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Stark systems over Gorenstein local rings

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In this paper, we define a Stark system over a complete Gorenstein local ring with a finite residue field. Under some standard assumptions, we show that the module of Stark systems is free of rank 1 and that these systems control all the higher Fitting ideals of the Pontryagin dual of the dual Selmer group. This is a generalization of the theory, developed by B. Mazur and K. Rubin, on Stark (or Kolyvagin) systems over principal ideal local rings. Applying our result to a certain Selmer structure over the cyclotomic Iwasawa algebra, we propose a new method for controlling Selmer groups using Euler systems.

1. Introduction

Euler systems were introduced by V. A. Kolyvagin in order to study Selmer groups. An Euler system is a collection of cohomology classes with certain norm relations. He constructed a collection of cohomology classes which is called Kolyvagin's derivative classes from an Euler system. He used these classes which come from the Euler system of Heegner points to bound the order of the Selmer group of certain elliptic curves over imaginary quadratic fields. Note that the Euler system of Heegner points does not fit in the formalism of this paper; ring class fields of an imaginary quadratic field are used to define this Euler system, while we consider an Euler system with respect to ray class fields in this paper. By using the Euler system of the cyclotomic units, K. Rubin gave another proof of the classical Iwasawa main conjecture over $\mathbb{Q}$, which had been proved by B. Mazur and A. Wiles. After that, many mathematicians, especially K. Kato, B. Perrin-Riou, and K. Rubin, have studied Selmer groups using Euler systems.

The images of Kolyvagin's derivative classes in the local cohomology groups satisfy certain relations which come from the norm relations of Euler systems. B. Mazur and K. Rubin noticed that, after minor modification, these classes satisfy certain additional local conditions. In [Mazur and Rubin 2004], they defined a Kolyvagin system to be a system of cohomology classes satisfying these good interrelations.

To explain their work in [Mazur and Rubin 2004; 2016], we introduce some notation. Let K be a number field and G_K denote the absolute Galois group of K. Let R be a complete noetherian local ring with a finite residue field of odd characteristic and T be a free R-module with an R-linear continuous G_K-action which is unramified outside a finite set of places of K. Let $\mathcal{F}$ be a Selmer structure on T; see Definition 3.1 or [Mazur and Rubin 2004, Definition 2.1.1]. Suppose that T satisfies Hypothesis 3.12 and $\mathcal{F}$ is cartesian (see Definition 3.8). Let $\chi(\mathcal{F}) \in \mathbb{Z}$ be the core rank of $\mathcal{F}$ (see Definition 3.19).

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Suppose that R is a principal ideal local ring and $\chi(\mathcal{F}) = 1$. In [Mazur and Rubin 2004], the authors proved that the module of Kolyvagin systems for $\mathcal{F}$ is free of rank 1 and that the R-module structure of the dual Selmer group associated with $\mathcal{F}$ is determined by these systems. Furthermore, if R is the ring of integers in a finite extension of the field $\mathbb{Q}_p$, then the authors constructed a canonical morphism from the module of Euler systems to the module of Kolyvagin systems for the canonical Selmer structure (see Example 3.4).

We still assume that R is a principal ideal local ring. Suppose that $\chi(\mathcal{F}) > 0$. T. Sano [2014] and B. Mazur and K. Rubin [2016] defined Stark systems, independently (Sano called them unit systems). Mazur and Rubin [2016] removed the assumption $\chi(\mathcal{F}) = 1$ by using Stark systems instead of Kolyvagin systems. Namely, they proved that the R-module structure of the dual Selmer group associated with $\mathcal{F}$ is determined by Stark systems. Furthermore, they showed that the module of Stark systems is free of rank 1 and that there is an isomorphism from the module of Stark systems to the module of Kolyvagin systems if $\chi(\mathcal{F}) = 1$.

To attempt Iwasawa main conjectures using an Euler system, we need a Stark system over a complete regular local ring. In [Mazur and Rubin 2016], the elementary divisor theorem is used to define the module of Stark systems. Hence Stark systems are not defined in that paper when R is not a principal ideal ring. We therefore have two important problems. The first problem is to define a Stark system over an arbitrary complete noetherian local ring. The second problem is to control Selmer groups using Stark systems. We obtain the following results in relation to these problems.

Stark systems over Gorenstein local rings. Suppose that R is a zero-dimensional Gorenstein local ring. To define a Stark system over R, we use a natural generalization of Rubin's lattice, called the exterior bidual, instead of the exterior power. In [Burns et al. 2016], D. Burns, M. Kurihara, and T. Sano used the exterior bidual to state some conjectures about Rubin–Stark elements.

By using the exterior bidual, we generalize some linear-algebraic lemmas in [Mazur and Rubin 2016] to zero-dimensional Gorenstein local rings (see Section 2). Thus, in the same way as that paper, we are able to define a Stark system over R. Furthermore, under the assumptions that T satisfies Hypothesis 3.12 and that $\mathcal{F}$ is cartesian (which are precisely the same kind of the assumptions in [Mazur and Rubin 2004; 2016]), we prove that the module of Stark systems over R is free of rank 1 (Theorem 4.7).

When R is an arbitrary complete Gorenstein local ring, we can construct an inverse system of the modules of Stark systems over some zero-dimensional Gorenstein quotients of R. We define the module of Stark systems over R to be its inverse limit and also prove that this module is free of rank 1 (Theorem 5.4).

Remark 1.1. When R is the cyclotomic Iwasawa algebra, $\mathcal{F}$ is the canonical Selmer structure, and $\chi(\mathcal{F}) = 1$, K. Büyükboduk [2011] proved that the module of Kolyvagin systems for $\mathcal{F}$ is free of rank 1. More generally, when R is a certain Gorenstein local ring, $\mathcal{F}$ is a certain Selmer structure, and $\chi(\mathcal{F}) = 1$, he showed that the module of Kolyvagin systems for $\mathcal{F}$ is free of rank 1 in [Büyükboduk 2016]. In this case, we are able to construct an isomorphism from the module of Kolyvagin systems to the module of Stark systems. Hence Theorem 5.4 is a generalization of these results.

Controlling Selmer groups using Stark systems. Under the same assumptions as above, we prove that all the higher Fitting ideals of the Pontryagin dual of the dual Selmer group associated with $\mathcal{F}$ are determined by Stark systems for $\mathcal{F}$ when R is a complete Gorenstein local ring (Theorem 5.6).

Remark 1.2. If R is a zero-dimensional principal ideal ring and M is a finitely generated R-module, then there is a noncanonical isomorphism $M \simeq \mathrm{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ as R-modules, which implies

$$\mathrm{Fitt}^i_R(M) = \mathrm{Fitt}^i_R(\mathrm{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p))$$

for any nonnegative integer i. Thus Theorem 5.6 is a generalization of [Mazur and Rubin 2016, Theorem 8.6] (see Proposition 4.17 and Remark 5.7).

Applying our result to a certain Selmer structure over the cyclotomic Iwasawa algebra Λ, we prove that all the higher Fitting ideals of the Pontryagin dual of the Λ-adic dual Selmer group are determined by Λ-adic Stark systems (Theorem 6.10). In particular, we show that Λ-adic Stark systems determine not only the characteristic ideal of this Λ-module but also its pseudoisomorphism class. In [Mazur and Rubin 2004], the authors showed that a primitive Λ-adic Kolyvagin system controls the characteristic ideal of the Pontryagin dual of the Λ-adic dual Selmer group when the core rank is 1. We will show that Theorem 6.10 is a generalization of this result (Proposition 6.11). However, our proof of Theorem 6.10 is different from the proof in [Mazur and Rubin 2004]. Furthermore, there is a canonical map from the module of Euler systems to the module of Λ-adic Stark systems when the core rank is 1. Hence we propose a new method for controlling Selmer groups using Euler systems.

Remark 1.3. After the author had almost all the results in this paper, T. Sano told the author that D. Burns and he also were studying Kolyvagin and Stark systems over zero-dimensional Gorenstein rings using an exterior bidual; see [Burns and Sano 2016].

Notation. Let p be an odd prime number and K a number field. Let K_v denote the completion of K at a place v. For a field L, let $\bar{L}$ denote a fixed separable closure of L and $G_L := \mathrm{Gal}(\bar{L}/L)$ the absolute Galois group of L.

Throughout this paper, R denotes a complete noetherian local ring with a finite residue field $\Bbbk$ of characteristic p. Let $\mathfrak{m}_R$ be the maximal ideal of R. In addition, T denotes a free R-module of finite rank with an R-linear continuous G_K-action which is unramified outside a finite set of places of K. For any nonnegative integer i, let $H^i(K, T)$ be the i-th continuous cohomology group of G_K with coefficient T.

If A is a commutative ring and I is an ideal of A, then define

$$M^* := \mathrm{Hom}_A(M, A)$$

and $M[I] := \{x \in M \mid Ix = 0\}$ for any A-module M. When $I = (x_1, \dots, x_n)$, we write $M[x_1, \dots, x_n]$ instead of $M[I]$. Let $\bigwedge^r_A M$ denote the r-th exterior power of M in the category of A-modules. We denote by $\ell_A(M) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ the A-length of M.

For any topological $\mathbb{Z}_p$-module M, we define the Pontryagin dual $M^\vee$ of M to be

$$M^\vee := \mathrm{Hom}_{\mathrm{cont}}(M, \mathbb{Q}_p/\mathbb{Z}_p).$$

2. Preliminaries

Let A be a commutative ring, M an A-module, and r a nonnegative integer. We define the r-th exterior bidual $\bigcap_A^r M$ of M by

$$\bigcap_A^r M := \Big(\bigwedge_A^r M^*\Big)^* = \mathrm{Hom}_A\Big(\bigwedge_A^r \mathrm{Hom}_A(M, A), A\Big).$$

The pairing

$$\bigwedge_A^r M \times \bigwedge_A^r M^* \to A, \quad (m_1 \wedge \cdots \wedge m_r, f_1 \wedge \cdots \wedge f_r) \mapsto \det(f_i(m_j)),$$

induces an A-homomorphism

$$\xi_M^r : \bigwedge_A^r M \to \bigcap_A^r M.$$

Note that the map ξ_M^r is an isomorphism if M is a finitely generated projective A-module. However ξ_M^r is, in general, neither injective nor surjective.

Lemma 2.1. *Let $F \xrightarrow{h} M \xrightarrow{g} N \to 0$ be an exact sequence of A-modules and r a nonnegative integer. If F is a free A-module of rank $s \leq r$, then there is a unique A-homomorphism*

$$\phi : \det(F) \otimes_A \bigwedge_A^{r-s} N \to \bigwedge_A^r M$$

such that

$$\phi(a \otimes \wedge^{r-s} g(b)) = (\wedge^s h(a)) \wedge b$$

for any $a \in \det(F)$ and $b \in \bigwedge_A^{r-s} M$.

Proof. Since g is surjective and $\bigwedge_A^{s+1} F = 0$, the map ϕ is well-defined and characterized by the relation $\phi(a \otimes \wedge^{r-s} g(b)) = (\wedge^s h(a)) \wedge b$. □

Lemma 2.2. *Let r be a nonnegative integer. Suppose that we have the following commutative diagram of A-modules*:

$$\begin{array}{ccccccc} F & \xrightarrow{h} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ F' & \xrightarrow{h'} & M' & \xrightarrow{g'} & N' & \longrightarrow & 0 \end{array}$$

where the horizontal rows are exact and both F and F' are free A-modules of rank $s \leq r$. Then the following diagram is commutative:

$$\begin{array}{ccc} \det(F) \otimes_A \bigwedge_A^{r-s} N & \xrightarrow{\phi} & \bigwedge_A^r M \\ \downarrow{\scriptstyle \wedge^s \alpha \otimes \wedge^{r-s} \gamma} & & \downarrow{\scriptstyle \wedge^r \beta} \\ \det(F') \otimes_A \bigwedge_A^{r-s} N' & \xrightarrow{\phi'} & \bigwedge_A^r M' \end{array}$$

where ϕ and ϕ' are the maps defined in Lemma 2.1.

Proof. Let $a \in \det(F)$ and $b \in \bigwedge_A^{r-s} M$. By Lemma 2.1, we have

$$\begin{aligned}(\phi' \circ (\wedge^s \alpha \otimes \wedge^{r-s} \gamma))(a \otimes \wedge^{r-s} g(b)) &= \phi'\big(\wedge^s \alpha(a) \otimes \wedge^{r-s}(g' \circ \beta)(b)\big) \\ &= (\wedge^s (h' \circ \alpha)(a)) \wedge (\wedge^{r-s} \beta(b)) \\ &= \wedge^r \beta((\wedge^s h(a)) \wedge b) \\ &= (\wedge^r \beta \circ \phi)(a \otimes \wedge^{r-s} g(b)).\end{aligned}$$

□

Suppose that A is a zero-dimensional Gorenstein local ring. Note that all free A-modules are injective by the definition of A. In particular, the functor $(-)^* := \mathrm{Hom}_A(-, A)$ is exact. Furthermore, by Matlis duality, we have $\ell_A(M) = \ell_A(M^*)$ and the canonical map $\xi_M^1 : M \to \bigcap_A^1 M$ is an isomorphism for any finitely generated A-module M.

For $i \in \{1, 2\}$, let M_i be an A-module and F_i a free A-module of rank s_i. Suppose that the following diagram is cartesian:

$$\begin{array}{ccc} M_1 & \hookrightarrow & M_2 \\ \downarrow & & \downarrow \\ F_1 & \hookrightarrow & F_2 \end{array}$$

where the horizontal maps are injective. Note that F_2/F_1 is a free A-module of rank $s_2 - s_1$ since F_1 is an injective A-module. Applying Lemma 2.1 to the exact sequence

$$(F_2/F_1)^* \to M_2^* \to M_1^* \to 0,$$

we obtain an A-homomorphism

$$\widetilde{\Phi} : \det((F_2/F_1)^*) \otimes_A \bigcap\nolimits_A^r M_2 \to \bigcap\nolimits_A^{r-s_2+s_1} M_1$$

for any nonnegative integer $r \geq s_2 - s_1$. Therefore we get the following map, which plays an important role in this paper.

Definition 2.3. For any cartesian diagram as above and a nonnegative integer $r \geq s_2 - s_1$, we define an A-homomorphism

$$\begin{aligned}\Phi : \det(F_2^*) \otimes_A \bigcap\nolimits_A^r M_2 &\xrightarrow{\sim} \det((F_2/F_1)^*) \otimes_A \det(F_1^*) \otimes_A \bigcap\nolimits_A^r M_2 \\ &\longrightarrow \det(F_1^*) \otimes_A \bigcap\nolimits_A^{r-s_2+s_1} M_1,\end{aligned}$$

where the first map is induced by the isomorphism

$$\det((F_2/F_1)^*) \otimes_A \det(F_1^*) \xrightarrow{\sim} \det(F_2^*), \quad a \otimes b \longrightarrow a \wedge \tilde{b},$$

where $\tilde{b}$ is a lift of b in $\bigwedge_A^{s_1} F_2^*$ and the second map is induced by $\widetilde{\Phi}$.

The map Φ is a generalization of the map defined in [Mazur and Rubin 2016, Proposition A.1]. Furthermore, we have the following proposition which is a generalization of [loc. cit., Proposition A.2] to zero-dimensional Gorenstein local rings.

Proposition 2.4. *Let A be a zero-dimensional Gorenstein local ring. Suppose that we have the following commutative diagram of A-modules*:

$$\begin{array}{ccccc} M_1 & \hookrightarrow & M_2 & \hookrightarrow & M_3 \\ \downarrow & & \downarrow & & \downarrow \\ F_1 & \hookrightarrow & F_2 & \hookrightarrow & F_3 \end{array}$$

where F_1, F_2, and F_3 are free of finite rank and the two squares are cartesian. Let $s_i = \mathrm{rank}_A(F_i)$ and $r \geq s_3 - s_1$. If $i, j \in \{1, 2, 3\}$ and $j < i$, we denote by

$$\Phi_{ij} : \det(F_i^*) \otimes_A \bigcap\nolimits_A^{r-s_3+s_i} M_i \to \det(F_j^*) \otimes_A \bigcap\nolimits_A^{r-s_3+s_j} M_j$$

the map given by Definition 2.3. Then we have

$$\Phi_{31} = \Phi_{21} \circ \Phi_{32}.$$

Proof. We may assume that $F_1 = 0$. Hence we have $s_1 = 0$. For $i \in \{1, 2\}$, applying Lemma 2.1 to the exact sequence $(F_{i+1}/F_i)^* \xrightarrow{h_i} M_{i+1}^* \xrightarrow{g_i} M_i^* \to 0$, we have a map

$$\phi_i : \det((F_{i+1}/F_i)^*) \otimes_A \bigwedge\nolimits_A^{r-s_3+s_i} M_i^* \to \bigwedge\nolimits_A^{r-s_3+s_{i+1}} M_{i+1}^*.$$

Let

$$\phi_3 : \det(F_3^*) \otimes_A \bigwedge\nolimits_A^{r-s_3} M_1^* \to \bigwedge\nolimits_A^{r} M_3^*$$

be the map defined by the exact sequence $F_3^* \xrightarrow{h_3} M_3^* \xrightarrow{g_3} M_1^* \to 0$ using Lemma 2.1. Put $q : F_3^* \to F_2^*$. Let $x \in \det((F_3/F_2)^*)$, $y \in \bigwedge_A^{s_2} F_3^*$, and $z \in \bigwedge_A^{r-s_3} M_3^*$. Then we compute

$$\begin{aligned}(\phi_2 \circ (\mathrm{id} \otimes \phi_1))(x \otimes \wedge^{s_2} q(y) \otimes \wedge^{r-s_3} g_3(z)) &= \phi_2\big(x \otimes ((\wedge^{s_2}(h_1 \circ q)(y)) \wedge (\wedge^{r-s_3} g_2(z)))\big) \\ &= (\wedge^{s_3-s_2} h_2(x)) \wedge (\wedge^{s_2} h_3(y)) \wedge z \\ &= (\wedge^{s_3} h_3(x \wedge y)) \wedge z \\ &= \phi_3\big((x \wedge y) \otimes \wedge^{r-s_3} g_3(z)\big).\end{aligned}$$

Since the image of $x \otimes \wedge^{s_2} q(y)$ under the isomorphism

$$\det((F_3/F_2)^*) \otimes_A \det(F_2^*) \xrightarrow{\sim} \det(F_3^*)$$

in Definition 2.3 is $x \wedge y$, by taking the dual, we get the desired equality. □

3. Selmer structures

In this section, we review the results of [Mazur and Rubin 2004; 2016]. Recall that p is an odd prime number, K is a number field, $(R, \mathfrak{m}_R)$ is a complete noetherian local ring with finite residue field $\Bbbk := R/\mathfrak{m}_R$ of characteristic p, and T is a free R-module of finite rank with an R-linear continuous G_K-action which is unramified outside a finite set of places of K.

Throughout this paper, we assume that the field $\overline{K}$ is contained in the complex number field $\mathbb{C}$ and that the fixed separable closure $\overline{K}_{\mathfrak{q}}$ of $K_{\mathfrak{q}}$ contains $\overline{K}$ for each prime $\mathfrak{q}$ of K. Let $K(\mathfrak{q})$ denote the p-part of the ray class field of K modulo $\mathfrak{q}$ and $K(\mathfrak{q})_{\mathfrak{q}}$ the closure of $K(\mathfrak{q})$ in $\overline{K}_{\mathfrak{q}}$. Let $\mathcal{D}_{\mathfrak{q}} := \mathrm{Gal}(\overline{K}_{\mathfrak{q}}/K_{\mathfrak{q}})$ be the decomposition group at $\mathfrak{q}$ in G_K. Set $H^1(K_{\mathfrak{q}}, T) = H^1(\mathcal{D}_{\mathfrak{q}}, T)$. Let $\mathcal{I}_{\mathfrak{q}} \subseteq \mathcal{D}_{\mathfrak{q}}$ be the inertia group at $\mathfrak{q}$ and $\mathrm{Fr}_{\mathfrak{q}}$ the Frobenius element of $\mathcal{D}_{\mathfrak{q}}/\mathcal{I}_{\mathfrak{q}}$. We write $\mathrm{loc}_{\mathfrak{q}}$ for the localization map $H^1(K, T) \to H^1(K_{\mathfrak{q}}, T)$. Define

$$H^1_{\mathrm{ur}}(K_{\mathfrak{q}}, T) := \ker(H^1(K_{\mathfrak{q}}, T) \to H^1(\mathcal{I}_{\mathfrak{q}}, T)),$$
$$H^1_{\mathrm{tr}}(K_{\mathfrak{q}}, T) := \ker(H^1(K_{\mathfrak{q}}, T) \to H^1(K(\mathfrak{q})_{\mathfrak{q}}, T))$$

for each prime $\mathfrak{q}$ of K. Furthermore, we set $H^1_f(K_{\mathfrak{q}}, T) = H^1_{\mathrm{ur}}(K_{\mathfrak{q}}, T)$ if T is unramified at a prime $\mathfrak{q}$ of K.

Definition 3.1. A Selmer structure $\mathcal{F}$ on T is a collection of the following data:

- a finite set $\Sigma(\mathcal{F})$ of places of K, including all the infinite places, all the primes above p, and all the primes where T is ramified,
- a choice of R-submodule $H^1_{\mathcal{F}}(K_{\mathfrak{q}}, T) \subseteq H^1(K_{\mathfrak{q}}, T)$ for each $\mathfrak{q} \in \Sigma(\mathcal{F})$.

Put $H^1_{\mathcal{F}}(K_{\mathfrak{q}}, T) := H^1_{\mathrm{ur}}(K_{\mathfrak{q}}, T)$ for each prime $\mathfrak{q} \notin \Sigma(\mathcal{F})$. We call $H^1_{\mathcal{F}}(K_{\mathfrak{q}}, T)$ the local condition of $\mathcal{F}$ at a prime $\mathfrak{q}$ of K.

Since p is an odd prime, we have $H^1(K_v, T) = 0$ for any infinite place v of K. Thus we ignore the local condition at any infinite place of K.

Remark 3.2. Let $\mathcal{F}$ be a Selmer structure on T and $R \to S$ a surjective ring homomorphism. Then $\mathcal{F}$ induces a Selmer structure on the S-module $T \otimes_R S$, which we will denote by $\mathcal{F}_S$. If there is no risk of confusion, then we write $\mathcal{F}$ instead of $\mathcal{F}_S$.

Definition 3.3. Let $\mathcal{F}$ be a Selmer structure on T. Set

$$H^1_{/\mathcal{F}}(K_{\mathfrak{q}}, T) := H^1(K_{\mathfrak{q}}, T)/H^1_{\mathcal{F}}(K_{\mathfrak{q}}, T)$$

for any prime $\mathfrak{q}$ of K. We define the Selmer group $H^1_{\mathcal{F}}(K, T) \subseteq H^1(K, T)$ associated with $\mathcal{F}$ to be the kernel of the direct sum of localization maps

$$H^1_{\mathcal{F}}(K, T) := \ker\Big(H^1(K, T) \to \bigoplus_{\mathfrak{q}} H^1_{/\mathcal{F}}(K_{\mathfrak{q}}, T)\Big),$$

where $\mathfrak{q}$ runs through all the primes of K.

Example 3.4. Suppose that R is p-torsion-free. Then we define a canonical Selmer structure $\mathcal{F}_{\mathrm{can}}$ on T by the following data:

- $\Sigma(\mathcal{F}_{\mathrm{can}}) := \{\mathfrak{q} \mid T \text{ is ramified at } \mathfrak{q}\} \cup \{\mathfrak{p} \mid p\} \cup \{v \mid \infty\}$.
- We define a local condition at a prime $\mathfrak{q} \nmid p$ to be
$$H^1_{\mathcal{F}_{\mathrm{can}}}(K_{\mathfrak{q}}, T) := H^1_f(K_{\mathfrak{q}}, T) := \ker(H^1(K_{\mathfrak{q}}, T) \to H^1(\mathcal{I}_{\mathfrak{q}}, T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)).$$
- We define a local condition at a prime $\mathfrak{p} \mid p$ to be $H^1_{\mathcal{F}_{\mathrm{can}}}(K_{\mathfrak{p}}, T) := H^1(K_{\mathfrak{p}}, T)$.

Note that $H^1_{\mathrm{ur}}(K_{\mathfrak{q}}, T) = H^1_f(K_{\mathfrak{q}}, T)$ for any prime $\mathfrak{q}$ where T is unramified since the map $H^1(\mathcal{I}_{\mathfrak{q}}, T) = \mathrm{Hom}(\mathcal{I}_{\mathfrak{q}}, T) \to \mathrm{Hom}(\mathcal{I}_{\mathfrak{q}}, T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) = H^1(\mathcal{I}_{\mathfrak{q}}, T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ is injective.

Definition 3.5. Let $\mathcal{F}$ be a Selmer structure on T and let $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{c}$ be pairwise relatively prime integral ideals of K. Define a Selmer structure $\mathcal{F}^{\mathfrak{a}}_{\mathfrak{b}}(\mathfrak{c})$ on T by the following data:

- $\Sigma(\mathcal{F}^{\mathfrak{a}}_{\mathfrak{b}}(\mathfrak{c})) := \Sigma(\mathcal{F}) \cup \{\mathfrak{q} \mid \mathfrak{abc}\}$.
- Define $H^1_{\mathcal{F}^{\mathfrak{a}}_{\mathfrak{b}}(\mathfrak{c})}(K_{\mathfrak{q}}, T) := \begin{cases} H^1(K_{\mathfrak{q}}, T) & \text{if } \mathfrak{q} \mid \mathfrak{a}, \\ 0 & \text{if } \mathfrak{q} \mid \mathfrak{b}, \\ H^1_{\mathrm{tr}}(K_{\mathfrak{q}}, T) & \text{if } \mathfrak{q} \mid \mathfrak{c}, \\ H^1_{\mathcal{F}}(K_{\mathfrak{q}}, T) & \text{otherwise.} \end{cases}$

Definition 3.6. For any positive integer n, let μ_{p^n} denote the group of all p^n-th roots of unity. The Cartier dual of T is defined by

$$T^\vee(1) := T^\vee \otimes_{\mathbb{Z}_p} \varprojlim_n \mu_{p^n} = \mathrm{Hom}_{\mathrm{cont}}(T, \mathbb{Q}_p/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \varprojlim_n \mu_{p^n}.$$

Then we have the local Tate paring

$$\langle \cdot, \cdot \rangle_{\mathfrak{q}} : H^1(K_{\mathfrak{q}}, T) \times H^1(K_{\mathfrak{q}}, T^\vee(1)) \to \mathbb{Q}_p/\mathbb{Z}_p$$

for each prime $\mathfrak{q}$ of K. Let $\mathcal{F}$ be a Selmer structure on T. Put

$$H^1_{\mathcal{F}^*}(K_{\mathfrak{q}}, T^\vee(1)) := \{x \in H^1(K_{\mathfrak{q}}, T^\vee(1)) \mid \langle y, x \rangle_{\mathfrak{q}} = 0 \text{ for any } y \in H^1_{\mathcal{F}}(K_{\mathfrak{q}}, T)\}.$$

In this manner, the Selmer structure $\mathcal{F}$ on T gives rise to a Selmer structure $\mathcal{F}^*$ on $T^\vee(1)$. We define the dual Selmer group $H^1_{\mathcal{F}^*}(K, T^\vee(1))$ associated with $\mathcal{F}$ to be the kernel of the sum of localization maps

$$H^1_{\mathcal{F}^*}(K, T^\vee(1)) := \ker\Bigl(H^1(K, T^\vee(1)) \to \bigoplus_{\mathfrak{q}} H^1_{/\mathcal{F}^*}(K_{\mathfrak{q}}, T^\vee(1))\Bigr),$$

where $\mathfrak{q}$ runs through all the primes of K.

Theorem 3.7 (Poitou–Tate global duality). *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be Selmer structures on T. If $H^1_{\mathcal{F}_1}(K_{\mathfrak{q}}, T) \subseteq H^1_{\mathcal{F}_2}(K_{\mathfrak{q}}, T)$ for any prime $\mathfrak{q}$ of K, then we have the exact sequence*

$$0 \to H^1_{\mathcal{F}_1}(K, T) \to H^1_{\mathcal{F}_2}(K, T) \to \bigoplus_{\mathfrak{q}} H^1_{\mathcal{F}_2}(K_{\mathfrak{q}}, T)/H^1_{\mathcal{F}_1}(K_{\mathfrak{q}}, T) \to H^1_{\mathcal{F}_1^*}(K, T^\vee(1))^\vee \to H^1_{\mathcal{F}_2^*}(K, T^\vee(1))^\vee \to 0$$

where $\mathfrak{q}$ runs through all the primes of K which satisfy $H^1_{\mathcal{F}_1}(K_{\mathfrak{q}}, T) \neq H^1_{\mathcal{F}_2}(K_{\mathfrak{q}}, T)$.

Proof. This is [Mazur and Rubin 2004, Theorem 2.3.4]. □

Definition 3.8 (cartesian condition). Suppose that R is a zero-dimensional Gorenstein local ring. We fix an injective map $\Bbbk \to R$. It induces an injective map $T/\mathfrak{m}_R T \to T$. We say that a Selmer structure $\mathcal{F}$ on T is cartesian if the map

$$H^1_{/\mathcal{F}_{\Bbbk}}(K_{\mathfrak{q}}, T/\mathfrak{m}_R T) \to H^1_{/\mathcal{F}}(K_{\mathfrak{q}}, T)$$

induced by the map $T/\mathfrak{m}_R T \to T$ is injective for any prime $\mathfrak{q} \in \Sigma(\mathcal{F})$.

Remark 3.9. When R is a principal artinian local ring, the definitions of a cartesian Selmer structure in this paper and [Mazur and Rubin 2004] are equivalent. Furthermore, under the setting of [Büyükboduk 2011], we see that the conditions (C2) and (C3) in Definition 2.5 of that paper are equivalent to the cartesian condition in the sense of this paper.

Remark 3.10. Suppose that R is a zero-dimensional Gorenstein local ring. By the definition of Gorenstein ring, we have $\dim_{\Bbbk} \mathrm{Hom}_R(\Bbbk, R) = 1$. Hence the definition of cartesian condition is independent of the choice of the injective map $\Bbbk \to R$.

Let R and S be zero-dimensional Gorenstein local rings and $\pi : R \to S$ a surjective ring homomorphism. Put $I = \ker(\pi)$. Since R is an injective R-module, $R[I] \simeq \mathrm{Hom}_R(S, R)$ is an injective S-module. Hence we conclude that there is an isomorphism $S \xrightarrow{\sim} R[I]$ as R-modules since S is an injective S-module. In particular, there is an injective R-module homomorphism $S \to R$.

Lemma 3.11. *Let R and S be zero-dimensional Gorenstein local rings and $\pi : R \to S$ a surjective ring homomorphism. If a Selmer structure $\mathcal{F}$ on T is cartesian, then so is $\mathcal{F}_S$.*

Proof. Let $S \to R$ and $\Bbbk \to S$ be injective R-module homomorphisms and $\mathfrak{q} \in \Sigma(\mathcal{F})$. Then these maps induce maps $H^1_{/\mathcal{F}_S}(K_\mathfrak{q}, T \otimes_R S) \to H^1_{/\mathcal{F}}(K_\mathfrak{q}, T)$ and $H^1_{/\mathcal{F}_\Bbbk}(K_\mathfrak{q}, T/\mathfrak{m}_R T) \to H^1_{/\mathcal{F}_S}(K_\mathfrak{q}, T \otimes_R S)$. By the definition of cartesian condition and Remark 3.10, the composition of maps

$$H^1_{/\mathcal{F}_\Bbbk}(K_\mathfrak{q}, T/\mathfrak{m}_R T) \to H^1_{/\mathcal{F}_S}(K_\mathfrak{q}, T \otimes_R S) \to H^1_{/\mathcal{F}}(K_\mathfrak{q}, T)$$

is injective, and thus so is $H^1_{/\mathcal{F}_\Bbbk}(K_\mathfrak{q}, T/\mathfrak{m}_R T) \to H^1_{/\mathcal{F}_S}(K_\mathfrak{q}, T \otimes_R S)$. □

Let $\mathcal{H}$ be the Hilbert class field of K and $\mathcal{O}_K$ the ring of integers of K. Set $\mu_{p^\infty} := \bigcup_{n\geq 0} \mu_{p^n}$ and $\mathcal{H}_\infty := \mathcal{H}(\mu_{p^\infty}, (\mathcal{O}_K^\times)^{p^{-\infty}})$. Here $(\mathcal{O}_K^\times)^{p^{-n}} := \{x \in \overline{K} \mid x^{p^n} \in \mathcal{O}_K^\times\}$ and $(\mathcal{O}_K^\times)^{p^{-\infty}} := \bigcup_{n\geq 0} (\mathcal{O}_K^\times)^{p^{-n}}$. In order to use the results of [Mazur and Rubin 2004; 2016], we will usually assume the following additional conditions.

Hypothesis 3.12. (H.1) $(T/\mathfrak{m}_R T)^{G_K} = (T^\vee(1)[\mathfrak{m}_R])^{G_K} = 0$ and $T/\mathfrak{m}_R T$ is an irreducible $\Bbbk[G_K]$-module.

(H.2) There is a $\tau \in \mathrm{Gal}(\overline{K}/\mathcal{H}_\infty)$ such that $T/(\tau - 1)T \simeq R$ as R-modules.

(H.3) $H^1(\mathcal{H}_\infty(T)/K, T/\mathfrak{m}_R T) = H^1(\mathcal{H}_\infty(T)/K, T^\vee(1)[\mathfrak{m}_R]) = 0$, where $\mathcal{H}_\infty(T)$ is the fixed field of the kernel of the map $\mathrm{Gal}(\overline{K}/\mathcal{H}_\infty) \to \mathrm{Aut}(T)$.

Lemma 3.13. *Suppose that R is a zero-dimensional Gorenstein local ring and $(T/\mathfrak{m}_R T)^{G_K} = 0$. Let $\mathcal{F}$ be a cartesian Selmer structure on T. Then an injective map $\Bbbk \to R$ induces isomorphisms*

(1) $H^1(K, T/\mathfrak{m}_R T) \xrightarrow{\sim} H^1(K, T)[\mathfrak{m}_R]$,

(2) $H^1_{\mathcal{F}}(K, T/\mathfrak{m}_R T) \xrightarrow{\sim} H^1_{\mathcal{F}}(K, T)[\mathfrak{m}_R]$.

Proof. Since R is a zero-dimensional Gorenstein local ring, an injective map $\Bbbk \to R$ induces an isomorphism $T/\mathfrak{m}_R T \xrightarrow{\sim} T[\mathfrak{m}_R]$ as $R[G_K]$-modules. Hence the proof of assertion (1) is the same as the first part of the proof of [Mazur and Rubin 2004, Lemma 3.5.3]. Furthermore, the proof of assertion (2) is the same as the last part of the proof of [loc. cit., Lemma 3.5.3] since $\mathcal{F}$ is a cartesian Selmer structure. □

Lemma 3.14. *Let $\mathcal{F}$ be a Selmer structure on T and I an ideal of R. Suppose that R is an artinian local ring and $(T^\vee(1)[\mathfrak{m}_R])^{G_K}=0$. Then the inclusion map $T^\vee(1)[I]\to T^\vee(1)$ induces isomorphisms*

(1) $H^1(K,T^\vee(1)[I]) \xrightarrow{\sim} H^1(K,T^\vee(1))[I]$,

(2) $H^1_{\mathcal{F}^*}(K,T^\vee(1)[I]) \xrightarrow{\sim} H^1_{\mathcal{F}^*}(K,T^\vee(1))[I]$.

Proof. This is [Mazur and Rubin 2004, Lemma 3.5.3]. □

Definition 3.15. Let $\mathcal{F}$ be a Selmer structure on T and n a positive integer. Define a set $\mathcal{P}(\mathcal{F})$ of primes of K by

$$\mathcal{P}(\mathcal{F}) := \{\mathfrak{q} \mid \mathfrak{q}\notin\Sigma(\mathcal{F})\}.$$

Let τ be as in (H.2) and q the order of the residue field $\Bbbk$ of R. Set $\mathcal{H}_n := \mathcal{H}(\mu_{q^n},(\mathcal{O}_K^\times)^{q^{-n}})$. Here $(\mathcal{O}_K^\times)^{q^{-n}} := \{x\in\overline{K}\mid x^{q^n}\in\mathcal{O}_K^\times\}$. Note that $\mathfrak{q}\in\mathcal{P}(\mathcal{F})$ is unramified in $\mathcal{H}_n(T/\mathfrak{m}_R^nT)/K$. We define a set $\mathcal{P}_n(\mathcal{F})$ of primes of K as

$$\mathcal{P}_n(\mathcal{F}) := \{\mathfrak{q}\in\mathcal{P}(\mathcal{F}) \mid \ \mathrm{Fr}_\mathfrak{q} \text{ is conjugate to } \tau \text{ in } \mathrm{Gal}(\mathcal{H}_n(T/\mathfrak{m}_R^nT)/K)\}.$$

Remark 3.16. If $\pi: R\to S$ is a surjective ring homomorphism, then we have $\mathcal{P}_n(\mathcal{F})\subseteq\mathcal{P}_n(\mathcal{F}_S)$ for any positive integer n.

Let $\mathfrak{q}$ be a prime of K where T is unramified. We define the singular quotient at $\mathfrak{q}$ as

$$H^1_{/f}(K_\mathfrak{q},T) := H^1(K_\mathfrak{q},T)/H^1_f(K_\mathfrak{q},T).$$

We denote by $\mathrm{loc}_\mathfrak{q}^{/f}: H^1(K,T)\to H^1_{/f}(K_\mathfrak{q},T)$ the composition of the localization map at $\mathfrak{q}$ and a surjective map $H^1(K_\mathfrak{q},T)\to H^1_{/f}(K_\mathfrak{q},T)$.

Let $\mathcal{N}(\mathcal{P})$ denote the set of square-free products of primes in a set $\mathcal{P}$ of primes of K. By considering the empty product, the trivial ideal 1 is contained in $\mathcal{N}(\mathcal{P})$.

Lemma 3.17. *Suppose that R is an artinian ring and T satisfies (H.2). If $\mathfrak{q}\in\mathcal{P}_{\ell_R(R)}(\mathcal{F})$, then both $H^1_f(K_\mathfrak{q},T)$ and $H^1_{/f}(K_\mathfrak{q},T)$ are free R-modules of rank 1 and the composition of maps*

$$H^1_{\mathrm{tr}}(K_\mathfrak{q},T)\to H^1(K_\mathfrak{q},T)\xrightarrow{\mathrm{loc}_\mathfrak{q}^{/f}} H^1_{/f}(K_\mathfrak{q},T)$$

is an isomorphism. Furthermore, we have canonical isomorphisms

(1) $H^1_f(K_\mathfrak{q},T)\otimes_R R/I \xrightarrow{\sim} H^1_f(K_\mathfrak{q},T/IT)$,

(2) $H^1_{/f}(K_\mathfrak{q},T)\otimes_R R/I \xrightarrow{\sim} H^1_{/f}(K_\mathfrak{q},T/IT)$

for any ideal I of R. In particular, the natural map $H^1(K_\mathfrak{q},T)\otimes_R R/I\to H^1(K_\mathfrak{q},T/IT)$ is an isomorphism.

Proof. This lemma follows from [Mazur and Rubin 2004, Lemmas 1.2.3 and 1.2.4]. □

Corollary 3.18. *Suppose that R is a zero-dimensional Gorenstein local ring and T satisfies* (H.2). *Let $\mathfrak{a},\mathfrak{b},\mathfrak{c}\in\mathcal{N}(\mathcal{P}_{\ell_R(R)}(\mathcal{F}))$ with $\mathfrak{abc}\in\mathcal{N}(\mathcal{P}_{\ell_R(R)}(\mathcal{F}))$. If a Selmer structure $\mathcal{F}$ on T is cartesian, then so is $\mathcal{F}^\mathfrak{a}_\mathfrak{b}(\mathfrak{c})$.*

Proof. We fix an injective map $i : \Bbbk \to R$. Since $\mathcal{F}$ is cartesian, we only need to show that the map

$$H^1_{/\mathcal{F}^{\mathfrak{a}}_{\mathfrak{b}}(\mathfrak{c})}(K_{\mathfrak{q}}, T/\mathfrak{m}_R T) \to H^1_{/\mathcal{F}^{\mathfrak{a}}_{\mathfrak{b}}(\mathfrak{c})}(K_{\mathfrak{q}}, T)$$

induced by the map $i : \Bbbk \to R$ is injective for any prime $\mathfrak{q} \mid \mathfrak{abc}$ of K. By the definition of the Selmer structure $\mathcal{F}^{\mathfrak{a}}_{\mathfrak{b}}(\mathfrak{c})$ and Lemma 3.17, the map $H^1(K_{\mathfrak{q}}, T) \to H^1(K_{\mathfrak{q}}, T/\mathfrak{m}_R T)$ induces an isomorphism $H^1_{/\mathcal{F}^{\mathfrak{a}}_{\mathfrak{b}}(\mathfrak{c})}(K_{\mathfrak{q}}, T) \otimes_R \Bbbk \xrightarrow{\sim} H^1_{/\mathcal{F}^{\mathfrak{a}}_{\mathfrak{b}}(\mathfrak{c})}(K_{\mathfrak{q}}, T/\mathfrak{m}_R T)$. Since $H^1_{/\mathcal{F}^{\mathfrak{a}}_{\mathfrak{b}}(\mathfrak{c})}(K_{\mathfrak{q}}, T)$ is a free R-module by Lemma 3.17 and the composition of the maps

$$H^1_{/\mathcal{F}^{\mathfrak{a}}_{\mathfrak{b}}(\mathfrak{c})}(K_{\mathfrak{q}}, T) \otimes_R \Bbbk \xrightarrow{\sim} H^1_{/\mathcal{F}^{\mathfrak{a}}_{\mathfrak{b}}(\mathfrak{c})}(K_{\mathfrak{q}}, T/\mathfrak{m}_R T) \longrightarrow H^1_{/\mathcal{F}^{\mathfrak{a}}_{\mathfrak{b}}(\mathfrak{c})}(K_{\mathfrak{q}}, T)$$

is $\mathrm{id}_{H^1_{/\mathcal{F}^{\mathfrak{a}}_{\mathfrak{b}}(\mathfrak{c})}(K_{\mathfrak{q}}, T)} \otimes i$, the map $H^1_{/\mathcal{F}^{\mathfrak{a}}_{\mathfrak{b}}(\mathfrak{c})}(K_{\mathfrak{q}}, T/\mathfrak{m}_R T) \to H^1_{/\mathcal{F}^{\mathfrak{a}}_{\mathfrak{b}}(\mathfrak{c})}(K_{\mathfrak{q}}, T)$ is injective. □

Definition 3.19. We define the core rank $\chi(\mathcal{F})$ of a Selmer structure $\mathcal{F}$ on T by

$$\chi(\mathcal{F}) := \dim_{\Bbbk} H^1_{\mathcal{F}}(K, T/\mathfrak{m}_R T) - \dim_{\Bbbk} H^1_{\mathcal{F}^*}(K, T^\vee(1)[\mathfrak{m}_R]).$$

Example 3.20. Let R be the ring of integers of a finite extension of the field $\mathbb{Q}_p$. Suppose that $(T/\mathfrak{m}_R T)^{G_K} = (T^\vee(1)[\mathfrak{m}_R])^{G_K} = 0$. Then by the same proof as [Mazur and Rubin 2004, Theorem 5.2.15], we have

$$\chi(\mathcal{F}_{\mathrm{can}}) = \sum_{v \mid \infty} \mathrm{rank}_R(H^0(K_v, T^\vee(1))^\vee) + \sum_{\mathfrak{p} \mid p} \mathrm{rank}_R(H^2(K_{\mathfrak{q}}, T)),$$

where $\mathcal{F}_{\mathrm{can}}$ is the canonical Selmer structure defined in Example 3.4.

Corollary 3.21. *Suppose that R is an artinian local ring and T satisfies* (H.2). *Let $\mathcal{F}$ be a Selmer structure on T and $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathcal{N}(\mathcal{P}_{\ell_R(R)}(\mathcal{F}))$ with $\mathfrak{abc} \in \mathcal{N}(\mathcal{P}_{\ell_R(R)}(\mathcal{F}))$. Then we have*

$$\chi(\mathcal{F}^{\mathfrak{a}}_{\mathfrak{b}}(\mathfrak{c})) = \chi(\mathcal{F}) + \nu(\mathfrak{a}) - \nu(\mathfrak{b}),$$

where $\nu(\mathfrak{n})$ denotes the number of prime factors of $\mathfrak{n} \in \mathcal{N}(\mathcal{P}_{\ell_R(R)}(\mathcal{F}))$.

Proof. Applying Theorem 3.7 with $\mathcal{F}_1 = \mathcal{F}$ and $\mathcal{F}_2 = \mathcal{F}^{\mathfrak{ac}}$, we have an exact sequence

$$0 \to H^1_{\mathcal{F}}(K, T/\mathfrak{m}_R T) \to H^1_{\mathcal{F}^{\mathfrak{ac}}}(K, T/\mathfrak{m}_R T) \to \bigoplus_{\mathfrak{q} \mid \mathfrak{ac}} H^1_{/f}(K_{\mathfrak{q}}, T/\mathfrak{m}_R T)$$
$$\to H^1_{\mathcal{F}^*}(K, (T/\mathfrak{m}_R T)^\vee(1))^\vee \to H^1_{(\mathcal{F}^{\mathfrak{ac}})^*}(K, (T/\mathfrak{m}_R T)^\vee(1))^\vee \to 0.$$

Since $(T/\mathfrak{m}_R T)^\vee = T^\vee[\mathfrak{m}_R]$, by the definition of core rank and Lemma 3.17, we have

$$\chi(\mathcal{F}^{\mathfrak{ac}}) = \chi(\mathcal{F}) + \sum_{\mathfrak{q} \mid \mathfrak{ac}} \dim_{\Bbbk} H^1_{/f}(K_{\mathfrak{q}}, T/\mathfrak{m}_R T) = \chi(\mathcal{F}) + \nu(\mathfrak{a}) + \nu(\mathfrak{c}).$$

Again using Lemma 3.17 and Theorem 3.7 with $\mathcal{F}_1 = \mathcal{F}^{\mathfrak{a}}_{\mathfrak{b}}(\mathfrak{c})$ and $\mathcal{F}_2 = \mathcal{F}^{\mathfrak{ac}}$, we see that $\chi(\mathcal{F}^{\mathfrak{a}}_{\mathfrak{b}}(\mathfrak{c})) = \chi(\mathcal{F}^{\mathfrak{ac}}) - \nu(\mathfrak{b}) - \nu(\mathfrak{c}) = \chi(\mathcal{F}) + \nu(\mathfrak{a}) - \nu(\mathfrak{b})$. □

Proposition 3.22. *Suppose that R is an artinian local ring and T satisfies Hypothesis 3.12. Let $c \in H^1(K, T)$ and $d \in H^1(K, T^\vee(1))$ be nonzero elements and n a positive integer. Then there is a set $\mathcal{Q} \subseteq \mathcal{P}_n(\mathcal{F})$ of positive density such that* $\mathrm{loc}_{\mathfrak{q}}(c) \neq 0$ *and* $\mathrm{loc}_{\mathfrak{q}}(d) \neq 0$ *for every* $\mathfrak{q} \in \mathcal{Q}$.

Proof. The proof of this proposition is the same as that of [Mazur and Rubin 2004, Proposition 3.6.1]. □

Remark 3.23. Proposition 3.22 is a weaker version of [Mazur and Rubin 2004, Proposition 3.6.1]. Thus we do not need the assumption (H.4) introduced in [Mazur and Rubin 2004; 2016] to prove Proposition 3.22.

Lemma 3.24. *Suppose that R is an artinian local ring and T satisfies Hypothesis 3.12. Let $c \in H^1(K, T)$ with $Rc \simeq R$ as R-modules and n a positive integer. Then there are infinitely many primes $\mathfrak{q} \in \mathcal{P}_n(\mathcal{F})$ such that $R \cdot \mathrm{loc}_{\mathfrak{q}}(c) = H^1_f(K_{\mathfrak{q}}, T)$.*

Proof. We may assume that $n > \ell_R(R)$. Note that $\mathrm{loc}_{\mathfrak{q}}(c) \in H^1_f(K_{\mathfrak{q}}, T)$ for all but finitely many primes $\mathfrak{q}$ of K and that $H^1_f(K_{\mathfrak{q}}, T)$ is free of rank 1 for any prime $\mathfrak{q} \in \mathcal{P}_n(\mathcal{F})$ by Lemma 3.17.

Since R is an artinian local ring, there is an element $m \in R$ such that $\mathfrak{m}_R = R[m]$. Put $c' := mc$. Since $Rc \simeq R$ as R-modules, the element c' is nonzero. Hence by Proposition 3.22, there are infinitely many primes $\mathfrak{q} \in \mathcal{P}_n(\mathcal{F})$ such that $\mathrm{loc}_{\mathfrak{q}}(c') \neq 0$ and $\mathrm{loc}_{\mathfrak{q}}(c) \in H^1_f(K_{\mathfrak{q}}, T)$. Since $\mathfrak{m}_R = R[m]$ and $H^1_f(K_{\mathfrak{q}}, T) \simeq R$ as R-modules, we have $R \cdot \mathrm{loc}_{\mathfrak{q}}(c) = H^1_f(K_{\mathfrak{q}}, T)$. □

Lemma 3.25. *Suppose that R is an artinian local ring and T satisfies Hypothesis 3.12. Let M be a free R-submodule of $H^1(K, T)$ of rank $s \geq 0$ and $R \to A$ a ring homomorphism. Then the composition of maps*

$$M \otimes_R A \to H^1(K, T) \otimes_R A \to H^1(K, T \otimes_R A)$$

is a split injection. Here $T \otimes_R A$ is equipped with the discrete topology.

Proof. Set $n = l_R(R)$. By Lemma 3.24, there are primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_s \in \mathcal{P}_n(\mathcal{F})$ such that the sum of localization maps $\bigoplus_{i=1}^s \mathrm{loc}_{\mathfrak{q}_i} : H^1(K, T) \to \bigoplus_{i=1}^s H^1(K_{\mathfrak{q}_i}, T)$ induces an isomorphism $M \xrightarrow{\sim} \bigoplus_{i=1}^s H^1_f(K_{\mathfrak{q}}, T)$. For any $1 \leq i \leq s$, T is unramified at $\mathfrak{q}_i$, which implies

$$\begin{aligned} H^1_f(K_{\mathfrak{q}_i}, T) \otimes_R A &\xrightarrow{\sim} (T/(\mathrm{Fr}_{\mathfrak{q}_i} - 1)T) \otimes_R A \\ &\xrightarrow{\sim} (T \otimes_R A)/((\mathrm{Fr}_{\mathfrak{q}_i} - 1)(T \otimes_R A)) \\ &\xleftarrow{\sim} H^1_f(K_{\mathfrak{q}_i}, T \otimes_R A). \end{aligned}$$

Therefore the natural map $M \otimes_R A \to \bigoplus_{i=1}^s H^1_f(K_{\mathfrak{q}}, T \otimes_R A)$ is an isomorphism. □

4. Stark systems over zero-dimensional Gorenstein local rings

We will now define a Stark system over a zero-dimensional Gorenstein local ring. Under Hypothesis 3.12, we will prove that the module of Stark systems associated with a cartesian Selmer structure is free of rank 1 and control all the higher Fitting ideals of the Pontryagin dual of dual Selmer groups using Stark systems (Theorems 4.7 and 4.10). Furthermore, we will show that these results generalize [Mazur and Rubin 2004; 2016].

Throughout this section, we assume that R is a zero-dimensional Gorenstein local ring and that $\mathcal{F}$ is a Selmer structure on T. Suppose that T satisfies Hypothesis 3.12.

For simplicity, we write $\mathcal{P} = \mathcal{P}_{\ell_R(R)}(\mathcal{F})$ and $\mathcal{N} = \mathcal{N}(\mathcal{P}_{\ell_R(R)}(\mathcal{F}))$. Let $\nu(\mathfrak{n})$ denote the number of prime factors of $\mathfrak{n} \in \mathcal{N}$.

Stark systems over zero-dimensional Gorenstein local rings. Let r be a nonnegative integer and $\mathfrak{n} \in \mathcal{N}$. Define $W_\mathfrak{n} := \bigoplus_{\mathfrak{q} \mid \mathfrak{n}} H^1_{/f}(K_\mathfrak{q}, T)^* = \bigoplus_{\mathfrak{q} \mid \mathfrak{n}} \mathrm{Hom}_R(H^1_{/f}(K_\mathfrak{q}, T), R)$ and

$$X_\mathfrak{n}(T, \mathcal{F}) := \det(W_\mathfrak{n}) \otimes_R \bigcap\nolimits_R^{r+\nu(\mathfrak{n})} H^1_{\mathcal{F}^\mathfrak{n}}(K, T).$$

If there is no risk of confusion, $X_\mathfrak{n}(T, \mathcal{F})$ is abbreviated to $X_\mathfrak{n}$. For an ideal $\mathfrak{m} \mid \mathfrak{n}$, we have the following cartesian diagram:

$$\begin{array}{ccc} H^1_{\mathcal{F}^\mathfrak{m}}(K, T) & \hookrightarrow & H^1_{\mathcal{F}^\mathfrak{n}}(K, T) \\ \downarrow & & \downarrow \\ \bigoplus_{\mathfrak{q} \mid \mathfrak{m}} H^1_{/f}(K_\mathfrak{q}, T) & \hookrightarrow & \bigoplus_{\mathfrak{q} \mid \mathfrak{n}} H^1_{/f}(K_\mathfrak{q}, T) \end{array}$$

Hence by Definition 2.3, we obtain a map

$$\Phi_{\mathfrak{n},\mathfrak{m}} : X_\mathfrak{n} \to X_\mathfrak{m}.$$

Lemma 4.1. *We have* $\Phi_{\mathfrak{n}_1,\mathfrak{n}_3} = \Phi_{\mathfrak{n}_2,\mathfrak{n}_3} \circ \Phi_{\mathfrak{n}_1,\mathfrak{n}_2}$ *for any* $\mathfrak{n}_1 \in \mathcal{N}$, $\mathfrak{n}_2 \mid \mathfrak{n}_1$, *and* $\mathfrak{n}_3 \mid \mathfrak{n}_2$.

Proof. This lemma follows from Proposition 2.4. □

Definition 4.2. Let $\mathcal{Q}$ be a subset of $\mathcal{P}$. We define the module $\boldsymbol{SS}_r(T, \mathcal{F}, \mathcal{Q})$ of Stark systems of rank r for $(T, \mathcal{F}, \mathcal{Q})$ by

$$\boldsymbol{SS}_r(T, \mathcal{F}, \mathcal{Q}) := \varprojlim_{\mathfrak{n} \in \mathcal{N}(\mathcal{Q})} X_\mathfrak{n},$$

where the inverse limit is taken with respect to the maps $\Phi_{\mathfrak{n},\mathfrak{m}}$. We call an element of $\boldsymbol{SS}_r(T, \mathcal{F}, \mathcal{Q})$ a Stark system of rank r associated with the tuple $(T, \mathcal{F}, \mathcal{Q})$.

Definition 4.3. We say that an ideal $\mathfrak{n} \in \mathcal{N}$ is a core vertex (for $\mathcal{F}$) if

$$H^1_{(\mathcal{F}^\mathfrak{n})^*}(K, T^\vee(1)) = H^1_{\mathcal{F}^*_\mathfrak{n}}(K, T^\vee(1)) = 0.$$

For any ideal $\mathfrak{n} \in \mathcal{N}$ we have $H^1_{\mathcal{F}^*_\mathfrak{n}}(K, T^\vee(1)) \subseteq H^1_{\mathcal{F}^*(\mathfrak{n})}(K, T^\vee(1))$. Hence a core vertex in the sense of [Mazur and Rubin 2004; 2016] is a core vertex in the sense of this paper.

Proposition 4.4. *The set* $\mathcal{N}(\mathcal{P}_n(\mathcal{F}))$ *has a core vertex for any positive integer* n.

Proof. Let n be a positive integer. If $H^1_{\mathcal{F}^*}(K, T^\vee(1)) \neq 0$, by Proposition 3.22, there is a prime $\mathfrak{q} \in \mathcal{P}_n(\mathcal{F})$ such that $\mathrm{loc}_\mathfrak{q}(H^1_{\mathcal{F}^*}(K, T^\vee(1))) \neq 0$. Then we have

$$\ell_R(H^1_{\mathcal{F}^*_\mathfrak{q}}(K, T^\vee(1))) < \ell_R(H^1_{\mathcal{F}^*}(K, T^\vee(1))) < \infty.$$

By repeating this argument, we get an ideal $\mathfrak{n} \in \mathcal{N}(\mathcal{P}_n(\mathcal{F}))$ such that $H^1_{\mathcal{F}^*_\mathfrak{n}}(K, T^\vee(1)) = 0$. □

Remark 4.5. (1) If $\mathfrak{q} \in \mathcal{P}$ and $\mathfrak{n} \in \mathcal{N}(\mathcal{P} \setminus \{\mathfrak{q}\})$ is a core vertex, then $\mathfrak{n}\mathfrak{q} \in \mathcal{N}$ is also a core vertex.

(2) Let $\mathfrak{n} \in \mathcal{N}$ be a core vertex, S a zero-dimensional Gorenstein local ring, and $\pi : R \to S$ a surjective ring homomorphism. Then $\mathfrak{n}$ is also a core vertex for $\mathcal{F}_S$ by Lemma 3.14.

Lemma 4.6. *Suppose that the Selmer structure $\mathcal{F}$ on T is cartesian. If $\mathfrak{n} \in \mathcal{N}$ is a core vertex, then the R-module $H^1_{\mathcal{F}^{\mathfrak{n}}}(K, T)$ is free of rank $\chi(\mathcal{F}) + \nu(\mathfrak{n})$.*

Proof. Let $\mathfrak{n} \in \mathcal{N}$ be a core vertex. We take an ideal $\mathfrak{a} \in \mathcal{N}$ such that $(\mathfrak{a}, \mathfrak{n}) = 1$ and $\nu(\mathfrak{a}) = \chi(\mathcal{F})$. By Corollary 3.21, we have $\chi(\mathcal{F}_{\mathfrak{a}}) = \chi(\mathcal{F}) - \nu(\mathfrak{a}) = 0$. Then by [Mazur and Rubin 2016, Theorem 11.6], there is an ideal $\mathfrak{m} \in \mathcal{N}$ such that $(\mathfrak{a}\mathfrak{n}, \mathfrak{m}) = 1$ and $H^1_{\mathcal{F}_{\mathfrak{a}}(\mathfrak{m})^*}(K, T^\vee(1)[\mathfrak{m}_R]) = 0$. Then we have

$$\dim_{\Bbbk} H^1_{\mathcal{F}_{\mathfrak{a}}(\mathfrak{m})}(K, T/\mathfrak{m}_R T) = \chi(\mathcal{F}_{\mathfrak{a}}(\mathfrak{m})) = \chi(\mathcal{F}_{\mathfrak{a}}) = 0,$$

where the first equality follows from $H^1_{\mathcal{F}_{\mathfrak{a}}(\mathfrak{m})^*}(K, T^\vee(1)[\mathfrak{m}_R]) = 0$ and the second equality follows from Corollary 3.21. Since $\mathcal{F}$ is cartesian, it follows from Lemmas 3.13, 3.14, and Corollary 3.18 that

$$H^1_{\mathcal{F}_{\mathfrak{a}}(\mathfrak{m})}(K, T) = H^1_{\mathcal{F}_{\mathfrak{a}}(\mathfrak{m})^*}(K, T^\vee(1)) = 0.$$

Applying Theorem 3.7 with $\mathcal{F}_1 = \mathcal{F}_{\mathfrak{a}}(\mathfrak{m})$ and $\mathcal{F}_2 = \mathcal{F}^{\mathfrak{m}\mathfrak{n}}$, we see that the canonical map

$$H^1_{\mathcal{F}^{\mathfrak{m}\mathfrak{n}}}(K, T) \to \bigoplus_{\mathfrak{q} \mid \mathfrak{a}} H^1_f(K_{\mathfrak{q}}, T) \oplus \bigoplus_{\mathfrak{q} \mid \mathfrak{m}} H^1(K_{\mathfrak{q}}, T)/H^1_{\mathrm{tr}}(K_{\mathfrak{q}}, T) \oplus \bigoplus_{\mathfrak{q} \mid \mathfrak{n}} H^1_{/f}(K_{\mathfrak{q}}, T)$$

is an isomorphism. Hence by Lemma 3.17, $H^1_{\mathcal{F}^{\mathfrak{m}\mathfrak{n}}}(K, T)$ is a free R-module of rank $\chi(\mathcal{F}) + \nu(\mathfrak{m}) + \nu(\mathfrak{n})$. Since $\mathfrak{n}$ is a core vertex, by Theorem 3.7, we have an exact sequence

$$0 \to H^1_{\mathcal{F}^{\mathfrak{n}}}(K, T) \to H^1_{\mathcal{F}^{\mathfrak{m}\mathfrak{n}}}(K, T) \to \bigoplus_{\mathfrak{q} \mid \mathfrak{m}} H^1_{/f}(K_{\mathfrak{q}}, T) \to 0.$$

Again by Lemma 3.17, $\bigoplus_{\mathfrak{q} \mid \mathfrak{m}} H^1_{/f}(K, T)$ is a free R-module of rank $\nu(\mathfrak{m})$, which implies $H^1_{\mathcal{F}^{\mathfrak{n}}}(K, T)$ is free of rank $\chi(\mathcal{F}) + \nu(\mathfrak{n})$. □

Theorem 4.7. *Suppose that $\mathcal{F}$ is cartesian and $r = \chi(\mathcal{F}) \geq 0$. Let $\mathcal{Q}$ be a subset of $\mathcal{P}$. If the set $\mathcal{N}(\mathcal{Q})$ has a core vertex, then the projection map*

$$\boldsymbol{SS}_r(T, \mathcal{F}, \mathcal{Q}) \to X_{\mathfrak{n}}(T, \mathcal{F})$$

is an isomorphism for any core vertex $\mathfrak{n} \in \mathcal{N}(\mathcal{Q})$. In particular, the module $\boldsymbol{SS}_r(T, \mathcal{F}, \mathcal{Q})$ is a free R-module of rank 1.

Proof. Let $\mathfrak{n} \in \mathcal{N}(\mathcal{Q})$ be a core vertex. We only need to show that the map $\Phi_{\mathfrak{n}\mathfrak{q},\mathfrak{n}}$ is an isomorphism for any prime $\mathfrak{q} \in \mathcal{Q}$ with $\mathfrak{q} \nmid \mathfrak{n}$. Note that $\mathfrak{n}\mathfrak{q}$ is also a core vertex. By Theorem 3.7 and Lemma 4.6, we have the exact sequence of free R-modules

$$0 \to H^1_{/f}(K_{\mathfrak{q}}, T)^* \to H^1_{\mathcal{F}^{\mathfrak{n}\mathfrak{q}}}(K, T)^* \to H^1_{\mathcal{F}^{\mathfrak{n}}}(K, T)^* \to 0.$$

Since $\mathrm{rank}_R(H^1_{\mathcal{F}^{\mathfrak{n}\mathfrak{q}}}(K, T)) = r + \nu(\mathfrak{n}\mathfrak{q})$ and $\mathrm{rank}_R(H^1_{\mathcal{F}^{\mathfrak{n}}}(K, T)) = r + \nu(\mathfrak{n})$, the map

$$H^1_{/f}(K_{\mathfrak{q}}, T)^* \otimes_R \bigwedge\nolimits_R^{r+\nu(\mathfrak{n})} H^1_{\mathcal{F}^{\mathfrak{n}}}(K, T)^* \to \bigwedge\nolimits_R^{r+\nu(\mathfrak{n}\mathfrak{q})} H^1_{\mathcal{F}^{\mathfrak{n}\mathfrak{q}}}(K, T)^*$$

defined by Lemma 2.1 is an isomorphism, and thus so is $\Phi_{\mathfrak{n}\mathfrak{q},\mathfrak{n}}$. □

Lemma 4.8. *Let s and t be nonnegative integers and*

$$0 \to N \to R^{s+t} \to R^s \to M \to 0$$

an exact sequence of R-modules. If $\phi \in \mathrm{Hom}_R\big(\bigwedge_R^t N^*, R\big)$ is the image of a basis of $\det((R^s)^*) \otimes_R \bigcap_R^{s+t} R^{s+t}$ under the map

$$\det((R^s)^*) \otimes_R \bigcap_R^{s+t} R^{s+t} \to \bigcap_R^t N = \mathrm{Hom}_R\Big(\bigwedge_R^t N^*, R\Big)$$

defined in Definition 2.3, then we have $\mathrm{im}(\phi) = \mathrm{Fitt}_R^0(M)$.

Proof. Let $r_1, \dots, r_{s+t}$ be the standard basis of R^{s+t} and $r'_1, \dots, r'_s$ be the standard basis of R^s. We denote by $r_1^*, \dots, r_{s+t}^*$ the dual basis of $r_1, \dots, r_{s+t}$ and by $r_1'^*, \dots, r_s'^*$ the dual basis of $r'_1, \dots, r'_s$. Let $A = (a_{ij})$ be the $s \times (s+t)$-matrix defined by the map $R^{s+t} \to R^s$.

Applying Lemma 2.1 to the exact sequence $(R^s)^* \to (R^{s+t})^* \to N^* \to 0$, we have a map

$$\det((R^s)^*) \otimes_R \bigwedge_R^t N^* \longrightarrow \bigwedge_R^{s+t} (R^{s+t})^* \xrightarrow{\sim} R$$

and its image is equal to $\mathrm{im}(\phi)$. Put

$$\Psi : \det((R^s)^*) \otimes_R \bigwedge_R^t (R^{s+t})^* \to \det((R^s)^*) \otimes_R \bigwedge_R^t N^* \to R.$$

Since a natural map $(R^{s+t})^* \to N^*$ is surjective, we have $\mathrm{im}(\phi) = \mathrm{im}(\Psi)$. If $\{i_1 < \dots < i_t\} \subseteq \{1, \dots, s+t\}$, then we compute

$$\Psi((r_1'^* \wedge \dots \wedge r_s'^*) \otimes (r_{i_1}^* \wedge \dots \wedge r_{i_t}^*)) = \pm \det((a_{kj_\nu})_{k,\nu}),$$

where $\{j_1 < \dots < j_s\}$ is the complement of $\{i_1 < \dots < i_t\}$ in $\{1, \dots, s+t\}$. By the definition of Fitting ideals, we conclude that $\mathrm{im}(\phi) = \mathrm{im}(\Psi) = \mathrm{Fitt}_R^0(M)$. □

Definition 4.9. Let $\mathcal{Q}$ be a subset of $\mathcal{P}$ and $\epsilon \in \boldsymbol{SS}_r(T, \mathcal{F}, \mathcal{Q})$. Fix an isomorphism $H^1_{/f}(K_\mathfrak{q}, T) \simeq R$ for each prime $\mathfrak{q} \in \mathcal{Q}$. Using these isomorphisms, we have an isomorphism

$$i_\mathfrak{n} : X_\mathfrak{n} \xrightarrow{\sim} \bigcap_R^{r+\nu(\mathfrak{n})} H^1_{\mathcal{F}^\mathfrak{n}}(K, T) = \mathrm{Hom}_R\Big(\bigwedge_R^{r+\nu(\mathfrak{n})} H^1_{\mathcal{F}^\mathfrak{n}}(K, T)^*, R\Big)$$

for each ideal $\mathfrak{n} \in \mathcal{N}(\mathcal{Q})$. Then we define an ideal $I_i(\epsilon) \subseteq R$ by

$$I_i(\epsilon) := \sum_{\mathfrak{n} \in \mathcal{N}(\mathcal{Q}), \nu(\mathfrak{n}) = i} \mathrm{im}(i_\mathfrak{n}(\epsilon_\mathfrak{n}))$$

for any nonnegative integer i. It is easy to see that the ideal $I_i(\epsilon)$ is independent of the choice of the isomorphisms $H^1_{/f}(K_\mathfrak{q}, T) \simeq R$.

Theorem 4.10. *Suppose that $\mathcal{F}$ is cartesian and $r = \chi(\mathcal{F}) \geq 0$. Let $\mathcal{Q}$ be an infinite subset of $\mathcal{P}$ such that $\mathcal{N}(\mathcal{Q})$ has a core vertex. If $\epsilon = \{\epsilon_\mathfrak{n}\}_{\mathfrak{n} \in \mathcal{N}(\mathcal{Q})}$ is a basis of $\boldsymbol{SS}_r(T, \mathcal{F}, \mathcal{Q})$, then we have*

$$I_i(\epsilon) = \mathrm{Fitt}_R^i(H^1_{\mathcal{F}^*}(K, T^\vee(1))^\vee)$$

for any nonnegative integer i.

Proof. Let i be a nonnegative integer. Since $\mathcal{N}(\mathcal{Q})$ has a core vertex and $\mathcal{Q}$ is an infinite set, we can take a core vertex $\mathfrak{n} \in \mathcal{N}(\mathcal{Q})$ with $\nu(\mathfrak{n}) \geq i$. Fix an isomorphism $H^1_{/f}(K_\mathfrak{q}, T) \simeq R$ for each prime $\mathfrak{q} \in \mathcal{Q}$. These isomorphisms induce an isomorphism

$$i_\mathfrak{m} : X_\mathfrak{m} \xrightarrow{\sim} \bigcap\nolimits_R^{r+\nu(\mathfrak{m})} H^1_{\mathcal{F}^\mathfrak{m}}(K, T)$$

for each ideal $\mathfrak{m} \in \mathcal{N}(\mathcal{Q})$. Since $\mathfrak{n}$ is a core vertex, by Theorem 3.7, we have an exact sequence

$$0 \to H^1_{\mathcal{F}^\mathfrak{m}}(K, T) \to H^1_{\mathcal{F}^\mathfrak{n}}(K, T) \to \bigoplus_{\mathfrak{q} \mid \mathfrak{m}^{-1}\mathfrak{n}} H^1_{/f}(K_\mathfrak{q}, T) \to H^1_{\mathcal{F}^*_\mathfrak{m}}(K, T^\vee(1))^\vee \to 0$$

for an ideal $\mathfrak{m} \mid \mathfrak{n}$. We denote by $e(\mathfrak{m})$ this exact sequence. Note that $H^1_{\mathcal{F}^\mathfrak{n}}(K, T)$ is free of rank $r + \nu(\mathfrak{n})$ by Lemma 4.6 and that $H^1_{/f}(K_\mathfrak{q}, T)$ is free of rank 1 for any prime $\mathfrak{q} \in \mathcal{Q}$ by Lemma 3.17. Since $\mathfrak{n}$ is a core vertex, by Theorem 4.7, the element $\epsilon_\mathfrak{n}$ is a basis of $X_\mathfrak{n}$. Thus for any ideal $\mathfrak{m} \mid \mathfrak{n}$, the exact sequence $e(\mathfrak{m})$ and the element $i_\mathfrak{m}(\epsilon_\mathfrak{m})$ satisfy the assumptions in Lemma 4.8. Hence we have

$$\mathrm{im}(i_\mathfrak{m}(\epsilon_\mathfrak{m})) = \mathrm{Fitt}^0_R(H^1_{\mathcal{F}^*_\mathfrak{m}}(K, T^\vee(1))^\vee).$$

By comparing the exact sequences $e(1)$ and $e(\mathfrak{m})$ for any $\mathfrak{m} \mid \mathfrak{n}$ with $\nu(\mathfrak{m}) = i$, we see that

$$\mathrm{Fitt}^i_R(H^1_{\mathcal{F}^*}(K, T^\vee(1))^\vee) = \sum_{\mathfrak{m} \mid \mathfrak{n}, \nu(\mathfrak{m})=i} \mathrm{Fitt}^0_R(H^1_{\mathcal{F}^*_\mathfrak{m}}(K, T^\vee(1))^\vee).$$

Thus we have

$$\mathrm{Fitt}^i_R(H^1_{\mathcal{F}^*}(K, T^\vee(1))^\vee) = \sum_{\mathfrak{m} \mid \mathfrak{n}, \nu(\mathfrak{m})=i} \mathrm{im}(i_\mathfrak{m}(\epsilon_\mathfrak{m})) \subseteq I_i(\epsilon)$$

for any core vertex $\mathfrak{n} \in \mathcal{N}(\mathcal{Q})$ with $\nu(\mathfrak{n}) \geq i$. Let $\mathfrak{m} \in \mathcal{Q}$ with $\nu(\mathfrak{m}) = i$. Then there is a core vertex $\mathfrak{n} \in \mathcal{N}(\mathcal{Q})$ with $\mathfrak{m} \mid \mathfrak{n}$. Thus we have

$$\mathrm{im}(i_\mathfrak{m}(\epsilon_\mathfrak{m})) \subseteq \sum_{\mathfrak{m} \mid \mathfrak{n}, \nu(\mathfrak{n})=i} \mathrm{im}(i_\mathfrak{m}(\epsilon_\mathfrak{m})) = \mathrm{Fitt}^i_R(H^1_{\mathcal{F}^*}(K, T^\vee(1))^\vee).$$

Hence we get the desired equality. □

The following proposition will be used in Section 6.

Proposition 4.11. *Suppose that $\mathcal{F}$ is cartesian and $r = \chi(\mathcal{F}) \geq 0$. Let $\mathcal{Q}$ be a subset of $\mathcal{P}$ such that $\mathcal{N}(\mathcal{Q})$ has a core vertex. Let $\epsilon = \{\epsilon_\mathfrak{n}\}_{\mathfrak{n} \in \mathcal{N}(\mathcal{Q})}$ be a basis of $\mathbf{SS}_r(T, \mathcal{F}, \mathcal{Q})$. For $\mathfrak{n} \in \mathcal{N}(\mathcal{Q})$, let*

$$\xi_\mathfrak{n}^{r+\nu(\mathfrak{n})} : \bigwedge\nolimits_R^{r+\nu(\mathfrak{n})} H^1_{\mathcal{F}^\mathfrak{n}}(K, T) \to \bigcap\nolimits_R^{r+\nu(\mathfrak{n})} H^1_{\mathcal{F}^\mathfrak{n}}(K, T)$$

denote the canonical map.

If there is a free R-submodule M of $H^1_\mathcal{F}(K, T)$ with a basis $c_1, \ldots, c_r$, then there is a unique element $\theta(\epsilon) \in R$ such that $\theta(\epsilon)R = \mathrm{Fitt}^0_R(H^1_{\mathcal{F}^*}(K, T^\vee(1))^\vee)$ and $\epsilon_1 = \xi_1^r(\theta(\epsilon)c_1 \wedge \cdots \wedge c_r)$. In particular, we have $\epsilon_1 = 0$ if there is an injective map $R^{r+1} \to H^1_\mathcal{F}(K, T)$.

Proof. Let $c := c_1 \wedge \cdots \wedge c_r$. Note that M is an injective R-module since R is a zero-dimensional Gorenstein local ring. Since the map $M \to H^1_{\mathcal{F}}(K, T)$ is a split injection, the composition of maps

$$\bigwedge\nolimits^r_R M \xrightarrow{\sim} \bigcap\nolimits^r_R M \longrightarrow \bigcap\nolimits^r_R H^1_{\mathcal{F}}(K, T)$$

is injective. Thus uniqueness follows from $\epsilon_1 = \xi^r_1(\theta(\epsilon)c)$.

Since $H^1_{\mathcal{F}^{\mathfrak{n}}}(K, T)$ is a free R-module of rank $r+\nu(\mathfrak{n})$ by Lemma 4.6, we can take elements $d_1, \ldots, d_{\nu(\mathfrak{n})} \in H^1_{\mathcal{F}^{\mathfrak{n}}}(K, T)$ such that $c_1, \ldots, c_r, d_1, \ldots, d_{\nu(\mathfrak{n})}$ is a basis of $H^1_{\mathcal{F}^{\mathfrak{n}}}(K, T)$. Let $N = \sum_{i=1}^{\nu(\mathfrak{n})} Rd_i$ and $d := d_1 \wedge \cdots \wedge d_{\nu(\mathfrak{n})}$. We also take a basis $w^* \in \det(W_{\mathfrak{n}})$ such that $w^* \otimes \xi^{r+\nu(\mathfrak{n})}_{\mathfrak{n}}(d \wedge c) = \epsilon_{\mathfrak{n}}$. Put $W^*_{\mathfrak{n}} := \bigoplus_{\mathfrak{q} \mid \mathfrak{n}} H^1_{/f}(K_{\mathfrak{q}}, T)$ and $\varphi : N \hookrightarrow H^1_{\mathcal{F}^{\mathfrak{n}}}(K, T) \to W^*_{\mathfrak{n}}$. We denote by $w \in \det(W^*_{\mathfrak{n}})$ the dual basis of w^*.

Since $\mathfrak{n}$ is a core vertex and $c_1, \ldots, c_r \in H^1_{\mathcal{F}}(K, T)$, by Theorem 3.7, we have a canonical isomorphism $\operatorname{coker}(\varphi) \xrightarrow{\sim} H^1_{\mathcal{F}^*}(K, T^\vee(1))^\vee$. We define an element $\theta(\epsilon) \in R$ by the relation $\wedge^{\nu(\mathfrak{n})}\varphi(d) = \theta(\epsilon)w$. Then by the definition of Fitting ideals, we have

$$\theta(\epsilon)R = \operatorname{Fitt}^0_R(H^1_{\mathcal{F}^*}(K, T^\vee(1))^\vee).$$

Applying Lemma 2.2 to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W_{\mathfrak{n}} & \longrightarrow & H^1_{\mathcal{F}^{\mathfrak{n}}}(K, T)^* & \longrightarrow & H^1_{\mathcal{F}}(K, T)^* & & \\ & & \downarrow{\scriptstyle \varphi^*} & & \downarrow{\scriptstyle \simeq} & & \downarrow & & \\ 0 & \longrightarrow & N^* & \longrightarrow & N^* \oplus M^* & \longrightarrow & M^* & \longrightarrow & 0 \end{array}$$

we get the commutative diagram

$$\begin{array}{ccc} \bigcap^{r+\nu(\mathfrak{n})}_R (N \oplus M) & \xrightarrow{\simeq} & \det(N) \otimes_R \bigcap^r_R M \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\ \bigcap^{r+\nu(\mathfrak{n})}_R H_{\mathcal{F}^{\mathfrak{n}}}(K, T) & \longrightarrow & \det(W^*_{\mathfrak{n}}) \otimes_R \bigcap^r_R H^1_{\mathcal{F}}(K, T) \end{array}$$

Since the right vertical map sends $d \otimes \xi^r_M(c)$ to $w \otimes \xi^r_1(\theta(\epsilon)c)$, we have

$$\epsilon_1 = \Phi_{\mathfrak{n},1}(\epsilon_{\mathfrak{n}}) = \Phi_{\mathfrak{n},1}(w^* \otimes \xi^r_1(d \wedge c)) = \xi^r_1(\theta(\epsilon)c). \qquad \square$$

Let $\mathcal{Q}$ be a subset of $\mathcal{P}$ such that $\mathcal{N}(\mathcal{Q})$ has a core vertex and let S be a zero-dimensional Gorenstein local ring. Suppose that $\mathcal{F}$ is cartesian and that there is a surjective ring homomorphism $\pi : R \to S$. Take a core vertex $\mathfrak{n} \in \mathcal{N}(\mathcal{Q})$ for $\mathcal{F}$. By Lemma 3.11, Remark 4.5(2), and Lemma 4.6, we obtain a map

$$\begin{aligned} \bigcap\nolimits^{r+\nu(\mathfrak{n})}_R H^1_{\mathcal{F}^{\mathfrak{n}}}(K, T) \xleftarrow{\sim} & \bigwedge\nolimits^{r+\nu(\mathfrak{n})}_R H^1_{\mathcal{F}^{\mathfrak{n}}}(K, T) \\ & \longrightarrow \bigwedge\nolimits^{r+\nu(\mathfrak{n})}_R H^1_{\mathcal{F}^{\mathfrak{n}}_S}(K, T \otimes_R S) \\ & \xrightarrow{\sim} \bigcap\nolimits^{r+\nu(\mathfrak{n})}_R H^1_{\mathcal{F}^{\mathfrak{n}}_S}(K, T \otimes_R S). \end{aligned}$$

Hence if $r = \chi(\mathcal{F}) \geq 0$, we get a map

$$\phi_\pi : \boldsymbol{SS}_r(T, \mathcal{F}, \mathcal{Q}) \xrightarrow{\sim} X_{\mathfrak{n}}(T, \mathcal{F}) \longrightarrow X_{\mathfrak{n}}(T \otimes_R S, \mathcal{F}_S) \xleftarrow{\sim} \boldsymbol{SS}_r(T \otimes_R S, \mathcal{F}, \mathcal{Q})$$

by Theorem 4.7. It is easy to see that the map ϕ_π is independent of the choice of the core vertex $\mathfrak{n} \in \mathcal{N}(\mathcal{Q})$.

Proposition 4.12. *Suppose that $\mathcal{F}$ is cartesian and $r = \chi(\mathcal{F}) \geq 0$. Let $\mathcal{Q}$ be a subset of $\mathcal{P}$ such that $\mathcal{N}(\mathcal{Q})$ has a core vertex, let S be a zero-dimensional Gorenstein local ring, and let $\pi : R \to S$ be a surjective ring homomorphism:*

(1) *The map ϕ_π induces an isomorphism*

$$\boldsymbol{SS}_r(T, \mathcal{F}, \mathcal{Q}) \otimes_R S \xrightarrow{\sim} \boldsymbol{SS}_r(T \otimes_R S, \mathcal{F}_S, \mathcal{Q}).$$

(2) *We have $I_i(\epsilon)S = I_i(\phi_\pi(\epsilon))$ for any $\epsilon \in \boldsymbol{SS}_r(T, \mathcal{F}, \mathcal{Q})$ and nonnegative integer i.*

Proof. Let $\mathfrak{n} \in \mathcal{N}(\mathcal{Q})$ be a core vertex. Note that $\mathcal{F}_S$ is cartesian by Lemma 3.11. Hence by Lemma 3.25 and Lemma 4.6, the map $X_{\mathfrak{n}}(T, \mathcal{F}) \to X_{\mathfrak{n}}(S, \mathcal{F}_S)$ induces an isomorphism $X_{\mathfrak{n}}(T, \mathcal{F}) \otimes_R S \xrightarrow{\sim} X_{\mathfrak{n}}(S, \mathcal{F}_S)$. Hence the map ϕ_π induces an isomorphism $\boldsymbol{SS}_r(T, \mathcal{F}, \mathcal{Q}) \otimes_R S \xrightarrow{\sim} \boldsymbol{SS}_r(S, \mathcal{F}_S, \mathcal{Q})$ by Theorem 4.7.

We will prove assertion (2). Let $\mathfrak{m} \in \mathcal{N}(\mathcal{Q})$. Since $\mathcal{N}(\mathcal{Q})$ has a core vertex, there is a core vertex $\mathfrak{n} \in \mathcal{N}(\mathcal{Q})$ with $\mathfrak{m} \mid \mathfrak{n}$. We fix an isomorphism $H^1_{/f}(K_{\mathfrak{q}}, T) \simeq R$ for each prime $\mathfrak{q} \mid \mathfrak{n}$. Let $? \in \{\mathfrak{m}, \mathfrak{n}\}$. Using these isomorphisms, we regard the element $\epsilon_?$ as an element of $\bigcap_R^{r+\nu(?)} H^1_{\mathcal{F}^?}(K, T)$ and $\phi_\pi(\epsilon)_?$ as an element of $\bigcap_S^{r+\nu(?)} H^1_{\mathcal{F}_S^?}(K, T \otimes_R S)$. We put $H_? := H^1_{\mathcal{F}^?}(K, T)^*$ and $H'_? := H^1_{\mathcal{F}_S^?}(K, T \otimes_R S)^*$. Applying Lemma 2.1 to the exact sequence $W_{\mathfrak{m}^{-1}\mathfrak{n}} \to H_{\mathfrak{n}} \to H_{\mathfrak{m}} \to 0$, we get a map

$$\psi : \bigwedge\nolimits_R^{r+\nu(\mathfrak{m})} H_{\mathfrak{m}} \xrightarrow{\sim} \det(W_{\mathfrak{m}^{-1}\mathfrak{n}}) \otimes_R \bigwedge\nolimits_R^{r+\nu(\mathfrak{m})} H_{\mathfrak{m}} \longrightarrow \bigwedge\nolimits_R^{r+\nu(\mathfrak{n})} H_{\mathfrak{n}},$$

where the first isomorphism is induced by the fixed isomorphisms $H^1_{/f}(K_{\mathfrak{q}}, T) \simeq R$. In the same way, we also get a map $\psi' : \bigwedge_S^{r+\nu(\mathfrak{m})} H'_{\mathfrak{m}} \to \bigwedge_S^{r+\nu(\mathfrak{n})} H'_{\mathfrak{n}}$. Then we have the following commutative diagram:

$$\begin{array}{ccccccc}
\bigwedge_R^{r+\nu(\mathfrak{m})} H_{\mathfrak{n}} & \xrightarrow{j} & \bigwedge_R^{r+\nu(\mathfrak{m})} H_{\mathfrak{m}} & \xrightarrow{\psi} & \bigwedge_R^{r+\nu(\mathfrak{n})} H_{\mathfrak{n}} & \xrightarrow{\epsilon_{\mathfrak{n}}} & R \\
\downarrow{\scriptstyle s} & & & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \pi} \\
\bigwedge_S^{r+\nu(\mathfrak{m})} H'_{\mathfrak{n}} & \xrightarrow{j'} & \bigwedge_S^{r+\nu(\mathfrak{m})} H'_{\mathfrak{m}} & \xrightarrow{\psi'} & \bigwedge_S^{r+\nu(\mathfrak{n})} H'_{\mathfrak{n}} & \xrightarrow{\phi_\pi(\epsilon)_{\mathfrak{n}}} & S
\end{array}$$

where the maps s and t are induced by a surjective homomorphism

$$H_{\mathfrak{n}} \longrightarrow H_{\mathfrak{n}} \otimes_R S \xrightarrow{\sim} (H^1_{\mathcal{F}^{\mathfrak{n}}}(K, T) \otimes_R S)^* \xleftarrow{\sim} H'_{\mathfrak{n}},$$

the map j is induced by the surjective map $H_{\mathfrak{n}} \to H_{\mathfrak{m}}$, and j' is induced by the surjective map $H'_{\mathfrak{n}} \to H'_{\mathfrak{m}}$. By the definition of Stark system, we have $\phi_\pi(\epsilon)_{\mathfrak{m}} = \phi_\pi(\epsilon)_{\mathfrak{n}} \circ \psi'$ and $\epsilon_{\mathfrak{m}} = \epsilon_{\mathfrak{n}} \circ \psi$. Since the maps s, j, and j' are surjective, we have

$$\mathrm{im}(\phi_\pi(\epsilon)_{\mathfrak{m}}) = \mathrm{im}(\phi_\pi(\epsilon)_{\mathfrak{m}} \circ j' \circ s) = \mathrm{im}(\pi \circ \epsilon_{\mathfrak{m}} \circ j) = \mathrm{im}(\epsilon_{\mathfrak{m}})S. \qquad \square$$

Stark systems over principal Artinian local rings. Suppose that R is a principal artinian local ring, $\mathcal{F}$ is cartesian, and $r = \chi(\mathcal{F}) \geq 0$. We will show that there is a canonical isomorphism from the module of Stark systems defined in [Mazur and Rubin 2016] to the module of Stark systems defined in the previous subsection.

Let $\mathcal{Q}$ be an infinite subset of $\mathcal{P}_{\ell_R(R)}(\mathcal{F})$ such that $\mathcal{N}(\mathcal{Q})$ has a core vertex. We put

$$W'_{\mathfrak{n}} := \bigoplus_{\mathfrak{q} \mid \mathfrak{n}} H^1_{\mathrm{tr}}(K_{\mathfrak{q}}, T)^* = \bigoplus_{\mathfrak{q} \mid \mathfrak{n}} \mathrm{Hom}_R(H^1_{\mathrm{tr}}(K_{\mathfrak{q}}, T), R)$$

for each ideal $\mathfrak{n} \in \mathcal{N}(\mathcal{Q})$. By Lemma 3.17, the composition of maps

$$H^1_{\mathrm{tr}}(K_{\mathfrak{q}}, T) \to H^1(K, T) \xrightarrow{\mathrm{loc}_{\mathfrak{q}}^{/f}} H^1_{/f}(K_{\mathfrak{q}}, T)$$

is an isomorphism for any prime $\mathfrak{q} \in \mathcal{Q}$. For any ideal $\mathfrak{n} \in \mathcal{N}(\mathcal{Q})$, we denote by $j_{\mathfrak{n}} : W'_{\mathfrak{n}} \xrightarrow{\sim} W_{\mathfrak{n}}$ the inverse of a natural isomorphism $W_{\mathfrak{n}} \xrightarrow{\sim} W'_{\mathfrak{n}}$.

Let $Y_{\mathfrak{n}} := \det(W'_{\mathfrak{n}}) \otimes_R \bigwedge_R^{r+\nu(\mathfrak{n})} H^1_{\mathcal{F}^{\mathfrak{n}}}(K, T)$ and let $\Psi_{\mathfrak{n},\mathfrak{m}} : Y_{\mathfrak{n}} \to Y_{\mathfrak{m}}$ be the map defined in [Mazur and Rubin 2016, Definition 6.3]. Then the module $\boldsymbol{SS}_r(T, \mathcal{F}, \mathcal{Q})'$ of Stark systems defined in that paper is the inverse limit

$$\boldsymbol{SS}_r(T, \mathcal{F}, \mathcal{Q})' := \varprojlim_{\mathfrak{n} \in \mathcal{N}(\mathcal{Q})} Y_{\mathfrak{n}}$$

with respect to the maps $\Psi_{\mathfrak{n},\mathfrak{m}}$.

Lemma 4.13. *Let $0 \to N \xrightarrow{g} M \xrightarrow{h} F$ be an exact sequence of finitely generated R-modules, F a free R-module of rank 1, and s a positive integer. Then the following diagram commutes*:

$$\begin{array}{ccc} \bigwedge_R^s M & \xrightarrow{\hat{h}} & F \otimes_R \bigwedge_R^{s-1} N \\ \downarrow{\scriptstyle \xi_M^s} & & \downarrow{\scriptstyle \mathrm{id}_F \otimes \xi_N^{s-1}} \\ \bigcap_R^s M & \xrightarrow{\tilde{h}} & F \otimes_R \bigcap_R^{s-1} N \end{array}$$

where $\hat{h}$ is the map defined in [Mazur and Rubin 2016, Proposition A.1] and $\tilde{h}$ is the dual of the map defined by the exact sequence $F^* \xrightarrow{h^*} M^* \xrightarrow{g^*} N^* \to 0$ using Lemma 2.1.

Proof. We may assume that $F = R$. Since R is a principal artinian local ring, we can write $M = Rm \oplus N_0$ and $N = Im \oplus N_0$ for some element m of M and ideal I of R. Then $\bigwedge_R^s M$ is generated by the set $E = \{m_1 \wedge \cdots \wedge m_s \mid m_2, \ldots, m_s \in N_0, m_1 \in \{m\} \cup N_0\}$. Let $m = m_1 \wedge \cdots \wedge m_s \in E$, $f = f_2 \wedge \cdots \wedge f_s \in \bigwedge_R^{s-1} M^*$, and $x = \wedge^{s-1} g^*(f)$. Note that $m_2 \wedge \cdots \wedge m_s \in \bigwedge_R^{s-1} N$. Put $f_1 = h$. Then we have

$$\begin{aligned} \tilde{h}(\xi_M^s(m))(x) &= \det(f_i(m_j)) \\ &= h(m_1) \det((f_i(m_j))_{2 \leq i \leq s, 2 \leq j \leq s}) \\ &= h(m_1) \xi_N^{s-1}(m_2 \wedge \cdots \wedge m_s)(x) \\ &= \xi_N^{s-1}(\hat{h}(m))(x), \end{aligned}$$

where the second equality follows from $f_s(m_i) = h(m_i) = 0$ for $2 \le i \le s$ and the last equality follows from $\hat{h}(m) = h(m_1)m_2 \wedge \cdots \wedge m_s$. □

Let $\mathfrak{n} \in \mathcal{N}(\mathcal{Q})$ and $\xi_{\mathfrak{n}} : \bigwedge_R^{r+\nu(\mathfrak{n})} H^1_{\mathcal{F}^{\mathfrak{n}}}(K, T) \to \bigcap_R^{r+\nu(\mathfrak{n})} H^1_{\mathcal{F}^{\mathfrak{n}}}(K, T)$ be the canonical map. Define a map $C_{\mathfrak{n}} : Y_{\mathfrak{n}} \to X_{\mathfrak{n}}$ by $C_{\mathfrak{n}} = j_{\mathfrak{n}} \otimes \xi_{\mathfrak{n}}$.

Proposition 4.14. *The maps $C_{\mathfrak{n}} : Y_{\mathfrak{n}} \to X_{\mathfrak{n}}$ induce an isomorphism*

$$C : \boldsymbol{SS}_r(T, \mathcal{F}, \mathcal{Q})' \xrightarrow{\sim} \boldsymbol{SS}_r(T, \mathcal{F}, \mathcal{Q}).$$

Proof. By Lemma 4.13 and the definition of the transition maps $\Psi_{\mathfrak{n},\mathfrak{m}}$ and $\Phi_{\mathfrak{n},\mathfrak{m}}$, the maps $C_{\mathfrak{n}}$ induce a map $C : \boldsymbol{SS}_r(T, \mathcal{F}, \mathcal{Q})' \to \boldsymbol{SS}_r(T, \mathcal{F}, \mathcal{Q})$. By Lemma 4.6 or [Mazur and Rubin 2016, Proposition 3.3], the map $C_{\mathfrak{n}} : Y_{\mathfrak{n}} \to X_{\mathfrak{n}}$ is an isomorphism for any core vertex $\mathfrak{n} \in \mathcal{N}(\mathcal{Q})$. Thus by Theorem 4.7 and [loc. cit., Theorem 6.7], the map C is an isomorphism. □

Lemma 4.15. *Let A be a zero-dimensional Gorenstein local ring, M a finitely generated A-module, and s a positive integer. Let $\xi_M^s : \bigwedge_A^s M \to \bigcap_A^s M = \mathrm{Hom}_R\big(\bigwedge_A^s M^*, A\big)$ denote the canonical map and $x \in \bigwedge_A^s M$:*

1. *Let J be an ideal of A. If $x \in J \bigwedge_A^s M$, then $\mathrm{im}(\xi_M^r(x)) \subseteq J$.*
2. *Suppose that there is a free A-submodule $M' \subseteq M$ of rank s such that $x \in \mathrm{im}\big(\bigwedge_A^s M' \to \bigwedge_A^s M\big)$. Then $x \in \mathrm{im}(\xi_M^s(x)) \bigwedge_A^s M$.*

Proof. Since $\mathrm{im}(\xi_M^s(ay)) = \mathrm{im}(a \cdot \xi_M^s(y)) = a \cdot \mathrm{im}(\xi_M^s(y))$ for any $y \in \bigwedge_A^s M$ and $a \in A$, assertion (1) holds.

We will show assertion (2). Since A is a zero-dimensional Gorenstein local ring, we can write $M = M' \oplus N$ for some A-submodule $N \subseteq M$. Let $y_1, \dots, y_s$ be a basis of M'. We take the element $y_i^* \in M^*$ such that $y_i^*(y_i) = 1$ and $y_i^*(y_j) = 0$ if $i \neq j$. We write $x = ay_1 \wedge \cdots \wedge y_s$ for some $a \in A$. Then we have $\xi_M^s(x)(y_1^* \wedge \cdots \wedge y_s^*) = a$. □

Lemma 4.16. *Let $0 \to N \to M \xrightarrow{h} F$ be an exact sequence of finitely generated R-modules, F a free R-module of rank 1, and s a positive integer. Let $\hat{h} : \bigwedge_R^s M \to F \otimes_R \bigwedge_R^{s-1} N$ denote the map defined in [Mazur and Rubin 2016, Proposition A.1] and $x \in \bigwedge_R^s M$.*

Suppose that there is a free R-submodule $M' \subseteq M$ of rank s such that $x \in \mathrm{im}\big(\bigwedge_R^s M' \to \bigwedge_R^s M\big)$. Then there is a free R-submodule $N' \subseteq M' \cap N$ of rank $s-1$ such that

$$\hat{h}(y) \in F \otimes_R \mathrm{im}\bigg(\bigwedge\nolimits_R^{s-1} N' \to \bigwedge\nolimits_R^{s-1} N\bigg) \subseteq F \otimes_R \bigwedge\nolimits_R^{s-1} N.$$

Proof. Since R is a principal artinian local ring, there is a basis $y_1, \dots, y_s$ of M' such that $y_2, \dots, y_s \in N$. Put $N' := \sum_{i=2}^s Ry_i$. Then N' is a free R-submodule of $M' \cap N$ of rank $s-1$ and $\hat{h}(y_1 \wedge \cdots \wedge y_s) = h(y_1) \otimes y_2 \wedge \cdots \wedge y_s \in F \otimes \mathrm{im}\big(\bigwedge_R^{s-1} N' \to \bigwedge_R^{s-1} N\big)$. □

The following proposition says that Theorem 4.10 is a generalization of [Mazur and Rubin 2016, Theorem 8.5].

Proposition 4.17. *Let ϵ' be a basis of $\boldsymbol{SS}_r(T, \mathcal{F}, \mathcal{Q})'$ and $k = \ell_R(R)$. Then we have*

$$\partial\varphi_{\epsilon'}(t) = \max\{n \in \{0, 1, \dots, k-1, \infty\} \mid I_t(C(\epsilon')) \subseteq \mathfrak{m}_R^n\}.$$

Here $\partial\varphi_{\epsilon'} : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ is the map defined in [Mazur and Rubin 2016, Definition 8.1].

Proof. We fix an isomorphism $H^1_{/f}(K_\mathfrak{q}, T) \simeq R$ for each prime $\mathfrak{q} \in \mathcal{Q}$. Then we get an isomorphism $i_\mathfrak{n} : X_\mathfrak{n} \xrightarrow{\sim} \bigcap_R^{r+\nu(\mathfrak{n})} H^1_{\mathcal{F}^\mathfrak{n}}(K, T)$ for each ideal $\mathfrak{n} \in \mathcal{N}(\mathcal{Q})$. Let c be a nonnegative integer and $\mathfrak{m} \in \mathcal{N}(\mathcal{Q})$. Then we only need to show that the following assertions are equivalent:

(i) $\epsilon'_\mathfrak{m} \in \mathfrak{m}_R^c Y_\mathfrak{m}$.

(ii) $\mathrm{im}(i_\mathfrak{m}(C(\epsilon')_\mathfrak{m})) = \mathrm{im}(i_\mathfrak{m}(C_\mathfrak{m}(\epsilon'_\mathfrak{m}))) \subseteq \mathfrak{m}_R^c$.

By Lemma 4.15(1), (i) implies (ii). We will show that (ii) implies (i). Since we assume that $\mathcal{N}(\mathcal{Q})$ has a core vertex, there is a core vertex $\mathfrak{n} \in \mathcal{N}(\mathcal{Q})$ with $\mathfrak{m} \mid \mathfrak{n}$. Since $H^1_{\mathcal{F}^\mathfrak{n}}(K, T)$ is a free R-module of rank $r + \nu(\mathfrak{n})$, by Lemma 4.16, there is a free R-submodule $M \subseteq H^1_{\mathcal{F}^\mathfrak{m}}(K, T)$ of rank $r + \nu(\mathfrak{m})$ such that $\epsilon'_\mathfrak{m} \in \det(W'_\mathfrak{m}) \otimes_R \mathrm{im}\big(\bigwedge_R^{r+\nu(\mathfrak{m})} M \to \bigwedge_R^{r+\nu(\mathfrak{m})} H^1_{\mathcal{F}^\mathfrak{m}}(K, T)\big)$. Thus by Lemma 4.15(2), we see that (ii) implies (i). □

5. Stark systems over Gorenstein local rings

In this section, we assume that R is a complete Gorenstein local ring and that T satisfies Hypothesis 3.12.

Since R is a complete Gorenstein local ring, there is an increasing sequence $J_1 \subseteq J_2 \subseteq J_3 \subseteq \cdots$ of ideals of R such that R/J_n is a zero-dimensional Gorenstein local ring and a natural map $R \to \varprojlim_n R/J_n$ is an isomorphism. In fact, since R is Cohen–Macaulay, there is a regular sequence $x_1, \dots, x_d \in \mathfrak{m}_R$ such that $\sqrt{(x_1, \dots, x_d)} = \mathfrak{m}_R$. Put $J_n = (x_1^n, \dots, x_d^n)$ for each positive integer n. Since $x_1^n, \dots, x_d^n$ is a regular sequence and $\sqrt{J_n} = \mathfrak{m}_R$, R/J_n is a zero-dimensional Gorenstein local ring. Furthermore the map $R \to \varprojlim_n R/J_n$ is an isomorphism since R is a complete noetherian local ring. Throughout this section, we fix such a sequence $\{J_n\}_{n\geq 1}$ of ideals of R.

Let n be a positive integer and $\mathcal{F}_n$ a Selmer structure on T/J_nT such that

$$H^1_{\mathcal{F}_n}(K_\mathfrak{q}, T/J_nT) = H^1_{(\mathcal{F}_{n+1})_{R/J_n}}(K_\mathfrak{q}, T/J_nT)$$

for any prime $\mathfrak{q}$ of K. Then we have $r := \chi(\mathcal{F}_n) = \chi(\mathcal{F}_{n+1})$ for any positive integer n. In order to use the results of the previous section, we assume that $r \geq 0$ and that the Selmer structure $\mathcal{F}_n$ is cartesian for all positive integers n. Put $\mathcal{F} := \{\mathcal{F}_n\}_{n\geq 1}$.

Remark 5.1. Let S be a zero-dimensional Gorenstein local ring and $\pi : R \to S$ a surjective ring homomorphism. Since the kernel of π is an open ideal, there is a positive integer n such that $I_n \subseteq \ker(\pi)$. Hence the collection of the Selmer structures $\mathcal{F}$ defines a Selmer structure $\mathcal{F}_S$ on $T \otimes_R S$. Note that $\chi(\mathcal{F}_S) = r$ and $\mathcal{F}_S$ is cartesian by Remark 3.10.

Remark 5.2. Let S be a complete Gorenstein local ring and $\pi : R \to S$ a surjective ring homomorphism. Take an increasing sequence $\{J_{S,n}\}_{n\geq 1}$ of ideals of S such that $S/J_{S,n}$ is a zero-dimensional Gorenstein local ring and the map $S \to \varprojlim_n S/J_{S,n}$ is an isomorphism. By Remark 5.1, we get a Selmer structure $\mathcal{F}_{S,n}$ on $T \otimes_R S/J_{S,n}$ for each positive integer n. Then we have $\chi(\mathcal{F}_{S,n}) = r$, the Selmer structure $\mathcal{F}_{S,n}$ is cartesian, and

$$H^1_{\mathcal{F}_{S,n}}(K_{\mathfrak{q}}, T \otimes_R S/J_{S,n}S) = H^1_{(\mathcal{F}_{S,n+1})_{R/J_n}}(K_{\mathfrak{q}}, T \otimes_R S/J_{S,n}S)$$

for any prime $\mathfrak{q}$ of K and positive integer n. We denote by $\mathcal{F}_S$ the collection $\{\mathcal{F}_{S,n}\}_{n\geq 1}$ of the Selmer structures.

Example 5.3. The following example is closely related to the results in [Büyükboduk 2014, Appendix A; Büyükboduk and Lei 2015, Appendix A]. Let $\mathcal{O}$ be the ring of integers of a finite extension of the field $\mathbb{Q}_p$ and $\mathcal{T}$ a free $\mathcal{O}$-module of finite rank with an $\mathcal{O}$-linear continuous G_K-action which is unramified outside a finite set of places of K. Let d be a positive integer and L a free $R := \mathcal{O}[[X_1, \dots, X_d]]$-module of rank 1 with an R-linear continuous G_K-action which is unramified outside primes above p. Put $T := \mathcal{T} \otimes_{\mathcal{O}} L$ and $J_n := (p^n, X_1^n, \dots, X_d^n)$. Suppose that the following conditions hold:

- The module $\mathcal{T}$ satisfies Hypothesis 3.12.
- The module $L \otimes_R R/(X_1, \dots, X_d)$ has trivial G_K-action.
- The module $(\mathcal{T} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^{\mathcal{I}_{\mathfrak{q}}}$ is divisible for every prime $\mathfrak{q} \nmid p$ of K.
- $H^2(K_{\mathfrak{p}}, \mathcal{T}) = 0$ for each prime $\mathfrak{p} \mid p$ of K.

For a positive integer n, we define a Selmer structure $\mathcal{F}_n$ on T/J_nT by the following data:

- $\Sigma(\mathcal{F}_n) := \{v \mid p\infty\} \cup \{\mathfrak{q} \mid \mathcal{T} \text{ is ramified at } \mathfrak{q}\}$.
- $H^1_{\mathcal{F}_n}(K_{\mathfrak{q}}, T/J_nT) := H^1_{\mathrm{ur}}(K_{\mathfrak{q}}, T/J_nT)$ for each prime $\mathfrak{q} \nmid p$ of K.
- $H^1_{\mathcal{F}_n}(K_{\mathfrak{p}}, T/J_nT) := H^1(K_{\mathfrak{p}}, T/J_nT)$ for each prime $\mathfrak{p} \mid p$ of K.

Since $(\mathcal{T} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^{\mathcal{I}_{\mathfrak{q}}}$ is divisible and $L^{\mathcal{I}_{\mathfrak{q}}} = L$ for every prime $\mathfrak{q} \nmid p$ of K, a canonical map $(T/J_{n+1}T)^{\mathcal{I}_{\mathfrak{q}}} \to (T/J_nT)^{\mathcal{I}_{\mathfrak{q}}}$ is surjective. Note that the cohomological dimension of $\mathcal{D}_{\mathfrak{q}}/\mathcal{I}_{\mathfrak{q}} \simeq \widehat{\mathbb{Z}}$ is 1. Therefore a canonical map $H^1_{\mathrm{ur}}(K_{\mathfrak{q}}, T/J_{n+1}T) \to H^1_{\mathrm{ur}}(K_{\mathfrak{q}}, T/J_nT)$ is surjective. By the inflation-restriction exact sequence, we have an isomorphism

$$H^1_{/\mathcal{F}_n}(K_{\mathfrak{q}}, T/J_nT) \xrightarrow{\sim} H^1(\mathcal{I}_{\mathfrak{q}}, T/J_nT)^{\mathrm{Fr}_{\mathfrak{q}}=1}.$$

Hence we see that the map

$$H^1_{/\mathcal{F}_n}(K_{\mathfrak{q}}, T/\mathfrak{m}_RT) \to H^1_{/\mathcal{F}_n}(K_{\mathfrak{q}}, T/J_nT)$$

induced by an injection $\Bbbk \to R$ is injective. Let $\mathfrak{p} \mid p$ be a prime of K. Since the cohomological dimension of $\mathcal{D}_{\mathfrak{p}}$ is 2, we have

$$H^2(K_{\mathfrak{p}}, T) \otimes_R R/(X_1, \dots, X_d) \xrightarrow{\sim} H^2(K_{\mathfrak{p}}, \mathcal{T}).$$

Because $H^2(K_\mathfrak{p}, \mathcal{T}) = 0$, we have $H^2(K_\mathfrak{p}, T) = 0$ by Nakayama's lemma, and a canonical map $H^1(K_\mathfrak{p}, T/J_{n+1}T) \to H^1(K_\mathfrak{p}, T/J_nT)$ is surjective for any positive integer n. Furthermore, by [Mazur and Rubin 2016, Theorem 5.4], we have

$$\chi(\mathcal{F}_n) = \sum_{v \mid \infty} \operatorname{rank}_{\mathcal{O}}(H^0(K_v, \mathcal{T}^\vee(1))^\vee) \geq 0,$$

where v runs through all the infinite places of K. Hence the collection of the Selmer structures $\{\mathcal{F}_n\}_{n\geq 1}$ satisfies all the conditions in the paragraph before Remark 5.1.

Stark systems over Gorenstein local rings. We fix a decreasing sequence $\mathcal{Q}_1 \supseteq \mathcal{Q}_2 \supseteq \mathcal{Q}_3 \supseteq \cdots$ such that $\mathcal{Q}_n$ is an infinite subset of $\mathcal{P}_{\ell_R(R/J_n)}(\mathcal{F}_n)$ and $\mathcal{N}(\mathcal{Q})$ has a core vertex for $\mathcal{F}_n$. For example, we take $\mathcal{Q}_n = \mathcal{P}_{\ell_R(R/J_n)}(\mathcal{F}_n) \setminus (\Sigma(\mathcal{F}_1) \cup \cdots \cup \Sigma(\mathcal{F}_n))$ for each positive integer n. We denote by $\mathcal{Q}$ the collection of the sequences $\{\mathcal{Q}_n\}_{n\geq 1}$.

Let n be a positive integer. Since $\mathcal{N}(\mathcal{Q}_{n+1})$ has a core vertex for $\mathcal{F}_{n+1}$, the map $R/J_{n+1} \to R/J_n$ induces a map $\boldsymbol{SS}_r(T/J_{n+1}T, \mathcal{F}_{n+1}, \mathcal{Q}_{n+1}) \to \boldsymbol{SS}_r(T/J_nT, \mathcal{F}_n, \mathcal{Q}_{n+1})$ as in the paragraph before Proposition 4.12. Since, by Theorem 4.7, a restriction map $\boldsymbol{SS}_r(T/J_nT, \mathcal{F}_n, \mathcal{Q}_n) \to \boldsymbol{SS}_r(T/J_nT, \mathcal{F}_n, \mathcal{Q}_{n+1})$ is an isomorphism, we get a map

$$\phi_{n+1,n} : \boldsymbol{SS}_r(T/J_{n+1}T, \mathcal{F}_{n+1}, \mathcal{Q}_{n+1}) \to \boldsymbol{SS}_r(T/J_nT, \mathcal{F}_n, \mathcal{Q}_n).$$

We define the module $\boldsymbol{SS}_r(T, \mathcal{F}, \mathcal{Q})$ of Stark systems of rank r for $\mathcal{F}$ by

$$\boldsymbol{SS}_r(T, \mathcal{F}, \mathcal{Q}) := \varprojlim_{n\geq 1} \boldsymbol{SS}_r(T/J_nT, \mathcal{F}_n, \mathcal{Q}_n),$$

where the inverse limit is taken with respect to the maps $\phi_{n+1,n}$.

Let S be a complete Gorenstein local ring and $\pi : R \to S$ a surjective ring homomorphism. Take an increasing sequence $J_{S,1} \subseteq J_{S,2} \subseteq J_{S,3} \subseteq \cdots$ of ideals of S such that $S/J_{S,n}$ is a zero-dimensional Gorenstein local ring and the map $S \to \varprojlim_n S/J_{S,n}$ is an isomorphism. Let $\mathcal{F}_S$ be the collection of the Selmer structures defined in Remark 5.2. For a positive integer n, we fix a positive integer n_S such that $n_S \geq n$ and such that the map $\pi : R \to S$ induces a map $\pi_n : R/J_{n_S} \to S/J_{S,n}$. Then we have the map

$$\phi_{\pi_n} : \boldsymbol{SS}_r(T/J_{n_S}T, \mathcal{F}_{n_S}, \mathcal{Q}_{n_S}) \to \boldsymbol{SS}_r(T \otimes_R S/J_{S,n}, \mathcal{F}_{S,n}, \mathcal{Q}_{n_S}).$$

Hence we get a canonical map

$$\phi_\pi : \boldsymbol{SS}_r(T, \mathcal{F}, \mathcal{Q}) \to \boldsymbol{SS}_r(T \otimes_R S, \mathcal{F}_S, \{\mathcal{Q}_{n_S}\}_{n\geq 1}).$$

For simplicity, we also denote by $\mathcal{Q}$ the subsequence $\{\mathcal{Q}_{n_S}\}_{n\geq 1}$ of $\mathcal{Q} = \{\mathcal{Q}_n\}_{n\geq 1}$.

Theorem 5.4. (1) *The R-module $\boldsymbol{SS}_r(T, \mathcal{F}, \mathcal{Q})$ is free of rank 1.*

(2) *Let S be a complete Gorenstein local ring and $\pi : R \to S$ a surjective ring homomorphism. Then the map ϕ_π induces an isomorphism*

$$\boldsymbol{SS}_r(T, \mathcal{F}, \mathcal{Q}) \otimes_R S \xrightarrow{\sim} \boldsymbol{SS}_r(T \otimes_R S, \mathcal{F}_S, \mathcal{Q}).$$

Proof. For any positive integers $n_1 < n_2$, by Proposition 4.12(1), a canonical map

$$\boldsymbol{SS}_r(T/J_{n_2}T, \mathcal{F}_{n_2}, \mathcal{Q}_{n_2}) \otimes_{R/J_{n_2}} R/J_{n_1} \to \boldsymbol{SS}_r(T/J_{n_1}T, \mathcal{F}_{n_1}, \mathcal{Q}_{n_1})$$

is an isomorphism. Furthermore, for any positive integer n, the module $\boldsymbol{SS}_r(T/J_nT, \mathcal{F}_n, \mathcal{Q}_n)$ is a free R/J_n-module of rank 1 by Theorem 4.7. Hence by definition, $\boldsymbol{SS}_r(T, \mathcal{F}, \mathcal{Q})$ is free of rank 1, so (1) holds. Note that assertion (1) and Proposition 4.12(1) imply $\boldsymbol{SS}_r(T, \mathcal{F}, \mathcal{Q}) \otimes_R R/J_n \xrightarrow{\sim} \boldsymbol{SS}_r(T/J_nT, \mathcal{F}_n, \mathcal{Q}_n)$. Again by Proposition 4.12(1), the map ϕ_{π_n} induces an isomorphism

$$\boldsymbol{SS}_r(T/J_{n_S}T, \mathcal{F}_{n_S}, \mathcal{Q}_{n_S}) \otimes_{R/J_{n_S}} S/J_{S,n} \xrightarrow{\sim} \boldsymbol{SS}_r(T \otimes_R S/J_{S,n}, \mathcal{F}_{S,n}, \mathcal{Q}_{n_S}).$$

Assertion (2) follows from these isomorphisms. □

Let i be a nonnegative integer and $\epsilon \in \boldsymbol{SS}_r(T, \mathcal{F}, \mathcal{Q})$. For a positive integer n, let

$$\phi_n : \boldsymbol{SS}_r(T, \mathcal{F}, \mathcal{Q}) \to \boldsymbol{SS}_r(T/J_nT, \mathcal{F}_n, \mathcal{Q}_n)$$

denote the projection map. By Proposition 4.12(2), we can define an ideal $I_i(\epsilon)$ of R by

$$I_i(\epsilon) := \varprojlim_{n \geq 1} I_i(\phi_n(\epsilon)) = \varprojlim_{n \geq 1} \Bigg(\sum_{\mathfrak{n} \in \mathcal{N}(\mathcal{Q}_n), \nu(\mathfrak{n}) = i} \operatorname{im}(\phi_n(\epsilon)_{\mathfrak{n}}) \Bigg),$$

where the inverse limit is taken with respect to the natural maps $R/J_{n+1} \to R/J_n$.

Definition 5.5. We define the dual Selmer group $H^1_{\mathcal{F}^*}(K, T^\vee(1))$ associated with the collection of the Selmer structures $\mathcal{F}$ by

$$H^1_{\mathcal{F}^*}(K, T^\vee(1)) := \varinjlim_{n \geq 1} H^1_{\mathcal{F}^*_n}(K, (T/J_nT)^\vee(1)),$$

where the injective limit is taken with respect to the maps induced by the Cartier duals of the maps $T/J_{n+1}T \to T/J_nT$.

The following theorem is the main result of this paper.

Theorem 5.6. *Let i be a nonnegative integer*:

(1) *If ϵ is a basis of $\boldsymbol{SS}_r(T, \mathcal{F}, \mathcal{Q})$, then we have*

$$I_i(\epsilon) = \operatorname{Fitt}^i_R(H^1_{\mathcal{F}^*}(K, T^\vee(1))^\vee).$$

(2) *Let S be a complete Gorenstein local ring and $\pi : R \to S$ a surjective ring homomorphism. Then we have $I_i(\epsilon)S = I_i(\phi_\pi(\epsilon))$ for any $\epsilon \in \boldsymbol{SS}_r(T, \mathcal{F}, \mathcal{Q})$.*

Proof. By Lemma 3.14, the canonical map

$$H^1_{\mathcal{F}^*}(K, T^\vee(1))^\vee \otimes_R R/J_n \to H^1_{\mathcal{F}^*_n}(K, (T/J_nT)^\vee(1))^\vee$$

is an isomorphism for any positive integer n. Thus we have

$$\operatorname{Fitt}^i_R(H^1_{\mathcal{F}^*}(K, T^\vee(1))^\vee)R/J_n = \operatorname{Fitt}^i_{R/J_n}(H^1_{\mathcal{F}^*_n}(K, (T/J_nT)^\vee(1))^\vee) = I_i(\phi_n(\epsilon)) = I_i(\epsilon)R/J_n,$$

where the second equality follows from Theorem 4.10, and the third follows from Proposition 4.12(2). Since R is a complete noetherian local ring, all the ideals of R are closed. Hence we have $I_i(\epsilon) = \mathrm{Fitt}^i_R(H^1_{\mathcal{F}^*}(K, T^\vee(1))^\vee)$. The second assertion follows from Proposition 4.12(2). □

Remark 5.7. Suppose that R is a discrete valuation ring. Let $\boldsymbol{SS}_r(T, \mathcal{F}, \mathcal{Q})'$ be the module of Stark systems defined in [Mazur and Rubin 2016, Definition 7.1] and n a positive integer. Let

$$C_n : \boldsymbol{SS}_r(T/\mathfrak{m}_R^n T, \mathcal{F}_n, \mathcal{Q}_n)' \to \boldsymbol{SS}_r(T/\mathfrak{m}_R^n T, \mathcal{F}_n, \mathcal{Q}_n)$$

be the isomorphism defined in Proposition 4.14. By the definition of the map C_n, we have the following commutative diagram:

$$\begin{array}{ccc} \boldsymbol{SS}_r(T/\mathfrak{m}_R^{n+1}T, \mathcal{F}_{n+1}, \mathcal{Q}_{n+1})' & \xrightarrow{C_{n+1}} & \boldsymbol{SS}_r(T/\mathfrak{m}_R^{n+1}T, \mathcal{F}_n, \mathcal{Q}_{n+1}) \\ \downarrow & & \downarrow \\ \boldsymbol{SS}_r(T/\mathfrak{m}_R^n T, \mathcal{F}_n, \mathcal{Q}_n)' & \xrightarrow{C_n} & \boldsymbol{SS}_r(T/\mathfrak{m}_R^n T, \mathcal{F}_n, \mathcal{Q}_n) \end{array}$$

Thus by [loc. cit., Proposition 7.3], the maps C_n induce an isomorphism

$$C : \boldsymbol{SS}_r(T, \mathcal{F}, \mathcal{Q})' \xrightarrow{\sim} \boldsymbol{SS}_r(T, \mathcal{F}, \mathcal{Q}).$$

Furthermore, if $\epsilon' \in \boldsymbol{SS}_r(T, \mathcal{F}, \mathcal{Q})'$ is nonzero, we see that

$$\partial\varphi_{\epsilon'}(t) = \max\{n \in \mathbb{Z}_{\geq 0} \cup \{\infty\} \mid I_t(C(\epsilon')) \subseteq \mathfrak{m}_R^n\}$$

for any nonnegative integer t by Proposition 4.17. This shows that Theorem 5.6 implies [loc. cit., Theorem 8.7].

6. Controlling Selmer groups using Λ-adic Stark systems

Let $\mathcal{O}$ be the ring of integers of a finite extension of the field $\mathbb{Q}_p$ and $\mathcal{T}$ a free $\mathcal{O}$-module of finite rank with an $\mathcal{O}$-linear continuous G_K-action which is unramified outside a finite set of places of K. We write K_∞ for the cyclotomic $\mathbb{Z}_p$-extension of K. Fix a topological generator γ of $\mathrm{Gal}(K_\infty/K)$. Let R be the ring of formal power series $\mathcal{O}[[X]]$. By using the element γ, we get an isomorphism $\mathcal{O}[[\mathrm{Gal}(K_\infty/K)]] \xrightarrow{\sim} R$, $\gamma \mapsto 1 + X$. It induces an R-module structure on $\mathcal{O}[[\mathrm{Gal}(K_\infty/K)]]$.

Following the notation in [Mazur and Rubin 2004, Section 5.3], we write Λ instead of R and put $\mathbb{T} := \mathcal{T} \otimes_\Lambda \mathcal{O}[[\mathrm{Gal}(K_\infty/K)]]$. We assume the following conditions:

- $(\mathcal{T}/\varpi\mathcal{T})^{G_K} = (\mathcal{T}^\vee(1)[\varpi])^{G_K} = 0$ and $\mathcal{T}/\varpi\mathcal{T}$ is an absolutely irreducible $\Bbbk[G_K]$-module, where ϖ is a uniformizer of $\mathcal{O}$.
- The module $\mathcal{T}$ satisfies conditions (H.2) and (H.3).
- The module $(\mathcal{T} \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{\mathcal{I}_\mathfrak{q}}$ is divisible for every prime $\mathfrak{q} \nmid p$ of K.
- $H^2(K_\mathfrak{p}, \mathcal{T}) = 0$ for each prime $\mathfrak{p} \mid p$ of K.

Let n be a positive integer and $J_n := (p^n, X^n)$. Let $\mathcal{F}_n$ be the Selmer structure on $\mathbb{T}/J_n\mathbb{T}$ defined in Example 5.3. Note that $\chi(\mathcal{F}_n) = \sum_{v \mid \infty} \mathrm{rank}_{\mathcal{O}}(H^0(K_v, \mathcal{T}^\vee)^\vee)$, where v runs through all the infinite places of K. Set $\mathcal{F} = \{\mathcal{F}_n\}_{n\geq 1}$ and $r = \chi(\mathcal{F}_n)$. We can define the module of Stark systems by

$$\boldsymbol{SS}_r(\mathbb{T}) := \boldsymbol{SS}_r(\mathbb{T}, \mathcal{F}, \{\mathcal{P}_{n^2}(\mathcal{F}_n)\}_{n\geq 1}) = \varprojlim_{n\geq 1} \boldsymbol{SS}_r(\mathbb{T}/J_n\mathbb{T}, \mathcal{F}_n, \mathcal{P}_{n^2}(\mathcal{F}_n)).$$

Note that $\boldsymbol{SS}_r(\mathbb{T})$ is a free Λ-module of rank 1 by Theorem 5.4. We call an element of $\boldsymbol{SS}_r(\mathbb{T})$ a Λ-adic Stark system. We say that a Λ-adic Stark system is primitive if it is a basis of $\boldsymbol{SS}_r(\mathbb{T})$.

Remark 6.1. Assume that $r = 1$. Let $\boldsymbol{KS}(\mathbb{T})$ be the module of Λ-adic Kolyvagin systems defined in [Mazur and Rubin 2004, Chapter 5; Büyükboduk 2011]. Note that the module of Λ-adic Stark systems is not defined in [Mazur and Rubin 2016] even if $r = 1$. We can construct a canonical isomorphism $\boldsymbol{SS}_1(\mathbb{T}) \xrightarrow{\sim} \boldsymbol{KS}(\mathbb{T})$; see [Büyükboduk 2011, Section 3.1.2]. Thus we have a natural map from the module of Euler systems to the module of Λ-adic Stark systems $\boldsymbol{SS}_1(\mathbb{T})$ by [Mazur and Rubin 2004, Theorem 5.3.3].

Example 6.2. Suppose that K is a totally real number field. Let $\chi : G_K \to \mathcal{O}^\times$ be an even character of finite prime-to-p order and let $\mathfrak{f}_\chi$ be the conductor of χ. Put $\mathcal{T} := \mathcal{O}(1) \otimes_{\mathcal{O}} \chi^{-1}$. Assume the following hypotheses:

- $(p, \mathfrak{f}_\chi) = 1$.
- K is unramified at all primes above p.
- $\chi(\mathrm{Frob}_{\mathfrak{q}}) \neq 1$ for any prime $\mathfrak{p}$ of K above p.

Then $\mathcal{T}$ satisfies all the conditions in this section and we have $r = [K : \mathbb{Q}]$ by [Mazur and Rubin 2016, Corollary 5.6]. Thus if $K = \mathbb{Q}$, then we get a Λ-adic Stark system from the Euler systems of cyclotomic units. By using the analytic class number formula, we see that this Stark system is primitive. More generally, Büyükboduk constructed an $\mathbb{L}$-restricted Kolyvagin system from conjectural Rubin–Stark units in [Büyükboduk 2009]. By [Büyükboduk 2011, Section 3.1.2], this Kolyvagin system gives rise to an element of $SS_1(\mathbb{T}, \mathcal{F}_{\mathbb{L}_\infty}, \{\mathcal{P}_{n^2}(\mathcal{F}_n)\}_{n\geq 1})$. Here $\mathcal{F}_{\mathbb{L}_\infty}$ is the $\mathbb{L}_\infty$-modified Selmer structure defined in [loc. cit., Definition 4.8]. Moreover, we see that there is a canonical isomorphism $SS_r(\mathbb{T}) \xrightarrow{\sim} SS_1(\mathbb{T}, \mathcal{F}_{\mathbb{L}_\infty}, \{\mathcal{P}_{n^2}(\mathcal{F}_n)\}_{n\geq 1})$. Hence we get a Λ-adic Stark system $\epsilon^{\mathrm{R\text{-}S}} = \{\epsilon_{\mathrm{n}}^{\mathrm{R\text{-}S}}\}_{\mathrm{n}}$ such that the leading term $\epsilon_1^{\mathrm{R\text{-}S}}$ is an inverse limit of Rubin–Stark elements.

Furthermore, when K is a CM-field, Büyükboduk [2014; 2018] constructed an $\mathbb{L}$-restricted Kolyvagin system from conjectural Rubin–Stark units. In this case, we also get a primitive Λ-adic Stark system such that the leading term is an inverse limit of Rubin–Stark elements.

Example 6.3. Suppose that $\mathcal{O} = \mathbb{Z}_p$. Let A be an abelian variety of dimension d defined over K and let $\mathcal{T}$ be the Tate module $T_p(A) := \varprojlim A[p^n]$. Assume the following hypotheses:

- A has good reduction at all the primes of K above p.
- The image of G_K in $\mathrm{Aut}(A[p]) \simeq \mathrm{GL}_{2d}(\mathbb{F}_p)$ contains $\mathrm{Sp}_{2d}(\mathbb{F}_p)$.

- $p \nmid 6\prod_{\mathfrak{q}\nmid p} c_{\mathfrak{q}}$, where $c_{\mathfrak{q}} \in \mathbb{Z}_{>0}$ is the Tamagawa factor at $\mathfrak{q}\nmid p$.
- $\widehat{A}(K_{\mathfrak{q}})[p] = 0$ for any prime $\mathfrak{p} \mid p$, where $\widehat{A}$ is the dual abelian variety of A.

Then $\mathcal{T} = T_p(A)$ satisfies all the conditions in this section and we have $r = d[K:\mathbb{Q}]$ by [Mazur and Rubin 2016, Proposition 5.7]. If $K = \mathbb{Q}$ and $d = 1$, then Kato's Euler system gives rise to a Λ-adic Stark system; see [Mazur and Rubin 2004, Theorem 6.2.4 and Proposition 6.2.6; Büyükboduk 2011, Proposition 4.2]. Under the assumptions of and by [Büyükboduk 2011, Proposition 4.2], and by Theorem 5.6, we see that this Λ-adic Stark system determines all the higher Fitting ideals of the Pontryagin dual of the Λ-adic Selmer group.

Under slightly different hypotheses, there is a nontrivial conjectural example in the case of $r > 1$. Assume that $\mathcal{T}$ is the twist of $T_p(A)$ by certain character of G_K and A has supersingular reduction at p and a complex multiplication. Under the Perrin-Riou–Stark conjecture, see [Büyükboduk and Lei 2015, Conjecture 4.14], which is closely related to Rubin–Stark conjecture, Büyükboduk and Lei [2015] constructed an $\mathbb{L}$-restricted Kolyvagin system. For the same reason as Example 6.2, we see that this Kolyvagin system gives rise to a Λ-adic Stark system.

By using [Rubin 2000, Proposition B.3.4], we have $H^1(K_{\mathfrak{q}}, \mathbb{T}) = H^1_{\mathrm{ur}}(K_{\mathfrak{q}}, \mathbb{T})$ for each prime $\mathfrak{q}\nmid p$ of K. Hence we can define a Selmer structure $\mathcal{F}_\Lambda$ on $\mathbb{T}$ by the following data:

- $\Sigma(\mathcal{F}_\Lambda) = \Sigma := \{v \mid p\infty\} \cup \{\mathfrak{q} \mid \mathcal{T}$ is ramified at $\mathfrak{q}\}$.
- $H^1_{\mathcal{F}_\Lambda}(K_{\mathfrak{q}}, \mathbb{T}) = H^1(K_{\mathfrak{q}}, \mathbb{T})$ for each prime $\mathfrak{q}$ of K.

Remark 6.4. Since $H^1(K_{\mathfrak{q}}, \mathbb{T}) = H^1_{\mathrm{ur}}(K_{\mathfrak{q}}, \mathbb{T})$ for each prime $\mathfrak{q}\nmid p$ of K, we have

$$H^1_{\mathcal{F}_\Lambda}(K, \mathbb{T}) = H^1(K, \mathbb{T}) = H^1(K_\Sigma/K, \mathbb{T}).$$

Here K_Σ is the maximal extension of K unramified outside Σ and put

$$H^1(K_\Sigma/K, M) := H^1(\mathrm{Gal}(K_\Sigma/K), M)$$

for any continuous $\mathrm{Gal}(K_\Sigma/K)$-module M. Furthermore, the maps $\mathbb{T} \to \mathbb{T}/J_n\mathbb{T}$ induce isomorphisms

$$H^1(K, \mathbb{T}) \xrightarrow{\sim} \varprojlim_{n \geq 1} H^1_{\mathcal{F}_n}(K, \mathbb{T}/J_n\mathbb{T})$$

and $H^1_{\mathcal{F}^*}(K, \mathbb{T}^\vee(1)) = H^1_{\mathcal{F}^*_\Lambda}(K, \mathbb{T}^\vee(1))$. Note that $H^1(K_\Sigma/K, \mathbb{T})$, $H^2(K_\Sigma/K, \mathbb{T})$, and $H^1_{\mathcal{F}^*_\Lambda}(K, \mathbb{T}^\vee(1))^\vee$ are finitely generated Λ-modules; see [Perrin-Riou 1992, Section 3].

Remark 6.5. Since $\mathcal{T}$ satisfies condition (H.1), the $\mathcal{O}$-module $H^1(K, \mathbb{T}) \otimes_\Lambda \Lambda/(X)$ is free and the map $H^1(K, \mathbb{T}) \xrightarrow{\times X} H^1(K, \mathbb{T})$ is injective. Thus $H^1(K, \mathbb{T})$ is a free Λ-module.

Proposition 6.6. *Let $\mathfrak{q}$ be a height-1 prime of Λ with $\mathfrak{q} \neq p\Lambda$ and let $S_{\mathfrak{q}}$ denote the integral closure of $\Lambda/\mathfrak{q}$ in its field of fractions. If $H^2(K_\Sigma/K, \mathbb{T})[\mathfrak{q}]$ is finite, then we have*

$$\mathrm{rank}_\Lambda(H^1(K, \mathbb{T})) = r + \mathrm{rank}_{S_{\mathfrak{q}}}(H^1_{\mathcal{F}^*_{\mathrm{can}}}(K, (\mathbb{T} \otimes_\Lambda S_{\mathfrak{q}})^\vee(1))^\vee),$$

where $\mathcal{F}_{\mathrm{can}}$ is the canonical Selmer structure defined in Example 3.4.

Proof. Note that $\mathbb{T} \otimes_\Lambda S_\mathfrak{q}$ satisfies Hypothesis 3.12. Since $H^2(K_\Sigma/K, \mathbb{T})[\mathfrak{q}]$ is finite, by the same proof as [Mazur and Rubin 2004, Proposition 5.3.14], one can show that the map

$$H^1(K, \mathbb{T}) \otimes_\Lambda \Lambda/\mathfrak{q} \to H^1_{\mathcal{F}_{\mathrm{can}}}(K, \mathbb{T} \otimes_\Lambda S_\mathfrak{q})$$

induced by the map $\mathbb{T} \to \mathbb{T} \otimes_\Lambda S_\mathfrak{q}$ is injective and its cokernel is finite. Since $H^1(K, \mathbb{T})$ is a free Λ-module by Remark 6.5, we have

$$\mathrm{rank}_\Lambda(H^1(K, \mathbb{T})) = \mathrm{rank}_{S_\mathfrak{q}}(H^1_{\mathcal{F}_{\mathrm{can}}}(K, \mathbb{T} \otimes_\Lambda S_\mathfrak{q})).$$

Since $H^2(K_\mathfrak{p}, \mathcal{T}) = 0$ for any prime $\mathfrak{p} \mid p$ of K, the core rank of $\mathcal{F}_{\mathrm{can}}$ on $S_\mathfrak{q}$ is r by Example 3.20. Thus by the same proof as [Mazur and Rubin 2004, Corollary 5.2.6], we have

$$\mathrm{rank}_{S_\mathfrak{q}}(H^1_{\mathcal{F}_{\mathrm{can}}}(K, \mathbb{T} \otimes_\Lambda S_\mathfrak{q})) = r + \mathrm{rank}_{S_\mathfrak{q}}(H^1_{\mathcal{F}^*_{\mathrm{can}}}(K, (\mathbb{T} \otimes_\Lambda S_\mathfrak{q})^\vee(1))^\vee). \qquad \square$$

Proposition 6.7. *The Λ-module $H^1_{\mathcal{F}^*_\Lambda}(K, \mathbb{T}^\vee(1))^\vee$ is torsion if and only if $H^1(K, \mathbb{T})$ is a free Λ-module of rank r.*

Proof. Since $H^2(K_\Sigma/K, \mathbb{T})$ is a finitely generated Λ-module, there is a height-1 prime $\mathfrak{q} \neq p\Lambda$ of Λ such that $H^2(K_\Sigma/K, \mathbb{T})[\mathfrak{q}]$ is finite. Then by the same proof as [Mazur and Rubin 2004, Proposition 5.3.14], we see that the kernel and the cokernel of the map

$$H^1_{\mathcal{F}^*_\Lambda}(K, \mathbb{T}^\vee(1))^\vee \otimes_\Lambda \Lambda/\mathfrak{q} \to H^1_{\mathcal{F}^*_{\mathrm{can}}}(K, (\mathcal{T} \otimes_\mathcal{O} S_\mathfrak{q})^\vee(1))^\vee$$

induced by the map $(\mathbb{T} \otimes_\Lambda S_\mathfrak{q})^\vee(1) \to \mathbb{T}^\vee(1)$ are finite. Thus if $H^1(K, \mathbb{T})$ is free of rank r, then $H^1_{\mathcal{F}^*_\Lambda}(K, \mathbb{T}^\vee(1))^\vee \otimes_\Lambda \Lambda/\mathfrak{q}$ is finite by Proposition 6.6. Hence $H^1_{\mathcal{F}_\Lambda}(K, \mathbb{T}^\vee(1))^\vee$ is a torsion Λ-module. If $H^1_{\mathcal{F}^*_\Lambda}(K, \mathbb{T}^\vee(1))^\vee$ is a torsion Λ-module, then there is a height-1 prime $\mathfrak{q} \neq p\Lambda$ of Λ such that both $H^1_{\mathcal{F}^*_\Lambda}(K, \mathbb{T}^\vee(1))^\vee \otimes_\Lambda \Lambda/\mathfrak{q}$ and $H^2(K_\Sigma/K, \mathbb{T})[\mathfrak{q}]$ are finite. Thus by Remark 6.5 and Proposition 6.6, $H^1(K, \mathbb{T})$ is a free Λ-module of rank r. $\square$

Remark 6.8. The assertion that $H^1_{\mathcal{F}^*_\Lambda}(K, \mathbb{T}^\vee(1))^\vee$ is a torsion Λ-module is a form of the weak Leopoldt conjecture; see [Perrin-Riou 2000, Section 1.3]. When $T = \mathbb{Z}_p(1)$, this assertion is closely related to boundedness of $\{\delta(K_n)\}_{n \in \mathbb{Z}_{>0}}$; see [Neukirch et al. 2008, Theorem 10.3.22]. Here K_n is the n-th layer of K_∞/K and $\delta(K_n)$ is the Leopoldt defect of K_n.

Lemma 6.9. *Let n be a positive integer. Then we have an exact sequence*

$$0 \to H^2(K_\Sigma/K, \mathbb{T})[J_n] \to H^1(K, \mathbb{T}) \otimes_\Lambda \Lambda/J_n \to H^1(K_\Sigma/K, \mathbb{T}/J_n\mathbb{T}).$$

Proof. Note that, by using the exact sequence $0 \to \mathbb{T}/X^n\mathbb{T} \xrightarrow{\times p^n} \mathbb{T}/X^n\mathbb{T} \to \mathbb{T}/J_n\mathbb{T} \to 0$, we have an exact sequence

$$0 \to H^1(K_\Sigma/K, \mathbb{T}/X^n\mathbb{T}) \xrightarrow{\times p^n} H^1(K_\Sigma/K, \mathbb{T}/X^n\mathbb{T}) \to H^1(K_\Sigma/K, \mathbb{T}/J_n\mathbb{T})$$

since we assume $(\mathbb{T}/(\varpi, X)\mathbb{T})^{G_K} = (\mathcal{T}/\varpi\mathcal{T})^{G_K} = 0$, where ϖ is a uniformizer of $\mathcal{O}$. Furthermore, the exact sequence $0 \to \mathbb{T} \xrightarrow{\times X^n} \mathbb{T} \to \mathbb{T}/X^n\mathbb{T} \to 0$ induces an exact sequence

$$0 \to H^1(K_\Sigma/K, \mathbb{T}) \otimes_\Lambda \Lambda/X^n\Lambda \to H^1(K_\Sigma/K, \mathbb{T}/X^n\mathbb{T}) \to H^2(K_\Sigma/K, \mathbb{T})[X^n] \to 0.$$

Applying $- \otimes_{\Lambda/(X^n)} \Lambda/J_n$ to the above exact sequence, we get the exact sequence

$$0 \to H^2(K_\Sigma/K, \mathbb{T})[J_n] \to H^1(K, \mathbb{T}) \otimes_\Lambda \Lambda/J_n \to H^1(K_\Sigma/K, \mathbb{T}/X^n\mathbb{T}) \otimes_{\Lambda/(X^n)} \Lambda/J_n. \qquad \square$$

The following theorem is a generalization of [Mazur and Rubin 2004, Theorems 5.3.6 and 5.3.10]. However, the proof of this theorem is different from the proof in that paper.

Theorem 6.10. *If ϵ is a primitive Λ-adic Stark system and i is a nonnegative integer, then we have*

$$I_i(\epsilon) = \mathrm{Fitt}^i_\Lambda(H^1_{\mathcal{F}^*_\Lambda}(K, \mathbb{T}^\vee(1))^\vee).$$

In particular, $H^1_{\mathcal{F}^*_\Lambda}(K, \mathbb{T}^\vee(1))^\vee$ is a torsion Λ-module if and only if there is a Λ-adic Stark system $\eta = \{\eta^{(n)}\}_{n\geq 1}$ such that $\eta^{(n)}_1 \neq 0$ for some positive integer n.

Proof. The first assertion follows from $H^1_{\mathcal{F}^*}(K, \mathbb{T}^\vee(1)) = H^1_{\mathcal{F}^*_\Lambda}(K, \mathbb{T}^\vee(1))$ and Theorem 5.6. Since

$$\mathrm{Ann}_\Lambda(M)^m \subseteq \mathrm{Fitt}^0_\Lambda(M) \subseteq \mathrm{Ann}_\Lambda(M)$$

for any Λ-module M which is generated by m elements, $H^1_{\mathcal{F}^*_\Lambda}(K, \mathbb{T}^\vee(1))^\vee$ is a torsion Λ-module if and only if $I_0(\epsilon) \neq 0$. Furthermore, by Proposition 4.12(2), there is a Λ-adic Stark system $\eta = \{\eta^{(n)}\}_{n\geq 1}$ such that $\eta^{(n)}_1 \neq 0$ for some positive integer n if and only if $I_0(\epsilon) \neq 0$. $\square$

We will show that Theorem 6.10 implies [Mazur and Rubin 2004, Theorems 5.3.6 and 5.3.10]. Assume $r = 1$. Let n be a positive integer. By Matlis duality, the natural map

$$H^1_{\mathcal{F}_n}(K, \mathbb{T}/J_n\mathbb{T}) \to {\bigcap}^1_{\Lambda/J_n} H^1_{\mathcal{F}_n}(K, \mathbb{T}/J_n\mathbb{T})$$

is an isomorphism. Hence we have a map

$$\Theta^{(n)} : \boldsymbol{SS}_1(\mathbb{T}/J_n\mathbb{T}, \mathcal{F}_n, \mathcal{P}_{n^2}(\mathcal{F}_n)) \longrightarrow {\bigcap}^1_{\Lambda/J_n} H^1_{\mathcal{F}_n}(K, \mathbb{T}/J_n\mathbb{T}) \xleftarrow{\sim} H^1_{\mathcal{F}_n}(K, \mathbb{T}/J_n\mathbb{T}),$$

where the first map is a projection map. It is easy to see that the maps $\Theta^{(n)}$ induce a map $\Theta : \boldsymbol{SS}_1(\mathbb{T}) \to H^1(K, \mathbb{T})$.

Proposition 6.11. *Let $H^2(K_\Sigma/K, \mathbb{T})_{\mathrm{fin}}$ denote the maximal finite Λ-submodule of $H^2(K_\Sigma/K, \mathbb{T})$. Suppose that $r = 1$. If $\mathrm{im}(\Theta) \neq 0$, then $H^1(K, \mathbb{T})$ is free of rank 1 and $H^1_{\mathcal{F}^*_\Lambda}(K, \mathbb{T}^\vee(1))^\vee$ is a torsion Λ-module. Furthermore, we have*

$$\mathrm{Fitt}^0_\Lambda(H^1_{\mathcal{F}^*_\Lambda}(K, \mathbb{T}^\vee(1))^\vee) = \mathrm{char}_\Lambda(H^1(K, \mathbb{T})/\Lambda\Theta(\epsilon))\,\mathrm{Ann}_\Lambda(H^2(K_\Sigma/K, \mathbb{T})_{\mathrm{fin}})$$

for any primitive Λ-adic Stark system ϵ. In particular,

$$\mathrm{char}_\Lambda(H^1_{\mathcal{F}^*_\Lambda}(K, \mathbb{T}^\vee(1))^\vee) = \mathrm{char}_\Lambda(H^1(K, \mathbb{T})/\Lambda\Theta(\epsilon)).$$

Proof. Since the map $H^1_{\mathcal{F}_n}(K, \mathbb{T}/J_n\mathbb{T}) \to \bigcap^1_{\Lambda/J_n} H^1_{\mathcal{F}_n}(K, \mathbb{T}/J_n\mathbb{T})$ is an isomorphism for any positive integer n and $\mathrm{im}(\Theta) \neq 0$, the Λ-module $H^1_{\mathcal{F}^*_\Lambda}(K, \mathbb{T}^\vee(1))^\vee$ is a torsion by Theorem 6.10. Thus $H^1(K, \mathbb{T})$ is a free Λ-module of rank 1 by Proposition 6.7. Let $\epsilon = \{\epsilon^{(n)}\}_{n\geq 1}$ be a primitive Λ-adic Stark system. Put $\mathrm{Ind}(\epsilon) := \mathrm{char}_\Lambda(H^1(K, \mathbb{T})/\Lambda\Theta(\epsilon))$. Then we have

$$\mathrm{Ind}(\epsilon) = \{f(\Theta(\epsilon)) \mid f \in \mathrm{Hom}_\Lambda(H^1(K, \mathbb{T}), \Lambda)\}.$$

Let n be a positive integer and $\Lambda_n := \Lambda/J_n$. We will compute the ideal $I_0(\epsilon^{(n)}) \subseteq \Lambda_n$. Let M_n be the image of the map $H^1(K, \mathbb{T}) \to H^1_{\mathcal{F}_n}(K, \mathbb{T}/J_n\mathbb{T})$. Fix an isomorphism $H^1(K, \mathbb{T}) \simeq \Lambda$. Let Z_n be the image of the injective map $H^2(K_\Sigma/K, \mathbb{T})[J_n] \to H^1(K, \mathbb{T}) \otimes_\Lambda \Lambda_n \simeq \Lambda_n$. By Lemma 6.9, the fixed isomorphism $H^1(K, \mathbb{T}) \simeq \Lambda$ induces an isomorphism $\Lambda_n/Z_n \xrightarrow{\sim} M_n$. Let $\Theta(\epsilon)_n = \Theta(\epsilon) \bmod J_n \in H^1_{\mathcal{F}_n}(K, \mathbb{T}/J_n\mathbb{T})$. Since the map $H^1_{\mathcal{F}_n}(K, \mathbb{T}/J_n\mathbb{T})^* \to M_n^*$ is surjective and $\Theta(\epsilon)_n \in M_n$, we have

$$I_0(\epsilon^n) = \{f(\Theta(\epsilon)_n) \mid f \in M_n^*\} = \mathrm{Ind}(\epsilon)\Lambda_n[Z_n].$$

By the definition of the ideal Z_n, we have

$$\ker(\Lambda \to \Lambda_n/\Lambda_n[Z_n]) = \mathrm{Ann}_\Lambda(H^2(K_\Sigma/K, \mathbb{T})[J_n]).$$

Since the Λ-module $H^2(K_\Sigma/K, \mathbb{T})$ is of finite type, we have

$$\ker(\Lambda \to \Lambda_n/\Lambda_n[Z_n]) = \mathrm{Ann}_\Lambda(H^2(K_\Sigma/K, \mathbb{T})_{\mathrm{fin}})$$

for all sufficiently large integers n. Thus we conclude that

$$I_0(\epsilon) = \varprojlim_{n\geq 1} \mathrm{Ind}(\epsilon)\Lambda_n[Z_n] = \mathrm{Ind}(\epsilon)\,\mathrm{Ann}_\Lambda(H^2(K_\Sigma/K, \mathbb{T})_{\mathrm{fin}}).$$

Hence this proposition follows from Theorem 6.10. □

Proposition 6.12. *For a positive integer n, let*

$$\xi_n : \bigwedge\nolimits^r_\Lambda H^1(K, \mathbb{T}) \otimes_\Lambda \Lambda/J_n \to \bigcap\nolimits^r_{\Lambda/J_n} H^1_{\mathcal{F}_n}(K, \mathbb{T}/J_n\mathbb{T})$$

denote a natural map. If $H^2(K_\Sigma/K, \mathbb{T})_{\mathrm{fin}} = 0$, there is a unique Λ-module homomorphism

$$\Theta : \boldsymbol{SS}_r(\mathbb{T}) \to \bigwedge\nolimits^r_\Lambda H^1(K, \mathbb{T})$$

such that $\xi_n(\Theta(\epsilon) \bmod J_n) = \epsilon_1^{(n)}$ for any positive integer n and Λ-adic Stark system $\epsilon = \{\epsilon^{(n)}\}_{n\geq 1}$. Furthermore, if $\mathrm{im}(\Theta) \neq 0$, then $H^1(K, \mathbb{T})$ is free of rank r, $H^1_{\mathcal{F}^*}(K, \mathbb{T}^\vee(1))^\vee$ is a torsion Λ-module, and

$$\mathrm{Fitt}^0_\Lambda(H^1_{\mathcal{F}^*_\Lambda}(K, \mathbb{T}^\vee(1))^\vee) = \mathrm{char}_\Lambda\left(\bigwedge\nolimits^r_\Lambda H^1(K, \mathbb{T}) \Big/ \Lambda\Theta(\epsilon)\right)$$

for any primitive Λ-adic Stark system ϵ.

Proof. Let n be a positive integer and $s = \mathrm{rank}_\Lambda(H^1(K, \mathbb{T}))$. Note that $s \geq r$ by Proposition 6.6. Let $\Lambda_n := \Lambda/J_n$. By Lemma 6.9 and $H^2(K_\Sigma/K, \mathbb{T})_{\mathrm{fin}} = 0$, the map $H^1(K, \mathbb{T}) \otimes_\Lambda \Lambda_n \to H^1_{\mathcal{F}_n}(K, \mathbb{T}/J_n\mathbb{T})$ is a split injection. Thus $H^1_{\mathcal{F}_n}(K, \mathbb{T}/J_n\mathbb{T})$ has a free Λ_n-submodule of rank s and the map ξ_n is injective.

Let $\epsilon = \{\epsilon^{(n)}\}_{n\geq 1}$ be a Λ-adic Stark system. By Proposition 4.11, there is a unique element $\Theta(\epsilon)_n \in \bigwedge_\Lambda^r H^1(K, \mathbb{T}) \otimes_\Lambda \Lambda_n$ such that $\xi_n(\Theta(\epsilon)_n) = \epsilon_1^{(n)}$. Since the set $\{\Theta(\epsilon)_n\}_{n\geq 1}$ becomes an inverse system, we have the desired map $\Theta(\epsilon) := \varprojlim_{n\geq 1} \Theta(\epsilon)_n$.

Let ϵ be a primitive Λ-adic Stark system. If $\mathrm{im}(\Theta) \neq 0$, we have $s = r$ by Proposition 4.11. Hence $H^1_{\mathcal{F}^*_\Lambda}(K, \mathbb{T}^\vee(1))^\vee$ is a torsion Λ-module by Proposition 6.7. Furthermore, we have

$$\begin{aligned}\mathrm{Fitt}^0_\Lambda\Big(\bigwedge\nolimits_\Lambda^r H^1(K, \mathbb{T}) \Big/ \Lambda\Theta(\epsilon)\Big)\Lambda_n &= \mathrm{Fitt}^0_{\Lambda_n}\Big(\bigwedge\nolimits_\Lambda^r H^1(K, \mathbb{T}) \otimes_\Lambda \Lambda_n \Big/ \Lambda_n\Theta(\epsilon)_n\Big) \\ &= \mathrm{Fitt}^0_{\Lambda_n}(H^1_{\mathcal{F}^*_n}(K, (\mathbb{T}/J_n\mathbb{T})^\vee(1))^\vee) \\ &= \mathrm{Fitt}^0_\Lambda(H^1_{\mathcal{F}^*_\Lambda}(K, \mathbb{T}^\vee(1))^\vee)\Lambda_n,\end{aligned}$$

where the second equality follows from Proposition 4.11 and the third equality follows from Lemma 3.14. Since Λ is a complete noetherian local ring, this completes the proof. □

Remark 6.13. We use the same notation as in Example 6.2. If we assume the Rubin–Stark conjecture, then we have a primitive Λ-adic stark system $\epsilon^{\text{R-S}}$; see [Büyükboduk 2011, Section 4.3; 2009] and Example 6.2. Since $H^2(K_\Sigma/K, \mathbb{T})_{\mathrm{fin}} = 0$, we have

$$\mathrm{char}(H^1_{\mathcal{F}^*_\Lambda}(K, \mathbb{T}^\vee(1))^\vee) = \mathrm{char}\Big(\bigwedge\nolimits_\Lambda^r H^1(K, \mathbb{T}) \Big/ \Lambda\Theta(\epsilon^{\text{R-S}})\Big)$$

by Proposition 6.12. Let $H^1(K_p, \mathbb{T}) = \bigoplus_{\mathfrak{p}\mid p} H^1(K_\mathfrak{p}, \mathbb{T})$ and $\mathrm{loc}_p : H^1(K, \mathbb{T}) \to H^1(K_p, \mathbb{T})$ denote the localization map at p. Suppose that the map loc_p is injective. Then by Theorem 3.7, we obtain an exact sequence

$$0 \to H^1(K, \mathbb{T}) \to H^1(K_p, \mathbb{T}) \to H^1_{\mathcal{F}^*_{\mathrm{str}}}(K, \mathbb{T}^\vee(1))^\vee \to H^1_{\mathcal{F}^*_\Lambda}(K, \mathbb{T}^\vee(1))^\vee \to 0.$$

Here the Selmer structure $\mathcal{F}_{\mathrm{str}}$ is defined in [Büyükboduk 2011, Proposition 4.11]. Then we have $\mathrm{rank}_\Lambda(H^1(K, \mathbb{T})) = \mathrm{rank}_\Lambda(H^1(K_p, \mathbb{T})) = r$ by Proposition 6.12 and [loc. cit., Proposition 4.7(i)]. Thus we get

$$\mathrm{char}(H^1_{\mathcal{F}^*_{\mathrm{str}}}(K, \mathbb{T}^\vee(1))^\vee) = \mathrm{char}\Big(\bigwedge\nolimits_\Lambda^r H^1(K_p, \mathbb{T}) \Big/ \Lambda \cdot \mathrm{loc}_p(\Theta(\epsilon^{\text{R-S}}))\Big).$$

Therefore we see that Proposition 6.12 implies [loc. cit., Theorem 4.15].

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