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Proper G_a-actions on $\mathbb{C}^4$ preserving a coordinate

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Dedicated to my teachers, Vladimir Yakovlevich Lin and Evgeniy Alekseevich Gorin

We prove that the actions mentioned in the title are translations. We show also that for certain G_a-actions on affine fourfolds the categorical quotient of the action is automatically an affine algebraic variety and describe the geometric structure of such quotients.

Introduction

An algebraic action G_a of the additive group $\mathbb{C}_+$ of complex numbers on a complex algebraic variety X is free if it has no fixed points. When X is a Euclidean space $\mathbb{C}^n$ with a coordinate system $(x_1, \dots, x_n)$ the simplest example of such an action is a translation for which the action of an element $t \in \mathbb{C}_+$ is given by $(x_1, x_2, \dots, x_n) \mapsto (x_1 + t, x_2, \dots, x_n)$. It turns out that for $n \leq 3$ these notions are "essentially" the same. More precisely, when $n \leq 3$ every nontrivial free G_a-action on $\mathbb{C}^n$ in a suitable polynomial coordinate system is a translation[1] (see [Gutwirth 1961; Rentschler 1968] for $n = 2$ and [Kaliman 2004] for $n = 3$).

Starting with $n = 4$ the similar statement does not hold and the basic example of Winkelmann [1990] gives a triangular[2] free G_a-action which is not a translation. In his example the geometric quotient of the action is not Hausdorff while for a translation on $\mathbb{C}^n$, the geometric quotient is isomorphic to $\mathbb{C}^{n-1}$. Recently Dubouloz, Finston, and Jaradat [Dubouloz et al. 2014] proved that every triangular action on $\mathbb{C}^n$ which is proper (in particular, it is free and has a Hausdorff geometric quotient), is a translation in a suitable coordinate system. Note that every triangular action preserves at least one of the coordinates, and one of the aims of this paper is the following generalization of the Finston–Dubouloz–Jaradat result:

Theorem 0.1. *Every proper G_a-action on $\mathbb{C}^4$ that preserves a coordinate is a translation in a suitable polynomial coordinate system.*[3]

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[1] In fact, for $n \leq 3$ every connected one-dimensional unipotent algebraic subgroup of Cremona group of $\mathbb{C}^n$ is conjugate to such a translation [Popov 2015, Corollary 5].

[2] Recall that a G_a-action on $\mathbb{C}^n$ is triangular if in a suitable polynomial coordinate system it is of the form $(x_1, \dots, x_n) \mapsto (x_1, x_2 + tp_2(x_1), x_3 + tp_3(x_1, x_2), \dots, x_n + tp_n(x_1, \dots x_{n-1}))$, where each p_i is a polynomial. For $n \geq 3$ not every G_a-action on $\mathbb{C}^n$ is triangulable (i.e., triangular in a suitable polynomial coordinate system) [Bass 1984].

[3] Neena Gupta informed the author that she and S. M. Bhatwadekar had recently obtained an independent proof of Theorem 0.1.

Its proof involves investigation of categorical quotients of G_a-actions on $\mathbb{C}^4$. Namely, let X be an affine algebraic variety equipped with a G_a-action $\Phi : G_a \times X \to X$. Denote by $\mathbb{C}[X]^\Phi$ the subring of Φ-invariant regular functions in the ring $\mathbb{C}[X]$ of regular functions on X and by $\operatorname{Spec}\mathbb{C}[X]^\Phi$ (resp. $\operatorname{Spec}\mathbb{C}[X]$) the spectrum of $\mathbb{C}[X]^\Phi$ (resp. $\mathbb{C}[X]$). Then the natural embedding $\mathbb{C}[X]^\Phi \hookrightarrow \mathbb{C}[X]$ induces a map $\operatorname{Spec}\mathbb{C}[X] \to \operatorname{Spec}\mathbb{C}[X]^\Phi$. The fact that $\mathbb{C}[X]^\Phi$ is finitely generated is equivalent to the fact that $\operatorname{Spec}\mathbb{C}[X]^\Phi$ can be viewed as an affine algebraic variety denoted by $X//\Phi$. It is called the categorical quotient of the action and the map of the spectra yields the quotient morphism $\varrho : X \to X//\Phi$ in the category of affine algebraic varieties. If $\dim X \leq 3$ then $\mathbb{C}[X]^\Phi$ is always finitely generated by a theorem of Zariski [1954]. In higher dimensions this fact is not necessarily true by Nagata's counterexample to the fourteenth Hilbert problem. Furthermore, extending Nagata's counterexample, Daigle and Freudenburg [1999] showed for a G_a-action on $\mathbb{C}^n$, the ring of invariant functions may not be finitely generated starting from dimension $n \geq 5$. In dimension 4 the same authors showed that the ring of regular functions invariant with respect to a triangular G_a-action on $\mathbb{C}^4$ is automatically finitely generated [Daigle and Freudenburg 2001] and later Bhatwadekar and Daigle [2009] proved that it remains finitely generated if one considers instead of triangular G_a-actions the wider class of G_a-actions preserving a coordinate.[4] In this paper we establish a stronger fact contained in the next theorem together with a generalization of Theorem 0.1.

Theorem 0.2. *Let $\varphi : X \to B$ be a surjective morphism of a factorial affine algebraic G_a-variety X (i.e., X is equipped with some G_a-action Φ) of dimension 4 into a smooth affine curve B. Suppose also that*

- *the action preserves each fiber of φ;*
- *the generic fiber of φ is a three-dimensional variety Y (over the field K of rational functions on B) for which the ring of invariants of the G_a-action (induced by Φ) on Y is the polynomial ring $K[z, w]$;*[5]
- *for every $b \in B$ the fiber $X_b = \varphi^{-1}(b)$ admits a nonconstant morphism into a curve if and only if this curve is a polynomial one (i.e., the normalization of the curve is the line $\mathbb{C}$).*

Then

(1) *the ring of Φ-invariant functions is finitely generated and, thus, it can be viewed as the ring of regular functions on an affine algebraic variety $Q = X//\Phi$;*

(2) *there is an affine modification $\psi : Q \to B \times \mathbb{C}^2$ such that for some nonempty Zariski dense subset $B^* \subset B$ the restriction of ψ over B^* is an isomorphism and every singular fiber of ψ is of form $C \times \mathbb{C}$ where C is a polynomial curve.*

(3) *Furthermore, if one requires additionally that*
   - *Φ is proper and X is Cohen–Macaulay,*
   - *each fiber X_b is normal,*

[4] The author is grateful to Neena Gupta for drawing his attention to the paper of Bhatwadekar and Daigle.

[5] This assumption that the ring of invariants of the induced action is isomorphic to $K[z, w]$ can be replaced by the following: for a general $b \in B$ the categorical quotient of the restriction of Φ to the fiber $X_b = \varphi^{-1}(b)$ is isomorphic to $\mathbb{C}^2$ (see Theorem 5.7).

- *and the restriction Φ_b of Φ to X_b is a translation (in particular, X_b is naturally isomorphic to a direct product $(X_b//\Phi_b) \times \mathbb{C}$),*

then the quotient Q is locally trivial $\mathbb{C}^2$-bundle over B (and in particular it is a vector bundle by [Bass et al. 1976/77]), *X is naturally isomorphic to $Q \times \mathbb{C}$, and Φ is generated by a translation on the second factor of $X \simeq Q \times \mathbb{C}$.*

Let us emphasize that the fact that X is a direct product $Q \times \mathbb{C}$ is a rather rare event for regular G_a-actions while in the category of rational actions of a connected linear algebraic group G on an algebraic variety Y (over an algebraically closed field of any characteristic) this variety is automatically birationally isomorphic to the product of $\mathbb{P}^s$ and the rational quotient of Y with respect to a Borel subgroup of G (see [Matsumura 1963; Popov 2016]). It is also worth mentioning that the requirement that the restriction of Φ to any X_b is a translation in (3) can be can be replaced by some topological assumptions. For instance, if each fiber X_b is smooth and factorial with trivial second and third homology groups then $\Phi|_{X_b}$ is automatically a translation by [Kaliman 2004, Theorem 5.1]. In particular, this is true when X_b is a smooth contractible threefold since by the Gurjar theorem [1980, Theorem 1] (see also [Fujita 1982]) such a threefold is factorial.

This leads to the following application of Theorem 0.2.

Corollary 0.3. *Let $\varphi : X \to B$ be a surjective morphism of a smooth factorial affine algebraic fourfold X into a smooth curve B such that X is equipped with a proper G_a-action Φ preserving every fiber $X_b = \varphi^{-1}(b),\ b \in B$ of φ. Suppose that each X_b is a smooth contractible threefold such that the restriction of Φ to X_b has the plane as the quotient.*[6]

Then there exits a categorical quotient of the action $Q = X//\Phi$ in the category of affine algebraic varieties. Furthermore, Q is a two-dimensional vector bundle over B and X is naturally isomorphic to $Q \times \mathbb{C}$ while Φ is generated by a translation on the second factor of $X \simeq Q \times \mathbb{C}$.

Indeed, it is straightforward that the assumptions of Corollary 0.3 imply all assumptions of Theorem 0.2 with a possible exception of the condition on the generic fiber of φ. But this last condition follows from a combination of the Kambayashi [Kambayashi 1975] and the Kraft–Russell [Kraft and Russell 2014] theorems (see Theorem 5.4 and Example 5.6 below for details).

When $X \simeq \mathbb{C}^4,\ B \simeq \mathbb{C}$, and φ is a coordinate function the assumptions of Corollary 0.3 are automatically true and we have Theorem 0.1.

Some of the results mentioned before (including Theorem 0.1) are extended in the last section of this paper where using the Lefschetz principle we show that similar facts (see, Theorems 9.11 and 9.12) remain valid if we consider varieties X and B not over the field of complex numbers $\mathbb{C}$ but over any (not necessarily algebraically closed) field of characteristic zero.

[6]For $X_b = \mathbb{C}^3$ the quotient of a nontrivial G_a-action is always isomorphic to $\mathbb{C}^2$ due to Miyanishi's theorem [1980] and this is also true for smooth contractible threefolds with negative logarithmic Kodaira dimension [Kaliman and Saveliev 2004] like the Russell cubic $\{x + x^2y + z^2 + t^3 = 0\} \subset \mathbb{C}^4_{x,y,z,t}$.

1. General facts

Theorem 1.1. *Let $\varphi : X \to Y$ be a birational morphism of irreducible affine algebraic varieties such that Y is normal and there is a subvariety Z of Y with codimension at least 2 for which the restriction of φ to $X \setminus \varphi^{-1}(Z)$ is a surjective morphism onto $Y \setminus Z$. Suppose also that for every point $y \in Y \setminus Z$ the preimage $\varphi^{-1}(y)$ is finite. Then $\varphi : X \to Y$ is an isomorphism.*

Proof. The Zariski Main theorem (e.g., see [Grothendieck 1964, Théorème 8.12.6] or [Hartshorne 1977, Chapter III, Corollary 11.4]) implies that the restriction of φ yields an isomorphism between $X \setminus \varphi^{-1}(Z)$ and $Y \setminus Z$. By the Hartogs theorem the composition of the inverse map $\varphi^{-1} : Y \setminus Z \to X \setminus \varphi^{-1}(Z)$ with the inclusion $X \setminus \varphi^{-1}(Z) \hookrightarrow X$ extends to a morphism $Y \to X$ and we are done. $\square$

Corollary 1.2. *Let $\varphi : X \to Y$ be a morphism of irreducible affine algebraic varieties, and $E \subset X$ and $D \subset Y$ be irreducible divisors such that the restriction of φ yields an isomorphism $X \setminus E \to Y \setminus D$. Suppose also that Y is normal and $\varphi(E)$ is Zariski dense in D. Then $\varphi : X \to Y$ is an isomorphism.*

Proof. The dimension argument implies that for a general point y in D the preimage $\varphi^{-1}(y)$ is finite. Hence it is finite outside a proper closed subvariety Z of D and we are done by Theorem 1.1. $\square$

The next fact and the modified proof of Theorem 1.4 below were suggested by the referee.

Proposition 1.3. *Let $\varrho : X \to Q$ be a morphism of irreducible affine algebraic varieties such that Q is normal and $R = Q \setminus \varrho(X)$ is of codimension at least 2 in Q. Then every regular function f on X which is constant on the general fibers of ϱ descends to a unique regular function g on Q.*

Proof. Let us show first that f is a lift of a rational function g on Q. That is, one needs to establish the regularity of g on a Zariski dense open subset Q_0 of $Q \setminus R$. We can suppose that the restriction of ϱ to $X_0 = \varrho^{-1}(Q_0)$ is flat by [Grothendieck 1964, Théorème 6.9.1] and, therefore, being surjective it is faithfully flat [Atiyah and Macdonald 1969, Ch. 3, Exercise 16]. Consider $Y = X_0 \times_{Q_0} X_0$. Then the two natural projections $Y \to X_0$ generate two homomorphisms $e_1 : \mathbb{C}[X_0] \to \mathbb{C}[Y]$ and $e_2 : \mathbb{C}[X_0] \to \mathbb{C}[Y]$. Since f is constant on general fibers of ϱ we see that $f|_{X_0} \in \operatorname{Ker}(e_1 - e_2)$. In combination with the faithful flatness of the natural morphism $\mathbb{C}[Q_0] \to \mathbb{C}[X_0]$ this implies that f must be a lift of a regular function $g|_{Q_0}$ [Fantechi et al. 2005, Lemma 2.61]. Hence g is a rational function on Q.

Assume that there exists a divisor T in Q such that a general point $t_0 \in T$ does not belong to the indeterminacy set of g and $g(t_0) = \infty$. We can suppose that also $t_0 \notin R$, i.e., $\varrho^{-1}(t_0) \neq \varnothing$. Hence for a germ of a curve C in X through $x_0 \in \varrho^{-1}(t_0)$ one has $f(c) \to f(x_0)$ as $c \in C$ approaches x_0. This implies that $g(\varrho(c)) \to f(x_0) \neq \infty$ as $\varrho(c)$ approaches t_0, a contradiction. That is, g is regular on Q outside a subvariety of codimension at least 2. Since Q is normal, the Hartogs theorem implies that g is regular on Q and we are done. $\square$

Theorem 1.4. *Let $\varrho : X \to Q$ be a morphism of irreducible affine algebraic varieties and $\Phi : G_a \times X \to X$ be a nontrivial G_a-action on X which preserves each fiber of ϱ. Let Q be normal and P be a subvariety of Q with codimension at least 2 such that $Q \setminus P \subset \varrho(X)$. Suppose also that for every point $q \in Q \setminus P$*

the preimage $\varrho^{-1}(q)$ is a curve and for a general point $q \in Q \setminus P$ the preimage $\varrho^{-1}(q)$ is an irreducible curve. Then $\varrho : X \to Q$ is the categorical quotient morphism in the category of affine algebraic varieties. In particular, the subring of G_a-invariants in the ring of regular functions on X is finitely generated.

Proof. Note that by the assumption every general fiber of ϱ is nothing but an orbit of Φ. Thus every Φ-invariant function f is constant on the general fibers of ϱ. By Proposition 1.3, f is a lift of a regular function on Q. Hence the ring of regular functions on Q coincides with the ring $\mathbb{C}[X]^{\Phi}$ of invariants and we are done. □

Now the dimension argument as in the proof of Corollary 1.2 implies the following:

Corollary 1.5. *Let $\varrho : X \to Q$ be a morphism of irreducible affine algebraic varieties such that Q is normal and let $\Phi : G_a \times X \to X$ be a nontrivial G_a-action on X preserving every fiber of ϱ. Let $E \subset X$ and $D \subset Q$ be irreducible divisors such that $X \setminus E$ and $Q \setminus D$ are affine algebraic varieties for which $\varrho(X \setminus E) = Q \setminus D$ and $\varrho|_{X \setminus E)} : X \setminus E \to Q \setminus D$ is the categorical quotient morphism of the action $\Phi|_{X \setminus E}$. Suppose also that $\varrho(E)$ is Zariski dense in D.*

Then $\varrho : X \to Q$ is the categorical quotient morphism of Φ.

Proposition 1.6 (cf., [Kaliman 2004, Lemma 2.1]). *Let $\varrho : X \to Q$ be a dominant morphism of normal affine algebraic varieties. Suppose that the general fibers of ϱ are irreducible and there are no nonconstant invertible regular functions on such fibers. Suppose also that $Q \setminus \varrho(X)$ is of codimension at least 2 in Q. Then*

(1) *for every principal irreducible divisor D in X, which does not meet general fibers of ϱ, the closure of $\varrho(D)$ is the support of a principal irreducible divisor in Q;*

(2) *if X is a factorial variety so is Q;*

(3) *if X is factorial the preimage of any irreducible reduced divisor T in Q is an irreducible reduced divisor in X.*

Proof. In (1) D is the zero locus of a regular function f. Since f does not vanish on general fibers and they are irreducible we see that f is constant on each general fiber. By Proposition 1.3, $f = g \circ \varrho$ where $g \in \mathbb{C}[Q]$. Let us show that the zero locus of g coincides with the closure of $\varrho(D)$. Assume to the contrary that there is a divisor $F \subset g^{-1}(0) \setminus \varrho(D)$ in Q. Choose a rational function h on Q whose poles are contained in the closure T of F and a regular function e on Q that vanishes on $T \cap \varrho(D) = T \cap \varrho(X)$ but not at general points of T. Then for sufficiently large k the function $(e^k h) \circ \varrho$ is regular on X. Since it is constant on every general fiber of X we conclude by Proposition 1.3 that $e^k h$ is regular on Q contrary to the fact that this function has poles on T. Thus we have (1).

Let T be an irreducible reduced Weil divisor in Q and let D be an irreducible component of $\varrho^{-1}(T)$ that is a divisor in X (such a component exists because $Q \setminus \varrho(X)$ is of codimension at least 2 in Q). Under the assumption of (2), $D = f^*(0)$ for a regular function f on X since X is factorial. By (1), $T = \overline{\varrho(D)}$ and it coincides with the zeros of $g \in \mathbb{C}[X]$ where $f = g \circ \varrho$. Furthermore, if e is another regular function on Q that vanishes on T then it is divisible by g since $e \circ \varrho$ is divisible by f, and by Proposition 1.3,

$e \circ \varrho / g \circ \varrho$ is the lift of a regular function on Q. That is, $T = g^*(0)$. This implies that every irreducible element in $\mathbb{C}[Q]$ has a zero locus which is reduced irreducible and principal. In particular, this element is prime and, therefore, we have (2).

Assume that D_1 and D_2 are reduced irreducible components of $\varrho^{-1}(T)$. Since X is factorial $D_i = f_i^*(0)$. By the argument above $f_i = g_i \circ \varrho$ for regular functions g_i on Q. Furthermore g_i vanishes on T only and thus g_1/g_2 is an invertible regular function. This implies that $D_1 = D_2$ and we have (3). $\square$

Corollary 1.7. *Let X be a normal affine algebraic variety and $\varrho : X \to Q$ be a quotient morphism of a nontrivial G_a-action on X (in the category of affine algebraic varieties). Then $Q \setminus \varrho(X)$ has codimension at least 2 in Q. Furthermore, if X is factorial so is Q.*

Proof. Note that Q is normal since X is. If $Q \setminus \varrho(X)$ contains a divisor F of Q then as in Statement (1) of Proposition 1.6 we can construct a rational function g on Q with poles on F whose lift to X is regular. By construction this lift is G_a-invariant, i.e., g must be regular. A contradiction. This implies the first statement while the second one follows from Proposition 1.6. $\square$

2. Basic definitions and properties of affine modifications

Definition 2.1. Recall that an affine modification is any birational morphism $\sigma : \hat{X} \to X$ of affine algebraic varieties [Kaliman and Zaidenberg 1999]. In particular, there exists a divisor $D \subset X$ such that for $\hat{D} = \sigma^{-1}(D)$ the restriction of σ yields an isomorphism $\hat{X} \setminus \hat{D} \to X \setminus D$. There is some freedom in the choice of D and we can always suppose that D is a principal effective divisor given by zeros of a regular function $f \in A := \mathbb{C}[X]$. This is the case which we consider in the present paper. When such f is fixed we call D the divisor of modification, $\hat{D}$ is called the exceptional divisor of modification, and the closure Z of $\sigma(\hat{D})$ in X is called the center of modification. The advantage of a principal divisor D is that the algebra $\hat{A} = \mathbb{C}[\hat{X}]$ can be viewed as a subalgebra $A[I/f]$ in the field $\operatorname{Frac}(A)$ of fractions of A where I is an ideal in A (see [Kaliman and Zaidenberg 1999]).

It is worth mentioning that I is not determined uniquely, i.e., one can find another ideal $J \subset A$ for which $\hat{A} = A[J/f]$. We suppose further that I is the largest among such ideals J and we call I the ideal of the modification. (Treating A as a subalgebra of $\hat{A}$ one can see that I is the intersection of A and the principal ideal generated by f in $\hat{A}$.)

Notation 2.2. The symbols X, Z, D, $\hat{X}$, $\hat{D}$, I, f in this section have the same meaning as in Definition 2.1.

Example 2.3. Let I be generated by regular functions $f, g_1, \dots, g_n \in \mathbb{C}[X]$. Consider the closed subspace Y of $X \times \mathbb{C}^n_{v_1,\dots,v_n}$ given by the system of equations $f v_j = g_j$ $(j = 1, \dots, n)$ and the proper transform $\hat{X}$ of X under the natural projection $Y \to X$ (i.e., $\hat{X}$ is the only irreducible component of Y whose image in X under the projection is dense in X). Then the restriction $\sigma : \hat{X} \to X$ of the natural projection is our affine modification. Note that D (resp. $\hat{D}$) coincides with the zero locus of f (resp. $f \circ \sigma$) and the center Z is given by the equations $f = g_1 = \cdots = g_n = 0$.

Remark 2.4. (1) The geometrical construction behind the modification in Example 2.3 is the following. Consider blowing up $\tau : \tilde{X} \to X$ of X with respect to the ideal sheaf generated by f and $g_1, \dots, g_n$. Delete from $\tilde{X}$ divisors on which the zero multiplicity of $f \circ \tau$ is more than the zero multiplicity of at least one of the functions $g_j \circ \tau$. The resulting variety is $\hat{X}$.

(2) Note that the replacement in Example 2.3 of functions $f, g_1, \dots, g_n$ by functions $hf, hg_1, \dots, hg_n$ respectively (where $h \in \mathbb{C}[X]$ is nonzero) does not change the modification $\sigma : \hat{X} \to X$. In order to avoid this ambiguity we have to fix f. We are not going to specify such f's for affine modifications considered below since the choice of f will be clear from the context in each particular case.

Definition 2.5. (1) Let the center of modification Z in Example 2.3 be a set-theoretical complete intersection in X given by the zeros $f = g_1 = \cdots = g_n = 0$ (i.e., Z coincides with the set of common zeros of these functions and the codimension of Z in X is $n+1$). Then we call such $\sigma : \hat{X} \to X$ a Davis modification. Its main property is that $\hat{D}$ is naturally isomorphic to $Z \times \mathbb{C}^n$ and that the support of $\hat{X}$ coincides with the support of Y.[7]

(2) Let Z be a strict complete intersection in X given by $f = g_1 = \cdots = g_n = 0$; that is, Z is not only a set-theoretical complete intersection but also the defining ideal of the (reduced) subvariety Z in X coincides with the ideal I generated by $f, g_1, \dots, g_n$. Then we call σ a simple modification. In this case $\hat{X}$ coincides with Y as a scheme. Note also that the zero multiplicity of $f \circ \sigma$ is 1 at general points of $\hat{D}$.

Here are some useful properties of simple modifications from [Kaliman 2002] which we shall need later.

Proposition 2.6. *Let $\sigma : \hat{X} \to X$ be a simple modification. Then*

(1) *$\hat{X}$ is smooth over points from $Z_{\mathrm{reg}} \cap D_{\mathrm{reg}} \cap X_{\mathrm{reg}}$*;

(2) *$\hat{X}$ is Cohen–Macauley provided X is Cohen–Macauley*;

(3) *furthermore*, *if in* (2) *X is normal and none of irreducible components of the center Z of σ is contained in the singularities of X or D then $\hat{X}$ is normal.*

3. Pseudoaffine modifications

From a geometrical point of view it is sometimes convenient for us to consider a neighborhood $U \subset X$ of some point $z \in Z$ in the standard topology (we call such U a Euclidean neighborhood) and the restriction $\sigma|_{\hat{U}} : \hat{U} \to U$ where $\hat{U}$ is any connected component of $\sigma^{-1}(U)$. Since U and $\hat{U}$ are not algebraic varieties but only complex spaces let us consider the analogue of affine modifications in the analytic setting.[8]

[7] Indeed, the preimage of Z in Y is naturally isomorphic to $Z \times \mathbb{C}^n$ and therefore has dimension $\dim X - 1$. On the other hand counting the number of equations defining Y in $X \times \mathbb{C}^n$ we see that every irreducible component of Y has dimension at least $\dim X$. This implies that Y is irreducible (since all irreducible components of Y but one are contained in the preimage of Z) and thus $\hat{D} \simeq Z \times \mathbb{C}^n$.

[8] One can adhere to the algebraic setting by viewing U as an étale neighborhood of z in X.

Definition 3.1. Let X be an irreducible complex Stein space and $\psi : X \to W$ be a meromorphic map into a projective algebraic variety W with a fixed ample divisor H such that ψ is holomorphic over $W \setminus H$. A minimal resolution $\pi : \tilde{X} \to X$ of indeterminacy points for ψ leads to a holomorphic map $\tilde{\psi} : \tilde{X} \to W$. Removing from $\tilde{X}$ the preimage of H we obtain $\hat{X}$ which, together with the natural projection $\sigma : \hat{X} \to X$, will be called a pseudoaffine modification. Consider the Weil divisor $\tilde{\psi}^*(H)$ and its pushforward (as a cycle) D by $\pi : \tilde{X} \to X$. Then D is the divisor of the modification, $\hat{D} = \sigma^{-1}(D) \subset \hat{X}$ is its exceptional divisor, and the closure Z of $\sigma(\hat{D})$ is its center. The restriction of σ induces, of course, a biholomorphism between $\hat{X} \setminus \hat{D}$ and $X \setminus D$. Note also that similarly to the algebraic setting $\sigma|_{\hat{D}} : \hat{D} \to \sigma(\hat{D})$ is, by construction, the restriction of the proper morphism $\pi|_T : T \to Z$ where T is the closure of $\hat{D}$ in $\tilde{X}$. The latter observation is important for the next remark.

Remark 3.2. Though we mostly omit explicit formulations it should be emphasized that practically all the facts valid for affine modifications have similar analytic analogues for pseudoaffine modifications (e.g., see Proposition 3.6 below).

Example 3.3. Let us switch to the analytic setting in Example 2.3 by assuming that X is a Stein variety, $f, g_1, \dots, g_n$ are holomorphic functions on X. As before, Y is given in $X \times \mathbb{C}^n_{v_1,\dots,v_n}$ by equations

$$f v_j = g_j, \quad j = 1, \dots, n$$

and $\hat{X}$ is the proper transform of X under the natural projection $Y \to X$. Then the restriction $\sigma : \hat{X} \to X$ of the natural projection is a pseudoaffine modification with D (resp. $\hat{D}$) being the zero locus of f (resp. $f \circ \sigma$) and the center Z given by the equations $f = g_1 = \cdots = g_n = 0$.

Definition 3.4. If in Example 3.3 Z is a set-theoretical complete intersection in X given by the zeros

$$f = g_1 = \cdots = g_n = 0,$$

then we call σ a Davis pseudoaffine modification. If furthermore Z is a strict complete intersection given by these function then σ is a simple pseudoaffine modification.

Remark 3.5. (1) For any pair U, $\hat{U}$ as in the beginning of this section the restriction $\hat{U} \to U$ of a simple (resp. Davis) affine modification $\sigma : \hat{X} \to X$ is automatically a simple (resp. Davis) pseudoaffine modification.

(2) Note that when X, D, and Z are smooth for a simple pseudoaffine modification then for every point $z \in Z$ the collection $\{f, g_1, \dots, g_n\}$ can be extended to a local coordinate system in a Euclidean neighborhood U of z in X. In particular, if $n = 1$ we can treat U as a germ of $\mathbb{C}^n$ at the origin with coordinates $(u, v, w_1, \dots, w_{n-2})$ such that D is given in U by $u = 0$, Z by $u = v = 0$, and the preimage of U in $\hat{X}$ is viewed as a subvariety of $U \times \mathbb{C}_w$ given by the equation $uw = v$.

(3) Similarly to the algebraic setting the exceptional divisor of a Davis pseudoaffine modification is a the product of its center and a Euclidean space.

The following analytic version of Proposition 2.6 for simple pseudoaffine modifications remains valid with a verbatim proof.

Proposition 3.6. *Let $\sigma : \hat{X} \to X$ be a simple pseudoaffine modification of irreducible Stein spaces. Then*

(1) *$\hat{X}$ is smooth over smooth points from Z that are not contained in the singularities of D or X;*

(2) *$\hat{X}$ is Cohen–Macaulay provided X is Cohen–Macaulay;*

(3) *furthermore, if in* (2) *X is normal and none of irreducible components of the center Z of σ is contained in the singularities of X or D then $\hat{X}$ is normal.*

Lemma 3.7. *Let $\psi : Y \to X$ be a holomorphic map of irreducible normal Stein spaces such that for some principal effective reduced divisor $D \subset X$ and every $x \in X \setminus D$ the preimage $\psi^{-1}(x)$ is a curve. Suppose that the divisor $E = \psi^*(D)$ is reduced and irreducible, $\psi(E)$ is not contained in the singularities of X or D, and for a general point $y \in E$ the variety $\psi^{-1}(\psi(y))$ is a surface. Then for such a point $y \in Y$ (resp. for $z = \psi(y) \in X$), there exists a local analytic coordinate system $(u', v', v'', \bar{w}')$ on Y at y (resp. $(u, v, \bar{w})$ on X at z) where $\bar{w}' = (w_1', \dots, w_{n-2}')$ (resp. $\bar{w} = (w_1, \dots, w_{n-2})$) for which the local coordinate form of ψ is given by*

$$(u', v', v'', \bar{w}') \mapsto (u, v, \bar{w}) = (u', (u')^l q(u', v', v'', \bar{w}'), \bar{w}'),$$

where $l \geq 1$ is the minimal zero multiplicity of the $(n \times n)$-minors in the Jacobi matrix of ψ at y and the function $q(0, v', v'', \bar{w}')$ depends on v' or v''.

Proof. Since y is a general point of E we see that it is a smooth point of E and X and by the assumption z is a smooth point of $Z = \overline{\psi(E)}$, D, and X. Thus locally D is given by $u = 0$ and Z by $u = v = 0$, where the holomorphic functions u and v can be included in a local coordinate system $(u, v, \bar{w})$ on X at z. If $u' = u \circ \psi$ then by the assumption E is given locally near y by $u' = 0$. Since $\dim E - \dim Z = 2$, by [Chirka 1989, Appendix, Theorem 2] there exists a local coordinate system $(v', v'', \bar{w}')$ on E such that the coordinate form of $\psi|_E : E \to D$ is given by $v = 0$ and $\bar{w} = \bar{w}'$. Extending functions v', v'', and $\bar{w}'$ holomorphically to a neighborhood of y in Y we get a local coordinate system $(u', v', v'', \bar{w}')$ in this neighborhood. Furthermore, the extension $\bar{w}'$ can be chosen as $\bar{w}' = \bar{w} \circ \psi$. Hence ψ is given locally by $u = u'$, $\bar{w} = \bar{w}'$, and $v = (u')^l q(u', v', v'', \bar{w}')$, where $q(0, v', v'', \bar{w}')$ is not identically zero. If $q(0, v', v'', \bar{w}')$ is independent of v' or v'', then replacing v by $v - u^l q(0, \bar{w})$, we increase l (which is at least 1 since otherwise $\psi(E)$ is not contained in Z). On the other hand l cannot be larger than the minimal zero multiplicity of the $(n \times n)$-minors in the Jacobi matrix of ψ at y. Hence we can suppose that $q(0, v', v'', \bar{w}')$ depends on v' or v'' in which case l is such a multiplicity. □

Proposition 3.8. *Let the assumptions of Lemma 3.7 hold. Suppose also that*

$$\longrightarrow X_m \xrightarrow{\sigma_m} X_{m-1} \xrightarrow{\sigma_{m-1}} \cdots \xrightarrow{\sigma_2} X_1 \xrightarrow{\sigma_1} X_0 := X$$

is a sequence of simple pseudoaffine modifications such that

(i) *there exists a holomorphic map $\psi_m : Y \to X_m$ for which $\psi = \sigma_1 \circ \cdots \circ \sigma_m \circ \psi_m$;*

(ii) *for $\psi_i = \sigma_{i+1} \circ \cdots \circ \sigma_m \circ \psi_m$ the closure of $\psi_i(E)$ coincides with $Z_i \subset D_i$, where Z_i and D_i are the center and the divisor of σ_{i+1} respectively;*

(iii) *general points of Z_i are contained in the smooth parts of X_i and D_i.*[9]

Then such a sequence cannot be extended to the left indefinitely.

Proof. Let y be a general point in E such that $\psi_i(y) = z_i \in Z_i$ which implies that near these points Z_i, D_i, E, Y, X are smooth. By Lemma 3.7 we can consider local coordinate systems $(u', v', v'', \bar{w}')$ on Y at y and $(u_i, v_i, \bar{w}_i)$ on X_i at z_i such that ψ_i is given locally by $u_i = u'$, $\bar{w}_i = \bar{w}'$, and $v_i = (u')^l q(u', v', v'', \bar{w}')$. By Remark 3.5 (2) for a local coordinate system $(u_{i+1}, v_{i+1}, \bar{w}_{i+1})$ on X_{i+1} the coordinate form of σ_{i+1} is $(u_i, v_i, \bar{w}_i) = (u_{i+1}, u_{i+1}v_{i+1}, \bar{w}_{i+1})$. Hence a local form of ψ_{i+1} is $u_{i+1} = u'$, $\bar{w}_{i+1} = \bar{w}'$, and $v_{i+1} = (u')^{l_i-1} q(u', v', v'', \bar{w}')$. That is, in this construction $l_{i+1} = l_i - 1$. Since such powers cannot be negative we get the desired conclusion. □

Remark 3.9. In fact, we showed that m in Proposition 3.8 cannot exceed the minimal zero multiplicity l of the $(n \times n)$-minors in the Jacobi matrix of ψ from Lemma 3.7.

Similarly, we have the following.

Proposition 3.10. *Let $\sigma : \hat{X} \to X$ be a pseudoaffine modification with center Z, divisor D, and exceptional divisor $\hat{D}$. Suppose that none of components of Z is contained in $X_{\text{sing}} \cup D_{\text{sing}}$, and the image of every component of $\hat{D}$ is of codimension* 1 *in D. Let*

$$\longrightarrow X_n \xrightarrow{\sigma_n} X_{n-1} \xrightarrow{\sigma_{n-1}} \cdots \xrightarrow{\sigma_2} X_1 \xrightarrow{\sigma_1} X_0 := X$$

be a sequence of simple modifications with similar conditions on centers and such that σ factors through the composition of these simple modifications. Then such a sequence cannot be extended to the left indefinitely without violating the fact that σ factors through the composition.

Proof. For local analytic coordinate systems at general points $y \in \hat{D}$ and $z = \sigma(y) \in Z \subset X$, consider the zero multiplicity k of the Jacobian of σ at y (clearly this multiplicity is independent of the choice of local coordinate system and the choice of a general point y) and let k_i be the similar multiplicity for the modification $\sigma_i \circ \cdots \circ \sigma_1 : X_i \to X$. Since $\sigma_{i+1} : X_{i+1} \to X_i$ contracts the exceptional divisor in X_{i+1} we see that $k_{i+1} > k_i$. On the other hand if σ factors through $\sigma_i \circ \cdots \circ \sigma_1$ one must have $k_i \leq k$. This yields the desired conclusion. □

Definition 3.11. (1) Given two (pseudo)affine modifications $\sigma : \hat{X} \to X$ and $\delta : \tilde{X} \to X$ with the same center Z and divisor $D \subset X$, we say that σ dominates δ if it factors through δ. For instance, consider the normalization $\nu : \tilde{X}' \to \tilde{X}$ and let $\delta' = \delta \circ \nu : \tilde{X}' \to X$. Then δ is dominated by δ'.

(2) Let f, g, and h be holomorphic functions on a Stein manifold X and $\sigma : \hat{X} \to X$ be a simple pseudoaffine modification with a smooth divisor $D = f^*(0)$ and a smooth center Z given by a strict complete intersection $f = h = 0$. Suppose that Z is also a set-theoretical complete intersection given by $f^k = g = 0$ and $\tilde{X} \subset X \times \mathbb{C}_w$ is given by $f^k w = g$. Then we call the Davis modification $\delta : \tilde{X} \to X$ (induced by the natural projection) homogeneous (of degree k) if $f I^{k-1}$ and g generate the ideal I^k where I is the defining ideal of Z in the ring of holomorphic functions on X.

[9] Actually, (iii) follows automatically from the Proposition 3.6.

Lemma 3.12. *Let X be a germ of $\mathbb{C}^n$ at the origin with coordinates $(u, v, w_1, \ldots, w_{n-2})$ and let $\sigma : \hat{X} \to X$ and I be as in* Definition 3.11 (2) *with $f = u$ and $h = v$. Suppose also that g and δ are as in* Definition 3.11 (2) *without δ being a priori homogeneous. Then δ is homogeneous if and only if and only if the function g is of the form $g = ev^k + a$, where a is in fI^{k-1} and e is an invertible holomorphic function.*

Proof. Since $f = u$ and I is generated by u and v one can see that fI^{k-1} is generated by functions $u^k, u^{k-1}v, \ldots, uv^{k-1}$. That is, the ideal I^k is generated by fI^{k-1} and v^k and also by fI^{k-1} and g. This is possible if and only if $g = ev^k + a$, where $a \in fI^{k-1}$ and e is an invertible holomorphic function. □

Proposition 3.13. *Let $\sigma : \hat{X} \to X$ and $\delta : \tilde{X} \to X$ be as in* Definition 3.11 (2). *Then $\hat{X}$ is a normalization of $\tilde{X}$ and in particular σ dominates δ.*

Proof. Since normalization is a local operation, by Remark 3.5 (2), we can view X as a germ of $\mathbb{C}^n$ at the origin with coordinates $(u, v, w_1, \ldots, w_{n-2})$ such that D is given by $u = 0$, Z by $u = v = 0$ (i.e., $\hat{X}$ can be viewed as the hypersurface in $X \times \mathbb{C}_w$ given by the equation $uw = v$). By Lemma 3.12 $g = ev^k + a$, where $a \in fI^{k-1}$ and e is an invertible holomorphic function. Changing e we can suppose that the Taylor series of a does not contain monomials divisible by v^k. Hence $u = g/f^k = g/u^k = ew^k + c$, where c is a polynomial in w of degree at most $k - 1$ whose coefficients are holomorphic functions on X. Thus w is integral over the ring of holomorphic functions on $\tilde{X}$ which concludes the proof. □

Remark 3.14. (1) Note that for a simple pseudoaffine modification from Definition 3.11 (2) with a given divisor D the function f is determined uniquely up to an invertible factor. This implies that for a given $k \geq 1$ the notion of a homogeneous modification of degree k from Definition 3.11 (2) is determined not as much by f and h but by D and the defining ideal I (or equivalently the center Z).

(2) In particular if $\delta : \tilde{X} \to X$ is a pseudoaffine modification with a divisor D and a center Z such that $\dim Z = \dim X - 2$ then for a smooth point z of Z that is not in $X_{\mathrm{sing}} \cup D_{\mathrm{sing}}$ we can say whether δ is locally homogeneous at z or not (where in the former case there exists a Euclidean neighborhood U of z in X such that for every connected component $\tilde{U}$ of $\delta^{-1}(U)$ the modification $\delta|_{\tilde{U}} : \tilde{U} \to U$ is homogeneous).

Proposition 3.15. *Let X be a normal irreducible Stein space which is Cohen–Macaulay, $\delta : \tilde{X} \to X$ be a Davis modification with an irreducible divisor D and a center Z of* $\operatorname{codim}_X Z = 2$ *such that none of the irreducible components of Z is contained in $X_{\mathrm{sing}} \cup D_{\mathrm{sing}}$. Suppose also that Z is a strict complete intersection of the form $f = h = 0$, where $D = f^*(0)$ and that at general points of Z this modification δ is locally homogeneous. Let $\sigma : \hat{X} \to X$ be a simple pseudoaffine modification associated with divisor D and center Z. Then $\hat{X}$ is a normalization of $\tilde{X}$ and thus σ dominates δ.*

Proof. First note that $\hat{X}$ is a normal Stein space by Proposition 3.6. Let z be a general point of Z, i.e., z is a smooth point of Z and it is not contained in some subvariety $P \subset Z$ of $\operatorname{codim}_Z P \geq 1$ such that $Z \cap (X_{\mathrm{sing}} \cup D_{\mathrm{sing}}) \subset P$. Then by Proposition 3.13 there exists a Euclidean neighborhood U of z in X such that $\sigma^{-1}(U)$ is a normalization of $\delta^{-1}(U)$. That is, we have a biholomorphism ψ between $\sigma^{-1}(X \setminus P)$ and a normalization of $\delta^{-1}(X \setminus P)$. Since P is of codimension at least 3 in X and δ is a Davis modification the codimension of $\delta^{-1}(P)$ in $\tilde{X}$ is at least of 2 (because by Remark 3.5(3) $\delta^{-1}(P) \simeq P \times \mathbb{C}$). Hence the Hartogs theorem implies that ψ extends to a biholomorphism between $\hat{X}$ and a normalization of $\tilde{X}$ which is the desired conclusion. $\square$

4. Semifinite modifications

Notation 4.1. Let T be a germ of an analytic set at the origin o of $\mathbb{C}^m$, $\mathrm{Hol}(T)$ be the ring of holomorphic functions on T, and $D = T \times \mathbb{C}$. We consider a hypersurface Z in D such that $\pi|_Z : Z \to T$ is finite where $\pi : D \to T$ is the natural projection. We also suppose that Z coincides (as a set) with the zeros of an analytic function h.

Lemma 4.2. *Let Notation 4.1 hold and w be a coordinate on the second factor of $D = T \times \mathbb{C}$. Then h can be chosen as a monic polynomial in w with coefficients from* $\mathrm{Hol}(T)$, *i.e.*, $h \in \mathrm{Hol}(T)[w]$.

Proof. Let $\pi^{-1}(o) \cap Z$ consist of points $z_1, \dots, z_n$, where $z_i = (o, w_i) \in T \times \mathbb{C}$. By the Weierstrass preparation theorem, for every z_i there is a Euclidean neighborhood in D in which h coincides with $e_i h_i(w)$, where e_i is an invertible function and $h_i \in \mathrm{Hol}(T)[w]$ is a monic polynomial in $w - w_i$. Note that Z coincides with the zeros of the product $h_1 h_2 \cdots h_m$ because of the finiteness of $\pi|_Z : Z \to T$. Thus replacing h with the product $h_1 h_2 \cdots h_m$ we get the desired conclusion. $\square$

Lemma 4.3. *Let Notation 4.1 hold and the zero multiplicity of h at general points of Z be n. Then $g = h^{1/n}$ is a holomorphic function on D and in particular the defining ideal of Z in the ring* $\mathrm{Hol}(T)[w]$ *is the principal ideal generated by g.*

Proof. By Lemma 4.2 we can suppose that h is a monic polynomial in w. Consider the restriction of h to any fiber. It is a polynomial whose roots have multiplicities divisible by n (since for general fibers such multiplicities are exactly n). Thus the restriction of g to any fiber of $\pi : D \to T$ can be chosen as a nonzero monic polynomial in w. Therefore, choosing such a restriction on $\pi^{-1}(o)$ we define by continuity a unique branch of $h^{1/n}$ on D as g. Note that g is holomorphic outside a subset $K = Z \cap \pi^{-1}(T_{\mathrm{sing}})$ and because of the assumption on h we see that $g = w^k + r_{k-1} w^{k-1} + \dots + r_1 w + r_0$, where each r_i is a continuous function on T holomorphic on $T \setminus K$. Hence the first statement will follow from the claim.

Claim. Let g be a (not a priori continuous) function on D, holomorphic outside a proper analytic subset K that does not contain any fiber of π. Suppose also that g is a polynomial in w. Then g is holomorphic on D.

Taking a smaller T, if necessary, one can suppose that the image K_0 of K under the natural projection $D \to \mathbb{C}_w$ is relatively compact. In particular, the function $r_k w_0^k + r_{k-1} w_0^{k-1} + \dots + r_1 w_0 + r_0$ is holomorphic on T for every w_0 in $\mathbb{C} \setminus K_0$. Since we have an infinite number of such w_0's, every coefficient r_i is also holomorphic on T, which concludes the proof of the Claim.

To see that Z is a principal divisor in D consider any function $f \in \mathrm{Hol}(T)[w]$ vanishing on Z. Then the quotient f/g is again holomorphic on D by the Claim which yields the desired conclusion. $\square$

Lemma 4.4. *Let Notation 4.1 hold and T be singular. Then so is Z.*

Proof. Assume that Z is smooth at some point $z \in Z \cap \pi^{-1}(o)$. Let g be a function as in Lemma 4.3. Extend g to a function $\tilde{g} \in \mathrm{Hol}(\mathbb{C}^m, o)[w]$ whose zeros define an extension $\tilde{Z}$ of Z. Note that if the partial derivative of $\tilde{g}$ with respect to w is nonzero at z then by the implicit function theorem $\tilde{Z}$ is biholomorphic to $(\mathbb{C}^m, o)$ and therefore Z is locally biholomorphic to T which implies that Z is singular. Hence we assume that this derivative is zero.

Let I be the defining ideal of T in $\mathrm{Hol}(\mathbb{C}^m, o)$ and k be the dimension of T. Choose any $m-k$ elements from I and consider their Jacobi matrix (with respect to coordinates $z_1, \dots, z_m$ of $\mathbb{C}^m$). Since T is singular at o, any $(m-k)$-minor M of this matrix vanishes at o. By Lemma 4.3 the defining ideal L of Z in $(\mathbb{C}^m, o) \times \mathbb{C}_w$ is generated by I and $\tilde{g}$. Take any $m+1-k$ elements from L and consider their Jacobi matrix with respect to the coordinates $(w, z_1, \dots, z_m)$. It follows from the previous argument about the partial derivative of $\tilde{g}$ with respect to w and about the $(m-k)$-minor M that any $(m+1-k)$-minor of this new Jacobi matrix vanishes at o. This is contrary to the fact that Z is smooth at z which concludes the proof. □

Example 4.5. Lemma 4.4 is one of the central technical results in this paper. The assumption that Z coincides with zeros of a global analytic function is very important. Indeed, consider the case when T is a semicubic parabola $x^2 - y^3 = 0$ in $\mathbb{C}^2_{x,y}$. Then there is a closed immersion of $\mathbb{C}$ into $D = T \times \mathbb{C} \subset \mathbb{C}^3_{x,y,w}$ given by $t \mapsto (t^3, t^2, t)$ such that the image is a smooth Weil divisor whose projection to the singular T is finite. Lemma 4.4 is not applicable here because this Weil divisor is not $\mathbb{Q}$-Cartier.

Remark 4.6. If T is not unibranch at o then the argument is much easier and, furthermore, Z is not unibranch at any point z_0 above o as well. Indeed, let T_1 and T_2 be distinct irreducible components of T at o. Then D has irreducible branches $T_1 \times \mathbb{C}$ and $T_2 \times \mathbb{C}$ meeting along the line $L = o \times \mathbb{C}$. Since the zeros of h contain the point $z_0 \in L$ but not the line L itself the zeros Z_i of h in $T_i \times \mathbb{C}$ produce two different branches Z_1 and Z_2 of Z at z_0.

Proposition 4.7. *Let T be an affine algebraic variety, $D = T \times \mathbb{C}$, and Z be an algebraic hypersurface in D such that $\pi|_Z : Z \to T$ is finite where $\pi : D \to T$ is the natural projection. Suppose that for every point $t_0 \in T$ there is a Euclidean neighborhood $U \subset T$ such that $Z \cap \pi^{-1}(U)$ is an analytic $\mathbb{Q}$-principal divisor in $\pi^{-1}(U)$, i.e., for some natural $k > 0$ and a holomorphic function g on $\pi^{-1}(U)$ the divisor $kZ \cap \pi^{-1}(U)$ coincides with $g^*(0)$.*

Then Z is an effective reduced principal divisor in D.

Proof. Let w be a function on $D = T \times \mathbb{C}$ induced by a coordinate on the second factor. The field $\mathbb{C}(Z)$ of rational functions on Z is a finite separable extension of the field $\mathbb{C}(T)$ and it is generated by $w|_Z$ over $\mathbb{C}(T)$. Let $g(w) = w^n + r_{n-1}(t)w^{n-1} + \cdots + r_1(t)w + r_0(t)$ be the minimal monic polynomial for w over $\mathbb{C}(T)$. In particular $Z \cap \pi^{-1}(U)$ is given by the zeros of this rational function g. By Lemma 4.3 $Z \cap \pi^{-1}(U)$ is a principal divisor in $\pi^{-1}(U)$ which implies that the function g is holomorphic. Therefore g is regular (e.g., see [Kaliman 1991, Theorem 5]). This yields the desired conclusion. □

Definition 4.8. Let $\sigma : \hat{X} \to X$ be an affine modification of normal affine algebraic varieties such that X is Cohen–Macaulay and

(a) its divisor $D = f^*(0)$ (where $f \in \mathbb{C}[X]$) is a reduced principal divisor in X isomorphic to a direct product $D \simeq T \times \mathbb{C}$;

(b) the restriction $\pi|_Z : Z \to T$ of the natural projection $\pi : D \to T$ to the center Z of σ is finite;

(c) none of irreducible components of Z is contained in the singularities of X (note that for the singularities of D a similar fact also holds);

(d) for any point z of Z there is a Euclidean neighborhood U in X for which the restriction $\sigma|_{\hat{U}} : \hat{U} \to U$ of σ to $\hat{U} = \sigma^{-1}(U)$ factors through some Davis pseudoaffine modification $\delta : \tilde{U} \to U$ with the following property: the restriction of δ over a neighborhood $U' \subset U$ of any general point $z' \in Z \cap U$ is homogeneous of degree k (where k does not depend on different irreducible components of $Z \cap U$).

Then we call such a σ a semifinite affine modification.

Proposition 4.9. *Let $\sigma : \hat{X} \to X$ be a semifinite affine modification with $D \simeq T \times \mathbb{C}$ and Z as in Definition 4.8. Then σ dominates a simple modification* (*with the same center and divisor*) *and if T is singular so is the center Z.*

Proof. Let a Davis modification $\delta : \tilde{U} \to U$ from Definition 4.8 (d) be defined by the ideal (f^k, g). By condition (d), δ is locally homogeneous at general points of $Z \cap U$ which in combination with Lemma 3.12 implies that for some $k \geq 1$ the divisor $kZ \cap U$ coincides with $g^*(0) \cap U$. That is, we are under the assumptions of Proposition 4.7. Hence $Z = h^*(0)$, where $h \in \mathbb{C}[D]$. Thus extending h to a regular function on X (denoted by the same symbol) we see that Z is a strict complete intersection given by $f = h = 0$. These functions f and h induce a simple affine modification $\tau : X' \to X$ with divisor D and center Z such that X' is a normal affine algebraic variety by Proposition 2.6. Furthermore, $\hat{X}$ and X' are also normal as analytic sets by [Zariski and Samuel 1960, Chapter 13, Theorem 32]. By Proposition 3.15 $\tau^{-1}(U)$ is an analytic normalization of $\tilde{U}$. Since $\hat{X}$ is normal then the holomorphic map $\sigma^{-1}(U) \to \tilde{U}$ (induced by the domination of δ by σ) factors through the normalization of $\tilde{U}$. This yields a desired holomorphic map from $\hat{X}$ to X' which is a morphism since its restriction over $X \setminus D$ is an algebraic isomorphism. That is, σ dominates τ. The second statement follows from Lemma 4.4. □

5. Applications of Kambayashi's theorem

Notation 5.1. In this section $\varphi : X \to B$ will be a morphism of complex factorial affine algebraic varieties, Φ will be a G_a-action on X which preserves the fibers of φ. We do not assume a priori that the $\mathbb{C}[B]$-algebra $\mathbb{C}[X]^\Phi$ of Φ-invariant regular functions is finitely generated.

However, such an algebra can be viewed (and will be viewed) as a direct limit $\varinjlim A_\alpha$ of its finitely generated $\mathbb{C}[B]$-subalgebras A_α (with respect to the partial order generated by inclusions) where α belongs to some index set and each A_α can be treated as the ring $\mathbb{C}[Q_\alpha]$ of regular functions of some affine algebraic variety Q_α. Replacing A_α with its integral closure we suppose that each of these Q_α is normal (the fact that this transition preserves affineness is a standard result, e.g., [Eisenbud 1995, Theorem 4.14]). Furthermore, by the Rosenlicht Theorem (e.g., see [Vinberg and Popov 1989, Theorem 2.3]) Q_α can be chosen so that the morphism $\varrho_\alpha : X \to Q_\alpha$ induced by the natural embedding $A_\alpha \subset \mathbb{C}[X]$ separates general orbits of Φ. As in [Flenner et al. 2016] we introduce the following.

Definition 5.2. A normal affine variety Q_α as before will be called a *partial quotient* of X by Φ and the morphism $\varrho_\alpha : X \to Q_\alpha$ (separating general orbits of Φ) will be called *a partial quotient morphism.*

Remark 5.3. Note that including the coordinate functions of the morphism φ into a ring A_α we can always choose a partial quotient morphism that is constant on the fibers of φ.

Theorem 5.4. *Let $\varphi : X \to B$ and Φ be as in Notation 5.1 and the generic fiber of φ be a three-dimensional variety Y (over the field K of rational functions on B). Let $\tilde{K}$ be the algebraic closure K and $\tilde{Y}$ be the variety over $\tilde{K}$ obtained from Y by the field extension. Suppose that $\varrho : X \to Q$ is a partial quotient morphism such that φ factors through ϱ. Suppose also that the ring of invariants of the G_a-action on $\tilde{Y}$ (induced by Φ) is a polynomial ring in two variables over $\tilde{K}$. Then there is a morphism $\psi : Q \to B \times \mathbb{C}^2$ such that for some nonempty Zariski dense open subset $B^* \subset B$ the restriction of ψ over B^* is an isomorphism.*

Proof. By the Kambayashi theorem [1975] the ring of invariants of the induced action on Y is also a polynomial ring in two variables over K. Hence for some B^* as above and $X^* = \varphi^{-1}(B^*)$, the induced action on X^* has the categorical quotient isomorphic to $B^* \times \mathbb{C}^2$. Assume that $B \setminus B^*$ is a principal effective divisor, which can be done without loss of generality. Let f be a regular function on B with zero locus $B \setminus B^*$ (we denote the lifts of f to X or Q by the same symbol). For every regular function $h \in \mathbb{C}[X^*]$ there exists natural k for which $f^k h$ extends to a regular function on X. Hence for sufficiently large k_1 and k_2 the composition of the quotient morphism $\varrho_0 : X^* \to B^* \times \mathbb{C}^2_{u_1,u_2}$ with the isomorphism $\kappa : B^* \times \mathbb{C}^2_{u_1,u_2} \to B^* \times \mathbb{C}^2_{u_1,u_2}$ given by $(b, u_1, u_2) \mapsto (b, f^{k_1}u_1, f^{k_2}u_2)$ extends to a morphism $\tau : X \to B \times \mathbb{C}^2$ (note that $\tau|_{X^*} : X^* \to B^* \times \mathbb{C}^2$ can be viewed now as the quotient morphism). Similarly, since $f \in \mathbb{C}[X]$ and each $u_i \in \mathbb{C}[X^*]$ are Φ-invariant we see that $f^{k_1}u_1$ and $f^{k_2}u_2$ can be viewed as regular functions on the normal variety Q. Thus $\tau = \psi \circ \theta$, where $\theta : X \to Q$ and $\psi : Q \to B \times \mathbb{C}$ are morphisms. By the universal property of quotient morphisms, $\theta|_{X^*} : X^* \to Q \setminus f^{-1}(0)$ factors through $\tau|_{X^*}$, which implies that ψ is invertible over B^*. This yields the desired conclusion. $\square$

Remark 5.5. By construction ψ is an affine modification. Another observation is that if the general fibers of φ do not admit nonconstant invertible functions then for every irreducible divisor T in B the variety $\varphi^{-1}(T)$ is reduced and irreducible by Proposition 1.6. Actually, in this case there is no need to assume that B is factorial since it follows automatically from the fact that X is factorial.

Example 5.6. Let Notation 5.1 hold and every general fiber of φ be isomorphic to the same affine algebraic threefold V. Suppose that Φ induces an action on each general fiber of φ with the categorical quotient isomorphic to $\mathbb{C}^2$. Then we claim that the above conclusion about the partial quotient Q as a modification of $B \times \mathbb{C}^2$ over B is valid.

Indeed, by [Kraft and Russell 2014] there exists a Zariski dense open subset $B^* \subset B$ and an unramified covering $\hat{B}^* \to B^*$ such that $\hat{X}^* = X \times_{B^*} \hat{B}^*$ is naturally isomorphic to $\hat{B}^* \times V$ over $\hat{B}^*$. In particular, the induced action on the generic fiber of $\hat{X}^* \to \hat{B}^*$ has the ring of invariants isomorphic to the polynomial ring $\hat{K}[x_1, x_2]$ where $\hat{K}$ is the field of rational functions on $\hat{B}^*$. Note that this ring is obtained from the ring of invariants on the generic fiber Y of $X^* \to B^*$ via a field extension $[\hat{K} : K]$. Hence by the Kambayashi theorem the latter ring of invariants is $K[x_1, x_2]$ and we are under the assumption of Theorem 5.4.

In fact, as suggested by the referee, we can strengthen Theorem 5.4 and Example 5.6 as follows:

Theorem 5.7. *Let Notation 5.1 hold and let Φ induce an action on each general fiber of φ with the categorical quotient isomorphic to $\mathbb{C}^2$. Let Y and K be as in Theorem 5.4. Then the ring of invariants of the G_a-action on Y (induced by Φ) is a polynomial ring in two variables over K. In particular, for a partial quotient morphism $\varrho : X \to Q$ for which φ factors through ϱ (i.e., for some morphism $\kappa : Q \to B$ one has $\varphi = \kappa \circ \varrho$) there is a morphism $\psi : Q \to B \times \mathbb{C}^2$ over B such that for a nonempty Zariski dense open subset $B^* \subset B$ the restriction of ψ over B^* is an isomorphism.*

In preparation for the proof of this theorem we need the next (certainly well-known) fact.

Proposition 5.8. *Let $\kappa : Q \to B$ be a dominant morphism of algebraic varieties such that Q is normal. Then for a general point $b \in B$ the variety $\kappa^{-1}(b)$ is normal.*

Proof. Replacing B by a Zariski dense open subset (and Q by its preimage) we can suppose that κ is flat [Grothendieck 1964, Théorème 6.9.1]. Let ω be a generic point of B. Then the fiber of κ over ω is normal (e.g., see [Atiyah and Macdonald 1969, Proposition 5.13]) which is equivalent to the fact that this fiber is geometrically normal[10] since we work over a field of characteristic zero. On the other hand the set of (not necessarily closed) points in B for which the fibers over them are geometrically normal is open [Grothendieck 1966, Théorème 12.1.6]. Since this set is nonempty it contains general (closed) points of B and we are done. $\square$

Proof of Theorem 5.7. Let Q_b be the fiber $\kappa^{-1}(b)$ over a general point $b \in B$ and $Q^b \simeq \mathbb{C}^2$ be the categorical quotient of the action $\Phi|_{\varphi^{-1}(b)}$. By the universal property of quotient morphisms one has a morphism $\psi_b : Q^b \to Q_b$. Since ϱ separates general orbits ψ_b must be birational.

Assume that for a curve $C_b \subset Q^b$ the image $\psi_b(C_b)$ is a point $q_b \in Q_b$. By Corollary 1.7 the preimage of C_b in $\varphi^{-1}(b)$ is a surface S_b, i.e., $\varrho(S_b) = q_b$. By [Shafarevich 1994, Chapter I, Section 6.3, Corollary] the closure T of the set $\{q \in Q \mid \dim \varrho^{-1}(q) = 2\}$ is a subvariety. Let L be the union of the one-dimensional components of T (it is nonempty since $q_b \in T$ for general b). Note that $\varrho^{-1}(L)$ is a closed Φ-stable threefold V. By the Hironaka flattening theorem [Hironaka 1975] there exists of a proper birational morphism $\pi : \hat{Q} \to Q$ such that for the irreducible component $\hat{X}$ of $X \times_Q \hat{Q}$ dominant over X the natural projection $\hat{\varrho} : \hat{X} \to \hat{Q}$ is flat, i.e., all of its fibers are one-dimensional or empty. This implies $\pi^{-1}(L)$ contains a surface $\hat{P} = \hat{\varrho}(\hat{V})$ where $\hat{V}$ is the proper transform of V in $\hat{X}$. Choose a curve $\hat{C} \subset \hat{P}$ whose image in B is dense and such that $\hat{\varrho}^{-1}(\hat{c}) \neq \varnothing$ for a general point $\hat{c} \in \hat{C}$. Let $\hat{R}$ be a closed

[10] Recall that a scheme Z over a field k is geometrically normal if it is normal and, furthermore, it remains normal under any field extension of k. In the case of a perfect field k (e.g., a field of characteristic zero) every normal scheme is automatically geometrically normal.

surface in $\hat{Q}$ that meets $\hat{P}$ along $\hat{C}$ and let R be the closure of $\pi(\hat{R})$ (note that R contains L). Then the closure of $\varrho^{-1}(R \setminus L)$ contains a Φ-stable threefold W that meets V over general points of B (since $\hat{\varrho}^{-1}(\hat{c}) \neq \varnothing$). Because X is factorial the threefold W is the zero locus of a regular function $h \in \mathbb{C}[X]$ which, by construction, does not vanish on general fibers of ϱ. That is, it is constant on general fibers and, therefore, Φ-invariant. By construction h is not constant on the surface $V \cap \varphi^{-1}(b)$ for general b (this follows from the fact that W meets $V \cap \varphi^{-1}(b)$ along a curve). Enlarging a set of generators of the ring $\mathbb{C}[Q]$ by h (and taking the integral closure to preserve the normality of Q assumed in Definition 5.2) we obtain another partial quotient for which the image of V is a surface. That is, the set T as before becomes at most finite and for a general $b \in B$ this curve C_b does not exist.

Hence the morphism $\psi_b : Q^b \to Q_b$ is quasifinite now in addition to being birational. Assume that it is not an embedding. Then by the Zariski main theorem Q_b is not normal for general $b \in B$, contrary to Proposition 5.8. Hence $\psi_b : Q^b \to Q_b$ is an embedding.

Let D_b be the complement to the image of Q^b in Q_b. Suppose that D_b is a curve for a general $b \in B$ (since Q_b and Q^b are affine the alternative is an empty D_b). The set $Q_b \setminus \varrho(\varphi^{-1}(b))$ consists of D_b and a finite set by Corollary 1.7. Since $\varrho(X)$ is a constructible set by [Hartshorne 1977, Chap. II, Exercise 3.19] we see that $Q \setminus \varrho(X)$ is an algebraic variety. Consider the irreducible components of this variety which are dominant over B and remove those of them that have dimension $\dim B$. Then we are left with $D := \bigcup_{b \in B} D_b$ because $Q_b \setminus (\varrho(\varphi^{-1}(b)) \cup D_b)$ is finite. That is, D is an algebraic variety and $Q \setminus D$ is a quasiaffine variety. The general fibers $Q_b \setminus D_b$ of the natural morphism $Q \setminus D \to B$ are isomorphic to $\mathbb{C}^2$. By [Kaliman and Zaidenberg 2001] for a Zariski dense open subset $B^* \subset B$ the variety $Q^* = \kappa^{-1}(B^*) \setminus D$ naturally isomorphic to $B^* \times \mathbb{C}^2$. Hence for $X^* = \varphi^{-1}(B^*)$ we get a partial quotient morphism $\varrho|_{X^*} : X^* \to B^* \times \mathbb{C}^2$.

Let $\varrho' : X^* \to Q'$ be another partial quotient morphism for $\Phi|_{X^*}$ into a normal variety Q' (over B^*) such that ϱ factors through ϱ', i.e., $\varrho = \theta \circ \varrho'$ for a morphism $\theta : Q' \to B^* \times \mathbb{C}^2$. Suppose that Q'_b is the fiber of the natural morphism $Q' \to B^*$ over a point $b \in B^*$, i.e., the restriction of θ yields a morphism $\theta_b : Q'_b \to Q_b \simeq Q^b \simeq \mathbb{C}^2$. By the universal property of quotient morphisms we have a natural morphism $Q^b \to Q'_b$, i.e., θ_b is invertible. Hence θ is bijective. By the Zariski main theorem, θ is an isomorphism. This implies that $\varrho|_{X^*} : X^* \to B^* \times \mathbb{C}^2$ is the categorical quotient morphism. In particular, for Y and K as in Theorem 5.4 the ring of invariants of the G_a-action on Y (induced by Φ) is a polynomial ring in two variables over K. Now the desired conclusion follows from Theorem 5.4. □

6. Criterion for existence of an affine quotient

Notation 6.1. Let B be a unibranch germ of a smooth complex algebraic curve at point o (i.e., o is the zero locus of some $f \in \mathbb{C}[B]$) and $\varphi : X \to B$ be a morphism from a complex factorial affine algebraic variety X equipped with a G_a-action Φ which preserves each fiber of φ. We suppose also that $\varrho : X \to Q$ is a partial quotient morphism of the action Φ. Let $\psi : Q \to Q_0$ be a morphism over B into a smooth affine algebraic variety Q_0 over B and $\tau = \psi \circ \varrho$. The divisor in X (resp. Q, resp. Q_0) over o will be denoted by E (resp. D, resp. D_0). We suppose that the morphism $Q_0 \to B$ is smooth (and in particular D_0 is a smooth reduced divisor) and that ψ induces an isomorphism $Q \setminus D \to Q_0 \setminus D_0$. We suppose also that $E = \varphi^*(o)$ is reduced and denote by Z_0 the closure of $\tau(E)$ in D_0. We consider the Stein factorization (e.g., see [Hartshorne 1977, Chapter III, Corollary 11.5]) of a proper extension $\overline{E} \to Z_0$ of the morphism $\tau|_E : E \to Z_0$. Its restriction to E enables us to treat $\tau|_E : E \to Z_0$ as a composition of a surjective morphism $\lambda : E \to V$ with connected general fibers and a quasifinite morphism $\theta : V \to Z_0$.

Theorem 1.1 implies the following:

Lemma 6.2. *Let Notation 6.1 hold and Z_0 (and therefore $\psi(D)$) be Zariski dense in D_0. Then ψ is an isomorphism.*

Convention 6.3. With an exception of Corollary 6.10 we suppose throughout this section that Z_0 is a divisor in D_0 and furthermore

(i) the morphism $\theta : V \to Z_0$ from Notation 6.1 is in fact *finite*;

(ii) for every point $z_0 \in Z_0$ the preimage $\tau^{-1}(z_0)$ is a *surface*.

Remark 6.4. Note that Convention 6.3 holds automatically when E is isomorphic to $\mathbb{C}^3$ (under the assumption that Z_0 is a curve). Indeed, $\tau^{-1}(z_0)$ cannot be three-dimensional (otherwise it coincides with E and $\tau(E)$ is a point, not a curve). Then both V and Z_0 must be polynomial curves and a nonconstant morphism of polynomial curve is always finite. In fact, this argument works not only when $E \simeq \mathbb{C}^3$. It is enough to assume that there is no nonconstant morphism from E into any nonpolynomial curve.

Notation 6.5. Let Notation 6.1 hold. Our aim is to present $\psi : Q \to Q_0$ over B as a composition

$$Q =: Q_n \xrightarrow{\sigma_n} Q_{n-1} \xrightarrow{\sigma_{n-1}} \cdots \xrightarrow{\sigma_2} Q_1 \xrightarrow{\sigma_1} Q_0$$

of simple affine modifications $\sigma_i : Q_i \to Q_{i-1}$ over B. Let $\psi_i = \sigma_1 \circ \cdots \circ \sigma_i : Q_i \to Q_0$ and suppose that ψ can be presented as $\psi = \psi_i \circ \delta_i$ for a modification $\delta_i : Q \to Q_i$. Let $\tau_i = \delta_i \circ \varrho$ and thus $\tau = \psi_i \circ \tau_i$. We suppose that the zero locus of $f \circ \psi_i$ is reduced and coincides with $D_i := \psi_i^{-1}(D_0)$ (which is nothing but the divisor of modification σ_{i+1}) while the center Z_i of σ_{i+1} coincides with $\tau_i(E)$. Note that under such assumptions every Q_i is normal and Cohen–Macaulay by Proposition 2.6 and induction starting from smooth Q_0 and D_0.

Lemma 6.6. *Suppose that for some i one has a sequence of simple affine modifications*

$$Q_i \xrightarrow{\sigma_i} Q_{i-1} \xrightarrow{\sigma_{i-1}} \cdots \xrightarrow{\sigma_2} Q_1 \xrightarrow{\sigma_1} Q_0$$

such that $\psi : Q \to Q_0$ factors through $\psi_i : Q_i \to Q_0$ and, therefore, $\psi = \psi_i \circ \delta_i$. Let $\delta_i : Q \to Q_i$ be a semifinite affine modification. Then the sequence can be extended to a sequence of simple affine modifications

$$Q_{i+1} \xrightarrow{\sigma_{i+1}} Q_i \xrightarrow{\sigma_i} \cdots \xrightarrow{\sigma_2} Q_1 \xrightarrow{\sigma_1} Q_0$$

such that D_i (resp. Z_i) is the divisor (resp. center) of σ_{i+1} and $\psi : Q \to Q_0$ factors through $\psi_{i+1} : Q_{i+1} \to Q_0$, i.e., $\psi = \psi_{i+1} \circ \delta_{i+1}$.

Proof. The statement follows from Proposition 4.9. □

In order to establish when δ_i is semifinite we need the following technical fact.

Lemma 6.7. *For any point $x_0 \in E$ there is the analytic germ S at x_0 of an algebraic subvariety of X of dimension* $\dim X - 2$ *such that*

(i) *$S_0 := \tau(S)$ is a germ of an analytic hypersurface in Q_0 at $z_0 = \tau(x_0)$;*

(ii) *the map $\tau|_S : S \to S_0$ is finite;*

(iii) *for every general point $z \in Z_0$ near z_0 there is a local coordinate system $(u, v, \bar{w})$ on Q_0 at z for which D_0 is given locally by $u = 0$, Z_0 by $u = v = 0$, and every irreducible branch of the germ of S_0 at z by an equation of form $e := v - u^l d(u, \bar{w}) = 0$ where d is holomorphic and l is the zero multiplicity of the function $e \circ \tau$ at general points of E.*

Proof. Consider two regular functions h and e on E with the set P of common zeros such that near any general point P is locally a strict complete intersection given by $h = e = 0$ and for $z_0 = \tau(x_0)$ the set $P \cap \tau^{-1}(z_0)$ contains x_0 as an isolated component (which is possible since $\tau^{-1}(z_0)$ is a surface). Extend h and e regularly to X. Without loss of generality we can suppose that the set of common zeros of these extensions is an $(n-2)$-dimensional subvariety T of X where $n = \dim X$. Denote by S (resp. R) the analytic germ of T (resp. P) at x_0 and by T_0 the closure of $\tau(T)$ in Q. Let S_0 be the analytic germ of the hypersurface $T_0 \subset Q_0$ at z_0. For the restriction $\kappa : S \to T_0$ of τ the preimage $\kappa^{-1}(z_0) = x_0$ is a singleton. Hence [Grauert and Remmert 1979, Chapter 1, Section 3, Theorem 2] implies that the holomorphic map $\tau|_S : S \to S_0$ is finite. Thus we have (i) and (ii).

By construction S meets E transversely at a general point x of R. Let $z = \tau(x)$ and $(u', v', v'', \bar{w}')$ (resp. $(u, v, \bar{w})$) be a local analytic coordinate system at $x \in X$ (resp. $z \in Q_0$) as in Lemma 3.7. That is, locally E (resp. D_0, resp. Z_0) is given by $u' = 0$ (resp. $u = 0$, resp. $u = v = 0$) and τ is given by $u = u'$, $\bar{w} = \bar{w}'$, and $v = (u')^l q(u', v', v'', \bar{w}')$.

The finiteness of $\tau|_S$ implies that $\tau|_{S\cap E} : S \cap E \to S_0 \cap D_0$ is étale over the general point $z \in Z_0$. Thus $E \cap S$ is given locally by equations of form $v' = h_1(\bar{w}')$ and $v'' = h_2(\bar{w}')$, where h_1 and h_2 are holomorphic functions. Since S meets E transversely we also see that S must be given locally by equations of form $v' = h_1(\bar{w}') + u' g_1(u', v', v'', \bar{w}')$ and $v'' = h_2(\bar{w}') + u' g_2(u', v', v'', \bar{w}')$ where g_1 and g_2 are holomorphic functions. Furthermore, by the implicit function theorem these equations can be rewritten as $v' = h_1(\bar{w}') + u'\tilde{g}_1(u', \bar{w}')$ and $v'' = h_2(\bar{w}') + u'\tilde{g}_2(u', \bar{w}')$ for some other holomorphic functions $\tilde{g}_1$ and $\tilde{g}_2$. Plugging these expressions for v' and v'' with $u' = u$ and $\bar{w} = \bar{w}'$ into equation $v = (u')^l q(u', v', v'', \bar{w}')$ we get the desired form of equation $e = 0$ in (iii). Note that the function $e \circ \tau$ is given by

$$e \circ \tau(u', v', v'', \bar{w}') = (u')^l [q(u', v', v'', \bar{w}') h(u', \bar{w}')].$$

Since $q(0, v', v'', \bar{w}')$ depends on v' or v'' by Lemma 3.7, we see that the expression in the brackets is not identically zero on E, which shows that l is the zero multiplicity of $e \circ \tau$ at a general point of E. □

Lemma 6.8. *The morphism δ_i is always semifinite unless $Z_i = \tau_i(E)$ is dense in D_i.*

Proof. As we mentioned in Notation 6.5, the variety Q_i is normal and Cohen–Macaulay, and the divisor $D_i = (f \circ \psi_i)^*(0)$ is principal. Furthermore, by construction, $D_i = Z_{i-1} \times \mathbb{C}$ for $i \geq 1$. Thus we have condition (a) from Definition 4.8. The finite morphism $V \to Z_0$ (from Notation 6.1 and Convention 6.3) factors through maps $\theta_i : V \to Z_i,\ Z_i \to Z_{i-1}$ and $Z_{i-1} \to Z_0$ each of which must be, therefore, finite, i.e., we have condition (b) from Definition 4.8. Condition (c) follows from construction and Proposition 2.6. It remains to check condition (d).

Consider the following construction of an analytic germ of an algebraic variety $\mathcal{S}$ in X of $\dim \mathcal{S} = n-2$ where $\dim X = n$. Let M be the set of points in V above z_0 (by Convention 6.3 M is finite and nonempty). For every $y \in M$ choose any $x \in E$ above y. Consider the analytic germ $S(y)$ of an $(n-2)$-dimensional algebraic subvariety of X at x such that $S_0(y) := \tau(S(y))$ is an analytic hypersurface in Q_0 and the restriction of τ yields a finite map $S(y) \to S_0(y)$ (the existence of such an $S(y)$ is provided by Lemma 6.7).

Let $V(y)$ be the analytic germ of V at y and $k(y)$ be the degree of the finite morphism $E \cap S(y) \to V(y)$ induced by $\lambda : E \to V$ from Notation 6.1. In general $k(y)$ may depend on y but we allow $S(y)$ to be nonreduced and then, replacing each $S(y)$ (and, therefore, $S_0(y)$) with a multiple of it, we can suppose that $k(y) = k$ for every $y \in M$.[11] Since the variety Q_0 is smooth every $S_0(y)$ coincides with $h_{0y}^*(0)$ for some holomorphic function h_{0y}. By Lemma 6.7 for a local analytic coordinate system $(u, v, \bar{w})$ at a general point $z \in Z_0$ this function h_{0y} can be viewed as $(v - u^l d(u, \bar{w}))^k$ where l is the zero multiplicity of $(v - u^l d) \circ \tau$ on E.

Let $\mathcal{S} = \bigcup_{y \in M} S(y)$, $\mathcal{S}_0 = \bigcup_{y \in M} S_0(y)$, and $h_0 = \prod_{y \in M} h_{0y}$, i.e., $\mathcal{S}_0 = h_0^*(0)$. Then by construction $\tau|_{\mathcal{S}} : \mathcal{S} \to \mathcal{S}_0$ is finite and $\mathcal{S}_0$ meets D_0 along the analytic germ of Z_0 at z_0. For m being the degree of the finite morphism $\theta : V \to Z_0$ from Convention 6.3, this implies that near point z the function h_0 can be viewed as a product of km factors of the form $v - u^l d(u, \bar{w})$. Hence the zero multiplicity of $h_0 \circ \tau$ at general points of E is klm.

Let $S_i(y) = \tau_i(S(y))$ and $\mathcal{S}_i = \tau_i(\mathcal{S}) = \bigcup_{y \in M} S_i(y)$. Note that the finiteness of $\tau|_{\mathcal{S}} : \mathcal{S} \to \mathcal{S}_0$ implies that $\mathcal{S}_i$ is an analytic set in Q_i and $(\tau_i)|_{\mathcal{S}} : \mathcal{S} \to \mathcal{S}_i$ is also finite. Furthermore, by construction $\mathcal{S}_i$ meets D_i along the union of analytic germs of Z_i at the points from $\psi_i^{-1}(z_0) \cap \mathcal{S}_i$. Assume by induction that for a germ of some holomorphic function h_i on Q_i the variety $\mathcal{S}_i$ coincides with $h_i^*(0)$ and the zero multiplicity of $h_i \circ \tau_i$ at general points of E is $kl_i m_i$ where m_i is the degree of the finite morphism $\theta_i : V \to Z_i$ and $l_i = l - i$ (note that l_i must be greater than zero since otherwise Z_i is dense in D_i).

Hence the function h_i / f^{km_i} has a holomorphic lift to the normal variety X. Furthermore, it is constant along each fiber of ϱ and it can be also pushed to a holomorphic function on the normal variety Q. This implies that δ_i factors through the Davis modification $\kappa : Q' \to Q_i$ of Q_i along D_i with ideal generated by f^{km_i} and h_i. By Lemma 6.7 for a general point z' of Z_i there is a Euclidean neighborhood with a local coordinate system $(u', v', \bar{w}')$ such that D_i is given locally by $u' = 0$, Z_i by $u' = v' = 0$, and up to an invertible factor h_i can be presented locally as a product of km_i factors of form $v' - u' d(u', \bar{w}')$. Hence

[11] We use here the following definition: if r is the degree of a finite morphism $W \to U$ of reduced analytic sets and W_m is the m-multiple of W then we put the degree of the morphism $W_m \to U$ equal to mr. Treating W_m as a union of m disjoint samples of W one can see that this extended notion of degree has all the properties of the standard degree.

by Lemma 3.12 κ is locally homogeneous at z'. Thus δ_i is semifinite and there is a simple modification $\sigma_{i+1} : Q_{i+1} \to Q_i$ as required in Lemma 6.6.

In order to finish induction it remains to prove the existence of a function h_{i+1}. Note that over z' the function $h_i \circ \sigma_{i+1}$ has zero multiplicity km_i at general points of $D_{i+1} \subset Q_{i+1}$. Thus $h_{i+1} = h_i \circ \sigma_{i+1}/f^{km_i}$ is a regular function on the normal variety Q_{i+1} that does not vanish at general points of D_{i+1}. In particular h_{i+1} vanishes only on the proper transform of $\mathcal{S}_i$ which is, by construction, $\mathcal{S}_{i+1}$. This is the desired function and we are done. □

Theorem 6.9. *Let Notation 6.1 and Convention 6.3 hold. Then $\mathbb{C}[X]^\Phi$ is finitely generated and $Q = X//\Phi$ coincides with the affine algebraic variety Q_n from Notation 6.5 for some n.*

Proof. By Proposition 3.8 the sequence

$$Q_n \xrightarrow{\sigma_n} Q_{n-1} \xrightarrow{\sigma_i} \cdots \xrightarrow{\sigma_2} Q_1 \xrightarrow{\sigma_1} Q_0$$

of simple modifications from Notation 6.5 cannot be extended indefinitely to the left. Hence by Lemmas 6.6 and 6.8 we can suppose that for some n the image $\tau_n(E)$ is dense in D_n. Suppose that $\varrho : X \to Q$ is any partial quotient morphism over Q_n. Then $\delta_n(D)$ is dense in D_n since τ_n factors through δ_n. Then by Lemma 6.2 Q is isomorphic to Q_n.

Recall that $\mathbb{C}[X]^\Phi = \varinjlim \mathbb{C}[Q_\alpha]$, where Q_α is a partial quotient. Since every such a quotient over Q_n is Q_n itself we see that $\mathbb{C}[X]^\Phi = \mathbb{C}[Q_n]$, which yields the desired conclusion. □

Corollary 6.10. *Let X be four-dimensional and let Notation 6.1 hold without assuming Convention 6.3. Suppose that E admits a nonconstant morphism into a curve if and only if this curve is a polynomial one.*

Then Convention 6.3 holds and thus $\mathbb{C}[X]^\Phi$ is finitely generated. In particular Statement (1) *of Theorem 0.2 is true.*

Proof. By Remark 6.4, Convention 6.3 is true if Z_0 is not a point. Assume it is a point z_0. Since Q_0 and $D_0 = f^*(0)$ are smooth we can suppose that z_0 is locally a strict complete intersection given by $f = g_1 = g_2 = 0$. Note that $g_i \circ \tau$ vanishes on E. Hence $(g_i/f) \circ \tau$ is a regular G_a-invariant function on the normal variety X. This implies that $\psi : Q \to Q_0$ factors through $\sigma_1 : Q_1 \to Q_0$, where σ_1 is the simple modification along D_0 with the ideal generated by f, g_1 and g_2. By construction Q_1 is smooth and the exceptional divisor is $D_1 \simeq \mathbb{C}^2$. Thus we can replace the pair (Q_0, D_0) with (Q_1, D_1). If the center of ψ after this replacement is still a point we observe that it is again a strict complete intersection $f = h_1 = h_2 = 0$ with $h_i = (g_i/f) \circ \sigma_1$. That is, the zero multiplicity of the lift h_i (to X) on E is less than the one of $g_i \circ \tau$. Continue this procedure. As soon as one of these multiplicities becomes zero the variety Z_0 becomes at least a curve. Thus we are done. □

Another consequence of the equality $Q = Q_n$ in Theorem 6.9 is that the fiber of the morphism $Q \to B$ over o is $D_n \simeq Z_{n-1} \times \mathbb{C}$. By the assumption of Theorem 0.2, Z_{n-1} is a polynomial curve. Hence:

Corollary 6.11. *Statement* (2) *of Theorem 0.2 is true.*

7. Proper actions

Definition 7.1. Recall that an action of an algebraic group G on a variety X is proper if the morphism $G \times X \to X \times X$ that sends $(g, x) \in G \times X$ to $(x, g.x) \in X \times X$ is proper.

It is well known that every proper G_a-action is automatically free. Some geometrical properties of such actions are described below.

Proposition 7.2. *Let X be a normal affine algebraic variety equipped with a proper G_a-action Φ and let C_i be a sequence of general orbits of Φ. Suppose that $C \subset X$ is a closed curve such that for every $c \in C$ and every neighborhood $V \subset X$ of c (in the standard topology) there exists i_0 for which C_i meets V whenever $i \geq i_0$. Then C is an orbit of Φ, i.e., $C \simeq \mathbb{C}$ (in particular it is smooth and connected).*

Proof. Assume that C meets two disjoint orbits C' and C'' of Φ and choose points $c_i', c_i'' \in C_i$ such that $c_i' \to c' \in C' \cap C$ and $c_i'' \to c'' \in C'' \cap C$ as $i \to \infty$. Note that $c_i'' = t_i.c_i'$. This sequence of numbers $\{t_i\}$ in the group $\mathbb{C}_+$ goes to ∞. Indeed, otherwise switching to a subsequence we can suppose that $\lim_{i \to \infty} t_i = t$ and by continuity $c'' = t.c'$, which is impossible since the last two points are in distinct orbits. But this implies that the preimage (in $G \times X$) of a small neighborhood of the point $(c', c'') \in X \times X$ contains points of form $(t_i, c_i') \in \mathbb{C}_+ \times X$ going to infinity, contrary to properness.

Hence C meets only one orbit O of Φ. Hence $C \setminus O = \varnothing$, since otherwise C meets other orbits. Therefore, $C = O \cap C$ which implies that $C = O \simeq \mathbb{C}$ (in particular, Φ is free) and we are done. ☐

Corollary 7.3. *Suppose that for some $q \in Q$ the fiber $\varrho^{-1}(q)$ is a curve. Then $\varrho^{-1}(q) \simeq \mathbb{C}$.*

Let us fix notation for the rest of this section.

Notation 7.4. We suppose that B is a unibranch germ of a smooth complex algebraic curve at point $o = f^*(0)$ (where $f \in \mathbb{C}[B]$) and $\varphi : X \to B$ is a morphism from a complex factorial affine algebraic variety X equipped with a G_a-action Φ which preserves each fiber of φ and such that the fiber $E = \varphi^*(o)$ is reduced.

We suppose also that Convention 6.3 is true for some partial quotient morphism $\varrho : X \to Q$ of the action Φ and a morphism $\psi : Q \to Q_0$ into a smooth affine algebraic variety Q_0 over B described in Notation 6.1. Recall that in this case by Theorem 6.9 $\mathbb{C}[X]^\Phi$ is finitely generated and $Q = X//\Phi$ coincides with an affine algebraic variety Q_n from Notation 6.5 for some n, and, particular, in that notation $\tau_n : X \to Q_n$ is the quotient morphism. Furthermore, let D_n be the divisor in Q_n over $o \in B$. Then it follows from the construction of Q_n that $D_n \simeq Z_{n-1} \times \mathbb{C}$ and from the proof of Theorem 6.9 that $\tau_n(E)$ is dense in D_n.

Proposition 7.5. *Let $\alpha : E \to R$ be the quotient morphism of the restriction of Φ to E. Then there is a natural affine modification $\beta : R \to D_n$ over Z_{n-1}.*

Proof. The density of $\tau_n(E)$ in D_n implies that for a general point $q \in D_n$ its preimage $\tau_n^{-1}(q)$ is a curve. By Corollary 7.3 this curve is isomorphic to $\mathbb{C}$.

By the universal property of quotient morphisms the map $\tau_n|_E : E \to D_n$ factors through α, i.e., there exists $\beta : R \to D_n$ for which $\tau_n|_E = \beta \circ \alpha$. Since τ_{n-1} (which maps E onto Z_{n-1}) factors through τ_n we see that β is a morphism over Z_{n-1}. Note that $\beta^{-1}(q)$ is a point since $\tau_n^{-1}(q)$ is a connected curve. Thus β is birational. Any birational morphism of affine algebraic varieties is an affine modification [Kaliman and Zaidenberg 1999] and we are done. □

Remark 7.6. Proposition 7.5 implies that β is an isomorphism over a Zariski dense open subset D_n^* of D_n. Furthermore removing a proper subvariety from D_n^* one can suppose that for every point in $\beta^{-1}(D_n^*)$ the fiber of α over this point is an orbit of Φ. Let $Q' = (Q \setminus D_n) \cup D_n^*$. Note that by construction every fiber of ϱ over point of Q_* is an orbit of Φ and the codimension of $Q \setminus Q' = D_n \setminus D_n^*$ in Q is at least 2.

Recall that when X is four-dimensional and there is no nonconstant morphism from E into any nonpolynomial curve then by Remark 6.4, Convention 6.3 holds, Z_{n-1} is a polynomial curve, and thus the normalization of $D_n = Z_{n-1} \times \mathbb{C}$ is $\mathbb{C}^2$. In particular, if E is normal then R is normal (being the quotient space of normal E) and we can mention the following interesting fact (which won't be used later).

Corollary 7.7. *Let the assumption of Corollary 6.10 hold and E be normal. Then R is an affine modification of $\mathbb{C}^2$. Furthermore, for a factorial E one has $R \simeq \mathbb{C}^2$ and normalization of morphism β is a birational morphism $\beta' : \mathbb{C}^2 \to \mathbb{C}^2$.*

Proof. Since β is a morphism over the curve Z_{n-1} we see that $\beta' : R \to \mathbb{C}^2$ is a morphism over the normalization $Z_{n-1}^{\text{norm}} \simeq \mathbb{C}$ of Z_{n-1}. Note that by Corollary 1.7 R is factorial, being the quotient space of the factorial threefold E. This implies that all fibers of the $\mathbb{C}$-fibration $R \to Z_{n-1}^{\text{norm}}$ are reduced and irreducible. Hence $R \simeq \mathbb{C}^2$ and we are done. □

Proposition 7.8. *Let β be as in Proposition 7.5 and suppose that one of the following conditions is satisfied*:

(a) *β is finite, E is smooth outside codimension* 2, *and every fiber of $\alpha : E \to R$ is a curve with a possible exception of a finite number of such fibers*;

(b) *β is quasifinite and the assumption of Theorem 0.2* (3) *holds* (*in particular*, *X is Cohen–Macaulay*, *Φ is proper*, *each fiber X_b of $\varphi : X \to B$* (*including E*) *is normal*, *and*, *being the image of E*, *the curve Z_{n-1} is a polynomial one*).

Then Z_{n-1} (*and therefore $D_n = Z_{n-1} \times \mathbb{C}$*) *is smooth and β is an isomorphism.*

Proof. The assumptions on α and β imply that there is a finite subset $M \subset D_n$ such that for every $q \in D_n \setminus M$ the preimage $\tau_n^{-1}(q)$ is a curve.

Recall that X is Cohen–Macaulay which implies that the morphism ϱ is faithfully flat over $Q \setminus M$ (e.g., see [Matsumura 1970, Chapter 2 (3.J) and Theorem 3] and [Eisenbud 1995, Theorem 18.6]). Hence Φ is locally a translation over $Q \setminus M$ by [Deveney et al. 1994, Theorem 2.8] (see also [Deveney and Finston 1999, Theorem 1.2]). This implies that Z_{n-1} is smooth since otherwise $D_n \setminus M$ and therefore

$(D_n \setminus M) \times \mathbb{C} \subset E$ are not smooth outside codimension 2. It remains to note that, being birational and finite, β is an isomorphism by the Zariski main theorem.

For the second statement, consider the normalization $\kappa : R^{\mathrm{norm}} \to D_n^{\mathrm{norm}} \simeq \mathbb{C}^2$ of β. Since β is birational by Proposition 7.5 and quasifinite by the assumption, the morphism κ is an embedding. Since both R^{norm} and D_n^{norm} are affine we see that $D_n^{\mathrm{norm}} \setminus \kappa(R^{\mathrm{norm}})$ is either empty or a Cartier divisor. Since $D_n^{\mathrm{norm}} \simeq \mathbb{C}^2$, such a divisor must be given by zeros of a regular function g. Then g admits a regular lift to the normal variety E that does not vanish. This contradicts the assumption that E does not admit nonconstant morphisms into a nonpolynomial curve. Thus $\kappa(R^{\mathrm{norm}}) = D_n^{\mathrm{norm}}$ and as before the assumption on α implies that there is a finite subset $M \subset D_n$ such that for every $q \in D_n \setminus M$ the preimage $\tau_n^{-1}(q)$ is a curve. Hence we can repeat the previous argument and get the desired conclusion. □

Remark 7.9. (1) Let Q' be as in Remark 7.6. Note that exactly the same argument as in Proposition 7.8 for the set $Q \setminus M$ implies that the action Φ over Q' is locally trivial[12] even without the assumption that β is quasifinite.

(2) Actually, following [Kaliman 2002] one can extract Sathaye's theorem [1983] from the argument in the proof of Proposition 7.8. Indeed, under the assumption of Sathaye's theorem, $D_n = \mathbb{C}^2$ and thus $Z_{n-1} \simeq \mathbb{C}$. Since Z_{n-1} is smooth, every polynomial curve Z_i is smooth by Lemma 4.4 and therefore $Z_i \simeq \mathbb{C}$. Hence every $D_i = Z_{i-1} \times \mathbb{C}$ is the plane and by the Abhyankar–Moh–Suzuki theorem Z_i can be viewed as a coordinate line in D_i. It is a straightforward fact that a modification $\sigma_{i+1} : Q_{i+1} \to Q_i$ with such center Z_i and divisor D_i leads to Q_{i+1} isomorphic to Q_i which implies that $Q_n \simeq B \times \mathbb{C}^2$.

(3) The assumption on properness of Φ is crucial in Proposition 7.8. In the absence of properness Winkelmann [1990] constructed a free action on $X = \mathbb{C}^4_{x_1,x_2,x_3,x_4}$ such that the quotient Q is isomorphic to the hypersurface in $\mathbb{C}^4_{x_1,u,v,w}$ given by

$$x_1 w = v^2 - u^2 - u^3.$$

Both X and Q are considered over the curve $B = \mathbb{C}_{x_1}$ and in particular the zero fiber D of the morphism $Q \to B$ is given by $v^2 - u^2 - u^3 = 0$ in $\mathbb{C}^3_{u,v,w}$. That is, D is not a plane but it is the product of $\mathbb{C}$ and Z_{n-1} where Z_{n-1} is a polynomial curve with one node as singularity (in accordance with Theorem 0.2 (2)).

8. Some facts about fibered products

In order to establish quasifiniteness required in Proposition 7.8 we need some auxiliary facts.

Notation 8.1. Let $\varrho : X \to Q$ be a dominant morphism of normal quasiprojective algebraic varieties and S be an irreducible closed subvariety of X such that for a Zariski dense open subset S' of S the restriction $\varrho|_{S'} : S' \to Q$ is quasifinite. Suppose that $k = \dim X - \dim S$ is the dimension of general fibers of ϱ.

[12] That is, $\varrho^{-1}(Q')$ can be covered by Φ-stable affine open subsets on each of which the action admits an equivariant trivialization.

Lemma 8.2. *Consider an irreducible germ $\Gamma \subset S$ of a general curve through $s \in S$ (in particular, $\Gamma \setminus s \subset S'$ and the dimension of $\varrho^{-1}(\varrho(\Gamma \setminus s))$ is $k+1$). Then the dimension of the fiber F in the closure of $\varrho^{-1}(\varrho(\Gamma \setminus s))$ (in X) over s is k.*

Proof. By the semicontinuity theorem [Shafarevich 1994, Chapter I, Section 6.3, Corollary] the dimension of F is at least k. On the other hand it cannot be greater than k since otherwise it is not contained in the closure of the $(k+1)$-dimensional variety $\varrho^{-1}(\varrho(\Gamma \setminus s))$. $\square$

Note that for a fixed s this fiber F may in general depend on the choice of Γ, i.e., $F = F(\Gamma)$.

Definition 8.3. Suppose that $F_0(\Gamma)$ is the component of $F(\Gamma)$ containing s. We say that the data in Notation 8.1 satisfies Condition (A) if for every $s \in S$ the set $\bigcup_\Gamma F_0(\Gamma)$ consists of a finite number of k-dimensional varieties.

Example 8.4. (1) If $X = S$ then Condition (A) holds automatically.

(2) Suppose that $\varrho : X \to Q$ is the quotient morphism of a proper G_a-action Φ and S is a hypersurface of X such that $\varrho(S)$ is Zariski dense in Q. Then for a general germ Γ as in Definition 8.3 and every point $s' \in \Gamma \setminus s$ the fiber $\varrho^{-1}(\varrho(s'))$ is a union of orbits of Φ. The set $\bigcup_\Gamma F_0(\Gamma)$ contains, of course, the orbit of Φ through s and by Proposition 7.2 it contains nothing else. Thus Condition (A) holds.

Proposition 8.5. *Let Notation 8.1 hold and Condition* (A) *be satisfied. Suppose that $\tilde{X}$ is the irreducible component of $X \times_Q S$ such that it contains $\tilde{S} = \{(s, s) \mid s \in S\}$. Denote by $\lambda : \tilde{X} \to S$ the restriction to $\tilde{X}$ of the natural projection.*

Then for every irreducible subvariety $P \subset S \setminus S'$ the dimension of $\lambda^{-1}(P)$ coincides with $m + k$ where $m = \dim P$.

Proof. Suppose that for $l \geq 0$ the subvariety $P(l) \subset P$ consists of points $s \in P$ for which $\dim \lambda^{-1}(s) = k + l$. Our aim is to show that $\dim P(l) \leq m - l$ and thus $\dim \lambda^{-1}(P) = m + k$.

Let $s \in S$, Γ, F be as in Definition 8.3 and let the closure of $\varrho^{-1}(\varrho(\Gamma \setminus s)) \cap S$ meets F at points $s_1, \dots, s_r$ (i.e., each irreducible component of F passes at least through one of these points). Note that for a fixed s this set $\{s_1, \dots, s_r\}$ may in general depend on the choice of Γ, i.e., $s_i = s_i(\Gamma)$. Suppose that $M(s)$ is the union $\bigcup_i \bigcup_\Gamma s_i(\Gamma)$. By definition of $\tilde{X}$ for every $s' \in \Gamma \setminus s$ one has $\lambda^{-1}(s') \simeq \varrho^{-1}((\varrho(s'))$. Hence by Lemma 8.2 and Condition (A), the variety $\lambda^{-1}(s) = \bigcup_\Gamma F(\Gamma)$ has dimension at most $\dim M(s) + k$. In particular $P(l)$ is contained in the set

$$\{s \in P \mid \dim M(s) \geq l\}$$

and we need to estimate the dimension of this last set.

For this we need to use the Hironaka flattening theorem [1975] which implies the existence of a proper birational morphism $\hat{Q} \to Q$ such that for the union $\hat{S}$ of irreducible components of $S \times_Q \hat{Q}$ with the dominant projections to S the induced morphism $\hat{S} \to \hat{Q}$ is quasifinite. We have the commutative diagram

$$\begin{array}{ccc} \hat{S} & \xrightarrow{\beta} & S \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\gamma} \\ \hat{Q} & \xrightarrow{\alpha} & Q \end{array}$$

where the morphisms α and β are birational and the morphism θ and $\gamma|_{S'}$ are quasifinite.

Let $P = \bigcup_i P_i$, where P_i are disjoint (not necessarily closed) irreducible varieties such that for any irreducible component V of $\beta^{-1}(P_i)$ all fibers of the morphism $\beta|_V : V \to P_i$ are of the same dimension. Varying i and V consider the collection $C_1, C_2, \ldots$ of irreducible subvarieties of $\hat{Q}$ that are of the form $C_j = \overline{\theta(V)}$. Observe that, stratifying the varieties P_i's further, one can suppose that for indices $j \neq l$ the varieties C_j and C_l are either equal or disjoint. Denote by I the set of all pairs (i, j) for which there exist $V \subset \beta^{-1}(P_i)$ with $\theta(V) = C_j$ and put $C_{ij} = V$ (in particular $\dim C_j = \dim C_{ij}$ because of quasifiniteness of θ).

Let n_{ij} be the dimension of a fiber $F_{ij} \subset \beta^{-1}(s)$ of the morphism $\beta|_{C_{ij}} : C_{ij} \to P_i$. In particular $\dim P_i = \dim C_j - n_{ij} \leq m$. Hence for $n_j = \min_i \{n_{ij} | (i, j) \in I\}$ one has $\dim P_i \leq m + n_j - n_{ij}$ for any $(i, j) \in I$.

Let $s \in P_i$, i.e., $\beta^{-1}(s) = \bigcup_j F_{ij}$. For every where $(tj) \in I$ we put $F_{ij}^t = \theta^{-1}(\theta(F_{ij})) \cap C_{tj}$, i.e., $\dim F_{ij}^t = n_{ij}$ because of quasifiniteness. Note that

$$\dim \beta(F_{ij}^t) = \max(0, n_{ij} - n_{tj}) \leq n_{ij} - n_j \leq \max_j (n_{ij} - n_j).$$

Thus $l := \dim \beta(\bigcup_t F_{ij}^t) \leq \max_j (n_{ij} - n_j)$ while $\dim P_i \leq m - \max_j (n_{ij} - n_j) \leq m - l$.

Since we started with a point $s \in P_i$ for the desired inequality $\dim P(l) \leq m - l$ it suffices to show that $M(s) \subset \bigcup_j \beta(\bigcup_t F_{ij}^t)$ (since the last set is contained in P_i). But this follows from the fact that the proper transform of Γ (mentioned in the definition of $M(s)$) in $\hat{S}$ meets $\beta^{-1}(s)$ and therefore meets some F_{ij}. By quasifiniteness the proper transform (in $\hat{S}$) of every component in the closure of $\varrho^{-1}(\varrho(\Gamma \setminus s)) \cap S$ meets some F_{ij}^t. This implies that this closure passes through a point in $\beta(F_{ij}^t)$ which yields $M(s) \subset \bigcup_j \beta(\bigcup_t F_{ij}^t)$ and we get $\dim \lambda^{-1}(P) \leq m + k$, i.e., $\dim \lambda^{-1}(P) \leq \dim \varrho^{-1}(\varrho(P))$. $\square$

Corollary 8.6. *If Condition* (A) *holds and* $\varrho^{-1}(\varrho(P))$ *is at least of codimension* 2 *in* X *then* $\lambda^{-1}(P)$ *has codimension at least* 2 *in* $\tilde{X}$.

9. Main theorems

Notation 9.1. Up to the proof of Theorem 0.2 below we adhere to the assumptions of Proposition 7.5 and, in particular, Notation 6.1. That is, B is the germ of a smooth algebraic curve at point $o = f^*(0)$, $\varphi : X \to B$ is a morphism of factorial varieties, Φ is a G_a-action on X, $Q = Q_n$ is the categorical quotient $X//\Phi$ in the category of affine algebraic varieties, $\tau_n = \varrho$, and $\varrho(E)$ is dense in $D_n =: D$. Since β is birational by Proposition 7.5, there is a Zariski dense open $R^* \subset R$ such that the restriction of $\beta|_{R^*} : R^* \to D \subset Q$ to R^* is an embedding.

Suppose also (as in Theorem 0.2 (3)) that the restriction of our proper G_a-action Φ to E is a translation. In particular, the quotient morphism $\alpha : E \to R$ admits a section whose image in E, by abuse of notation, will be denoted by R. The image of R^* in this section will keep notation R^* as well (i.e., $\varrho|_{R^*} : R^* \to D$ is an embedding).

The section R coincides with the zero locus of a regular function on E which has simple zeros at every point of R. Extend this function to a regular function h on X and consider the zero locus $h^{-1}(0) =: S \subset X$ of this extension.

Lemma 9.2. *The function h from Notation 9.1 can be perturbed so that for $S^* = S \setminus R$ and $Q^* = Q \setminus D$ the restriction $\varrho|_{S^*} : S^* \to Q^*$ is finite. In particular, for $S' = S^* \cup R^*$ the restriction $\varrho|_{S'} : S' \to Q$ is quasifinite.*

Proof. Recall that $X \setminus E \simeq Q^* \times \mathbb{C}$. Let w be a coordinate on the second factor, i.e., $h|_{X\setminus E}$ can be treated as a polynomial in w with coefficients in $\mathbb{C}[Q^*]$. Let n be the degree of this polynomial and f be as in Notation 6.1. Then for sufficiently large k the function $h+f^k w^{n+1}$ is regular on X and the restriction of $h+f^k w^{n+1}$ to E has only simple zeros on its zero locus R. This $h+f^k w^{n+1}$ yields the desired perturbation. □

Proposition 9.3. *Suppose that $\tilde{X}$ is the irreducible component of $X \times_Q S$ containing $\tilde{S} = \{(s, s) \mid s \in S\}$. Then $\tilde{X}$ is naturally isomorphic to $S \times \mathbb{C}$, with $\tilde{S}$ being the section of the natural projection $\lambda : \tilde{X} \to S$.*

Proof. Let S' be as in Lemma 9.2 and $s \in S'$. Note that $\varrho^{-1}(\varrho(s))$ is a disjoint union of orbits of Φ because of quasifiniteness of $\varrho|_{S'}$. One of these orbits O_s passes through s. By construction $\lambda^{-1}(\lambda(s)) \subset \varrho^{-1}(\varrho(s)) \times s$. Since $O_s \times s$ is the only component of $\varrho^{-1}(\varrho(s)) \times s$ that meets $\tilde{S}$ we see that $\lambda^{-1}(\lambda(s)) = O_s \times s$, which is an orbit of the free action $\tilde{\Phi}$ on $\tilde{X}$ induced by Φ (in particular $\tilde{\Phi}$ preserves the fibers of λ). The variety $\tilde{S}' = \{(s, s) \mid s \in S'\}$ is a section of $\tilde{\Phi}$ over S'. Hence $\lambda^{-1}(S')$ is naturally isomorphic to $S' \times \mathbb{C}$ because the action $\tilde{\Phi}$ is free. By construction $S \setminus S'$ is of codimension at least 2 in S. Hence the same is true for the subvariety $(S \setminus S') \times \mathbb{C}$ in $S \times \mathbb{C}$. By Corollary 8.6 and Example 8.4 (2) $\tilde{X} \setminus \lambda^{-1}(S')$ is of codimension at least 2 in $\tilde{X}$. By the Hartogs theorem the isomorphism $\lambda^{-1}(S') \simeq S' \times \mathbb{C}$ extends to an isomorphism $\tilde{X} \simeq S \times \mathbb{C}$ since both varieties are affine. This is the desired conclusion. □

Lemma 9.4. *For $\tilde{X}$ from Proposition 9.3 the restriction $\kappa : \tilde{X} \to X$ of the natural projection $X \times_Q S \to X$ is quasifinite.*

Proof. The restriction of κ to $\lambda^{-1}(S')$ is quasifinite because of quasifiniteness of $\varrho|_{S'}$. Since $\tilde{X} \setminus \lambda^{-1}(S') = (S \setminus S') \times \mathbb{C} = R \times \mathbb{C}$ it suffices to check that the restriction of κ yields a quasifinite morphism $R \times \mathbb{C} \to E$. The last map is an isomorphism since R is a section of $\Phi|_E$ and we are done. □

Notation 9.5. By the Grothendieck version of the Zariski main theorem (see [Grothendieck 1964, Théorème 8.12.6]) there is an embedding $\tilde{X} \to \hat{X}$ of $\tilde{X}$ into an algebraic variety $\hat{X}$ such that κ can be extended to a finite morphism $\chi : \hat{X} \to X$. Note that the finiteness implies that $\hat{X}$ is affine since X is. Furthermore, replacing if necessary $\hat{X}$, $\tilde{X}$, and S with their normalizations, we suppose that these varieties are normal.

Denote the lift of $f \in \mathbb{C}[B]$ from Notation 9.1 to X (resp. $\tilde{X}$, resp. $\hat{X}$) by the same letter f (resp. $\tilde{f}$, resp. $\hat{f}$). Let ν be the locally nilpotent vector field on X associated with the action Φ. Since $f \in \operatorname{Ker} \nu$ the field $f^k \nu$ is also locally nilpotent for every $k > 0$ and it is associated with a G_a-action Φ_k on X which has the same quotient morphism $\varrho : X \to Q$. The lift of ν to $\tilde{X}$ (resp. $\hat{X}$) will be denoted by $\tilde{\nu}$ (resp. $\hat{\nu}$). By construction $\tilde{\nu}$ (and, hence, $\tilde{f}^k\tilde{\nu}$) is locally nilpotent on $\tilde{X}$ and $\hat{\nu}$ is a rational vector field on $\hat{X}$. Since $\hat{X} \setminus \hat{f}^{-1}(0) \subset \tilde{X}$ we see that for sufficiently large k the field $\hat{f}^k\hat{\nu}$ is locally nilpotent on $\hat{X}$. We denote by $\hat{\Phi}_k$ the G_a-action associated with the last field.

Proposition 9.6. *The action $\hat{\Phi}_k$ admits a quotient morphism $\hat{\varrho} : \hat{X} \to \hat{Q}$ in the category of affine algebraic varieties such that the commutative diagram*

$$\begin{tikzcd} \hat{X} \arrow[r, "\hat{\varrho}"] \arrow[d, "\chi"] & \hat{Q} \arrow[d, "\tau"] \\ X \arrow[r, "\varrho"] & Q \end{tikzcd}$$

holds, where τ is a finite morphism.

Proof. For every function $g \in \mathbb{C}[\tilde{X}]^{\tilde{\Phi}}$ and sufficiently large $m > 0$, the function $\tilde{f}^m g$ has a regular extension to $\hat{X}$, i.e., it can be viewed as a function from $\mathbb{C}[\hat{X}]^{\hat{\Phi}_k}$. Since by construction $\hat{X} \setminus \hat{f}^{-1}(0) = \tilde{X} \setminus \tilde{f}^{-1}(0) = \lambda^{-1}(S \setminus R)$, where $\lambda : \tilde{X} \to S$ is the quotient morphism of $\tilde{\Phi}$ (by Proposition 9.3) we see now that a finite number of functions from $\mathbb{C}[\hat{X}]^{\hat{\Phi}_k}$ separate the orbits of $\hat{\Phi}_k$ contained in $\lambda^{-1}(S \setminus R)$. Consider a partial quotient morphism $\hat{\varrho} : \hat{X} \to \hat{Q}$ whose coordinates include these functions. Furthermore, since the lift of every function from $\mathbb{C}[Q]$ to $\hat{X}$ is a function from $\mathbb{C}[\hat{X}]^{\hat{\Phi}_k}$ we can include the lifts of the coordinate functions of ϱ into the set of coordinate functions of $\hat{\varrho}$. Then the commutative diagram as before make sense.

Let us show that τ is finite. Indeed, by finiteness of χ every function g from $\mathbb{C}[\hat{X}]^{\hat{\Phi}_k}$ is integral over $\mathbb{C}[X]$. That is, g is a root of an irreducible monic polynomial $P(g) := g^n + a_{n-1}g^{n-1} + \cdots + a_1 g + a_0$, where each $a_i \in \mathbb{C}[X]$. Since g is $\hat{\Phi}_k$-invariant it satisfies also an equation $g^n + a_{n-1}^t g^{n-1} + \cdots + a_1^t g + a_0^t = 0$, where a_i^t is the result of the action of $t \in \mathbb{C}_+$ (induced by $\hat{\Phi}_k$) on a_i. However, the minimal polynomial P should be unique, which implies that a_i are $\hat{\Phi}_k$-invariant and thus $\mathbb{C}[\hat{X}]^{\hat{\Phi}_k}$ is integral over $\mathbb{C}[Q]$. That is, every function from $\mathbb{C}[\hat{Q}]$ is integral over $\mathbb{C}[Q]$ and hence τ is finite.

This leads to the commutative diagram

$$\begin{tikzcd} \hat{E} \arrow[r] \arrow[d] & \hat{D} \arrow[d] \\ E \arrow[r] & D \end{tikzcd}$$

where $\hat{D} := \tau^{-1}(D)$, $\hat{E} = \hat{f}^{-1}(0)$, and the vertical morphisms are finite. Since for a general point $q \in D$ its preimage $\varrho^{-1}(q)$ is a curve in E the diagram implies that for a general point $\hat{q} \in \hat{D}$ its preimage $\hat{\varrho}^{-1}(\hat{q})$ is a curve in $\hat{E}$. Thus, by Theorem 1.4 $\hat{\varrho}$ is a quotient morphism which is the desired conclusion. □

Lemma 9.7. *Let $Q' = Q^* \cup D^*$ be as in Remark 7.9. Then*

(i) $\hat{X} \setminus (\varrho \circ \chi)^{-1}(Q')$ *has codimension at least 2 in* $\hat{X}$;

(ii) *for every point in Q' there is a Zariski neighborhood $V \subset Q'$ such that $(\varrho \circ \chi)^{-1}(V)$ is naturally isomorphic to $\hat{V} \times \mathbb{C}$ where $\hat{V} = \tau^{-1}(V)$.*

Proof. Recall that $\varrho^{-1}(Q \setminus Q') = E \setminus (\varrho|_E)^{-1}(D^*)$ is of codimension at least 2 in X. Hence (i) is a consequence of finiteness of χ.

By Remark 7.9 Φ is a locally trivial over Q'. In particular, for every point in Q' there is a Zariski neighborhood $V \subset Q'$ such that $\varrho^{-1}(V)$ is naturally isomorphic to $V \times \mathbb{C}$ where the projection $\varrho^{-1}(V) \to V$ is the restriction of ϱ. Note that the restriction of the commutative diagram in Notation 9.5 yields another commutative diagram

$$\begin{tikzcd} \hat{W} \arrow[r] \arrow[d] & \hat{V} \arrow[d] \\ \varrho^{-1}(V) \arrow[r] & V \end{tikzcd}$$

where $\hat{W} = (\varrho \circ \chi)^{-1}(V)$. Hence we have $\hat{W} \simeq \hat{V} \times \mathbb{C}$ since $\varrho^{-1}(V) \simeq V \times \mathbb{C}$. This yields (ii). $\square$

Proposition 9.8. *Let $\hat{S}$ be the closure of $\tilde{S}$ in $\hat{X}$. Then $\hat{S}$ is a section of the quotient morphism $\hat{\varrho} : \hat{X} \to \hat{Q}$.*

Proof. If suffices to construct a $\hat{\Phi}_k$-equivariant morphism $\psi : \hat{X} \to \hat{S}$ from $\hat{X}$ to $\hat{S}$. Indeed, then by the universal properties of quotient morphisms this morphism factors through $\hat{Q}$ and, therefore, $\hat{\varrho}|_{\hat{S}} : \hat{S} \to \hat{Q}$ is invertible.

Choose as the restriction of such a ψ to $\tilde{X}$ the natural projection $\lambda : \tilde{X} = S \times \mathbb{C} \to S$. Note that for $\hat{V} \simeq V \times \mathbb{C}$ from Lemma 9.7(ii) λ agrees on $\tilde{X} \cap \hat{V}$ with the natural projection $\hat{V} \to V$. Hence we can extend ψ to $\tilde{X} \cup \chi^{-1}(Q')$. Now because of Lemma 9.7(i) and the Hartogs theorem this restriction extends to $\hat{X}$ and we are done. $\square$

Remark 9.9. The fact that the morphism $\alpha : E \to R$ has a section may be made weaker for Proposition 9.8. It suffices, say, to require that there exists is a reduced irreducible principal effective divisor $T = h^*(0)$ in E such that the restriction $\alpha|_T : T \to R$ is surjective and quasifinite. Then one can extend h to a regular function on X and the same proofs as before imply that the result remains valid.

Corollary 9.10. *The morphism $\beta : R \to D$ is quasifinite.*

Proof. Let $\tilde{R}$ be the preimage of $R \subset S$ in $\tilde{S}$. Note that $\tilde{R}$ is a subvariety of $\tilde{X} \subset \hat{X}$. On the other hand since $\hat{Q}$ is isomorphic to $\hat{S} \supset \tilde{S}$ by Proposition 9.8, we can treat $\tilde{R}$ as a subvariety of $\hat{Q}$. Furthermore, by construction the quotient morphism $\hat{\varrho}$ yields an automorphism of $\tilde{R}$. Hence the restriction of the morphism $\tau \circ \hat{\varrho} = \varrho \circ \chi$ to $\tilde{R} \subset \hat{X}$ yields a quasifinite morphism $\tilde{R} \to D$ (where τ, χ, $\hat{\varrho}$ are from Proposition 9.6). Since morphism $\varrho \circ \chi|_{\tilde{R}}$ factors through $\beta = \varrho|_R : R \to D$, the latter is also quasifinite and we are done. $\square$

Proof of Theorem 0.2. Note that Convention 6.3 is valid by Remark 6.4. Hence Theorem 6.9 and thus Proposition 7.5 is applicable. For the time being consider the case when B is a germ of a smooth curve at a point o and denote by E the fiber in X over o. Since $\Phi|_E$ is a translation in Theorem 0.2 (3), for every $r \in R$ the preimage $\alpha^{-1}(r)$ is a curve. Thus, taking into consideration Corollary 9.10, we see that the assumptions of Proposition 7.8 are valid. This implies that the polynomial curve Z_{n-1} is smooth and we have $Z_{n-1} \simeq \mathbb{C}$ and $D_n \simeq \mathbb{C}^2$. Hence the polynomial curve Z_{n-1} is smooth by Proposition 7.8 and we have $Z_{n-1} \simeq \mathbb{C}$ and $D_n \simeq \mathbb{C}^2$. By assumption D_n is also a smooth reduced fiber of the morphism $Q_n \to B$ and $Q_n \setminus D_n \simeq B^* \times \mathbb{C}^2$. Sathaye's theorem [1983] implies now that Q_n is naturally isomorphic to $B \times \mathbb{C}^2$. Similarly $X = Q_n \times \mathbb{C}$ when B is a germ of a curve.

Let us return now to the general case when B is a smooth affine curve. The local statement over germ of B at o implies the first global claim of Theorem 0.2 (3) while the second one is the consequence of the fact (see [Serre 1955]) that every affine manifold with a free G_a-action and an affine geometrical quotient is the direct product of that quotient and $\mathbb{C}$. Thus Statement (3) of Theorem 0.2 is valid while Statements (1) and (2) are true by Corollaries 6.10 and 6.11. Hence we are done with the proof of Theorem 0.2. □

General case of an affine algebraic variety X over a field* k *of characteristic zero. Recall that there is a one-to-one correspondence between the set of G_a-actions on an affine algebraic variety X and the set of locally nilpotent derivations (LNDs) on the algebra $\mathcal{B}$ of regular functions on X (e.g., see [Freudenburg 2006]). Suppose that ν is such an LND associated with a proper G_a-action Φ on X. Then condition that Φ is free (resp. a translation) is equivalent to the fact that the ideal generated by $\nu(\mathcal{B})$ coincides with $\mathcal{B}$ (resp. $\nu(\mathcal{B}) = \mathcal{B}$). To show the validity of Theorem 0.1 for any $\mathbf{k}$ of characteristic zero one needs to repeat the argument of Daigle from [Daigle and Kaliman 2009, Theorem 3.2].

Namely, if a fact is true for varieties over $\mathbb{C}$ it is true in the case when X is considered over a field which is a universal domain[13] [Eklof 1973]. Thus consider a field extension $\mathbf{k}'/\mathbf{k}$ where $\mathbf{k}'$ is a universal domain. Then ν extends to an LND ν' on $\mathcal{B}' = \mathbf{k}' \otimes_{\mathbf{k}} \mathcal{B}$ associated with a G_a-action Φ'. Note that ν' is free since ν is. Similarly, Φ' is proper, since properness survives base extension [Hartshorne 1977, Corollary 4.8]. Thus under the assumptions of Theorem 0.1 (with $\mathbb{C}$ replaced by $\mathbf{k}'$) Φ' is a translation by the argument above, i.e., $\nu' : \mathcal{B}' \to \mathcal{B}'$ is surjective. Since ν' is obtained by applying the functor $\mathbf{k}' \otimes_{\mathbf{k}} -$ to ν and since $\mathbf{k}'$ is a faithfully flat k-module we see that $\nu : \mathcal{B} \to \mathcal{B}$ is also surjective. Therefore Φ is a translation and we have:

Theorem 9.11. *Let* $\mathbf{k}$ *be a field of characteristic zero and* Φ *be a proper* G_a*-action on the four-space* $\mathbb{A}^4_{\mathbf{k}}$ *preserving a coordinate. Then* Φ *is a translation in a suitable polynomial coordinate system.*

Similarly the argument before implies the following.

Theorem 9.12. *Let the assumptions of Theorem 0.2 hold with the only change that* X *and* B *are varieties over some field* $\mathbf{k}$ *of characteristic zero* (*and not over* $\mathbb{C}$). *Suppose also that* Φ *is proper and the restriction of* Φ *to every fiber is a translation* (*as in Theorem 0.2* (3)). *Then* Φ *is a translation.*

Conclusive Remark. In dimension 5 the analogue of Theorem 0.1 is not true by the second of Winkelmann's examples in [Winkelmann 1990]. This raises the question of whether the properness is the right condition for this type of problems. The author suspects that another condition may be more effective at least in dimension 4, which leads to the following question:

[13] Recall that a universal domain is an algebraically closed field containing $\mathbb{Q}$ such that it has an infinite transcendence degree over $\mathbb{Q}$.

Question. Suppose that the quotient of a free G_a-action Φ on $\mathbb{C}^4$ is isomorphic to $\mathbb{C}^3$. Is it true that Φ is a translation?

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