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Nonemptiness of Newton strata of Shimura varieties of Hodge type

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For a Shimura variety of Hodge type with hyperspecial level at a prime p, the Newton stratification on its special fiber at p is a stratification defined in terms of the isomorphism class of the rational Dieudonné module of parameterized abelian varieties endowed with a certain fixed set of Frobenius-invariant crystalline tensors ("$G_{\mathbb{Q}_p}$-isocrystal"). There has been a conjectural group-theoretic description of the F-isocrystals that are expected to show up in the special fiber. We confirm this conjecture. More precisely, for any $G_{\mathbb{Q}_p}$-isocrystal that is expected to appear (in a precise sense), we construct a special point whose reduction has associated F-isocrystal equal to the given one.

1. Introduction

Fix a prime $p > 0$, and for $g \geq 1$, let $\mathcal{A}_g$ be the moduli space of principally polarized abelian varieties of dimension g in characteristic p. Then the Newton stratification on $\mathcal{A}_g$ is the stratification such that each stratum consists of points $x = (A_x, \lambda_x)$ whose associated p-divisible group $(A_x[p^\infty], \lambda_x[p^\infty])$ with quasipolarization is quasi-isogenous to a fixed one. The rational Dieudonné module $\mathbb{D}(X)$ of a p-divisible group X over a perfect field k of characteristic $p > 0$ provides an equivalence of categories between the isogeny category of (quasipolarized) p-divisible groups and the category of (quasipolarized) F-isocrystals. An *F-isocrystal* (or simply *isocrystal*) over a perfect field k of characteristic p is a finite-dimensional $L(k)$-vector space M with a σ-linear bijective operator $\Phi : M \to M$, where $L(k) := \mathrm{Frac}(W(k))$ and σ is its Frobenius automorphism.

According to the Dieudonné–Manin classification, an isocrystal over an algebraically closed field is determined by the slope sequence of Φ. The latter combinatorial datum in turn can be faithfully represented by a lower-convex (piecewise-linear) polygon with integral break points lying in the first quadrant of $\mathbb{Z}^2$, which will be called the *Newton polygon* hereafter. In other words, the Newton polygon of $\mathbb{D}(A_x[p^\infty])$ determines the quasi-isogeny class of $A_x[p^\infty]$. The existence of a quasipolarization forces the Newton polygon of $\mathbb{D}(A_x[p^\infty])$ to be "symmetric". Then, it is a natural question to ask which symmetric Newton polygon can be the Newton polygon of a point of $\mathcal{A}_g$. In fact, the Newton polygons arising from points of $\mathcal{A}_g$ in the way just described meet two more restrictions. First, when the initial point is located at the origin, the end point is always $(2g, g) \in \mathbb{Z}^2$. Secondly, it "lies above" the *ordinary* Newton polygon, which by definition is the Newton polygon having two slopes $(0, 1)$ with the same multiplicity g. Then, it

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has long been known (after it was conjectured by Manin) that *any* Newton polygon subject to these two restraints is the Newton polygon of a point on $\mathcal{A}_g$; a proof can be found, e.g., in [Oort 2013, Corollary 7.8], which uses the Honda–Tate theory, and thus gives examples defined over finite fields.

The main result of this article is a generalization of this fact to more general moduli spaces of abelian varieties, namely to Shimura varieties of Hodge type. First, we explain how each point in characteristic $p > 0$ of a Shimura variety of Hodge type gives an F-isocrystal with certain additional structure. A detailed discussion will appear in Section 2.

Let (G, X) be a Shimura datum of Hodge type. This means that there exists an embedding of Shimura data $(G, X) \hookrightarrow (\mathrm{GSp}(W, \psi), \mathfrak{H}_g^{\pm})$ into a Siegel Shimura datum, in the sense that there exists an embedding $\rho_W : G \hookrightarrow \mathrm{GSp}(W, \psi)$ of $\mathbb{Q}$-groups which sends each morphism in X to a member of $\mathfrak{H}_g^{\pm}$; we fix such an embedding. For a compact open subgroup $K \subset G(\mathbb{A}_f)$ (which will be tacitly assumed sufficiently small), the associated (canonical model over the reflex field $E(G, X)$ of the) Shimura variety $\mathrm{Sh}_K(G, X)$ parameterizes polarized abelian varieties endowed with a fixed set of (absolute) Hodge cycles and a K-level structure, and carries a universal family of abelian varieties $A \to S_K := \mathrm{Sh}_K(G, X)$ equipped with a similar set of (absolute Hodge) tensors on $H^1_{\mathrm{dR}}(A/S_K)$. To study its reduction modulo p, we fix a prime $\wp$ of $E(G, X)$ and let $\mathcal{O} := \mathcal{O}_{(\wp)}$ be the localization at $\wp$ of the ring of integers of $E(G, X)$ with residue field $\kappa(\wp)$. To obtain an integral model over $\mathcal{O}$ with good reduction, we assume that $G_{\mathbb{Q}_p}$ is unramified, which is equivalent to the existence of a reductive group scheme $G_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$ with generic fiber $G_{\mathbb{Q}_p}$. We choose one such model $G_{\mathbb{Z}_p}$ and set $K_p := G_{\mathbb{Z}_p}(\mathbb{Z}_p)$ (such compact open subgroups of $G(\mathbb{Q}_p)$ are called *hyperspecial*). For K, we take $K = K_p \times K^p$ for a (sufficiently small) compact open subgroup $K^p \subset G(\mathbb{A}_f^p)$. Then, by [Vasiu 1999; Kisin 2010], it is known that there exists a smooth integral model $\mathcal{S}_K := \mathcal{S}_K(G, X)$ over $\mathcal{O}$ with generic fiber S_K, which is furthermore uniquely characterized by a "Neron-extension property". Also, this integral model carries a universal abelian scheme $\mathcal{A}$ over it by construction. Then each point on the reduction $\mathcal{S}_K \otimes_{\mathcal{O}} \kappa(\wp)$ gives an F-isocrystal as follows.

Let z be a point of $\mathcal{S}_K \otimes_{\mathcal{O}} \kappa(\wp)$ defined over an algebraically closed field k, assumed (for simplicity) to be of finite transcendence degree over $\overline{\mathbb{F}}_p$. By smoothness of $\mathcal{S}_K$ and the moduli interpretation of S_K, the Dieudonné module $\mathbb{D}(\mathcal{A}_z)$ is supplied with a set of Frobenius-invariant tensors $\{s_{\alpha,0,z}\}_{\alpha\in\mathcal{J}}$, and there exists an $L(k)$-isomorphism between the dual space $W^\vee \otimes L(k)$ and $\mathbb{D}(\mathcal{A}_z)$ which matches the G-invariant tensors on $W^\vee$ and the tensors $\{s_{\alpha,0,z}\}_{\alpha\in\mathcal{J}}$ on $\mathbb{D}(\mathcal{A}_z)$, which is thus canonically determined up to the action of $G_{L(k)}$ on $W^\vee_{L(k)}$ (Lemma 2.2.5). Choosing such an isomorphism and transporting the Frobenius operator Φ to $W^\vee_{L(k)}$, we get an element $b \in G(L(k))$ such that $\Phi = \rho_{W^\vee}(b)(\mathrm{id}_{W^\vee} \otimes \sigma)$, where $\rho_{W^\vee} : G \hookrightarrow \mathrm{GL}(W^\vee)$ is the contragredient representation of the symplectic representation ρ_W fixed in the beginning. Recall that two elements g_1, g_2 of $G(L(k))$ are called *σ-conjugate* if there exists $h \in G(L(k))$ with $g_2 = hg_1\sigma(h)^{-1}$. Then the σ-conjugacy class $[b]$ of b is independent of the choice of an isomorphism $W^\vee_{L(k)} \to \mathbb{D}(A_x)$ just explained, and the isomorphism class of the F-isocrystal $(\mathbb{D}(\mathcal{A}_z), \Phi)$ equipped with the Frobenius-invariant tensors $\{s_{\alpha,0,z}\}_{\alpha\in\mathcal{J}}$, called *$G_{\mathbb{Q}_p}$-isocrystal*, corresponds to a unique σ-conjugacy class of elements of $G(L(k))$. If $B(G_{\mathbb{Q}_p})$ denotes the set of the σ-conjugacy classes of elements of $G(L(k))$, there exists a subset $B(G_{\mathbb{Q}_p}, X)$ of $B(G_{\mathbb{Q}_p})$ which is expected to be the set of isocrystals with

$G_{\mathbb{Q}_p}$-structure coming from points on $\mathcal{S}_K \otimes_{\mathcal{O}} \kappa(\wp)$. We remark that this is a subset of $B(G_{\mathbb{Q}_p})$ defined by certain two conditions that are analogous to those discussed above in the Siegel case, and it turns out that for any point of $\mathcal{S}_K \otimes_{\mathcal{O}} \kappa(\wp)$, its associated $G_{\mathbb{Q}_p}$-isocrystal belongs to this subset $B(G_{\mathbb{Q}_p}, X)$.

For $[b] \in B(G_{\mathbb{Q}_p}, X)$, let $\mathcal{S}_{[b]}$ be the subset of points of $\mathcal{S}_K \otimes_{\mathcal{O}} \kappa(\wp)$ whose associated $G_{\mathbb{Q}_p}$-isocrystal is $[b]$. The main result of this paper is then that for every $[b] \in B(G_{\mathbb{Q}_p}, X)$, $\mathcal{S}_{[b]}$ is nonempty. Our proof consists of finding a special point whose reduction has F-isocrystal equal to a given one. We recall that a point $[h, g_f \cdot K] \in \mathrm{Sh}_K(G, X)(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$ is called *special* if $h \in X$ factors through $T_{\mathbb{R}}$ for some maximal $\mathbb{Q}$-torus T of G. Special points are known to be defined over $\overline{\mathbb{Q}} \subset \mathbb{C}$ and extend over the valuation ring of an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ for any reasonable integral model.

Theorem 1.0.1 (Theorem 3.2.1). *Let (G, X) be a Shimura datum and p a rational prime. Assume that $G_{\mathbb{Q}_p}$ is unramified. Fix a hyperspecial subgroup K_p of $G(\mathbb{Q}_p)$ and a sufficiently small compact open subgroup $K^p \subset G(\mathbb{A}_f^p)$; put $K = K_p \times K^p$. Choose a prime $\wp$ of $E(G, X)$ above p, and an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ inducing $\wp$.*

(1) *For every $[b] \in B(G_{\mathbb{Q}_p}, X)$, there exists a special Shimura subdatum $(T, h \in \mathrm{Hom}(\mathbb{S}, T_{\mathbb{R}}) \cap X)$ such that $T_{\mathbb{Q}_p}$ is unramified and the σ-conjugacy class of $\mu_h^{-1}(p) \in G(\mathbb{Q}_p^{\mathrm{ur}})$ equals $[b]$, and further, such that the (unique) hyperspecial subgroup of $T(\mathbb{Q}_p)$ is contained in K_p.*

(2) *If (G, X) is of Hodge type, for every special point $[h, g_f]_K \in \mathrm{Sh}_K(G, X)(\overline{\mathbb{Q}})$ (defined for any $g_f \in G(\mathbb{A}_f)$), its reduction has F-isocrystal equal to $[b]$.*

The statement (1) here is in fact a combination of two statements (1) and (3) of Theorem 3.2.1.

The nonemptiness of Newton strata has been conjectured by Fargues [2004, Conjecture 3.1.1] and Rapoport [2005, Conjecture 7.1]. Some partial cases confirming this conjecture were known. We only mention previous works in two directions. For PEL-type Shimura varieties (which form a subclass of Hodge-type Shimura varieties), this conjecture (i.e., nonemptiness of *all* Newton strata in $B(G_{\mathbb{Q}_p}, X)$) was proved by C.-F. Yu [2005] in the Lie-type C cases, then by Viehmann and Wedhorn [2013, Theorem 1.6] in general, and by Kret [2012] for some simple groups of Lie type A or C. For a general Hodge-type Shimura variety, to the best of the author's knowledge, there are two partial results. First, Wortmann [2013] showed nonemptiness of the *μ-ordinary locus*, a generalization of the (usual) ordinary locus in $\mathcal{A}_g$. For *projective* Shimura varieties of Hodge type, Koskivirta [2016] proved the conjecture (nonemptiness of *all* Newton strata).

While all these previous results on the nonemptiness of Newton strata, except for that of Kret, are obtained by algebro-geometric methods in positive characteristic, which are sometimes of limited applicability for Hodge-type Shimura varieties, we point out that the first claim of our theorem above is a purely group-theoretic statement on Shimura *data* without any reference to the geometry of the Shimura variety. When the Shimura datum is of Hodge type, this group-theoretic statement carries the geometric meaning of the second claim (by Lemma 3.1.1). In this geometric interpretation of the group-theoretic statement of the first claim, one does not need a fine geometric property of the integral canonical model $\mathcal{S}_K$.

We also point out that our method of proof of the theorem, more precisely that of constructing a special point with certain prescribed properties, allows us to obtain some finer results. First, as is already apparent, we prove more than just nonemptiness of Newton strata; namely, we show existence of a special point whose reduction lies in any given Newton stratum. As another example, we also prove a generalization of a result of Wedhorn [1999, (1.6.3)] (the first statement of the next result in the PEL-type cases of Lie type A or C), using an argument different from the original one of Wedhorn (see Remark 3.2.5).

Corollary 1.0.2 (Corollary 3.2.3). *Let (G, X) be a Shimura datum of Hodge type. Suppose that $G_{\mathbb{Q}_p}$ is unramified and choose a hyperspecial subgroup K_p of $G(\mathbb{Q}_p)$. Let $\wp$ be a prime of $E(G, X)$ above p. Then the reduction $\mathcal{S}_{K_p}(G, X) \times \kappa(\wp)$ has nonempty ordinary locus if and only if $\wp$ has absolute height one (that is, $E(G, X)_\wp = \mathbb{Q}_p$). In this case, if the chosen embedding $G \hookrightarrow \mathrm{GSp}(W, \psi)$ induces an embedding $G_{\mathbb{Z}_p} \hookrightarrow \mathrm{GL}(W_{\mathbb{Z}_p})$ of $\mathbb{Z}_p$-group schemes, there exists a special point which is the Serre–Tate canonical lifting of its reduction.*

We remark (see 2.2.1) that the constructions of the integral canonical model by Vasiu and Kisin both require this condition on the embedding $G \hookrightarrow \mathrm{GSp}(W, \psi)$ to be satisfied. Finally, we remark that since our methods of proof of the above results are purely group-theoretic (based on Galois cohomology theory of algebraic groups), they are very likely to apply to more general Shimura varieties, particularly to Shimura varieties of abelian-type and/or with nonhyperspecial level. We do not pursue such ideas in this article, though.

This article is organized as follows. In Section 2, we review the construction of the canonical integral model $\mathcal{S}_K(G, X)$ of a Hodge-type Shimura variety $\mathrm{Sh}_K(G, X)$ with hyperspecial level at p, due to Vasiu and Kisin, and of the Newton stratification on its (good) reduction $\mathcal{S}_K(G, X) \otimes \kappa(\wp)$. We also recall the definitions of Newton map and Kottwitz map for reductive groups over p-adic fields, which are needed to define Newton stratification. Section 3 is devoted to the proof of the nonemptiness of Newton strata, by constructing a special point with prescribed F-isocrystal. For that, we generalize (in Hodge-type situation) a result of Kottwitz which identifies the $G_{\mathbb{Q}_p}$-isocrystal of the reduction of a special point in terms of the defining special Shimura datum. As a corollary of our method, we also obtain a criterion for when the good reduction of a Shimura variety with hyperspecial level has an ordinary point, and prove that in such cases, there always exists a special point which is the Serre–Tate canonical lifting of its reduction.

Notation. Throughout this paper, $\overline{\mathbb{Q}}$ denotes the algebraic closure of $\mathbb{Q}$ inside $\mathbb{C}$ (so $\overline{\mathbb{Q}}$ has a privileged embedding into $\mathbb{C}$). For a (connected) reductive group G over a field, we let G^{sc} be the universal covering of its derived group G^{der}, and for a (general linear algebraic) group G, $Z(G)$ and G^{ad} denote its center and the adjoint group $G/Z(G)$, respectively.

2. Newton stratification on good reduction of Shimura varieties

In this section, we give an account of the construction of Newton stratification on the reduction of the integral canonical model of a Hodge-type Shimura variety with hyperspecial level.

First, we begin with a brief review of the Newton map and Kottwitz map, which are defined for connected reductive groups over p-adic fields. For a reductive group G over a local field F and a conjugacy class $\mathcal{C}$ of cocharacters of $G_{\overline{F}}$, we define a certain subset $B(G, \mathcal{C})$ of $B(G)$, the set of all G-isocrystals.

2.1. *G-isocrystals and the set B(G, C).* In this subsection only, all algebraic groups considered will be defined over p-adic fields.

Let k be an algebraically closed field of characteristic p, and let $W(k)$ be its Witt vector ring and $K_0 = L(k)$ the fraction field of $W(k)$. Fix an algebraic closure $\overline{K_0}$ of K_0. Let F be a finite extension of $\mathbb{Q}_p$ in $\overline{K_0}$, and denote by L the composite of K_0 and F in $\overline{K_0}$; for our application to Shimura varieties, the most interesting case will be when $k = \overline{\mathbb{F}}_p$ and $F = \mathbb{Q}_p$, so that $L = L(\overline{\mathbb{F}}_p)$. Denote by σ the Frobenius automorphism of L/F (i.e., the automorphism of L fixing F and inducing the q-th power map on the residue field k of L, where $q \in \mathbb{N}$ is the cardinality of the residue field of F). Let G be a connected reductive group over F. Put $\mathrm{Gal}_F := \mathrm{Gal}(\overline{F}/F)$.

2.1.1. Let $B(G)$ be the set of σ-conjugacy classes in $G(L)$:

$$B(G) = G(L)/\sim,$$

where two elements x, y of $G(L)$ are σ-conjugated, denoted by $x \sim y$, if $x = gy\sigma(g)^{-1}$ for some $g \in G(L)$; the σ-conjugacy class of $b \in G(L)$ is denoted by $[b]$. An element of $B(G)$ is called an *(F-)isocrystal with G-structure*.

To relate this notion to the previously introduced notion of F-isocrystal (i.e., a finite-dimensional vector space U over L endowed with a σ-linear bijection $\Phi : U \to U$), let $\mathcal{REP}_F(G)$ be the category of finite-dimensional F-linear representations of G, and $\mathcal{CRYS}$ the category of F-isocrystals. Both are in a natural manner F-linear Tannakian categories (for $\mathcal{CRYS}$, see [Kottwitz 1985, §3]). An element $[b]$ of $B(G)$ can equivalently be considered as an F-linear exact (faithful) tensor functor

$$\mathcal{REP}_F(G) \to \mathcal{CRYS} : (\rho, V) \mapsto (V \otimes L, \rho(b) \cdot (\mathrm{id}_V \otimes \sigma)),$$

for any representative $b \in G(L)$ of $[b] \in B(G)$; see [Rapoport and Richartz 1996, Remark 3.4(i)]. If $[b_1] = [b_2]$, there exists a natural transformation between the corresponding tensor functors. Then any F-isocrystal (U, Φ) of height n (i.e., $\dim_L U = n$) gives rise to an F-isocrystal with GL_n-structure [Rapoport and Richartz 1996, Remarks 3.4(ii)]. So, $B(\mathrm{GL}_n)$ classifies the isomorphism classes of F-isocrystals (U, Φ) of height n.

2.1.2. Let $\mathbb{D}$ be the pro-algebraic torus over $\mathbb{Q}_p$ with character group $\mathbb{Q}$. We define $\mathcal{N}(G)$ to be the set of σ-invariants in the set of conjugacy classes of homomorphisms $\mathbb{D}_L \to G_L$:

$$\mathcal{N}(G) = (\mathrm{Int}\, G(L) \setminus \mathrm{Hom}_L(\mathbb{D}, G))^{\langle\sigma\rangle},$$

where $\langle\sigma\rangle$ denotes the infinite cyclic group generated by σ (and is endowed with the discrete topology). For example, when $G = T$ is a torus, we have $\mathcal{N}(T) = X_*(T)_{\mathbb{Q}}^{\mathrm{Gal}_F}$.

If $T \subset G$ is a maximal F-torus with (absolute) Weyl group Ω, there is a natural identification

$$\mathcal{N}(G) = (X_*(T)_{\mathbb{Q}}/\Omega)^{\mathrm{Gal}_F},$$

where the Galois action on $X_*(T)_{\mathbb{Q}}/\Omega$ is the canonical action.

Kottwitz [1985, §4] (cf. [Rapoport and Richartz 1996, Theorem 1.8]) constructs a map

$$\nu = \nu_G : G(L) \to \mathrm{Hom}_L(\mathbb{D}, G)$$

which is functorial in G. We call the induced functorial map $\bar{\nu}$, defined on the category of connected reductive groups over F, the *Newton map*:

$$\bar{\nu} : B(\,\cdot\,) \to \mathcal{N}(\,\cdot\,); \qquad \bar{\nu}_G([b]) = \overline{\nu_G(b)}, \quad b \in [b].$$

Here, $b \in G(L)$ is a representative of $[b]$ and $\overline{\nu_G(b)}$ is the conjugacy class of $\nu_G(b)$.

When $G = \mathrm{GL}_n$, the Newton map sends an F-isocrystal (U, Φ) of height n to its Newton polygon, which is represented in $\mathcal{N}(\mathrm{GL}_n)$ by the corresponding slope homomorphism; see Example 1.10 of [Rapoport and Richartz 1996].

2.1.3. Let $\pi_1(G) = \pi_1(G, T) := X_*(T)/\sum_{\alpha \in R^*} \mathbb{Z}\alpha^\vee$ be the algebraic fundamental group à la Borovoi, where T is a maximal torus of $G_{\bar{F}}$ and $R^* \subset X^*(T)$ is the set of roots of (G, T). This is a Gal_F-module in a natural manner, and (as the notation suggests) is canonically attached to G only. The functor $G \mapsto \pi_1(G)$ is an exact functor from the category of connected reductive groups over F to the category of finitely generated discrete Gal_F-modules; see [Rapoport and Richartz 1996, 1.13]. Kottwitz [1990, §6] constructs a natural transformation

$$\kappa : B(\,\cdot\,) \to \pi_1(\,\cdot\,)_{\mathrm{Gal}_F} \tag{2.1.3.1}$$

of set-valued functors on the category of connected reductive groups over F, where $\pi_1(\,\cdot\,)_{\mathrm{Gal}_F}$ is the group of coinvariants for the canonical Galois action [Rapoport and Richartz 1996, 1.13]. For tori, for which $B(\,\cdot\,)$ has an obvious abelian group structure, this map is an isomorphism [Kottwitz 1985, (5.5.1)] (cf. [Rapoport and Richartz 1996, Theorem 1.15]).

2.1.4. In the following discussion, suppose given a $G(\bar{F})$-conjugacy class $\mathcal{C}$ of cocharacters into $G_{\bar{F}}$. We define a finite subset $B(G, \mathcal{C})$ of $B(G)$, following Kottwitz [1997, §6] (cf. [Rapoport 2005, §4]).

Choose a Borel pair (T, B) (i.e., T is a maximal $\bar{F}$-torus and B is a Borel subgroup of $G_{\bar{F}}$ containing T). Let $\mathcal{BR}(G) = (X^*(T), R^*, X_*(T), R_*, \Delta)$ be the associated based root datum and $\bar{C} \subset X_*(T)_{\mathbb{R}}$ the closed Weyl chamber associated with the root base $\Delta \subset R^*$.

Denoting by $\mu_{\bar{C}} = \mu_{\bar{C}}(G, \mathcal{C})$ the (unique) representative in $\bar{C}$ of $\mathcal{C}$, we set

$$\bar{\mu}(G, \mathcal{C}) := |\mathrm{Gal}_F \cdot \mu_{\bar{C}}|^{-1} \sum_{\mu' \in \mathrm{Gal}_F \cdot \mu_{\bar{C}}} \mu' \in \bar{C}.$$

Here, the action of Gal_F on $\bar{C}$ is the canonical action. The $G(\bar{F})$-conjugacy class of cocharacters containing $\bar{\mu}(G, \mathcal{C})$ depends only on the pair $(G, \mathcal{C})$ (i.e., is independent of the choice of a Borel pair (T, B)).

Let $\mu^\natural = \mu^\natural(\mathcal{C})$ denote the image of $\mu \in X_*(T)$ under the natural map

$$X_*(T) \to \pi_1(G)_{\mathrm{Gal}_F} = \Big(X_*(T) \,\Big/ \sum_{\alpha \in R^*} \mathbb{Z}\alpha^\vee\Big)_{\mathrm{Gal}_F}$$

for any cocharacter μ of $T_{\overline{F}}$ lying in the conjugacy class $\mathcal{C}$; $\mu^\natural$ depends only on the conjugacy class $\mathcal{C}$, and neither on the choice of a representative μ in $T_{\overline{F}}$ nor on that of T.

For a connected reductive group G over F and a $G(\overline{F})$-conjugacy class $\mathcal{C}$ of cocharacters into $G_{\overline{F}}$, we define

$$B(G,\mathcal{C}) := \{[b] \in B(G) \mid \kappa_G([b]) = \mu^\natural,\ \bar{\nu}_G([b]) \preceq \bar{\mu}(G,\mathcal{C})\},$$

where $\preceq$ is the natural partial order on the closed Weyl chamber $\overline{C}$ defined so that $\nu \preceq \nu'$ if $\nu' - \nu$ is a nonnegative linear combination with *real* coefficients of simple coroots in $X_*(T)_{\mathbb{R}}$ [Rapoport and Richartz 1996, Lemma 2.2]. Also, here we use the canonical identification of $X_*(T)_{\mathbb{Q}}/\Omega$ as a subset of $\overline{C}$.

One knows by [Kottwitz 1997, 4.13] that the map $(\bar{\nu}, \kappa) : B(G) \to \mathcal{N}(G) \times \pi_1(G)_{\mathrm{Gal}_F}$ is injective; hence $B(G,\mathcal{C})$ can be identified with a subset of $\mathcal{N}(G)$.

2.2. ***Integral canonical model and Newton stratification.*** Let (G, X) be a Shimura datum of Hodge type. We assume given an embedding $\rho : (G, X) \to (\mathrm{GSp}(W, \psi), \mathfrak{H}^{\pm})$ of Shimura data (i.e., there exists an embedding $\rho_W : G \hookrightarrow \mathrm{GSp}(W, \psi)$ of $\mathbb{Q}$-algebraic groups which sends each morphism in X to a member of $\mathfrak{H}_g^{\pm}$). Consider a compact open subgroup K of $G(\mathbb{A}_f)$ of the form $K = K_p K^p$, where K_p and K^p are compact open subgroups of $G(\mathbb{Q}_p)$ and $G(\mathbb{A}_f^p)$, respectively. Choose a prime $\wp$ of $E(G, X)$ above p and let $\mathcal{O}_{(\wp)}$ be the localization at $\wp$ of the ring of integers $\mathcal{O}_{E(G,X)}$ of $E(G, X)$. We also use $\mathrm{Sh}_K(G, X)$ to denote the canonical model over $E(G, X)$ [Deligne 1979, 2.2].

2.2.1. When K_p is hyperspecial, Vasiu [1999] (cf. [Vasiu 2007a; 2007b]) and Kisin [2010] independently constructed a smooth integral model $\mathcal{S}_K(G, X)$ over $\mathcal{O}_{(\wp)}$ of the canonical model $\mathrm{Sh}_K(G, X)$. The projective limit $\mathcal{S}_{K_p}(G, X)$ of $\mathcal{S}_{K_p K^p}(G, X)$ over varying $\{K^p\}$ is characterized, as a scheme over $\mathcal{O}_{(\wp)}$, uniquely by the extension property: for any regular, formally smooth $\mathcal{O}_{(\wp)}$-scheme S, every $E(G, X)$-morphism $S_{E(G,X)} \to \mathrm{Sh}_{K_p}(G, X)$ extends uniquely over $\mathcal{O}_{(\wp)}$.

Remark 2.2.2. This definition of the extension property is due to Kisin [2010, Theorem 2.3.8]. It differs slightly from that of Vasiu, who uses, as test schemes, healthy regular schemes over $\mathcal{O}_{(\wp)}$ [Vasiu 1999, Section 3.2] instead of regular, formally smooth $\mathcal{O}_{(\wp)}$-schemes. But, when $G_{\mathbb{Q}_p}$ is unramified, every regular, formally smooth scheme over $\mathcal{O}_{(\wp)}$ is also healthy regular over $\mathcal{O}_{(\wp)}$, because $\mathcal{O}_{(\wp)}$ is unramified over $\mathbb{Z}_{(p)}$ [Milne 1994, Corollary 4.7(a)] and every regular, formally smooth scheme is healthy regular (if $p > 2$) over any d.v.r. unramified over $\mathbb{Z}_{(p)}$, according to a lemma of Faltings (see [Moonen 1998, Lemma 3.6]). Consequently, this difference (and related other minor differences) will not matter when the extension property is invoked for such schemes.

In our work, we need a more precise description of the integral canonical model. Let $G_{\mathbb{Z}_p}$ be a reductive group scheme over $\mathbb{Z}_p$ with generic fiber $G_{\mathbb{Q}_p}$ and such that $G_{\mathbb{Z}_p}(\mathbb{Z}_p) = K_p$. By [Kisin 2010, (2.3)], there exists a lattice $W_{\mathbb{Z}}$ of W such that the embedding $\rho_{\mathbb{Q}_p} : G_{\mathbb{Q}_p} \hookrightarrow \mathrm{GL}(W_{\mathbb{Q}_p})$ is induced by a closed embedding

$$G_{\mathbb{Z}_p} \hookrightarrow \mathrm{GL}(W_{\mathbb{Z}_p}),$$

where $W_{\mathbb{Z}_p} := W_{\mathbb{Z}} \otimes \mathbb{Z}_p$. For K^p stabilizing $W_{\widehat{\mathbb{Z}}} = W_{\mathbb{Z}} \otimes \widehat{\mathbb{Z}}$, one can find a compact open subgroup $H = H^p H_p \subset \mathrm{GSp}(W, \psi)(\mathbb{A}_f)$ with $\rho(K) \subset H$ such that H stabilizes $W_{\widehat{\mathbb{Z}}}$ and also that $\rho : G \hookrightarrow \mathrm{GSp}(W, \psi)$ induces an embedding (of weakly canonical models of Shimura varieties)

$$\mathrm{Sh}_K(G, X) \hookrightarrow \mathrm{Sh}_H(\mathrm{GSp}(W_{\mathbb{Z}}, \psi), \mathfrak{H}^{\pm}) \otimes_{\mathbb{Q}} E(G, X).$$

For $d \in \mathbb{N}$ and sufficiently small H, let $\mathcal{A}_{g,d,H}$ denote the Mumford (fine) moduli scheme over $\mathbb{Z}[1/d]$ which parametrizes abelian schemes endowed with a polarization of degree d and a level H-structure. By replacing $W_{\mathbb{Z}}$ by a scalar multiple of it, we may and do assume that ψ is $\mathbb{Z}$-valued on $W_{\mathbb{Z}}$; let $d := [W_{\mathbb{Z}}^* : W_{\mathbb{Z}}]$ for the dual $W_{\mathbb{Z}}^* \subset W$. Then, by taking sufficiently small K^p (so that H^p is also so), we get an embedding of schemes

$$\mathrm{Sh}_H(\mathrm{GSp}(W_{\mathbb{Z}}, \psi), \mathfrak{H}^{\pm}) \hookrightarrow \mathcal{A}_{g,d,H}.$$

By construction, $\mathcal{S}_K(G, X)$ is the normalization of the Zariski closure of the image of the resulting embedding

$$\mathrm{Sh}_K(G, X) \hookrightarrow \mathcal{A}_{g,d,H} \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{(\wp)}. \tag{2.2.2.1}$$

From now on, when we talk about the canonical integral model $\mathcal{S}_{K_p K^p}$ (with hyperspecial K_p), we will tacitly assume that K^p is small enough that the above conditions imposed are achieved.

Let $\mathcal{S}_K \otimes \kappa(\wp)$ be the reduction of the integral canonical model $\mathcal{S}_K = \mathcal{S}_K(G, X)$ at $\wp$. Next, we define the Newton stratification on this scheme, following [Rapoport 2005; Vasiu 2008] (cf. [Wortmann 2013]). We fix an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ which induces the chosen prime $\wp$ on $E(G, X)$; such choice amounts to fixing an embedding $E(G, X)_{\wp} \hookrightarrow \overline{\mathbb{Q}}_p$ together with an embedding ι_p of $E(G, X)$-algebras.

2.2.3. For a vector space V, let $\mathcal{T}(V)$ be the tensor space attached to V:

$$\mathcal{T}(V) = \bigoplus_{(s,t) \in \mathbb{N}^2} V^{\otimes s} \otimes (V^{\vee})^{\otimes t}.$$

An element of $\mathcal{T}(V)$ is called a *tensor on* V. Let $\{s_\alpha\}_{\alpha \in \mathcal{J}}$ be the set of tensors on $W_{\mathbb{Q}}$ fixed under the extended action of $G_{\mathbb{Q}} \hookrightarrow \mathrm{GL}(W)$ on $\mathcal{T}(W)$.

Let $(\pi : A \to \mathrm{Sh}_K, \lambda_A)$ be the pullback to $\mathrm{Sh}_K = \mathrm{Sh}_K(G, X)$ of the universal abelian scheme with a polarization of degree d over $\mathcal{A}_{g,d,H}$; as usual, we assume K^p to be sufficiently small for this to make sense. We have the local system $\mathcal{W} := R^1(\pi^{\mathrm{an}})_* \mathbb{Q}$ of $\mathbb{Q}$-vector spaces and the (analytic) vector bundle $\mathbb{W}_{\mathbb{C}}^{\mathrm{an}} := R^1(\pi^{\mathrm{an}})_* \Omega^\bullet_{A_{\mathbb{C}}/\mathrm{Sh}_K}$ with Gauss–Manin connection (which is an integrable connection with regular singularities). By Deligne's theorem, the latter is the analytification of a unique algebraic vector bundle $\mathbb{W}$ over Sh_K (in this case, the relative de Rham cohomology $H^1_{\mathrm{dR}}(A/\mathrm{Sh}_K) := R^1\pi_* \Omega^\bullet_{A/\mathrm{Sh}_K}$) with integrable connection. Each tensor s_α defines a global section $s_{\alpha,\mathrm{B}}$ of the local system $\mathcal{T}(\mathcal{W})$ and also a global section $s_{\alpha,\mathrm{dR}}$ of the vector bundle $\mathcal{T}(H^1_{\mathrm{dR}}(A/\mathrm{Sh}_K))$ [Kisin 2010, (2.2)].

Now, let $F \supset E(G, X)$ be an extension which can be embedded in $\mathbb{C}$; we fix an embedding $\sigma_\infty : \bar{F} \hookrightarrow \mathbb{C}$ which extends the given embedding $E(G, X) \hookrightarrow \mathbb{C}$. For $x \in \mathrm{Sh}_K(G, X)(F)$, let A_x be the corresponding abelian variety over F, and let $H^m_{\mathrm{dR}}(A_x/F)$ and $H^m_{\text{ét}}(A_x \otimes_F \bar{F}, \mathbb{Q}_p)$ denote the de Rham cohomology of A_x/F and the étale cohomology of $A_x \otimes_F \bar{F}$, respectively. For each tensor $s_\alpha \in \mathcal{T}(W)$, let $s_{\alpha,\mathrm{B},\sigma_\infty(x)}$ in $\mathcal{T}(H^1_{\mathrm{B}}(\sigma_\infty(A_x), \mathbb{Q}))$ denote the fiber of $s_{\alpha,\mathrm{B}}$ at $\sigma_\infty(x) \in \mathrm{Sh}_K(G, X)(\mathbb{C})$. By the moduli interpretation of the complex points of $\mathrm{Sh}_K(G, X)$ [Milne 1994, Proposition 3.9], there is an isomorphism $W^\vee \xrightarrow{\sim} H^1_{\mathrm{B}}(A_{\sigma_\infty(x)}, \mathbb{Q})$, uniquely determined up to action of $G(\mathbb{Q})$, where G acts on $W^\vee$ by the contragredient representation. Moreover, $s_{\alpha,\mathrm{B},\sigma_\infty(x)}$ is the image of s_α by any such isomorphism (in particular, it does not depend on the choice of such isomorphism). For each prime l, let $s_{\alpha,l,\sigma_\infty(x)}$ be the tensor in $\mathcal{T}(H^1_{\text{ét}}(A_x \otimes_F \bar{F}, \mathbb{Q}_l))$ which is the preimage of $s_{\alpha,\mathrm{B},\sigma_\infty(x)}$ under the canonical isomorphism

$$H^1_{\text{ét}}(A_x \otimes_F \bar{F}, \mathbb{Q}_l) \xrightarrow{\sim} H^1_{\text{ét}}(\sigma_\infty(A_x \otimes_F \bar{F}), \mathbb{Q}_l) = H^1_{\mathrm{B}}(\sigma_\infty(A_x), \mathbb{Q}) \otimes \mathbb{Q}_l, \tag{2.2.3.1}$$

where the first isomorphism σ_∞ is the proper base change isomorphism in étale cohomology. The image of $s_{\alpha,\mathrm{B},\sigma_\infty(x)}$ under the comparison isomorphism

$$H^1_{\mathrm{B}}(\sigma_\infty(A_x), \mathbb{Q}) \otimes \mathbb{C} = H^1_{\mathrm{dR}}(A_x \otimes_{F,\sigma_\infty} \mathbb{C}/\mathbb{C})$$

is the stalk $s_{\alpha,\mathrm{dR},\sigma_\infty(x)}$ at $\sigma_\infty(x)$ of the section $s_{\alpha,\mathrm{dR}}$ of $\mathcal{T}(H^1_{\mathrm{dR}}(A/\mathrm{Sh}_K))$. As the algebraic vector bundle $H^1_{\mathrm{dR}}(A/\mathrm{Sh}_K)$ is defined over $E(G, X)$ [Kisin 2010, Corollary 2.2.2], so is $\sigma_\infty(s_{\alpha,\mathrm{dR},x})$, the base change via σ_∞ of the stalk $s_{\alpha,\mathrm{dR},x}$ of $s_{\alpha,\mathrm{dR}}$ at x. Set $H^m_{\mathbb{A}}(A_x) := H^m_{\mathrm{dR}}(A_x/F) \times \prod_l H^m_{\text{ét}}(A_x, \mathbb{Q}_l)$ and $\mathbb{Q}(1) := F \times \mathbb{A}_f(1)$, where $\mathbb{A}_f(1) := (\varprojlim_r \mu_r) \otimes_{\mathbb{Z}} \mathbb{Q}$ (as usual, $\mu_r := \{\zeta \in \bar{F} \mid \zeta^r = 1\}$), and let $\mathcal{T}(H^1_{\mathbb{A}}(A_x), \mathbb{Q}(1))$ be the tensor space generated by these $F \times \mathbb{A}_f$-modules $H^m_{\mathbb{A}}(A_x)$, $\mathbb{Q}(1)$ (and their duals) [Deligne et al. 1982, Chapter I, §1]. Then, the element

$$(s_{\alpha,\mathrm{dR},x}, s_{\alpha,\text{ét},\sigma_\infty(x)} := (s_{\alpha,l,\sigma_\infty(x)})_l)$$

of $\mathcal{T}(H^1_{\mathbb{A}}(A_x), \mathbb{Q}(1))$ is an absolute Hodge cycle on A_x [Deligne et al. 1982, Chapter I, §2].

2.2.4. Now we show that every geometric point $z \in \mathcal{S}_K \otimes \bar{\mathbb{F}}_p(k)$ (with $k = \bar{k}$, $\mathrm{char}\, k = p$) gives rise to a $G_{\mathbb{Q}_p}$-isocrystal over k, i.e., a σ-conjugacy class of elements in $G(L)$. Here, we assume that $L(k)$ can be embedded in $\mathbb{C}$; for example, this is the case if k has finite transcendence degree over $\bar{\mathbb{F}}_p$. The p-divisible group $(A_z[p^\infty], \lambda_{A_z}[p^\infty])$ with quasipolarization gives rise to an F-isocrystal over k equipped with a nondegenerate alternating pairing. More precisely, the crystalline cohomology $M := H^1_{\mathrm{cris}}(A_z/W(k))$ of A_z with quasipolarization λ_z is a quasipolarized F-crystal $(M, \phi, \langle\ ,\ \rangle)$: M is a free $W(k)$-module of rank $2 \dim A_z$, $\phi \in \mathrm{End}_{\mathbb{Z}_p}(M)$ is a σ-linear endomorphism such that $pM \subset \phi(M)$, and $\langle\ ,\ \rangle : M \times M \to W(k)$ is an alternating form with the property that $\langle \phi(v_1), \phi(v_2)\rangle = p\langle v_1, v_2\rangle^\sigma$ for $v_1, v_2 \in M$.

To proceed, we fix an embedding $E(G, X)_\wp \hookrightarrow L(k)$, and thus an identification $E(G, X)_\wp = L(\mathbb{F}_q)$ as well, where $\mathbb{F}_q$ is the residue field of $\mathcal{O}_{(\wp)}$. Then there exists a lift of $z : \mathrm{Spec}(k) \to \mathcal{S}_K \otimes_\kappa k$ to $\mathrm{Spec}(W(k))$, since $\mathcal{S}_K(G, X)$ is smooth over $\mathcal{O}_{(\wp)}$ and $\mathcal{O}_{(\wp)}$ is unramified over $\mathbb{Z}_{(p)}$ [Milne 1994, Corollary 4.7]. We choose one, say $x : \mathrm{Spec}(W(k)) \to \mathcal{S}_K$, and set $(A_x, \lambda_x) := x^*(A, \lambda_A)$.

Lemma 2.2.5. *For each $\alpha \in \mathcal{J}$, let $s_{\alpha,0,x} \in \mathcal{T}(M \otimes_{W(k)} L(k))$ denote the image of $s_{\alpha,\mathrm{dR},x}$ (the stalk at x of $s_{\alpha,\mathrm{dR}}$) under the canonical isomorphism $H^1_{\mathrm{dR}}(A_x/L(k)) = M \otimes_{W(k)} L(k)$.*

(1) *The tensor $s_{\alpha,0,x}$ is a* crystalline cycle, *namely $s_{\alpha,0,z}$ belongs to the F^0-filtration of $\mathcal{T}(M \otimes L(k))$ and is fixed under the Frobenius ϕ, where the filtration on $\mathcal{T}(M \otimes L(k))$ is the one transported from the Hodge filtration on $\mathcal{T}(H^1_{\mathrm{dR}}(A_x/L(k))$ by the canonical isomorphism $M \otimes L(k) = H^1_{\mathrm{dR}}(A_x/L(k))$.*

(2) *The triple*

$$(M \otimes F, \phi, (s_{\alpha,0,x})_{\alpha\in\mathcal{J}})$$

depends only on z, not on the lift x; let us write $s_{\alpha,0,z}$ for $s_{\alpha,0,x}$. There exists an $L(k)$-isomorphism

$$W^\vee \otimes_{\mathbb{Q}} L(k) \xrightarrow{\sim} M \otimes_{W(k)} L(k) \tag{2.2.5.1}$$

which maps s_α to $s_{\alpha,0,z}$ for every $\alpha \in \mathcal{J}$.

Proof. (1) This is well-known. Originally, this statement was proved by Blasius, Ogus, and Wintenberger independently (see [Blasius 1994, §5]) when x was defined over a number field, as a consequence of the fact that the Hodge cycles on an abelian variety over a number field are *de Rham* and the compatibility of the crystalline and de Rham comparison isomorphisms [Blasius 1994, 5.1(5)]. Then Vasiu [2008, Section 8] generalized it to arbitrary fields.

(2) As before, we choose an embedding $\sigma_\infty : \overline{L(k)} \hookrightarrow \mathbb{C}$ of $E(G, X)$-algebras. We have seen that for each $\alpha \in \mathcal{J}$, the image of s_α under the comparison isomorphism

$$W^\vee \otimes \mathbb{C} \xrightarrow{\sim} H^1_{\mathrm{B}}(\sigma_\infty(A_x), \mathbb{Q}) \otimes \mathbb{C} \xrightarrow{\sim} H^1_{\mathrm{dR}}(A_x/L(k)) \otimes_{L(k),\sigma_\infty} \mathbb{C} \tag{2.2.5.2}$$

is $\sigma_\infty(s_{\alpha,\mathrm{dR},x}) = s_{\alpha,\mathrm{dR},x} \otimes 1$ (a section of $\mathcal{T}(H^1_{\mathrm{dR}}(A_{x'}/L(k))) \otimes_{L(k),\sigma_\infty} \mathbb{C}$); recall that the first isomorphism is unique up to the action of $G(\mathbb{Q})$ on $W^\vee$. Next, for any two lifts x, x' over $W(k)$ of z, the composite of the canonical maps

$$H^1_{\mathrm{dR}}(A_x/L(k)) \xrightarrow{\sim} H^1_{\mathrm{cris}}(A_z/W(k)) \otimes L(k) \xrightarrow{\sim} H^1_{\mathrm{dR}}(A_{x'}/L(k))$$

is given by parallel transport with respect to the Gauss–Manin connection [Berthelot and Ogus 1983, Remark 2.9]. As $\sigma_\infty(s_{\alpha,\mathrm{dR},y})$ for $y = x, x'$ are the stalks of a horizontal global section $s_{\alpha,\mathrm{B}}$ of the local system $\mathcal{T}(\mathcal{W}_{\mathbb{C}})$, we see that under the composite map at hand, $\sigma_\infty(s_{\alpha,\mathrm{dR},x})$ must map to $\sigma_\infty(s_{\alpha,\mathrm{dR},x'})$. This proves that the triple $(M \otimes L(k), \phi, (s_{\alpha,0,x})_{\alpha\in\mathcal{J}})$ is independent of the choice of lift x. Let us write $s_{\alpha,0,z}$ for $s_{\alpha,0,x}$ ($\alpha \in \mathcal{J}$).

Now, the functor defined on $L(k)$-algebras

$$R \mapsto \mathrm{Isom}_R((W^\vee_{L(k)} \otimes R, \{s_\alpha \otimes 1\}), (M \otimes R, \{s_{\alpha,0,z} \otimes 1\}))$$

of isomorphisms between $W^\vee \otimes_{\mathbb{Q}} L(k)$ and $M = H^1_{\mathrm{cris}}(A_z/L(k))$ taking s_α to $s_{\alpha,0,z}$ for every $\alpha \in \mathcal{J}$ is represented by a scheme which is nonempty since it has a $\mathbb{C}$-valued point, as was just seen, and thus is a torsor over $L(k)$ under $G_{L(k)}$. Since $H^1(L(k), G_{L(k)}) = \{0\}$ (Steinberg's theorem), this torsor is trivial, namely has an $L(k)$-valued point. □

Therefore, when one chooses an isomorphism as in (2.2.5.1) and transports the σ-linear map ϕ to $W^\vee \otimes L(k)$, we obtain an element $b \in G(L(k))$ with

$$\phi = \rho_{W^\vee}(b)(\mathrm{id}_{W^\vee} \otimes \sigma),$$

where $\rho_{W^\vee} : G \hookrightarrow \mathrm{GL}(W^\vee)$ denotes the contragredient representation: indeed, both ϕ and $\mathrm{id}_{W^\vee} \otimes \sigma$ (thus $\rho_{W^\vee}(b)$ as well) fix each $s_\alpha \otimes 1 \in \mathcal{T}(W^\vee \otimes L(k))$. Although b depends on the choice of an isomorphism (2.2.5.1), its σ-conjugacy class $[b] \in B(G_{\mathbb{Q}_p})$ is independent of such choice. This shows that the F-isocrystal over k attached to z is an $G_{\mathbb{Q}_p}$-isocrystal. For an arbitrary point z of $\mathcal{S}_K(G, X) \otimes \kappa(\wp)$, we define the F-isocrystal attached to z to be the F-isocrystal of the geometric point in $\mathcal{S}_K(G, X)(k)$ induced from z for *any* algebraically closed field k containing the residue field $\kappa(z)$ of z. The resulting F-isocrystal does not depend on the choice of k'; see [Viehmann and Wedhorn 2013, §8; Rapoport and Richartz 1996, Lemma 1.3]. In this way, we obtain a (set-theoretic) map

$$\Theta : \mathcal{S}_K(G, X) \otimes \kappa(\wp) \to B(G_{\mathbb{Q}_p}).$$

By the same argument (applied to $\mathrm{Sh}(T, h)$ instead of $\mathrm{Sh}(G, X)$), we see that if $\sigma_\infty(x)$ is a special point, i.e., $\sigma_\infty(x) = [h, g_f] \in \mathrm{Sh}_K(G, X)(\sigma_\infty(F))$ for some $h \in X$ factoring through a maximal $\mathbb{Q}$-torus T of G, the $G_{\mathbb{Q}_p}$-isocrystal (attached to its reduction $z \in \mathcal{S}_K(G, X)$) has a representative in $T(L)$.

Finally, we remark that although our definition of the F-isocrystal attached to a point of $\mathcal{S}_K(G, X)(k)$ uses *cohomology* spaces, one can equally work with homology spaces, as adopted by other people (such as Viehmann and Wedhorn [2013]). This does not alter the definition of the map Θ.

2.2.6. With every Shimura datum (G, X), there is associated a natural $G(\mathbb{C})$-conjugacy class $c(G, X)$ of cocharacters of $G_{\mathbb{C}}$, namely that containing the Hodge cocharacter $\mu_h^{-1} := h^{-1}|_{\mathbb{G}_m}$, where $\mathbb{G}_m$ refers to the factor of $(\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m)_{\mathbb{C}} = \prod_{\mathrm{Gal}(\mathbb{C}/\mathbb{R})} \mathbb{G}_m$ corresponding to the identity embedding of $\mathbb{C}$. Recall that we fixed an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ inducing the chosen prime $\wp$ on $E(G, X)$ and $\overline{\mathbb{Q}}$ is given as a subfield of $\mathbb{C}$. As is well-known, this choice allows us to consider $c(G, X)$ as a $G(\overline{\mathbb{Q}}_p)$-conjugacy class of cocharacters of $G_{\overline{\mathbb{Q}}_p}$; we continue to denote it by $c(G, X)$.

Put $B(G_{\mathbb{Q}_p}, X) := B(G_{\mathbb{Q}_p}, c(G, X))$ (the subset of $B(G_{\mathbb{Q}_p})$ defined in 2.1.4 for $\mathcal{C} = c(G, X)$) and $\bar{\mu}(G_{\mathbb{Q}_p}, X) := \bar{\mu}(G_{\mathbb{Q}_p}, c(G, X))$.

Proposition 2.2.7. *Let (G, X) be a Shimura datum of Hodge type and $K = K_pK^p \subset G(\mathbb{A}_f)$ a compact open subgroup. Suppose that G is unramified over $\mathbb{Q}_p$ and $K_p \subset G(\mathbb{Q}_p)$ is hyperspecial.*

(1) *The image of $\Theta : \mathcal{S}_K(G, X) \otimes \kappa(\wp) \to B(G_{\mathbb{Q}_p})$ is contained in $B(G_{\mathbb{Q}_p}, X)$.*

(2) *For $[b] \in B(G_{\mathbb{Q}_p})$, the subset*

$$\mathcal{S}_{[b]} := \Theta^{-1}([b])$$

is locally closed inside $\mathcal{S}_K(G, X) \otimes \overline{\mathbb{F}}_p$.

For (1), see, e.g., [Viehmann 2015, Theorem 3.7], and for (2), [Vasiu 2011, 5.3.1].

Endowed with reduced induced subscheme structure, the subvarieties $\mathcal{S}_{[b]}$ of $\mathcal{S}_K(G, X) \otimes \overline{\mathbb{F}}_p$ are called the *Newton strata* of $\mathcal{S}_K(G, X)$.

Rapoport [2005, Conjecture 7.1] conjectures the following.

Conjecture 2.2.8. $$\mathrm{Im}(\Theta) = B(G_{\mathbb{Q}_p}, X).$$

Remark 2.2.9. (1) It is known (see [Rapoport and Richartz 1996; Chai 2000]) that with respect to the partial order $\preceq$ on $B(G_{\mathbb{Q}_p}, X)$, there exist a unique maximal element, called the *μ-ordinary* element, and a unique minimal element, called the *basic* element. The μ-ordinary element is just (the σ-conjugacy class in $B(G_{\mathbb{Q}_p}, X)$ with Newton point) $\bar{\mu}(G_{\mathbb{Q}_p}, X)$; it is clear from the definition that $\bar{\mu}(G_{\mathbb{Q}_p}, X) \in B(G_{\mathbb{Q}_p}, X)$. Previously, nonemptiness is known for these two special strata. For the μ-ordinary locus, this was first proved by Wedhorn [1999] in the PEL-type cases, using an equicharacteristic deformation argument (which is thus yet unavailable for Hodge type Shimura varieties) and by Wortmann [2013] for general Hodge-type cases, along the lines of [Viehmann and Wedhorn 2013] comparing the Newton stratification with the Ekedahl–Oort stratification. There was also a group-theoretic approach of [Bültel 2001]. Nonemptiness of the basic locus in the PEL-type cases was shown in [Fargues 2004].

(2) The Newton stratification on Shimura varieties has been a research topic of intensive study with constant progress and outputs. We refer the reader to the recent survey article [Viehmann 2015] on the current status of research (updating the reports by Rapoport [2003; 2005]). Here, we just mention one question directly related to our work, namely the dimension of Newton strata (the result on which either assumes or subsumes the nonemptiness of stratum). There is a conjectural formula for the (co)dimension of each Newton stratum [Chai 2000, Question 7.6; Rapoport 2005, p. 296]. This conjecture was inspired by the result of Oort [2001, Theorem 4.2] confirming it in the Siegel case, and was proved, among others, in the general PEL-type case by Hamacher [2015]; with our nonemptiness result, his recent work [Hamacher 2017] also establishes this formula in the general Hodge-type case. Also, the result of [Kret 2012], alluded to in the introduction, in fact establishes nonemptiness of Newton strata by proving this formula in the simple PEL-type cases of Lie type A or C; his result, however, depends on the resolution of the Langlands–Rapoport conjecture [Kisin 2017] and the stabilization of the twisted trace formula. There is also a work of Scholze and Shin [2013, Corollary 8.4] of similar flavor.

(3) We do not know yet if our definition of the $G_{\mathbb{Q}_p}$-isocrystal is a $G_{\mathbb{Q}_p}$-isocrystal on $\mathcal{S}_K \otimes \kappa(\wp)$ in the sense of [Rapoport and Richartz 1996, §3]. Due to this flaw, some basic properties on Newton stratification that are known for PEL-type Shimura varieties, such as the Grothendieck specialization theorem, are not yet established for Hodge type Shimura varieties. For some known properties, we refer to [Rapoport 2003; Vasiu 2008].

3. Proof of nonemptiness of Newton strata

We keep the notation from the previous section; in particular, we chose an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ inducing $\wp$ on $E(G, X)$. In this section, we prove the main result, namely that for any $G_{\mathbb{Q}_p}$-isocrystal $[b]$ lying in $B(G_{\mathbb{Q}_p}, X)$, there exists a special point in $\mathrm{Sh}_K(G, X)(\overline{\mathbb{Q}})$ such that the F-isocrystal of its reduction is $[b]$. For that, we give a generalization of a result of Kottwitz [1992, Lemma 13.1] (in the

PEL-cases) which identifies the $G_{\mathbb{Q}_p}$-isocrystal of the reduction of a special point in terms of the special Shimura subdatum defining the point.

3.1. ***$G_{\mathbb{Q}_p}$-isocrystal attached to special points.*** We recall that for a torus T over a nonarchimedean local field F, there exists a unique maximal compact subgroup of $T(F)$ (which is also open). In fact, T has a canonical integral form $\mathcal{T}$ over $\mathcal{O}_F$ such that for every finite extension F' of F, the maximal compact subgroup of $T(F')$ is $\mathcal{T}(\mathcal{O}_{F'})$. If E is a finite Galois extension of F splitting T, the maximal compact subgroup of $T(E) = \mathrm{Hom}(X^*(T), E^\times)$ ($\simeq (E^\times)^{\dim T}$) is $\mathcal{T}(\mathcal{O}_E) = \mathrm{Hom}(X^*(T), \mathcal{O}_E^\times)$ ($\simeq (\mathcal{O}_E^\times)^{\dim T}$); furthermore, we have $\mathcal{T}(\mathcal{O}_F) = \mathrm{Hom}_{\mathrm{Gal}(E/F)}(X^*(T), \mathcal{O}_E^\times)$. If T is unramified over F, $\mathcal{T}(\mathcal{O}_F)$ is the unique hyperspecial subgroup of $T(F)$ (see Theorem 1 and the discussion after Proposition 2 in §10.3 of [Voskresenskiĭ 1998]; also see [Tits 1979]). We will also write simply $T(\mathcal{O}_F)$ for $\mathcal{T}(\mathcal{O}_F)$.

Now, let $x = [h, k_f] \in \mathrm{Sh}_K(G, X)(\overline{\mathbb{Q}})$ be a special point, namely $h \in \mathrm{Hom}(\mathbb{S}, T_{\mathbb{R}}) \cap X$ for a maximal $\mathbb{Q}$-torus T of G and $k_f \in G(\mathbb{A}_f)$. Then, $\iota_p(x) \in \mathrm{Sh}_K(G, X)(\overline{\mathbb{Q}}_p)$ extends (uniquely) to an $\mathcal{O}_{\iota_p}$-valued point of the integral canonical model $\mathcal{S}_K$, where $\mathcal{O}_{\iota_p}$ is the valuation ring of $\overline{\mathbb{Q}}$ defined by ι_p. This is due to the *construction* of $\mathcal{S}_K$ as the normalization of the Zariski closure of the image of a natural embedding $\mathrm{Sh}_K(G, X) \hookrightarrow \mathcal{A}_{g,d,H} \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{(\wp)}$ (as provided by (2.2.2.1)). Indeed, the image of x under such an embedding is a polarized abelian variety of CM-type; thus, it extends to an $\mathcal{O}_{\iota_p}$-valued point of $\mathcal{A}_{g,d.H}$. Since the morphism $\mathcal{S}_K(G, X) \to \mathcal{A}_{g,d,H} \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{(\wp)}$ of $\mathcal{O}_{(\wp)}$-schemes is finite (as the target is an excellent scheme), the claim follows by the valuative criterion of properness applied to a suitable normal model over $\mathcal{O}_{(\wp)}$ of $\mathrm{Sh}_{K'}(T, h)$ for $K' \subset T(\mathbb{A}_f) \cap K$ a compact open subgroup. Set $L := L(\overline{\mathbb{F}}_p)$.

Lemma 3.1.1. (1) *The $G_{\mathbb{Q}_p}$-isocrystal of the reduction z of $\iota_p(x) \in \mathrm{Sh}_K(G, X)(\mathcal{O}_{\iota_p})$ is the image of $-\underline{\mu_h} \in X_*(T_{\mathbb{Q}_p})_{\mathrm{Gal}_{\mathbb{Q}_p}}$ under the natural map $X_*(T_{\mathbb{Q}_p})_{\mathrm{Gal}_{\mathbb{Q}_p}} \xrightarrow{\sim} B(T_{\mathbb{Q}_p}) \to B(G_{\mathbb{Q}_p})$, where the first isomorphism is the inverse of the isomorphism $\kappa_{T_{\mathbb{Q}_p}} : B(T_{\mathbb{Q}_p}) \xrightarrow{\sim} X_*(T_{\mathbb{Q}_p})_{\mathrm{Gal}_{\mathbb{Q}_p}}$* (2.1.3.1).

(2) *If $T_{\mathbb{Q}_p}$ is unramified and the* (*unique*) *hyperspecial subgroup $T(\mathbb{Z}_p)$ of $T(\mathbb{Q}_p)$ is contained in K_p, then $\iota_p(x)$ is defined over an unramified extension E' of $E(G, X)_\wp$ and extends uniquely over $\mathcal{O}_{E'}$.*

It follows readily from [Kottwitz 1985, 2.5] that the $G_{\mathbb{Q}_p}$-isocrystal of the reduction z is represented by

$$\mathrm{Nm}_{E/E_0}(\mu(\pi_E))^{-1} \in T(L),$$

where $E = E(T, h)_{\mathfrak{p}}$ for the prime $\mathfrak{p}$ of the reflex field $E(T, h)$ induced by ι_p, $E_0 \subset E$ is the maximal unramified subextension of $\mathbb{Q}_p$, $\mathrm{Nm}_{E/E_0} : T(E) \to T(E_0)$ is the Norm map, and π_E is a uniformizer of E.

Proof. (1) This was proved by Kottwitz [1992, Lemma 13.1] in his PE(L)-type situation. In our general Hodge-type situation, we adapt his arguments; incidentally. We already observed that the F-isocrystal of z is represented by an element of $T(L)$. Before entering into a detailed proof, we briefly explain the strategy of Kottwitz's proof [1992, §12–13]. It consists of two parts. First, with any $\mathbb{Q}_p$-torus $(T', \mu' \in X_*(T'))$ endowed with a cocharacter, he constructs a certain T'-isocrystal, i.e., a $\mathbb{Q}_p$-linear exact tensor functor

$$\Xi_{T',\mu'} : \mathcal{REP}_{\mathbb{Q}_p}(T') \longrightarrow \mathcal{CRYS} \qquad (3.1.1.1)$$

(see Section 2.1) and identifies it (via the isomorphism $\kappa_{T'}: B(T') \xrightarrow{\sim} X_*(T')_{\mathrm{Gal}_{\mathbb{Q}_p}}$ (2.1.3.1)); this part of his argument applies to any torus over a p-adic field endowed with a cocharacter. Next, he specializes the pair (T', μ') to the one $(T_{\mathbb{Q}_p}, \mu_h)$ coming from a special Shimura subdatum (T, h) of $(\mathrm{GSp}(W, \psi), \mathfrak{H}^{\pm})$, and in this case obtains the geometric interpretation $\Xi_{T_{\mathbb{Q}_p}, \mu_h}(W_{\mathbb{Q}_p}) = H_1^{\mathrm{cris}}(A_z/L)$, where $W_{\mathbb{Q}_p}$ refers to the representation of $T_{\mathbb{Q}_p}$ given by the chosen embedding $T \hookrightarrow \mathrm{GSp}(W, \psi)$ and $H_1^{\mathrm{cris}}(A_z/L)$ is the F-isocrystal dual to $H^1_{\mathrm{cris}}(A_z/L)$ for the reduction z of $\iota_p(x)$ [Kottwitz 1992, Lemma 13.1]. Here, Kottwitz works with special Shimura-subdatum defined by a PEL-datum, but one easily checks that this statement continues to hold for any special Shimura subdatum of $(\mathrm{GSp}(W, \psi), \mathfrak{H}^{\pm})$ (we will justify this shortly). Finally, a standard argument of Tannakian theory gives an isomorphism

$$f(W_{\mathbb{Q}_p}) : H_1^{\mathrm{cris}}(A_z/L) \xrightarrow{\sim} W_{\mathbb{Q}_p} \otimes L$$

of L-vector spaces; this is obtained by identifying some two L-valued fiber functors of the Tannakian category $\mathcal{REP}_{\mathbb{Q}_p}(T')$, from which it follows (cf. Section 2.1) that the Frobenius automorphism $H_1^{\mathrm{cris}}(A_z/L)$ transfers to $b\sigma$ with $b \in T'(L)$ via such isomorphism. Our main job then consists in verifying that in a general Hodge-type situation, such an isomorphism $f(W_{\mathbb{Q}_p})$ still qualifies as an isomorphism of (2.2.5.1), namely that under the induced map $f(W_{\mathbb{Q}_p})^{\vee} : W_L^{\vee} \xrightarrow{\sim} H^1_{\mathrm{cris}}(A_z/L)$, for each $\alpha \in \mathcal{J}$, the tensor s_α goes over to the crystalline cycle $s_{\alpha,0,z}$ of Lemma 2.2.5(2).

To justify these claims in detail, we begin by recalling the construction of the functor $\Xi_{T_{\mathbb{Q}_p}, \mu_h}$ (3.1.1.1). Let $F \subset \overline{\mathbb{Q}}_p$ be a finite extension of $\mathbb{Q}_p$ and $\mathcal{T}^F$ the category of tori over $\mathbb{Q}_p$ split by F. The functor $T \mapsto X_*(T)$ is an equivalence of categories between $\mathcal{T}^F$ and the category of $\mathrm{Gal}(F/\mathbb{Q}_p)$-modules that are free of finite rank as abelian groups. Then, for any object T' of $\mathcal{T}^F$ endowed with a cocharacter $\mu' \in X_*(T')$, one defines a $\mathbb{Q}_p$-linear exact tensor functor

$$\Xi_{T', \mu'} : \mathcal{REP}_{\mathbb{Q}_p}(T') \to \mathcal{CRYS} \tag{3.1.1.2}$$

as the composite of two $\mathbb{Q}_p$-linear exact tensor functors

$$\rho^*_{T', \mu'} : \mathcal{REP}_{\mathbb{Q}_p}(T') \to \mathcal{CR}(F), \qquad \mathbb{D} : \mathcal{CR}(F) \to \mathcal{CRYS},$$

where $\mathcal{CR}(F)$ is the neutral $\mathbb{Q}_p$-linear Tannakian category of finite-dimensional crystalline representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F^{\mathrm{ur}})$, where $F^{\mathrm{ur}} \subset \overline{\mathbb{Q}}_p$ is the maximal unramified subextension of F. Here, the first functor $\rho^*_{T', \mu'}$ is dual to a certain natural homomorphism

$$\rho_{T', \mu'} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/F^{\mathrm{ur}}) \to T'(\mathbb{Q}_p) \tag{3.1.1.3}$$

that is constructed from (T', μ') in [Kottwitz 1992, §12]. The second functor $\mathbb{D}$ is the Dieudonné functor in the Fontaine–Messing theory: for a finite unramified extension E $(\subset F^{\mathrm{ur}})$ of F and a crystalline representation V of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/E)$,

$$\mathbb{D}(V) = (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/E)},$$

where B_{cris} is the crystalline period ring of [Fontaine and Messing 1987]. Then, in view of [Kottwitz 1985, 2.5], Lemma 12.1 of [Kottwitz 1992] says that, regarding $\Xi_{T',\mu'}$ as an element of $B(T')$, one has the equality

$$\Xi_{T',\mu'} = -\underline{\mu}'$$

under the isomorphism $\kappa_{T'} : B(T') \xrightarrow{\sim} X_*(T')_{\mathrm{Gal}_{\mathbb{Q}_p}}$ (2.1.3.1).

In the geometric situation where $(T', \mu') = (T_{\mathbb{Q}_p}, \mu_h)$, we claim that one has a canonical isomorphism of F-isocrystals

$$\Xi_{T_{\mathbb{Q}_p},\mu_h}(W_{\mathbb{Q}_p}) = H_1^{\mathrm{cris}}(A_z/L),$$

where $H_1^{\mathrm{cris}}(A_z/L)$ is the F-isocrystal dual to $H^1_{\mathrm{cris}}(A_z/L)$, i.e., the linear dual $\mathrm{Hom}_L(H^1_{\mathrm{cris}}(A_z/L), L)$ equipped with the Frobenius operator $\Phi_1(f)(v) := p^{-1} \cdot {}^{\sigma}f(Vv)$ for $f \in H_1^{\mathrm{cris}}(A_z/L)$ and $v \in H^1_{\mathrm{cris}}(A_z/L)$ (V being the Verschiebung operator on $H^1_{\mathrm{cris}}(A_z/L)$). In view of the comparison theorem of [Fontaine and Messing 1987] and [Faltings 1989], it is sufficient to show that the (crystalline) representation $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F^{\mathrm{ur}}) \to T_{\mathbb{Q}_p} \hookrightarrow \mathrm{GL}(W_{\mathbb{Q}_p})$ (3.1.1.3) is isomorphic, via some isomorphism $W^\vee_{\mathbb{Q}_p} \xrightarrow{\sim} H^1_{\text{ét}}(\iota_p(A_x), \mathbb{Q}_p)$, to the canonical Galois representation $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F^{\mathrm{ur}}) \to \mathrm{GL}(H_1^{\text{ét}}(\iota_p(A_x), \mathbb{Q}_p))$, where $H_1^{\text{ét}}(\iota_p(A_x), \mathbb{Q}_p)$ is $H^1_{\text{ét}}(\iota_p(A_x), \mathbb{Q}_p)^\vee$ (linear dual). The moduli interpretation of $\mathrm{Sh}_K(G, X)(\mathbb{C})$ provides an isomorphism $W^\vee \xrightarrow{\sim} H^1_{\mathrm{B}}(A_x, \mathbb{Q})$, uniquely determined up to action of $G(\mathbb{Q})$. There exists such an isomorphism under which the Hodge structure $h : \mathbb{C}^\times \to \mathrm{GL}(W)_{\mathbb{R}}$ corresponds to the defining Hodge structure $h_x : \mathbb{C}^\times \to \mathrm{GL}(H^1_{\mathrm{B}}(A_x, \mathbb{Q}))$ of A_x; we fix one, say θ. The induced isomorphism

$$W^\vee_{\mathbb{Q}_p} \xrightarrow{\sim} H^1_{\mathrm{B}}(A_x, \mathbb{Q}) \otimes \mathbb{Q}_p \xrightarrow{\sim} H^1_{\text{ét}}(\iota_p(A_x), \mathbb{Q}_p) \tag{3.1.1.4}$$

will be what we want. Under such isomorphism, the Galois representation $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F^{\mathrm{ur}}) \to T_{\mathbb{Q}_p}$ (3.1.1.3) is identified with the one attached to the pair (T_x, h_x) by the same procedure in [Kottwitz 1992, §12], where $T_x := \mathrm{Int}(\theta)(T) \subset \mathrm{GL}(H^1_{\mathrm{B}}(A_x, \mathbb{Q}))$. Therefore, given these, the claim at hand follows readily from a result in the theory of complex multiplication of Shimura and Taniyama, which says that when $\iota_p(A_x)$ is defined over a finite extension F of $\mathbb{Q}_p$, the canonical Galois representation $\mathrm{Gal}(F^{\mathrm{ab}}/F) \to \mathrm{GL}(H_1^{\text{ét}}(\iota_p(A_x), \mathbb{Q}_p))$ composed with the isomorphism $F^\times \xrightarrow{\sim} W_F^{\mathrm{ab}}$ in local class field theory (normalized in such a way that the geometric Frobenius automorphisms map to uniformizers) is

$$r_{T_x,\mu_{h_x}}|_{E(T,h)^\times_{\mathfrak{p}}} \circ \mathrm{Nm}_{F/E(T,h)_{\mathfrak{p}}} : F^\times \to E(T,h)^\times_{\mathfrak{p}} \to (T_x)_{\mathbb{Q}_p},$$

where $E(T, h)$ is the reflex field of (T, μ_h) and $\mathfrak{p}$ is the place of $E(T, h)$ defined by $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ (so that $E(T, h)_{\mathfrak{p}} \subset F$); see [Chai et al. 2014, A.2.5.3, A.2.5.8, (1), and A.2.4.5].

From $\Xi_{T_{\mathbb{Q}_p},\mu_h}$ (defined for $(T', F, \mu') = (T_{\mathbb{Q}_p}, F, \mu_h)$), and the obvious fiber functor $\mathcal{CRYS} \to \mathcal{VEC}_L : (U, \Phi) \mapsto U$, we get a fiber functor over L: $\omega_1 : \mathcal{REP}_{\mathbb{Q}_p}(T_{\mathbb{Q}_p}) \to \mathcal{VEC}_L$. By Steinberg's theorem ($H^1(L, T_L) = \{0\}$), this nonstandard fiber functor is isomorphic to the standard fiber functor $\omega_0 : \mathcal{REP}_{\mathbb{Q}_p}(T_{\mathbb{Q}_p}) \to \mathcal{VEC}_L : (\rho, V) \mapsto V \otimes_{\mathbb{Q}_p} L$; we fix such an isomorphism $f : \omega_1 \xrightarrow{\sim} \omega_0$. Then, we have seen that there exists $b \in T(L)$ such that

$$(H_1^{\mathrm{cris}}(A_z/L), \Phi_1) \simeq (W \otimes L, \rho_W(b)(\mathrm{id}_W \otimes \sigma)),$$

via the induced isomorphism $f(W_{\mathbb{Q}_p}) : \omega_1(W_{\mathbb{Q}_p}) = H_1^{\mathrm{cris}}(A_z/L) \xrightarrow{\sim} \omega_0(W_{\mathbb{Q}_p}) = W_{\mathbb{Q}_p} \otimes L$, and that $[b] = -\underline{\mu}' \in X_*(T')_{\mathrm{Gal}_{\mathbb{Q}_p}}$ under the isomorphism $\kappa_{T'} : B(T') \xrightarrow{\sim} X_*(T')_{\mathrm{Gal}_{\mathbb{Q}_p}}$ (2.1.3.1). Also note in passing that $H^1_{\mathrm{cris}}(A_x/L) \simeq (W^\vee \otimes L, \rho_{W^\vee}(b)(\mathrm{id}_{W^\vee} \otimes \sigma))$. Therefore, to prove statement (1), we only need to show that for any isomorphism $f : \omega_1 \xrightarrow{\sim} \omega_0$ of fiber functors, $f(W_{\mathbb{Q}_p}) : H_1^{\mathrm{cris}}(A_z/L) \xrightarrow{\sim} W_L$ qualifies as an isomorphism of (2.2.5.1) (i.e., an isomorphism compatible with tensors). We remark that in the PEL-type setting, the tensors involved were morphisms in (abelian) categories, so this was obvious. In a general Hodge-type case, however, the proof of this fact requires a nontrivial fact [Blasius 1994] that the Hodge cycles on abelian varieties over number fields are de Rham.

In more detail, we consider each tensor s_α as a morphism $s_\alpha : \mathbf{1} \to W^{\otimes}_{\mathbb{Q}_p}$ in $\mathcal{REP}_{\mathbb{Q}_p}(T_{\mathbb{Q}_p})$, where $\mathbf{1}$ is the trivial representation $\mathbb{Q}_p$ (which is an identity object of the tensor category $\mathcal{REP}_{\mathbb{Q}_p}(T_{\mathbb{Q}_p})$) and $W^{\otimes}_{\mathbb{Q}_p}$ denotes some specific object of $\mathcal{REP}_{\mathbb{Q}_p}(T_{\mathbb{Q}_p})$. Applying the functor $\Xi_{T_{\mathbb{Q}_p},\mu_h}$, we get a morphism $s'_\alpha : \Xi_{T_{\mathbb{Q}_p},\mu_h}(\mathbf{1}) \to \Xi_{T_{\mathbb{Q}_p},\mu_h}(W^{\otimes}_{\mathbb{Q}_p})$ in $\mathcal{CRYS}$. As $\Xi_{T_{\mathbb{Q}_p},\mu_h}(\mathbf{1})$ is the trivial F-isocrystal (L, σ), again this is equivalently regarded as a crystalline tensor on $\Xi_{T_{\mathbb{Q}_p},\mu_h}(W_{\mathbb{Q}_p}) = H_1^{\mathrm{cris}}(A_z/L)$. Moreover, as $f : \omega_1 \to \omega_0$ is an isomorphism of fiber functors, we have a commutative diagram of L-vector spaces:

$$\begin{array}{ccc} \omega_1(\mathbf{1}) & \xrightarrow{s'_\alpha} & \omega_1(W^{\otimes}_{\mathbb{Q}_p}) \\ {\scriptstyle f(\mathbf{1})}\downarrow \simeq & & \simeq \downarrow {\scriptstyle f(W^{\otimes}_{\mathbb{Q}_p})} \\ \omega_0(\mathbf{1}) & \xrightarrow{s_\alpha} & \omega_0(W^{\otimes}_{\mathbb{Q}_p}) \end{array}$$

So, it remains to check that the crystalline tensor s'_α on $H^1_{\mathrm{cris}}(A_z/L)$ equals the crystalline cycle $s_{\alpha,0,z}$, i.e., the de Rham cycle $s_{\alpha,\mathrm{dR},x}$ on $H^1_{\mathrm{dR}}(A_x/L)$ (under the canonical isomorphism $H^1_{\mathrm{cris}}(A_z/L) = H^1_{\mathrm{dR}}(A_x/L)$). First, it follows from the previous discussion that the image of the tensor $s_\alpha : \mathbf{1} \to W^{\otimes}_{\mathbb{Q}_p}$ under the functor $\rho^*_{T_{\mathbb{Q}_p},\mu_h}$ is its image under our chosen identification $W^\vee_{\mathbb{Q}_p} \xrightarrow{\sim} H^1_{\text{ét}}(\iota_p(A_x), \mathbb{Q}_p)$ (3.1.1.4). As this identification is induced from an isomorphism $W^\vee \xrightarrow{\sim} H^1_{\mathrm{B}}(A_x, \mathbb{Q})$ provided by the moduli interpretation of $\mathrm{Sh}_{K_pK^p}(G, X)(\mathbb{C})$, the image tensor equals the tensor $s_{\alpha,p,\sigma_\infty(x)} : \mathbf{1} \to H_1^{\text{ét}}(\iota_p(A_x), \mathbb{Q}_p)^{\otimes}$. This is the p-adic component of the absolute Hodge cycle $(s_{\alpha,\mathrm{dR},x}, (s_{\alpha,l,\sigma_\infty(x)})_l)$. Then, since the étale and the de Rham component of an absolute Hodge cycle on an abelian variety over a number field match under the p-adic comparison isomorphism, from which the second functor $\mathbb{D}$ is induced [Blasius 1994, Theorem 0.3], the image of $s_{\alpha,p,\sigma_\infty(x)}$ under the functor $\mathbb{D}$ must be $s_{\alpha,\mathrm{dR},x} : \mathbf{1} \to H_1^{\mathrm{dR}}(A_x/L)^{\otimes}$. So, we have just proved that $s'_\alpha = \Xi_{\mu_h}(s_\alpha) = s_{\alpha,\mathrm{dR},x}$, and consequently statement (1).

(2) The statement on the field of definition is an easy consequence of the reciprocity law characterizing the canonical model of Shimura varieties [Deligne 1979, 2.2] and local class field theory. For any compact open subgroup K' contained in $T(\mathbb{A}_f) \cap K$, there is a natural map $\mathrm{Sh}_{K'}(T, h) \to \mathrm{Sh}_K(G, X) \otimes_{E(G,X)} E(T, h)$ of weakly canonical models of Shimura varieties [Deligne 1979, 2.2.5]. Hence, it suffices to show that for the compact open subgroup $K' = T(\mathbb{Z}_p) \times K'^p$ for any $K'^p \subset T(\mathbb{A}^p) \cap K^p$, the connected components of the finite scheme $\mathrm{Sh}_{K'}(T, h)$ over $E(T, h)$ are defined over an (abelian) extension E' of $E(T, h)$ such that the prime $\mathfrak{p}'$ of E' induced by ι_p is unramified over $\wp$. The action of $\mathrm{Gal}(\overline{\mathbb{Q}}/E(T, h))$ (which factors through

the topological abelianization $\mathrm{Gal}(E(T,h)^{\mathrm{ab}}/E(T,h)))$ on the group $\mathrm{Sh}_{K'}(T,h)(\overline{\mathbb{Q}}) = T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K'$ is given by (the adelic points of) the reciprocity map [Deligne 1979, 2.2]

$$r_{T,\mu_h} : \mathrm{Res}_{E(T,h)/\mathbb{Q}}(\mathbb{G}_{m E(T,h)}) \xrightarrow{\mathrm{Res}_{E(T,h)/\mathbb{Q}}(\mu_h)} \mathrm{Res}_{E(T,h)/\mathbb{Q}}(T_{E(G,h)}) \xrightarrow{N_{E(T,h)/\mathbb{Q}}} T \tag{3.1.1.5}$$

via class field theory: for $\rho \in \mathrm{Gal}(E(T,h)_{\mathfrak{p}}^{\mathrm{ab}}/E(T,h)_{\mathfrak{p}}) = E(T,h)_{\mathfrak{p}}^{\times}$ and $[t]_{K'} \in \mathrm{Sh}_{K'}(T,h)(\overline{E_{\mathfrak{p}}}) = T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K'$, one has

$$\rho[t] = [t \cdot r_{T,\mu_h}(\rho)^{-1}],$$

where one uses the convention that under the identification $\mathrm{Gal}(E(T,h)_{\mathfrak{p}}^{\mathrm{ab}}/E(T,h)_{\mathfrak{p}}) = E(T,h)_{\mathfrak{p}}^{\times}$, the *geometric Frobenius* corresponds to a uniformizer of $E(T,h)_{\mathfrak{p}}$. Obviously, the image in $T(\mathbb{Q}_p)$ under the map r_{T,μ_h} of the unit group $(\mathcal{O}_{E(T,h)} \otimes \mathbb{Z}_p)^{\times}$ is contained in the maximal compact open subgroup $T(\mathbb{Z}_p)$. As $\mathrm{Gal}(E(T,h)_{\mathfrak{p}}^{\mathrm{ab}}/E(T,h)_{\mathfrak{p}}^{\mathrm{ur}}) = (\mathcal{O}_{E(T,h)})_{\mathfrak{p}}^{\times} \subset (\mathcal{O}_{E(T,h)} \otimes \mathbb{Z}_p)^{\times}$, this proves the claim. Then, by the extension property [Kisin 2010, 2.3.7] and Remark 2.2.2 of the integral canonical model, x extends to an $\mathcal{O}_F$-valued point of $\mathcal{S}_K$. □

Note that when the conditions of (2) hold, the "good reduction at ι_p" of the special point follows from an intrinsic property of the integral canonical model (i.e., the extension property), rather than its construction.

3.2. *Construction of special Shimura data with prescribed F-isocrystals.*

Theorem 3.2.1. *Let (G,X) be a Shimura datum (not necessarily of Hodge type) such that $G_{\mathbb{Q}_p}$ is unramified and $K_p = G_{\mathbb{Z}_p}(\mathbb{Z}_p)$ a hyperspecial subgroup of $G(\mathbb{Q}_p)$.*

(1) *For every $[b] \in B(G_{\mathbb{Q}_p}, X)$, there exists a special Shimura subdatum $(T, h : \mathbb{S} \to T_{\mathbb{R}})$ such that*

$$\bar{\nu}_{G_{\mathbb{Q}_p}}([b]) = -\frac{1}{[F:\mathbb{Q}_p]}\mathrm{Nm}_{F/\mathbb{Q}_p}(\mu_h).$$

Here, the right-hand side is the image of the corresponding quasicocharacter under the canonical map $X_*(T)_{\mathbb{Q}}^{\mathrm{Gal}_{\mathbb{Q}_p}} = \mathcal{N}(T_{\mathbb{Q}_p}) \to \mathcal{N}(G_{\mathbb{Q}_p})$ and $F \subset \overline{\mathbb{Q}}_p$ is the field of definition of $\mu \in X_*(T_{\mathbb{Q}_p})$.

(2) *If (G,X) is of Hodge type, for every $[b] \in B(G_{\mathbb{Q}_p}, X)$, there exists a special Shimura subdatum (T,h) such that for any $g_f \in G(\mathbb{A}_f)$, the reduction (at ι_p) of the special point $[h, g_f]_{K_p} \in \mathrm{Sh}_{K_p}(G,X)(\overline{\mathbb{Q}})$ has the F-isocrystal equal to $[b]$.*

(3) *Furthermore, we may find such a special Shimura subdatum (T,h) with the additional properties that $T_{\mathbb{Q}_p}$ is unramified and $T(\mathbb{Z}_p) \subset K_p$.*

The additional properties in statement (3) specify the prime-to-p isogeny class of the reduction of a special point defined by such (T,h), not just its $\mathbb{Q}$-isogeny class; such finer information has an amusing application (see Corollary 3.2.3).

Proof. (1) Our proof consists of two steps. Fix a rational prime $l \neq p$.

Step 1. We claim that for an *arbitrary* maximal $\mathbb{Q}_p$-torus T' of $G_{\mathbb{Q}_p}$, there exist a maximal $\mathbb{Q}$-torus T_0 of G such that $(T_0)_{\mathbb{Q}_v}$ is elliptic in $G_{\mathbb{Q}_v}$ for $v = \infty, l$ and $(T_0)_{\mathbb{Q}_p} = \mathrm{Int}(g_p)(T')$ for some $g_p \in G(\mathbb{Q}_p)$. Then, for such T_0, we will shortly find $\mu \in X_*(T_0)$ satisfying the following conditions:

$$\mu \in X_*(T_0) \cap c(G, X) \quad \text{and} \quad \bar{\nu}_{G_{\mathbb{Q}_p}}([b]) = \frac{1}{[F : \mathbb{Q}_p]} \mathrm{Nm}_{F/\mathbb{Q}_p}(\mu). \tag{3.2.1.1}$$

Here, in the equation on the right, F is any extension of $\mathbb{Q}_p$ splitting $(T_0)_{\mathbb{Q}_p}$; note that the quasicocharacter itself remains the same if we take F to be a field of definition of μ.

First, to find a maximal $\mathbb{Q}$-torus T_0 of G with the required properties, we consider the following three sets:

$X_v = \{x \in G^{\mathrm{sc}}(\mathbb{Q}_v) \mid x \text{ is regular semisimple and } \mathrm{Cent}_G(x) \text{ is elliptic in } G_{\mathbb{Q}_v}\}, \quad (v = \infty, l);$

$X_p = \{x \in G^{\mathrm{sc}}(\mathbb{Q}_p) \mid x \text{ is regular semisimple and } \mathrm{Cent}_G(x) = \mathrm{Int}(g_p)(T') \text{ for some } g_p \in G(\mathbb{Q}_p)\}.$

These sets are all nonempty: for $v = \infty$, this follows from Deligne's condition on Shimura data [Deligne 1979, (2.1.1.2)], and for $v = l$, from [Platonov and Rapinchuk 1994, Theorem 6.21]. They are also open in the real, l-adic, and p-adic topology, respectively: for two sufficiently close regular semisimple elements, their centralizers are conjugated (cf. [Bültel 2001, proof of Lemma 3.1]). So, by the weak approximation theorem, there exists an element $x \in G^{\mathrm{sc}}(\mathbb{Q})$ which lies in all of them. Its centralizer $T_0 := \mathrm{Cent}_G(x)$ is then the desired maximal torus.

Secondly, existence of a cocharacter $\mu \in X_*(T_0)$ satisfying the conditions (3.2.1.1) is established in Lemma 5.11 of [Langlands and Rapoport 1987]. (See [Lee 2016, Proposition 4.2.4] for another proof.) We remark that this proof of Langlands–Rapoport uses, as the input condition, nonemptiness of the affine Deligne–Lusztig variety $X(c(G, X), b)_{K_p}$ attached to the tuple $(G_{\mathbb{Q}_p}, c(G, X), K_p, [b])$:

$$X(c(G, X), b)_{K_p} := \{g G_{\mathbb{Z}_p}(\mathcal{O}_L) \in G(L)/G_{\mathbb{Z}_p}(\mathcal{O}_L) \mid g^{-1} b \sigma(g) \in G_{\mathbb{Z}_p}(\mathcal{O}_L)\mu(p) G_{\mathbb{Z}_p}(\mathcal{O}_L)\},$$

where μ is any morphism $\mathbb{G}_{\mathrm{m},\mathcal{O}_L} \to G_{\mathbb{Z}_p} \otimes \mathcal{O}_L$ lying in $c(G, X)$. But [Wintenberger 2005] shows that $[b] \in B(G_{\mathbb{Q}_p}, c(G, X))$ implies such nonemptiness.

Step 2. Next, for any maximal $\mathbb{Q}$-torus T_0 of G, elliptic at $\mathbb{R}$, and each cocharacter $\mu \in X_*(T_0) \cap c(G, X)$, we claim that there exists $u \in G(\overline{\mathbb{Q}})$ such that

(i) $\mathrm{Int}\, u : (T_0)_{\overline{\mathbb{Q}}} \hookrightarrow G_{\overline{\mathbb{Q}}}$ is defined over $\mathbb{Q}$ and the $\mathbb{Q}$-torus $T := \mathrm{Int}\, u(T_0)$ is elliptic in G over $\mathbb{R}$,

(ii) $\mathrm{Int}\, u(\mu)$ equals μ_h^{-1} for some $h \in \mathrm{Hom}(\mathbb{S}, T_{\mathbb{R}}) \cap X$.[1]

Moreover, if $(T_0)_{\mathbb{Q}_l}$ is elliptic in $G_{\mathbb{Q}_l}$ for some $l \neq p$, there exist such $u \in G(\overline{\mathbb{Q}})$ and $h \in X$ which satisfy, in addition to these properties, that

(iii) there exists $y \in G(\mathbb{Q}_p)$ such that $(T_{\mathbb{Q}_p}, \mu_h^{-1}) = \mathrm{Int}\, y((T_0)_{\mathbb{Q}_p}, \mu)$.

Our proof is an adaptation of the argument of [Langlands and Rapoport 1987, Lemma 5.12]. Let T_0^{sc} be the inverse image of $(T_0 \cap G^{\mathrm{der}})^0$ under the canonical isogeny $G^{\mathrm{sc}} \to G^{\mathrm{der}}$, and pick any morphism $h_0 \in X$

[1] Recall our convention that $c(G, X)$ is the conjugacy class of cocharacters into $G_{\mathbb{C}}$ containing μ_h^{-1} for some $h \in X$.

factoring through $(T_0)_{\mathbb{R}}$, which exists since $(T_0)_{\mathbb{R}}$ is elliptic in $G_{\mathbb{R}}$ and any two elliptic maximal tori in any connected reductive real algebraic group H are conjugate under $H(\mathbb{R})$. Choose $w \in N_G(T_0)(\mathbb{C})$ such that $\mu = w(\mu_{h_0}^{-1})$; we may find such a w in $N_{G^{\mathrm{sc}}}(T_0^{\mathrm{sc}})(\mathbb{C})$ as the Weyl groups of G and G^{sc} are the same. This gives us a cocycle $\alpha^\infty \in Z^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), G^{\mathrm{sc}}(\mathbb{C}))$ defined by

$$\alpha^\infty_\iota = w \cdot \iota(w^{-1})$$

for all $\iota \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. As a matter of fact, this has values in $T_0^{\mathrm{sc}}(\mathbb{C})$. Indeed, by [Shelstad 1979, Proposition 2.2], the automorphism $\mathrm{Int}(w^{-1})$ of $(T_0^{\mathrm{sc}})_{\mathbb{C}}$ is defined over $\mathbb{R}$, so

$$\mathrm{Int}(w^{-1})(\iota(t)) = \iota(\mathrm{Int}(w^{-1})t) = \mathrm{Int}(\iota(w^{-1}))(\iota(t))$$

for all $t \in T_0^{\mathrm{sc}}(\mathbb{C})$, i.e., $\iota(w)w^{-1} \in \mathrm{Cent}_{G^{\mathrm{sc}}}(T_0^{\mathrm{sc}})(\mathbb{C}) = T_0^{\mathrm{sc}}(\mathbb{C})$, and so is $w\iota(w^{-1}) = \iota(\iota(w)w^{-1})$. Then, by Lemma 7.16 of [Langlands 1983], there exists a global cocycle

$$\alpha \in Z^1(\mathbb{Q}, T_0^{\mathrm{sc}})$$

mapping to $\alpha^\infty \in H^1(\mathbb{R}, T_0^{\mathrm{sc}})$. When $(T_0)_{\mathbb{Q}_l}$ is elliptic in $G_{\mathbb{Q}_l}$ for some $l \neq p$, then the following lemma ensures the existence of a cocycle $\alpha \in Z^1(\mathbb{Q}, T_0^{\mathrm{sc}})$ mapping to $\alpha^\infty \in H^1(\mathbb{R}, T_0^{\mathrm{sc}})$ and further to zero in $H^1(\mathbb{Q}_p, T_0^{\mathrm{sc}})$. For a finitely generated abelian group A, let A_{tors} be the subgroup of its torsion elements.

Lemma 3.2.2. (1) *For a maximal $\mathbb{Q}$-torus T of G which is elliptic at $l \neq p$, the natural map*

$$(\pi_1(T^{\mathrm{sc}})_{\Gamma(l)})_{\mathrm{tors}} \to (\pi_1(T^{\mathrm{sc}})_{\Gamma})_{\mathrm{tors}}$$

is surjective, where we write $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $\Gamma(l) = \mathrm{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q}_l)$.

(2) *There exists $\alpha \in Z^1(\mathbb{Q}, T_0^{\mathrm{sc}})$ mapping to $\alpha^\infty \in H^1(\mathbb{R}, T_0^{\mathrm{sc}})$ and to zero in $H^1(\mathbb{Q}_p, T_0^{\mathrm{sc}})$.*

Proof of lemma. First, we recall some (standard) notations: for a torus T over a field F, $\widehat{T}$ denotes the dual torus of T, and for an abelian topological group A, A^D denotes its dual group $\mathrm{Hom}_{\mathrm{cont}}(A, \mathbb{C}^\times)$ (group of continuous homomorphisms).

(1) Since this map is induced (by restriction) by the natural surjection $\pi_1(T^{\mathrm{sc}})_{\Gamma(l)} \twoheadrightarrow \pi_1(T^{\mathrm{sc}})_\Gamma$, it suffices to prove that $\pi_1(T^{\mathrm{sc}})_{\Gamma(l)}$ is a torsion group. But, since $T^{\mathrm{sc}}_{\mathbb{Q}_l}$ is anisotropic, $\widehat{T^{\mathrm{sc}}}^{\Gamma(l)}$ is a finite group, and so is

$$\pi_1(T^{\mathrm{sc}})_{\Gamma(l)} = X^*(\widehat{T^{\mathrm{sc}}}^{\Gamma(l)}) = \mathrm{Hom}(\widehat{T^{\mathrm{sc}}}^{\Gamma(l)}, \mathbb{C}^\times).$$

(2) For any place v of $\mathbb{Q}$, there exists a canonical isomorphism

$$H^1(F, T^{\mathrm{sc}}) \xrightarrow{\sim} \mathrm{Hom}(\pi_0(\widehat{T^{\mathrm{sc}}}^{\Gamma(v)}), \mathbb{C}^\times) = ((\widehat{T^{\mathrm{sc}}}^{\Gamma(v)})^D)_{\mathrm{tors}} \cong (\pi_1(T^{\mathrm{sc}})_{\Gamma(v)})_{\mathrm{tors}}$$

[Kottwitz 1984, (3.3.1); Rapoport and Richartz 1996, 1.14], and a short exact sequence

$$H^1(\mathbb{Q}, T^{\mathrm{sc}}) \to H^1(\mathbb{Q}, T^{\mathrm{sc}}(\overline{\mathbb{A}})) := \bigoplus_v H^1(\mathbb{Q}_v, T^{\mathrm{sc}}) \xrightarrow{\theta} \pi_0(\widehat{T^{\mathrm{sc}}}^{\Gamma})^D = (\pi_1(T^{\mathrm{sc}})_\Gamma)_{\mathrm{tors}}$$

[Kottwitz 1986, Proposition 2.6]. Here, θ is the composite

$$\bigoplus_v H^1(\mathbb{Q}_v, T^{\mathrm{sc}}) \xrightarrow{\sim} \bigoplus_v \pi_0(\widehat{T^{\mathrm{sc}}}^{\Gamma(v)})^D \to \pi_0(\widehat{T^{\mathrm{sc}}}^{\Gamma})^D,$$

of which the second map is identified with the natural map $\bigoplus_v (\pi_1(T^{\mathrm{sc}})_{\Gamma(v)})_{\mathrm{tors}} \twoheadrightarrow (\pi_1(T^{\mathrm{sc}})_{\Gamma})_{\mathrm{tors}}$ (thus, it equals the direct sum of the maps considered in the statement of (1)). Hence, by (1), there exists a class $\alpha^l \in H^1(\mathbb{Q}_l, T^{\mathrm{sc}})$ such that $\theta(\alpha^l) = -\theta(\alpha^\infty)$, and thus the element $(\beta^v)_v \in H^1(\mathbb{Q}, T^{\mathrm{sc}}(\overline{\mathbb{A}}))$ defined by the condition that $\beta^v = \alpha^v$ for $v = \infty, l$ and $\beta^v = 0$ for $v \neq l, \infty$ in $\pi_0(\widehat{T^{\mathrm{sc}}}^{\Gamma})^D$. By exactness of the sequence, we conclude existence of $\alpha \in Z^1(\mathbb{Q}, T^{\mathrm{sc}})$ whose cohomology class maps to $(\beta^v)_v$. □

Now, by changing α^∞ and w further, we may assume that α^∞ *is* the restriction of α to $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Then, using the fact that the restriction map $H^1(\mathbb{Q}, G^{\mathrm{sc}}) \to H^1(\mathbb{R}, G^{\mathrm{sc}})$ is injective (which follows from the Hasse principle [Platonov and Rapinchuk 1994, Theorem 6.6] and the fact that $H^1(\mathbb{Q}_v, G^{\mathrm{sc}}) = 0$ for any nonarchimedean place v [Platonov and Rapinchuk 1994, Theorem 6.4]), we see that α becomes trivial as a cocycle with values in $G^{\mathrm{sc}}(\overline{\mathbb{Q}})$. We summarize this discussion in the diagram:

$$\begin{array}{ccc} H^1(\mathbb{R}, T_0^{\mathrm{sc}}) & \longrightarrow & H^1(\mathbb{R}, G^{\mathrm{sc}}) \\ \uparrow & & \uparrow \\ H^1(\mathbb{Q}, T_0^{\mathrm{sc}}) & \longrightarrow & H^1(\mathbb{Q}, G^{\mathrm{sc}}) \end{array} \qquad \begin{array}{ccc} \alpha^\infty & \longmapsto & 0 \\ \uparrow & & \uparrow \\ \exists\, \alpha & \longmapsto & \alpha' \end{array} \quad \Rightarrow \quad \alpha' = 0.$$

So there exists $u \in G^{\mathrm{sc}}(\overline{\mathbb{Q}})$ such that for all $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$,

$$\alpha_\rho = u^{-1}\rho(u).$$

Then the homomorphism $\mathrm{Int}\, u : (T_0)_{\overline{\mathbb{Q}}} \to G_{\overline{\mathbb{Q}}} : t \mapsto t' = utu^{-1}$ is defined over $\mathbb{Q}$; in particular, $T := uT_0u^{-1}$ is a torus defined over $\mathbb{Q}$, and as the restriction of $\mathrm{Int}\, u$ to $Z(G)$ is the identity, $T_{\mathbb{R}}$ is also elliptic in $G_{\mathbb{R}}$. Moreover, since $u^{-1}\iota(u) = w\iota(w^{-1})$ for $\iota \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$, one has that $uw \in G^{\mathrm{sc}}(\mathbb{R})$ and

$$\mathrm{Int}\, u(\mu) = \mathrm{Int}(uw)\mu_{h_0}^{-1} = \mu_h^{-1}$$

for $h := \mathrm{Int}(uw)(h_0) \in X \cap \mathrm{Hom}(\mathbb{S}, T)$. This establishes the claims (i) and (ii). The proof of claim (iii) is similar. As the restriction of α to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is trivial, there exists $x \in T(\overline{\mathbb{Q}}_p)$ such that $x\rho(x^{-1}) = \alpha_\rho = u^{-1}\rho(u)$ for all $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, i.e., $ux \in G(\mathbb{Q}_p)$. But the homomorphism $\mathrm{Int}\, u : (T_0)_{\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$ also equals $\mathrm{Int}\, u = \mathrm{Int}\, ux$. This proves (iii).

Now we complete the proof of Theorem 3.2.1(1). Let $[b] \in B(G_{\mathbb{Q}_p}, X)$. For the first statement, we take an arbitrary maximal $\mathbb{Q}$-torus T_0 of G that is elliptic over $\mathbb{R}$, and find a cocharacter μ of T_0 having the properties (3.2.1.1); we remark that Lemma 5.1 of [Langlands and Rapoport 1987], which was used to find such a μ_0, works for any maximal $\mathbb{Q}_p$-torus of $G_{\mathbb{Q}_p}$ (in particular, for $(T_0)_{\mathbb{Q}_p}$). From such a pair (T_0, μ), we get a special Shimura subdatum (T, h) with the properties (i) and (ii) of Step 2. This special Shimura subdatum (T, h) satisfies the condition of the first statement, since $\mathrm{Int}(u) : T_0 \to T$ is defined over $\mathbb{Q}$ (for $u \in G(\overline{\mathbb{Q}})$ as chosen in the proof) so that $\mathrm{Int}(u)(\mathrm{Nm}_{F/\mathbb{Q}_p}(\mu)) = \mathrm{Nm}_{F/\mathbb{Q}_p}(\mu_h^{-1})$.

(2) Note that the quasicocharacter $-\frac{1}{[F:\mathbb{Q}_p]}\mathrm{Nm}_{F/\mathbb{Q}_p}(\mu_h)$ equals the image of $-\underline{\mu_h}$ under the natural map $B(T_{\mathbb{Q}_p}) \cong X_*(T)_{\mathrm{Gal}_{\mathbb{Q}_p}} \to \mathcal{N}(T_{\mathbb{Q}_p}) = X_*(T)_{\mathbb{Q}}^{\mathrm{Gal}_{\mathbb{Q}_p}}$: use the fact that there exists a commutative diagram

$$\begin{array}{ccc} B(G) & \xrightarrow{\bar{\nu}_G} & \mathcal{N}(G) \\ \kappa_G \downarrow & & \downarrow \delta_G \\ \pi_1(G)_{\mathrm{Gal}_F} & \xrightarrow{\alpha_G} & \pi_1(G)_{\mathbb{Q}}^{\mathrm{Gal}_F} \end{array}$$

(for any reductive group G over a p-adic field F). Here, the map α_G in the bottom is given by

$$\bar{\mu} \mapsto |\mathrm{Gal}_F \cdot \mu|^{-1} \sum_{\mu' \in \mathrm{Gal}_F \cdot \mu} \mu'$$

(cf. [Kottwitz 1985, 2.8; Rapoport and Richartz 1996, Theorem 1.15]). Therefore, since the map $(\bar{\nu}, \kappa) : B(G_{\mathbb{Q}_p}) \to \mathcal{N}(G_{\mathbb{Q}_p}) \times \pi_1(G)_{\mathrm{Gal}_{\mathbb{Q}_p}}$ is injective [Kottwitz 1997, 4.13], the second statement follows from the first statement and Lemma 3.1.1.

(3) Finally, for the existence of a special Shimura subdatum (T, h) with the additional property on the position of the hyperspecial subgroup $T(\mathbb{Z}_p)$, we begin with an unramified maximal $\mathbb{Q}_p$-torus T' of $G_{\mathbb{Q}_p}$ such that the unique hyperspecial subgroup $T'(\mathbb{Z}_p)$ of $T'(\mathbb{Q}_p)$ is contained in K_p: such a torus exists by Lemma 3.2.4 below. Then, from such T', we can find (by Step 1) a pair (T_0, μ) having the properties (3.2.1.1) of Step 1 and, further, such that $(T_0)_{\mathbb{Q}_v}$ is elliptic for $v = \infty$, (some) $l \neq p$ and also that the unique hyperspecial subgroup of $T_0(\mathbb{Q}_p)$ is contained in $\mathrm{Int}(g_p)(K_p)$ for some $g_p \in G(\mathbb{Q}_p)$. For such (T_0, μ), again Step 2 produces a special Shimura datum (T_1, h_1) satisfying the properties (i)–(iii) of Step 2. Now, since $G(\mathbb{Q})$ is dense in $G(\mathbb{Q}_p)$ [Milne 1994, Lemma 4.10], there exists $g \in G(\mathbb{Q}) \cap K_p \cdot g_p^{-1}$. Then the new special Shimura datum $(T, h) := \mathrm{Int}(g)(T_1, h_1)$ still satisfies the condition in the first statement, and further, the unique hyperspecial subgroup of $T(\mathbb{Q}_p)$ is contained in K_p. □

Corollary 3.2.3. *Let (G, X) be a Shimura datum of Hodge type. Choose an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and let $\wp$ be the prime of $E(G, X)$ induced by ι_p. Suppose that $G_{\mathbb{Q}_p}$ is unramified and choose a hyperspecial subgroup K_p of $G(\mathbb{Q}_p)$. Then the reduction $\mathcal{S}_{K_p}(G, X) \times \kappa(\wp)$ has nonempty ordinary locus if and only if $\wp$ has absolute height one* (*i.e.*, *$E(G, X)_\wp = \mathbb{Q}_p$*). *In this case, if $G_{\mathbb{Z}_p} \hookrightarrow \mathrm{GL}(W_{\mathbb{Z}_p})$ is an embedding of $\mathbb{Z}_p$-group schemes* (see [Kisin 2010, (2.3.1)]), *there exists a special point which is the canonical lifting of its reduction.*

Recall from 2.2.1 that in the construction of the integral canonical model, an embedding $G \hookrightarrow \mathrm{GSp}(W, \psi)$ is chosen so as to satisfy the condition in the corollary.

Proof. The necessity of $E(G, X)_\wp = \mathbb{Q}_p$ was proved in [Bültel 2001, Section 3]. For sufficiency, we note that it suffices to find a special Shimura subdatum (T, h) such that

there exists a Borel subgroup B over $\mathbb{Q}_p$ of $G_{\mathbb{Q}_p}$ containing $T_{\mathbb{Q}_p}$ and such that $\mu_h \in X_*(T)$ lies in the closed Weyl chamber determined by $(T_{\mathbb{Q}_p}, B)$. (∗)

Indeed, then the canonical Galois action of $\mathrm{Gal}_{\mathbb{Q}_p}$ on $X_*(T)$ coincides with the naive Galois action; hence, one has that $E(G, X)_\wp = E(T, h)_\mathfrak{p}$, where $\mathfrak{p}$ is the prime of $E(T, h)$ induced by ι_p. Given this, the corollary then follows from [Bültel 2001, Lemma 2.2]: for a CM-abelian variety A over $\overline{\mathbb{Q}}$ $(\subset \mathbb{C})$ with CM-type (F, Φ), where F is a CM-algebra and Φ is a subset of $\mathrm{Hom}(F, \mathbb{C})$ with $\Phi \sqcup \iota \circ \Phi = \mathrm{Hom}(F, \mathbb{C})$, the reduction of A at $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ is an ordinary abelian variety if and only if the prime of the reflex field $E = E(F, \Phi)$ induced by ι_p is of absolute height one; when A is defined by a special Shimura datum (T, h), the reflex field $E(F, \Phi)$ equals the reflex field $E(T, h)$ defined earlier.

Now, pick an arbitrary maximal $\mathbb{Q}_p$-torus T' of $G_{\mathbb{Q}_p}$ containing a maximal $\mathbb{Q}_p$-split torus. As $G_{\mathbb{Q}_p}$ is quasisplit, there exists a Borel subgroup B' defined over $\mathbb{Q}_p$ which contains T'. Let μ' be the cocharacter in $c(G, X)$ factoring through T' and lying in the closed Weyl chamber determined by (T', B'). Let (T, h) be a special Shimura subdatum produced from (T', μ') as in the proof of Theorem 3.2.1 and such that $(T_{\mathbb{Q}_p}, \mu_h)$ is conjugate to (T', μ') under $G(\mathbb{Q}_p)$. Then, for such (T, μ_h), the condition $(*)$ continues to hold, which proves the claim.

For the second statement, there exists a special Shimura subdatum (T, h) as in the last statement of Theorem 3.2.1. Then, as μ_h is a morphism of $\mathbb{Z}_p$-group schemes $\mathbb{G}_\mathrm{m} \to G_{\mathbb{Z}_p} \hookrightarrow \mathrm{GL}(W_{\mathbb{Z}_p})$, any special point $[h, 1 \times g_f^p] \in \mathrm{Sh}_{K_p K^p}(G, X)(\overline{\mathbb{Q}})$ (with arbitrary $g_f^p \in G(\mathbb{A}_f^p)$) is the canonical lifting of its reduction, according to [Milne 2006, Proposition 9.24]. □

Lemma 3.2.4. *Let F be a p-adic field with ring of integers $\mathcal{O}_F$. For any reductive group scheme $H_{\mathcal{O}_F}$ over $\mathcal{O}_F$, there exists a reductive $\mathcal{O}_F$-subgroup scheme $S_{\mathcal{O}_F}$ of $H_{\mathcal{O}_F}$ whose generic fiber is a maximal F-split torus of $H := H_{\mathcal{O}_F} \otimes F$. Its centralizer $T'_{\mathcal{O}_F} := \mathrm{Cent}_{H_{\mathcal{O}_F}}(S_{\mathcal{O}_F})$ is a smooth, closed subscheme of $H_{\mathcal{O}_F}$.*

Note that as H is quasisplit, the generic fiber T' of $T'_{\mathcal{O}_F}$ is a maximal F-torus of H.

Proof. Let $\mathcal{B}(H, F)$ denote the Bruhat–Tits building of H over F. Choose a maximal F-split F-torus S of H such that the hyperspecial point x fixed by $K_p := H_{\mathcal{O}_F}(\mathcal{O}_F)$ is contained in the apartment associated with S. Let $S_{\mathcal{O}_F}$ be the (unique) reductive $\mathcal{O}_F$-group scheme with generic fiber S. Then, $S_{\mathcal{O}_F}$ is a closed subscheme of $H_{\mathcal{O}_F}$. Indeed, we first observe that $S_{\mathcal{O}_F}(\mathcal{O}_L)$ maps to $H_{\mathcal{O}_F}(\mathcal{O}_L)$ under the embedding $S \hookrightarrow H$, where L is the completion of the maximal unramified extension (in an algebraic closure $\overline{F}$) of F and $\mathcal{O}_L$ is the ring of integers of L. This follows from these two facts: first, the hyperspecial point x of $\mathcal{B}(H, F)$ is also a hyperspecial point of $\mathcal{B}(H, L)$ with the associated $\mathcal{O}_L$-group scheme being $(H_{\mathcal{O}_F})_{\mathcal{O}_L}$ [Tits 1979, 3.4, 3.8], and secondly, for a maximal L-split torus S_1 of H containing S and defined over F (which exists by [Bruhat and Tits 1984, 5.1.12]), $x \in \mathcal{B}(H, L)$ lies in the apartment corresponding to S_1 [Tits 1979, 1.10]. So, x is fixed under the maximal bounded subgroup $S_1(\mathcal{O}_L)$ of $S_1(L)$, and thus under $S_{\mathcal{O}_F}(\mathcal{O}_L)$ as well. Then, according to [Vasiu 1999, 3.1.2.1], this implies that the F-embedding $S \hookrightarrow H$ extends to an $\mathcal{O}_F$-embedding $S_{\mathcal{O}_F} \hookrightarrow H_{\mathcal{O}_F}$. The fact that $T'_{\mathcal{O}_F} := \mathrm{Cent}_{H_{\mathcal{O}_F}}(S_{\mathcal{O}_F})$ is a smooth closed group $\mathcal{O}_F$-subscheme of $H_{\mathcal{O}_F}$ is well-known [SGA 3_{II} 1970, XI, Corollaire 5.3; Conrad 2014, 2.2]. Moreover, one knows that it represents the functor on $\mathcal{O}_F$-schemes

$$R' \mapsto \{g \in H(R') \mid g(S_{\mathcal{O}_F})_{R'} g^{-1} = (S_{\mathcal{O}_F})_{R'}\},$$

and its generic fiber $T' := \mathrm{Cent}_H(S)$ is a maximal torus of H which splits over L, since H is quasisplit [Tits 1979, 1.10]. □

Remark 3.2.5. (1) In fact, for any special Shimura subdatum (T, h) with the property $(*)$ in the proof, every special point of $\mathrm{Sh}_{K_p}(G, X)$ defined by it has μ-ordinary reduction at ι_p. So, we get a more direct proof of nonemptiness of μ-ordinary locus (without invoking Lemma 5.11 of [Langlands and Rapoport 1987] used in the proof of Theorem 3.2.1).

(2) This corollary was proved in the PEL-type cases of Lie type A or C by Wedhorn [1999]. His argument was to first establish nonemptiness of the μ-ordinary locus, and then to check that the μ-ordinary locus equals the (usual) ordinary locus when $E(G, X)_{\wp} = \mathbb{Q}_p$ (in the PEL-type cases of Lie type A or C). But to check the second statement, he relied on a case-by-case analysis. In contrast, we observe that in view of (1), the equality of the μ-ordinary Newton point and the ordinary Newton point when $E(G, X)_{\wp} = \mathbb{Q}_p$ is simply a consequence of the existence of a special Shimura subdatum (T, h) with the property $(*)$ in the proof (since the reduction of such a special point attains simultaneously the two Newton points when $E(G, X)_{\wp} = \mathbb{Q}_p$).

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