_c \Gamma)$ indexed by $i \in F$ and coefficients $r_i \in R$ indexed by $i \in F$ such that

$$x = \sum_{i \in F} r_i 1_{B_i}.$$

For each B_i, we write $B_i = \bigcup_{k \in F_i} B_{ik} \times \{\delta_{ik}\}$ such that F_i is a finite set and the δ_{ik} indexed by $k \in F_i$ are distinct. Set

$$\Delta = \{\delta_{ik} \mid i \in F, k \in F_i\}.$$

For each $\delta \in \Delta$, let $F_\delta \subseteq F$ be the collection $F_\delta = \{i \in F \mid \delta \in \{\delta_{ik} \mid k \in F_i\}\}$. Then

$$\phi(x) = \sum_{i \in F} r_i \phi(1_{B_i}) = \sum_{i \in F} \sum_{k \in F_i} r_i 1_{B_{ik}} p_{\delta_{ik}} = \sum_{\delta \in \Delta} \sum_{i \in F_\delta} r_i 1_{B_{i,k(\delta)}} p_\delta = 0.$$

For any $\delta \in \Delta$, we obtain $\sum_{i \in F_\delta} r_i 1_{B_{i,k(\delta)}} = 0$. Since the B_i are mutually disjoint, for any element $g \in \mathcal{G}$, we have

$$\left(\sum_{i \in F_\delta} r_i 1_{B_{i,k(\delta)}} \right)(g) = \begin{cases} r_i & \text{if } g \in B_{i,k(\delta)} \text{ for some } i \in F_\delta, \\ 0 & \text{otherwise.} \end{cases}$$

Then $r_i = 0$ for any $i \in F_\delta$, giving $x = 0$. $\qquad \square$

3D. C^*-algebras and crossed-products. In the groupoid-C^*-algebra literature, it is well-known that if $\mathcal{G}$ is a Γ-graded groupoid, and Γ is abelian, then the C^*-algebra $C^*(\mathcal{G} \times \Gamma)$ of the skew-product groupoid is isomorphic to the crossed product C^*-algebra $C^*(\mathcal{G}) \times_{\alpha^c} \widehat{\Gamma}$, where α^c is the action of the Pontryagin dual $\widehat{\Gamma}$ such that $\alpha_\chi^c(f)(g) = \chi(c(g)) f(g)$ for $f \in C_c(\mathcal{G})$, $\chi \in \widehat{\Gamma}$, and $g \in \mathcal{G}$. This extends to nonabelian Γ via the theory of C^*-algebraic coactions.

In this subsection, we reconcile this result with Theorem 3.4 by showing that there is a natural embedding of $A_{\mathbb{C}}(\mathcal{G}) \# \Gamma$ into $C^*(\mathcal{G}) \times_{\alpha^c} \widehat{\Gamma}$ when Γ is abelian.

Lemma 3.5. *Suppose that Γ is a discrete abelian group and that $\mathcal{G}$ is a Γ-graded groupoid with grading cocycle $c : \mathcal{G} \to \Gamma$. For $a \in A_{\mathbb{C}}(\mathcal{G})$ and $\gamma \in \Gamma$, define $a \cdot \hat{\gamma} \in C(\widehat{\Gamma}, C^*(\mathcal{G})) \subseteq C^*(\mathcal{G}) \times_{\alpha^c} \widehat{\Gamma}$ by*

$$(a \cdot \hat{\gamma})(\chi) = \chi(\gamma)a.$$

Then there is a homomorphism $A_{\mathbb{C}}(\mathcal{G}) \# \Gamma \hookrightarrow C^*(\mathcal{G}) \times_{\alpha^c} \widehat{\Gamma}$ that carries ap_γ to $a \cdot \hat{\gamma}$.

Proof. The multiplication in the crossed-product C^*-algebra is given on elements of $C(\widehat{\Gamma}, C^*(\mathcal{G}))$ by $(F * G)(\chi) = \int_{\widehat{\Gamma}} F(\rho) \alpha_\rho^c (G(\rho^{-1}\chi)) \, d\mu(\rho)$, where μ is Haar measure on $\widehat{\Gamma}$.

The action of $\widehat{\Gamma}$ induces a Γ-grading of $C^*(\mathcal{G}) \times_{\alpha^c} \widehat{\Gamma}$ such that for $a \in C^*(\mathcal{G}) \times_{\alpha^c} \widehat{\Gamma}$ and $\gamma \in \Gamma$, the corresponding homogeneous component a_γ of a is given by

$$a_\gamma = \int_{\widehat{\Gamma}} \overline{\chi(\gamma)} \alpha_\chi^c(a) \, d\mu(\chi).$$

There is certainly a linear map $i : A_{\mathbb{C}}(\mathcal{G}) \# \Gamma \to C^*(\mathcal{G}) \times_{\alpha^c} \widehat{\Gamma}$ satisfying $i(ap_\gamma) = a \cdot \hat{\gamma}$; we just have to check that it is multiplicative. For this, fix $a, b \in A_{\mathbb{C}}(\mathcal{G})$ and $\gamma, \beta \in \Gamma$ and $\chi \in \widehat{\Gamma}$, and calculate

$$\left(i(ap_\gamma)i(bp_\beta)\right)(\chi) = \int_{\widehat{\Gamma}} i(ap_\gamma)(\rho) \alpha_\rho^c (i(bp_\beta)(\rho^{-1}\chi)) \, d\mu(\rho) = \int_{\widehat{\Gamma}} a \cdot \hat{\gamma}(\rho) \alpha_\rho^c (b\hat{\beta}(\rho^{-1}\chi)) \, d\mu(\rho)$$

$$= \int_{\widehat{\Gamma}} \rho(\gamma) a (\rho^{-1}\chi)(\beta) \alpha_\rho^c(b) \, d\mu(\rho) = \chi(\beta) a \int_{\widehat{\Gamma}} \overline{\rho(\gamma^{-1}\beta)} \alpha_\rho^c(b) \, d\mu(\rho)$$

$$= \chi(\beta) ab_{\gamma^{-1}\beta} = (ab_{\gamma^{-1}\beta}) \cdot \hat{\beta} = i(ab_{\gamma^{-1}\beta} p_\beta) = i(ap_\gamma bp_\beta).$$

So i is multiplicative as required. $\qquad\square$

4. Nonstable graded K-theory

For a unital ring A, we denote by $\mathcal{V}(A)$ the abelian monoid of isomorphism classes of finitely generated projective left A-modules under direct sum. In general for an abelian monoid M and elements $x, y \in M$, we write $x \leq y$ if $y = x + z$ for some $z \in M$. An element $d \in M$ is called *distinguished* (or an *order unit*) if for any $x \in M$, we have $x \leq nd$ for some $n \in \mathbb{N}$. A monoid is called *conical*, if $x + y = 0$ implies $x = y = 0$. Clearly $\mathcal{V}(A)$ is conical with a distinguished element $[A]$. For a finitely generated conical abelian monoid M containing a distinguished element d, Bergman constructed a "universal" K-algebra B (here K is a field) for which there is an isomorphism $\phi : \mathcal{V}(B) \to M$, such that $\phi([B]) \to d$ ([Bergman 1974, Theorem 6.2]).

For a (finite) directed graph E, one defines an abelian monoid M_E generated by the vertices, identifying a vertex with the sum of vertices connected to it by edges (see Section 5C). The Bergman universal algebra associated to this monoid (with the sum of vertices as a distinguished element) is the Leavitt path algebra $L_K(E)$ associated to the graph E, i.e., $\mathcal{V}(L_K(E)) \cong M_E$. Leavitt path algebras of directed graphs have been studied intensively since their introduction [Abrams and Aranda Pino 2005; Ara et al. 2007]. The classification of such algebras is still a major open topic and one would like to find a complete invariant for such algebras. Due to the success of K-theory in the classification of graph C^*-algebras [Phillips 2000], one would hope that the Grothendieck group K_0 with relevant ingredients might act as a complete

invariant for Leavitt path algebras; particularly since $K_0(L_K(E))$ is the group completion of $\mathcal{V}(L_K(E))$. However, unless the graph consists of only cycles with no exit, $\mathcal{V}(L_K(E))$ is not a cancellative monoid (Lemma 5.5) and thus $\mathcal{V}(L_K(E)) \to K_0(L_K(E))$ is not injective, reflecting that K_0 might not capture all the properties of $L_K(E)$.

For a graded ring A one can consider the abelian monoid of isomorphism classes of graded finitely generated projective modules denoted by $\mathcal{V}^{\mathrm{gr}}(A)$. Since a Leavitt path algebra has a canonical $\mathbb{Z}$-graded structure, one can consider $\mathcal{V}^{\mathrm{gr}}(L_K(E))$. One of the main aims of this paper is to show that the graded monoid carries substantial information about the algebra.

In Sections 5 and 6 we will use the results on smash products obtained in Section 3 to study the graded monoid of Leavitt path algebras and Kumjian–Pask algebras. In this section we collect the facts we need on the graded monoid of a graded ring with graded local units.

4A. *The monoid of a graded ring with graded local units.* For a ring A with unit, the monoid $\mathcal{V}(A)$ is defined as the set of isomorphism classes $[P]$ of finitely generated projective A-modules P, with addition given by $[P] + [Q] = [P \oplus Q]$.

For a nonunital ring A, we consider a unital ring $\tilde{A}$ containing A as a two-sided ideal and define

$$\mathcal{V}(A) = \big\{[P] \mid P \text{ is a finitely generated projective } \tilde{A}\text{-module and } P = AP\big\}. \tag{4-1}$$

This construction does not depend on the choice of $\tilde{A}$, as can be seen from the following alternative description: $\mathcal{V}(A)$ is the set of equivalence classes of idempotents in $M_\infty(A)$, where $e \sim f$ in $M_\infty(A)$ if and only if there are $x, y \in M_\infty(A)$ such that $e = xy$ and $f = yx$ ([Menal and Moncasi 1987, p. 296]).

When A has local units,

$$\mathcal{V}(A) = \big\{[P] \mid P \text{ is a finitely generated projective } A\text{-module in } A\text{-Mod}\big\}. \tag{4-2}$$

To see this, recall that the *unitisation ring* $\tilde{A}$ of a ring A is a copy of $\mathbb{Z} \times A$ with componentwise addition, and with multiplication given by

$$(n, a)(m, b) = (nm, ma + nb + ab) \quad \text{for all } n, m \in \mathbb{Z} \text{ and } a, b \in A.$$

The forgetful functor provides a category isomorphism from $\tilde{A}$-Mod to the category of arbitrary left A-modules [Faith 1973, Proposition 8.29B]. Any A-module N can be viewed as a $\tilde{A}$-module via $(m, b)x = mx + bx$ for $(m, b) \in \tilde{A}$ and $x \in N$. By [Ara and Goodearl 2012, Lemma 10.2], the projective objects in A-Mod are precisely those which are projective as $\tilde{A}$-modules; that is, the projective $\tilde{A}$-modules P such that $AP = P$. Any finitely generated $\tilde{A}$-module M with $AM = M$ is a finitely generated A-module. In fact, suppose that M is generated as an $\tilde{A}$-module by $x_1, \ldots, x_n$. Since $AM = M$, each x_i can be written as $x_i = \sum_{j=1}^{t_i} b_j x_{ij}$ for some $b_j \in A$ and $x_{ij} \in M$. Now any $m \in M$ can be written

$$m = \sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} \sum_{j=1}^{t_i} a_i b_j x_{ij}$$

So $\{x_{ij} \mid 1 \le i \le n \text{ and } 1 \le j \le t_i\}$ generates M as an A-module. Clearly any finitely generated A-module is a finitely generated $\tilde{A}$-module. So the definitions of $\mathcal{V}(A)$ in (4-1) and (4-2) coincide.

We need a graded version of (4-2) as this presentation will be used to study the monoid associated to the Leavitt path algebras of arbitrary graphs.

Recall that for a group Γ and a Γ-graded ring A with unit, the monoid $\mathcal{V}^{\mathrm{gr}}(A)$ consists of isomorphism classes $[P]$ of graded finitely generated projective A-modules with the direct sum $[P] + [Q] = [P \oplus Q]$ as the addition operation.

For a nonunital graded ring A, a similar construction as in (4-1) can be carried over to the graded setting (see [Hazrat 2016, §3.5]). Let $\tilde{A}$ be a Γ-graded ring with identity such that A is a graded two-sided ideal of A. For example, consider $\tilde{A} = \mathbb{Z} \times A$. Then $\tilde{A}$ is Γ-graded with

$$\tilde{A}_0 = \mathbb{Z} \times A_0, \qquad \text{and} \qquad \tilde{A}_\gamma = 0 \times A_\gamma \quad \text{for } \gamma \ne 0.$$

Define

$$\mathcal{V}^{\mathrm{gr}}(A) = \big\{ [P] \mid P \text{ is a graded finitely generated projective } \tilde{A}\text{-module and } AP = P \big\}, \qquad (4\text{-}3)$$

where $[P]$ is the class of graded $\tilde{A}$-modules, graded isomorphic to P, and addition is defined via direct sum. Then $\mathcal{V}^{\mathrm{gr}}(A)$ is isomorphic to the monoid of equivalence classes of graded idempotent matrices over A [Hazrat 2016, p. 146].

Let A be a Γ-graded ring with graded local units. We will show that

$$\mathcal{V}^{\mathrm{gr}}(A) = \big\{ [P] \mid P \text{ is a graded finitely generated projective } A\text{-module in } A\text{-Gr} \big\}. \qquad (4\text{-}4)$$

For this we need to relate the graded projective modules to modules which are projective. A graded A-module P in A-Gr is called a graded projective A-module if for any epimorphism $\pi : M \to N$ of graded A-modules in A-Gr and any morphism $f : P \to N$ of graded A-modules in A-Gr, there exists a morphism $h : P \to M$ of graded A-modules such that $\pi \circ h = f$.

In the case of unital rings, a module is graded projective if and only if it is graded and projective [Hazrat 2016, Proposition 1.2.15]. We need a similar statement in the setting of rings with local units.

Lemma 4.1. *Let A be a Γ-graded ring with graded local units and P a graded unital left A-module. Then P is a graded projective left A-module in A-Gr if and only if P is a graded left A-module which is projective in A-Mod.*

Proof. First suppose that P is a graded projective A-module in A-Gr. It suffices to prove that P is projective in A-Mod. For any homogeneous element $p \in P$ of degree δ_p, there exists a homogeneous idempotent $e_p \in A$ such that $e_p p = p$. Let $\bigoplus_{p \in P^h} A e_p(-\delta_p)$ be the direct sum of graded A-modules where $\deg(e_p) = \delta_p$ and P^h is the set of homogeneous elements of P. Then there exists a surjective graded A-module homomorphism

$$f : \bigoplus_{p \in P^h} A e_p(-\delta_p) \to P$$

such that $f(ae_p) = ae_p p = ap$ for $a \in Ae_p$. Since P is graded projective, there exists a graded A-module homomorphism $g : P \to \bigoplus_{p \in P^h} Ae_p(-\delta_p)$ such that $fg = \mathrm{Id}_P$. Forgetting the grading, P is a direct summand of $\bigoplus_{p \in P^h} Ae_p$ as an A-module. By [Wisbauer 1991, §49.2(3)], $\bigoplus_{p \in P^h} Ae_p$ is projective in A-Mod. So P is projective in A-Mod.

Conversely, suppose that P is a graded and projective A-module. Let $\pi : M \to N$ be an epimorphism of graded A-modules in A-Gr and $f : P \to N$ a morphism of graded A-modules in A-Gr. We first claim that any epimorphism $\pi : M \to N$ of graded A-modules in A-Gr is surjective. To prove the claim, write A^h for the set of all homogeneous elements of A. Let $X = \{x \in N \mid A^h x \subseteq \pi(M)\} \subseteq N$ (compare [Goodearl 2009, §5.3]). Then X is a graded submodule of N. We denote by $q : N \to N/X$ the quotient map. Then $q \circ \pi = 0$. Hence, $q = 0$, giving $N = X$. It follows that $N = \pi(M)$. So the epimorphism $\pi : M \to N$ of graded A-modules in A-Gr is surjective. Forgetting the grading, $\pi : M \to N$ is a surjective morphism of A-modules in A-Mod. Since P is projective in A-Mod, there exists $h : P \to M$ such that $\pi \circ h = f$. By [Hazrat 2016, Lemma 1.2.14], there exists a morphism $h' : P \to M$ of graded A-modules such that $\pi \circ h' = f$. Thus, P is a graded projective left A-module in A-Gr. $\square$

Thus for a Γ-graded ring A with graded local units, combining Lemma 4.1 with [Ara and Goodearl 2012, Lemma 10.2] (i.e., projective objects in A-Mod are precisely those that are projective as $\tilde{A}$-modules), P is a graded finitely generated projective $\tilde{A}$-module with $AP = P$ if and only if P is a finitely generated A-module which is graded projective in A-Gr. This shows that the definitions of $\mathcal{V}^{\mathrm{gr}}(A)$ by (4-3) and (4-4) coincide.

5. Application: Leavitt path algebras

In this section we study the monoid $\mathcal{V}^{\mathrm{gr}}(L_K(E))$ of the Leavitt path algebra of a graph E (4-4). Using the results on smash products of Steinberg algebras obtained in Section 3, we give a presentation for this monoid in line with M_E (see Section 5C). Using this presentation we show that $\mathcal{V}^{\mathrm{gr}}(L_K(E))$ is a cancellative monoid and thus the natural map $\mathcal{V}^{\mathrm{gr}}(L_K(E)) \to K_0^{\mathrm{gr}}(L_K(E))$ is injective (Corollary 5.8). It follows that there is a lattice correspondence between the graded ideals of $L_K(E)$ and the graded ordered ideals of $K_0^{\mathrm{gr}}(L_K(E))$ (Theorem 5.11). This is further evidence for the conjecture that the graded Grothendieck group K_0^{gr} may be a complete invariant for Leavitt path algebras [Hazrat 2013b].

5A. *Leavitt path algebras modelled as Steinberg algebras.* We briefly recall the definition of Leavitt path algebras and establish notation.

A directed graph E is a tuple (E^0, E^1, r, s), where E^0 and E^1 are sets and r, s are maps from E^1 to E^0. We think of each $e \in E^1$ as an arrow pointing from $s(e)$ to $r(e)$. We use the convention that a (finite) path p in E is a sequence $p = \alpha_1 \alpha_2 \cdots \alpha_n$ of edges α_i in E such that $r(\alpha_i) = s(\alpha_{i+1})$ for $1 \le i \le n-1$. We define $s(p) = s(\alpha_1)$, and $r(p) = r(\alpha_n)$. If $s(p) = r(p)$, then p is said to be closed. If p is closed and $s(\alpha_i) \ne s(\alpha_j)$ for $i \ne j$, then p is called a cycle. An edge α is an exit of a path $p = \alpha_1 \cdots \alpha_n$ if there exists i such that $s(\alpha) = s(\alpha_i)$ and $\alpha \ne \alpha_i$. A graph E is called acyclic if there is no closed path in E.

A directed graph E is said to be *row-finite* if for each vertex $u \in E^0$, there are at most finitely many edges in $s^{-1}(u)$. A vertex u for which $s^{-1}(u)$ is empty is called a *sink*, whereas $u \in E^0$ is called an *infinite emitter* if $s^{-1}(u)$ is infinite. If $u \in E^0$ is neither a sink nor an infinite emitter, then it is called a *regular vertex*.

Definition 5.1. Let E be a directed graph and R a commutative ring with unit. The *Leavitt path algebra* $L_R(E)$ of E is the R-algebra generated by the set $\{v \mid v \in E^0\} \cup \{e \mid e \in E^1\} \cup \{e^* \mid e \in E^1\}$ subject to the following relations:

(1) $uv = \delta_{u,v} v$ for every $u, v \in E^0$;

(2) $s(e)e = er(e) = e$ for all $e \in E^1$;

(3) $r(e)e^* = e^* = e^* s(e)$ for all $e \in E^1$;

(4) $e^* f = \delta_{e,f} r(e)$ for all $e, f \in E^1$; and

(5) $v = \sum_{e \in s^{-1}(v)} ee^*$ for every regular vertex $v \in E^0$.

Let Γ be a group with identity ε, and let $w : E^1 \to \Gamma$ be a function. Extend w to vertices and ghost edges by defining $w(v) = \varepsilon$ for $v \in E^0$ and $w(e^*) = w(e)^{-1}$ for $e \in E^1$. The relations in Definition 5.1 are compatible with w, so there is a Γ-grading on $L_R(E)$ such that $e \in L_R(E)_{w(e)}$ and $e^* \in L_R(E)_{w(e)^{-1}}$, for all $e \in E^1$, and $v \in L_R(E)_\varepsilon$, for all $v \in E^0$. The set of all finite sums of distinct elements of E^0 is a set of graded local units for $L_R(E)$ [Abrams and Aranda Pino 2005, Lemma 1.6]. Furthermore, $L_R(E)$ is unital if and only if E^0 is finite.

Leavitt path algebras associated to arbitrary graphs can be realised as Steinberg algebras. We recall from [Clark and Sims 2015, Example 2.1] the construction of the groupoid $\mathcal{G}_E$ from an arbitrary graph E, which was introduced in [Kumjian et al. 1997] for row-finite graphs and generalised to arbitrary graphs in [Paterson 2002]. We then realise the Leavitt path algebra $L_R(E)$ as the Steinberg algebra $A_R(\mathcal{G})$. This allows us to apply Theorem 3.4 to the setting of Leavitt path algebras.

Let $E = (E^0, E^1, r, s)$ be a directed graph. We denote by E^∞ the set of infinite paths in E and by E^* the set of finite paths in E. Set

$$X := E^\infty \cup \{\mu \in E^* \mid r(\mu) \text{ is not a regular vertex}\}.$$

Let

$$\mathcal{G}_E := \{(\alpha x, |\alpha| - |\beta|, \beta x) \mid \alpha, \beta \in E^*, x \in X, r(\alpha) = r(\beta) = s(x)\}.$$

We view each $(x, k, y) \in \mathcal{G}_E$ as a morphism with range x and source y. The formulas $(x, k, y)(y, l, z) = (x, k+l, z)$ and $(x, k, y)^{-1} = (y, -k, x)$ define composition and inverse maps on $\mathcal{G}_E$ making it a groupoid with $\mathcal{G}_E^{(0)} = \{(x, 0, x) \mid x \in X\}$ which we identify with the set X.

Next, we describe a topology on $\mathcal{G}_E$. For $\mu \in E^*$ define

$$Z(\mu) = \{\mu x \mid x \in X, r(\mu) = s(x)\} \subseteq X.$$

For $\mu \in E^*$ and a finite $F \subseteq s^{-1}(r(\mu))$, define

$$Z(\mu \setminus F) = Z(\mu) \setminus \bigcup_{\alpha \in F} Z(\mu\alpha).$$

The sets $Z(\mu \setminus F)$ constitute a basis of compact open sets for a locally compact Hausdorff topology on $X = \mathcal{G}_E^{(0)}$ (see [Webster 2014, Theorem 2.1]).

For $\mu, \nu \in E^*$ with $r(\mu) = r(\nu)$, and for a finite $F \subseteq E^*$ such that $r(\mu) = s(\alpha)$ for $\alpha \in F$, we define

$$Z(\mu, \nu) = \{(\mu x, |\mu| - |\nu|, \nu x) \mid x \in X, r(\mu) = s(x)\},$$

and then

$$Z((\mu, \nu) \setminus F) = Z(\mu, \nu) \setminus \bigcup_{\alpha \in F} Z(\mu\alpha, \nu\alpha).$$

The sets $Z((\mu, \nu) \setminus F)$ constitute a basis of compact open bisections for a topology under which $\mathcal{G}_E$ is a Hausdorff ample groupoid. By [Clark and Sims 2015, Example 3.2], the map

$$\pi_E : L_R(E) \to A_R(\mathcal{G}_E) \tag{5-1}$$

defined by $\pi_E\big(\mu\nu^* - \sum_{\alpha \in F} \mu\alpha\alpha^*\nu^*\big) = 1_{Z((\mu, \nu) \setminus F)}$ extends to a $\mathbb{Z}$-graded algebra isomorphism. We observe that the isomorphism of algebras in (5-1) satisfies

$$\pi_E(v) = 1_{Z(v)}, \quad \pi_E(e) = 1_{Z(e, r(e))}, \quad \pi_E(e^*) = 1_{Z(r(e), e)}, \tag{5-2}$$

for each $v \in E^0$ and $e \in E^1$.

If $w : E^1 \to \Gamma$ is a function, we extend w to E^* by defining $w(v) = 0$ for $v \in E^0$, and $w(\alpha_1 \cdots \alpha_n) = w(\alpha_1) \cdots w(\alpha_n)$. Thus $L_R(E)$ is a Γ-graded ring. On the other hand, defining $\widetilde{w} : \mathcal{G}_E \to \Gamma$ by

$$\widetilde{w}(\alpha x, |\alpha| - |\beta|, \beta x) = w(\alpha)w(\beta)^{-1}, \tag{5-3}$$

gives a cocycle ([Kumjian and Pask 1999, Lemma 2.3]) and thus $A_R(\mathcal{G})$ is a Γ-graded ring as well. A quick inspection of isomorphism (5-1) shows that π_E respects the Γ-grading.

5B. *Covering of a graph.* In this section we show that the smash product of a Leavitt path algebra is graded isomorphic to the Leavitt path algebra of its covering graph. We briefly recall the concept of skew product or covering of a graph (see [Green 1983, §2; Kumjian and Pask 1999, Definition 2.1]).
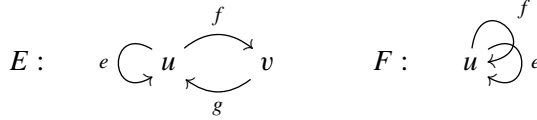
Let Γ be a group and $w : E^1 \to \Gamma$ a function. As in [Green 1983, §2], the *covering graph* $\bar{E}$ of E with respect to w is given by

$$\begin{aligned}
\bar{E}^0 &= \{v_\alpha \mid v \in E^0 \text{ and } \alpha \in \Gamma\}, & \bar{E}^1 &= \{e_\alpha \mid e \in E^1 \text{ and } \alpha \in \Gamma\}, \\
s(e_\alpha) &= s(e)_\alpha, & r(e_\alpha) &= r(e)_{w(e)^{-1}\alpha}.
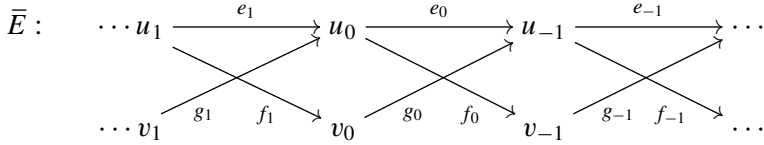\end{aligned} \tag{5-4}$$

Example 5.2. Let E be a graph and define $w : E^1 \to \mathbb{Z}$ by $w(e) = 1$ for all $e \in E^1$. Then $\bar{E}$ (sometimes denoted $E \times_1 \mathbb{Z}$) is given by

$$\bar{E}^0 = \{v_n \mid v \in E^0 \text{ and } n \in \mathbb{Z}\}, \qquad \bar{E}^1 = \{e_n \mid e \in E^1 \text{ and } n \in \mathbb{Z}\},$$

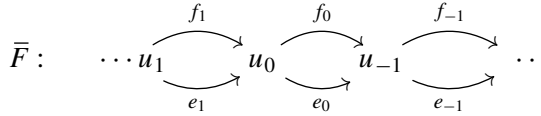$$s(e_n) = s(e)_n, \qquad\qquad r(e_n) = r(e)_{n-1}.$$

As examples, consider the following graphs

Then

and

Since $\widetilde{w} : \mathcal{G}_E \to \Gamma$ is a cocycle (5-3), by Lemma 3.3 the skew-product groupoid $\mathcal{G}_E \times \Gamma$ is a Γ-graded ample groupoid. For each (possibly infinite) path $x = e^1 e^2 e^3 \cdots$ of E and each $\gamma \in \Gamma$, there is a path x_γ of $\bar{E}$ given by

$$x_\gamma = e^1_\gamma e^2_{w(e^1)^{-1}\gamma} e^3_{w(e^1 e^2)^{-1}\gamma} \cdots. \tag{5-5}$$

There is an isomorphism

$$f : \mathcal{G}_E \times \Gamma \to \mathcal{G}_{\bar{E}} \tag{5-6}$$

of groupoids such that $f((x, k, y), \gamma) = (x_{\widetilde{w}(x,k,y)\gamma}, k, y_\gamma)$ (see [Kumjian and Pask 1999, Theorem 2.4]). The grading of the skew-product $\mathcal{G}_E \times \Gamma$ induces a grading of $\mathcal{G}_{\bar{E}}$, and the isomorphism f respects the gradings of the two groupoids, and so induces a Γ-graded isomorphism of Steinberg algebras

$$\tilde{f} : A_R(\mathcal{G}_E \times \Gamma) \to A_R(\mathcal{G}_{\bar{E}}).$$

Set $g = \tilde{f}^{-1} : A_R(\mathcal{G}_{\bar{E}}) \to A_R(\mathcal{G}_E \times \Gamma)$. Then

$$g(1_{Z(v_\gamma)}) = 1_{Z(v) \times \{\gamma\}} \qquad\qquad \text{for } v \in E^0 \text{ and } \gamma \in \Gamma,$$

$$g(1_{Z(e_\alpha, r(e)_{w(e)^{-1}\alpha})}) = 1_{Z(e, r(e)) \times \{w(e)^{-1}\alpha\}} \quad \text{for } e \in E^1 \text{ and } \alpha \in \Gamma, \tag{5-7}$$

$$g(1_{Z(r(e)_{w(e)^{-1}\alpha}, e_\alpha)}) = 1_{Z(r(e), e) \times \{\alpha\}} \qquad \text{for } e \in E^1 \text{ and } \alpha \in \Gamma.$$

Let $\phi : A_R(\mathcal{G}_E \times \Gamma) \to A_R(\mathcal{G}_E) \# \Gamma$ be the isomorphism of Theorem 3.4, let $g : A_R(\mathcal{G}_{\bar{E}}) \to A_R(\mathcal{G}_E \times \Gamma)$ be the isomorphism (5-7), let $\pi_E : L_R(E) \to A_R(\mathcal{G}_E)$ and $\pi_{\bar{E}} : L_R(\bar{E}) \to A_R(\mathcal{G}_{\bar{E}})$ be as in (5-1), and

let $\widetilde{\pi}_E : L_R(E) \# \Gamma \to A_R(\mathcal{G}_E) \# \Gamma$ be given by $\widetilde{\pi}_E(xp_\gamma) = \pi_E(x)p_\gamma$, for $x \in L_R(E)$ and $\gamma \in \Gamma$. Define $\phi' := \widetilde{\pi}_E^{-1} \circ \phi \circ g \circ \pi_{\bar{E}}$. Then we have the following commuting diagram of Γ-graded isomorphisms:

$$
\begin{array}{ccc}
L_R(\bar{E}) & \xrightarrow{\phi'} & L_R(E) \# \Gamma \\
\Big\downarrow{\scriptstyle \pi_{\bar{E}}} & & \Big\downarrow{\scriptstyle \widetilde{\pi}_E} \\
A_R(\mathcal{G}_{\bar{E}}) & \xrightarrow{\phi \circ g} & A_R(\mathcal{G}_E) \# \Gamma
\end{array}
\tag{5-8}
$$

Recall from (5-6) that $\mathcal{G}_{\bar{E}}$ is Γ-graded. Then the Steinberg algebra $A_R(\mathcal{G}_{\bar{E}})$ is Γ-graded. By the algebra isomorphism $\pi_{\bar{E}} : L_R(\bar{E}) \cong A_R(\mathcal{G}_{\bar{E}})$, $L_R(\bar{E})$ has a Γ-grading such that the grading map $w' : \bar{E}^1 \to \Gamma$ is given by $w'(e_\alpha) = w(e)$, for $e \in E^1$ and $\alpha \in \Gamma$.

Corollary 5.3. *The map $\phi' : L_R(\bar{E}) \to L_R(E) \# \Gamma$ is an isomorphism of Γ-graded algebras such that $\phi'(v_\beta) = vp_\beta$, $\phi'(e_\alpha) = ep_{w(e)^{-1}\alpha}$ and $\phi'(e_\alpha^*) = e^*p_\alpha$, for $v \in E^0$, $e \in E^1$, and $\alpha, \beta \in \Gamma$.*

Proof. Since all the homomorphisms in the diagram (5-8) preserve gradings of algebras, the map $\phi' : L_R(\bar{E}) \to L_R(E) \# \Gamma$ is an isomorphism of Γ-graded algebras. For each vertex $v_\gamma \in \bar{E}^0$ and each edge $e_\alpha \in \bar{E}^1$, we have

$$
\phi'(v_\gamma) = (\widetilde{\pi}_E^{-1} \circ \phi \circ g)(1_{Z(v_\gamma)}) = (\widetilde{\pi}_E^{-1} \circ \phi)(1_{Z(v) \times \{\gamma\}}) = \widetilde{\pi}_E^{-1}(1_{Z(v)}p_\gamma) = vp_\gamma,
$$

$$
\phi'(e_\alpha) = (\widetilde{\pi}_E^{-1} \circ \phi \circ g)(1_{Z(e_\alpha, r(e)_{w(e)^{-1}\alpha})}) = (\widetilde{\pi}_E^{-1} \circ \phi)(1_{Z(e, r(e)) \times \{w(e)^{-1}\alpha\}}) = \widetilde{\pi}_E^{-1}(1_{Z(e, r(e))}p_{w(e)^{-1}\alpha})
$$

$$
= ep_{w(e)^{-1}\alpha},
$$

$$
\phi'(e_\alpha^*) = (\widetilde{\pi}_E^{-1} \circ \phi \circ g)(1_{Z(r(e)_{w(e)^{-1}\alpha}, e_\alpha)}) = (\widetilde{\pi}_E^{-1} \circ \phi)(1_{Z(r(e), e) \times \{\alpha\}}) = \widetilde{\pi}_E^{-1}(1_{Z(r(e), e)}p_\alpha) = e^*p_\alpha. \qquad \square
$$

Kumjian and Pask [1999] show that given a free action of a group Γ on a graph E, the crossed product $C^*(E) \times \Gamma$ by the induced action is strongly Morita equivalent to $C^*(E/\Gamma)$, where E/Γ is the quotient graph and obtained an isomorphism similar to Corollary 5.3 for graph C^*-algebras. Corollary 5.3 shows that this isomorphism already occurs on the algebraic level (see Section 3D), so the following diagram commutes:

$$
\begin{array}{ccc}
L_{\mathbb{C}}(\bar{E}) & \longrightarrow & L_{\mathbb{C}}(E) \# \Gamma \\
\Big\downarrow & & \Big\downarrow \\
C^*(\bar{E}) & \longrightarrow & C^*(E) \times \Gamma
\end{array}
$$

Remark 5.4. Green [1983] showed that the theory of coverings of graphs with relations and the theory of graded algebras are essentially the same. For a Γ-graded path algebra A, Green constructed a covering of the quiver of A and showed that the category of representations of the covering satisfying a certain set of relations is equivalent to the category of finite dimensional graded A-modules.

For any graph E and a function $w : E^1 \to \Gamma$, we consider the smash product of a quotient algebra of the path algebra of E with the group Γ. Let K be a field, E a graph and $w : E^1 \to \Gamma$ a weight map. Denote by KE the path algebra of E. A *relation in E* is a K-linear combination $\sum_i k_i q_i$ with q_i paths in

E having the same source and range. Let r be a set of relations in E and $\langle r \rangle$ the two sided ideal of KE generated by r. Set

$$\mathcal{A}_r(E) = KE/\langle r \rangle.$$

We denote by $\bar{r}$ the lifting of r in $\bar{E}$ (see (5-4)): For each finite path $p = e^1 e^2 \cdots e^n$ in E and $\gamma \in \Gamma$, there is a path p^γ of $\bar{E}$ given by

$$p^\gamma = e^1_{\prod_{i=1}^n w(e^i)\gamma} \cdots e^{n-1}_{\prod_{i=n-1}^n w(e^i)\gamma} e^n_{w(e^n)\gamma},$$

similar to (5-5). Then for each relation $\sum_i k_i q_i \in r$ and each $\gamma \in \Gamma$, we have

$$\sum_i k_i q_i^\gamma \in \bar{r}.$$

Now set

$$\mathcal{A}_{\bar{r}}(\bar{E}) = K\bar{E}/\langle \bar{r} \rangle.$$

We prove that $\mathcal{A}_{\bar{r}}(\bar{E}) \cong \mathcal{A}_r(E) \# \Gamma$. Define $h : K\bar{E} \to \mathcal{A}_r(E) \# \Gamma$ by $h(v_\gamma) = v p_\gamma$ and $h(e_\alpha) = e p_{w(e)^{-1}\alpha}$, for $v \in E^0$, $e \in E^1$ and $\alpha, \gamma \in \Gamma$. Since h annihilates the relations $\bar{r}$, it induces a homomorphism

$$\bar{h} : \mathcal{A}_{\bar{r}}(\bar{E}) \to \mathcal{A}_r(E) \# \Gamma.$$

We show that $\bar{h}$ is an isomorphism. For injectivity, suppose that $x = \sum_{i=1}^m \lambda_i \xi_i \in \mathcal{A}_{\bar{r}}(\bar{E})$ with $\lambda_i \in K$ and ξ_i pairwise distinct paths in $\bar{E}$. Each ξ_i has the form of $(\xi_i')^{\alpha_i}$ for some $\xi_i' \in E^*$ and $\alpha_i \in \Gamma$. If $\bar{h}(x) = 0$, then $h(x) = \sum_{i=1}^m \lambda_i \xi_i' p_{\alpha_i} = 0$. Suppose that the α_i are not distinct; so by rearranging, we can assume that $\alpha_1 = \cdots = \alpha_k$ for some $k \le m$. Then $\sum_{i=1}^k \lambda_i \xi_i' = 0$ in $\mathcal{A}_r(E)$. Observe that $\sum_{i=1}^k \lambda_i \xi_i' = 0$ in $\mathcal{A}_r(E)$ implies $\sum_{i=1}^k \lambda_i \xi_i = 0$ in $\mathcal{A}_{\bar{r}}(\bar{E})$. Hence $x = 0$, implying $\bar{h}$ is injective. For surjectivity, fix η in E^* and $\gamma \in \Gamma$. Then $h(\eta^\gamma) = \eta p_\gamma$ by definition. Since the elements $\{\eta p_\gamma \mid \eta \in E^*, \gamma \in \Gamma\}$ span $\mathcal{A}_r(E) \# \Gamma$, we deduce that $\bar{h}$ is surjective. Thus $\bar{h}$ is an isomorphism as claimed.

5C. *The monoid* $\mathcal{V}^{\mathrm{gr}}(L_K(E))$. In this subsection, we consider the Leavitt path algebra $L_K(E)$ over a field K. Ara, Moreno and Pardo [Ara et al. 2007] showed that for a Leavitt path algebra associated to the graph E, the monoid $\mathcal{V}(L_K(E))$ is entirely determined by elementary graph-theoretic data. Specifically, for a row-finite graph E, we define M_E to be the abelian monoid generated by E^0 subject to

$$v = \sum_{e \in s^{-1}(v)} r(e), \tag{5-9}$$

for every $v \in E^0$ that is not a sink. Theorem 3.5 of [Ara et al. 2007] says that $\mathcal{V}(L_K(E)) \cong M_E$.

There is an explicit description [Ara et al. 2007, §4] of the congruence on the free abelian monoid given by the defining relations of M_E. Let F be the free abelian monoid on the set E^0. The nonzero elements of F can be written in a unique form up to permutation as $\sum_{i=1}^n v_i$, where $v_i \in E^0$. Define a binary relation $\to_1$ on $F \setminus \{0\}$ by $\sum_{i=1}^n v_i \to_1 \sum_{i \ne j} v_i + \sum_{e \in s^{-1}(v_j)} r(e)$, whenever $j \in \{1, \ldots, n\}$ and v_j is not a sink. Let $\to$ be the transitive and reflexive closure of $\to_1$ on $F \setminus \{0\}$ and $\sim$ the congruence on F generated by the relation $\to$. Then $M_E = F/\sim$.

Ara and Goodearl [2012, Theorem 4.3] defined analogous monoids $M(E, C, S)$ and constructed natural isomorphisms $M(E, C, S) \cong \mathcal{V}(\mathrm{CL}_K(E, C, S))$ for arbitrary separated graphs. The nonseparated case reduces to that of ordinary Leavitt path algebras, and extends the result of [Ara et al. 2007] to non-row-finite graphs.

Following [Ara and Goodearl 2012; 2015], we recall the definition of M_E when E is not necessarily row-finite. In [Ara and Goodearl 2015, §4.1] the generators $v \in E^0$ of the abelian monoid M_E for E are supplemented by generators q_Z as Z runs through all nonempty finite subsets of $s^{-1}(v)$ for infinite emitters v. The relations are

(1) $v = \sum_{e \in s^{-1}(v)} r(e)$ for all regular vertices $v \in E^0$;

(2) $v = \sum_{e \in Z} r(e) + q_Z$ for all infinite emitters $v \in E^0$; and

(3) $q_{Z_1} = \sum_{e \in Z_2 \setminus Z_1} r(e) + q_{Z_2}$ for all nonempty finite sets $Z_1 \subseteq Z_2 \subseteq s^{-1}(v)$, where $v \in E^0$ is an infinite emitter.

An abelian monoid M is *cancellative* if it satisfies full cancellation, namely, $x + z = y + z$ implies $x = y$, for any $x, y, z \in M$. In order to prove that the *graded* monoid associated to any Leavitt path algebra is cancellative (Corollary 5.8), we will need to know that the monoid associated to Leavitt path algebras of acyclic graphs are cancellative.

Lemma 5.5. *Let E be an arbitrary graph. The monoid M_E is cancellative if and only if no cycle in E has an exit. In particular, if E is acyclic, then M_E is cancellative.*

Proof. We first claim that M_E is cancellative for any row-finite acyclic graph E. By [Ara et al. 2007, Lemma 3.1], the row-finite graph E is a direct limit of a directed system of its finite complete subgraphs $\{E_i\}_{i \in X}$. In turn, the monoid M_E is the direct limit of $\{M_{E_i}\}_{i \in X}$ ([Ara et al. 2007, Lemma 3.4]). We claim that M_E is cancellative. Let $x + u = y + u$ in M_E, where x, y, u are sum of vertices in E. By [Ara et al. 2007, Lemma 4.3], there exists $b \in F$ (sum of vertices in E) such that $x + u \to b$ and $y + u \to b$. Observe that vertices involved in this transformations are finite. Thus there is a finite graph E_i such that all these vertices are in E_i. It follows that in M_{E_i} we have $x + u \to b$ and $y + u \to b$. Thus $x + u = y + u$ in M_{E_i}. Since the subgraph E_i of E is finite and acyclic, M_{E_i} is a direct sum of copies of $\mathbb{N}$ (as $L_K(E_i)$ is a semisimple ring) and thus is cancellative. So $x = y$ in M_{E_i} and so the same in M_E. Hence, M_E is cancellative.

We now show that it suffices to consider the case where E is a row-finite graph in which no cycle has an exit. To see this, let E be any graph, and let E_d be its Drinen–Tomforde desingularisation [Drinen and Tomforde 2005], which is row-finite. Then $L_K(E)$ and $L_K(E_d)$ are Morita equivalent, and so $M_E \cong M_{E_d}$ [Abrams and Aranda Pino 2008, Theorem 5.6]. So M_E has cancellation if and only if M_{E_d} has cancellation. Since no cycle in E has an exit if and only if E_d has the same property, it therefore suffices to prove the result for row-finite graph E in which no cycle has an exit.

Finally, we show that for any row-finite graph E in which no cycle has an exit, the monoid M_E is cancellative. For this, fix a set $C \subseteq E^1$ such that C contains exactly one edge from every cycle in E [Sims 2010]. Let F be the subgraph of E obtained by removing all the edges in C. We claim that

$M_F \cong M_E$. To see this, observe that they have the same generating set $F^0 = E^0$, and the generating relation $\xrightarrow{F}_1$ is contained in $\xrightarrow{E}_1$. So it suffices to show that $\xrightarrow{E}_1 \subseteq \xrightarrow{F}$. For this, note that for $v \in E^0$, we have $s_E^{-1}(v) = s_F^{-1}(v)$ unless $v = s(e)$ for some $e \in C$, in which case $s_E^{-1}(v) = \{e\}$ and $s_F^{-1}(v) = \varnothing$. So it suffices to show that for $e \in C$, we have $s(e) \xrightarrow{F} r(e)$. Let $p = e\alpha_2\alpha_3 \cdots \alpha_n$ be the cycle in E containing e. Then

$$r(e) \xrightarrow{F}_1 s(\alpha_2) \xrightarrow{F}_1 r(\alpha_2) \xrightarrow{F}_1 s(\alpha_3) \xrightarrow{F}_1 r(\alpha_3) \xrightarrow{F}_1 \cdots \xrightarrow{F}_1 s(\alpha_n) \xrightarrow{F}_1 r(\alpha_n) \xrightarrow{F}_1 s(e).$$

So $M_F \cong M_E$ as claimed. So the preceding paragraphs show that M_E is cancellative.

Now suppose that E has a cycle with an exit; say $p = \alpha_1 \cdots \alpha_n$ has an exit α. Without loss of generality, $s(\alpha) = s(\alpha_n)$ and $\alpha \neq \alpha_n$. Write

$$s(p)E^{\leq n} = \{q \in E^* \mid s(q) = s(p), \text{ and } |q| = n \text{ or } |q| < n \text{ and } r(q) \text{ is not regular}\}.$$

Let $p' := \alpha_1 \cdots \alpha_{n-1}\alpha$ and $X := s(p)E^{\leq n} \setminus \{p, p'\}$. A simple induction shows that

$$s(p) \to \sum_{q \in s(\alpha)E^{\leq n}} r(q) = r(p) + r(p') + \sum_{q \in X} r(q) = s(p) + r(p') + \sum_{q \in X} r(q).$$

Since $r(p') \neq 0$ in M_E, it follows that M_E does not have cancellation. $\qquad\square$

In order to compute the monoid $\mathcal{V}^{\mathrm{gr}}(L_K(E))$ for an arbitrary graph E, we define an abelian monoid M_E^{gr} such that the generators $\{a_v(\gamma) \mid v \in E^0, \gamma \in \Gamma\}$ are supplemented by generators $b_Z(\gamma)$ as $\gamma \in \Gamma$ and Z runs through all nonempty finite subsets of $s^{-1}(u)$ for infinite emitters $u \in E^0$. The relations are:

(1) $a_v(\gamma) = \sum_{e \in s^{-1}(v)} a_{r(e)}(w(e)^{-1}\gamma)$ for all regular vertices $v \in E^0$ and $\gamma \in \Gamma$;

(2) $a_u(\gamma) = \sum_{e \in Z} a_{r(e)}(w(e)^{-1}\gamma) + b_Z(\gamma)$ for all $\gamma \in \Gamma$, infinite emitters $u \in E^0$ and nonempty finite subsets $Z \subseteq s^{-1}(u)$;

(3) $b_{Z_1}(\gamma) = \sum_{e \in Z_2 \setminus Z_1} a_{r(e)}(w(e)^{-1}\gamma) + b_{Z_2}(\gamma)$ for all $\gamma \in \Gamma$, infinite emitters $u \in E^0$ and nonempty finite subsets $Z_1 \subseteq Z_2 \subseteq s^{-1}(u)$.

The group Γ acts on the monoid M_E^{gr} as follows. For any $\beta \in \Gamma$,

$$\beta \cdot a_v(\gamma) = a_v(\beta\gamma) \quad \text{and} \quad \beta \cdot b_Z(\gamma) = b_Z(\beta\gamma). \tag{5-10}$$

There is a surjective monoid homomorphism $\pi : M_E^{\mathrm{gr}} \to M_E$ such that $\pi(a_v(\gamma)) = v$ and $\pi(b_Z(\gamma)) = q_Z$ for $v \in E^0$ and nonempty finite subset $Z \subset s^{-1}(u)$, where u is an infinite emitter. π is Γ-equivariant in the sense that $\pi(\beta \cdot x) = \pi(x)$ for all $\beta \in \Gamma$ and $x \in M_E^{\mathrm{gr}}$.

Recall the covering graph $\bar{E}$ from Section 5B. Let $L_K(\bar{E})$-Mod be the category of unital left $L_K(\bar{E})$-modules and $L_K(E)$-Gr the category of graded unital left $L_K(E)$-modules. The isomorphism $\phi' : L_K(\bar{E}) \xrightarrow{\sim} L_K(E) \# \Gamma$ of Corollary 5.3 and Proposition 2.5 yield an isomorphism of categories

$$\Phi : L_K(E)\text{-Gr} \to L_K(\bar{E})\text{-Mod}. \tag{5-11}$$

Lemma 5.6. *Let E be an arbitrary graph, Γ a group and $w : E^1 \to \Gamma$ a function.*

(1) *Fix a path η in E, and $\beta \in \Gamma$, and let $\bar{\eta} = \eta_{\beta^{-1}}$ be the path in $\bar{E}$ defined at (5-5). Then*

$$\Phi\big((L_K(E)\eta\eta^*)(\beta)\big) \cong L_K(\bar{E})\bar{\eta}\bar{\eta}^*.$$

In particular, $\Phi((L_K(E)v)(\beta)) \cong L_K(\bar{E})v_{\beta^{-1}}$.

(2) *Let $u \in E^0$ be an infinite emitter, and let $Z \subseteq s_E^{-1}(u)$ be a nonempty finite set. Fix $\beta \in \Gamma$, and let $\bar{Z} = \{e_{\beta^{-1}} \mid e \in Z\}$. Then $u_{\beta^{-1}}$ is an infinite emitter in $\bar{E}$ and $\bar{Z}$ is a nonempty finite subset of $s_{\bar{E}}^{-1}(u_{\beta^{-1}})$. Moreover, $\Phi(L_K(E)(u - \sum_{e \in Z} ee^*)(\beta)) \cong L_K(\bar{E})(u_{\beta^{-1}} - \sum_{f \in \bar{Z}} ff^*)$.*

Proof. We prove (1). By the isomorphism of algebras in Corollary 5.3, we have

$$L_K(\bar{E})\bar{\eta}\bar{\eta}^* \cong (L_K(E) \# \Gamma)\eta\eta^* p_{\beta^{-1}}.$$

We claim that $f : \Phi((L_K(E)\eta\eta^*)(\beta)) \to (L_K(E)\#\Gamma)\eta\eta^* p_{\beta^{-1}}$ given by $f(y) = y p_{\beta^{-1}}$ is an isomorphism of left $L_K(\bar{E})$-modules. It is clearly a group isomorphism. To see that it is an $L_K(\bar{E})$-module morphism, note that $(r p_\gamma) y = r y_\gamma$ for $y \in (L_K(E)\eta\eta^*)(\beta)$ and y_γ a homogeneous element of degree γ. We have $y \in L_K(E)_{\gamma\beta}\eta\eta^*$, yielding $f((r p_\gamma) y) = r y_\gamma p_{\beta^{-1}} = (r p_\gamma)(y p_{\beta^{-1}}) = r p_\gamma f(y)$. The proof for (2) is similar. $\square$

Recall from Section 2B that there is a shift functor $\tilde{\mathcal{S}}_\alpha$ on $L_K(E) \# \Gamma$-Mod for each $\alpha \in \Gamma$. So the isomorphism $\phi' : L_K(\bar{E}) \xrightarrow{\sim} L_K(E)\#\Gamma$ of Corollary 5.3 yields a shift functor $\mathcal{T}_\alpha$ on $L_K(\bar{E})$-Mod. This in turn induces a homomorphism $\mathcal{T}_\alpha : \mathcal{V}(L_K(\bar{E})) \to \mathcal{V}(L_K(\bar{E}))$, giving a Γ-action on the monoid $\mathcal{V}(L_K(\bar{E}))$.

Fix $v_\gamma \in \bar{E}^0$, an infinite emitter $u_\beta \in \bar{E}^0$, and a finite $Z \subseteq s_{\bar{E}}^{-1}(u_\beta)$. Write $Z \cdot \alpha^{-1} = \{e_{\beta\alpha^{-1}} \mid e_\beta \in Z\}$. We claim that

$$\mathcal{T}_\alpha([L_K(\bar{E})v_\gamma]) = [L_K(\bar{E})v_{\gamma\alpha^{-1}}] \quad \text{and} \quad \mathcal{T}_\alpha([L_K(\bar{E})(u_\beta - \sum_{e \in Z} ee^*)]) = [L_K(\bar{E})(u_{\beta\alpha^{-1}} - \sum_{f \in Z \cdot a^{-1}} ff^*)].$$

(5-12)

To establish the first equality in (5-12), we use Lemma 5.6 to see that

$$\Phi(L_K(E)v(\gamma^{-1})) = L_K(\bar{E})v_\gamma \quad \text{and} \quad \Phi(L_K(E)v(\alpha\gamma^{-1})) = L_K(\bar{E})v_{\gamma\alpha^{-1}}.$$

Using the commutative diagram (2-3) at the second equality, we see that

$$\mathcal{T}_\alpha(L_K(\bar{E})v_\gamma) = (\mathcal{T}_\alpha \circ \Phi)(L_K(E)v(\gamma^{-1})) = (\Phi \circ \mathcal{T}_\alpha)(L_K(E)v(\gamma^{-1})) = \Phi(L_K(E)v(\alpha\gamma^{-1}))$$
$$= L_K(\bar{E})v_{\gamma\alpha^{-1}}.$$

The proof for the second equality in (5-12) is similar.

The group Γ acts on the monoid $M_{\bar{E}}$ as follows. Again fix $v_\gamma \in \bar{E}^0$, an infinite emitter $u_\beta \in \bar{E}^0$, and a finite $Z \subseteq s_{\bar{E}}^{-1}(u_\beta)$, and write $Z \cdot \alpha^{-1} = \{e_{\beta\alpha^{-1}} \mid e_\beta \in Z\}$. Then

$$\alpha \cdot v_\gamma = v_{\gamma\alpha^{-1}} \quad \text{and} \quad \alpha \cdot q_Z = q_{Z \cdot \alpha^{-1}}.$$

(5-13)

Proposition 5.7. *Let E be an arbitrary graph, K a field, Γ a group and $w : E^1 \to \Gamma$ a function. Let $\mathcal{A} = L_K(E)$ and $\bar{\mathcal{A}} = L_K(\bar{E})$. Then the monoid $\mathcal{V}^{\mathrm{gr}}(\mathcal{A})$ is generated by $[\mathcal{A}v(\alpha)]$ and $[\mathcal{A}(u - \sum_{e \in Z} ee^*)(\beta)]$, where $v \in E^0$, $\alpha, \beta \in \Gamma$ and Z runs through all nonempty finite subsets of $s^{-1}(u)$ for infinite emitters*

$u \in E^0$. *Given an infinite emitter* $u \in E^0$, *a finite nonempty set* $Z \subseteq s^{-1}(u)$, *and* $\beta \in \Gamma$, *write* $Z_{\beta^{-1}} :=$ $\{e_{\beta^{-1}} : e \in Z\} \subseteq s_{\bar{E}}^{-1}(u_{\beta^{-1}})$. *Then there are* Γ-*module isomorphisms*

$$\mathcal{V}^{\mathrm{gr}}(\mathcal{A}) \cong \mathcal{V}(\bar{\mathcal{A}}) \cong M_{\bar{E}} \cong M_E^{\mathrm{gr}}, \tag{5-14}$$

that satisfy

$$[\mathcal{A}v(\alpha)] \mapsto [\bar{\mathcal{A}}v_{\alpha^{-1}}] \mapsto [v_{\alpha^{-1}}] \mapsto [a_v(\alpha)],$$

for all $v \in E^0$ *and* $\alpha \in \Gamma$, *and*

$$\left[\mathcal{A}\left(u - \sum_{e \in Z} ee^*\right)(\beta)\right] \mapsto \left[\bar{\mathcal{A}}\left(u_{\beta^{-1}} - \sum_{\bar{e} \in Z_{\beta^{-1}}} \bar{e}\bar{e}^*\right)\right] \mapsto [q_{Z_{\beta^{-1}}}] \mapsto [b_Z(\beta)],$$

for every infinite emitter u, *finite nonempty* $Z \subseteq s^{-1}(u)$, *and* $\beta \in \Gamma$.

Proof. Let P be a graded finitely generated projective left $\mathcal{A}$-module. We claim that the isomorphism $\Phi : \mathcal{A}\text{-Gr} \to \bar{\mathcal{A}}\text{-Mod}$ in (5-11) preserves the finitely generated projective objects. Since Φ is an isomorphism of categories, $\Phi(P)$ is projective. Observe that P has finite number of homogeneous generators $x_1, \ldots, x_n$ of degree γ_i. By the $\bar{\mathcal{A}}$-action of $\Phi(P)$, we have the following equalities:

(1) if $v \in E^0$ and $\gamma \in \Gamma$, then

$$v_\gamma x_i = v p_\gamma x_i = \begin{cases} v x_i & \text{if } \gamma_i = \gamma, \\ 0 & \text{otherwise}; \end{cases} \tag{5-15}$$

(2) if $e : u \to v \in E^1$, $w(e) = \beta$ and $\gamma \in \Gamma$, then

$$e_\gamma x_i = e p_{\beta^{-1}\gamma} x_i = \begin{cases} e x_i & \text{if } \gamma_i = \beta^{-1}\gamma, \\ 0 & \text{otherwise}; \text{ and} \end{cases} \tag{5-16}$$

(3) if $e : u \to v \in E^1$, $w(e) = \beta$ and $\gamma \in \Gamma$, then

$$e_\gamma^* x_i = e^* p_\gamma x_i = \begin{cases} e^* x_i & \text{if } \gamma_i = \gamma, \\ 0 & \text{otherwise}. \end{cases} \tag{5-17}$$

So for $y \in \Phi(P)$, we can express $y = \sum_{i=1}^n r_i x_i$ for some $r_i \in \mathcal{A}$. Fix $i \leq n$ and paths η, τ in E satisfying $r(\eta) = r(\tau)$. Then (5-15), (5-16), and (5-17) give

$$\tau \eta^* x_i = \tau_{w(\tau)w(\eta)^{-1}\gamma_i}(\eta_{\gamma_i})^* x_i. \tag{5-18}$$

Since $y = \sum_{i=1}^n r_i x_i = \sum_{i=1}^n \sum_{h \in \Gamma} r_{i,h} x_i$ with $r_{i,h}$ a homogeneous element of degree h, Equation (5-18) gives $y \in \bar{\mathcal{A}}(\Phi(P))$. Thus $\Phi(P)$ is a finitely generated projective $\bar{\mathcal{A}}$-module.

By (4-2) and (4-4), there exists a homomorphism $\mathcal{V}^{\mathrm{gr}}(\mathcal{A}) \to \mathcal{V}(\bar{\mathcal{A}})$ sending $[P]$ to $[\Phi(P)]$ for a graded finitely generated projective left $\mathcal{A}$-module P. Applying [Ara and Goodearl 2012, Theorem 4.3] for the nonseparated case, we obtain the second monoid isomorphism $\mathcal{V}(\bar{\mathcal{A}}) \xrightarrow{\sim} M_{\bar{E}}$ in (5-14). Then for each graded finitely generated projective left $\mathcal{A}$-module P, the module $\Phi(P)$ in $\bar{\mathcal{A}}$-Mod is generated by the elements $\bar{\mathcal{A}}v_\alpha$ and $\bar{\mathcal{A}}(u_\beta - \sum_{e \in Z'} ee^*)$ that it contains. Combining this with Lemma 5.6 gives the first

isomorphism of monoids. The last monoid isomorphism $M_{\bar{E}} \cong M_E^{\mathrm{gr}}$ follows directly by their definitions. By (5-10), (5-12) and (5-13), the monoid isomorphisms in (5-14) are Γ-module isomorphisms. $\qquad\square$

Recall the following classification conjecture [Abrams 2015; Ara and Pardo 2014; Hazrat 2013b]. Let E and F be finite graphs. Then there is an order preserving $\mathbb{Z}[x, x^{-1}]$-module isomorphism $\phi :$ $K_0^{\mathrm{gr}}(L_K(E)) \to K_0^{\mathrm{gr}}(L_K(F))$ if and only if $L_K(E)$ is graded Morita equivalent to $L_K(F)$. Furthermore, if $\phi([L_K(E)]) = [L_K(F)]$ then $L_K(E) \cong_{\mathrm{gr}} L_K(F)$.

Note that $K_0(L_K(E))$ and $K_0^{\mathrm{gr}}(L_K(E))$ are the group completions of $\mathcal{V}(L_K(E))$ and $\mathcal{V}^{\mathrm{gr}}(L_K(E))$, respectively. Let $\Gamma = \mathbb{Z}$ and let $w : E^1 \to \mathbb{Z}$ be the function assigning 1 to each edge. Then Proposition 5.7 implies that there is an order preserving $\mathbb{Z}[x, x^{-1}]$-module isomorphism $K_0^{\mathrm{gr}}(L_K(E)) \cong K_0(L_K(\bar{E}))$, thus relating the study of a Leavitt path algebra over an arbitrary graph to the case of acyclic graphs (see Example 5.2).

The following corollary is the first evidence that $K_0^{\mathrm{gr}}(L_K(E))$ preserves all the information of the graded monoid.

Corollary 5.8. *Let E be an arbitrary graph. Consider $L_K(E)$ as a graded ring with the grading determined by the function $w : E^1 \to \mathbb{Z}$ such that $w(e) = 1$ for all e. Then $\mathcal{V}^{\mathrm{gr}}(L_K(E))$ is cancellative.*

Proof. By Proposition 5.7, we have $\mathcal{V}^{\mathrm{gr}}(L_K(E)) \cong M_{\bar{E}}$. Since $\bar{E} = E \times \mathbb{Z}$ is an acyclic graph, the monoid $M_{\bar{E}}$ is cancellative by Lemma 5.5. Hence $\mathcal{V}^{\mathrm{gr}}(L_K(E))$ is cancellative. $\qquad\square$

For the next result we need to recall the notion of order-ideals of a monoid. An *order-ideal* of a monoid M is a submonoid I of M such that $x + y \in I$ implies $x, y \in I$. Equivalently, an order-ideal is a submonoid I of M that is hereditary in the sense that $x \le y$ and $y \in I$ implies $x \in I$. The set $\mathcal{L}(M)$ of order-ideals of M forms a (complete) lattice (see [Ara et al. 2007, §5]). Given a subgroup I of $K_0^{\mathrm{gr}}(A)$, we write $I^+ = I \cap K_0^{\mathrm{gr}}(A)^+$. We say that I is a *graded ordered ideal* if I is closed under the action of $\mathbb{Z}[x, x^{-1}]$, $I = I^+ - I^+$, and I^+ is an order-ideal.

Let E be a graph. Recall that a subset $H \subseteq E^0$ is said to be hereditary if for any $e \in E^1$ we have that $s(e) \in H$ implies $r(e) \in H$. A hereditary subset $H \subseteq E^0$ is called saturated if whenever $0 < |s^{-1}(v)| < \infty$, then $\{r(e) \mid e \in E^1 \text{ and } s(e) = v\} \subseteq H$ implies $v \in H$. If H is a hereditary subset, a breaking vertex of H is a vertex $v \in E^0 \setminus H$ such that $|s^{-1}(v)| = \infty$ but $0 < |s^{-1}(v) \setminus r^{-1}(H)| < \infty$. We write $B_H := \{v \in E^0 \setminus H \mid v \text{ is a breaking vertex of } H\}$. We call (H, S) an *admissible pair* in E^0 if H is a saturated hereditary subset of E^0 and $S \subseteq B_H$.

Let E be a row-finite graph. Isomorphisms between the lattice of saturated hereditary subsets of E^0, the lattice $\mathcal{L}(M_E)$, and the lattice of graded ideals of $L_K(E)$ were established in [Ara et al. 2007, Theorem 5.3]. Tomforde [2007, Theorem 5.7] used the admissible pairs (H, S) of vertices to parametrise the graded ideals of $L_K(E)$ for a graph E which is not row-finite. In analogy, Ara and Goodearl [2012] proved that the lattice of those ideals of Cohn–Leavitt algebras $\mathrm{CL}_K(E, C, S)$ generated by idempotents is isomorphic to a certain lattice $\mathcal{A}_{C,S}$ of admissible pairs (H, G), where $H \subseteq E^0$ and $G \subseteq C$ (see [Ara and Goodearl 2012, Definition 6.5] for the precise definition). There is also a lattice isomorphism between

$\mathcal{A}_{C,S}$ and the lattice $\mathcal{L}(M(E, C, S))$ of order-ideals of $M(E, C, S)$. Specialising to the nonseparated graph E, there is a lattice isomorphism

$$\mathcal{H} \cong \mathcal{L}(M_E) \tag{5-19}$$

between the lattice $\mathcal{H}$ of admissible pairs (H, S) of E^0 and the lattice $\mathcal{L}(M_E)$ of order-ideals of the monoid M_E.

Let E be a finite graph with no sinks. There is a one-to-one correspondence [Hazrat 2013a, Theorem 12] between the set of hereditary and saturated subsets of E^0 and the set of graded ordered ideals of $K_0^{\mathrm{gr}}(L_K(E))$. The main theorem of this section describes a one-to-one correspondence between the set of admissible pairs (H, S) of vertices and the set of graded ordered ideals of $K_0^{\mathrm{gr}}(L_K(E))$ for an arbitrary graph E. To prove it, we first need to extend [Ara et al. 2007, Lemma 4.3] to arbitrary graphs. This may also be useful in other situations.

Lemma 5.9. *Let E be an arbitrary graph and denote by F the free abelian group generated by $E^0 \cup \{q_Z\}$, where Z ranges over all the nonempty finite subsets of $s^{-1}(v)$ for infinite emitters v. Let $\sim$ be the congruence on F such that $F/\!\sim = M_E$. Let $\rightarrow_1$ be the relation on F defined by $v + \alpha \rightarrow_1 \sum_{e \in s^{-1}(v)} r(e) + \alpha$ if v is a regular vertex in E, $v + \alpha \rightarrow_1 r(z) + q_{\{z\}} + \alpha$ if $v \in E^0$ is an infinite emitter and $z \in s^{-1}(v)$, and also $q_Z + \alpha \rightarrow_1 r(z) + q_{Z \cup \{z\}} + \alpha$, if Z is a nonempty finite subset of $s^{-1}(v)$ for an infinite emitter v and $z \in s^{-1}(v) \setminus Z$. Let $\rightarrow$ be the transitive and reflexive closure of $\rightarrow_1$. Then $\alpha \sim \beta$ in F if and only if there is a $\gamma \in F$ such that $\alpha \rightarrow \gamma$ and $\beta \rightarrow \gamma$.*

Proof. As in [Ara and Goodearl 2015, Alternative proof of Theorem 4.1], we write $M_E = \lim M(E', C', T')$, where E' ranges over all the finite complete subgraphs of E and

$$C' = \{s_{E'}^{-1}(v) \mid v \in (E')^0, \ |s_{E'}^{-1}(v)| > 0\}, \quad T' = \{s_{E'}^{-1}(v) \in C' \mid v \in (E')^0, \ 0 < |s_E^{-1}(v)| < \infty\}.$$

Applying [Ara and Goodearl 2012, Construction 5.3], we get that $M(E', C', T') = M_{\widetilde{E}}$ for some finite graph $\widetilde{E}$. The vertices of $\widetilde{E}$ are the vertices of E and the elements of the form q_Z, where $Z \in C' \setminus T'$, and there is a new edge $e_Z : v \rightarrow q_Z$ if the source of Z is v. If $\alpha \sim \beta$ in F, then $[\alpha] = [\beta]$ in M_E, and so there is (E', C', T') as above such that $[\alpha] = [\beta]$ in $M(E', C', T')$. But since $M(E', C', T') = M_{\widetilde{E}}$, and $\widetilde{E}$ is finite, we conclude from [Ara et al. 2007, Lemma 4.3] that there is an element γ in the free monoid on $(E')^0 \cup \{q_Z \mid Z \in C' \setminus T'\}$ such that $\alpha \rightarrow \gamma$ and $\beta \rightarrow \gamma$. This implies that $\alpha \rightarrow \gamma$ and $\beta \rightarrow \gamma$ in F. $\square$

Lemma 5.10. *Let E be an arbitrary graph and K a field. Consider $L_K(E)$ as a graded ring with the grading determined by the function $w : E^1 \rightarrow \mathbb{Z}$ such that $w(e) = 1$ for all e. Let $\mathcal{L}^c(M_E^{\mathrm{gr}})$ be the set of order-ideals of M_E^{gr} which are closed under the $\mathbb{Z}$-action. Let $\pi : M_E^{\mathrm{gr}} \rightarrow M_E$ be the canonical surjective homomorphism. Then the map $\phi : \mathcal{L}(M_E) \rightarrow \mathcal{L}^c(M_E^{\mathrm{gr}})$ defined by $\phi(I) = \pi^{-1}(I)$ is a lattice isomorphism.*

Proof. It is easy to show that the map ϕ is well-defined. The key to show the result is to prove the equality $\pi^{-1}(\pi(J)) = J$ for any $J \in L^c(M_E^{\mathrm{gr}})$. The inclusion $J \subseteq \pi^{-1}(\pi(J))$ is obvious. To show the reverse inclusion $\pi^{-1}(\pi(J)) \subseteq J$, denote by F the free abelian group on $E^0 \cup \{q_Z\}$, where Z ranges over all the

nonempty finite subsets of $s^{-1}(v)$ for infinite emitters v. Take $z \in \pi^{-1}(\pi(J))$. Then there is $y \in J$ such that $\pi(z) = \pi(y)$. Now write

$$z = \sum_i a_{v_i}(\gamma_i) + \sum_j b_{Z_j}(\lambda_j), \quad y = \sum_i a_{v_i'}(\gamma_i') + \sum_j b_{Z_j'}(\lambda_j').$$

Then we have $\sum_i v_i + \sum_j q_{Z_j} = \pi(z) = \pi(y) = \sum_i v_i' + \sum_j q_{Z_j'}$. By Lemma 5.9, there is $x = \sum_i w_i + \sum_j q_{W_j}$ such that $\pi(z) \to x$ and $\pi(y) \to x$ in F. Now using the same changes than in the paths $\pi(y) \to x$ and $\pi(z) \to x$, but lifted to M_E^{gr}, we obtain that $y = \sum_i a_{w_i}(\eta_i) + \sum_j b_{W_j}(v_j)$ in M_E^{gr} and $z = \sum_i a_{w_i}(\eta_i') + \sum_j b_{W_j}(v_j')$ in M_E^{gr}. But now $y \in J$ and J is an order ideal of M_E^{gr}, so it follows that $a_{w_i}(\eta_i) \in J$ for all i and $b_{W_j}(v_j) \in J$ for all j. Using that J is invariant, we obtain $a_{w_i}(\eta_i') \in J$ for all i and $b_{W_j}(v_j') \in J$ for all j. Thus $z = \sum_i a_{w_i}(\eta_i') + \sum_j b_{W_j}(v_j') \in J$ and we conclude the proof.

Now using that $J = \pi^{-1}(\pi(J))$, we can easily show that $\pi(J)$ is an order-ideal of M_E and that the map ϕ is bijective, with $\phi^{-1}(J) = \pi(J)$. $\qquad \square$

We can now state the main theorem of this section, which indicates that the graded K_0-group captures the lattice structure of graded ideals of a Leavitt path algebra.

Theorem 5.11. *Let E be an arbitrary graph and K a field. Consider $L_K(E)$ as a graded ring with the grading determined by the function $w : E^1 \to \mathbb{Z}$ such that $w(e) = 1$ for all e. Then there is a one-to-one correspondence between the admissible pairs of E^0 and the graded ordered ideals of $K_0^{\mathrm{gr}}(L_K(E))$.*

Proof. Let $\mathcal{H}$ be the set of all admissible pairs of E^0 and $\mathcal{L}(K_0^{\mathrm{gr}}(\mathcal{A}))$ the set of all graded ordered ideals of $K_0^{\mathrm{gr}}(\mathcal{A})$, where $\mathcal{A} = L_K(E)$. We first claim that there is a one-to-one correspondence between the order-ideals of M_E and order-ideals of M_E^{gr} which are closed under the $\mathbb{Z}$-action. Let $\mathcal{L}^c(M_E^{\mathrm{gr}})$ be the set of order-ideals of M_E^{gr} which are closed under the $\mathbb{Z}$-action.

The map $\phi : \mathcal{L}(M_E) \to \mathcal{L}^c(M_E^{\mathrm{gr}})$ has been defined in Lemma 5.10, where it is proved that it is a lattice isomorphism.

By Corollary 5.8, we have an injective homomorphism $\mathcal{V}^{\mathrm{gr}}(\mathcal{A}) \to K_0^{\mathrm{gr}}(\mathcal{A})$. By Proposition 5.7, there is a one-to-one correspondence between the order-ideals of M_E^{gr} which are closed under the $\mathbb{Z}$-action and the graded ordered ideals of $K_0^{\mathrm{gr}}(\mathcal{A})$. Finally by (5-19), we have lattice isomorphisms

$$\mathcal{H} \cong \mathcal{L}(M_E) \cong \mathcal{L}^c(M_E^{\mathrm{gr}}) \cong \mathcal{L}(K_0^{\mathrm{gr}}(\mathcal{A})). \qquad \square$$

6. Application: Kumjian–Pask algebras

In this section we will use our result on smash products (Theorem 3.4) to study the structure of Kumjian–Pask algebras [Aranda Pino et al. 2013] and their graded K-groups. We will see that the graded K_0-group remains a useful invariant for studying Kumjian–Pask algebras. We deal exclusively with row-finite k-graphs with no sources: our analysis for arbitrary graphs relied on constructions like desingularisation that are not available in general for k-graphs. We briefly recall the definition of Kumjian–Pask algebras and establish our notation. We follow the conventions used in the literature of this topic (in particular the paths are written from right to left).

Recall that a *graph of rank k* or *k-graph* is a countable category $\Lambda = (\Lambda^0, \Lambda, r, s)$ together with a functor $d : \Lambda \to \mathbb{N}^k$, called the *degree map*, satisfying the following factorisation property: if $\lambda \in \Lambda$ and $d(\lambda) = m + n$ for some $m, n \in \mathbb{N}^k$, then there are unique $\mu, \nu \in \Lambda$ such that $d(\mu) = m, d(\nu) = n$, and $\lambda = \mu\nu$. We say that Λ is *row finite* if $r^{-1}(v) \cap d^{-1}(n)$, abbreviated $v\Lambda^n$ is finite for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$; we say that Λ has *no sources* if each $v\Lambda^n$ is nonempty.

An important example is the *k-graph* Ω_k defined as a set by $\Omega_k = \{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k \mid m \leq n\}$ with $d(m, n) = n - m$, $\Omega_k^0 = \mathbb{N}^k$, $r(m, n) = m$, $s(m, n) = n$ and $(m, n)(n, p) = (m, p)$.

Definition 6.1. Let Λ be a row-finite k-graph without sources and K a field. The *Kumjian–Pask K-algebra* of Λ is the K-algebra $\mathrm{KP}_K(\Lambda)$ generated by $\Lambda \cup \Lambda^*$ subject to the relations

(KP1) $\{v \in \Lambda^0\}$ is a family of mutually orthogonal idempotents satisfying $v = v^*$,

(KP2) for all $\lambda, \mu \in \Lambda$ with $r(\mu) = s(\lambda)$, we have

$$\lambda\mu = \lambda \circ \mu, \quad \mu^*\lambda^* = (\lambda \circ \mu)^*, \quad r(\lambda)\lambda = \lambda = \lambda s(\lambda), \quad s(\lambda)\lambda^* = \lambda^* = \lambda^* r(\lambda),$$

(KP3) for all $\lambda, \mu \in \Lambda$ with $d(\lambda) = d(\mu)$, we have

$$\lambda^*\mu = \delta_{\lambda,\mu} s(\lambda),$$

(KP4) for all $v \in \Lambda^0$ and all $n \in \mathbb{N}^k \setminus \{0\}$, we have

$$v = \sum_{\lambda \in v\Lambda^n} \lambda\lambda^*.$$

Let Λ be a row-finite k-graph without sources and $\mathrm{KP}_K(\Lambda)$ the Kumjian–Pask algebra of Λ. Following [Kumjian and Pask 2000, §2], an infinite path in Λ is a degree-preserving functor $x : \Omega_k \to \Lambda$. Denote the set of all infinite paths by Λ^∞. We define the relation of *tail equivalence* on the space of infinite path Λ^∞ as follows: for $x, y \in \Lambda^\infty$, we say x is tail equivalent to y, denoted, $x \sim y$, if $x(n, \infty) = y(m, \infty)$, for some $n, m \in \mathbb{N}^k$. This is an equivalence relation. For $x \in \Lambda^\infty$, we denote by $[x]$ the equivalence class of x, i.e., the set of all infinite paths which are tail equivalent to x. An infinite path x is called *aperiodic* if $x(n, \infty) = x(m, \infty)$, $n, m \in \mathbb{N}^k$, implies $n = m$.

We can form the skew-product k-graph, or covering graph, $\bar{\Lambda} = \Lambda \times_d \mathbb{Z}^k$ which is equal as a set to $\Lambda \times \mathbb{Z}^k$, has degree map given by $\bar{d}(\lambda, n) = d(\lambda)$, range and source maps $r(\lambda, n) = (r(\lambda), n)$ and $s(\lambda, n) = (s(\lambda), n + d(\lambda))$ and composition given by $(\lambda, n)(\mu, n + d(\lambda)) = (\lambda\mu, n)$.

As in the theory of Leavitt path algebras, one can model Kumjian–Pask algebras as Steinberg algebras via the infinite-path groupoid of the k-graph (see [Clark and Pangalela 2017, Proposition 5.4]). For the k-graph Λ,

$$\mathcal{G}_\Lambda = \{(x, l - m, y) \in \Lambda^\infty \times \mathbb{Z}^k \times \Lambda^\infty \mid x(l, \infty) = y(m, \infty)\}.$$

Define range and source maps $r, s : \mathcal{G}_\Lambda \to \Lambda^\infty$ by $r(x, n, y) = x$ and $s(x, n, y) = y$. The multiplication and inverse are given for $(x, n, y), (y, l, z) \in \mathcal{G}_\Lambda$, by $(x, n, y)(y, l, z) = (x, n + l, z)$ and $(x, n, y)^{-1} = (y, -n, x)$. $\mathcal{G}_\Lambda$ is a groupoid with $\Lambda^\infty = \mathcal{G}_\Lambda^{(0)}$ under the identification $x \mapsto (x, 0, x)$. For

$\mu, \nu \in \Lambda$ with $s(\mu) = s(\nu)$, let $Z(\mu, \nu) := \{(\mu x, d(\mu) - d(\nu), \nu x) \mid x \in \Lambda^\infty, x(0) = s(\mu)\}$. Then the sets $Z(\mu, \nu)$ comprise a basis of compact open sets for an ample Hausdorff topology on $\mathcal{G}_\Lambda$. There is a cocycle $c : \mathcal{G}_\Lambda \to \mathbb{Z}^k$ given by $c(x, m, y) = m$.

For the skew-product k-graph $\bar{\Lambda} = \Lambda \times_d \mathbb{Z}^k$, we have $\mathcal{G}_{\bar{\Lambda}} \cong \mathcal{G}_\Lambda \times_c \mathbb{Z}^k$ (see [Kumjian and Pask 2000, Theorem 5.2]). Thus specialising Theorem 3.4 to this setting, we have

$$\mathrm{KP}_K(\bar{\Lambda}) \cong \mathrm{KP}_K(\Lambda) \# \mathbb{Z}^k. \tag{6-1}$$

We will show that $\mathrm{KP}_K(\bar{\Lambda})$ is an ultramatricial algebra.

Given a set X and a ring R, $M_X(R)$ denotes the collection of finitely supported $X \times X$ matrices with values in R; that is, $M_X(R)$ consists of finitely supported functions from $X \times X$ to R such that the multiplication is given by $(ab)(x, y) = \sum_{z \in X} a(x, z) b(z, y)$.

Lemma 6.2. *For $n \in \mathbb{Z}^k$ define $B_n \subseteq \mathrm{KP}_K(\bar{\Lambda})$ by*

$$B_n = \mathrm{span}_K \big\{ (\lambda, n - d(\lambda))(\mu, n - d(\mu))^* \mid \lambda, \mu \in \Lambda, s(\lambda) = s(\mu) \big\}.$$

Then B_n is a subalgebra of $\mathrm{KP}_K(\bar{\Lambda})$ and there is an isomorphism $B_n \cong \bigoplus_{v \in \Lambda^0} M_{\Lambda v}(K)$ that carries $(\lambda, n - d(\lambda))(\mu, n - d(\mu))^*$ to the matrix unit $e_{\lambda, \mu}$.

Proof. For the first statement we just have to show that for any $\lambda, \mu, \eta, \zeta \in \Lambda$ we have

$$(\lambda, n - d(\lambda))(\mu, n - d(\mu))^*(\eta, n - d(\eta))(\zeta, n - d(\zeta))^* \in B_n.$$

This follows from the argument of [Kumjian and Pask 2000, Lemma 5.4]. To wit, we have

$$(\mu, n - d(\mu))^*(\eta, n - d(\eta)) = 0 \quad \text{unless } r(\mu, n - d(\mu)) = r(\eta, n - d(\eta)),$$

which in turn forces $d(\mu) = d(\eta)$. But then $\bar{d}(\mu, n - d(\mu)) = \bar{d}(\eta, n - d(\eta))$, and then the Cuntz–Krieger relation forces $(\mu, n - d(\mu))^*(\eta, n - d(\eta)) = \delta_{\mu, \eta}(s(\mu), n)$. Hence

$$(\lambda, n - d(\lambda))(\mu, n - d(\mu))^*(\eta, n - d(\eta))(\zeta, n - d(\zeta))^* = \delta_{\mu, \eta}(\lambda, n - d(\lambda))(\zeta, n - d(\zeta))^* \in B_n.$$

For each $v \in \Lambda^0$, $M_{\bar{\Lambda}(v,n)}(K) \cong M_{\Lambda v}(K)$. So the elements $(\lambda, n - d(\lambda))(\mu, n - d(\mu))^*$ satisfy the same multiplication formula as the matrix units $e_{\lambda, \mu}$ in $\bigoplus_{v \in \Lambda^0} M_{\Lambda v}(K)$. Hence the uniqueness of the latter shows that there is an isomorphism as claimed. $\square$

Lemma 6.3. *For $m \le n \in \mathbb{Z}^k$, we have $B_m \subseteq B_n$, and in particular for each $v \in \Lambda^0$, we have $(v, m) = \sum_{\alpha \in v \Lambda^{n-m}} (\alpha, m)(\alpha, m)^*$.*

Proof. Again, this follows from the proof of [Kumjian and Pask 2000, Lemma 5.4]. We just apply the Cuntz–Krieger relation, using at the first equality that Λ has no sources:

$$(\lambda, m - d(\lambda))(\mu, m - d(\mu))^* = (\lambda, m - d(\lambda)) \bigg(\sum_{\alpha \in s(\lambda) \Lambda^{n-m}} (\alpha, m)(\alpha, m)^* \bigg)(\mu, m - d(\mu))^*$$

$$= \sum_{\alpha \in s(\lambda) \Lambda^{n-m}} (\lambda \alpha, m - d(\lambda))(\mu \alpha, m - d(\mu))^* \in B_n.$$

This gives the first assertion, and the second follows by taking $\lambda = \mu = v$.   □

Theorem 6.4. *Let Λ be a row-finite k-graph with no sources and K a field. Then the Kumjian–Pask algebra $\mathrm{KP}_K(\Lambda)$ is a graded von Neumann regular ring.*

Proof. Lemma 2.3 shows that $\mathrm{KP}_K(\Lambda)$ is graded regular if and only if $\mathrm{KP}_K(\Lambda) \# \mathbb{Z}^k$ is graded regular. By (6-1) $\mathrm{KP}_K(\Lambda) \# \mathbb{Z}^k \cong \mathrm{KP}_K(\bar{\Lambda})$ and the latter is an ultramatricial algebra by Lemma 6.3. Since ultramatricial algebras are regular, the theorem follows.   □

Since $\mathrm{KP}_K(\Lambda)$ is graded von Neumann regular, we immediately obtain the following statements.

Theorem 6.5. *Let Λ be a row-finite k-graph with no sources and K a field. Then the Kumjian–Pask algebra $A = \mathrm{KP}_K(\Lambda)$ has the following properties:*

 (1) *any finitely generated right (left) graded ideal of A is generated by one homogeneous idempotent;*

 (2) *any graded right (left) ideal of A is idempotent;*

 (3) *any graded ideal is graded semiprime;*

 (4) *$J(A) = J^{\mathrm{gr}}(A) = 0$; and*

 (5) *there is a one-to-one correspondence between the graded right (left) ideals of A and the right (left) ideals of A_0.*

Proof. All the assertions are the properties of a graded von Neumann regular ring [Hazrat 2016, §1.1.9], so the result follows from Theorem 6.4.   □

For the next result, given a k-graph Λ, and given $m \le n \in \mathbb{Z}^k$, we define $\phi_{m,n} : \mathbb{N}\Lambda^0 \to \mathbb{N}\Lambda^0$ by $\phi_{m,n}(v) = \sum_{w \in \Lambda^0} |v\Lambda^{n-m}w| w$. Here, $\mathbb{N}\Lambda^0$ is the abelian monoid freely generated by Λ^0, and $\phi_{m,n}$ is the unique monoid homomorphism determined by the above rule.

Corollary 6.6. *Let Λ be a row-finite k-graph with no sources and K a field. There is an isomorphism*

$$\mathcal{V}(\mathrm{KP}_K(\bar{\Lambda})) \cong \varinjlim_{\mathbb{Z}^k} \left(\mathbb{N}\Lambda^0, \phi_{m,n} \right)$$

that carries $[(v, n)]$ to the copy of v in the n-th copy of $\mathbb{N}\Lambda^0$. Furthermore, the monoid $\mathcal{V}(\mathrm{KP}_K(\bar{\Lambda}))$ is cancellative.

Proof. It is standard that there is an isomorphism $\mathcal{V}\left(\bigoplus_{v \in \Lambda^0} M_{\Lambda v}(K)\right) \cong \mathbb{N}\Lambda^0$ that takes $e_{\lambda,\lambda}$ to $s(\lambda)$ for all λ. So Lemma 6.2 implies that there is an isomorphism $\mathcal{V}(B_n) \to \mathbb{N}\Lambda^0$ that carries $[(\lambda, n-d(\lambda))(\lambda, n-d(\lambda))^*]$ to $s(\lambda)$ for all λ. Let S_n be a copy $\mathbb{N}\Lambda^0 \times \{n\}$ of the monoid $\mathbb{N}\Lambda^0$ (so $(a, n) + (b, n) = (a + b, n)$ in S_n). Lemma 6.3 shows that these isomorphisms of monoids carry the inclusions $B_m \hookrightarrow B_n$ to the maps $(v, m) \mapsto \sum_{\lambda \in v\Lambda^{n-m}} (s(\lambda), n)$, which is precisely given by the formula $\phi_{m,n}$ for $m \le n \in \mathbb{Z}^k$. Since the monoid of a direct limit is the direct limit of the monoids of the approximating algebras, we have an isomorphism $\mathcal{V}(\mathrm{KP}_K(\bar{\Lambda})) \cong \varinjlim_{\mathbb{Z}^k} S_n$, which sends $[(v, n)]$ to $(v, n) \in S_n$.

Suppose that $x + z = y + z$ in $\mathcal{V}(\mathrm{KP}_K(\bar{\Lambda}))$. By the isomorphism $\mathcal{V}(\mathrm{KP}_K(\bar{\Lambda})) \cong \varinjlim_{\mathbb{Z}^k} S_n$, there exist images x', y', z' of x, y, z, respectively, in $S_{n_0} = \mathbb{N}\Lambda^0 \times \{n_0\}$ for some $n_0 \in \mathbb{Z}^k$ such that $x' + z' = y' + z'$. The monoid $\mathbb{N}\Lambda^0$ is cancellative, so $\mathcal{V}(\mathrm{KP}_K(\bar{\Lambda}))$ is too.   □

Corollary 6.7. *Let Λ be a row-finite k-graph with no sources and K a field. Then*

$$\mathcal{V}^{\mathrm{gr}}(\mathrm{KP}_K(\Lambda)) \cong \varinjlim_{\mathbb{Z}^k}(\mathbb{N}\Lambda^0, \phi_{m,n}).$$

Proof. Recall from (6-1) that $\mathrm{KP}_K(\bar{\Lambda}) \cong \mathrm{KP}_K(\Lambda) \# \mathbb{Z}^k$. Specialising Proposition 2.5 to Kumjian–Pask algebras, we have the isomorphism of categories $\Psi : \mathrm{KP}_K(\Lambda)\text{-}\mathrm{Gr} \xrightarrow{\sim} \mathrm{KP}_K(\bar{\Lambda})\text{-}\mathrm{Mod}$. We argue as in the directed-graph situation that Ψ preserves finitely generated projective objects. By (4-2) and (4-4), we have $\mathcal{V}^{\mathrm{gr}}(\mathrm{KP}_K(\Lambda)) \cong \mathcal{V}(\mathrm{KP}_K(\bar{\Lambda}))$. $\qquad\square$

7. The graded representations of the Steinberg algebra

In this section, for a Γ-graded groupoid $\mathcal{G}$ and its associated Steinberg algebra $A_R(\mathcal{G})$, we construct graded simple $A_R(\mathcal{G})$-modules. Specialising our results to the trivial grading, we obtain irreducible representations of (ungraded) Steinberg algebras. We determine the ideals arising from these representations and prove that these ideals relate to the effectiveness or otherwise of the groupoid.

7A. *Representations of a Steinberg algebra.* Let $\mathcal{G}$ be an ample Hausdorff groupoid, let Γ be a discrete group with identity ε, and let $c : \mathcal{G} \to \Gamma$ be a cocycle. A subset U of the unit space $\mathcal{G}^{(0)}$ of $\mathcal{G}$ is *invariant* if $d(\gamma) \in U$ implies $r(\gamma) \in U$; equivalently,

$$r(d^{-1}(U)) = U = d(r^{-1}(U)).$$

Given an element $u \in \mathcal{G}^{(0)}$, we denote by $[u]$ the smallest invariant subset of $\mathcal{G}^{(0)}$ which contains u. Then

$$r(d^{-1}(u)) = [u] = d(r^{-1}(u)).$$

That is, for any $v \in [u]$, there exists $x \in \mathcal{G}$ such that $d(x) = u$ and $r(x) = v$; equivalently, for any $w \in [u]$, there exists $y \in \mathcal{G}$ such that $d(y) = w$ and $r(y) = u$. Thus for any $v, w \in [u]$, there exists $x \in \mathcal{G}$ such that $d(x) = v$ and $r(x) = w$. We call $[u]$ an *orbit*. Observe that an invariant subset $U \subseteq \mathcal{G}^{(0)}$ is an orbit if and only if for any $v, w \in U$, there exists $x \in \mathcal{G}$ such that $d(x) = v$ and $r(x) = w$.

Lemma 7.1. *Let $u_1, u_2, \ldots, u_n$ be pairwise distinct elements of $\mathcal{G}^{(0)}$ with $n \geq 2$. Then there exist disjoint compact open bisections $B_i \subseteq \mathcal{G}^{(0)}$ such that $u_i \in B_i$ for each $i = 1, \ldots, n$.*

Proof. Since $\mathcal{G}^{(0)}$ is a Hausdorff space, there exist disjoint open subsets X_i of $\mathcal{G}^{(0)}$ such that $u_i \in X_i$ for all i. Since $\mathcal{G}$ is ample, we can choose compact open bisections $B_i \subseteq X_i$ such that $u_i \in B_i$ for all i. $\qquad\square$

The *isotropy group* at a unit u of $\mathcal{G}$ is the group $\mathrm{Iso}(u) = \{\gamma \in \mathcal{G} \mid d(\gamma) = r(\gamma) = u\}$. A unit $u \in \mathcal{G}^{(0)}$ is called Γ-*aperiodic* if $\mathrm{Iso}(u) \subseteq c^{-1}(\varepsilon)$, otherwise u is called Γ-*periodic*. For an invariant subset $W \subseteq \mathcal{G}^{(0)}$, we denote by W_{ap} the collection of Γ-aperiodic elements of W and by W_{p} the collection of Γ-periodic elements of W. Then

$$W = W_{\mathrm{ap}} \bigsqcup W_{\mathrm{p}}.$$

If $W = W_{\mathrm{ap}}$, we say that W is Γ-*aperiodic*; If $W = W_{\mathrm{p}}$, we say that W is Γ-*periodic*.

Remark 7.2. Let E be a directed graph. Let $\mathcal{G}_E$ be the associated graph groupoid and $c : \mathcal{G}_E \to \mathbb{Z}$ the canonical cocycle $c(x, m, y) = m$. It was shown in [Kumjian et al. 1997] that $c^{-1}(0)$ is a principal groupoid, in the sense that $\mathrm{Iso}(c^{-1}(0)) = \mathcal{G}_E^{(0)}$. Hence $x \in \mathcal{G}_E^{(0)} = E^\infty$ is $\mathbb{Z}$-aperiodic if and only if $\mathrm{Iso}(x) = \{x\}$. It is standard that $\mathrm{Iso}(x) = \{x\}$ if and only if $x \neq \mu \lambda^\infty$ for any cycle λ in E. So x is $\mathbb{Z}$-aperiodic if and only if $x \neq \mu \lambda^\infty$ for any cycle λ.

Lemma 7.3. *Let $W \subseteq \mathcal{G}^{(0)}$ be an invariant subset. Then W_{ap} and W_{p} are both invariant subsets of $\mathcal{G}^{(0)}$.*

Proof. For $x \in \mathcal{G}$, let $u = d(x)$ and $v = r(x)$. Suppose that $u \in W_{\mathrm{ap}}$. If $c(y) \neq \varepsilon$ for some $y \in \mathrm{Iso}(v)$, then $x^{-1}yx \in \mathrm{Iso}(u)$ and $\varepsilon \neq c(y) = c(x)c(x^{-1}yx)c(x)^{-1}$, forcing $c(x^{-1}yx) \neq \varepsilon$, a contradiction. Hence, $v = r(x)$ is Γ-aperiodic. Since W is invariant, we have $v \in W_{\mathrm{ap}}$. So W_{ap} is invariant. Since $W = W_{\mathrm{ap}} \sqcup W_{\mathrm{p}}$, it follows that W_{p} is also invariant. $\qquad\square$

By the proof of Lemma 7.3, $u \in \mathcal{G}^{(0)}$ is Γ-aperiodic if and only if its orbit $[u]$ is Γ-aperiodic.

Example 7.4. In this example we construct a $\mathbb{Z}$-aperiodic invariant subset which is neither open nor closed in $\mathcal{G}^{(0)}$. Let E be the following directed graph.
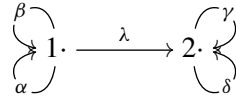
Let u be the infinite path $\alpha\beta\alpha^2\beta\alpha^3\beta\cdots$. Then u is an element in $\mathcal{G}_E^{(0)}$. The orbit $[u]$ consists of all infinite paths tail equivalent to u. So $\alpha^n u \in [u]$ for all $n \in \mathbb{N}$. The sequence $\alpha^n u$ converges to α^∞, which does not belong to $[u]$. So $[u]$ is not closed. Similarly, the points $u_n := \alpha\beta\alpha^2\beta\cdots\alpha^n\beta\alpha^\infty$ all belong to $\mathcal{G}^{(0)} \setminus [u]$, but $u_n \to u$, so $[u]$ is not open. In particular, neither $[u]$ nor its complement is the invariant subset of $\mathcal{G}^{(0)}$ corresponding to any saturated hereditary subset of E^0.

We will employ Γ-aperiodic invariant subsets of $\mathcal{G}^{(0)}$ to obtain graded representations for the Steinberg algebra $A_R(\mathcal{G})$. For any invariant subset $U \subseteq \mathcal{G}^{(0)}$ and a unital commutative ring R, we denote by RU the free R-module with basis U. For every compact open bisection $B \subseteq \mathcal{G}$, there is a function $f_B : \mathcal{G}^{(0)} \to RU$ which has support contained in $d(B) \cap U$ and $f_B(d(\gamma)) = r(\gamma)$ for all $\gamma \in B \cap d^{-1}(U)$. There is a unique representation $\pi_U : A_R(\mathcal{G}) \to \mathrm{End}_R(RU)$ such that

$$\pi_U(1_B)(u) = f_B(u), \tag{7-1}$$

for every compact open bisection B and $u \in U$. This representation makes RU an $A_R(\mathcal{G})$-module (see [Brown et al. 2014, Proposition 4.3]). An $A_R(\mathcal{G})$-submodule $V \subseteq RU$ is called a *basic submodule* of RU if whenever $r \in R \setminus \{0\}$ and $ru \in V$, we have $u \in V$. We say an $A_R(\mathcal{G})$-module is *basic simple* if it has no nontrivial basic submodules.

We can state one of the main results of this section.

Theorem 7.5. *Let U be an invariant subset of $\mathcal{G}^{(0)}$. Then U is a Γ-aperiodic orbit if and only if RU is a graded basic simple $A_R(\mathcal{G})$-module. Furthermore, RU is a graded basic simple $A_R(\mathcal{G})$-module if and only if it is graded and basic simple.*

Proof. Suppose that $u \in \mathcal{G}^{(0)}$ satisfies $U = [u]$, and that $[u]$ is a Γ-aperiodic orbit. We first show that $R[u]$ is a Γ-graded $A_R(\mathcal{G})$-module. For any $\gamma \in \Gamma$, set

$$[u]_\gamma = \{v \in [u] \mid \text{ there exists } x \in \mathcal{G} \text{ such that } c(x) = \gamma, \ d(x) = u \text{ and } r(x) = v\}.$$

We claim that $[u]_\gamma \cap [u]_{\gamma'} \neq \varnothing$ implies $\gamma = \gamma'$. Indeed, if $v \in [u]_\gamma \cap [u]_{\gamma'}$, then there exist $x \in c^{-1}(\gamma)$ and $y \in c^{-1}(\gamma')$ such that $d(x) = d(y) = u$ and $r(x) = r(y) = v$. Now $x^{-1}y \in \mathrm{Iso}(u)$. Since u is Γ-aperiodic this forces $\gamma^{-1}\gamma' = c(x^{-1}y) = \varepsilon$, and so $\gamma = \gamma'$. This gives a partition $[u] = \bigsqcup_{\gamma \in \Gamma}[u]_\gamma$. Therefore $A_R(\mathcal{G})$-module $R[u]$ has a decomposition of R-modules

$$R[u] = \bigoplus_{\gamma \in \Gamma}(R[u])_\gamma,$$

where $(R[u])_\gamma$ is a free R-module with basis $[u]_\gamma$.

We show that $A_R(\mathcal{G})_\alpha \cdot (R[u])_\gamma \subseteq (R[u])_{\alpha\gamma}$, for $\alpha, \gamma \in \Gamma$. Fix $v \in [u]_\gamma$ and $B \in B_\alpha^{\mathrm{co}}(\mathcal{G})$. We use $\cdot$ to denote the action of $A_R(\mathcal{G})$ on RU. We have

$$1_B \cdot v = \begin{cases} r(b) & \text{if } b \in B \text{ satisfies } d(b) = v, \\ 0 & \text{if } v \notin d(B). \end{cases}$$

Clearly $0 \in (R[u])_{\alpha\gamma}$, so suppose that $b \in B$ satisfies $d(b) = v$. Since $v \in [u]_\gamma$, there exists $x \in \mathcal{G}$ such that $c(x) = \gamma$, $d(x) = u$, and $r(x) = v$. Now $d(bx) = u$, $r(bx) = r(b)$, and $c(bx) = c(b)c(x) = \alpha\gamma$. So $r(b) \in [u]_{\alpha\gamma}$. Since elements of the form 1_B where $B \in B_\alpha^{\mathrm{co}}(\mathcal{G})$ span $A_R(\mathcal{G})_\alpha$, we deduce that $A_R(\mathcal{G})_\alpha \cdot (R[u])_\gamma \subseteq (R[u])_{\alpha\gamma}$ as claimed.

Next we show that $R[u]$ is a basic simple $A_R(\mathcal{G})$-module. Suppose that $V \neq 0$ is a basic $A_R(\mathcal{G})$-submodule of $R[u]$. Take a nonzero element $x \in V$. Fix nonzero elements $r_i \in R$ and pairwise distinct $u_i \in [u]$ such that $x = \sum_{i=1}^m r_i u_i$. By Lemma 7.1, there exist disjoint compact open bisections $B_i \subseteq \mathcal{G}^{(0)}$ such that $u_i \in B_i$ for all $i = 1, \ldots, m$. Now

$$1_{B_1} \cdot x = 1_{B_1} \cdot \sum_{i=1}^m r_i u_i = \sum_{i=1}^m r_i(1_{B_1} \cdot u_i) = r_1 f_{B_1}(u_1).$$

Thus $u_1 = f_{B_1}(u_1) \in V$, because V is a basic submodule. Fix $v \in [u]$ and choose $x \in \mathcal{G}$ such that $d(x) = u_1$ and $r(x) = v$. Fix a compact open bisection D containing x. Then $1_D \cdot u_1 = f_D(u_1) = r(x) = v \in V$, giving $V = R[u]$. Thus $R[u]$ is basic simple, and consequently graded basic simple.

For the converse suppose that RU is a graded basic simple $A_R(\mathcal{G})$-module. We first show that U is Γ-aperiodic. Let $u \in U$. We claim that there exists $r \in R \setminus \{0\}$ such that ru is a homogeneous element of RU. To see this, express $u = \sum_{i=1}^l h_i$, where $h_i \neq u$ are homogeneous elements. For each i, express $h_i = \sum_{j=1}^{s_i} \lambda_{ij}u_{ij}$ with $\lambda_{ij} \in R \setminus \{0\}$ and the $u_{ij} \in U$ pairwise distinct. We first show that $u \in \{u_{ij} \mid i = 1, \ldots, l; \ j = 1, \ldots, s_i\}$; for if not, then Lemma 7.1 gives compact open bisections B, B_{ij} such that $u \in B$ and $u \notin B_{ij}$ for all i, j. So $1_B \cdot u \neq 0$, whereas

$$1_B \cdot u = 1_B \cdot \sum_{i=1}^l h_i = 1_B \cdot \sum_{i=1}^l \sum_{j=1}^{s_i} \lambda_{ij}u_{ij} = \sum_{i=1}^l \sum_{j=1}^{s_i} \lambda_{ij}1_B \cdot u_{ij} = 0.$$

This is a contradiction. So $u = u_{ij}$ for some i, j as claimed; without loss of generality, $u = u_{11}$. Hence $h_1 = \lambda_{11} u + \sum_{j=2}^{s_1} \lambda_{1j} u_{1j}$. There exist compact open bisections B', $B'_{1j} \subseteq \mathcal{G}^{(0)} \subseteq c^{-1}(\varepsilon)$ such that $u \in B'$ but $u \notin B'_{1j}$ for $j \neq 1$. Hence $r := \lambda_{11}$ belongs to $R \setminus \{0\}$, and

$$ru = \lambda_{11} 1_{B'} \cdot u = 1_{B'} \cdot h_1$$

is homogeneous as claimed. Now suppose that u is not Γ-aperiodic. Then there exists $x \in \mathrm{Iso}(u)$ with $c(x) \neq \varepsilon$. Fix $D \in B^{\mathrm{co}}_{c(x)}(\mathcal{G})$ containing x. Then $1_D \cdot ru = r 1_D \cdot u = ru$ is homogeneous. Thus $1_D \in A_R(\mathcal{G})_\varepsilon$, forcing $c(x) = \varepsilon$. This is a contradiction. Thus U is Γ-aperiodic.

For the last part of the theorem we prove that U is an orbit. If not then there exist $u, v \in \mathcal{G}^{(0)}$ with $[u] \cap [v] = \varnothing$ and $[u] \sqcup [v] \subseteq U$. Hence $R[u] \subseteq RU \setminus R[v]$ is a nontrivial proper graded basic submodule of RU by the first part of the theorem. This is a contradiction. So U is an orbit. The last statement of the theorem follows from the first part of the proof. $\qquad\square$

Corollary 7.6. *Let $\mathcal{G}$ be an ample Hausdorff groupoid, and U be an invariant subset of $\mathcal{G}^{(0)}$. Then U is an orbit of $\mathcal{G}^{(0)}$ if and only if RU is a basic simple $A_R(\mathcal{G})$-module.*

Proof. Apply Theorem 7.5 with $c : \mathcal{G} \to \{\varepsilon\}$ the trivial grading. $\qquad\square$

Specialising Theorem 7.5 to the case of Leavitt path algebras we obtain irreducible representations for these algebras.

Let K be a field. For an infinite path p in a graph E, Chen constructed the left $L_K(E)$-module $\mathcal{F}_{[p]}$ of the space of infinite paths tail-equivalent to p and proved that it is an irreducible representation of the Leavitt path algebra (see [Chen 2015, Theorem 3.3]). These were subsequently called Chen simple modules and further studied in [Abrams et al. 2015; Ara and Rangaswamy 2014; 2015; Hazrat and Rangaswamy 2016; Rangaswamy 2016]. In the groupoid setting, the infinite path p is an element in $\mathcal{G}_E^{(0)}$. Thus q belongs to the orbit $[p]$ if and only if q is tail-equivalent to p. Applying Corollary 7.6, we immediately obtain that $K[p] = \mathcal{F}_{[p]}$ is an irreducible representation of the Leavitt path algebra. Furthermore, by Theorem 7.5, p is an aperiodic infinite path (irrational path) if and only if $\mathcal{F}_{[p]}$ is a graded module (see [Hazrat and Rangaswamy 2016, Proposition 3.6]).

Recall from [Chen 2015, Theorem 3.3] that $\mathrm{End}_{L_K(E)}(\mathcal{F}_{[p]}) \cong K$. We claim that $\mathrm{End}_{A_R(\mathcal{G})}(R[u]) \cong R$ for $u \in \mathcal{G}_E^{(0)}$. Indeed, let $f : R[u] \to R[u]$ be a nonzero homomorphism of $A_R(\mathcal{G})$-modules. Then $\mathrm{Ker} f$ is a basic submodule of $R[u]$. Since $R[u]$ is basic simple, we deduce that f is injective. For $v \in [u]$, we write $f(v) = \sum_{i=1}^n r_i v_i$ with $0 \neq r_i \in R$ and v_i are distinct. We prove that $n = 1$ and $v = v_1$. For if not, then we may assume that $v \neq v_1$. By Lemma 7.1, there exist disjoint compact open bisections B, $B_1 \subseteq \mathcal{G}^{(0)}$ such that $v \in B$, $v_1 \in B_1$ and $v_i \notin B_1$ for $i \neq 1$. Then $1_{B_1} \cdot f(v) = f(1_{B_1} \cdot v) = 0$. But, $1_{B_1} \cdot f(v) = 1_{B_1} \cdot \sum_{i=1}^n r_i v_i = r_1 v_1$ which is a contradiction.

Likewise, Theorem 7.5 specialises to k-graph groupoids, giving new information about Kumjian–Pask algebras.

Corollary 7.7. *Let Λ be a row-finite k-graph without sources and $\mathrm{KP}_K(\Lambda)$ the Kumjian–Pask algebra of Λ. Then:*

(1) *for an infinite path $x \in \Lambda^\infty$, $K[x]$ is a simple left $\mathrm{KP}_K(\Lambda)$-module;*

(2) *for $x, y \in \Lambda^\infty$, we have $K[x] \cong K[y]$ if and only if $x \sim y$; and*

(3) *for $x \in \Lambda^\infty$, $K[x]$ is a graded module if and only if x is an aperiodic path.*

Proof. For (1), the equivalence class of x is the orbit of $\mathcal{G}_\Lambda^{(0)}$ which contains x. By (7-1) and Corollary 7.6, the statement follows directly. For (2), let $\phi : F([x]) \to F([y])$ be an isomorphism. Write $\phi(x) = \sum_{i=1}^l r_i y_i$, where $y_i \sim y$ are all distinct. If $x = y_i$, for some i, then by transitivity of $\sim$, $x \sim y$ and we are done. Otherwise one can choose $n \in \mathbb{N}^k$ such that all $y_i(0, n)$ and $x(0, n)$ are distinct. Setting $a = y_1(0, n)$, we have $0 = \phi(a^* x) = a^* \phi(x) = y_1(n, \infty)$, which is not possible unless $x = y_1$ and $l = 1$. This gives that $x \sim y$. The converse is clear. The statement (3) follows immediately by Theorem 7.5. □

7B. *The annihilator ideals and effectiveness of groupoids.* In this section, we describe the annihilator ideals of the graded modules over a Steinberg algebra and prove that these ideals reflect the effectiveness of the groupoid.

As in previous sections, we assume that $\mathcal{G}$ is a Γ-graded ample Hausdorff groupoid which has a basis of graded compact open bisections. Let R be a commutative ring with identity and $A_R(\mathcal{G})$ the Γ-graded Steinberg algebra associated to $\mathcal{G}$.

Let $W \subseteq \mathcal{G}^{(0)}$ be an invariant subset. We write $\mathcal{G}_W := d^{-1}(W)$ which coincides with the restriction $\mathcal{G}|_W = \{x \in \mathcal{G} \mid d(x) \in W, r(x) \in W\}$. Notice that $\mathcal{G}_W$ is a groupoid with unit space W.

Observe that the interior W° of an invariant subset W is invariant. Indeed, $r(d^{-1}(W^\circ))$ is an open subset of $\mathcal{G}^{(0)}$, since W° is an open subset of $\mathcal{G}^{(0)}$. Since W is invariant, $r(d^{-1}(W^\circ)) \subseteq W$. Thus $r(d^{-1}(W^\circ)) \subseteq W^\circ$. It follows that the closure W^- of W is also an invariant subset of $\mathcal{G}^{(0)}$, since $W^- = \mathcal{G}^{(0)} \setminus (\mathcal{G}^{(0)} \setminus W)^\circ$.

Recall from (7-1) that

$$\pi_W : A_R(\mathcal{G}) \to \mathrm{End}_R(RW)$$

makes RW an $A_R(\mathcal{G})$-module.

Lemma 7.8. *Let $W \subseteq \mathcal{G}^{(0)}$ be an invariant subset of the unit space of $\mathcal{G}$, and let $U = (\mathcal{G}^{(0)} \setminus W)^\circ$. Then*

$$A_R(\mathcal{G}_U) \subseteq \mathrm{Ann}_{A_R(\mathcal{G})}(RW).$$

Proof. For any $f \in A_R(\mathcal{G}_U)$, we write $f = \sum_{k=1}^m r_k 1_{B_k}$ with $B_k \subseteq \mathcal{G}_U$ compact open bisections of $\mathcal{G}$ and $r_k \in R$ nonzero scalars. Since $d(B_k) \subseteq U$, we have $d(B_k) \cap W = \varnothing$. Thus $f \cdot w = 0$ for any $w \in W$, and hence $f \in \mathrm{Ann}_{A_R(\mathcal{G})}(RW)$. □

From now on, $W \subseteq \mathcal{G}^{(0)}$ is a Γ-aperiodic invariant subset. We have

$$W = \bigcup_{u \in W} [u].$$

Of course, two elements of W may belong to the same orbit.

Recall from Theorem 7.5 that if $u \in \mathcal{G}^{(0)}$ is Γ-aperiodic, then $R[u]$ is a Γ-graded $A_R(\mathcal{G})$-module. Therefore RW is a Γ-graded $A_R(\mathcal{G})$-module. In order to construct graded representations for $A_R(\mathcal{G})$, we need to consider the "closed" subgroups of $\mathrm{End}_R(FW)$ defined in (7-1). Namely, we consider the subgroup $\mathrm{END}_R(RW) = \bigoplus_{\gamma \in \Gamma} \mathrm{Hom}_R(RW, RW)_\gamma$, where each component $\mathrm{Hom}_R(RW, RW)_\gamma$ consists of R-maps of degree γ.

Then the map

$$\pi_W : A_R(\mathcal{G}) \to \mathrm{END}_R(RW) \tag{7-2}$$

given by the $A_R(\mathcal{G})$-module action is a homomorphism of Γ-graded algebras. To prove that π_W preserves the grading, fix $\alpha \in \Gamma$ and $B \in B_\alpha^{\mathrm{co}}(\mathcal{G})$. Take $u \in W$ and $v \in [u]$. Fix $x \in \mathcal{G}$ with $d(x) = u$ and $r(x) = v$, and put $\beta = c(x)$ so that $v \in [u]_\beta$. Then

$$\pi_W(1_B)(v) = \begin{cases} r(\gamma) & \text{if } v = d(\gamma) \text{ for some } \gamma \in B, \\ 0 & \text{otherwise.} \end{cases}$$

Since $c(\gamma x) = \alpha\beta$, we obtain $\pi_W(1_B) \in \mathrm{Hom}_R(RW, RW)_\alpha$.

Recall that an ample Hausdorff groupoid $\mathcal{G}$ is *effective* if $\mathrm{Iso}(\mathcal{G})^\circ = \mathcal{G}^{(0)}$, where $\mathrm{Iso}(\mathcal{G}) = \bigsqcup_{u \in \mathcal{G}^{(0)}} \mathrm{Iso}(u)$. It follows that $\mathcal{G}$ is effective if and only if for any nonempty $B \in B_*^{\mathrm{co}}(\mathcal{G})$ with $B \cap \mathcal{G}^{(0)} = \varnothing$, we have $B \not\subseteq \mathrm{Iso}(\mathcal{G})$ (see [Brown et al. 2014, Lemma 3.1] for other equivalent conditions).

We need the following graded uniqueness theorem for Steinberg algebras.

Lemma 7.9 [Clark and Edie-Michell 2015, Theorem 3.4]. *Let $\mathcal{G}$ be a Γ-graded ample Hausdorff groupoid such that $c^{-1}(\varepsilon)$ is effective. If $\pi : A_R(\mathcal{G}) \to A$ is a graded R-algebra homomorphism with $\mathrm{Ker}(\pi) \neq 0$ then there is a compact open subset $B \subseteq \mathcal{G}^{(0)}$ and $r \in R \setminus \{0\}$ such that $\pi(r1_B) = 0$.*

The following key lemma will be used to determine the annihilator ideal of the $A_R(\mathcal{G})$-module RW. This is a generalisation of [Brown et al. 2014, Proposition 4.4] adapted to the graded setting. Recall that if $\mathcal{G}$ is a graded groupoid with grading given by the cocycle $c : \mathcal{G} \to \Gamma$, then $c^{-1}(\varepsilon)$ is a (trivially graded) clopen subgroupoid of $\mathcal{G}$.

Lemma 7.10. *Let $W \subseteq \mathcal{G}^{(0)}$ be a Γ-aperiodic invariant subset and $\pi_W : A_R(\mathcal{G}) \to \mathrm{END}_R(RW)$ the homomorphism of Γ-graded algebras given in (7-2). Then π_W is injective if and only if W is dense in $\mathcal{G}^{(0)}$ and $c^{-1}(\varepsilon)$ is effective.*

Proof. Suppose π_W is injective and there exists an open subset K of $\mathcal{G}^{(0)}$ such that $K \cap W = \varnothing$. We have $K = \bigcup_i B_i$, where B_i are compact open bisections of $\mathcal{G}$. So $B_i \cap W = \varnothing$ for each i, giving $\pi_W(1_{B_i}) = 0$, a contradiction. Thus for any open subset K of $\mathcal{G}^{(0)}$, $K \cap W \neq \varnothing$. Therefore W is dense in $\mathcal{G}^{(0)}$.

Suppose now that $c^{-1}(\varepsilon)$ is not effective. Then there exists a nonempty compact open bisection $B \subseteq c^{-1}(\varepsilon) \setminus \mathcal{G}^{(0)}$ such that $d(b) = r(b)$ for all $b \in B$. We have that $d(B) \neq B$ and that B is a compact open bisection of $\mathcal{G}$. Thus $1_B - 1_{d(B)} \in \mathrm{Ker}(\pi_W)$. This is a contradiction. Hence, $c^{-1}(\varepsilon)$ is effective.

For the converse, Lemma 7.9 implies that it suffices to prove that for any compact open subset $B \subseteq \mathcal{G}^{(0)}$ and $r \in R \setminus \{0\}$, $\pi_W(r1_B) \neq 0$. Since W is dense in $\mathcal{G}^{(0)}$, we have $B \cap W \neq \varnothing$. There exists $w \in B \cap W$ such that $\pi_W(r1_B)(w) \neq 0$, proving $\pi_W(r1_B) \neq 0$. $\qquad\square$

If the group Γ is trivial, then by Lemma 7.10, for an invariant subset $W \subseteq \mathcal{G}^{(0)}$, the homomorphism $\pi_W : A_R(\mathcal{G}) \to \mathrm{End}_R(RW)$ is injective if and only if W is dense in $\mathcal{G}^{(0)}$ and the groupoid $\mathcal{G}$ is effective.

The following is the main result of this section.

Theorem 7.11. *Let $\mathcal{G}$ be a Γ-graded ample Hausdorff groupoid, R a commutative ring with identity and $A_R(\mathcal{G})$ the Steinberg algebra associated to $\mathcal{G}$. The following statements are equivalent*:

(i) *Let $W \subseteq \mathcal{G}^{(0)}$ be a Γ-aperiodic invariant subset and W^- the closure of W. Then the groupoid $(c|_{\mathcal{G}_{W^-}})^{-1}(\varepsilon)$ is effective*;

(ii) *For any Γ-aperiodic invariant subset $W \subseteq \mathcal{G}^{(0)}$,*

$$\mathrm{Ann}_{A_R(\mathcal{G})}(RW) = A_R(\mathcal{G}_U),$$

where $U = (\mathcal{G}^{(0)} \setminus W)^\circ$ is the interior of the invariant subset $\mathcal{G}^{(0)} \setminus W$.

Proof. (i) $\Rightarrow$ (ii). Let $W \subseteq \mathcal{G}^{(0)}$ be a Γ-aperiodic invariant subset. By Theorem 7.5, RW is a graded $A_R(\mathcal{G})$-module. By Lemma 7.8, we have $A_R(\mathcal{G}_U) \subseteq \mathrm{Ann}_{A_R(\mathcal{G})}(RW)$ with $U = (\mathcal{G}^{(0)} \setminus W)^\circ$. It follows that RW is an $A_R(\mathcal{G})/A_R(\mathcal{G}_U)$-module. By [Clark et al. 2016, Lemma 3.6], we have an exact sequence of canonical ring homomorphisms

$$0 \to A_R(\mathcal{G}_U) \to A_R(\mathcal{G}) \to A_R(\mathcal{G}_D) \to 0,$$

where $D = \mathcal{G}^{(0)} \setminus U$. The homomorphisms are induced by extensions from $\mathcal{G}_U$ to $\mathcal{G}$ and restrictions from $\mathcal{G}$ to $\mathcal{G}_D$, respectively. One can easily check that the homomorphisms are graded. It therefore follows that the quotient algebra $A_R(\mathcal{G})/A_R(\mathcal{G}_U)$ is graded isomorphic to $A_R(\mathcal{G}_D)$. It follows that RW is a Γ-graded $A_R(\mathcal{G}_D)$-module (this also follows from Theorem 7.5). We denote by $\hat{\pi}_W : A_R(\mathcal{G}_D) \to \mathrm{END}_R(RW)$ the induced graded homomorphism. Observe that $(\mathcal{G}_D)^{(0)} = D$ is the closure of W. Thus by Lemma 7.10, the homomorphism $\hat{\pi}_W$ is injective. This implies that RW is a faithful $A_R(\mathcal{G}_D)$-module. Hence, the annihilator ideal of RW as an $A_R(\mathcal{G})$-module is $A_R(\mathcal{G}_U)$.

(ii) $\Leftarrow$ (i). Let D denote the closure of W in $\mathcal{G}^{(0)}$. Then RW is a faithful $A_R(\mathcal{G}_D)$-module. So the result follows from Lemma 7.10. $\square$

Recall that a groupoid $\mathcal{G}$ is *strongly effective* if for every nonempty closed invariant subset D of $\mathcal{G}^{(0)}$, the groupoid $\mathcal{G}_D$ is effective.

Remark 7.12. (1) If $c^{-1}(\varepsilon)$ is strongly effective, then Theorem 7.11(i) holds. In fact, a closed invariant subset D of the unit space of $\mathcal{G}$ is in particular a closed $c^{-1}(\varepsilon)$-invariant subset of $\mathcal{G}^{(0)}$. We have $c^{-1}(\varepsilon)_D = c^{-1}(\varepsilon) \cap \mathcal{G}_D = (c|_{\mathcal{G}_D})^{-1}(\varepsilon)$. Hence, Theorem 7.11(i) follows directly. Example 7.13 below, on the other hand, shows that Theorem 7.11(i) does not imply that $c^{-1}(\varepsilon)$ is strongly effective.

(2) Resume the notation of Example 7.4, so $u = \alpha\beta\alpha^2\beta\cdots \in E^\infty$. Let D be the closure of the $\mathbb{Z}$-aperiodic invariant subset $[u] \subseteq \mathcal{G}_E^{(0)}$. As we saw in that example, D is not itself $\mathbb{Z}$-aperiodic, because it contains α^∞.

Example 7.13. It is easy to construct examples of Γ-graded groupoids with no Γ-aperiodic points. For example, let X be the Cantor set. Regard $\mathcal{G} = X \times \mathbb{Z}^2$ as a groupoid with unit space $X \times \{0\}$ identified with X by setting $r(x, m) = x = d(x, m)$ and defining composition and inverses by $(x, n)(x, m) = (x, m + n)$ and $(x, m)^{-1} = (x, -m)$. The map $c : \mathcal{G} \to \mathbb{Z}$ given by $c(x, (m_1, m_2)) = m_1$ is a cocycle. We have $c^{-1}(0) = X \times (\{0\} \times \mathbb{Z})$, which is not effective (for example $X \times \{(0, 1)\}$ is a compact open bisection contained in the isotropy subgroupoid of $c^{-1}(0)$). Moreover, $\mathcal{G}^{(0)}$ has no $\mathbb{Z}$-aperiodic points because $\{u\} \times (\mathbb{Z} \times \{0\}) \subseteq \mathrm{Iso}(u) \setminus c^{-1}(0)$ for all $u \in \mathcal{G}^{(0)}$; so every $u \in \mathcal{G}^{(0)}$ is $\mathbb{Z}$-periodic.

Applying Theorem 7.11 to the trivial grading, we obtain a new characterisation of strong effectiveness.

Corollary 7.14. *Let $\mathcal{G}$ be an ample Hausdorff groupoid*, *and R be a commutative ring with identity. Then $\mathcal{G}$ is strongly effective if and only if for any invariant subset W of $\mathcal{G}^{(0)}$, the annihilator of the $A_R(\mathcal{G})$-module RW is $A_R(\mathcal{G}_U)$, where $U = (\mathcal{G}^{(0)} \setminus W)^\circ$.*

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