Closures in varieties of representations
and irreducible components

Kenneth R. Goodearl and Birge Huisgen-Zimmermann

msp

Closures in varieties of representations and irreducible components

Kenneth R. Goodearl and Birge Huisgen-Zimmermann

Dedicated to the memory of Peter Gabriel

For any truncated path algebra Λ of a quiver, we classify, by way of representation-theoretic invariants, the irreducible components of the parametrizing varieties $\mathbf{Rep}_d(\Lambda)$ of the Λ-modules with fixed dimension vector $\boldsymbol{d}$. In this situation, the components of $\mathbf{Rep}_d(\Lambda)$ are always among the closures $\overline{\mathbf{Rep}\,\mathbb{S}}$, where $\mathbb{S}$ traces the semisimple sequences with dimension vector $\boldsymbol{d}$, and hence the key to the classification problem lies in a characterization of these closures.

Our first result concerning closures actually addresses arbitrary basic finite-dimensional algebras over an algebraically closed field. In the general case, it corners the closures $\overline{\mathbf{Rep}\,\mathbb{S}}$ by means of module filtrations "governed by $\mathbb{S}$"; when Λ is truncated, it pins down the $\overline{\mathbf{Rep}\,\mathbb{S}}$ completely.

The analysis of the varieties $\overline{\mathbf{Rep}\,\mathbb{S}}$ leads to a novel upper semicontinuous module invariant which provides an effective tool towards the detection of components of $\mathbf{Rep}_d(\Lambda)$ in general. It detects all components when Λ is truncated.

1. Introduction

By strong consensus, a classification of all indecomposable finite-dimensional representations of a finite-dimensional algebra Λ is an unattainable goal in general. A far more promising alternative to this impossibly comprehensive problem is that of *generically* classifying the finite-dimensional Λ-modules. This amounts to understanding the generic structure of the modules in the irreducible components of the varieties $\mathbf{Rep}_d(\Lambda)$ which parametrize the Λ-modules with dimension vector $\boldsymbol{d}$. By its very nature, this quest comes paired with the task of pinning down the irreducible components of the $\mathbf{Rep}_d(\Lambda)$ in representation-theoretic terms.

In the present article, the component problem is solved for arbitrary truncated path algebras Λ over an algebraically closed field K. In tandem, significant headway is made towards determining the generic features of the modules in the components.

The classification of the components, in turn, relies on a characterization of the modules in the closures of certain representation-theoretically defined locally closed subvarieties of $\mathbf{Rep}_d(\Lambda)$. Our initial round of results regarding such closures, including the description of an associated upper semicontinuous module invariant which serves to test for inclusions, holds for arbitrary basic finite-dimensional K-algebras. The findings lead to partial lists of components in this broad scenario. The results become tight on specialization to the truncated case.

Throughout, we assume K to be an algebraically closed field and Λ a basic finite-dimensional K-algebra. This means that, up to isomorphism, $\Lambda = KQ/I$ for a quiver Q and an admissible ideal I in the path algebra. The maximal length of a path in $KQ \setminus I$ will be denoted by L; in other words, L is minimal with respect to $J^{L+1} = 0$, where J is the Jacobson radical of Λ. Consequently, the *radical layering* $\mathbb{S}(M)$ of a Λ-module M has no more than $L+1$ nonzero entries: $\mathbb{S}(M) = (J^l M/J^{l+1}M)_{0 \le l \le L}$. By $\mathbf{Rep}_d(\Lambda)$, we denote the standard affine variety parametrizing the Λ-modules with dimension vector d. This variety is partitioned into finitely many locally closed subvarieties $\mathbf{Rep}\, \mathbb{S}$ corresponding to the *semisimple sequences* $\mathbb{S}$ *with dimension vector d*; these are the sequences $\mathbb{S} = (\mathbb{S}_0, \ldots, \mathbb{S}_L)$ of (isomorphism classes of) semisimple Λ-modules with $\underline{\dim}\, \mathbb{S} := \sum_{0 \le l \le L} \underline{\dim}\, \mathbb{S}_l = d$; here $\mathbf{Rep}\, \mathbb{S}$ consists of those points x in $\mathbf{Rep}_d(\Lambda)$ which represent modules M_x with $\mathbb{S}(M_x) = \mathbb{S}$.

The closures $\overline{\mathbf{Rep}\, \mathbb{S}}$ are relevant to the problem of describing the irreducible components of $\mathbf{Rep}_d(\Lambda)$: indeed, it is readily seen that the components of the ambient variety are always among those of the $\overline{\mathbf{Rep}\, \mathbb{S}}$, where $\mathbb{S}$ traces the d-dimensional semisimple sequences. Less obviously, the components of the subvarieties $\mathbf{Rep}\, \mathbb{S}$, and hence those of their closures, may be obtained from Q and I by way of a straightforward algorithm, each component tagged by a "generic minimal projective presentation" of the modules it encodes (see [Babson et al. 2009] and [Huisgen-Zimmermann 2009]). Identifying the components of $\mathbf{Rep}_d(\Lambda)$ thus amounts to a sorting problem: for which components $\mathcal{C}$ of $\mathbf{Rep}\, \mathbb{S}$ is the closure $\overline{\mathcal{C}}$ *maximal* among the irreducible subsets of $\mathbf{Rep}_d(\Lambda)$? This is an extremely taxing question in general, calling for a thorough understanding of the boundaries of the varieties $\mathbf{Rep}\, \mathbb{S}$.

Our strategy consists of moving back and forth between the varieties $\mathbf{Rep}_d(\Lambda)$ and $\mathrm{GRASS}_d(\Lambda)$; the latter is a closed subvariety of a vector space Grassmannian which parametrizes the modules with dimension vector d by suitable submodules of a projective cover of the semisimple module with this dimension vector (see Section 2 and [Huisgen-Zimmermann 2009; Huisgen-Zimmermann and Goodearl 2012]). The irreducible components of the projective variety $\mathrm{GRASS}_d(\Lambda)$ may be studied by "spreading them out" within a suitable flag variety (Theorem 3.9), and the subsequent transfer of information $\mathrm{GRASS}_d(\Lambda) \longleftrightarrow \mathbf{Rep}_d(\Lambda)$ is modeled on the influential work [Gabriel 1975]. In a first step, we show:

Theorem A (cf. Theorem 3.8 and Theorem 4.3; see also Remark 3.7(4)). *Let* $\Lambda = KQ/I$ *be a path algebra modulo relations,* $L + 1$ *its Loewy length, and* $\mathbb{S} = (\mathbb{S}_0, \ldots, \mathbb{S}_L)$ *a **d**-dimensional semisimple sequence in* Λ-mod. *Then every module in the closure* $\overline{\textbf{Rep}\,\mathbb{S}}$ *has a filtration by submodules,*

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_{L+1} = 0,$$

which is "governed by $\mathbb{S}$*" in the sense that each quotient* M_l/M_{l+1} *is isomorphic to* $\mathbb{S}_l$ $(0 \le l \le L)$. *In fact, the set* $\textbf{Filt}\,\mathbb{S}$ *consisting of those points in* $\textbf{Rep}_d(\Lambda)$ *that correspond to modules with at least one filtration governed by* $\mathbb{S}$ *is always closed.*

If Λ *is a truncated path algebra, i.e.,* $\Lambda = KQ/\langle$ *all paths of length* $L + 1\rangle$*, and* $\textbf{Rep}\,\mathbb{S}$ *is nonempty, then*

$$\overline{\textbf{Rep}\,\mathbb{S}} = \textbf{Filt}\,\mathbb{S}.$$

For general Λ, the inclusion $\overline{\textbf{Rep}\,\mathbb{S}} \subseteq \textbf{Filt}\,\mathbb{S}$ may be proper. The question of whether a point in $\textbf{Rep}_d(\Lambda)$ belongs to $\textbf{Filt}\,\mathbb{S}$ may be answered by testing for similarity of certain matrices. By contrast, to date, there is no algorithm for deciding whether a module belongs to $\overline{\textbf{Rep}\,\mathbb{S}}$.

A semisimple sequence $\mathbb{S}$ is called *realizable* if $\textbf{Rep}\,\mathbb{S} \ne \varnothing$. (When Λ is a truncated path algebra, realizability is checked via mere inspection of the quiver; see [Huisgen-Zimmermann 2016, Criterion 3.2] and Realizability Criterion 4.1 below.)

Corollary B (cf. Corollary 3.11). *For* $M \in \Lambda$-mod, *let* $\Gamma(M)$ *be the number of those realizable semisimple sequences that govern at least one filtration of* M*. Then*

$$\Gamma_\bullet : \textbf{Rep}_d(\Lambda) \to \mathbb{N}, \quad x \mapsto \Gamma(M_x),$$

is an upper semicontinuous function.

In particular, whenever $\mathcal{C}$ *is an irreducible component of some* $\textbf{Rep}\,\mathbb{S}$ *such that* $1 \in \Gamma_\bullet(\mathcal{C})$*, the closure* $\overline{\mathcal{C}}$ *is an irreducible component of* $\textbf{Rep}_d(\Lambda)$*.*

In the second part of the paper, we derive consequences for truncated path algebras. As is suggested by Theorem A, the component problem simplifies considerably in this situation. Notably, the subvarieties $\overline{\textbf{Rep}\,\mathbb{S}}$ are all irreducible, and generic minimal projective presentations of the modules in $\textbf{Rep}\,\mathbb{S}$ are immediate from quiver and Loewy length (see [Babson et al. 2009, Section 5] and Section 5A below). In some prominent special cases, particularly manageable solutions to the problem of sifting out the inclusion-maximal ones among the closures $\overline{\textbf{Rep}\,\mathbb{S}}$ are already available (see [Huisgen-Zimmermann 2016; Huisgen-Zimmermann and Shipman 2017]): for instance, if Λ is either local or based on an acyclic quiver Q, the semisimple sequences singled out by the minimal values of the following upper semicontinuous map furnish a complete, nonrepetitive parametrization of the

components $\overline{\mathbf{Rep}\, \mathbb{S}}$ of $\mathbf{Rep}_d(\Lambda)$:

$$\Theta = (\mathbb{S}_{\bullet}, \mathbb{S}_{\bullet}^*) : \mathbf{Rep}_d(\Lambda) \to \mathbf{Seq}(d) \times \mathbf{Seq}(d), \quad x \mapsto (\mathbb{S}(M_x), \mathbb{S}^*(M_x)); \quad (1\text{-}1)$$

here the codomain of Θ is partially ordered by the componentwise dominance order on the set $\mathbf{Seq}(d)$ of all d-dimensional semisimple sequences (see Section 2), and $\mathbb{S}^*(M_x)$ stands for the socle layering of the module M_x (the dual of the radical layering). The unique minimal sequence $\mathbb{S}^*(M_x)$ attained on $\mathbf{Rep}\, \mathbb{S}$, that is, the generic socle layering of the modules in $\overline{\mathbf{Rep}\, \mathbb{S}}$, is supplied by a closed formula based on $\mathbb{S}$, Q and L [Huisgen-Zimmermann and Shipman 2017, Theorem 3.8], which makes the Θ-test very user-friendly. But for general truncated Λ, the map Θ fails to detect all components, even when supplemented by further standard semicontinuous module invariants, such as path ranks or assortments of annihilator dimensions. The map $\Gamma_{\bullet}$, on the other hand, compensates for the blind spots of Θ:

Theorem C (cf. Theorem 4.5). *If Λ is any truncated path algebra, the irreducible components of $\mathbf{Rep}_d(\Lambda)$ are precisely those closures $\overline{\mathbf{Rep}\, \mathbb{S}}$ on which $\Gamma_{\bullet}$ attains the value* 1.

In other words, $\overline{\mathbf{Rep}\, \mathbb{S}}$ is maximal among the irreducible subsets of $\mathbf{Rep}_d(\Lambda)$ if and only if there exists a module N in $\mathbf{Rep}\, \mathbb{S}$ such that $N \supseteq JN \supseteq \cdots \supseteq J^{L+1}N$ is the only filtration of N which is governed by a realizable semisimple sequence.

In deciding which semisimple sequences $\mathbb{S}$ are the generic radical layerings of the irreducible components of $\mathbf{Rep}_d(\Lambda)$, Theorem C thus permits exclusive reliance on $\Gamma_{\bullet}$. However, in practice, combining $\Gamma_{\bullet}$ with the test map Θ is considerably more efficient.

In the pursuit of a generic approach to the structure of Λ-modules, the hereditary case, pioneered in [Kac 1980; 1982] and [Schofield 1992], serves as a model. We further point to a selection of existing contributions to the component problem over nonhereditary algebras: General tools were developed in [Crawley-Boevey and Schröer 2002] and [Babson et al. 2009]. Solutions to the problem over specific classes of tame algebras were given in [Barot and Schröer 2001; Carroll and Weyman 2013; Donald and Flanigan 1977; Geiss and Schröer 2003; 2005; Morrison 1980; Riedtmann et al. 2011; Schröer 2004] for instance; solutions for certain classes of wild nonhereditary algebras can be found in [Bleher et al. 2015; Huisgen-Zimmermann 2016; Huisgen-Zimmermann and Shipman 2017]. As is to be expected, meaningful classifications of the irreducible components of $\mathbf{Rep}_d(\Lambda)$ in the quoted instances are throughout obtained via partial lists of generic properties of the modules in the components. For a more detailed discussion of prior work on the topic we refer to the introduction of [Huisgen-Zimmermann 2016].

We add a few comments on the foundational nature of truncated path algebras with respect to the component problem. Clearly, given an arbitrary basic K-algebra

$\Lambda = KQ/I$, there is a unique truncated path algebra Λ_{trunc} having the same quiver and Loewy length as Λ. In the general situation, the varieties $\overline{\mathbf{Rep}\,\mathbb{S}}$ typically break up into multiple components. Given that all of them are contained in irreducible components of $\mathbf{Rep}_d(\Lambda_{\text{trunc}})$, it is advantageous to first determine the latter, say

$$\overline{\mathbf{Rep}_{\Lambda_{\text{trunc}}}\,\mathbb{S}^{(1)}} = \mathbf{Filt}_{\Lambda_{\text{trunc}}}(\mathbb{S}^{(1)}), \quad \dots, \quad \overline{\mathbf{Rep}_{\Lambda_{\text{trunc}}}\,\mathbb{S}^{(m)}} = \mathbf{Filt}_{\Lambda_{\text{trunc}}}(\mathbb{S}^{(m)}),$$

before aiming at the irreducible components of $\mathbf{Rep}_d(\Lambda)$. Indeed, this confines the need for size comparisons among the closures of components of the varieties $\mathbf{Rep}_\Lambda\,\mathbb{S}$ to the subvarieties $\mathbf{Filt}_{\Lambda_{\text{trunc}}}\,\mathbb{S}^{(j)} \cap \mathbf{Rep}_d(\Lambda)$; see Section 6B.

Overview. In Section 2, we provide background for the proofs of the main results and introduce a recurring example. Section 3 addresses the general case, where Λ is basic but otherwise unrestricted. In Sections 4 and 5, we apply the findings to truncated path algebras. Section 4 contains the announced classification of the irreducible components of $\mathbf{Rep}_d(\Lambda)$, while in Section 5, we discuss generic modules and apply the results of Section 4 to exhibit interconnections among the components. Section 6, finally, illustrates the theory and addresses the interplay $\mathbf{Rep}_d(\Lambda) \longleftrightarrow \mathbf{Rep}_d(\Lambda_{\text{trunc}})$.

2. Conventions and prerequisites

To repeat: throughout, we assume $\Lambda = KQ/I$ to be a basic finite-dimensional algebra over $K = \overline{K}$ with Jacobson radical J and Loewy length $L + 1$. The composition pq of paths stands for "p after q" when $\text{start}(p) = \text{end}(q)$, while $pq = 0$ in KQ otherwise. By Λ_{trunc} we denote the truncated path algebra associated to Λ, namely,

$$\Lambda_{\text{trunc}} = KQ/\langle\text{the paths of length } L + 1\rangle;$$

we make no notational distinction between the Λ- and Λ_{trunc}-structures of the objects in Λ-mod. The vertices $e_1, \dots, e_n$ of Q will be identified with the paths of length zero in KQ, as well as with the corresponding primitive idempotents in Λ. An element x of a Λ-module M is said to be *normed by* e_i if $x = e_i x$, and a normed element in $M \setminus JM$ is called a *top element* of M. A *full sequence of top elements* of M is a generating set of M consisting of top elements which are K-linearly independent modulo JM. The simple module $\Lambda e_i / J e_i$ corresponding to the vertex e_i will be denoted by S_i, and isomorphic semisimple modules will be identified.

The *dominance order* on the set $\mathbf{Seq}(d)$ of all semisimple sequences with dimension vector d is defined as follows:

$$(\mathbb{S}_0, \dots, \mathbb{S}_L) \le (\mathbb{S}'_0, \dots, \mathbb{S}'_L) \quad \Longleftrightarrow \quad \bigoplus_{0 \le j \le l} \mathbb{S}_j \subseteq \bigoplus_{0 \le j \le l} \mathbb{S}'_j \quad \text{for } 0 \le l \le L.$$

Recall that the radical and socle layerings of a Λ-module M are denoted by $\mathbb{S}(M)$ and $\mathbb{S}^*(M)$. For basic properties of these semisimple sequences, we refer to [Huisgen-Zimmermann 2016, Section 2.B].

We fix our notation for the parametrizing varieties of the $\boldsymbol{d}$-dimensional Λ-modules. The affine variety $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ is

$$\left\{ (x_\alpha)_{\alpha \in Q_1} \in \prod_{\alpha \in Q_1} \mathrm{Hom}_K(K^{d_{\mathrm{start}(\alpha)}}, K^{d_{\mathrm{end}(\alpha)}}) \,\Big|\, \text{the } x_\alpha \text{ satisfy all relations in } I \right\},$$

where Q_1 is the set of arrows of Q. The orbits of the obvious conjugation action on $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ by the group $\mathrm{GL}(\boldsymbol{d}) := \prod_{1 \le i \le n} \mathrm{GL}_{d_i}(K)$ are in natural bijection with the isomorphism classes of the $\boldsymbol{d}$-dimensional Λ-modules. Given $\mathbb{S} \in \mathbf{Seq}(\boldsymbol{d})$, we denote by $\mathbf{Rep}\,\mathbb{S}$ the locally closed subvariety of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ which consists of the points x for which the corresponding module M_x has radical layering $\mathbb{S}$. Clearly, the varieties $\mathbf{Rep}\,\mathbb{S}$, where $\mathbb{S}$ traces the semisimple sequences with $\underline{\dim}\,\mathbb{S} = \boldsymbol{d}$, partition $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$. However, in general, this (finite) partition falls short of being a stratification of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ in the strict sense, in that closures of strata need not be unions of strata.

To introduce the projective parametrizing variety $\mathrm{GRASS}_{\boldsymbol{d}}(\Lambda)$, we fix a projective Λ-module $\boldsymbol{P}$ whose top $\boldsymbol{P}/J\boldsymbol{P}$ has dimension vector $\boldsymbol{d}$, and set $d = |\boldsymbol{d}|$. The variety $\mathrm{GRASS}_{\boldsymbol{d}}(\Lambda)$ is the closed subvariety of the vector space Grassmannian $\mathrm{Gr}((\dim \boldsymbol{P} - d), \boldsymbol{P})$ consisting of those points $C \in \mathrm{Gr}((\dim \boldsymbol{P} - d), \boldsymbol{P})$ which are Λ-submodules of $\boldsymbol{P}$ with the property that $\underline{\dim}(\boldsymbol{P}/C) = \boldsymbol{d}$. This time, the group action whose orbits determine the isomorphism classes of the quotients $\boldsymbol{P}/C$ in Λ-mod is the canonical action of $\mathrm{Aut}_\Lambda(\boldsymbol{P})$ on $\mathrm{GRASS}_{\boldsymbol{d}}(\Lambda)$. The role played by $\mathbf{Rep}\,\mathbb{S}$ in the affine setting is taken over by $\mathrm{GRASS}(\mathbb{S})$, the locally closed subvariety consisting of those $C \in \mathrm{GRASS}_{\boldsymbol{d}}(\Lambda)$ for which $\mathbb{S}(\boldsymbol{P}/C) = \mathbb{S}$.

The following connection between the affine and projective parametrizing varieties was proved in [Bongartz and Huisgen-Zimmermann 2001, Proposition C]; it was inspired by [Gabriel 1975], as is explained in some detail in Remark 3 of [Bongartz and Huisgen-Zimmermann 2001, Section 2]. We restate the result for convenient reference.

Proposition 2.1. *Consider the natural isomorphism from the lattice of* $\mathrm{GL}(\boldsymbol{d})$-*stable subsets of* $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ *on one hand to the lattice of* $\mathrm{Aut}_\Lambda(\boldsymbol{P})$-*stable subsets of* $\mathrm{GRASS}_{\boldsymbol{d}}(\Lambda)$ *on the other, which pairs orbits encoding isomorphic modules. This correspondence preserves and reflects openness, closures, irreducibility, and smoothness.*

In describing generic projective resolutions of the modules in an irreducible component of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$, a key invariant of a $\boldsymbol{d}$-dimensional Λ-module M is its set of skeleta. These skeleta live in a projective cover of M in Λ_{trunc}-mod. In the following definitions, we fix a semisimple sequence $\mathbb{S}$ with $\underline{\dim}\,\mathbb{S} = \boldsymbol{d}$.

Definitions 2.2 (coordinatized projective modules and skeleta).

(1) Let P_{trunc} be a projective cover of $\mathbb{S}_0$ in Λ_{trunc}-mod. This cover is referred to as a *coordinatized* projective module when it comes equipped with a fixed full sequence of top elements $z_1, \ldots, z_t$, where $t = \dim \mathbb{S}_0$. In particular, we obtain a decomposition $P_{\mathrm{trunc}} = \bigoplus_{1 \le r \le t} \Lambda_{\mathrm{trunc}} z_r$. A *path of length l* in the coordinatized projective module P_{trunc} is any nonzero element $\boldsymbol{p} = p z_r$ where p is a path of length l in Q; thus each z_r is now viewed as a path of length zero. Note that we have a well-defined concept of path length in Λ_{trunc}, and hence also in P_{trunc}. Clearly, each path $\boldsymbol{p} = p z_r \in P_{\mathrm{trunc}}$ is normed by a primitive idempotent, namely by $\mathrm{end}(p)$, and the primitive idempotent norming z_r is $\mathrm{start}(p)$.

(2) An (abstract) *skeleton with layering* $\mathbb{S}$ is a set σ consisting of paths in P_{trunc} which satisfies the following two conditions:

- It is closed under initial subpaths, i.e., whenever $p z_r \in \sigma$, and q is an initial subpath of p (meaning $p = q'q$ for some path q'), the path $q z_r$ again belongs to σ.

- For $0 \le l \le L$, the number of those paths of length l in σ which end in a given vertex e_i coincides with the multiplicity of S_i in the semisimple module $\mathbb{S}_l$.

Note that any skeleton σ with layering $\mathbb{S}$ includes the paths $z_1, \ldots, z_t$ of length zero.

(3) Let $M \in \Lambda$-mod. An abstract skeleton σ is a *skeleton of M* if M has a full sequence $z_1, \ldots, z_t$ of top elements, each z_r normed by the same vertex as z_r, such that

- $\{p z_r \mid p z_r \in \sigma\}$ is a K-basis for M, and

- the layering of σ coincides with the radical layering $\mathbb{S}(M)$ of M.

In this situation, we also say that σ is a skeleton of M *relative to $z_1, \ldots, z_t$.*

Clearly, the set of skeleta of any finite-dimensional Λ-module M is nonempty, and the set of all skeleta of modules with fixed dimension vector $\boldsymbol{d}$ is finite. The relevance of skeleta towards a generic understanding of the modules in the irreducible components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ is underlined by the following fact:

Observation 2.3. Let $\mathcal{P}$ be the power set of the set of all skeleta with dimension vector $\boldsymbol{d}$. Then the map

$$\mathbf{Rep}_{\boldsymbol{d}}(\Lambda) \to \mathcal{P}, \quad x \mapsto \{\text{skeleta of } M_x\},$$

is generically constant on each irreducible component of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$.

To see this, let $\mathcal{C} \subseteq \mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ be an irreducible component, and $\mathbb{S}$ the generic radical layering of its modules. Then $\mathcal{C} \cap \mathbf{Rep}\,\mathbb{S}$ is open in $\mathcal{C}$, and for any skeleton σ

with layering $\mathbb{S}$, the set

$$\mathbf{Rep}(\sigma) := \{x \in \mathbf{Rep}_d(\Lambda) \mid \sigma \text{ is a skeleton of } M_x\}$$

is an open subvariety of $\mathbf{Rep}\,\mathbb{S}$; see [Huisgen-Zimmermann 2007, Lemma 3.8]. Hence, a skeleton σ with layering $\mathbb{S}$ arises as a skeleton of the modules in a dense open subset of $\mathcal{C}$ precisely when $\mathcal{C} \cap \mathbf{Rep}(\sigma)$ is nonempty. Given that there are only finitely many eligible skeleta, this proves the claim.

Next, we recall more discerning graphical invariants associated to a finite-dimensional Λ-module, namely its hypergraphs; see Definition 3.9 of [Babson et al. 2009].

Definitions 2.4 (σ-critical paths and hypergraphs). Again, we let P_{trunc} be a co-ordinatized projective Λ_{trunc}-module with top $\mathbb{S}_0$ and assume $\sigma \subseteq P_{\text{trunc}}$ to be an abstract skeleton with layering $\mathbb{S}$. Recall that the distinguished top elements z_r of P_{trunc} coincide with the paths of length zero in σ.

(1) A σ-*critical path* is a path $\boldsymbol{q} \in P_{\text{trunc}} \setminus \sigma$ such that every <u>proper</u> initial subpath of $\boldsymbol{q}$ belongs to σ. Thus, $\boldsymbol{q} = \alpha \boldsymbol{q}'$, where $\boldsymbol{q}' \in \sigma$ and α is an arrow; in particular, length$(\boldsymbol{q}) > 0$. Given a σ-critical path $\boldsymbol{q}$, we define a subset $\sigma_{\boldsymbol{q}} \subseteq \sigma$ as follows:

$$\sigma_{\boldsymbol{q}} := \{\text{paths } \boldsymbol{p} \in \sigma \mid \text{length}(\boldsymbol{p}) \geq \text{length}(\boldsymbol{q}) \text{ and } \text{end}(\boldsymbol{p}) = \text{end}(\boldsymbol{q})\}.$$

The final condition in the definition of $\sigma_{\boldsymbol{q}}$ means that all paths in $\sigma_{\boldsymbol{q}}$ are normed (on the left) by the same vertex as $\boldsymbol{q}$.

(2) Suppose $M \in \Lambda$-mod has skeleton σ relative to a full sequence $z_1, \ldots, z_t$ of top elements. The Λ-structure of M is then determined by the family of expansion coefficients corresponding to the σ-critical paths $\boldsymbol{q} = q z_r \in P_{\text{trunc}}$, namely

$$q z_r = \sum_{\boldsymbol{p} = p z_s \in \sigma_{\boldsymbol{q}}} c_{\boldsymbol{q}, \boldsymbol{p}}\, p z_s \qquad (2\text{-}1)$$

for unique scalars $c_{\boldsymbol{p}, \boldsymbol{q}} \in K$.

(3) We refer to any pair

$$\mathcal{G} = (\sigma, (\tau_{\boldsymbol{q}})_{\boldsymbol{q}\, \sigma\text{-critical}}) \quad \text{with } \tau_{\boldsymbol{q}} \subseteq \sigma_{\boldsymbol{q}} \text{ for all } \sigma\text{-critical paths } \boldsymbol{q}$$

as an (undirected) *hypergraph in* P_{trunc}. The set $\tau_{\boldsymbol{q}}$ is called the *support set* of $\boldsymbol{q}$. Empty support sets are allowed.

In informal terms: the vertices of these hypergraphs are the elements of σ, and a typical (hyper)edge, labeled by an arrow $\gamma \in Q_1$, connects a vertex $\boldsymbol{p} \in \sigma$ to the vertex $\gamma \boldsymbol{p}$ if $\gamma \boldsymbol{p} \in \sigma$ and to the support set $\tau_{\gamma \boldsymbol{p}}$ of vertices if $\gamma \boldsymbol{p}$ is σ-critical.

(4) A hypergraph $\mathcal{G}$ as above is called a *hypergraph of* a Λ-module M (relative to a full sequence $z_1, \ldots, z_t$ of top elements of M) if σ is a skeleton of M and, in the expansion (2-1) above, $c_{q,p} \neq 0$ precisely when $p \in \tau_q$.

While hypergraphs pin down *families* of modules, as opposed to individual isomorphism classes, they provide a useful tool for communicating, in a visually suggestive format, the generic structure of the modules in the components. For our diagrammatic representations of hypergraphs, we refer to [Babson et al. 2009], [Derksen et al. 2014], and to the example below. This example will serve as a staple in the sequel.

Example 2.5. Let $\Lambda = KQ/\langle$the paths of length $4\rangle = \Lambda_{\mathrm{trunc}}$, where Q is the quiver
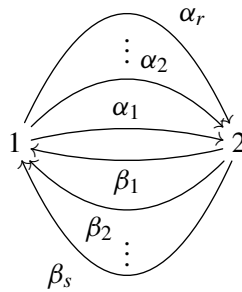
(a) First suppose that $r = 2$ and $s = 1$. Choose $\mathbb{S} := (S_1, S_2, S_1, S_2)$, and let $P_{\mathrm{trunc}} = \Lambda_{\mathrm{trunc}}z$ be the corresponding Λ_{trunc}-projective cover of $\mathbb{S}_0 = S_1$, coordinatized by a fixed top element z. Generically, the modules in **Rep** $\mathbb{S}$ then have a hypergraph of the form
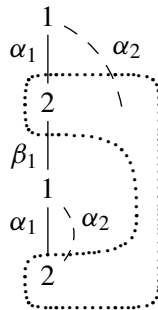
This diagram is to be read as follows: the radical layering of any module G having the above hypergraph (relative to a top element $z \in G$, say) is $\mathbb{S}$, and the skeleton chosen to represent G is $\sigma := \{z, \alpha_1 z, \beta_1 \alpha_1 z, \alpha_1 \beta_1 \alpha_1 z\}$; the edges corresponding to paths in the skeleton σ are drawn as solid edges, while the dashed edges stand for the terminal arrows of σ-critical paths. Moreover, the diagram contains the information that the support sets τ_q for the two σ-critical paths $q = \alpha_2 z$ and $q = \alpha_2 \beta_1 \alpha_1 z$ in P_{trunc} (in the sense of Definitions 2.4), are $\tau_{\alpha_2 z} = \{\alpha_1 z, \alpha_1 \beta_1 \alpha_1 z\}$ and $\tau_{\alpha_2 \beta_1 \alpha_1 z} = \{\alpha_1 \beta_1 \alpha_1 z\}$. Indeed, the "dotted pool" indicates that the element $\alpha_2 z$

of G is a K-linear combination of $\alpha_1 z$ and $\alpha_1 \beta_1 \alpha_1 z$ with coefficients in K^*; on the other hand, given that the set $\tau_{\alpha_2\beta_1\alpha_1 z}$ is a singleton, no extra pooling device is required to communicate the condition that $\alpha_2\beta_1\alpha_1 z \in G$ be a nonzero scalar multiple of $\alpha_1\beta_1\alpha_1 z$.

Next, we consider the semisimple sequence $\mathbb{S}' := (S_1^2, S_2^2, 0, 0)$. The modules in **Rep** $\mathbb{S}'$ generically look as follows, relative to top elements z_1, z_2, say:
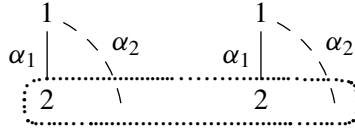
Here, the dotted pool serves double duty in indicating that both $\alpha_2 z_1$ and $\alpha_2 z_2$ are linear combinations of $\alpha_1 z_1$ and $\alpha_1 z_2$ with (unspecified) nonzero coefficients. In the sequel, we will use the fact that, generically, the modules in **Rep** $\mathbb{S}'$ decompose in the form

(b) Now let $r = 3$. The hypergraphs
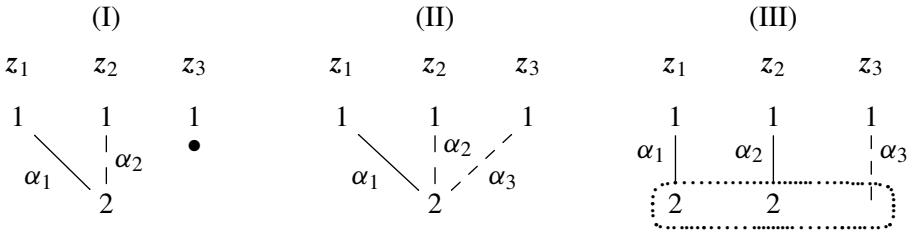
are hypergraphs of modules $M_i = \left(\bigoplus_{1 \le j \le 3} \Lambda z_j\right)/U_i$, where $z_j = e_1$ for $j = 1, 2, 3$. Here the submodule U_1 is generated by $\alpha_2 z_2 - \alpha_1 z_1$, $\alpha_3 z_3$ and $\alpha_j z_k$ for $j \ne k$, while U_2 is generated by $\alpha_2 z_2 - \alpha_1 z_1$, $\alpha_3 z_3 - \alpha_1 z_1$ and $\alpha_j z_k$ for $j \ne k$; finally, U_3 is generated by $\alpha_3 z_3 - (\alpha_1 z_1 + \alpha_2 z_2)$ and $\alpha_j z_k$ for $j \ne k$. The chosen reference skeleton of M_1 and M_2 is $\sigma := \{z_1, z_2, z_3, \alpha_1 z_1\}$, and that of M_3 is $\sigma \cup \{\alpha_2 z_2\}$. Note that the dimension of JM_3 is 2, the number of displayed vertices in the second row of the hypergraph.

Generically, the modules with radical layering $\mathbb{S}' := (S_1^2, S_2^2, 0, 0)$ are indecomposable and have hypergraphs of the form
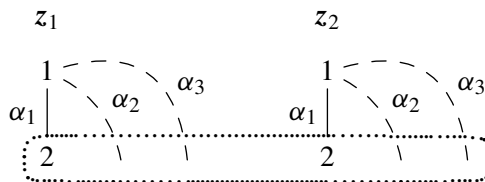
The modules in **Rep** $\mathbb{S}$, where $\mathbb{S} := (S_1, S_2, S_1, S_2)$, generically have a hypergraph akin to the first one shown in part (a). □

3. The main results for general Λ

3A. *Pared-down parametrizing varieties.* Towards a description of $\overline{\textbf{Rep}\,\mathbb{S}}$, we present lower-dimensional, more manageable varieties parametrizing the modules with radical layering $\mathbb{S}$.

Definition 3.1 (decompositions of $K^{|\boldsymbol{d}|}$ induced by semisimple sequences). Let $\mathbb{S} = (\mathbb{S}_0, \ldots, \mathbb{S}_L)$ be a realizable semisimple sequence in Λ-mod with $\underline{\dim}\,\mathbb{S} = \boldsymbol{d}$, and write $d = |\boldsymbol{d}|$. Consider a vector space decomposition of K^d which is *induced by* $\mathbb{S}$ in the following sense: namely,

$$K^d = \bigoplus_{\substack{0 \le l \le L \\ 1 \le i \le n}} \mathcal{K}_{(l,i)}$$

with the property that $\dim \mathcal{K}_{(l,i)} = \dim e_i \mathbb{S}_l$ for all eligible indices l and i. Set $\mathcal{K}_l = \bigoplus_{1 \le i \le n} \mathcal{K}_{(l,i)}$ for $l \le L$, and $\mathcal{K}_{L+1} = \mathcal{K}_{(L+1,i)} = 0$. Given a family $(f_\alpha)_{\alpha \in Q_1}$ of K-endomorphisms of K^d, the following notation will be convenient: whenever $p = \alpha_l \cdots \alpha_1$ is a path of positive length l in Q, we set $f_p = f_{\alpha_l} \circ \cdots \circ f_{\alpha_1}$; if p is a path of length 0, say $p = e_i$, then f_p is defined to be the canonical projection $K^d \to \bigoplus_{0 \le l \le L} \mathcal{K}_{(l,i)} \subseteq K^d$ relative to the above decomposition. Thus, we obtain a K-algebra homomorphism $KQ \to \mathrm{End}_K(K^d)$ such that $p \mapsto f_p$ for all paths p in Q.

By $Q_{\ge l}$ we denote the set of paths of length at least l in Q. The following lemma is an upgraded version of [Huisgen-Zimmermann 2016, Lemma 5.1] and is proved analogously.

Lemma 3.2 (triangular points in $\textbf{Rep}_d(\Lambda)$). *We refer to the above notation. Suppose that $f = (f_\alpha)_{\alpha \in Q_1}$ is a family of K-linear maps $K^d \to K^d$ satisfying the following three conditions: For any arrow α from e_i to e_j and any index $l \in \{0, \ldots, L\}$,*

(i) $f_\alpha(\mathcal{K}_{(l,r)}) = 0$ *for all* $r \ne i$;

(ii) $f_\alpha(\mathcal{K}_{(l,i)}) \subseteq \bigoplus_{l+1 \le m \le L} \mathcal{K}_{(m,j)}$;

(iii) *whenever $c_1, \ldots, c_m \in K$ and $p_1, \ldots, p_m$ are paths of length $\le L$ in Q, which have a common starting vertex and a common terminal vertex,*

$$\sum_{1 \le j \le m} c_j p_j \in I \quad \Longrightarrow \quad \sum_{1 \le j \le m} c_j f_{p_j} = 0.$$

Then the following statements (I)–(III) *hold*:

(I) *The tuple f is a point in $\mathbf{Rep}_d(\Lambda)$, and the radical layering of the correspond-ing Λ-module M_f satisfies $\mathbb{S}(M_f) \geq \mathbb{S}$. Moreover, all Λ-modules with radical layering $\mathbb{S}$ are represented by suitable points $f \in \mathbf{Rep}_d(\Lambda)$ satisfying* (i)–(iii).

(II) $J^l M_f = \sum_{p \in Q_{\geq l}} \mathrm{Im}(f_p)$ *for all* $l \in \{0, \ldots, L\}$.

(III) $\mathbb{S}(M_f) = \mathbb{S}$ *precisely when, for each* $h \in \{0, \ldots, L\}$, *the linear map*

$$(\mathcal{K}_0)^{Q_{\geq h}} \to \bigoplus_{l \geq h} \mathcal{K}_l, \quad (x_q)_{q \in Q_{\geq h}} \mapsto \sum_q f_q(x_q),$$

has maximal rank, namely $\sum_{l \geq h} \dim \mathcal{K}_l$. □

The lemma prompts an analysis of the following two subvarieties of $\mathbf{Rep}_d(\Lambda)$.

3.3 (the varieties $\Delta\text{-}\mathbf{Rep}(\geq \mathbb{S})$ and $\Delta\text{-}\mathbf{Rep}\,\mathbb{S}$). Keep $\mathbb{S}$ and a decomposition of K^d induced by $\mathbb{S}$ fixed. The collection of all $f = (f_\alpha)$ satisfying conditions (i)–(iii) of Lemma 3.2 is a closed subvariety of $\mathbf{Rep}_d(\Lambda)$ which we denote by $\Delta\text{-}\mathbf{Rep}(\geq \mathbb{S})$. Indeed, the inclusion map

$$\Delta\text{-}\mathbf{Rep}(\geq \mathbb{S}) \hookrightarrow \mathbf{Rep}_d(\Lambda)$$

provided by part (I) of Lemma 3.2 is a closed immersion.

To see this, take $B_{(l,\mu)} = (b_{(l,\mu)}^1, \ldots, b_{(l,\mu)}^{d_{l,\mu}})$ to be an ordered basis for $\mathcal{K}_{(l,\mu)}$ and B to be the lexicographically ordered union of the $B_{(l,\mu)}$. Relative to this basis for K^d, the image of the above embedding consists of all those families (F_α) of matrices in $\mathbf{Rep}_d(\Lambda)$ such that each F_α has a strictly lower triangular form of the following ilk:

- the only nonzero entries in any column labeled $(l, \mu)^{(j)}$ are confined to posi-tions with lower label $(l + 1, \nu), \ldots, (L, \nu)$, provided α is an arrow $e_\mu \to e_\nu$, and

- condition (iii) of Lemma 3.2 is satisfied.

The latter requirement translates into polynomial equations for the entries of the F_α. This shows that the considered embedding is indeed a closed immersion.

Moreover, observe that, up to isomorphism, the variety $\Delta\text{-}\mathbf{Rep}(\geq \mathbb{S})$ is determined by $\mathbb{S}$, irrespective of the choice of a decomposition $K^d = \bigoplus_{l,i} \mathcal{K}_{(l,i)}$ induced by $\mathbb{S}$. Lemma and Definition 3.6 below will show that the $\mathrm{GL}(d)$-stable hull $\mathrm{GL}(d).(\Delta\text{-}\mathbf{Rep}(\geq \mathbb{S})) \subseteq \mathbf{Rep}_d(\Lambda)$ is, in fact, unique in the strict sense.

We will identify $\Delta\text{-}\mathbf{Rep}(\geq \mathbb{S})$ with its image under the above immersion whenever convenient. The subset of $\Delta\text{-}\mathbf{Rep}(\geq \mathbb{S})$ consisting of the points which correspond to modules with radical layering $\mathbb{S}$ will be denoted by $\Delta\text{-}\mathbf{Rep}\,\mathbb{S}$. In view of part (III) of Lemma 3.2, $\Delta\text{-}\mathbf{Rep}\,\mathbb{S}$ is an open subvariety of $\Delta\text{-}\mathbf{Rep}(\geq \mathbb{S})$.

Next, we consider the effect of conjugation by $\mathrm{GL}(d)$ on the varieties $\Delta\text{-}\mathbf{Rep}(\geq \mathbb{S})$ and $\Delta\text{-}\mathbf{Rep}(\mathbb{S})$.

3.4 (Δ-**Rep**($\geq\mathbb{S}$) under the GL($\boldsymbol{d}$)-action). Viewed as subvarieties of **Rep**$_{\boldsymbol{d}}(\Lambda)$, the varieties Δ-**Rep**($\geq\mathbb{S}$) and Δ-**Rep**($\mathbb{S}$) fail to be stable under the GL($\boldsymbol{d}$)-action in all nontrivial cases. However, each of these varieties carries a conjugation action by the subgroup GL($\mathbb{S}$) of GL($\boldsymbol{d}$) which consists of the sequences $(g_1, \ldots, g_n)$ with the property that each g_i leaves the subspaces $\bigoplus_{j\geq l} \mathcal{K}_{(j,i)}$ invariant for all l. Caveat: the GL($\mathbb{S}$)-action does not separate the isomorphism classes of the pertinent modules in general.

By part (I) of Lemma 3.2, the closure of Δ-**Rep**($\geq\mathbb{S}$) under the GL($\boldsymbol{d}$)-action on **Rep**$_{\boldsymbol{d}}(\Lambda)$ is contained in the closed subvariety $\bigcup_{\mathbb{S}'\geq\mathbb{S}}$ **Rep** $\mathbb{S}'$ of **Rep**$_{\boldsymbol{d}}(\Lambda)$. In fact, in view of the lemma,

$$\mathbf{Rep}\,\mathbb{S} = \mathrm{GL}(\boldsymbol{d}).(\Delta\text{-}\mathbf{Rep}(\mathbb{S})) \subseteq \mathrm{GL}(\boldsymbol{d}).(\Delta\text{-}\mathbf{Rep}(\geq\mathbb{S})) \subseteq \bigcup_{\mathbb{S}'\geq\mathbb{S}} \mathbf{Rep}\,\mathbb{S}'.$$

Either inclusion may be proper. This is obvious for the first. Regarding the second, let $\Lambda = KQ/\langle\beta^2\rangle$, for instance, where

$$Q := 1 \xrightarrow{\alpha} 2 \,\circlearrowright\, \beta.$$

Moreover, take $\mathbb{S} := (S_1^2, S_2^2)$ and $\widetilde{\mathbb{S}} := (S_1^2 \oplus S_2, S_2)$. Then $\widetilde{\mathbb{S}} \geq \mathbb{S}$, but the module $N := S_1^2 \oplus \Lambda e_2$ in **Rep**($\widetilde{\mathbb{S}}$) is not isomorphic to a module in Δ-**Rep**($\geq\mathbb{S}$). Indeed, since $\mathcal{K}_{(0,2)} = 0$ and $\dim\mathcal{K}_{(1,2)} = 2$ in the decomposition of K^4 induced by $\mathbb{S}$, we have $S_2^2 \subseteq \mathrm{soc}\,M$ for all M in Δ-**Rep**($\geq\mathbb{S}$), while this is not the case for N.

3B. *The closure of* **Rep** $\mathbb{S}$ *in* **Rep**$_{\boldsymbol{d}}(\Lambda)$. We start with an elementary lemma characterizing the modules corresponding to the points in Δ-**Rep**($\geq\mathbb{S}$). For a given realizable semisimple sequence $\mathbb{S} = (\mathbb{S}_0, \ldots, \mathbb{S}_L)$ with $\underline{\dim}\,\mathbb{S} = \boldsymbol{d}$, we fix a decomposition of $K^{|\boldsymbol{d}|}$ induced by $\mathbb{S}$ as in Definition 3.1. As we already pointed out, modulo isomorphism of varieties, this choice has no bearing on Δ-**Rep**($\geq\mathbb{S}$).

Definition 3.5 (filtrations governed by $\mathbb{S}$). Let M be a Λ-module. A *filtration of M governed by* $\mathbb{S}$ is any chain of submodules

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_{L+1} = 0$$

such that each factor M_l/M_{l+1} is isomorphic to $\mathbb{S}_l$; in other words, $JM_l \subseteq M_{l+1}$ and $\underline{\dim}\,M_l/M_{l+1} = \underline{\dim}\,\mathbb{S}_l$ for $0 \leq l \leq L$. Filtrations with these properties will also be referred to more briefly as $\mathbb{S}$-*filtrations*.

Lemma and Definition 3.6 (the variety **Filt** $\mathbb{S}$). *Let $\Lambda = KQ/I$ be an arbitrary basic finite-dimensional K-algebra. Moreover, let $\mathbb{S}$ be a semisimple sequence with $\underline{\dim}\,\mathbb{S} = \boldsymbol{d}$. Then the following conditions are equivalent for a Λ-module M:*

(1) *M belongs to $\mathrm{GL}(\boldsymbol{d}).(\Delta\text{-}\mathbf{Rep}(\geq\mathbb{S}))$, the $\mathrm{GL}(\boldsymbol{d})$-stable hull of Δ-$\mathbf{Rep}(\geq\mathbb{S})$.*

(2) *M has a filtration governed by $\mathbb{S}$.*

In particular, $\mathrm{GL}(\boldsymbol{d}).(\Delta\text{-}\mathbf{Rep}(\geq\mathbb{S}))$ *is independent of the choice of a decomposition of $K^{|\boldsymbol{d}|}$ induced by $\mathbb{S}$. Motivated by the above equivalence, we will denote this subvariety of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ by* $\mathbf{Filt}\,\mathbb{S}$.

Proof. (1) $\implies$ (2): Suppose that M is represented by some point $f = (f_\alpha) \in \Delta\text{-}\mathbf{Rep}(\geq\mathbb{S})$. This means that, up to isomorphism, M equals K^d, equipped with the Λ-module structure of Lemma 3.2. In particular, we obtain a filtration of M governed by $\mathbb{S}$ by setting $M_l = \bigoplus_{j \geq l,\, 1 \leq i \leq n} \mathcal{K}_{(j,i)}$.

(2) $\implies$ (1): Given an $\mathbb{S}$-filtration $(M_l)_{0 \leq l \leq L+1}$ of M, we take $M_{(l,i)}$ to be a vector space complement of $e_i M_{l+1}$ in $e_i M_l$ for $0 \leq l \leq L$. Moreover, we set $f = (f_\alpha)_{\alpha \in Q_1}$, where $f_\alpha(x) = \alpha x$ for $x \in M$. Then the decomposition $M = \bigoplus_{0 \leq l \leq L,\, 1 \leq i \leq n} M_{(l,i)}$ satisfies conditions (i)–(iii) of Lemma 3.2, and thus can be shifted to a decomposition $\bigoplus_{0 \leq l \leq L,\, 1 \leq i \leq n} \mathcal{K}_{(l,i)}$ of K^d induced by $\mathbb{S}$ via a suitable family $h = (h_{(l,i)})$ of isomorphisms $h_{(l,i)} : M_{(l,i)} \to \mathcal{K}_{(l,i)}$. We conclude that $hfh^{-1} \in \Delta\text{-}\mathbf{Rep}(\geq\mathbb{S})$ and that $M_{hfh^{-1}} \cong M$. $\qquad\square$

The upcoming remarks (1)–(3) will be tacitly used throughout the sequel.

Remarks 3.7. (1) $\mathbf{Filt}\,\mathbb{S}$ is always nonempty, irrespective of whether $\mathbb{S}$ is realizable. Indeed, the semisimple module $\bigoplus_{0 \leq l \leq L} \mathbb{S}_l$ has a filtration governed by $\mathbb{S}$.

(2) For any $M \in \Lambda\text{-mod}$, the chain $M \supseteq JM \supseteq \cdots \supseteq J^{L+1}M = 0$ is the only filtration of M governed by $\mathbb{S}(M)$; moreover, if $\mathbb{S}'$ is any semisimple sequence governing a filtration of M, then $\mathbb{S}' \leq \mathbb{S}(M)$.

(3) The socle layering $\mathbb{S}^*(M)$ of M governs the socle filtration, provided the traditional indexing of the latter is reversed; i.e., if $\mathbb{S}^*(M) = (\mathbb{S}_0^*, \ldots, \mathbb{S}_m^*, 0, \ldots, 0)$ with $\mathbb{S}_m^* \neq 0$, then the filtration

$$\mathrm{soc}_m M = M \supseteq \mathrm{soc}_{m-1} M \supseteq \cdots \supseteq \mathrm{soc}_0 M = \mathrm{soc}\, M \supseteq 0$$

is governed by the semisimple sequence $(\mathbb{S}_m^*, \ldots, \mathbb{S}_0^*, 0, \ldots, 0)$ (which is not necessarily realizable). In particular, $(\mathbb{S}_m^*, \ldots, \mathbb{S}_0^*, 0, \ldots, 0) \leq \mathbb{S}(M)$.

(4) K. Bongartz pointed out to us that the upcoming Theorem 3.8 may alternatively be derived from a useful result of Steinberg. We state it below, but omit detail. We do fully anchor our own steppingstone to Theorem 3.8 (namely Theorem 3.9), though. The embedding of $\mathrm{GRASS}(\mathbb{S})$ into a flag variety, as specified there, is instrumental in a further analysis of the closure of $\mathrm{GRASS}(\mathbb{S})$ in $\mathrm{GRASS}_{\boldsymbol{d}}(\Lambda)$.

Lemma [Steinberg 1974, Lemma 2, p. 68]. Let V be a quasiprojective variety carrying a morphic action by a connected linear algebraic group G. Moreover, let U be a closed subvariety of V which is stable under the action of some parabolic subgroup of G. Then the G-stable hull $G.U$ of U in V is in turn closed.

Theorem 3.8. *Let Λ be an arbitrary basic finite-dimensional algebra, and let $\mathbb{S}$ be a semisimple sequence in Λ-mod with $\underline{\dim}\,\mathbb{S} = \boldsymbol{d}$. Then the $\mathrm{GL}(\boldsymbol{d})$-stable set $\mathbf{Filt}\,\mathbb{S}$, which consists of the points in $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ encoding modules with $\mathbb{S}$-filtrations, is a closed subvariety of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$.*

In particular, $\overline{\mathbf{Rep}\,\mathbb{S}} \subseteq \mathbf{Filt}\,\mathbb{S}$, meaning that every module in $\overline{\mathbf{Rep}\,\mathbb{S}}$ has an $\mathbb{S}$-filtration.

To prove Theorem 3.8, we switch back and forth between the affine and projective settings, $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ and $\mathrm{GRASS}_{\boldsymbol{d}}(\Lambda)$, using Proposition 2.1 to transfer information from one to the other. Again, we denote by $\boldsymbol{P}$ the Λ-projective cover of $\bigoplus_{1 \le i \le n} S_i^{d_i}$ in whose submodule lattice the points of $\mathrm{GRASS}_{\boldsymbol{d}}(\Lambda)$ are located. We start by establishing a natural embedding of $\mathrm{GRASS}(\mathbb{S})$ into a projective variety consisting of submodule flags $D_{L+1} \subseteq D_L \subseteq \cdots \subseteq D_0 = \boldsymbol{P}$ of $\boldsymbol{P}$ which are governed by $\mathbb{S}$. It is this embedding which makes information about the closure of $\mathrm{GRASS}(\mathbb{S})$ in $\mathrm{GRASS}_{\boldsymbol{d}}(\Lambda)$ more accessible.

Theorem 3.9. *Consider the subset $\mathfrak{U}$ of the partial flag variety $\mathfrak{Flag}(\partial_0, \ldots, \partial_{L+1}, \boldsymbol{P})$ of $\boldsymbol{P}$, where $\partial_i := (\dim \boldsymbol{P} - |\boldsymbol{d}|) + \sum_{l=L+1-i}^{L} |\underline{\dim}\,\mathbb{S}_l|$, consisting of the Λ-submodule flags*

$$0 \subseteq D_{L+1} \subseteq D_L \subseteq \cdots \subseteq D_0 = \boldsymbol{P} \quad \text{with } D_l/D_{l+1} \cong \mathbb{S}_l \text{ for } 0 \le l \le L.$$

Then $\mathfrak{U}$ is closed, and there is a natural embedding of varieties

$$\Phi : \mathrm{GRASS}(\mathbb{S}) \to \mathfrak{U},$$

which induces an isomorphism onto its image.

Proof of Theorem 3.9. Recall that a module N belongs to $\mathrm{GRASS}(\mathbb{S})$, meaning that $N \cong \boldsymbol{P}/C$ with $C \in \mathrm{GRASS}(\mathbb{S})$, precisely when

$$\underline{\dim}\,\mathbb{S}_l = \underline{\dim}\,J^l N/J^{l+1}N = \underline{\dim}(C + J^l\boldsymbol{P})/(C + J^{l+1}\boldsymbol{P})$$

for all eligible l. Set $\boldsymbol{d}^{(L+1)} = \boldsymbol{d}$ and $\boldsymbol{d}^{(l)} = \boldsymbol{d} - \sum_{l \le r \le L} \underline{\dim}\,\mathbb{S}_r$ for $0 \le l \le L$. In particular, we obtain $\mathrm{GRASS}_{\boldsymbol{d}^{(L+1)}}(\Lambda) = \mathrm{GRASS}_{\boldsymbol{d}}(\Lambda)$, and $\mathrm{GRASS}_{\boldsymbol{d}^{(0)}}(\Lambda) = \{\boldsymbol{P}\}$.

Clearly, $\mathfrak{U}$ is a subset of the projective variety

$$\mathrm{GRASS}_{\boldsymbol{d}^{(L+1)}}(\Lambda) \times \mathrm{GRASS}_{\boldsymbol{d}^{(L)}}(\Lambda) \times \cdots \times \mathrm{GRASS}_{\boldsymbol{d}^{(0)}}(\Lambda);$$

namely, $\mathfrak{U}$ consists of those points $(D_{L+1}, \ldots, D_0)$ in the direct product that correspond to flags $D_{L+1} \subseteq D_L \subseteq \cdots \subseteq D_0 = \boldsymbol{P}$ of Λ-submodules of $\boldsymbol{P}$ satisfying

$$J D_l \subseteq D_{l+1} \quad \text{and} \quad \underline{\dim}\, D_l/D_{l+1} = \underline{\dim}\,\mathbb{S}_l \qquad \text{for } 0 \le l \le L. \tag{$\ddagger$}$$

To verify that the set $\mathfrak{U}$ is closed in the given direct product of module Grassmannians, note that the equalities under ($\ddagger$), specifying the dimension vectors of the consecutive quotients D_l/D_{l+1}, are actually automatic; this is due to the placement of the D_l in

$\mathrm{GRASS}_{\boldsymbol{d}^{(l)}}(\Lambda)$, respectively. As for the inclusions under (‡): it is well-known that, given any $f \in \mathrm{End}_K(\boldsymbol{P})$, the requirement "$f(D_l) \subseteq D_{l+1}$ for all l" cuts a closed subset out of the variety

$$\left\{ (D_l) \in \prod_{0 \leq l \leq L+1} \mathrm{GRASS}_{\boldsymbol{d}^{(l)}}(\Lambda) \,\Big|\, D_{l+1} \subseteq D_l \text{ for } 0 \leq l \leq L \right\}$$

of partial submodule flags. Applying this to the linear maps $\boldsymbol{P} \to \boldsymbol{P}$ given by $x \mapsto \alpha x$ for $\alpha \in Q_1$, and investing the fact that the displayed partial flag variety is closed in the given product of Grassmannians, one finds that $\mathfrak{U}$ is indeed closed. In particular, $\mathfrak{U}$ is a projective variety.

We have a natural embedding of $\mathrm{GRASS}(\mathbb{S})$ into $\mathfrak{U}$, namely

$$\Phi : \mathrm{GRASS}(\mathbb{S}) \to \mathfrak{U}, \quad C \mapsto (C + J^{L+1}\boldsymbol{P}, C + J^L \boldsymbol{P}, \ldots, C + J\boldsymbol{P}, C + J^0\boldsymbol{P}),$$

where the leftmost entry $C + J^{L+1}\boldsymbol{P}$ of the sequence equals C, and the rightmost entry equals $\boldsymbol{P}$.

To see that Φ is a morphism, we use the open affine cover $(\mathrm{GRASS}(\sigma))_\sigma$ of $\mathrm{GRASS}(\mathbb{S})$, where σ traces the skeleta with layering $\mathbb{S}$ and $\mathrm{GRASS}(\sigma) \neq \varnothing$. For that purpose, recall the following description of $\mathrm{GRASS}(\sigma)$ from [Huisgen-Zimmermann 2009]. We view the Λ-projective cover P of $\mathbb{S}_0$ as a direct summand of the projective cover $\boldsymbol{P} = \bigoplus_{1 \leq r \leq |\boldsymbol{d}|} \Lambda z_r$ of $\bigoplus_{0 \leq l \leq L} \mathbb{S}_l$, say $P = \bigoplus_{1 \leq r \leq t} \Lambda z_r$. On identifying the top elements z_r of P with those of P_{trunc} (see Definitions 2.2), we retrieve each of the subsets σ of P_{trunc} as a subset of P; as such, σ consists of $|\boldsymbol{d}|$ linearly independent elements of $\boldsymbol{P}$. Define $s := \dim \boldsymbol{P} - |\boldsymbol{d}|$, and let $\mathbf{Schu}(\sigma)$ be the big open Schubert cell of $\mathrm{Gr}(s, \boldsymbol{P})$ consisting of the vector space complements of the subspace $\bigoplus_{p \in \sigma} K p$ in $\boldsymbol{P}$. Then $\mathrm{GRASS}(\sigma) := \mathrm{GRASS}(\mathbb{S}) \cap \mathbf{Schu}(\sigma)$ is open in $\mathrm{GRASS}(\mathbb{S})$, and the union of the $\mathrm{GRASS}(\sigma)$, with σ as specified, equals $\mathrm{GRASS}(\mathbb{S})$; see [Huisgen-Zimmermann 2009, Observation 3.6]. By [ibid., Theorem 3.17], the $\mathrm{GRASS}(\sigma)$ are affine; in fact, they can readily be realized as closed subsets of the K-space $\bigwedge^s \boldsymbol{P}$ relative to the Plücker coordinates $[c_1 \wedge \cdots \wedge c_s]$ of $\mathbf{Schu}(\sigma)$.

Hence it suffices to show that, for each such skeleton σ, the restriction Φ_σ of Φ to $\mathrm{GRASS}(\sigma)$ is a morphism. For $0 \leq j \leq L$, let σ_j be the set of all paths of length j in σ. Enumerate the elements of σ so that increasing indices correspond to weakly decreasing lengths. If $t_l := |\sigma_l| + \cdots + |\sigma_L|$, we thus obtain $\bigsqcup_{l \leq j \leq L} \sigma_j$ in the form

$$\bigsqcup_{l \leq j \leq L} \sigma_j = \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{t_l}\} \quad \text{for } 0 \leq l \leq L.$$

We deduce that, given any K-basis $c_1, \ldots, c_s$ for a point $C \in \mathrm{GRASS}(\sigma)$, the elements $c_1, \ldots, c_s, \boldsymbol{p}_1, \ldots, \boldsymbol{p}_{t_l}$ form a K-basis for $C + J^l \boldsymbol{P}$: indeed, $J^l \boldsymbol{P}$ is generated by the paths in $\boldsymbol{P}$ of the form $q z_r$, where q is a path of length $\geq l$ in KQ

and $r \leq |\boldsymbol{d}|$. Moreover, by the definition of GRASS(σ), $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{t_l}$ induce a basis for $J^l(\boldsymbol{P}/C) = (J^l\boldsymbol{P} + C)/C$. This shows that the restriction Φ_σ sends any point $C \in \text{GRASS}(\sigma)$ to

$$\left([c_1 \wedge \cdots \wedge c_s], [c_1 \wedge \cdots \wedge c_s \wedge \boldsymbol{p}_1 \wedge \cdots \wedge \boldsymbol{p}_{t_L}], \ldots, [c_1 \wedge \cdots \wedge c_s \wedge \boldsymbol{p}_1 \wedge \cdots \wedge \boldsymbol{p}_{t_0}]\right),$$

whence Φ_σ is indeed a morphism.

Finally, we observe that Φ induces an isomorphism onto its image. Indeed, the inverse is the restriction to $\text{Im}(\Phi)$ of the projection onto the leftmost component of the direct product of the GRASS$_{\boldsymbol{d}^{(l)}}(\Lambda)$, namely the restriction of

$$\Psi : \prod_{0 \leq l \leq L+1} \text{GRASS}_{\boldsymbol{d}^{(l)}}(\Lambda) \to \text{GRASS}_{\boldsymbol{d}}(\Lambda), \quad (D_{L+1}, \ldots, D_0) \mapsto D_{L+1},$$

to $\text{Im}(\Phi)$. Therefore $\Phi^{-1} : \text{Im}(\Phi) \to \text{GRASS}(\mathbb{S})$ is a morphism. $\qquad\square$

Proof of Theorem 3.8. We refer to the notation in the proof of Theorem 3.9. Since $\mathfrak{U}$ is a projective variety, so is $\Psi(\mathfrak{U})$. In particular, $\Psi(\mathfrak{U})$ is closed in GRASS$_{\boldsymbol{d}}(\Lambda)$.

By condition (‡) spelled out in the proof of Theorem 3.9, the image $\Psi(\mathfrak{U}) \subseteq$ GRASS$_{\boldsymbol{d}}(\Lambda)$ consists precisely of those points $C \in \text{GRASS}_{\boldsymbol{d}}(\Lambda)$ which have the property that $\boldsymbol{P}/C$ has a filtration governed by $\mathbb{S}$; in particular $\Psi(\mathfrak{U})$ is stable under the $\text{Aut}_\Lambda(\boldsymbol{P})$-action of GRASS$_{\boldsymbol{d}}(\Lambda)$. In light of Lemma and Definition 3.6, Proposition 2.1 thus matches up $\Psi(\mathfrak{U})$ with the GL$(\boldsymbol{d})$-stable subset **Filt** $\mathbb{S}$ of **Rep**$_{\boldsymbol{d}}(\Lambda)$ and tells us that **Filt** $\mathbb{S}$ is in turn closed.

For the final claim, it suffices to observe that **Rep** $\mathbb{S} \subseteq$ **Filt** $\mathbb{S}$. $\qquad\square$

Theorem 3.8 prompts us to introduce a new module invariant, which will turn out to be highly informative in gauging the overlaps among the closed varieties $\overline{\textbf{Rep}\,\mathbb{S}}$.

Definition 3.10 (the module invariant Γ). For $M \in \Lambda\text{-mod}$, let $\Gamma(M)$ denote the number of realizable semisimple sequences which govern some filtration of M.

Corollary 3.11. *The map* $\Gamma_\bullet : \textbf{Rep}_{\boldsymbol{d}}(\Lambda) \to \mathbb{N}$ *sending* x *to* $\Gamma(M_x)$ *is upper semicontinuous.*

In particular, whenever $\mathcal{C}$ *is an irreducible component of some* **Rep** $\mathbb{S}$ *such that* $1 \in \Gamma_\bullet(\mathcal{C})$, *the closure* $\overline{\mathcal{C}}$ *is an irreducible component of* $\textbf{Rep}_{\boldsymbol{d}}(\Lambda)$.

Proof. Let $\mathcal{R}$ be the set of all realizable semisimple sequences with dimension vector $\boldsymbol{d}$. Moreover, for $a \in \mathbb{N}$, let $\mathcal{R}(a)$ be the collection of all those intersections $\bigcap_i \textbf{Filt}(\mathbb{S}^{(i)})$ which involve at least a distinct sequences $\mathbb{S}^{(i)} \in \mathcal{R}$. Then the preimage $\Gamma_\bullet^{-1}([a, \infty))$ is the union of the sets in $\mathcal{R}(a)$. Since each $\textbf{Filt}(\mathbb{S}^{(i)})$ is closed in $\textbf{Rep}_{\boldsymbol{d}}(\Lambda)$ by Theorem 3.8 and $\mathcal{R}(a)$ is finite, the union $\Gamma_\bullet^{-1}([a, \infty))$ is closed. This proves the claim regarding upper semicontinuity.

To justify the final assertion, suppose that $\overline{\mathcal{C}}$ is properly contained in some irreducible component $\mathcal{C}'$ of $\textbf{Rep}_{\boldsymbol{d}}(\Lambda)$. Then $\mathcal{C}'$ is an irreducible component of

some $\overline{\mathbf{Rep}\,\mathbb{S}'}$ with $\mathbb{S}' < \mathbb{S}$. Since $\overline{\mathbf{Rep}\,\mathbb{S}'} \subseteq \mathbf{Filt}(\mathbb{S}')$ by Theorem 3.8, all modules in $\bar{\mathcal{C}}$ have a filtration governed by $\mathbb{S}'$ in this situation, whence $\Gamma(M) > 1$ for all $M \in \mathbf{Rep}\,\mathbb{S}$. □

Now let $D = \mathrm{Hom}_K(-, K) : \Lambda\text{-mod} \to \text{mod-}\Lambda$ be the standard duality. Clearly, $M \in \Lambda\text{-mod}$ contains a descending submodule chain governed by $\mathbb{S} = (\mathbb{S}_0, \ldots, \mathbb{S}_L)$ if and only if $D(M)$ contains an ascending chain $M'_{-1} = 0 \subseteq M'_0 \subseteq \cdots \subseteq M'_L = D(M)$ which is *cogoverned* by $D(\mathbb{S}) = (D(\mathbb{S}_0), \ldots, D(\mathbb{S}_L))$, in the sense that each of the consecutive quotients M'_l / M'_{l-1} is isomorphic to $D(\mathbb{S}_l)$. We define $\mathbf{Cofilt}\,\mathbb{S}'$ to be the subset of $\mathbf{Rep}_d(\Lambda)$ whose points correspond to the modules which are cogoverned by a semisimple sequence $\mathbb{S}'$. The duality $\widehat{D} : \mathbf{Rep}_d(\Lambda\text{-mod}) \to \mathbf{Rep}_d(\text{mod-}\Lambda)$ of [Huisgen-Zimmermann and Shipman 2017, Section 2.C] thus yields the following dual of Theorem 3.8; we spell it out since, in size comparisons of $\overline{\mathcal{C}^{(i)}}$ versus $\overline{\mathcal{C}^{(j)}}$, for irreducible components $\mathcal{C}^{(k)}$ of $\mathbf{Rep}\,\mathbb{S}$, one gains mileage in combining Theorem 3.8 with its dual. (Recall that the process of filtering the irreducible components of $\mathbf{Rep}_d(\Lambda)$ out of

$$\{\bar{\mathcal{C}} \mid \mathcal{C} \text{ is a component of some } \mathbf{Rep}\,\mathbb{S} \text{ with } \underline{\dim}\,\mathbb{S} = d\}$$

rests on comparisons of this ilk.)

Theorem 3.12 (dual of Theorem 3.8). *If $\mathbb{S}^* = (\mathbb{S}_0^*, \ldots, \mathbb{S}_L^*)$ is a semisimple sequence in Λ-mod with dimension vector d, let $\mathbf{Corep}\,\mathbb{S}^*$ (resp. $\mathbf{Cofilt}\,\mathbb{S}^*$) be the set of all points in $\mathbf{Rep}_d(\Lambda)$ which correspond to modules with socle series $\mathbb{S}^*$ (resp. to modules with filtrations cogoverned by $\mathbb{S}^*$).*

Then $\mathbf{Cofilt}(\mathbb{S}^*)$ is a closed subvariety of $\mathbf{Rep}_d(\Lambda)$, and hence $\overline{\mathbf{Corep}\,\mathbb{S}^*} \subseteq \mathbf{Cofilt}\,\mathbb{S}^*$. In particular, if $\mathcal{C}$ is an irreducible component of $\mathbf{Rep}\,\mathbb{S}$ such that, generically, the modules in $\mathcal{C}$ have socle layering $\mathbb{S}^*$, then $\bar{\mathcal{C}} \subseteq \mathbf{Filt}\,\mathbb{S} \cap \mathbf{Cofilt}\,\mathbb{S}^*$. □

We close the section with an example to the effect that, in general, the inclusion $\overline{\mathbf{Rep}\,\mathbb{S}} \subseteq \mathbf{Filt}\,\mathbb{S}$ may be proper and the final implication of Corollary 3.11 need not be reversible. This contrasts with the situation where $\Lambda = \Lambda_{\mathrm{trunc}}$, as we will see in Section 4.

Example 3.13. Consider the quiver Q of Example 2.5 with $r = 2$ and $s = 1$, and set

$$\Lambda = KQ/\langle \beta_1\alpha_2,\ \alpha_2\beta_1, \text{ all paths of length } 4\rangle.$$

Let $d := (2, 2)$, $\mathbb{S} := (S_1, S_2, S_1, S_2)$, and $\mathbb{S}' := (S_1^2, S_2^2, 0, 0)$. Then the varieties $\mathbf{Rep}\,\mathbb{S}$ and $\mathbf{Rep}\,\mathbb{S}'$ are irreducible, and generically their modules have hypergraphs as shown in Figure 1, whence both are contained in $\mathbf{Filt}\,\mathbb{S}$. Clearly, $\overline{\mathbf{Rep}\,\mathbb{S}} \not\subseteq \overline{\mathbf{Rep}\,\mathbb{S}'}$, due to the generic Loewy lengths of the modules in $\mathbf{Rep}\,\mathbb{S}$ and $\mathbf{Rep}\,\mathbb{S}'$. By comparing generic α_2-ranks, one finds, moreover, that $\overline{\mathbf{Rep}\,\mathbb{S}'} \not\subseteq \overline{\mathbf{Rep}\,\mathbb{S}}$. In conclusion, both $\overline{\mathbf{Rep}\,\mathbb{S}}$ and $\overline{\mathbf{Rep}\,\mathbb{S}'}$ are components of $\mathbf{Filt}\,\mathbb{S}$. In fact, both of these closures are

$$
\begin{array}{c}
1 \\
\alpha_1 \Big| \diagdown \\
2 \quad \diagdown \\
\beta_1 \Big| \quad \Big) \alpha_2 \\
1 \quad \diagup \\
\alpha_1 \Big| \diagup \\
2
\end{array}
\qquad
\alpha_1 \left(\begin{array}{c} 1 \\ \Big) \alpha_2 \\ 2 \end{array} \right)
\;\oplus\;
\alpha_1 \left(\begin{array}{c} 1 \\ \Big) \alpha_2 \\ 2 \end{array} \right)
$$
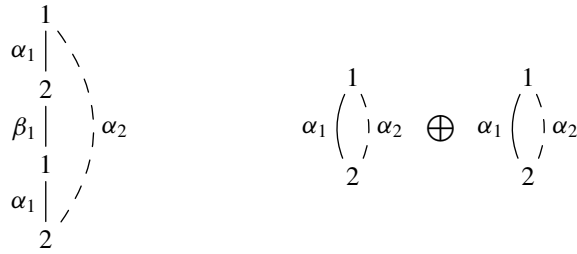
Figure 1. Hypergraphs of generic modules for **Rep** $\mathbb{S}$ (left) and
Rep $\mathbb{S}'$ (right) for Example 3.13.

even irreducible components of $\mathbf{Rep}_d(\Lambda)$, the latter failing to satisfy the sufficient condition of Corollary 3.11. Indeed, $\Gamma(M) = 2$ for all M in $\overline{\mathbf{Rep}\,\mathbb{S}'}$.

It is readily verified that the total number of components of $\mathbf{Rep}_d(\Lambda)$ is three, the remaining component being $\overline{\mathbf{Rep}\,\mathbb{S}''} = \mathbf{Filt}(\mathbb{S}'')$ for $\mathbb{S}'' = (S_2, S_1, S_2, S_1)$. By contrast, on replacing Λ by the associated truncated path algebra Λ_{trunc}, two of the three components of $\mathbf{Rep}_d(\Lambda)$ fuse into a single component of $\mathbf{Rep}_d(\Lambda_{\mathrm{trunc}})$; see Example 6.1(b) below. $\qquad\square$

4. The main results for truncated Λ

Throughout this section, Λ stands for a truncated path algebra of Loewy length $L + 1$, i.e., $\Lambda = \Lambda_{\mathrm{trunc}}$. In particular, the irreducible components of $\mathbf{Rep}_d(\Lambda)$ are among the $\overline{\mathbf{Rep}\,\mathbb{S}}$, where $\mathbb{S}$ traces the d-dimensional realizable semisimple sequences. The upcoming theory characterizes these components in terms of their generic radical layerings $\mathbb{S}$ (or, equivalently, in terms of their *generic modules* in the sense of Section 5 below). As in the special cases already mastered — the local case and that of an acyclic quiver Q — the classification may be implemented on a computer; see Section 5B. However, the general algorithm is considerably more labor-intensive than the Θ-test which applies to the local and acyclic cases.

As we will recall in Section 5, the generic properties of the modules in any component $\overline{\mathbf{Rep}\,\mathbb{S}}$ may be accessed via a single generic module $G(\mathbb{S})$. A key asset of the truncated situation lies in the fact that such a module $G(\mathbb{S})$ is available on sight from $\mathbb{S}$; detail will follow in Section 5A below.

Moreover, it is particularly easy to recognize realizability of semisimple sequences over truncated path algebras. We recall the following from [Huisgen-Zimmermann 2016, Criterion 3.2]:

Realizability Criterion 4.1. Let $B = (B_{ij})$ be the adjacency matrix of Q, i.e., B_{ij} is the number of arrows from e_i to e_j. Then $\mathbb{S} = (\mathbb{S}_0, \ldots, \mathbb{S}_L)$ is realizable if and only if $\underline{\dim}\,\mathbb{S}_l \leq (\underline{\dim}\,\mathbb{S}_{l-1}) \cdot B$ for all $1 \leq l \leq L$; the latter, in turn, is equivalent to realizability of the two-term sequences $(\mathbb{S}_l, \mathbb{S}_{l+1})$ in (Λ/J^2)-mod for $l < L$. $\qquad\square$

In more intuitive terms: $\mathbb{S}$ is realizable if and only if there exists an abstract skeleton with layering $\mathbb{S}$. Moreover, note that in the positive case, any such skeleton belongs to the *generic* set of skeleta of the modules in $\overline{\textbf{Rep}\,\mathbb{S}}$.

Next, we find that the description of $\Delta\text{-}\textbf{Rep}(\geq\mathbb{S})$ may be simplified in the truncated situation, in that requirement (iii) of Lemma 3.2 is now void.

Observation 4.2 ($\Delta\text{-}\textbf{Rep}(\geq\mathbb{S})$ is an affine space). Referring to the decomposition of K^d induced by $\mathbb{S}$ in Definition 3.1, we obtain that $\Delta\text{-}\textbf{Rep}(\geq\mathbb{S})$ consists of those points $f = (f_\alpha)_\alpha \in (\text{End}_\Lambda(K^d))^{Q_1}$ which satisfy the following conditions: for any arrow α from e_i to e_j,

- $f_\alpha(\mathcal{K}_{(l,r)}) = 0$ for all $r \neq i$, and
- $f_\alpha(\mathcal{K}_{(l,i)}) \subseteq \bigoplus_{l+1 \leq m \leq L} \mathcal{K}_{(m,j)}.$

In particular, $\Delta\text{-}\textbf{Rep}(\geq\mathbb{S})$ is a full affine space in this situation. Indeed, the image of the closed immersion $\Delta\text{-}\textbf{Rep}(\geq\mathbb{S}) \hookrightarrow \textbf{Rep}_d(\Lambda)$, which we presented in 3.3, consists of *all* sequences of $d_i \times d_i$ matrices of the described lower triangular format. Consequently, $\textbf{Filt}\,\mathbb{S}$, being a morphic image of $\text{GL}(\textbf{d}) \times \Delta\text{-}\textbf{Rep}(\geq\mathbb{S})$, is irreducible as well.

This observation, in turn, allows us to derive a full characterization of the modules in $\overline{\textbf{Rep}\,\mathbb{S}}$ from Theorem 3.8.

Theorem 4.3. *Suppose Λ is a truncated path algebra and $\mathbb{S}$ a realizable semisimple sequence. Then*

$$\overline{\textbf{Rep}\,\mathbb{S}} = \textbf{Filt}\,\mathbb{S}.$$

In other words, a module M belongs to $\overline{\textbf{Rep}\,\mathbb{S}}$ precisely when M has a filtration governed by $\mathbb{S}$.

Dually, $\overline{\textbf{Corep}\,\mathbb{S}^*} = \textbf{Cofilt}\,\mathbb{S}^*$, where $\mathbb{S}^*$ is the generic socle layering of the modules in $\textbf{Rep}\,\mathbb{S}$. If $\overline{\textbf{Rep}\,\mathbb{S}}$ is an irreducible component of $\textbf{Rep}_d(\Lambda)$, then

$$\overline{\textbf{Corep}\,\mathbb{S}^*} = \textbf{Cofilt}\,\mathbb{S}^* = \textbf{Filt}\,\mathbb{S} = \overline{\textbf{Rep}\,\mathbb{S}}.$$

Proof. Concerning the first equality: In light of Observation 4.2, the variety $\Delta\text{-}\textbf{Rep}(\geq\mathbb{S})$ is irreducible. Therefore the open subset $\Delta\text{-}\textbf{Rep}\,\mathbb{S}$ is dense in it, meaning that the closure $\overline{\Delta\text{-}\textbf{Rep}\,\mathbb{S}}$ in $\textbf{Rep}_d(\Lambda)$ contains $\Delta\text{-}\textbf{Rep}(\geq\mathbb{S})$. Moreover, $\Delta\text{-}\textbf{Rep}\,\mathbb{S} \subseteq \Delta\text{-}\textbf{Rep}(\geq\mathbb{S})$ by construction, whence we obtain

$$\Delta\text{-}\textbf{Rep}(\geq\mathbb{S}) \subseteq \overline{\Delta\text{-}\textbf{Rep}\,\mathbb{S}} \subseteq \overline{\textbf{Rep}\,\mathbb{S}}.$$

Given that $\overline{\textbf{Rep}\,\mathbb{S}}$ is $\text{GL}(\textbf{d})$-stable, it follows that $\textbf{Filt}\,\mathbb{S} \subseteq \overline{\textbf{Rep}\,\mathbb{S}}$ due to Lemma and Definition 3.6. The reverse inclusion was established in Theorem 3.8. The second assertion follows by duality (see Theorem 3.12 and [Huisgen-Zimmermann and Shipman 2017, Corollary 3.4.b]).

In particular, duality guarantees that the varieties $\overline{\mathbf{Corep}\,\mathbb{S}^*}$ are again irreducible. For arbitrary $\mathbb{S}$ we find, moreover, that $\mathbf{Rep}\,\mathbb{S} \subseteq \overline{\mathbf{Corep}\,\mathbb{S}^*}$, since the modules in a dense open subset of $\mathbf{Rep}\,\mathbb{S}$ have socle layering $\mathbb{S}^*$. Therefore, $\overline{\mathbf{Rep}\,\mathbb{S}} = \overline{\mathbf{Corep}\,\mathbb{S}^*}$ whenever $\overline{\mathbf{Rep}\,\mathbb{S}}$ is an irreducible component of $\mathbf{Rep}_d(\Lambda)$. □

The following consequence, addressing the relative sizes of the closures $\overline{\mathbf{Rep}\,\mathbb{S}}$, is now immediate. It was independently obtained by I. Shipman with different methods; he also developed an algorithm for checking the considered inclusion via matrices of dimension vectors (Shipman, personal communication, 2016). Algorithmic counterparts to the upcoming Corollary 4.4 and Theorem 4.5 will be addressed in Section 5B.

Corollary 4.4 (comparing the varieties $\overline{\mathbf{Rep}\,\mathbb{S}}$). *Let Λ be a truncated path algebra. Moreover, suppose that $\mathbb{S}$ and $\mathbb{S}'$ are realizable semisimple sequences with the same dimension vector. Then $\overline{\mathbf{Rep}\,\mathbb{S}} \subseteq \overline{\mathbf{Rep}\,\mathbb{S}'}$ if and only if* (generically) *the modules in $\overline{\mathbf{Rep}\,\mathbb{S}}$ have filtrations governed by $\mathbb{S}'$.* □

The upper semicontinuous map $\Gamma_\bullet : \mathbf{Rep}_d(\Lambda) \to \mathbb{N}$ of Corollary 3.11 detects all irreducible components of $\mathbf{Rep}_d(\Lambda)$. Indeed, $\mathbb{S}$ is the generic radical layering of an irreducible component of $\mathbf{Rep}_d(\Lambda)$ if and only if $\Gamma_\bullet$ attains the value 1 on $\mathbf{Rep}\,\mathbb{S}$. We record this as follows.

Theorem 4.5. *Let Λ be a truncated path algebra. If $\mathbb{S}^{(1)}, \ldots, \mathbb{S}^{(m)}$ are the distinct d-dimensional semisimple sequences $\mathbb{S}$ with $1 \in \Gamma_\bullet(\mathbf{Rep}\,\mathbb{S})$, then*

$$\mathbf{Filt}(\mathbb{S}^{(1)}) = \overline{\mathbf{Rep}\,\mathbb{S}^{(1)}}, \quad \ldots, \quad \mathbf{Filt}(\mathbb{S}^{(m)}) = \overline{\mathbf{Rep}\,\mathbb{S}^{(m)}}$$

are the distinct irreducible components of $\mathbf{Rep}_d(\Lambda)$.

Proof. Suppose $\mathbb{S}$ is a realizable d-dimensional semisimple sequence. If $1 \in \Gamma_\bullet(\mathbf{Rep}\,\mathbb{S})$, then $\mathbf{Rep}\,\mathbb{S} \not\subseteq \mathbf{Filt}(\mathbb{S}') = \overline{\mathbf{Rep}\,\mathbb{S}'}$ for any semisimple sequence $\mathbb{S}' \neq \mathbb{S}$, whence $\overline{\mathbf{Rep}\,\mathbb{S}}$ is an irreducible component of $\mathbf{Rep}_d(\Lambda)$.

If, on the other hand, $1 \notin \Gamma_\bullet(\mathbf{Rep}\,\mathbb{S})$, then every module in $\mathbf{Rep}\,\mathbb{S}$ is contained in some variety $\mathbf{Filt}\,\mathbb{S}'$, where $\mathbb{S}'$ is a realizable semisimple sequence different from $\mathbb{S}$. Therefore,

$$\overline{\mathbf{Rep}\,\mathbb{S}} \subseteq \bigcup_{\substack{\mathbb{S}'\ \text{realizable} \\ \mathbb{S}' \neq \mathbb{S}}} \mathbf{Filt}\,\mathbb{S}' = \bigcup_{\substack{\mathbb{S}'\text{realizable} \\ \mathbb{S}' \neq \mathbb{S}}} \overline{\mathbf{Rep}\,\mathbb{S}'},$$

the final equality being part of Theorem 4.3. Irreducibility of $\overline{\mathbf{Rep}\,\mathbb{S}}$ thus implies $\overline{\mathbf{Rep}\,\mathbb{S}} \subseteq \overline{\mathbf{Rep}\,\mathbb{S}'}$ for some $\mathbb{S}' \neq \mathbb{S}$, which shows that $\overline{\mathbf{Rep}\,\mathbb{S}}$ fails to be maximal irreducible. □

5. Applications of Section 4: Generic modules for
the components over truncated path algebras

Barring Example 5.2(b), Λ *will, throughout this section, stand for a truncated path algebra of Loewy length $L + 1$. Moreover, $\boldsymbol{d}$ will be a dimension vector of Λ.*

   If one extends the base field K of Λ to an algebraically closed field of infinite transcendence degree over its prime field K_0, neither the description of the components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ nor the generic properties of their modules will be affected; see [Huisgen-Zimmermann and Shipman 2017, Section 2.B]. This means that, in developing a generic representation theory for the irreducible components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$, one does not lose generality in assuming that $\mathrm{trdeg}(K : K_0) = \infty$.

5A. *Generic modules.* Assume that K has infinite transcendence degree over K_0, and let $\mathbb{S}$ be a realizable $\boldsymbol{d}$-dimensional semisimple sequence. Given that $\Lambda = \Lambda_{\mathrm{trunc}}$, we will denote the coordinatized projective Λ_{trunc}-projective cover $P_{\mathrm{trunc}} = \bigoplus_{1 \le r \le t} \Lambda z_r$ of $\mathbb{S}_0$ (see Section 2) more simply by P.

   Let σ be any skeleton with layering $\mathbb{S}$. Then the following module $G = G(\mathbb{S})$ is generic for $\overline{\mathbf{Rep}}\,\mathbb{S}$ in the strict sense of [Babson et al. 2009, Definition 4.2]:

$$G = P/C, \quad \text{where } C = \sum_{\boldsymbol{q} \ \sigma\text{-critical}} \Lambda\left(\boldsymbol{q} - \sum_{\boldsymbol{p} \in \sigma_{\boldsymbol{q}}} c_{\boldsymbol{q},\boldsymbol{p}}\boldsymbol{p}\right)$$

for some family $(c_{\boldsymbol{q},\boldsymbol{p}})_{\boldsymbol{q} \ \sigma\text{-critical}, \ \boldsymbol{p} \in \sigma_{\boldsymbol{q}}}$ of scalars which is algebraically independent over K_0. That G is *generic* means that G has all those generic properties of the modules in $\overline{\mathbf{Rep}}\,\mathbb{S}$ which are invariant under Morita self-equivalences $\Lambda\text{-mod} \to \Lambda\text{-mod}$ induced by automorphisms of K over $\overline{K_0}$. Moreover, G is unique relative to this property, up to such a Morita self-equivalence. We refer to [ibid., Theorem 5.12], and to [ibid., Section 4] for a more general statement addressing arbitrary path algebras modulo relations.

Filtrations of generic modules. In particular, the preceding comments ensure that tests for semisimple sequences which generically govern filtrations of the modules in $\overline{\mathbf{Rep}}\,\mathbb{S}$ may be confined to "the" generic module $G = G(\mathbb{S})$.

   Caveat: Suppose G is a generic module for an irreducible component of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$. While the combination of Corollary 3.11 and Theorem 4.5 guarantees that the radical layering $\mathbb{S}(G)$ is the only *realizable* semisimple sequence to govern a filtration of G, there will in general be further, nonrealizable, sequences governing suitable filtrations. For instance, let Q be the quiver $4 \longleftarrow 1 \overset{\alpha}{\longrightarrow} 2 \longrightarrow 3$ and Λ any truncated path algebra based on Q. If $\boldsymbol{d} = (0, 1, 1, 1)$, then $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ is irreducible with generic module $G = \Lambda\alpha \oplus S_4$ for any truncation Λ of KQ. In particular, $\mathbb{S}(G) = (S_2 \oplus S_4, S_3)$ is the only realizable semisimple sequence governing all modules with dimension vector $\boldsymbol{d}$. If Λ has Loewy length 2, the sequence $(S_2, S_3 \oplus S_4)$ also governs a

filtration of G; if the Loewy length of Λ is 3, then (S_2, S_3, S_4) and (S_4, S_2, S_3) are additional (nonrealizable) semisimple sequences governing filtrations of G.

5B. *Algorithmic aspect of Corollary 4.4 and Theorem 4.5.* Section 5A tells us that, for any two realizable $\boldsymbol{d}$-dimensional semisimple sequences $\mathbb{S}$ and $\mathbb{S}'$, we have

$$\mathbf{Rep}\,\mathbb{S} \subseteq \overline{\mathbf{Rep}\,\mathbb{S}'} \quad \Longleftrightarrow \quad G(\mathbb{S}) \in \mathbf{Filt}\,\mathbb{S}'.$$

From Lemma and Definition 3.6 we know, moreover, that $\mathbf{Filt}\,\mathbb{S}'$ is the $\mathrm{GL}(\boldsymbol{d})$-stable hull of $\Delta\text{-}\mathbf{Rep}(\geq \mathbb{S}')$. Hence, if the point $(G_\alpha)_{\alpha \in Q_1} \in \mathbf{Rep}\,\mathbb{S}$ represents the isomorphism class of $G(\mathbb{S})$, the question of whether $G(\mathbb{S})$ lies in $\mathbf{Filt}\,\mathbb{S}'$ boils down to the question of whether the matrices G_α are "simultaneously" similar (i.e., similar by way of a single element of $\mathrm{GL}(\boldsymbol{d})$) to matrices having the lower triangular format F_α characterizing the points in $\Delta\text{-}\mathbf{Rep}(\geq \mathbb{S}')$. This format is spelled out in 3.3.

Given that there are only finitely many $\boldsymbol{d}$-dimensional semisimple sequences to be compared, this means in particular that the decision of whether or not $\overline{\mathbf{Rep}\,\mathbb{S}}$ is a component of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ is algorithmic.

5C. *Interconnections among the components.* The following statement rephrases a result of Crawley-Boevey and Schröer [2002, Theorem 1.1] in terms of generic modules: if G is a generic module for an irreducible component $\mathcal{C}$ of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ and $G = \bigoplus_{1 \leq j \leq s} G_j$ is a decomposition into direct summands, then each G_j is generic for an irreducible component of $\mathbf{Rep}_{\underline{\dim} G_j}(\Lambda)$. Over a truncated path algebra, this result may be sharpened as follows.

Call a submodule M of N *layer-stably embedded* in N if $J^l M = M \cap J^l N$ for all $l \leq L$. As a consequence of Theorem 4.5, we obtain:

Theorem 5.1. *Suppose that Λ is a truncated path algebra and $\overline{\mathbf{Rep}\,\mathbb{S}}$ is an irreducible component of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ with generic module G. If $G' \subseteq G$ is a layer-stably embedded submodule of G with $\mathbb{S}(G') = \mathbb{S}'$ and $\underline{\dim} G' = \boldsymbol{d}'$, then $\overline{\mathbf{Rep}\,\mathbb{S}'}$ is an irreducible component of $\mathbf{Rep}_{\boldsymbol{d}'}(\Lambda)$ with generic module G'.*

Proof. Let $H := G'$ be layer-stably embedded in G. From [Huisgen-Zimmermann and Shipman 2017, Corollary 3.2] we know that H is generic for $\mathbf{Rep}\,\mathbb{S}' = \mathbf{Rep}\,\mathbb{S}(H)$. Thus only the status of $\overline{\mathbf{Rep}\,\mathbb{S}'}$ as a potential component of $\mathbf{Rep}_{\boldsymbol{d}'}(\Lambda)$ needs to be addressed.

Assume that $\mathbf{Rep}\,\mathbb{S}'$ fails to be an irreducible component of $\mathbf{Rep}_{\boldsymbol{d}'}(\Lambda)$. In view of Theorem 4.5, this means that H has a filtration governed by some realizable semisimple sequence $\mathbb{S}''$ which is strictly smaller than $\mathbb{S}'$, say $H = H_0 \supseteq H_1 \supseteq \cdots \supseteq H_L \supseteq H_{L+1} = 0$; by definition, $\mathbb{S}''_l = H_l/H_{l+1}$. We aim at constructing a submodule filtration $G = G_0 \supseteq \cdots \supseteq G_L \supseteq 0$ which, in turn, is governed by a realizable semisimple sequence $\widehat{\mathbb{S}}$ strictly smaller than $\mathbb{S}$. Another application of Theorem 4.5

will then show that $\overline{\mathbf{Rep}\,\mathbb{S}}$ is not an irreducible component of $\mathbf{Rep}_d(\Lambda)$, contrary to our hypothesis.

For $l \leq L$, let $\pi_l : G \to G/J^{l+1}G$ denote the quotient map. We recursively choose submodules U_l of J^lG such that

$$J^{l+1}G \subseteq U_l \subseteq J^lG, \quad U_l \subseteq JU_{l-1}, \quad J^lG/J^{l+1}G = \pi_l(J^lH) \oplus \pi_l(U_l). \quad (5\text{-}1)$$

First, semisimplicity of G/JG implies that $G/JG = \pi_0(H) \oplus \pi_0(U_0)$ for some $U_0 \subseteq G$, and since U_0 may be replaced by $U_0 + JG$, there is no loss of generality in assuming that $JG \subseteq U_0$. If $U_0, \ldots, U_k$ for some $k < L$ have been chosen so as to satisfy (5-1), we have $J^kG = J^kH + U_k$ by Nakayama's lemma, whence $J^{k+1}G = J^{k+1}H + JU_k$. Consequently, $J^{k+1}G/J^{k+2}G = \pi_{k+1}(J^{k+1}H) \oplus \pi_{k+1}(U_{k+1})$ for some $U_{k+1} \subseteq JU_k$. On replacing U_{k+1} by $U_{k+1} + J^{k+2}G$, we obtain (5-1) for $l = k+1$. Finally, set $U_{L+1} := 0$.

Now define $G_l := H_l + U_l$ for $l \leq L+1$. That the consecutive factors of the sequence

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_L \supseteq G_{L+1} = 0 \qquad (5\text{-}2)$$

are semisimple, i.e., $JG_l \subseteq G_{l+1}$ for $l \leq L$, is straightforward from our construction. Indeed, $JH_l \subseteq H_{l+1}$ and

$$JU_l \subseteq J^{l+1}G = J^{l+1}H + U_{l+1} \subseteq H_{l+1} + U_{l+1}.$$

Let $\widehat{\mathbb{S}}$ be the semisimple sequence governing the filtration (5-2). Remark 3.7(2) tells us that $\widehat{\mathbb{S}} \leq \mathbb{S}$.

Suppose m is minimal with the property that $J^mH \subsetneqq H_m$. Such an index m exists, since $\mathbb{S}'' < \mathbb{S}'$. Then $m \geq 1$. Using layer-stability of H in G, we derive

$$J^mG = J^mH + U_m \subsetneqq H_m + U_m = G_m.$$

On the other hand, $G_l = J^lG$ for $l < m$, so that the first discrepancy between the downward filtration (5-2) and the radical filtration of G occurs at $l = m$. More specifically,

$$\underline{\dim}\,\widehat{\mathbb{S}}_{m-1} = \underline{\dim}(G_{m-1}/G_m) < \underline{\dim}(G_{m-1}/J^mG) = \underline{\dim}\,J^{m-1}G/J^mG = \underline{\dim}\,\mathbb{S}_{m-1}.$$

This yields $\widehat{\mathbb{S}} < \mathbb{S}$.

It remains to be verified that $\widehat{\mathbb{S}}$ is realizable. To do so, we make repeated use of Realizability Criterion 4.1. Again, $\boldsymbol{B}$ is the adjacency matrix of Q. First we note that realizability of $\mathbb{S}$ and $\mathbb{S}''$ entails

$$\begin{aligned}
\underline{\dim}\,J^lG/J^{l+1}G &\leq (\underline{\dim}\,J^{l-1}G/J^lG) \cdot \boldsymbol{B}, \\
\underline{\dim}\,H_l/H_{l+1} &\leq (\underline{\dim}\,H_{l-1}/H_l) \cdot \boldsymbol{B}
\end{aligned} \qquad (5\text{-}3)$$

for $1 \leq l \leq L$. Therefore $\underline{\dim}\,G_l/G_{l+1} \leq (\underline{\dim}\,G_{l-1}/G_l) \cdot \boldsymbol{B}$ for $1 \leq l \leq m-2$.

Invoking (5-1), we find that, for $1 \leq l \leq L$,

$$G_l/J^{l+1}G = (H_l + J^{l+1}G)/J^{l+1}G \oplus (U_l/J^{l+1}G)$$

and

$$G_{l+1}/J^{l+1}G = (H_{l+1} + J^{l+1}G)/J^{l+1}G,$$

where the sum in the first equation is direct because $H_l \cap U_l \subseteq H \cap J^l G = J^l H$ implies $H_l \cap U_l = J^l H \cap U_l \subseteq J^{l+1}G$. We also have

$$(H_l + J^{l+1}G)/(H_{l+1} + J^{l+1}G) \cong H_l/H_{l+1},$$

since layer-stability of H in G guarantees that $H_l \cap J^{l+1}G \subseteq J^{l+1}H \subseteq H_{l+1}$. Consequently,

$$G_l/G_{l+1} \cong (H_l/H_{l+1}) \oplus (U_l/J^{l+1}G) \quad \text{for } 1 \leq l \leq L. \tag{5-4}$$

Since $U_l \subseteq JU_{l-1}$ we obtain, moreover, that

$$\underline{\dim} \, U_l/J^{l+1}G \leq \underline{\dim} \, JU_{l-1}/J(J^l G) \leq (\underline{\dim} \, U_{l-1}/J^l G) \cdot \boldsymbol{B} \quad \text{for } 1 \leq l \leq L. \tag{5-5}$$

Combining (5-5) with (5-3) and (5-4) yields $\underline{\dim} \, G_l/G_{l+1} \leq (\underline{\dim} \, G_{l-1}/G_l) \cdot \boldsymbol{B}$ for $1 \leq l \leq L$, which shows that $\widehat{\mathbb{S}}$ is realizable as required. $\qquad \square$

The following examples demonstrate: (a) that the conclusion of Theorem 5.1 does not extend to arbitrary top-stably embedded submodules G' of G, i.e., to submodules G' satisfying only $JG' = G' \cap JG$, and (b) that Theorem 5.1 has no analogue for nontruncated Λ in general.

Examples 5.2 (demonstrating the sharpness of Theorem 5.1). Consider the quivers
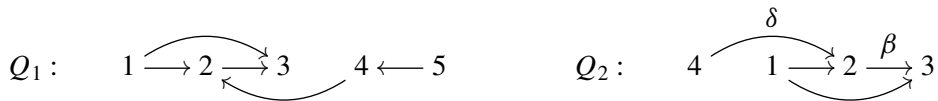
(a) Let Λ be the truncated path algebra of Loewy length 3 based on the quiver Q_1. For $\boldsymbol{d} = (1, 1, 1, 1, 1)$, the variety $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ has two irreducible components, with generic radical layerings

$$\mathbb{S}^{(1)} := (S_1 \oplus S_5, S_3 \oplus S_4, S_2) \quad \text{and} \quad \mathbb{S}^{(2)} := (S_1 \oplus S_5, S_2 \oplus S_4, S_3)$$

and generic modules G_1 and G_2 as graphed below:
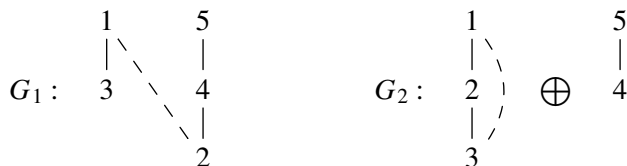
Clearly, the top-stably embedded submodule G' of G_1 generated by any element $z = e_1 z \in G_1$ has dimension vector $\boldsymbol{d}' := (1, 1, 1, 0, 0)$. On the other hand, the sequence $\mathbb{S}(G') = (S_1, S_2 \oplus S_3, 0)$ fails to be the generic radical layering of an irreducible component of $\mathbf{Rep}_{\boldsymbol{d}'}(\Lambda)$, the latter variety being irreducible with uniserial generic modules.

(b) Now let $\Lambda = KQ_2/\langle\beta\delta\rangle$ and $\boldsymbol{d} := (1, 1, 1, 1)$. Then, again, $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ consists of two irreducible components. Their generic modules are graphed below:

$$
\begin{array}{ccc}
G_1: &
\begin{array}{ccc}
& 1 & \quad 4 \\
& \diagup \ \diagdown & \diagup \\
3 & \quad 2 &
\end{array}
&
\qquad
G_2:
\begin{array}{ccc}
1 & & 4 \\
| \ \diagdown & & \bullet \\
2 \ \backslash & \oplus & \\
| \ \diagup & & \\
3 & &
\end{array}
\end{array}
$$
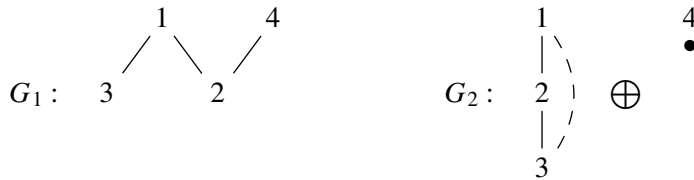
The submodule G' of G_1 generated by any element $z = e_1 z \in G_1$ has dimension vector $\boldsymbol{d}' := (1, 1, 1, 0)$ and is layer-stably embedded in G_1 this time. Nonetheless, $\overline{\mathbf{Rep}\,\mathbb{S}(G')}$ fails to be an irreducible component of $\mathbf{Rep}_{\boldsymbol{d}'}(\Lambda)$. Indeed, once again, $\mathbf{Rep}_{\boldsymbol{d}'}(\Lambda)$ is irreducible and its generic modules are uniserial. $\qquad\square$

6. Examples illustrating the theory. The interplay $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda) \longleftrightarrow \mathbf{Rep}_{\boldsymbol{d}}(\Lambda_{\mathrm{trunc}})$

6A. *Illustrations of the truncated case.* *In this subsection, Λ denotes a truncated path algebra.*

In sifting the radical layerings of the components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ out of the set $\mathbf{Seq}(\boldsymbol{d})$, it is computationally advantageous to supplement $\Gamma_{\bullet}$ by the map Θ of equation (1-1), or by the upgraded map Θ^+ to be introduced next.

Example 4.8 in [Huisgen-Zimmermann 2016] shows that Θ fails to detect all irreducible components in the general truncated case. However, in that instance (as in many others), supplementing Θ by path ranks compensates for the blind spots of Θ. Here the *path rank* of a finite-dimensional Λ-module M is the tuple $(\dim pM)_p \in \mathbb{Z}^\tau$, where τ is the set of paths in $KQ \setminus I$. Set $f(M) = (-\dim pM)_p$, and let $f^*(M)$ be the negative of the path rank of the right Λ-module $D(M)$. Clearly, the map

$$\Theta^+ : \mathbf{Rep}_{\boldsymbol{d}}(\Lambda) \to \mathbf{Seq}(\boldsymbol{d}) \times \mathbf{Seq}(\boldsymbol{d}) \times \mathbb{Z}^\tau \times \mathbb{Z}^\tau,$$

$$x \mapsto (\mathbb{S}(M_x), \mathbb{S}^*(M_x), f(M_x), f^*(M_x)),$$

is in turn upper semicontinuous. Therefore, it is generically constant on the varieties $\mathbf{Rep}\,\mathbb{S}$. In particular, those closures $\overline{\mathbf{Rep}\,\mathbb{S}}$ on which Θ^+ attains its minimal values (relative to the componentwise partial order on the codomain) are components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$. Yet, part (c) of the next example attests to the fact that the augmented

upper semicontinuous map Θ^+ still leaves certain components undetected in general. We use $\Gamma_\bullet$ to fill in what Θ^+ fails to pick up.

Example 6.1. Let Λ be the truncated path algebra of Loewy length 4 based on the quiver Q of Example 2.5, and take $\boldsymbol{d} = (2, 2)$. The semisimple sequences which are in the running as potential generic radical layerings of components of $\mathbf{Rep}_d(\Lambda)$ are

$$\mathbb{S}^{(1)} = (S_1, S_2, S_1, S_2), \quad \mathbb{S}^{(4)} = (S_2, S_1^2, S_2, 0), \quad \mathbb{S}^{(7)} = (S_1 \oplus S_2, S_1 \oplus S_2, 0, 0),$$

$$\mathbb{S}^{(2)} = (S_2, S_1, S_2, S_1), \quad \mathbb{S}^{(5)} = (S_1^2, S_2^2, 0, 0), \quad \mathbb{S}^{(8)} = (S_1 \oplus S_2, S_1, S_2, 0),$$

$$\mathbb{S}^{(3)} = (S_1, S_2^2, S_1, 0), \quad \mathbb{S}^{(6)} = (S_2^2, S_1^2, 0, 0), \quad \mathbb{S}^{(9)} = (S_1 \oplus S_2, S_2, S_1, 0).$$

The list excludes the sequences which are not realizable for any choice of r and s, such as $(S_1, S_1 \oplus S_2, S_2, 0)$ and $(S_1, S_2, S_1 \oplus S_2, 0)$, as well as the radical layering $\mathbb{S}^{(0)}$ of the semisimple module, given that $\mathbf{Rep}\,\mathbb{S}^{(0)}$ is contained in all nonempty varieties $\overline{\mathbf{Rep}\,\mathbb{S}}$. Except for $\mathbb{S}^{(3)}$ and $\mathbb{S}^{(4)}$, all sequences on the list are realizable for arbitrary positive integers r, s.

Theorem 4.5 allows us to discard $\mathbb{S}^{(j)}$ for $j = 7, 8, 9$ from the list of possible generic radical layerings of irreducible components: indeed, the modules in $\mathbf{Rep}\,\mathbb{S}^{(7)}$ are generically decomposable, which makes it evident that they have filtrations governed by both $\mathbb{S}^{(1)}$ and $\mathbb{S}^{(2)}$. Any generic module G_8 for $\mathbf{Rep}\,\mathbb{S}^{(8)}$ has hypergraph
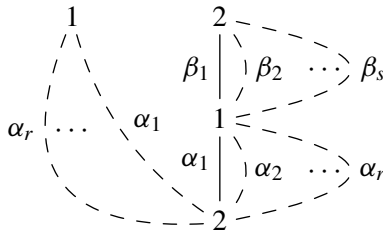
Clearly, G_8 is generated by elements $z_1 = e_1 z_1$ and $z_2 = e_2 z_2$, and the following submodule chain is governed by $\mathbb{S}^{(1)}$:

$$G_8 \supseteq \Lambda z_2 \supseteq \Lambda \beta_1 z_2 \supseteq \Lambda \alpha_1 \beta_1 z_2 \supseteq 0.$$

Consequently, $\mathbf{Rep}\,\mathbb{S}^{(8)} \subseteq \mathbf{Filt}\,\mathbb{S}^{(1)}$ by Corollary 4.4. An analogous argument shows $\mathbf{Rep}\,\mathbb{S}^{(9)} \subseteq \mathbf{Filt}\,\mathbb{S}^{(2)}$.

On the other hand, $\mathcal{C}_j := \overline{\mathbf{Rep}\,\mathbb{S}^{(j)}}$ for $j = 1, 2$ are components of $\mathbf{Rep}_d(\Lambda)$ for all choices of $r, s \geq 1$ by Theorem 4.5, since $\Gamma(U) = 1$ for any uniserial module U. Hence only the sequences $\mathbb{S}^{(j)}$ for $3 \leq j \leq 6$ require discussion by cases. We consider only the cases when $r \geq s$, due to the symmetry of the quiver Q.

(a) Let $r = s = 1$. Then $\mathbf{Rep}_d(\Lambda)$ has precisely two irreducible components, namely $\mathcal{C}_j = \overline{\mathbf{Rep}\,\mathbb{S}^{(j)}}$ for $j = 1, 2$. We rule out the remaining sequences. First, $\mathbb{S}^{(3)}$ and $\mathbb{S}^{(4)}$ fail to be realizable when $r = s = 1$. Generically, the modules in

Rep $\mathbb{S}^{(5)}$ are direct sums of two uniserials with radical layering $(S_1, S_2, 0, 0)$, and such a module has a filtration governed by $\mathbb{S}^{(1)}$. Thus, **Rep** $\mathbb{S}^{(5)} \subseteq$ **Filt** $\mathbb{S}^{(1)} = \mathcal{C}_1$. Similarly, **Rep** $\mathbb{S}^{(6)} \subseteq \mathcal{C}_2$.

(b) Let $r = 2$, $s = 1$. Then $\mathbf{Rep}_d(\Lambda)$ again has precisely two irreducible components, $\mathcal{C}_1$ and $\mathcal{C}_2$. Concerning $\mathbb{S}^{(3)}$: a generic module G_3 for **Rep** $\mathbb{S}^{(3)}$ has a hypergraph of the form

$$
\begin{array}{ccc}
 & 1 & \\
\alpha_1 \diagup & & \diagdown \alpha_2 \\
2 & & 2 \\
\beta_1 \diagdown & & \diagup \beta_1 \\
 & 1 &
\end{array}
$$
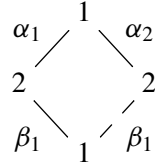
In particular, the socle of $G_3 = \Lambda z$ contains a copy of S_2, namely $\Lambda(\alpha_1 - k\alpha_2)z$ for a suitable scalar $k \in K^*$. We deduce that the submodule chain

$$G_3 \supseteq JG_3 \supseteq \Lambda(\alpha_1 - k\alpha_2)z + \Lambda\beta_1\alpha_1 z \supseteq \Lambda(\alpha_1 - k\alpha_2)z \supseteq 0$$

is governed by $\mathbb{S}^{(1)}$, showing **Rep** $\mathbb{S}^{(3)} \subseteq$ **Filt** $\mathbb{S}^{(1)} = \mathcal{C}_1$. (On the side, we mention that **Rep** $\mathbb{S}^{(3)}$ is not contained in $\mathcal{C}_2$ because the sequences $\mathbb{S}^{(2)}$ and $\mathbb{S}^{(3)}$ are not comparable under the dominance order.)

The sequence $\mathbb{S}^{(4)}$ fails to be realizable for $s = 1$. As for $\mathbb{S}^{(5)}$: generically, the modules in **Rep** $\mathbb{S}^{(5)}$ decompose in the form shown at the end of Example 2.5(a), whence **Rep** $\mathbb{S}^{(5)} \subseteq \mathcal{C}_1$. (Clearly, **Rep** $\mathbb{S}^{(5)} \not\subseteq \mathcal{C}_2$, because $\mathbb{S}^{(5)}$ is not comparable to $\mathbb{S}^{(2)}$.) A routine check shows that **Rep** $\mathbb{S}^{(6)}$ is contained in $\mathcal{C}_2$, but not in $\mathcal{C}_1$.

(c) Let $r \geq 3$, $s = 1$. Then the variety $\mathbf{Rep}_d(\Lambda)$ has three irreducible components, namely $\mathcal{C}_j = \overline{\mathbf{Rep}\,\mathbb{S}^{(j)}}$, for $j = 1, 2, 5$. The status of $\mathcal{C}_1, \mathcal{C}_2$ being clear, we focus on the variety **Rep** $\mathbb{S}^{(5)}$ with generic module G_5 as depicted at the end of Example 2.5(b). Again, we prove our claim regarding $\mathcal{C}_5$ via Theorem 4.5: to see that $\mathbb{S}^{(5)} = \mathbb{S}(G_5)$ is the only realizable semisimple sequence governing a filtration of G_5, we note that the only other realizable sequence not ruled out by Θ (i.e., with a Θ-value less than $\Theta(G_5)$) is $\mathbb{S}^{(1)}$. To verify, without computational effort, that $\mathbb{S}^{(1)}$ does not govern any filtration of G_5, it suffices to observe that, for any module N in **Filt** $\mathbb{S}^{(1)}$, we have $S_1 \subseteq N/\Lambda x$ for some $x \in e_2N$. On the other hand, it is readily checked that $S_1 \not\subseteq G_5/\Lambda x$ for all elements $x \in e_2G_5$, which shows $\Gamma(G_5) = 1$ as required. Finally, to link up with the remarks preceding Example 6.1, we point out that $\Theta^+(G_1) < \Theta^+(G_5)$, whence the Θ^+-test fails to detect the status of $\overline{\mathbf{Rep}\,\mathbb{S}^{(5)}}$ as an irreducible component of $\mathbf{Rep}_d(\Lambda)$.

To see that $\mathbb{S}^{(j)}$ for $j = 3, 4, 6$ do not arise as generic radical layerings of irreducible components of $\mathbf{Rep}_d(\Lambda)$, one may follow the patterns of part (b).

(d) Moving to $r \geq 3$ and $s = 2$ raises the number of irreducible components of $\mathbf{Rep}_d(\Lambda)$ to five. We first show that $\overline{\mathbf{Rep}\,\mathbb{S}^{(3)}}$ is now a component. Generically, the

modules in **Rep** $\mathbb{S}^{(3)}$ have hypergraph
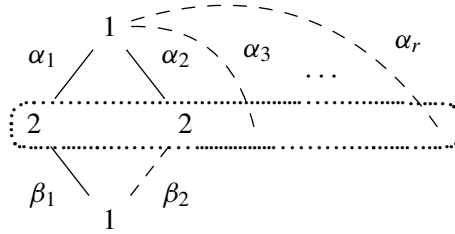
Again, the only $\overline{\textbf{Rep}\,\mathbb{S}^{(j)}}$ (for $j \leq 6$) potentially containing **Rep** $\mathbb{S}^{(3)}$ is $\overline{\textbf{Rep}\,\mathbb{S}^{(1)}} =$ **Filt** $\mathbb{S}^{(1)}$. Since the modules in **Filt** $\mathbb{S}^{(1)}$ clearly contain a copy of S_2 in their socle, while G_3 does not, this possibility is ruled out, and our claim is justified.

The discussion of $\overline{\textbf{Rep}\,\mathbb{S}^{(4)}}$ is analogous, in that the only $\overline{\textbf{Rep}\,\mathbb{S}^{(j)}}$ (for $j \leq 6$) potentially containing **Rep** $\mathbb{S}^{(4)}$ is $\overline{\textbf{Rep}\,\mathbb{S}^{(2)}} =$ **Filt** $\mathbb{S}^{(2)}$, and the modules in **Filt** $\mathbb{S}^{(2)}$ contain a copy of S_1 in their socle, while a generic module for **Rep** $\mathbb{S}^{(4)}$ does not.

As in part (c), one shows that $\overline{\textbf{Rep}\,\mathbb{S}^{(5)}}$ is a component of $\textbf{Rep}_d(\Lambda)$. On the other hand, $\overline{\textbf{Rep}\,\mathbb{S}^{(6)}}$ still fails to be a component; the argument used in part (b) (in that case, to exclude $\overline{\textbf{Rep}\,\mathbb{S}^{(5)}}$ from the list of components for $r = 2$) may now be applied to $s = 2$.

(e) Finally, let $r \geq 3$ and $s \geq 3$. Then all of the varieties $\overline{\textbf{Rep}\,\mathbb{S}^{(j)}}$ for $j = 1, \ldots, 6$ are irreducible components of $\textbf{Rep}_d(\Lambda)$. The argument backing the status of $\mathbb{S}^{(6)}$ follows the reasoning we used to confirm $\overline{\textbf{Rep}\,\mathbb{S}^{(5)}}$ as a component of $\textbf{Rep}_d(\Lambda)$ in part (c). For $r = s = 3$, hypergraphs of generic modules for the components $\overline{\textbf{Rep}\,\mathbb{S}^{(j)}}$ for $j = 1, 3, 5$ are shown below:
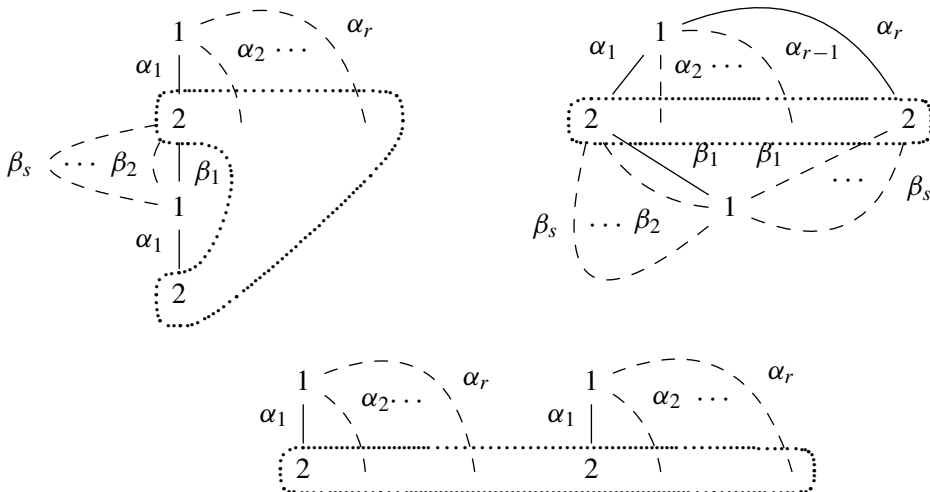
Due to symmetry, the generic structure of the modules in the remaining components is obtained by swapping the roles played by the vertices 1 and 2. $\square$

Consequences of the "truncated" theory, exemplified by Example 6.1.

(1) *Allocation of modules to the components.* Once the irreducible components **Rep** $\mathbb{S}^{(j)}$ of $\mathbf{Rep}_d(\Lambda)$ have been pinned down, by way of Theorem 4.5, say, one is in a position to list the components containing any given d-dimensional Λ-module M. Indeed, compiling this list amounts to deciding which of the $\mathbb{S}^{(j)}$ govern filtrations of M; as was pointed out in Section 5B, there is an algorithm for carrying out this task.

In Example 6.1 with $r = 3$ and $s \geq 1$, for instance, any module M with hypergraph
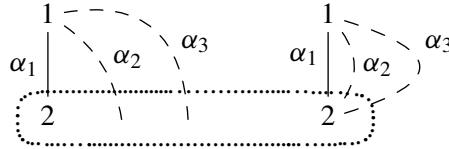
belongs to the components $\mathcal{C}_1 = \mathbf{Filt}\,\mathbb{S}^{(1)}$ and $\mathcal{C}_5 = \mathbf{Filt}\,\mathbb{S}^{(5)}$, but does not have a filtration governed by $\mathbb{S}^{(j)}$ for $j \in \{2, 3, 4, 6\}$. Therefore, M belongs to precisely two of the irreducible components of $\mathbf{Rep}_d(\Lambda)$, namely to $\mathcal{C}_1$ and $\mathcal{C}_5$.

(2) *Comparing the generic behavior of the finite-dimensional Λ-modules to that of the finite-dimensional KQ-modules.* Examples 6.1(a)–(e) place a spotlight on the fact that, in the presence of oriented cycles, the generic representation theory of the path algebra KQ may be "disjoint" from that of its truncations in the following sense: for $r, s \geq 1$, we have $J(KQ) = 0$, and for $d = (2, 2)$ the modules in the irreducible variety $\mathbf{Rep}_d(KQ)$ are generically simple. Since generically the latter modules are not annihilated by any path in KQ, we find the variety $\mathbf{Rep}_d(KQ/\langle\text{the paths of length }4\rangle)$ to be contained in the boundary of a dense open subset of $\mathbf{Rep}_d(KQ)$.

6B. *Information on the components of $\mathbf{Rep}_d(\Lambda)$ from those of $\mathbf{Rep}\,d(\Lambda_{\mathbf{trunc}})$.* We conclude with a first installment of observations on how to pull information about the components of $\mathbf{Rep}_d(\Lambda)$ from knowledge of the components of $\mathbf{Rep}_d(\Lambda_{\text{trunc}})$. Suppose that the distinct irreducible components of $\mathbf{Rep}_d(\Lambda_{\text{trunc}})$ are

$$\overline{\mathbf{Rep}_{\Lambda_{\text{trunc}}}}\,\mathbb{S}^{(1)} = \mathbf{Filt}_{\Lambda_{\text{trunc}}}(\mathbb{S}^{(1)}), \quad \ldots, \quad \overline{\mathbf{Rep}_{\Lambda_{\text{trunc}}}}\,\mathbb{S}^{(m)} = \mathbf{Filt}_{\Lambda_{\text{trunc}}}(\mathbb{S}^{(m)}).$$

Moreover, suppose that $\mathcal{C}$ is an irreducible component of some $\mathbf{Rep}_\Lambda\,\mathbb{S}$ with generic module G (recall that, for any Λ, these components and their generic modules may be algorithmically accessed from quiver and relations of Λ). To compare with $\mathbf{Rep}_d(\Lambda_{\text{trunc}})$, one first determines which among the $\mathbb{S}^{(j)}$ govern a filtration of G. Suppose the pertinent sequences are $\mathbb{S}^{(1)}, \ldots, \mathbb{S}^{(r)}$, that is, $\mathcal{C} \subseteq \mathbf{Filt}_\Lambda\,\mathbb{S}^{(j)}$ precisely when $j \leq r$.

Observation 6.2. The closure $\bar{\mathcal{C}}$ is an irreducible component of $\mathbf{Rep}_d(\Lambda)$ if and only if $\bar{\mathcal{C}}$ is maximal irreducible in $\mathbf{Filt}_\Lambda\,\mathbb{S}^{(j)}$ for all $j \leq r$.

Proof. The claim is immediate from the fact that every irreducible subvariety $\mathcal{D}$ of $\mathbf{Rep}_d(\Lambda)$ which contains $\overline{\mathcal{C}}$ is contained in one of the intersections

$$\mathbf{Rep}_d(\Lambda) \cap \mathbf{Filt}_{\Lambda_{\text{trunc}}} \mathbb{S}^{(j)} = \mathbf{Filt}_\Lambda \mathbb{S}^{(j)}. \qquad \square$$

This leads to a lower bound for the number of irreducible components of $\mathbf{Rep}_d(\Lambda)$. Computing it in specific instances typically requires a nonnegligible effort, as it is not simply based on the *number* of components of $\mathbf{Rep}_d(\Lambda_{\text{trunc}})$. The bound is sharp in general. Indeed, if Δ denotes the algebra of Example 6.1(e) and $\Lambda = \Delta/\langle \beta_i \alpha_j \beta_k \mid i, j, k \in \{1, 2, 3\}\rangle$, then $\Delta = \Lambda_{\text{trunc}}$ and the number of irreducible components of $\mathbf{Rep}_d(\Lambda)$ coincides with the lower bound given below.

Corollary 6.3. *Again, let d be a dimension vector of a basic K-algebra Λ, and adopt the above notation for the irreducible components of $\mathbf{Rep}_d(\Lambda_{\text{trunc}})$. Moreover, set*

$$A_j := \mathbf{Rep}_d(\Lambda_{\text{trunc}}) \setminus \bigcup_{\substack{i \leq m \\ i \neq j}} \mathbf{Filt}_{\Lambda_{\text{trunc}}} \mathbb{S}^{(i)} \quad \text{for } j \leq m.$$

Then the number of irreducible components of $\mathbf{Rep}_d(\Lambda)$ is bounded from below by the number of A_j which have nonempty intersection with $\mathbf{Rep}_d(\Lambda)$.

Proof. Suppose $A_1, \ldots, A_s$ are the A_j which intersect $\mathbf{Rep}_d(\Lambda)$ nontrivially, and let $\mathfrak{U}_j$ be an irreducible subvariety of $A_j \cap \mathbf{Rep}_d(\Lambda)$ for $j \leq s$. Among the $\mathbf{Filt}_{\Lambda_{\text{trunc}}} \mathbb{S}^{(i)}$, the variety $\mathbf{Filt}\,\mathbb{S}^{(j)}$ is then the only one to contain $\mathfrak{U}_j$. Consequently, any maximal irreducible subset $\mathcal{D}_j$ of $\mathbf{Rep}_d(\Lambda)$ containing $\mathfrak{U}_j$ is an irreducible component of $\mathbf{Rep}_d(\Lambda)$ by the preceding observation. By construction, the resulting $\mathcal{D}_j$ are pairwise different. $\qquad \square$

References

[Babson et al. 2009] E. Babson, B. Huisgen-Zimmermann, and R. Thomas, "Generic representation theory of quivers with relations", *J. Algebra* **322**:6 (2009), 1877–1918. MR Zbl

[Barot and Schröer 2001] M. Barot and J. Schröer, "Module varieties over canonical algebras", *J. Algebra* **246**:1 (2001), 175–192. MR Zbl

[Bleher et al. 2015] F. M. Bleher, T. Chinburg, and B. Huisgen-Zimmermann, "The geometry of finite dimensional algebras with vanishing radical square", *J. Algebra* **425** (2015), 146–178. MR Zbl

[Bongartz and Huisgen-Zimmermann 2001] K. Bongartz and B. Huisgen-Zimmermann, "Varieties of uniserial representations, IV: Kinship to geometric quotients", *Trans. Amer. Math. Soc.* **353**:5 (2001), 2091–2113. MR Zbl

[Carroll and Weyman 2013] A. T. Carroll and J. Weyman, "Semi-invariants for gentle algebras", pp. 111–136 in *Noncommutative birational geometry, representations and combinatorics*, edited by A. Berenstein and V. Retakh, Contemp. Math. **592**, Amer. Math. Soc., Providence, RI, 2013. MR Zbl

[Crawley-Boevey and Schröer 2002] W. Crawley-Boevey and J. Schröer, "Irreducible components of varieties of modules", *J. Reine Angew. Math.* **553** (2002), 201–220. MR Zbl

[Derksen et al. 2014] H. Derksen, B. Huisgen-Zimmermann, and J. Weyman, "Top-stable degenerations of finite dimensional representations, II", *Adv. Math.* **259** (2014), 730–765. MR Zbl

[Donald and Flanigan 1977] J. Donald and F. J. Flanigan, "The geometry of Rep(A, V) for a square-zero algebra", *Notices Amer. Math. Soc.* **24** (1977), A–416.

[Gabriel 1975] P. Gabriel, "Finite representation type is open", pp. 132–155 in *Representations of algebras* (Carleton Univ., Ottawa, 1974), edited by V. Dlab and P. Gabriel, Lecture Notes in Mathematics **488**, Springer, 1975. MR Zbl

[Geiss and Schröer 2003] C. Geiss and J. Schröer, "Varieties of modules over tubular algebras", *Colloq. Math.* **95**:2 (2003), 163–183. MR Zbl

[Geiss and Schröer 2005] C. Geiss and J. Schröer, "Extension-orthogonal components of preprojective varieties", *Trans. Amer. Math. Soc.* **357**:5 (2005), 1953–1962. MR Zbl

[Huisgen-Zimmermann 2007] B. Huisgen-Zimmermann, "Classifying representations by way of Grassmannians", *Trans. Amer. Math. Soc.* **359**:6 (2007), 2687–2719. MR Zbl

[Huisgen-Zimmermann 2009] B. Huisgen-Zimmermann, "A hierarchy of parametrizing varieties for representations", pp. 207–239 in *Rings, modules and representations*, edited by N. V. Dung et al., Contemp. Math. **480**, Amer. Math. Soc., Providence, RI, 2009. MR Zbl

[Huisgen-Zimmermann 2016] B. Huisgen-Zimmermann, "Irreducible components of varieties of representations: the local case", *J. Algebra* **464** (2016), 198–225. MR Zbl

[Huisgen-Zimmermann and Goodearl 2012] B. Huisgen-Zimmermann and K. R. Goodearl, "Irreducible components of module varieties: projective equations and rationality", pp. 141–167 in *New trends in noncommutative algebra*, edited by P. Ara et al., Contemp. Math. **562**, Amer. Math. Soc., Providence, RI, 2012. MR Zbl

[Huisgen-Zimmermann and Shipman 2017] B. Huisgen-Zimmermann and I. Shipman, "Irreducible components of varieties of representations: the acyclic case", *Math. Z.* **287**:3 (2017), 1083–1107. MR Zbl

[Kac 1980] V. G. Kac, "Infinite root systems, representations of graphs and invariant theory", *Invent. Math.* **56**:1 (1980), 57–92. MR Zbl

[Kac 1982] V. G. Kac, "Infinite root systems, representations of graphs and invariant theory, II", *J. Algebra* **78**:1 (1982), 141–162. MR Zbl

[Morrison 1980] K. Morrison, "The scheme of finite-dimensional representations of an algebra", *Pacific J. Math.* **91**:1 (1980), 199–218. MR Zbl

[Riedtmann et al. 2011] C. Riedtmann, M. Rutscho, and S. O. Smalø, "Irreducible components of module varieties: an example", *J. Algebra* **331** (2011), 130–144. MR Zbl

[Schofield 1992] A. Schofield, "General representations of quivers", *Proc. London Math. Soc.* (3) **65**:1 (1992), 46–64. MR Zbl

[Schröer 2004] J. Schröer, "Varieties of pairs of nilpotent matrices annihilating each other", *Comment. Math. Helv.* **79**:2 (2004), 396–426. MR Zbl

[Steinberg 1974] R. Steinberg, *Conjugacy classes in algebraic groups*, Lecture Notes in Mathematics **366**, Springer, 1974. MR Zbl

goodearl@math.ucsb.edu   *Department of Mathematics, University of California Santa Barbara, Santa Barbara, CA, United States*

birge@math.ucsb.edu   *Department of Mathematics, University of California Santa Barbara, Santa Barbara, CA, United States*

Algebra & Number Theory

msp.org/ant

Algebra & Number Theory

Volume 12   No. 2   2018