Algebra & Number Theory

Volume 12
2018
No. 3

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Annette Huber and Shane Kelly

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ALGEBRA AND NUMBER THEORY 12:3 (2018)
dx.doi.org/10.2140/ant.2018.12.649

Differential forms in positive characteristic, II: cdh-descent via functorial Riemann–Zariski spaces

Annette Huber and Shane Kelly

This paper continues our study of the sheaf associated to Kähler differentials in the cdh-topology and its cousins, in positive characteristic, without assuming resolution of singularities. The picture for the sheaves themselves is now fairly complete. We give a calculation $\mathcal{O}_{\mathrm{cdh}}(X) \cong \mathcal{O}(X^{\mathrm{sn}})$ in terms of the seminormalisation. We observe that the category of representable cdh-sheaves is equivalent to the category of seminormal varieties. We conclude by proposing some possible connections to Berkovich spaces and F-singularities in the last section. The tools developed for the case of differential forms also apply in other contexts and should be of independent interest.

1. Introduction

This paper continues the programme started in [Huber and Jörder 2014] (characteristic 0) and [Huber, Kebekus and Kelly 2017] (positive characteristic). For a survey see also [Huber 2016].

Programme. Let us quickly summarise the main idea. Sheaves of differential forms are very rich sources of invariants in the study of algebraic invariants of smooth algebraic varieties. However, they are much less well-behaved for singular varieties. In characteristic 0, the use of the h-topology — replacing Kähler differentials with their sheafification in this Grothendieck topology — is very successful. It unifies several ad hoc notions and simplifies arguments. In positive characteristic, resolution of singularities would imply that the cdh-sheafification could be used in a very similar way. Together with the results of [Huber, Kebekus and Kelly 2017], we now have a fairly complete unconditional picture, at least for the sheaves themselves. We refer to the follow-up [Huber and Kelly $\geq$ 2018] for results on cohomological descent, where, however, many questions remain open.

Results. There are a number of weaker cousins of the h-topology in the literature. They exclude the Frobenius morphism but still allow abstract blowups. The cdh-topology (see Section 2B) is the most well-established, appearing prominently in work on motives and K-theory, and having connections to rigid geometry; see [Friedlander and Voevodsky 2000; Voevodsky 2000; Suslin and Voevodsky 2000a; Cisinski and Déglise 2015; Cortiñas et al. 2008; Geisser and Hesselholt 2010; Cisinski 2013; Kerz et al. 2018; Morrow 2016], for example. We write Ω^n_τ for the sheafification of the presheaf $X \mapsto \Gamma(X, \Omega^n_{X/k})$ with respect to the topology τ.

MSC2010: primary 14G17; secondary 14F20.
Keywords: differential forms, cdh-topology, valuation rings, seminormalization, singularities.

Theorem 1.1. *Suppose k is a perfect field.*

(1) ([Huber, Kebekus and Kelly 2017], also Theorem 4.12) *For a smooth k-scheme X and $n \geq 0$,*

$$\Gamma(X, \Omega^n_{X/k}) = \Omega^n_{\mathrm{cdh}}(X).$$

(2) (Theorem 5.4, Proposition 5.9) *For any finite type separated k-scheme X, the restrictions $\Omega^n_{\mathrm{cdh}}|_{X_{\mathrm{Zar}}}$ and $\Omega^n_{\mathrm{cdh}}|_{X_{\mathrm{et}}}$ to the small Zariski and étale sites are coherent $\mathcal{O}_X$-modules.*

(3) (Proposition 6.2) *For functions, we have isomorphisms, functorial in X,*

$$\mathcal{O}_{\mathrm{cdh}}(X) = \mathcal{O}(X^{\mathrm{sn}}),$$

where X^{sn} is the seminormalisation of the variety X; see Section 2C.

(4) (Proposition 6.9) *For top degree differentials, we have*

$$\Omega^d_{\mathrm{cdh}}(X) \cong \varinjlim_{\substack{X' \to X \\ \text{proper birational}}} \Gamma(X', \Omega^d_{X'/k}), \qquad \dim X = d.$$

By combining (1) and (4) we deduce a corollary that involves only Kähler differentials. To our knowledge the formula is new:

Corollary 1.2 (Corollary 6.11). *Let k be a perfect field and X a smooth k-scheme. We have*

$$\Gamma(X, \Omega^d_{X/k}) \cong \varinjlim_{\substack{X' \to X \\ \text{proper birational}}} \Gamma(X', \Omega^d_{X'/k}), \qquad \dim X = d.$$

Note that it could also be deduced directly from [Huber, Kebekus and Kelly 2017, Theorem 5.8].

In contrast to the case of characteristic 0, the sheaves Ω^n_{cdh} are not torsion free (this was shown in [Huber, Kebekus and Kelly 2017, Example 3.6] by "pinching" along the Frobenius of a closed subscheme). So not only does lacking access to resolution of singularities cause proofs to become harder, the existence of inseparable field extensions actually changes some of the results.

Main tool. Our main tool, taking the role of a desingularisation of a variety X, is the category

$$\mathrm{val}(X),$$

a functorial variant of the Riemann–Zariski space, which we now discuss. Recall that the Riemann–Zariski space of an integral variety X is (as a set) given by the set of (all, not necessarily discrete) X-valuation rings of $k(X)$. The Riemann–Zariski space is only functorial for dominant morphisms of integral varieties. We replace it by the category $\mathrm{val}(X)$ (see Definition 2.21) of X-schemes of the form $\mathrm{Spec}(R)$ with R either a field of finite transcendence degree over k or a valuation ring of such a field. In [Huber, Kebekus and Kelly 2017], we were focussing on the discrete valuation rings — this turned out to be useful, but allowing nonnoetherian valuation rings yields a much better tool.

We define $\Omega^n_{\mathrm{val}}(X)$ as the set of global sections of the presheaf $\operatorname{Spec}(R) \mapsto \Omega^n_{R/k}$ on $\mathrm{val}(X)$; that is,

$$\Omega^n_{\mathrm{val}}(X) = \varprojlim_{R \in \mathrm{val}(X)} \Omega^n_{R/k}.$$

A global section admits the following very explicit description (Lemma 3.7). It is uniquely determined by specifying an element $\omega_x \in \Omega^n_{\kappa(x)/k}$ for every point $x \in X$ subject to two compatibility conditions: If R is an X-valuation ring of a residue field $\kappa(x)$, then ω_x has to be integral, i.e., contained in $\Omega^n_{R/k} \subseteq \Omega^n_{\kappa(x)/k}$. If ξ is the image of its special point in X, then $\omega_R|_{\operatorname{Spec}(R/\mathfrak{m})}$ has to agree with $\omega_\xi|_{\operatorname{Spec}(R/\mathfrak{m})}$ in $\Omega^n_{(R/\mathfrak{m})/k}$. The above mentioned results are deduced by establishing eh-descent (and therefore cdh- and rh-descent) for Ω^n_{val}. This is used to show the following:

Theorem 1.3 (Theorem 4.12). $\qquad \Omega^n_{\mathrm{rh}} = \Omega^n_{\mathrm{cdh}} = \Omega^n_{\mathrm{eh}} = \Omega^n_{\mathrm{val}}.$

This gives a very useful characterisation of Ω^n_{cdh}, and answers the open question [Huber, Kebekus and Kelly 2017, Proposition 5.13]. We also show that using other classes of valuation rings (e.g., rank one or strictly henselian or removing the transcendence degree bound) produces the same sheaf (see Proposition 3.11).

Special cases and the* sdh*-topology. There are two special cases, both of particular importance and with better properties: the case of 0-forms (i.e., functions) and the case of the "canonical sheaf" (i.e., d-forms on the category of k-schemes of dimension at most d). In both cases, the resulting sheaves even have descent for the sdh-topology introduced in [Huber, Kebekus and Kelly 2017]:

$$\mathcal{O}_{\mathrm{cdh}} = \mathcal{O}_{\mathrm{sdh}}, \qquad \Omega^d_{\mathrm{cdh}} = \Omega^d_{\mathrm{sdh}}$$

(see Remark 6.7 and Proposition 6.12). Recall that every variety is locally smooth in the sdh-topology by de Jong's theorem on alterations, and the hope was that requiring morphisms to be separably decomposed would prohibit pathologies caused by purely inseparable extensions. Unfortunately, $\Omega^n(X) \neq \Omega^n_{\mathrm{sdh}}(X)$ for general n and X smooth; see [Huber, Kebekus and Kelly 2017].

Seminormalisation. For functions, we have the explicit computation

$$\mathcal{O}_{\mathrm{cdh}}(X) = \mathcal{O}(X^{\mathrm{sn}}), \tag{1}$$

where X^{sn} is the seminormalisation of the variety X (see Proposition 6.2). Here, the seminormalisation $X^{\mathrm{sn}} \to X$ is the universal morphism, which induces isomorphisms of topological spaces and residue fields; see Section 2C. In fact, we have this result for all representable presheaves.

Theorem 1.4 (Proposition 6.14). *Suppose S is a noetherian scheme and the normalisation of every finite type S-scheme is also finite type (i.e., that S is Nagata, as defined in Remark 2.7). For instance, S might be the spectrum of a perfect field. Then for all separated finite type S-schemes X and Y, the canonical morphisms*

$$h^Y_{\mathrm{rh}}(X) = h^Y(X^{\mathrm{sn}}) \tag{2}$$

are isomorphisms, where $h^Y(-) = \hom_S(-, Y)$. *The natural maps*

$$h^Y_{\mathrm{rh}} \to h^Y_{\mathrm{cdh}} \to h^Y_{\mathrm{eh}} \to h^Y_{\mathrm{sdh}} \to h^Y_{\mathrm{val}} \tag{3}$$

are isomorphisms of presheaves on $\mathsf{Sch}^{\mathrm{ft}}_S$.

In characteristic 0, equation (2) is already formulated for the h-topology by Voevodsky [1996, Section 3.2], and generalised to algebraic spaces by Rydh [2010]. The theorem confirms that we have identified the correct analogy in positive characteristic.

As (2) confirms, the cdh-topology is not subcanonical. In fact, the full subcategory spanned by those cdh-sheaves which are sheafifications of representable sheaves is equivalent to the category of seminormal schemes (Corollary 6.17). In particular, the subcategory of smooth or even normal varieties remains unchanged; however, we lose information about certain singularities, e.g., cuspidal singularities are smoothened out. Depending on the question, this might be considered an advantage or a disadvantage. We strongly argue that it is an advantage for the natural questions of birational algebraic geometry, where differential forms are a main tool.

Recall that if X is integral, the ring of global sections of the structure sheaf of the normalisation $\widetilde{X}$ is the intersection of all X-valuation rings of the function field:

$$\Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = \bigcap_{\text{valuation rings } R \text{ of } k(X)} R.$$

In light of $\mathcal{O}_{\mathrm{cdh}} \cong \mathcal{O}_{\mathrm{val}}$, equation (1) has the following neat interpretation:

Scholium 1.5. *If X is reduced, the ring of global sections of the structure sheaf of the seminormalisation X^{sn} is the "intersection" of all X-valuation rings*:

$$\Gamma(X^{\mathrm{sn}}, \mathcal{O}_{X^{\mathrm{sn}}}) = \varprojlim_{\substack{\mathrm{Spec}(R) \to X \\ \text{with } R \text{ a valuation ring}}} R.$$

The seminormalisation was introduced and studied quite some time ago; see for example [Traverso 1970; Swan 1980], and for a historical survey see [Vitulli 2011]. The original motivation for considering the seminormalisation (or rather, the closely related and equivalent in characteristic 0 concept of weak normalisation) was to make the moduli space of positive analytic d-cycles on a projective variety "more normal" without changing its topology, i.e., without damaging too much the way that it solved its moduli problem [Andreotti and Norguet 1967]. Clearly, the cdh- and eh-topology are relevant to these moduli questions. Indeed, the cdh-topology already appears in the study of moduli of cycles in [Suslin and Voevodsky 2000a, Theorem 4.2.11]. In relation to this, let us point out that basically all of the present paper works for arbitrary unramified étale sheaves commuting with filtered limits of schemes, and in particular applying $(-)_{\mathrm{val}}$ to various Hilbert presheaves provides alternative constructions for Suslin and Voevodsky's relative cycle presheaves $z(X/S, r), z^{\mathrm{eff}}(X/S, r), c(X/S, r), c^{\mathrm{eff}}(X/S, r)$ introduced in [Suslin and Voevodsky 2000a], and heavily used in their work on motivic cohomology. Such matters, however, go beyond the scope of this current paper.

Outline of the paper. We start in Section 2 by collecting basic notation and facts on the Grothendieck topologies that we use, seminormality and valuation rings. In Section 2E we introduce our main tool, different categories of local rings above a given X, and discuss the relation to the Riemann–Zariski space.

In Section 3 we discuss and compare the presheaves on schemes of finite type over the base induced from presheaves on our categories of local rings. Everything is then applied to the case of differential forms.

In Section 4 we verify sheaf conditions on these presheaves. In Theorem 4.12, this culminates in our main comparison theorem on differential forms.

Section 5 establishes coherence of Ω^n_{rh} locally on the Zariski or even small étale site of any X. In Section 6 we turn to the special examples of $\mathcal{O}$, the canonical sheaf, and more generally the category of representable sheaves. Finally, in Section 7 we outline interesting open connections to the theory of Berkovich spaces and to F-singularities.

2. Commutative algebra and general definitions

2A. ***Notation.*** Throughout, k is assumed to be a perfect field. The case of interest is the case of positive characteristic. Sometimes we use a separated noetherian base scheme S. This includes the case $S = \mathrm{Spec}(k)$ of course.

The valuation rings we use are *not* assumed to be noetherian!

We denote by $\mathsf{Sch}^{\mathrm{ft}}_S$ the category of separated schemes of finite type over S, and write $\mathsf{Sch}^{\mathrm{ft}}_k$ when considering the case $S = \mathrm{Spec}(k)$.

We write $\Omega^n(X)$ for the vector space of k-linear n-differential forms on a k-scheme X, often denoted elsewhere by $\Gamma(X, \Omega^n_{X/S})$. Note the assignment $X \mapsto \Omega^n(X)$ is functorial in X.

Following [Stacks project, Tag 01RN], we call a morphism of schemes with finitely many irreducible components $f : X \to Y$ *birational* if it induces a bijection between the sets of irreducible components and an isomorphism on the residue fields of generic points. In the case of varieties (or more generally, if f is of finite presentation and X, Y are generically reduced [EGA IV$_3$ 1966, Théorème 8.10.5(i)]) this is equivalent to the existence of dense open subsets $U \subseteq X$ and $V \subset Y$ such that f induces an isomorphism $U \to V$.

2B. ***Topologies.***

Definition 2.1 (cdp-morphism). A morphism $f : Y \to X$ is called a cdp-*morphism* if it is proper and completely decomposed, where by "completely decomposed" we mean that for every (not necessarily closed) point $x \in X$ there is a point $y \in Y$ with $f(y) = x$ and $[k(y) : k(x)] = 1$.

These morphisms are also referred to as *proper* cdh-*covers* (e.g., by Suslin and Voevodsky [2000a]), or *envelopes* (e.g., by Fulton [1998]).

Remark 2.2 (rh, cdh and eh-topologies).

(1) Recall that the rh-topology on $\mathsf{Sch}^{\mathrm{ft}}_S$ is generated by the Zariski topology and cdp-morphisms [Goodwillie and Lichtenbaum 2001]. In a similar vein, the cdh-topology is generated by the

Nisnevich topology and cdp-morphisms [Suslin and Voevodsky 2000a, §5]. The eh-topology is generated by the étale topology and cdp-morphisms [Geisser 2006].

(2) We have an even stronger statement: every rh-, cdh- or eh- covering $U \to X$ in $\mathsf{Sch}^{\mathrm{ft}}_S$ admits a refinement of the form $W \to V \to X$, where $V \to X$ is a cdp-morphism and $W \to V$ is a Zariski, Nisnevich or étale covering, respectively [Suslin and Voevodsky 2000a, proof of Proposition 5.9].

2C. *Seminormality.* A special case of a cdp-morphism is the seminormalisation.

Definition 2.3 [Swan 1980]. A reduced ring A is *seminormal* if for all $b, c \in A$ satisfying $b^3 = c^2$, there is an $a \in A$ with $a^2 = b, a^3 = c$. Equivalently, every morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(\mathbb{Z}[t^2, t^3])$ factors through $\mathrm{Spec}(\mathbb{Z}[t]) \to \mathrm{Spec}(\mathbb{Z}[t^2, t^3])$.

Definition 2.4. Recall that an inclusion of rings $A \subseteq B$ is called *subintegral* if $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a completely decomposed homeomorphism [Swan 1980, §2]. In other words, an inclusion is subintegral if it induces an isomorphism on topological spaces, and residue fields.

Remark 2.5. Any (not necessarily injective) ring map $\phi : A \to B$ inducing a completely decomposed homeomorphism is *integral* in the sense that for every $b \in B$ there is a monic $f(x) \in A[x]$ such that b is a solution to the image of $f(x)$ in $B[x]$ [Stacks project, Tag 04DF]. Consequently, a posteriori, subintegral inclusions are contained in the normalisation.

We have the following nice properties.

Lemma 2.6. *Let A be a reduced ring and $(A_i)_I$ a (not necessarily filtered) diagram of reduced rings.*

(1) *If A is normal, it is seminormal.*

(2) [Swan 1980, Corollary 3.3] *If all A_i are seminormal, then so is $\varprojlim_{i \in I} A_i$.*

(3) [Swan 1980, Corollary 3.4] *If the total ring of fractions $Q(A)$ is a product of fields, then A is seminormal if and only if for every subintegral extension $A \subseteq B \subseteq Q(A)$ we have $A = B$. In particular, this holds if A is noetherian or A is a valuation ring.*

(4) [Swan 1980, §2, Theorem 4.1] *There exists a universal morphism $A \to A^{\mathrm{sn}}$ with target a seminormal reduced ring. The morphism $\mathrm{Spec}(A^{\mathrm{sn}}) \to \mathrm{Spec}(A)$ is subintegral.*

(5) [Swan 1980, Corollary 4.6] *For any multiplicative set $S \subseteq A$ we have $A^{\mathrm{sn}}[S^{-1}] = A[S^{-1}]^{\mathrm{sn}}$; in particular, if A is seminormal, so is $A[S^{-1}]$.*

Remark 2.7. Recall that a scheme S is called *Nagata* if it is locally noetherian, and if for every $X \in \mathsf{Sch}^{\mathrm{ft}}_S$, the normalisation $X^n \to X$ [Stacks project, Tag 035E] is finite. Well-known examples are fields or the ring of integers $\mathbb{Z}$; more generally, quasi-excellent rings are Nagata [Stacks project, Tag 07QV].

Actually, the definition of Nagata [Stacks project, Tag 033S] is: for every point $x \in X$, there is an open $U \ni x$ such that $R = \mathcal{O}_X(U)$ is a Nagata ring [Stacks project, Tag 032R], i.e., noetherian and for every prime $\mathfrak{p}$ of R the quotient $R/\mathfrak{p}$ is N-2 [Stacks project, Tag 032F], i.e., for every finite field extension $L/\mathrm{Frac}(R/\mathfrak{p})$, the integral closure of $R/\mathfrak{p}$ in L is finite over $R/\mathfrak{p}$. However, Nagata proved [Stacks project, Tag 0334] that Nagata rings are characterised by being noetherian and universally Japanese [Stacks project, Tag 032R], but being universally Japanese is the same as having the property that every finite type ring morphism $R \to R'$ with R' a domain, the integral closure of R' in its fraction field is finite over R' [Stacks project, Tag 032F, Tag 0351]. Since we can assume the U from above is affine, and finiteness of the normalisation is detected locally, one sees that our definition above is equivalent to the standard one.

Proposition 2.8. *Let X be a scheme. There exists a universal morphism*

$$X^{\mathrm{sn}} \to X$$

from a scheme whose structure sheaf is a sheaf of seminormal reduced rings, called the seminormalisation *of X.*

(1) *It is a homeomorphism of topological spaces.*

(2) *If $X = \operatorname{Spec}(A)$ then $X^{\mathrm{sn}} = \operatorname{Spec}(A^{\mathrm{sn}})$.*

(3) *If the normalisation $X^n \to X$ is finite (e.g., if $X \in \mathsf{Sch}^{\mathsf{ft}}_k$, or more generally, if $X \in \mathsf{Sch}^{\mathsf{ft}}_S$ with S a Nagata scheme), then $X^{\mathrm{sn}} \to X$ is a finite* cdp-*morphism.*

Proof. Replacing X with its associated reduced scheme X_{red} we can assume that X is reduced. Define X^{sn} to be the ringed space with the same underlying topological space as X, and structure sheaf the sheaf obtained from $U \mapsto \mathcal{O}(U)^{\mathrm{sn}}$. It satisfies the appropriate universality by the universal property of $(-)^{\mathrm{sn}}$ for rings.

Since $(-)^{\mathrm{sn}}$ commutes with inverse limits and localisation by Lemma 2.6 parts (2) and (5), for any reduced ring A the structure sheaf of $\operatorname{Spec}(A^{\mathrm{sn}})$ is a sheaf of seminormal rings, and we obtain a canonical morphism $\operatorname{Spec}(A^{\mathrm{sn}}) \to \operatorname{Spec}(A)^{\mathrm{sn}}$ of ringed spaces. By the universal properties, $(-)^{\mathrm{sn}}$ commutes with colimits, so for any point $x \in X$ we have $\mathcal{O}_{X^{\mathrm{sn}},x} = \mathcal{O}^{\mathrm{sn}}_{X,x}$, and consequently, $\operatorname{Spec}(A^{\mathrm{sn}}) \to \operatorname{Spec}(A)^{\mathrm{sn}}$ is an isomorphism of ringed spaces. From this we deduce that in general X^{sn} is a scheme, and $X^{\mathrm{sn}} \to X$ is a completely decomposed morphism of schemes (see Lemma 2.6(4)).

Finally, by the universal property, there is a factorisation $X^n \to X^{\mathrm{sn}} \to X$. So if the normalisation is finite, then so is the seminormalisation. □

The following well-known property explains the significance of the seminormalisation in our context.

Lemma 2.9. *Let $\mathcal{F}$ be an* rh-*sheaf on $\mathsf{Sch}^{\mathsf{ft}}_S$. Take $X \in \mathsf{Sch}^{\mathsf{ft}}_S$ and $X' \to X$ a completely decomposed homeomorphism. Then*

$$\mathcal{F}(X) \to \mathcal{F}(X^{\mathrm{red}}) \to \mathcal{F}(X')$$

are isomorphisms. In particular,

$$\mathcal{F}(X) \cong \mathcal{F}(X^{\mathrm{sn}})$$

if $X^{\mathrm{sn}} \in \mathsf{Sch}^{\mathsf{ft}}_S$ (for example, if S is Nagata).

Proof. The map $X^{\mathrm{red}} \to X$ is cdp and we have $(X^{\mathrm{red}} \times_X X^{\mathrm{red}}) = X^{\mathrm{red}}$. The isomorphism for X^{red} follows from the sheaf sequence. The same argument applies to X': Since $\mathcal{F}$ is an rh-sheaf we can assume X' is reduced. Then since $X' \to X$ is a finite [Stacks project, Tag 04DF] completely decomposed homeomorphism, so are the projections $X' \times_X X' \to X'$, and it follows that the diagonal induces an isomorphism $X' \xrightarrow{\sim} (X' \times_X X')^{\mathrm{red}}$. □

Lemma 2.10. *Suppose that*

$$\begin{array}{ccc} E & \xrightarrow{j} & X' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow{i} & X \end{array}$$

is a commutative square in $\mathrm{Sch}_S^{\mathrm{ft}}$, *with* i, j *closed immersions*, p, q *finite surjective*, *and* p *an isomorphism outside* $i(Z)$.

Then the pushout $Z \sqcup_E X'$ *exists in* $\mathrm{Sch}_S^{\mathrm{ft}}$, *is reduced if both* Z *and* X' *are reduced*, *and the canonical morphism* $Z \sqcup_E X' \to X$ *is a finite completely decomposed homeomorphism. In particular*,

$$Z \sqcup_E X' \cong X$$

if Z, X' *are reduced and* X *is seminormal.*

Proof. First consider the case where X is affine (so all four schemes are affine). The pushout exists by [Ferrand 2003, Scolie 4.3, Théorème 5.1] and is given explicitly by the spectrum of the pullback of the underlying rings. By construction it is reduced as Z and X' are. The underlying set of the pushout is the pushout of the underlying sets [Ferrand 2003, Scolie 4.3]. As p is an isomorphism outside $i(Z)$ and q is surjective, it follows that $Z \sqcup_E X' \to X$ is a bijection on the underlying sets. The existence of the (continuous) map (of topological spaces) $Z \sqcup_E X' \to X$ shows that every open set of X is an open set of $Z \sqcup_E X'$. Conversely, if W is a closed set of $Z \sqcup_E X'$ then its preimage in X' is closed. As p is proper, this implies that W is also closed in X. In other words, every open set of $Z \sqcup_E X'$ is an open set of X. Hence $Z \sqcup_E X' \to X$ is a homeomorphism with reduced source. Now $Z \sqcup X' \to X$ is finite and completely decomposed, so $Z \sqcup_E X' \to X$ is also finite and completely decomposed. To summarise, $Z \sqcup_E X' \to X$ is a finite completely decomposed homeomorphism. That is, if both are reduced it comes from a subintegral extension of rings. If in addition X is seminormal, this implies that it is an isomorphism (Lemma 2.6(3)).

For a general $X \in \mathrm{Sch}_S^{\mathrm{ft}}$, the pushout exists by [Ferrand 2003, Théorème 7.1] (one checks the condition (ii) easily by pulling back an open affine of X to X'). Then the properties of $Z \sqcup_E X' \to X$ claimed in the statement can be verified on an open affine cover of X. As long as

$$U \times_X (Z \sqcup_E X') \cong (U \times_X Z) \sqcup_{U \times_X E} (U \times_X X')$$

for every open affine $U \subseteq X$, it follows from the case where X is affine. This latter isomorphism can be checked in the category of locally ringed spaces using the explicit description of [Ferrand 2003, Scolie 4.3]; it is also a special case of [Ferrand 2003, Lemme 4.4]. □

2D. *Valuation rings.* Recall that an integral domain R is called a *valuation ring* if for all $x \in K = \mathrm{Frac}(R)$, at least one of x or x^{-1} is in $R \subseteq K$. If R contains a field k, we say that R is a *k-valuation ring*. We say R is a *valuation ring of K* to emphasise that $K = \mathrm{Frac}(R)$.

The name valuation ring comes from the fact that the abelian group $\Gamma_R = K^*/R^*$ equipped with the relation "$a \leq b$ if and only if $b/a \in (R - \{0\})/R^*$" is a totally ordered group, and the canonical homomorphism $v : K^* \to K^*/R^*$ is a valuation, in the sense that $v(a+b) \geq \min(v(a), v(b))$ for all $a, b \in K^*$. Conversely, for any valuation on a field, the set of elements with nonnegative value are a valuation ring in the above sense. If $\Gamma_R = K^*/R^*$ is isomorphic to $\mathbb{Z}$ we say that R is a *discrete valuation ring*. Every noetherian valuation ring is either a discrete valuation ring or a field.

One of the many, varied characterisations of valuation rings is the following.

Proposition 2.11 [Bourbaki 1964, Chapitre VI, §1.2, Théorème 1]. *Let $R \subseteq K$ be a subring of a field. Then R is a valuation ring if and only if its set of ideals is totally ordered.*

In particular, this implies that the maximal ideal is unique, that is, every valuation ring is a local ring. The cardinality of the set of nonzero prime ideals of a valuation ring is called its *rank*. As the set of primes is totally ordered, the rank agrees with the Krull dimension.

Corollary 2.12. *Let R be a valuation ring. If $\mathfrak{p} \subseteq R$ is a prime ideal, then both the quotient $R/\mathfrak{p}$ and the localisation $R_{\mathfrak{p}}$ are again valuation rings.*

Corollary 2.13. *Let R be a valuation ring and $S \subseteq R - \{0\}$ a multiplicative set. Then $R[S^{-1}]$ is a valuation ring. In fact, $\mathfrak{p} = \bigcup_{S \cap \mathfrak{q} = \varnothing} \mathfrak{q}$ is a prime and $R[S^{-1}] = R_{\mathfrak{p}}$.*

Proof. By Corollary 2.12 it suffices to show the second claim. Clearly $\mathfrak{p}$ is prime. Recall that the canonical inclusion $R \to R[S^{-1}]$ induces an isomorphism of *locally ringed spaces* between $\mathrm{Spec}(R[S^{-1}])$ and $\{\mathfrak{q} : \mathfrak{q} \cap S = \varnothing\} \subseteq \mathrm{Spec}(R)$. Since this set has a maximal element, namely $\mathfrak{p}$, the morphism $\mathrm{Spec}(R_{\mathfrak{p}}) \to \mathrm{Spec}(R[S^{-1}])$ is an isomorphism of *locally ringed spaces*. Applying $\Gamma(-, \mathcal{O}_{-})$ gives the desired ring isomorphism. $\square$

Given a prime $\mathfrak{p}$ of a valuation ring R, we can of course reconstruct R from the valuation rings $R_{\mathfrak{p}}$ and $R/\mathfrak{p}$ and their canonical maps to the residue field $\kappa(\mathfrak{p})$:

Lemma 2.14. *Let R be a valuation ring and $\mathfrak{p}$ a prime ideal. Then the diagram*

$$\begin{tikzcd} R \arrow[r, hook] \arrow[d] & R_{\mathfrak{p}} \arrow[d] \\ R/\mathfrak{p} \arrow[r, hook] & R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \cong \mathrm{Frac}(R/\mathfrak{p}) \end{tikzcd} \tag{4}$$

is cartesian and the canonical R-module morphism $\mathfrak{p} \to \mathfrak{p}R_{\mathfrak{p}}$ is an isomorphism.

Proof. We have to check that an element a/s of $R_{\mathfrak{p}}$ is in R if its reduction modulo $\mathfrak{p}R_{\mathfrak{p}}$ is in $R/\mathfrak{p}$. This amounts to showing that $\mathfrak{p} = \mathfrak{p}R_{\mathfrak{p}}$: if there is $b \in R$ which agrees with a/s mod $\mathfrak{p}R_{\mathfrak{p}}$, then $b - a/s \in \mathfrak{p}R_{\mathfrak{p}}$. But if $\mathfrak{p} = \mathfrak{p}R_{\mathfrak{p}}$, then $b - a/s \in \mathfrak{p} \subset R$, i.e., $a/s \in R \subseteq R_{\mathfrak{p}}$.

Let $x = a/s \in \mathfrak{p}R_{\mathfrak{p}}$ with $a \in \mathfrak{p}$ and $s \in R - \mathfrak{p}$. Let v be the valuation of R. We compare $v(a)$ and $v(s)$. If we had $v(s) \geq v(a)$, then $x^{-1} \in R$ and hence $s = x^{-1}a \in \mathfrak{p}$ because $\mathfrak{p}$ is an ideal. This is a contradiction. Hence $v(s) < v(a)$ and $x \in R$. Now we have the equation $sx = a$ in R; hence $sx \in \mathfrak{p}$. As $\mathfrak{p}$ is a prime ideal and $s \notin \mathfrak{p}$, this implies $x \in \mathfrak{p}$. □

Another one of the many characterisations of valuation rings is the following.

Proposition 2.15 [Stacks project, Tag 092S; Olivier 1983]. *A local ring R is a valuation ring if and only if every submodule of every flat R-module is again a flat R-module.*

From this we immediately deduce the following, which is crucial for Corollary 2.18.

Corollary 2.16. *Let R be a valuation ring. If $R \to A$ is a flat R-algebra with $A \otimes_R A \to A$ also flat, then for every prime ideal $\mathfrak{p} \subset A$ the localisation $A_{\mathfrak{p}}$ is again a valuation ring.*

Proof. If A satisfies the hypotheses, then so does $A_{\mathfrak{p}}$, so we can assume A is local, and it suffices to show that every sub-A-module of a flat A-module is a flat A-module. Recall that flatness of $R \to A$ implies the forgetful functor $U : A\text{-mod} \to R\text{-mod}$ preserves flatness, because $- \otimes_A A \otimes_R - \cong (U-) \otimes_R -$. Recall also that flatness of $A \otimes_R A \to A$ implies that U *detects* flatness, because we have isomorphisms

$$- \otimes_A - \cong (- \otimes_A (A \otimes_R A) \otimes_A -) \otimes_{(A \otimes_R A)} A \quad \text{and} \quad (- \otimes_A (A \otimes_R A) \otimes_A -) \cong (U-) \otimes_R (U-).$$

Since U preserves and detects flatness, and preserves monomorphisms, the claim now follows from Proposition 2.15. □

As one might expect, the rank is bounded by the transcendence degree.

Proposition 2.17 [Bourbaki 1964, Chapitre VI, §10.3, Corollaire 1; Gabber and Ramero 2003, 6.1.24; Engler and Prestel 2005, Corollary 3.4.2]. *Suppose that R' is a valuation ring and $K' = \mathrm{Frac}(R')$. Let K'/K be a field extension. Note that $R = K \cap R'$ is again a valuation ring, and the inclusion $R \subseteq R'$ induces a field extension of residue fields $\kappa \subseteq \kappa'$, and a morphism $\Gamma_R \to \Gamma_{R'}$ of totally ordered groups. With this notation, we have*

$$\mathrm{tr.d}(K'/K) + \mathrm{rk}\, R \geq \mathrm{tr.d}(\kappa'/\kappa) + \mathrm{rk}\, R'.$$

In particular, if R' is a k-valuation ring for some field k, we have

$$\mathrm{tr.d}(K'/k) \geq \mathrm{tr.d}(\kappa'/k) + \mathrm{rk}\, R',$$

and finiteness of $\mathrm{tr.d}(K'/k)$ *implies finiteness of* $\mathrm{rk}\, R'$.

Recall that a local ring R is called *strictly henselian* if every faithfully flat étale morphism $R \to A$ admits a retraction. Every local ring R admits a "smallest" local morphism towards a strictly henselian local ring, which is unique up to nonunique isomorphism. The target of any such morphism is called *the strict henselisation* and is denoted by R^{sh}. There are various ways to construct this. One standard construction is to choose a separable closure κ^s of the residue field κ and take the colimit

$$R^{\mathrm{sh}} = \varinjlim_{R \to A \to \kappa^s} A \tag{5}$$

over factorisations such that $R \to A$ is étale.

From Corollary 2.16 we deduce the following.

Corollary 2.18. *For any valuation ring R, the strict henselisation is again a valuation ring. If R is of finite rank, then the following hold*:

(1) *If $R \to A$ is an étale algebra, $R \to A_{\mathfrak{p}}$ is also an étale algebra for all $\mathfrak{p} \subset A$.*

(2) *In the colimit (5), it suffices to consider those étale R-algebras A which are valuation rings.*

(3) *Every étale covering $U \to \mathrm{Spec}(R)$ admits a Zariski covering $V \to U$ such that V is a disjoint union of spectra of valuation rings.*

This corollary actually holds whenever the primes of R are well-ordered by [Bourbaki 1964, Chapitre VI, §8.3, Théorème 1], and might very well be true in general, but finite rank suffices for our purposes.

Proof. Recall that the diagonal is open immersion in the case of an unramified morphism. Hence the assumptions of Corollary 2.16 are satisfied for all étale R-algebras and their cofiltered limits. In particular, R^{sh} is a valuation ring.

(1) It suffices to show that $A \to A_{\mathfrak{p}}$ is of finite type. First note that since $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ is étale, it is quasifinite, and so since R has finite rank, A has finitely many primes. For every prime $\mathfrak{q} \subset A$ such that $\mathfrak{q} \not\subseteq \mathfrak{p}$, choose one $\pi_{\mathfrak{q}} \in \mathfrak{q} \backslash (\mathfrak{q} \cap \mathfrak{p})$. Then $A_{\mathfrak{p}} = A[\pi_{\mathfrak{q}_1}^{-1}, \dots, \pi_{\mathfrak{q}_n}^{-1}]$ (see Corollary 2.13).

(2) We can replace each A with $A_{\mathfrak{p}}$ without affecting the colimit (see part (1)).

(3) Part (1) also implies that $\mathrm{Spec}(A_{\mathfrak{p}}) \to \mathrm{Spec}(A)$ is an open immersion. Each of the finitely many $A_{\mathfrak{p}}$ is a valuation ring by Corollary 2.16 □

Just as strictly henselian rings are "local rings" for the étale topology, strictly henselian valuation rings are the "local rings" for the eh-topology.

Proposition 2.19 [Gabber and Kelly 2015, Theorem 2.6]. *A k-ring R is a valuation ring* (*resp. henselian valuation ring*, *strictly henselian valuation ring*) *if and only if for every* rh-*covering* (*resp.* cdh-*covering*, eh-*covering*) $\{U_i \to X\}$ *in* $\mathrm{Sch}_k^{\mathrm{ft}}$, *the morphism of sets of k-scheme morphisms*

$$\coprod_{i \in I} \hom(\mathrm{Spec}(R), U_i) \to \hom(\mathrm{Spec}(R), X)$$

is surjective.

The key input into the present paper is the same as for [Huber, Kebekus and Kelly 2017].

Theorem 2.20. *For every finitely generated extension K/k and every k-valuation ring R of K the map*

$$\Omega^n(R) \to \Omega^n(K)$$

is injective for all $n \geq 0$, i.e., Ω^n is torsion free on $\mathrm{val}(k)$.

This is due to Gabber and Ramero [2003, Corollary 6.5.21] for $n = 1$. The general case is deduced in [Huber, Kebekus and Kelly 2017, Lemma A.4].

2E. *Presheaves on categories of local rings.* We fix a base scheme S. The case of main interest for the present paper is $S = \operatorname{Spec}(k)$ with k a field.

Definition 2.21. Let X/S be a scheme of finite type. We use the following notation:

- $\mathrm{val}^{\mathrm{big}}(X)$ is the category of X-schemes $\operatorname{Spec}(R)$ with R either a field extension of k or a k-valuation ring of such a field.
- $\mathrm{val}(X)$ is the full subcategory of $\mathrm{val}^{\mathrm{big}}(X)$ of those $\pi : \operatorname{Spec}(R) \to X$ such that the transcendence degree $\mathrm{tr.d}(k(\eta)/k(\pi(\eta)))$ is finite (but we do not demand the field extension to be finitely generated), where η is the generic point of $\operatorname{Spec}(R)$.
- $\mathrm{val}^{\leq r}(X)$ is the full subcategory of $\mathrm{val}(X)$ of those $\operatorname{Spec}(R) \to X$ such that rank of the valuation ring R is $\leq r$.
- $\mathrm{dvr}(X)$ is the full subcategory of $\mathrm{val}(X)$ of those $\operatorname{Spec}(R) \to X$ such that $R = \mathcal{O}_{Y,y}$ for some $Y \in \mathsf{Sch}^{\mathrm{ft}}_X$ and some point $y \in Y$ of codimension ≤ 1 which is regular.
- $\mathrm{shval}(X)$ is the full subcategory of $\mathrm{val}(X)$ of those $\operatorname{Spec}(R) \to X$ such that R is strictly henselian.
- $\mathrm{rval}(X)$ is the full subcategory of $\mathrm{val}(X)$ of those $\operatorname{Spec}(R) \to X$ such that R is a valuation ring of a residue field of X.

We generically denote one of the above categories of local rings by $\mathrm{loc}(X)$. We also write loc for $\mathrm{loc}(S)$.

Note that the morphisms in the above categories are not required to be induced by local homomorphisms of local rings. All X-morphisms are allowed.

Definition 2.22. Let $\mathcal{F}$ be a presheaf on loc. We say that $\mathcal{F}$ is *torsion free* if

$$\mathcal{F}(R) \to \mathcal{F}(\mathsf{Frac}(R))$$

is injective for all valuation rings R in loc and their field of fractions $\mathsf{Frac}(R)$.

We could have also called this property *separated*, since when $\mathcal{F}$ is representable, it is the valuative criterion for separatedness.

Lemma 2.23. *For any $X \in \mathsf{Sch}^{\mathrm{ft}}_S$, let $X' \to X$ be any completely decomposed homeomorphism (in particular, if $X^{\mathrm{sn}} \in \mathsf{Sch}^{\mathrm{ft}}_S$, for example if S is Nagata, we can take $X' = X^{\mathrm{sn}}$). Then*

$$\mathrm{loc}(X) = \mathrm{loc}(X')$$

for all of the above categories of local rings.

Proof. This follows from the universal property of $(-)^{\mathrm{sn}}$ since valuation rings are normal. $\square$

The category val should be seen as a fully functorial version of the Riemann–Zariski space.

Definition 2.24. Let X be an integral S-scheme of finite type with generic point η. As a set, the Riemann–Zariski space $\mathsf{RZ}(X)$, called the "Riemann surface" in [Zariski and Samuel 1960, §17, p. 110], is the set of (not necessarily discrete) valuation rings over X of the function field $k(X)$; see also [Temkin 2011, before Remark 2.1.1, after Remark 2.1.2, before Proposition 2.2.1, Proposition 2.2.1, Corollary 3.4.7]. We turn it into a topological space by using as a basis the sets of the form

$$E(A') = \{R \in \mathsf{RZ}(X) | A' \subset R\},$$

where $\operatorname{Spec}(A)$ is an affine open of X, and A' is a finitely generated sub-A-algebra of $k(X)$; see [Temkin 2011, before Lemma 3.1.1, before Lemma 3.1.8]. It has a canonical structure of locally ringed space induced by the assignment $U \mapsto \bigcap_{R \in U} R$ for open subsets $U \subseteq \mathsf{RZ}(X)$. One can equivalently define $\mathsf{RZ}(X)$ as the inverse limit of all proper birational morphisms $Y \to X$, taking the inverse limit in the category of locally ringed spaces. In particular, as a set it is the inverse limit of the underlying sets of the Y, equipped with the coarsest topology making the projections $\varprojlim Y \to Y$ continuous, and the structure sheaf is the colimit of the inverse images of the $\mathcal{O}_Y$ along the projections $\varprojlim Y \to Y$.

This topological space is quasicompact, in the sense that every open cover admits a finite subcover; see [Zariski and Samuel 1960, Theorem 40] for the case $S = \operatorname{Spec}(k)$, and [Temkin 2011, Proposition 3.1.10] for general S.

Note that the Riemann–Zariski space is functorial only for dominant morphisms. Our category $\operatorname{loc}(X)$ above is the functorial version: it is the union of the Riemann–Zariski-spaces of all integral X-schemes of finite type.

3. Presheaves on categories of valuation rings

3A. ***Generalities.*** We now introduce our main player. We fix a base scheme S.

Definition 3.1. Let X/S be of finite type. Let $\mathcal{F}$ be a presheaf on one of the categories of local rings $\operatorname{loc}(X)$ of Definition 2.21 over X.

We define $\mathcal{F}_{\mathrm{loc}}(X)$ as a global section of the presheaf $\operatorname{Spec}(R) \mapsto \mathcal{F}(\operatorname{Spec}(R))$ on $\operatorname{loc}(X)$, i.e., as the projective limit

$$\mathcal{F}_{\mathrm{loc}}(X) = \varprojlim_{\operatorname{loc}(X)} \mathcal{F}(\operatorname{Spec}(R))$$

over the respective categories.

Remark 3.2. This means that an element of $\mathcal{F}_{\mathrm{loc}}(X)$ is defined as a system of elements $s_R \in \mathcal{F}(\operatorname{Spec}(R))$ indexed by objects $\operatorname{Spec}(R) \to X \in \operatorname{loc}(X)$ which are compatible in the sense that for every morphism $\operatorname{Spec}(R) \to \operatorname{Spec}(R')$ in $\operatorname{loc}(X)$, we have $s_{R'}|_R = s_R$. Note that this is an abuse of notation, since the element s_R does *not only* depend on R but also on the structure map $\operatorname{Spec}(R) \to X$. Most of the time the structure map will be clear from the context.

Remark 3.3. We have the following equivalent definitions of $\mathcal{F}_{\mathrm{loc}}(X)$.

(1) $\mathcal{F}_{\mathrm{loc}}(X)$ is the equaliser of the canonical maps

$$\mathcal{F}(X) = \mathrm{eq}\Bigg(\prod_{\mathrm{Spec}(R)\to X\in \mathrm{loc}(X)} \mathcal{F}(R) \underset{d^1}{\overset{d^0}{\rightrightarrows}} \prod_{\mathrm{Spec}(R)\to \mathrm{Spec}(R')\to X\in \mathrm{loc}(X)} \mathcal{F}(R)\Bigg).$$

In particular, since valuation rings are the "local rings" for the rh-site [Gabber and Kelly 2015], the construction $\mathcal{F}_{\mathrm{val}}$ can be thought of as a naïve Godement sheafification (it differs in general from the Godement sheafification because colimits do not commute with infinite products).

(2) $\mathcal{F}_{\mathrm{loc}}$ is the (restriction to $\mathsf{Sch}_S^{\mathrm{ft}}$ of the) right Kan extension along the inclusion $\iota : \mathrm{loc} \subseteq \mathsf{Sch}_S$:

$$\mathcal{F}_{\mathrm{val}} = (\iota^! \mathcal{F})|_{\mathsf{Sch}_S^{\mathrm{ft}}}.$$

(3) $\mathcal{F}_{\mathrm{loc}}(X)$ is the set of natural transformations

$$\mathcal{F}_{\mathrm{loc}}(X) = \hom_{\mathsf{PreShv(loc)}}(h^X, \mathcal{F}),$$

where $h^X = \hom_{\mathsf{Sch}_S}(-, X)$.

Lemma 3.4. *Let $\mathcal{F}$ be a presheaf on* loc. *Then the assignment $X \mapsto \mathcal{F}_{\mathrm{loc}}(X)$ defines a presheaf $\mathcal{F}_{\mathrm{loc}}$ on $\mathsf{Sch}_S^{\mathrm{ft}}$.*

Proof. Composition of $\mathrm{Spec}(R) \to X$ with a morphism $f : X \to Y$ of schemes of finite type over k defines a functor $\mathrm{loc}(X) \to \mathrm{loc}(Y)$ and hence a homomorphism of limits $f^* : \mathcal{F}_{\mathrm{loc}}(Y) \to \mathcal{F}_{\mathrm{loc}}(X)$. □

Remark 3.5. We show in Proposition 3.11 that for $S = \mathrm{Spec}(k)$ with k a perfect field, we have

$$\Omega^n_{\mathrm{val}} = \Omega^n_{\mathrm{val}^{\mathrm{big}}} = \Omega^n_{\mathrm{val}^{\le 1}} = \Omega^n_{\mathrm{shval}}$$

as presheaves on $\mathsf{Sch}_k^{\mathrm{ft}}$. In [Huber, Kebekus and Kelly 2017], we systematically studied the case of the category dvr. If every $X \in \mathsf{Sch}_k^{\mathrm{ft}}$ admits a proper birational morphism from a smooth k-scheme, we also have

$$\Omega^n_{\mathrm{val}} = \Omega^n_{\mathrm{dvr}}$$

because both are equal to Ω^n_{cdh} in this case. In positive characteristic, the only cases that we know $\Omega^n_{\mathrm{val}}(X) = \Omega^n_{\mathrm{dvr}}(X)$ unconditionally are if either $n = 0$ (see Remark 6.7), $n = \dim X$ (see Proposition 6.12), $\dim X < n$ (in which case both are zero) or $\dim X \le 3$.

Lemma 3.6. *Let $\mathcal{F}$ be a presheaf on* $\mathrm{val}^{\mathrm{big}}$. *Then $\mathcal{F}_{\mathrm{val}} = \mathcal{F}_{\mathrm{val}^{\mathrm{big}}}$ as presheaves on $\mathsf{Sch}_S^{\mathrm{ft}}$. A section over X is uniquely determined by the value on the residue fields of X and their valuation rings, that is, the maps $\mathcal{F}_{\mathrm{loc}}(X) \to \prod_{\mathrm{rval}(X)} \mathcal{F}(R)$ are injective for* $\mathrm{loc} = \mathrm{val}, \mathrm{val}^{\mathrm{big}}$ *and all $X \in \mathsf{Sch}_S^{\mathrm{ft}}$.*

Proof. We begin with the second statement. Suppose $s, t \in \mathcal{F}_{\mathrm{loc}}(X)$ are sections such that $s|_R = t|_R$ for all $R \in \mathrm{rval}(X)$. If $R \subseteq K$ is a valuation ring, then by Proposition 2.17, the intersection $R' = R \cap \kappa(x) \subseteq K$ is a valuation ring. Then the sections s_R, t_R for $\mathrm{Spec}(R) \to X$ (see Remark 3.2) agree by the compatibility condition $s_R = s_{R'}|_R = t_{R'}|_R = t_R$.

Now the first statement. Since we have a factorisation

$$\mathcal{F}_{\mathrm{val}^{\mathrm{big}}}(X) \to \mathcal{F}_{\mathrm{val}}(X) \to \prod_{\mathrm{rval}(X)} \mathcal{F}(R),$$

it follows that the first map is injective, and we only need to show it is surjective.

Let $s \in \mathcal{F}_{\mathrm{val}}(X)$ be a global section. Defining $t_R := s_{R'}|_R$, with R and R' as above, we get a candidate element $t = (t_R) \in \prod_{\mathrm{val}^{\mathrm{big}}(X)} \mathcal{F}(R)$ which is potentially in $\mathcal{F}_{\mathrm{val}^{\mathrm{big}}}(X)$. Let $\mathrm{Spec}(R_1) \to \mathrm{Spec}(R_2)$ be a morphism in $\mathrm{val}^{\mathrm{big}}(X)$. Let R'_1, R'_2 be the valuation rings of residue fields of X corresponding to R_1, R_2 as above, but since there is not necessarily a morphism $\mathrm{Spec}(R'_1) \to \mathrm{Spec}(R'_2)$, we also set $\mathfrak{p}'_2 = \ker(R'_2 \to R_1)$ and $S = \kappa(\mathfrak{p}'_2) \cap R_1$, to obtain the commutative diagram

$$\begin{tikzcd}
& & \mathrm{Spec}(R'_1) \\
& \mathrm{Spec}(S) \arrow[ur] \arrow[dr] & \\
\mathrm{Spec}(R_1) \arrow[ur] \arrow[dr] & & \mathrm{Spec}(R'_2) \\
& \mathrm{Spec}(R_2) \arrow[ur] &
\end{tikzcd} \tag{6}$$

in $\mathrm{val}^{\mathrm{big}}(X)$, with $R'_1, R'_2, S \in \mathrm{val}(X)$ by Proposition 2.17. Now the result follows from a diagram chase: we have

$$t_{R_1} = s_{R'_1}|_{R_1} = s_{R'_1}|_S|_{R_1} = s_S|_{R_1} = s_{R'_2}|_S|_{R_1} = s_{R'_2}|_{R_2}|_{R_1} = t_{R_2}|_{R_1}. \qquad \square$$

Recall that a presheaf on loc is torsion free if it sends dominant morphisms to monomorphisms (see Definition 2.22).

Lemma 3.7. *Let* $\mathrm{loc} \in \{\mathrm{val}, \mathrm{val}^{\mathrm{big}}, \mathrm{val}^{\le 1}, \mathrm{rval}\}$. *Let* $\mathcal{F}$ *be a torsion free presheaf on* $\mathrm{loc}(X)$. *Then* $\mathcal{F}_{\mathrm{loc}}(X)$ *is canonically isomorphic to*

$$\left\{ (s_x) \in \prod_{x \in X} \mathcal{F}(x) \;\middle|\; \begin{array}{l} \textit{for every } T \to X \in \mathrm{loc}(X) \textit{ there exists} \\ \qquad\qquad s_T \in \mathcal{F}(T) \textit{ such that } (s_x)|_{\prod_{t \in T} F(t)} = s_R|_{\prod_{t \in T} F(t)} \end{array} \right\}. \tag{7}$$

It is perhaps worth noting that the description in (7) is basically the presheaf rs F from the proof of [Kelly 2012, Proposition 3.6.12].

Proof. By torsion freeness, the projection $\mathcal{F}_{\mathrm{loc}}(X) \to \mathcal{F}_{\mathrm{val}^{\le 0}}(X)$ is injective. By functoriality the map $\mathcal{F}_{\mathrm{val}^{\le 0}}(X) \to \prod_{x \in X} \mathcal{F}(x)$ is injective.

Assume conversely we are given a system of s_x as in (7). As in the proof of the previous lemma, this gives us a candidate section $t_R \in \mathcal{F}(\mathrm{Spec}(R))$ for all $\mathrm{Spec}(R) \in \mathrm{loc}(X)$. It remains to check compatibility of these sections. Let $\mathrm{Spec}(R_1) \to \mathrm{Spec}(R_2)$ be a morphism in $\mathrm{loc}(X)$. The generic point of $\mathrm{Spec}(R_1)$ maps to a point of $\mathrm{Spec}(R_2)$ corresponding to a prime ideal $\mathfrak{p}_2 \subset R_2$. By torsion freeness, we may replace R_1 by its field of fractions and R_2 by $(R_2)_{\mathfrak{p}_2}$. In other words, R_1 is a field containing the residue field of R_2. Now the same diagram chase as for (6) works. Since $S = R'_2/\mathfrak{m}$ and $x = \mathrm{Spec}(R'_1)$ is the image of $\mathrm{Spec}(R_1)$ in X, keeping in mind the condition of (7) above, we have

$$t_{R_1} = s_x|_{R_1} = s_x|_{R'_2/\mathfrak{m}}|_{R_1} = s_{\mathrm{Frac}(R'_2)}|_{R'_2/\mathfrak{m}}|_{R_1} = s_{\mathrm{Frac}(R'_2)}|_{R_2}|_{R_1} = t_{R_2}|_{R_1}. \qquad \square$$

3B. ***Reduction to strictly henselian valuation rings.*** The aim of this section is to establish that using strictly henselian local rings gives the same result.

Proposition 3.8. *Let $\mathcal{F}$ be a presheaf on* val *that commutes with filtered colimits and satisfies the sheaf condition for the étale topology. Then the canonical projection morphism*

$$\mathcal{F}_{\mathrm{val}}(X) \to \mathcal{F}_{\mathrm{shval}}(X)$$

is an isomorphism.

Proof. Let $s, t \in \mathcal{F}_{\mathrm{val}}(X)$ be sections such that the induced elements of $\mathcal{F}_{\mathrm{shval}}(X)$ agree. Let $x \in X$ be a point with residue field κ. The separable closure κ^s of κ is in the category shval. By assumption,

$$\mathcal{F}(\kappa^s) = \varinjlim_{\lambda/\kappa} \mathcal{F}(\kappa),$$

where λ runs through the finite extensions of κ contained in κ^s. The vanishing of s_{κ^s} implies that there is one such λ with $s_\lambda = 0$. As $\lambda \subseteq \kappa^s$, the morphism $\mathrm{Spec}(\lambda) \to \mathrm{Spec}(\kappa)$ is étale. As a consequence of étale descent, we know that the map $\mathcal{F}(\kappa) \to \mathcal{F}(\lambda)$ is injective, hence $s_\kappa = t_\kappa$. The same argument also applies to a valuation ring R of κ and its strict henselisation, viewed as the colimit of (5); see Corollary 2.18(2). Hence the morphism in the statement is injective.

Now let $t \in \mathcal{F}_{\mathrm{shval}}(X)$. For any morphism $\mathrm{Spec}(R) \to X$ in $\mathrm{val}(X)$ with strict henselisation R^{sh}, let $G = \mathrm{Gal}(\mathrm{Frac}(R^{\mathrm{sh}})/\mathrm{Frac}(R))$. Since $\mathrm{Frac}(R) = \mathrm{Frac}(R^{\mathrm{sh}})^G$, and $R = R^{\mathrm{sh}} \cap \mathrm{Frac}(R)$, we have $R = (R^{\mathrm{sh}})^G$. In particular, the element $t_{R^{\mathrm{sh}}}$ lifts to $\mathcal{F}(R) \subseteq \mathcal{F}(R^{\mathrm{sh}})$, as $t_{R^{\mathrm{sh}}}$ must be compatible with every morphism $\mathrm{Spec}(R^{\mathrm{sh}}) \to \mathrm{Spec}(R^{\mathrm{sh}})$ in $\mathrm{shval}(X)$. In this way, we obtain an element t_R for every $\mathrm{Spec}(R) \to X$ in $\mathrm{val}(X)$. We want to know that these form a section of $\mathcal{F}_{\mathrm{val}}(X)$. But compatibility with morphisms $\mathrm{Spec}(R) \to \mathrm{Spec}(R')$ of $\mathrm{val}(X)$ follows from the definition of the t_R, the fact that strict henselisations are functorial [Stacks project, Tag 08HR], and the morphisms $\mathcal{F}(R) \to \mathcal{F}(R^{\mathrm{sh}})$ being injective, which we just proved. □

3C. ***Reduction to rank one.*** The aim of this section is to establish that using rank one valuation rings gives the same result:

Proposition 3.9. *Let $\mathcal{F}$ be a torsion free presheaf on* val*. Assume that for every valuation ring R in* val *and prime ideal $\mathfrak{p}$ the diagram*

$$\begin{tikzcd} \mathcal{F}(R) \arrow[r, hook] \arrow[d] & \mathcal{F}(R_{\mathfrak{p}}) \arrow[d] \\ \mathcal{F}(R/\mathfrak{p}) \arrow[r, hook] & \mathcal{F}(\mathrm{Frac}(R/\mathfrak{p})) \end{tikzcd}$$

is cartesian. Then the natural restriction

$$\mathcal{F}_{\mathrm{val}} \to \mathcal{F}_{\mathrm{val}^{\leq 1}}$$

is an isomorphism.

Proof. Recall that $\mathrm{val}^{\leq r}$ is defined to be the subcategory of val using only valuations of rank at most r, and $\mathcal{F}_{\mathrm{val}^{\leq r}}$ denotes the presheaf obtained using only $\mathrm{val}^{\leq r}$. There are canonical morphisms

$$\mathcal{F}_{\mathrm{val}} \to \mathcal{F}_{\mathrm{val}^{\leq r}} \to \mathcal{F}_{\mathrm{val}^{\leq r-1}}$$

for all $r \geq 1$. By torsion freeness these are all subpresheafs of $\mathcal{F}_{\mathrm{val}^{\leq 0}}$, and so the two morphisms above are monomorphisms. Moreover, $\mathcal{F}_{\mathrm{val}} = \bigcap_r \mathcal{F}_{\mathrm{val}^{\leq r}}$ because by definition, all fields in val are of finite transcendence degree over k and hence all valuation rings have finite rank (see Proposition 2.17). Hence it suffices to show that $\mathcal{F}_{\leq r} \to \mathcal{F}_{\leq r-1}$ is an epimorphism for all $r \geq 2$.

Let $\mathfrak{p}$ be a prime ideal of a valuation ring R. By Corollary 2.12 both $R_{\mathfrak{p}}$ and $R/\mathfrak{p}$ are valuation rings and if $\mathfrak{p}$ is not maximal or zero, then $R_{\mathfrak{p}}$ and $R/\mathfrak{p}$ are of rank smaller than R. Indeed, $\mathrm{rk}\, R = \mathrm{rk}\, R_{\mathfrak{p}} + \mathrm{rk}\, R/\mathfrak{p}$ since the rank is equal to the Krull dimension and the set of ideals, and in particular prime ideals, of a valuation ring is totally ordered (see Proposition 2.11).

Let $t \in \mathcal{F}_{\mathrm{val}^{\leq r-1}}(X)$ be a section. If $r \geq 2$, we choose a canonical candidate $s \in \prod_{R \in \mathrm{val}^{\leq r}} \mathcal{F}(\mathrm{Spec}(R))$ for an element of $\mathcal{F}_{\mathrm{val}^{\leq r}}(X)$ in the preimage of t: for every valuation ring of rank r, take $\mathfrak{p}$ to be any prime ideal with $R_{\mathfrak{p}}, R/\mathfrak{p} \in \mathrm{val}^{\leq r-1}$ and construct a section over R using the cartesian square of the assumption. Since the morphisms $\mathcal{F}(\mathrm{Spec}(R)) \to \mathcal{F}(\mathrm{Spec}(R_{\mathfrak{p}}))$ are monomorphisms, the choice of $\mathfrak{p}$ does not matter.

It remains to check that this candidate section $s \in \prod_{\mathrm{val}^{\leq r}} \mathcal{F}(R)$ is actually a section of $\mathcal{F}_{\mathrm{val}^{\leq r}}(X)$, i.e., for any X-morphism of valuation rings $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ with R or R' (or both) of rank r, we want to know that the element s_R restricts to $s_{R'}$. The morphism $\mathcal{F}(R') \to \mathcal{F}(\mathsf{Frac}(R'))$ is injective, so it suffices to consider the case when R' is some field L. Then $R \to L$ factors as

$$R \to R/\mathfrak{p} \to \mathsf{Frac}(R/\mathfrak{p}) \to L,$$

where $\mathfrak{p}$ is the prime $\mathfrak{p} = \ker(R \to L)$. That s_R is sent to $s_{R/\mathfrak{p}}$ comes from the independence of the choice of $\mathfrak{p}$ that we used to construct s. For the same reason, $s_{R/\mathfrak{p}}$ is sent to $s_{\mathsf{Frac}(R/\mathfrak{p})}$. Finally, both $\mathsf{Frac}(R/\mathfrak{p})$ and L, being fields, are of rank zero, and so $s_{\mathsf{Frac}(R/\mathfrak{p})} = t_{\mathsf{Frac}(R/\mathfrak{p})}$ is sent to $s_L = t_L$ since t is already a section of $\mathcal{F}_{\mathrm{val}^{\leq r-1}}$. □

Corollary 3.10. *For any scheme Y, writing h^Y for the presheaf* $\mathrm{hom}_{\mathsf{Sch}_k}(-, Y)$, *the canonical maps are isomorphisms*

$$h^Y_{\mathrm{val}^{\mathrm{big}}} \cong h^Y_{\mathrm{val}} \cong h^Y_{\mathrm{val}^{\leq 1}} \cong h^Y_{\mathrm{shval}}.$$

Proof. The first isomorphism is Lemma 3.6. The second one follows from Proposition 3.9 — note that

$$\mathrm{Spec}(R) = \mathrm{Spec}(R/\mathfrak{p}) \amalg_{\mathrm{Spec}(\mathsf{Frac}(R/\mathfrak{p}))} \mathrm{Spec}(R_{\mathfrak{p}})$$

by [Ferrand 2003, Théorème 5.1]. The last one follows from Proposition 3.8. □

3D. ***The case of differential forms.*** Now we show that the previous material applies to the case of differential forms:

Proposition 3.11. *Let $S = \mathrm{Spec}(k)$ with k a perfect field. The canonical morphisms*

$$\begin{tikzcd}
 & & \Omega^n_{\mathrm{shval}} \\
\Omega^n_{\mathrm{val}^{\mathrm{big}}} \arrow[urr, bend left] \arrow[r, "\text{Lemma 3.6}", "\cong"'] \arrow[drr, bend right] & \Omega^n_{\mathrm{val}} \arrow[ur, "\text{Proposition 3.8}"', "\cong"] \arrow[dr, "\text{Proposition 3.9}", "\cong"'] & \\
 & & \Omega^n_{\mathrm{val}^{\le 1}}
\end{tikzcd}$$

are isomorphisms of presheaves on $\mathsf{Sch}^{\mathrm{ft}}_k$.

Proof. The presheaf Ω^n on $\mathrm{val}^{\mathrm{big}}$ is an étale sheaf and commutes with direct limits. Hence we may apply Proposition 3.8 in order to show the comparison to $\Omega^n_{\mathrm{shval}}$. For the final isomorphism we want to apply Proposition 3.9. The rest of this section is devoted to checking the necessary cartesian diagram. □

Lemma 3.12. *Let R be a k-valuation ring and $\mathfrak{p}$ a prime ideal. Then the diagram*

$$\begin{tikzcd}
\Omega^1(R) \arrow[r, hook] \arrow[d, two heads] & \Omega^1(R_{\mathfrak{p}}) \arrow[d, two heads] \\
\Omega^1(R/\mathfrak{p}) \arrow[r, hook] & \Omega^1(\mathsf{Frac}(R/\mathfrak{p}))
\end{tikzcd}$$

is cartesian.

Proof. The R-module $\Omega^1(R)$ is flat because it is torsion free over a valuation ring. We tensor the diagram (4) of Lemma 2.14 with the flat R-module $\Omega^1(R)$ and obtain the cartesian diagram

$$\begin{tikzcd}
\Omega^1(R) \arrow[r, hook] \arrow[d] & \Omega^1(R_{\mathfrak{p}}) \arrow[d] \\
\Omega^1(R)\otimes_R R/\mathfrak{p} \arrow[r, hook] & \Omega^1(R_{\mathfrak{p}})\otimes_{R_{\mathfrak{p}}} R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}
\end{tikzcd}$$

In the next step we use the fundamental exact sequence for differentials of a quotient [Matsumura 1970, Theorem 58, p. 187] and obtain the following diagram with exact columns:

$$\begin{tikzcd}
\mathfrak{p}/\mathfrak{p}^2 \arrow[r, "\text{epi}"] \arrow[d] & \mathfrak{p}R_{\mathfrak{p}}/\mathfrak{p}^2R_{\mathfrak{p}} \arrow[d] \\
\Omega^1(R)\otimes_R R/\mathfrak{p} \arrow[r, hook, "\text{monic}"] \arrow[d, two heads, "\text{epi}"'] & \Omega^1(R_{\mathfrak{p}})\otimes_{R_{\mathfrak{p}}} R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \arrow[d] \\
\Omega^1(R/\mathfrak{p}) \arrow[r, hook] \arrow[d] & \Omega^1(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}) \arrow[d] \\
0 & 0
\end{tikzcd} \tag{8}$$

The top horizontal arrow is an isomorphism, and in particular a surjection, because $\mathfrak{p} = \mathfrak{p}R_{\mathfrak{p}}$. A small diagram chase now shows that the second square is cartesian. Putting the two diagrams together, we get the claim. □

Lemma 3.13. *Let R be a k-valuation ring and $\mathfrak{p}$ a prime ideal. Then the diagram*

$$\begin{CD} \Omega^n(R) @>>> \Omega^n(R_{\mathfrak{p}}) \\ @VVV @VVV \\ \Omega^n(R/\mathfrak{p}) @>>> \Omega^n(\mathsf{Frac}(R/\mathfrak{p})) \end{CD}$$

is cartesian for all $n \geq 0$.

Proof. We have already done the cases $n = 0, 1$. We want to go from $n = 1$ to general n by taking exterior powers. We write $\Omega^1(R)$ as the union of its finitely generated sub-R-modules

$$\Omega^1(R) = \bigcup N.$$

We write $\overline{N}$ for the image of N in $\Omega^1(R/\mathfrak{p})$. Note that

$$\begin{aligned} \Omega^1(R/\mathfrak{p}) &= \bigcup \overline{N}, \\ \Omega^1(R_{\mathfrak{p}}) = \Omega^1(R)_{\mathfrak{p}} &= \bigcup N_{\mathfrak{p}}, \\ \Omega^1(R_{\mathfrak{p}}/\mathfrak{p}) = \Omega^1(R/\mathfrak{p})_{\mathfrak{p}} &= \bigcup \overline{N}_{\mathfrak{p}}. \end{aligned}$$

The module N is a torsion free finitely generated module over the valuation ring R, hence free. The $R_{\mathfrak{p}}$, $R/\mathfrak{p}$, $R_{\mathfrak{p}}/\mathfrak{p}$ modules $N_{\mathfrak{p}}$, $N/\mathfrak{p}$, $N_{\mathfrak{p}}/\mathfrak{p}$ are therefore also free, and the same is true for all the exterior powers. So for all $n \geq 0$ we get the following cartesian diagram:

$$\begin{CD} \bigwedge^n N @>>> \bigwedge^n N_{\mathfrak{p}} \\ @VVV @VVV \\ \bigwedge^n (N/\mathfrak{p}) @>>> \bigwedge^n (N_{\mathfrak{p}}/\mathfrak{p}) \end{CD}$$

Note that if the rank of N is one, and $n = 0$, this is just the cartesian diagram from Lemma 2.14. Note also that at this stage we are working with $N/\mathfrak{p}$ and $N_{\mathfrak{p}}/\mathfrak{p}$, instead of $\overline{N}$ and $\overline{N}_{\mathfrak{p}}$.

Passing to the direct limit, we have established as a first step that the diagram

$$\begin{CD} \Omega^n(R) @>>> \Omega^n(R_{\mathfrak{p}}) \\ @VVV @VVV \\ \Omega^n(R)/\mathfrak{p} @>>> \Omega^n(R_{\mathfrak{p}})/\mathfrak{p} \end{CD}$$

is cartesian.

Let $\pi : \Omega^n(R)/\mathfrak{p} \to \Omega^n(R/\mathfrak{p})$ and $\theta : \Omega^n(R_\mathfrak{p})/\mathfrak{p} \to \Omega^n(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})$ be the natural maps. We want to show that

$$\begin{tikzcd}
\Omega^n(R)/\mathfrak{p} \arrow[r] \arrow[d] & \Omega^n(R_\mathfrak{p})/\mathfrak{p} \arrow[d] \\
\Omega^n(R/\mathfrak{p}) \arrow[r] & \Omega^n(R_\mathfrak{p}/\mathfrak{p})
\end{tikzcd}$$

is also cartesian. Let $\pi^{-1}\overline{N} \subseteq \Omega^1(R)/\mathfrak{p}$ be the preimage of $\overline{N}$. Similarly, let $\theta^{-1}\overline{N}_\mathfrak{p} \subseteq \Omega^1(R_\mathfrak{p})/\mathfrak{p}$ be the preimage of $\overline{N}_\mathfrak{p}$ by $\theta : \Omega^1(R_\mathfrak{p})/\mathfrak{p} \to \Omega^1(R_\mathfrak{p}/\mathfrak{p})$. In particular, we have the following cube, for which the two side squares are cartesian by definition, the front square is the cartesian square from diagram (8) on page 666, and a diagram chase then shows that the back square is also cartesian. Note that the lower and upper faces are probably not cartesian, but this does not affect the argument.

$$\begin{tikzcd}
\pi^{-1}\overline{N} \arrow[rr, hook] \arrow[dr, hook] \arrow[dd, two heads, "\pi"'] & & \theta^{-1}\overline{N}_\mathfrak{p} \arrow[dr, hook] \arrow[dd, two heads, "\theta"' near start] & \\
& \Omega^1(R)/\mathfrak{p} \arrow[rr, hook] \arrow[dd, two heads] & & \Omega^1(R_\mathfrak{p})/\mathfrak{p} \arrow[dd, two heads] \\
\overline{N} \arrow[rr, hook] \arrow[dr, hook] & & \overline{N}_\mathfrak{p} \arrow[dr, hook] & \\
& \Omega^1(R/\mathfrak{p}) \arrow[rr, hook] & & \Omega^1(R_\mathfrak{p}/\mathfrak{p})
\end{tikzcd}$$

We claim that the back square stays cartesian when passing to higher exterior powers. We show this by comparing the kernels of $\bigwedge^n \pi$ and $\bigwedge^n \theta$, cf. the diagram chase of diagram (8) on page 666. More precisely, to show that the higher exterior powers of the back square are cartesian, it suffices to show that $\ker \bigwedge^n \pi \to \ker \bigwedge^n \theta$ is a surjection. We will show that it is an isomorphism.

Note that since the back face is cartesian, and the horizontal morphism is a monomorphism, we have $\ker(\pi) \cong \ker(\theta)$. Let X be this common kernel. The module $\overline{N} \subset \Omega^1(R/\mathfrak{p})$ is torsion free and finitely generated over the valuation ring $R/\mathfrak{p}$, and therefore it is free, and in particular, projective. Hence π admits a splitting σ.

The map $\iota = \sigma \otimes_{R/\mathfrak{p}} R_\mathfrak{p}/\mathfrak{p}$ is then a splitting of θ compatible with σ. In particular, we have compatible decompositions

$$\pi^{-1}\overline{N} = X \oplus \overline{N}, \qquad \theta^{-1}\overline{N}_\mathfrak{p} = X \oplus \overline{N}_\mathfrak{p}.$$

The X's are the same due to the square being cartesian. Hence

$$\bigwedge_{R/\mathfrak{p}}^{n} \pi^{-1}\overline{N} = \bigoplus_{i=0}^{n} \bigwedge_{R/\mathfrak{p}}^{i} X \underset{R/\mathfrak{p}}{\otimes} \bigwedge_{R/\mathfrak{p}}^{n-i} \overline{N},$$

$$\bigwedge_{R_\mathfrak{p}/\mathfrak{p}}^{n} \theta^{-1}\overline{N}_\mathfrak{p} = \bigoplus_{i=0}^{n} \bigwedge_{R_\mathfrak{p}/\mathfrak{p}}^{i} X \underset{R_\mathfrak{p}/\mathfrak{p}}{\otimes} \bigwedge_{R_\mathfrak{p}/\mathfrak{p}}^{n-i} \overline{N}_\mathfrak{p}.$$

The kernels of $\bigwedge^n \pi$ and $\bigwedge^n \pi_F$ are given by the analogous sums but indexed by $i = 1, \dots, n$. Note that X is an $R_\mathfrak{p}/\mathfrak{p} = \mathsf{Frac}(R/\mathfrak{p})$-vector space. Since $R_\mathfrak{p}/\mathfrak{p} = \mathsf{Frac}(R/\mathfrak{p})$, if A, B are $R_\mathfrak{p}/\mathfrak{p}$-vector spaces, then

$$A \otimes_{R/\mathfrak{p}} B = A \otimes_{R_\mathfrak{p}/\mathfrak{p}} B.$$

Hence

$$\bigwedge_{R/\mathfrak{p}}^{i} X = \bigwedge_{R_\mathfrak{p}/\mathfrak{p}}^{i} X$$

and finally

$$\Big(\bigwedge_S^i X\Big) \otimes_S \overline{N} = \Big(\bigwedge_S^i X\Big) \otimes_F F \otimes_S \overline{N} = \Big(\bigwedge_F^i X\Big) \otimes_F \overline{N}_\mathfrak{p},$$

where $S = R/\mathfrak{p}$ and $F = \mathsf{Frac}(R/\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}$. There are similar formulas for higher exterior powers of $\overline{N}_\mathfrak{p}$ and $\overline{N}$. So we have shown that $\ker \bigwedge^n \pi \to \ker \bigwedge^n \theta$ is an isomorphism as claimed. $\square$

Remark 3.14. This finishes the proof of Proposition 3.11.

4. Descent properties of Ω^n_{val}

Our presheaf of interest, the presheaf Ω^n, is a sheaf for the étale topology. This has far reaching consequences.

Remark 4.1. Let us point out that we have written Ω^n everywhere because this is our main object of study, but everything in this section is valid for any presheaf $\mathcal{F}$ on $\langle \mathrm{val}(S), \mathsf{Sch}^{\mathrm{ft}}_S\rangle$, the full subcategory of S-schemes whose objects are those of $\mathrm{val}(S)$ and $\mathsf{Sch}^{\mathrm{ft}}_S$, satisfying:

(Co) $\mathcal{F}$ commutes with filtered colimits,

(Et) $\mathcal{F}$ satisfies the sheaf condition for the étale topology,

(TF) $\mathcal{F}$ is torsion free in the sense that $\mathcal{F}(R) \to \mathcal{F}(\mathsf{Frac}(R))$ is injective for every valuation ring R.

For example, if Y is any scheme, then $\mathcal{F}(-) = h^Y(-) = \hom(-, Y)$ satisfies these conditions.

Proposition 4.2. *Suppose that $\mathcal{F}$ is a presheaf on* $\mathrm{shval}(X)$. *Then $\mathcal{F}_{\mathrm{shval}}$ is an* eh-*sheaf. Similarly, $\mathcal{F}_{\mathrm{val}}$ is an* rh-*sheaf for any presheaf $\mathcal{F}$ on* $\mathrm{val}(X)$.

Remark 4.3. If we had defined a category hval of henselian valuation rings, we could also have said that $\mathcal{F}_{\mathrm{hval}}$ is a cdh-sheaf for any presheaf $\mathcal{F}$ on $\mathrm{hval}(X)$.

Proof. We only give the proof for the $\mathrm{shval}(X)$, $\mathcal{F}_{\mathrm{shval}}$, eh case, as the same proof works for $\mathrm{val}(X)$, $\mathcal{F}_{\mathrm{val}}$, rh. Let $U \to X$ be an eh-cover. The map $\mathcal{F}_{\mathrm{shval}}(X) \to \mathcal{F}_{\mathrm{shval}}(U)$ is injective because by Proposition 2.19 every morphism $\mathrm{Spec}(R) \to X$ from a strictly henselian valuation ring factors through U.

Now suppose $s \in \mathcal{F}_{\mathrm{shval}}(U)$ satisfies the sheaf condition for the cover $U \to X$. Let $f : \mathrm{Spec}(R) \to X$ be in $\mathrm{shval}(X)$. By choosing a lift $g : \mathrm{Spec}(R) \to U \to X$, we obtain a candidate section $s_g \in \mathcal{F}(\mathrm{Spec}(R))$. We claim that it is independent of the choice of lift. Let g' be a second lift. The pair (g, g') defines a morphism $\mathrm{Spec}(R) \to U \times_X U$. By assumption, s is in the kernel of $\mathcal{F}_{\mathrm{shval}}(U) \xrightarrow{pr_1 - pr_2} \mathcal{F}_{\mathrm{shval}}(U \times_X U)$. In particular,

$s_g = s_{g'} \in \mathcal{F}(\mathrm{Spec}(R))$ in the (g, g')-component. Let $f_1 : \mathrm{Spec}(R_1) \to X$ and $f_2 : \mathrm{Spec}(R_2) \to X$ be in $\mathrm{loc}(X)$ and $\mathrm{Spec}(R_1) \to \mathrm{Spec}(R_2)$ an X-morphism. The choice of a lift $g_2 : \mathrm{Spec}(R_2) \to U$ also induces a lift g_1. The section s_{g_2} restricts to s_{g_1}, hence s_{f_2} restricts to s_{f_1}. Our candidate components define an element of $\mathcal{F}_{\mathrm{val}}(X)$. □

Corollary 4.4. *Let k be a perfect field. The presheaf Ω^n_{val} is an* eh*-sheaf on $\mathrm{Sch}^{\mathrm{ft}}_k$. More generally, $\mathcal{F}_{\mathrm{val}}$ is an* eh*-sheaf on $\mathrm{Sch}^{\mathrm{ft}}_S$ for any presheaf $\mathcal{F}$ on* $\mathrm{val}(S)$ *which commutes with filtered colimits and satisfies the sheaf condition for the étale topology.*

Proof. The proof follows from $\Omega^n_{\mathrm{val}} = \Omega^n_{\mathrm{shval}}$ (see Proposition 3.8). □

This immediately implies the following:

Corollary 4.5. *The map of presheaves $\Omega^n \to \Omega^n_{\mathrm{val}}$ on $\mathrm{Sch}^{\mathrm{ft}}_k$ induces maps of presheaves*

$$\Omega^n_{\mathrm{rh}} \to \Omega^n_{\mathrm{cdh}} \to \Omega^n_{\mathrm{eh}} \to \Omega^n_{\mathrm{val}}. \tag{9}$$

More generally, this is true for any presheaf $\mathcal{F}$ satisfying (Co) *and* (Et) *from Remark 4.1.*

We find it worthwhile to restate the following theorem from [Huber, Kebekus and Kelly 2017]. The original statement is for Ω^n and $S = \mathrm{Spec}(k)$, but one may check directly that the proof works for any Zariski sheaf and the relative Riemann–Zariski space (Definition 2.24) of [Temkin 2011].

Theorem 4.6 [Huber, Kebekus and Kelly 2017, Theorem A.3]. *For any presheaf $\mathcal{F}$ on $\mathrm{Sch}^{\mathrm{ft}}_S$ the following are equivalent*:

(1) cf. [Huber, Kebekus and Kelly 2017, Hypothesis H] *For every integral $X \in \mathrm{Sch}^{\mathrm{ft}}_S$ and $s, t \in \mathcal{F}(X)$ such that $s|_U = t|_U$ for some dense open $U \subseteq X$, there exists a proper birational morphism $X' \to X$ with $s|_{X'} = t|_{X'}$.*

(2) cf. [Huber, Kebekus and Kelly 2017, Hypothesis V] *For any $R \in \mathrm{val}(S)$ with fraction field K the morphism $\mathcal{F}'(R) \to \mathcal{F}'(K)$ is injective. That is, $\mathcal{F}'$ is torsion free (Definition 2.22).*

Here $\mathcal{F}'$ is the presheaf sending $R \in \mathrm{val}(S)$ to the colimit $\mathcal{F}'(R) = \varinjlim_{(R\downarrow \mathrm{Sch}^{\mathrm{ft}}_S)} \mathcal{F}(Y)$ over factorisations $\mathrm{Spec}(R) \to Y \to S$ *through $Y \in \mathrm{Sch}^{\mathrm{ft}}_S$. Clearly, $(\Omega^n)' = \Omega^n$.*

More generally, for any presheaf $\mathcal{F}$ satisfying (Co) *and* (Et) *from Remark 4.1, one has $\mathcal{F}' = \mathcal{F}$, and* (2) *is exactly the condition* (TF).

Corollary 4.7. *The maps $\Omega^n_{\mathrm{rh}} \to \Omega^n_{\mathrm{cdh}} \to \Omega^n_{\mathrm{eh}} \to \Omega^n_{\mathrm{val}}$ are injective. More generally, the same is true for any presheaf $\mathcal{F}$ satisfying* (Co), (Et) *and* (TF) *from Remark 4.1.*

Proof. See [Huber, Kebekus and Kelly 2017, Corollary 5.10, Proposition 5.12]. It suffices to show that for any $X \in \mathrm{Sch}^{\mathrm{ft}}_S$ and any $\omega, \omega' \in \Omega^n(X)$ with $\omega|_{\prod_{x\in X} \Omega^n(x)} = \omega'|_{\prod_{x\in X} \Omega^n(x)}$ there is a cdp-morphism $X' \to X$ with $\omega|_{X'} = \omega'|_{X'}$. By noetherian induction, it suffices to show that Theorem 4.6(1) is satisfied. But by Theorem 4.6, this is equivalent to being torsion free. □

In order to prove surjectivity, we need a strong compactness property.

Lemma 4.8. *Let $(Y_i)_{i\in I}$ be a filtered system of nonempty noetherian topological spaces, with transition morphisms $\pi_{ij} : Y_i \to Y_j$. Then there is a system of nonempty irreducible closed subsets $Z_i \subseteq Y_i$ such that $\overline{\pi_{ij}(Z_i)} = Z_j$.*

Proof. To every i we attach a finite set F_i as follows: Let $V_1, \dots, V_n$ be the irreducible components of Y_i. Let F_i be the set of the irreducible components of all multiple intersections $V_{m_1} \cap \dots \cap V_{m_k}$ for all $1 \le k \le n$ and all choices of m_j. We define a transition map $F_i \to F_j$ by mapping an element $W \in F_i$ to the smallest element of F_j containing $\pi_{ij}(W)$. This defines a filtered system of nonempty finite sets $(F_i)_{i\in I}$. Its projective limit is nonempty by [Stacks project, Tag 086J]. Let $(W_i)_{i\in I}$ be an element of the limit.

Now for each $j \in I$, consider the partially ordered set of closures of images $\{\overline{\pi_{ij}(W_i)} \subseteq Y_j : i \le j\}$. We claim that there is an i_j such that $\overline{\pi_{i'j}(W_{i'})} = \overline{\pi_{i_j j}(W_{i_j})}$ for all $i' \le i_j$. Indeed, if not, then we can construct a strictly decreasing sequence of closed subsets of Y_j, contradicting the fact that it is a noetherian topological space. Define $Z_j = \overline{\pi_{i_j j}(W_{i_j})} \subseteq Y_j$ for such an i_j. It follows from our definitions of the Z_j that for every $i, j \in I$ with $i \le j$, we have $\overline{\pi_{ij}(Z_i)} = Z_j$. □

Let X be integral and $Y \to X$ proper birational. Fix $\omega, \omega' \in \Omega^n_{\mathrm{val}}(X)$ (or in $\mathcal{F}_{\mathrm{val}}(X)$ if we are using an $\mathcal{F}$ as in Remark 4.1) and define $Y^{\omega\neq\omega'} \subset Y$ to be the subset of points $y \in Y$ for which $\omega_y \neq \omega'_y$. We view it as a topological space with the induced topology.

Lemma 4.9. *The topological space $Y^{\omega\neq\omega'}$ is noetherian. In addition, every irreducible closed subset of $Y^{\omega\neq\omega'}$ has a unique generic point.*

Proof. The topological space of Y is noetherian. Subspaces of noetherian topological spaces with the induced topology are noetherian (easy exercise), and hence $Y^{\omega\neq\omega'}$ is noetherian. Let $Z^{\omega\neq\omega'} \subseteq Y^{\omega\neq\omega'}$ be irreducible. Note its closure $Z \subseteq Y$ is also irreducible. Let η be the generic point of Z in the scheme Y. Assume $\eta \notin Z^{\omega\neq\omega'}$. By definition this means $\omega_\eta = \omega'_\eta$. Hence, since Ω^n commutes with filtered colimits, $\omega = \omega'$ also on some open dense $U \subseteq Z$ of the reduction of Z. As $Z^{\omega\neq\omega'}$ is dense in Z, the intersection $U \cap Z^{\omega\neq\omega'}$ is nonempty, implying the existence of a point $y \in Y^{\omega\neq\omega'}$ for which $\omega_y = \omega'_y$, and contradicting the definition of $Y^{\omega\neq\omega'}$. Hence $\eta \in Z^{\omega\neq\omega'}$ and we have found a generic point. Uniqueness follows easily from uniqueness of generic points in Y. □

We are implicitly using the Riemann–Zariski space of X (see Definition 2.24) in the following proof.

Lemma 4.10. *Let X be integral and $\omega, \omega' \in \Omega^n_{\mathrm{val}}(X)$ such that $\omega_{k(X)} = \omega'_{k(X)}$. Then there exists a proper birational $Y \to X$ such that $\omega|_Y = \omega'|_Y \in \Omega^n_{\mathrm{val}}(Y)$. More generally, this is true for a presheaf $\mathcal{F}$ satisfying* (Co), (Et) *and* (TF) *from Remark 4.1.*

Proof. Suppose the contrary — that $Y^{\omega\neq\omega'}$ is nonempty for all proper birational maps $Y \to X$. The Y's and hence also their subspaces $Y^{\omega\neq\omega'}$ form a filtered system of noetherian topological spaces. By Lemma 4.8, if all $Y^{\omega\neq\omega'}$ are nonempty, then there exists a system $Z_Y \subseteq Y^{\omega\neq\omega'}$ of irreducible subspaces of the $Y^{\omega\neq\omega'}$, such that the map $Z_{Y'} \to Z_Y$ induced by any birational $Y' \to Y \to X$ is dominant. Their generic points (which exist and are unique by Lemma 4.9) define a point $y = (y_i) \in \varprojlim Y$ for which

$\omega_{k(y_i)} \neq \omega'_{k(y_i)}$ for all y_i. But Ω^n commutes with cofiltered limits of rings, so $\omega|_{k(y)} \neq \omega'|_{k(y)}$, where $k(y)$ is the residue field of the valuation ring $R = \varinjlim \mathcal{O}_{Y_i,y_i}$ in the locally ringed space $\varprojlim Y$. But R is an X-valuation ring of $k(X)$, so by torsion freeness we must also have $\omega|_{k(X)} \neq \omega'|_{k(X)}$, contradicting our assumption that $\omega|_{k(X)} = \omega'|_{k(X)}$. □

Proposition 4.11. *The map $\Omega^n_{\mathrm{rh}} \to \Omega^n_{\mathrm{val}}$ is surjective. More generally, $\mathcal{F}_{\mathrm{rh}} \to \mathcal{F}_{\mathrm{val}}$ is surjective for any presheaf $\mathcal{F}$ satisfying* (Co), (Et) *and* (TF) *from Remark 4.1.*

Proof. As both are rh-sheaves, we may work rh-locally. In particular, without loss of generality X is integral with function field K. Choose $\omega = (\omega_R)_R \in \Omega^n_{\mathrm{val}}(X)$.

We start with an X-valuation ring $R \subset K$, i.e., $R \in \mathrm{RZ}(X)$. The form ω_R is already defined on some ring A_R of finite type over X. Let $\tilde{\omega}_R$ be this class. For all R' in the Zariski-open subset of $\mathrm{RZ}(X)$ defined by A_R (i.e., $A_R \subset R'$), we have $\omega_{R'} = \tilde{\omega}_R$ because $\Omega^n(R')$ is torsion free and the equality holds in $\Omega^n(K)$.

As $\mathrm{RZ}(X)$ is quasicompact, we may cover it by finitely many of these and obtain a finite set of rings $A_i \subset K$ which are of finite type over X. On each of these we are given a differential form $\omega_{A_i} \in \Omega^n(A_i)$ inducing all ω_R for $R \in \mathrm{RZ}(X)$ containing A_i. The forms $\omega|_{A_i}$ and ω_{A_i} in $\Omega^n_{\mathrm{val}}(\mathrm{Spec}(A_i))$ agree in $\Omega^n_{\mathrm{val}}(K)$. By Lemma 4.10 there exists a blowup $W_i \to \mathrm{Spec}(A_i)$ such that $\omega_{A_i}|_{W_i} = \omega|_{W_i}$. Hence, $\omega|_{W_i}$ is represented by a class ω_{W_i} in $\Omega^n_{\mathrm{rh}}(W_i) \subset \Omega^n_{\mathrm{val}}(W_i)$.

By Nagata compactification, there is a factorisation

$$W_i \to \overline{W}_i \to X,$$

where the first map is a dense open immersion and the second is proper and birational. Let V be the closure of $\mathrm{Spec}(K)$ in $\overline{W}_1 \times_X \cdots \times_X \overline{W}_n$. The canonical morphism $V \to X$ is proper and birational. Every point of V is dominated by a valuation ring of K [EGA II 1961, Proposition 7.1.7]. There is necessarily at least one of the A_i contained in it, and so the base changes $V_i = W_i \times_{\overline{W}_i} V$ form an open cover of V. Let $\omega_{V_i} \in \Omega^n_{\mathrm{rh}}(V_i)$ be the restriction of ω_{W_i}.

We have $\omega_{V_i} = \omega_{V_j}$ in $\Omega^n_{\mathrm{val}}(V_i \cap V_j)$ (and hence $\Omega^n_{\mathrm{rh}}(V_i \cap V_j)$ by Corollary 4.7) because both agree with the restriction of ω. By Zariski-descent, the ω_{V_i} glue to a global differential form $\omega_V \in \Omega^n_{\mathrm{rh}}(V)$ representing $\omega|_V$ in $\Omega^n_{\mathrm{rh}}(V) \subset \Omega^n_{\mathrm{val}}(V)$.

Now let $Z \subset X$ be the exceptional locus of $V \to X$ and let $E \subset V$ be its preimage. By induction on the dimension there is a class ω_Z in $\Omega^n_{\mathrm{rh}}(Z)$ mapping to $\omega|_Z \in \Omega^n_{\mathrm{val}}(Z)$. We know that ω_Z and ω_Y agree on E because $\Omega^n_{\mathrm{rh}}(E) \to \Omega^n_{\mathrm{val}}(E)$ is injective and both represent $\omega|_E$. By rh-descent this gives a class $\omega_X \in \Omega^n_{\mathrm{val}}(X)$ mapping to ω. □

Theorem 4.12. *The canonical morphisms of presheaves on $\mathrm{Sch}^{\mathrm{ft}}_k$*

$$\Omega^n_{\mathrm{rh}} \to \Omega^n_{\mathrm{cdh}} \to \Omega^n_{\mathrm{eh}} \to \Omega^n_{\mathrm{val}}$$

are isomorphisms. More generally, these are isomorphisms for any presheaf $\mathcal{F}$ satisfying (Co), (Et) *and*

(TF) *from Remark 4.1. Moreover,*

$$\Omega^n(X) = \Omega^n_{\mathrm{val}}(X)$$

when $S = \mathrm{Spec}(k)$ *for* X *smooth.*

Proof. Corollary 4.7 says that the maps are injective. As the composition is surjective, they are all isomorphisms. The smooth case is then a consequence of [Huber, Kebekus and Kelly 2017, Theorem 5.11], which says that $\Omega(X) \cong \Omega^n_{\mathrm{rh}}(X)$ for smooth X. □

Remark 4.13. This still leaves open whether $\Omega^n_{\mathrm{eh}} \to \Omega^n_{\mathrm{dvr}}$ is an isomorphism. Weak resolution of singularities would imply that this is an isomorphism in general; see [Huber, Kebekus and Kelly 2017, Proposition 5.13].

5. Quasicoherence

The sheaves $\Omega^n_{\mathrm{eh}}|_X$ are obviously sheaves of $\mathcal{O}_X$-modules. We want to show that they are coherent. The main step is actually quasicoherence.

5A. *Quasicoherence.*

Lemma 5.1. *Let* A *be a finite type* k*-algebra,* $f \in A$ *not nilpotent, and* $\omega \in \Omega^n_{\mathrm{val}}(A)$ *be such that* $\omega|_{A_f} = 0$. *Then* $f\omega = 0$. *Moreover, the map*

$$\Omega^n_{\mathrm{val}}(A)_f \to \Omega^n_{\mathrm{val}}(A_f)$$

is injective.

Note this would follow directly from quasicoherence of $\Omega^n_{\mathrm{val}}|_{\mathrm{Spec}(A)}$. We want to prove it directly in order to show quasicoherence down the line.

Proof. By torsion freeness it suffices to show that $f\omega_x = 0$ for all residue fields $\kappa(x)$ of points $x \in \mathrm{Spec}(A)$ (Lemma 3.7). It suffices to consider the case $f|_{\kappa(x)} \neq 0$. But then $A \to \kappa(x)$ factors through A_f, and the claim follows from the assumption $\omega|_{A_f} = 0$.

We now turn to injectivity. Let ω/f^N be in the kernel of

$$\Omega^n_{\mathrm{val}}(A)_f \to \Omega^n_{\mathrm{val}}(\mathrm{Spec}\, A_f).$$

This means that $\omega_R/f^N = 0$ for every $R \in \mathrm{val}(A_f)$. As f is invertible in this ring, this implies $\omega_R = 0$. That is, ω satisfies the assumption of the first assertion. Hence $f\omega = 0$ in $\Omega^n_{\mathrm{val}}(A)$ and this implies $\omega/f^N = 0$ in the localization. □

In order to proceed, we need a lemma from algebraic geometry.

Lemma 5.2. *Let* $U \subset X$ *be an open immersion of integral schemes, and let* $V \to U$ *be a* cdp*-morphism with integral connected components. Then there is a cartesian diagram*

$$\begin{array}{ccc} V & \xrightarrow{j} & Y \\ \downarrow & & \downarrow p \\ U & \longrightarrow & X \end{array}$$

with p *a* cdp*-morphism and* j *an open immersion.*

Proof. Let V' be an irreducible component of V. We factor $V' \to U \to X$ as a (dense) open immersion $V' \to Y'$ followed by a proper map. It is easy to see that $V' = Y' \times_X U$ because the map $V' \to U \times_X Y'$ is both proper and a (dense) open immersion. We define Y as the disjoint union of all Y' and $X \setminus U$. $\square$

Proposition 5.3. *Let X be an integral k-scheme of finite type. Then the restriction to the small Zariski site $\Omega^n_{\mathrm{rh}}|_{X_{\mathrm{Zar}}}$ is quasicoherent.*

Proof. It is sufficient to show that for every integral ring A and $f \in A$, the canonical morphism $\Omega^n_{\mathrm{rh}}(A)_f \to \Omega^n_{\mathrm{rh}}(A_f)$ is an isomorphism. By the last lemma and because $\Omega^n_{\mathrm{rh}} = \Omega^n_{\mathrm{val}}$, the map is injective.

To show that it is surjective, it suffices to check that for every

$$\omega \in \Omega^n_{\mathrm{rh}}(A_f)$$

there is N such that $f^N\omega$ lifts to $\Omega^n_{\mathrm{rh}}(A)$. We put $X = \mathrm{Spec}(A)$, $U = \mathrm{Spec}(A_f)$. There is an rh-cover $\{V_i \to U\}_{i=1}^m$ such that ω is represented by an algebraic differential form on $\coprod V_i$. The strategy is to show that, up to multiplication by f, the form ω is actually representable on a cdp-morphism of U, and then descend it to U. Lemma 5.2 is key.

We can choose the cover in the form

$$V_i \to V \to U \tag{10}$$

with $\pi : V \to U$ a cdp-morphism, V reduced and $V_i \to V$ open immersions. Moreover, we may assume that V is the disjoint union of its irreducible components and that they are birational over their image in U (because we will want to apply Lemma 5.2).

We now construct the following cartesian squares whose vertical morphisms are proper envelopes, and horizontal ones are open immersions:

$$\begin{tikzcd}
\widetilde{W}_i \cap \widetilde{W}_j \arrow[r] \arrow[d] & \widetilde{W}_i \arrow[r] \arrow[d] & \widetilde{W} \arrow[d] \\
\begin{array}{c} W_{ij} \\ \scriptstyle \omega_i - \omega_j = 0 \in \Omega^n(W_{ij}) \end{array} \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \widetilde{W}_{ij} \arrow[d, "p_{ij}"] \\
V_i \cap V_j \arrow[r] & \begin{array}{c} V_i \\ \scriptstyle \omega_i \in \Omega^n(V_i) \end{array} \arrow[r] & V
\end{tikzcd}$$

Let $\omega_i \in \Omega^n(V_i)$ be the representing form. Since ω_i came from an element $\omega \in \Omega^n_{\mathrm{rh}}(U)$, the differences $\omega_i - \omega_j$ vanish in $\Omega^n_{\mathrm{rh}}(V_i \times_U V_j)$ by the exact sequence

$$0 \to \Omega^n_{\mathrm{rh}}(U) \to \bigoplus \Omega^n_{\mathrm{rh}}(V_i) \to \bigoplus \Omega^n_{\mathrm{rh}}(V_i \times_U V_j),$$

and hence vanish in $\Omega^n_{\mathrm{rh}}(V_i \times_V V_j) = \Omega^n_{\mathrm{rh}}(V_i \cap V_j)$. Consequently, there is an rh-cover of $V_i \cap V_j$ on which $\omega_i - \omega_j$ vanishes as a section of the presheaf Ω^n. We can assume that this cover is again of the form (10). Since Ω^n is Zariski separated, we find that there is a cdp-morphism $W_{ij} \to V_i \cap V_j$ such that

$\omega_i - \omega_j$ vanishes on W_{ij}. By Lemma 5.2 there is a commutative diagram

$$\begin{tikzcd} W_{ij} \arrow[r, "j"] \arrow[d] & \widetilde{W}_{ij} \arrow[d, "p_{ij}"] \\ V_i \cap V_j \arrow[r] & V \end{tikzcd}$$

with an open immersion j and a proper envelope p_{ij}. Let $\widetilde{W} = \prod_V \widetilde{W}_{ij}$ be the fibre product of all $\widetilde{W}_{ij}$ over V. Hence $\widetilde{W} \to V$ is a proper envelope which factors through all p_{ij}. Let $\widetilde{W}_i$ be the preimage of V_i in $\widetilde{W}$. Now consider the above big diagram. The differences of the restrictions $\omega_i - \omega_j$ vanish in $\Omega^n(\widetilde{W}_i \cap \widetilde{W}_j)$, and $\{\widetilde{W}_i \to \widetilde{W}\}$, being the pullback of the Zariski cover $\{V_i \to V\}$ is also a Zariski cover. Hence, we can lift the restrictions $\widetilde{\omega}_i \in \Omega^n(\widetilde{W}_i)$ to a section

$$\omega_{\widetilde{W}} \in \Omega^n(\widetilde{W}).$$

We find that $\omega \in \Omega^n_{\mathrm{rh}}(U)$ and $\omega_{\widetilde{W}} \in \Omega^n(\widetilde{W})$ agree in $\Omega^n_{\mathrm{rh}}(\widetilde{W})$ since $\{\widetilde{W}_i \to \widetilde{W}\}$ is a Zariski cover, and they agree on $\widetilde{W}_i$. In other words, at this point, we have shown that $\omega|_{\widetilde{W}}$ is in the image of $\Omega^n \to \Omega^n_{\mathrm{rh}}$.

Again by Lemma 5.2 there is a cartesian diagram

$$\begin{tikzcd} \widetilde{W} \arrow[r, "j"] \arrow[d] & Y \arrow[d, "p"] \\ U \arrow[r] & X \end{tikzcd}$$

with j an open immersion and p a proper envelope. The open subset U in X is by definition the complement of $V(f)$, hence the same is true for $\widetilde{W}$ in Y. As Ω^n_Y is coherent, this implies that there is N such that $f^N \omega_{\widetilde{W}}$ extends to Y.

Let ω_Y be an extension of $f^N \omega_{\widetilde{W}}$ to $\Omega^n(Y)$. Let $Z = V(f) \subset X$ and E its preimage in Y. Consider the exact sequence

$$0 \to \Omega^n_{\mathrm{rh}}(X) \to \Omega^n_{\mathrm{rh}}(Y) \oplus \Omega^n_{\mathrm{rh}}(Z) \to \Omega^n_{\mathrm{rh}}(E).$$

The class $(f\omega_Y, 0)$ maps to $f\omega_Y|_E$. As $f = 0$ in all of E, this equals zero. Hence there is a class $\omega'_X \in \Omega^n_{\mathrm{rh}}(X)$ such that $\omega'_X|_{\widetilde{W}} = f^{N+1}\omega_{\widetilde{W}}$ in $\Omega^n_{\mathrm{rh}}(\widetilde{W})$. Recall that two paragraphs ago we mentioned that $\omega|_{\widetilde{W}} = \omega_{\widetilde{W}}$ in $\Omega^n_{\mathrm{rh}}(\widetilde{W})$. So in fact, we know that $\omega'_X|_{\widetilde{W}} = f^{N+1}\omega|_{\widetilde{W}}$ in $\Omega^n_{\mathrm{rh}}(\widetilde{W})$. But $\widetilde{W} \to U$ is a cdp-morphism, so it follows that $\omega'_X|_U = f^{N+1}\omega$ in $\Omega^n_{\mathrm{rh}}(U)$. □

5B. *Coherence.*

Theorem 5.4. *Let X be of finite type over k. Then $\Omega^n_{\mathrm{rh}}|_{X_{\mathrm{Zar}}}$ is coherent.*

Proof. We write $\Omega^n_{\mathrm{rh}}|_X = \Omega^n_{\mathrm{rh}}|_{X_{\mathrm{Zar}}}$ for briefness. Let $i : X_{\mathrm{red}} \to X$ be the reduction. We have $i_*(\Omega^n_{\mathrm{rh}}|_{X_{\mathrm{red}}}) = \Omega^n_{\mathrm{rh}}|_X$. Hence we may assume that X is reduced. Let $X = X_1 \cup \cdots \cup X_N$ be the decomposition into reduced irreducible components. Let $i_j : X_j \to X$ and $i_{jl} : X_j \cap X_l \to X$ be the closed immersions. By rh-descent, we have an exact sequence of sheaves of $\mathcal{O}_X$-modules

$$0 \to \Omega^n_{\mathrm{rh}}|_X \to \bigoplus_{j=1}^{N} i_{j*}(\Omega^n_{\mathrm{rh}}|_{X_j}) \to \bigoplus_{j,l=1}^{N} i_{jl*}(\Omega^n_{\mathrm{rh}}|_{X_j \cap X_l}).$$

By induction on the dimension it suffices to consider the irreducible case.

Let X be integral with function field K. Let $\widetilde{X}$ be its normalisation. The map $\pi : \widetilde{X} \to X$ is an isomorphism outside some closed proper subset $Z \subset X$. Let E be its preimage in $\widetilde{X}$. From the blowup sequence we obtain

$$0 \to \Omega^n_{\mathrm{rh}}|_X \to \pi_*(\Omega^n_{\mathrm{rh}}|_{\widetilde{X}}) \oplus i_{Z*}(\Omega^n_{\mathrm{rh}}|_Z) \to \pi_* i_{E*}(\Omega^n_{\mathrm{rh}}|_E).$$

Hence by induction on the dimension, we may assume that X is normal. We have shown in Proposition 5.3 that the sheaf $\Omega^n_{\mathrm{rh}}|_X$ is quasicoherent in the integral case. Let $j : X^{\mathrm{sm}} \to X$ be the inclusion of the smooth locus with closed complement Z. It is of codimension at least 2. Hence $j_*\Omega^n_{X^{\mathrm{sm}}}$ is coherent. By Theorem 4.12, $\Omega^n_{X^{\mathrm{sm}}} \cong \Omega^n_{\mathrm{rh}}|_{X^{\mathrm{sm}}}$ so we have a map $\Omega^n_{\mathrm{rh}}|_X \to j_*\Omega^n_{X^{\mathrm{sm}}}$. Its image is coherent because it is a quasicoherent subsheaf of a coherent sheaf. Its kernel $\mathcal{K}$ is also quasicoherent. We claim that it is a subsheaf of $i_{Z*}(\Omega^n_{\mathrm{rh}}|_Z)$. It suffices to prove that the canonical composition $\mathcal{K}(U) \to \Omega^n_{\mathrm{rh}}|_X(U) \to i_{Z*}(\Omega^n_{\mathrm{rh}}|_Z)(U)$ is a monomorphism for all open $U \subseteq X$. Replacing X with U, it suffices to consider the case $U = X$. Then this morphism is canonically identified with the morphism $\ker(\Omega^n_{\mathrm{val}}(X) \to \Omega^n_{\mathrm{val}}(X^{\mathrm{sm}})) \to \Omega^n_{\mathrm{val}}(Z)$, which is injective because $\Omega^n_{\mathrm{val}}(X) \to \Omega^n_{\mathrm{val}}(X^{\mathrm{sm}}) \times \Omega^n_{\mathrm{val}}(Z)$ is injective (Lemma 3.7). By induction on the dimension we can assume that $i_{Z*}\Omega^n_{\mathrm{rh}}|_Z$ is also coherent, so the kernel $\mathcal{K}$ is coherent as well. As a quasicoherent extension of two coherent sheaves the sheaf $\Omega^n_{\mathrm{rh}}|_X$ is coherent. □

5C. *Torsion.* We return to the question of torsion forms. As in [Huber, Kebekus and Kelly 2017], we denote by $\operatorname{tor}\Omega^n_{\mathrm{eh}}(X)$ the submodule of torsion sections, i.e., those vanishing on some dense open subset. There is an obvious source of torsion classes: Let $f : Y \to X$ be proper birational with centre $Z \subset X$ and preimage $E \subset Y$. Any $\omega \in \ker(\Omega^n_{\mathrm{eh}}(Z) \to \Omega^n_{\mathrm{eh}}(E))$ gives rise to a torsion class on X by the blowup sequence. By [Huber, Kebekus and Kelly 2017, Example 5.15] this kernel can indeed be nonzero. We have established in Lemma 4.10 that all torsion classes arise in this way.

Proposition 5.5. *Let X be of finite type over k.*

(1) *The presheaf $\mathcal{T}_X : U \mapsto \operatorname{tor}\Omega^n_{\mathrm{rh}}(U)$ is a coherent sheaf of $\mathcal{O}_X$-modules on X_{Zar}.*

(2) *There is a proper birational morphism $f : Y \to X$ such that*

$$\mathcal{T}_X = \ker(\Omega^n_{\mathrm{rh}}|_{X_{\mathrm{Zar}}} \to f_*\Omega^n_{\mathrm{rh}}|_{Y_{\mathrm{Zar}}}).$$

Proof. By reductions similar to those in the first paragraph of the proof of Theorem 5.4 it suffices to show coherence for X integral, and since quasicoherent subsheaves of coherent sheaves are coherent, it suffices to show quasicoherence for X integral.

Let $X = \mathrm{Spec}(A)$, $U = \mathrm{Spec}(A_f)$. We have to show that $\mathcal{T}_X(U) = \mathcal{T}_X(X)_f$. Let

$$\omega \in \mathcal{T}_X(U) \subset \Omega^n_{\mathrm{rh}}|_{X_{\mathrm{Zar}}}(U) = (\Omega^n_{\mathrm{rh}}(X))_f.$$

Then ω is of the form $\widetilde{\omega}/f^N$ with $\widetilde{\omega} \in \Omega^n_{\mathrm{rh}}(X)$. By assumption, ω vanishes at the generic point of U, which is equal to the generic point of X. Hence the same is true for $\widetilde{\omega}$. This finishes the proof of coherence.

Any form vanishing on a blowup is torsion, i.e., $\ker(\Omega^n_{\mathrm{rh}}|_{X_{\mathrm{Zar}}} \to f_*\Omega^n_{\mathrm{rh}}|_{Y_{\mathrm{Zar}}}) \subseteq \mathcal{T}_X$ for any proper birational $f : Y \to X$, and so our job is to find a Y for which this inclusion is surjective. By Theorem 4.12, $\Omega^n_{\mathrm{rh}} = \Omega^n_{\mathrm{val}}$, and so by Lemma 4.10, for any $\omega \in \mathcal{T}_X(X)$, there is a proper birational morphism $Y_\omega \to X$ for which $\omega \in \ker(\Omega^n_{\mathrm{rh}}(X) \to \Omega^n_{\mathrm{rh}}(Y))$.

If X is affine, the $\mathcal{O}(X)$-module $\mathcal{T}_X(X)$ is finitely generated. We can find a proper birational $Y \to X$ killing the generators and hence all of $\mathcal{T}_X(X)$, and by coherence even all of $\mathcal{T}_X$. If X is not affine, let $X = U_1 \cup \cdots \cup U_m$ be an affine cover, and $Y_i \to U_i$ proper birational morphisms killing all of $\mathcal{T}_{U_i}$. By Nagata compactification, there is a proper birational morphism $\overline{Y_i} \to X$ such that $Y_i = U_i \times_X \overline{Y_i}$. Let V be the closure of $\mathrm{Spec}(k(X))$ in $\overline{Y_1} \times_X \cdots \times_X \overline{Y_m}$. It is equipped with the open cover $\{U_i \times_X V \to V\}$. Moreover, each $U_i \times_X V \to V \to X$ factors through Y_i, so $\mathcal{T}_X = \bigcup_{i=1}^m \ker(\Omega^n_{\mathrm{rh}}|_{X_{\mathrm{Zar}}} \to f_*\Omega^n_{\mathrm{rh}}|_{(U_i \times_X V)_{\mathrm{Zar}}})$. Then by Zariski descent we deduce that $\mathcal{T}_X = \ker(\Omega^n_{\mathrm{rh}}|_{X_{\mathrm{Zar}}} \to f_*\Omega^n_{\mathrm{rh}}|_{(V)_{\mathrm{Zar}}})$. □

It now becomes an interesting question to understand whether a given X admits such a blowup $Y \to X$ such that there is a point ξ in the centre over which all residue fields of Y_ξ are inseparable over $\kappa(\xi)$. This does not happen in the smooth case: *any* blowup of a regular scheme is completely decomposed (unconditionally) [Huber, Kebekus and Kelly 2017, Proposition 2.12]. In the example in [Huber, Kebekus and Kelly 2017, Example 3.6] the point ξ had codimension 1 and Y was the normalisation. It induced a purely inseparable field extension of $k(\xi)$.

One might wonder if assuming X is normal is enough to avoid this pathology. Let us show that it is not.

Example 5.6. Here we give an example of a normal variety X over a perfect field k of positive characteristic p, a point $\xi \in X$ and a blowup $Y \to X$ such that for every point in the fibre $Y_\xi \to \xi$ the residue field extension is inseparable.

In particular, for every X-valuation ring R of $k(X)$ sending its special point to ξ, the field extension $k(\xi) \subseteq R/m$ is inseparable.

Our variety is

$$X = \mathrm{Spec}\left(\frac{k[s,t,x,y,z]}{z^p - sx^p - ty^p}\right),$$

from [Suslin and Voevodsky 2000b, Example 3.5.10]. Let Y be the blowup of this variety at the ideal (x, y, z). The blowup Y admits an open affine covering by affine schemes with rings

$$\frac{k\big[s,t,\frac{x}{z},\frac{y}{z},z\big]}{1 - s\big(\frac{x}{z}\big)^p - t\big(\frac{y}{z}\big)^p}, \qquad \frac{k\big[s,t,\frac{x}{y},y,\frac{z}{y}\big]}{\big(\frac{z}{y}\big)^p - s\big(\frac{x}{y}\big)^p - t}, \qquad \frac{k\big[s,t,x,\frac{y}{x},\frac{z}{x}\big]}{\big(\frac{z}{x}\big)^p - s - t\big(\frac{y}{x}\big)^p},$$

and the intersections of these open affines with the exceptional fibre E are

$$\frac{k\big[s,t,\frac{x}{z},\frac{y}{z}\big]}{1-s\big(\frac{x}{z}\big)^p-t\big(\frac{y}{z}\big)^p},\qquad \frac{k\big[s,t,\frac{x}{y},\frac{z}{y}\big]}{\big(\frac{z}{y}\big)^p-s\big(\frac{x}{y}\big)^p-t},\qquad \frac{k\big[s,t,\frac{y}{x},\frac{z}{x}\big]}{\big(\frac{z}{x}\big)^p-s-t\big(\frac{y}{x}\big)^p},$$

respectively, all lying over the singular locus $V(x,y,z)=\mathrm{Spec}(k[s,t])\subseteq X$. Our point ξ is the generic point $\xi=\mathrm{Spec}(k(s,t))$. Every point of the fibre E_ξ has residue field an inseparable extension of $k(\xi)$: Consider for example the fibre of the rightmost affine for concreteness (all three fibres are isomorphic up to an automorphism of $k(s,t)$). It factors as

$$\mathrm{Spec}\left(\frac{k(s,t)\big[\frac{y}{x},\frac{z}{x}\big]}{\big(\frac{z}{x}\big)^p-s-t\big(\frac{y}{x}\big)^p}\right)\to \mathrm{Spec}\big(k(s,t)\big[\tfrac{y}{x}\big]\big)\to \mathrm{Spec}(k(s,t)).$$

Any residue field $k(\zeta)$ of a point ζ of the leftmost affine scheme is a finite field extension of the residue field $k(\theta)$ of a point $\theta\in\mathbb{A}^1_{k(s,t)}$ in the middle. This latter $k(\theta)$ is generated over $k(s,t)$ by the image of $\frac{y}{x}$. This finite field extension $k(\zeta)/k(\theta)$ is purely inseparable so long as $s+t\big(\frac{y}{x}\big)^p$ is nonzero in $k(\theta)$. If $k(\theta)=k\big(s,t,\frac{y}{x}\big)$, then clearly this is the case. If $k(\theta)=k(s,t)\big[\frac{y}{x}\big]/f\big(\frac{y}{x}\big)$ for some irreducible polynomial $f\big(\frac{y}{x}\big)\in k(s,t)\big[\frac{y}{x}\big]$, then $s+t\big(\frac{y}{x}\big)^p$ is nonzero if and only if $\big(\frac{y}{x}\big)^p=-\frac{s}{t} \bmod f\big(\frac{y}{x}\big)$. But since $\big(\frac{y}{x}\big)^p+\frac{s}{t}$ is irreducible in $k(s,t)\big[\frac{y}{x}\big]$, this would imply $f\big(\frac{y}{x}\big)=\big(\frac{y}{x}\big)^p+\frac{s}{t}$, in which case the subextension $k(\zeta)/k(\theta)/k(s,t)$ is already purely inseparable.

Now the blowup $Y\to X$ is birational and proper, and so any X-valuation ring R of $k(X)$ is uniquely a Y-valuation ring of $k(Y)$, and if the special point of R is sent to ξ, then the lift sends this special point to some point $\eta\in E_\xi$. But any field extension which contains an inseparable field extension is inseparable, and $k(\xi)\to R/\mathfrak{m}$ contains $k(\xi)\to k(\eta)$, hence, $k(\xi)\to R/\mathfrak{m}$ is inseparable.

Remark 5.7. Let us also observe that the rightmost open affine contains the point $\xi'=V\big(\frac{y}{x}\big)$ of E_ξ with residue field $k(s,t)\big[\frac{z}{x}\big]/\big(\big(\frac{z}{x}\big)^p-s\big)\cong k(s^{1/p},t)$. In particular, we have produced a blowup $Y\to X$, a point $\xi\in X$, and a point $\xi'\in Y$ over it for which $\Omega^1(k(\xi))\to\Omega^1(k(\xi'))$ is neither injective nor zero.

5D. *The étale case.*

We recall:

Definition 5.8. Let $X\in\mathsf{Sch}_k^{\mathrm{ft}}$. A presheaf $\mathcal{F}$ of $\mathcal{O}$-modules on the small étale site X_{et} is called *coherent* if for all X' étale over X, the sheaf $\mathcal{F}|_{X'_{\mathrm{Zar}}}$ is a coherent sheaf for the Zariski-topology and, in addition, for all $\pi: X_1\to X_2$ étale, the natural map

$$\big(\pi^*\mathcal{F}|_{X_{2,\mathrm{Zar}}}\big)\otimes_{\pi^*\mathcal{O}_{X_2}}\mathcal{O}_{X_1}\to\mathcal{F}|_{X_{1,\mathrm{Zar}}}$$

is an isomorphism.

The left-hand side is nothing but the pullback in the category of quasicoherent sheaves. We reserve the notation π^* for the pullback of abelian sheaves.

Proposition 5.9. *For all $X\in\mathsf{Sch}_k^{\mathrm{ft}}$, the sheaf $\Omega^n_{\mathrm{eh}}|_{X_{\mathrm{et}}}$ is coherent.*

Proof. We already know coherence for the Zariski-topology. By replacing X_2 by X, it suffices to check the condition for all étale morphisms $X'\to X$. Both sides are sheaves for the Zariski-topology; hence it suffices to consider the case where X and X' are both affine and $\pi : X' \to X$ is surjective. (Indeed, first assume X and X' affine, then cover the image of X' by affines U_i and replace X by U_i and X' by the preimage of U_i.) We fix such a π.

We need to show that

$$\Omega^n_{\mathrm{eh}}(X) \otimes_{\mathcal{O}(X)} \mathcal{O}(X') \to \Omega^n_{\mathrm{eh}}(X') \tag{11}$$

is an isomorphism. Note that the Ω^n analogue of this assertion holds.

Step 1: Consider the morphism of presheaves on the category of separated schemes of finite type over X

$$Y \mapsto \big(\Omega^n(Y) \otimes_{\mathcal{O}(X)} \mathcal{O}(X') \to \Omega^n(Y \times_X X')\big). \tag{12}$$

We claim that it is an isomorphism. Indeed, both sides are sheaves for the Zariski-topology (the left-hand side because $\mathcal{O}(X) \to \mathcal{O}(X')$ is flat). Hence it suffices to consider the case where Y is affine. Let $Y' = Y \times_X X'$. It is also affine. Then the map (12) identifies as

$$\Omega^n(Y) \otimes_{\mathcal{O}(X)} \mathcal{O}(X') = \Omega^n(Y) \otimes_{\mathcal{O}(Y)} \mathcal{O}(Y') \to \Omega^n(Y'),$$

and hence it is an isomorphism.

Step 2: We now sheafify the morphism of presheaves with respect to the eh-topology. As $\mathcal{O}(X) \to \mathcal{O}(X')$ is flat, it commutes with sheafification, and we get an isomorphism of eh-sheaves

$$\Omega^n_{\mathrm{eh}} \otimes_{\mathcal{O}(X)} \mathcal{O}(X') \to (\Omega^n(- \times_X X'))_{\mathrm{eh}}. \tag{13}$$

Step 3: We claim that

$$(\Omega^n(- \times_X X'))_{\mathrm{eh}}(Y) = \Omega^n_{\mathrm{eh}}(Y \times_X X'). \tag{14}$$

It suffices to show that every eh-cover of $Y' = Y \times_X X'$ can be refined by the pullback of an eh-cover of Y. Let $Z \to Y'$ be an eh-cover. The composition $Z \to Y' \to Y$ is also an eh-cover. The pullback $Z \times_Y Z \to Z \to Y'$ is the required refinement.

Combining the isomorphisms (13) and (14) in the case $Y = X$ gives the desired isomorphism (11). □

6. Special cases

The cases of forms of degree zero or top degree are easier to handle than the general case. In this section we study these special cases.

6A. 0-*differentials.* This section is about the sheafification of the presheaf $\Omega^0 = \mathcal{O}$. For expositional reasons we work over a field. For more general bases, and more general representable presheaves, see Section 6C.

In contrast to the general case, $\mathcal{O}_{\mathrm{val}}$ is torsion free.

Lemma 6.1. *For every $X \in \mathsf{Sch}^{\mathrm{ft}}_k$, every dense open immersion $U \subseteq X$ in $\mathsf{Sch}^{\mathrm{ft}}_k$ induces an inclusion $\mathcal{O}_{\mathrm{val}}(X) \subseteq \mathcal{O}_{\mathrm{val}}(U)$.*

Proof. First suppose it is true for irreducible schemes, and let $X_1, \dots, X_n$ be the irreducible components of X. Let $U_1, \dots, U_n$ be their intersections with U. Each U_i is dense in X_i. Let $f, g \in \mathcal{O}_{\mathrm{val}}(X)$ such that $f|_U = g|_U$. Then $f|_{U_i} = g|_{U_i}$. By the irreducible case, $f|_{X_i} = g|_{X_i}$. By cdh-descent

$$\mathcal{O}_{\mathrm{val}}(X) \subseteq \mathcal{O}_{\mathrm{val}}(X_i)$$

and hence $f = g$.

Now consider X irreducible. Since $\mathcal{O}_{\mathrm{val}}(X) = \mathcal{O}_{\mathrm{val}}(X_{\mathrm{red}})$ we can assume X is integral. Consider the description of Lemma 3.7. If two sections $(s_x), (t_x) \in \mathcal{O}_{\mathrm{val}}(X)$ are equal on a dense open, then $s_\eta = t_\eta$ where η is the generic point of X. Consequently, for any valuation ring R of the form $\eta \to \mathrm{Spec}(R) \to X$, the lifts $s_{\mathrm{Spec}(R)}, t_{\mathrm{Spec}(R)}$ are also equal, as well as their images in R/m, and from there we deduce that $s_x = t_x$ where x is the image of $\mathrm{Spec}(R/\mathfrak{m}) \to X$. But for every point $x \in X$ there is a valuation ring $R_x \subset k(X)$ such that the special point of $\mathrm{Spec}(R)$ maps to x [EGA II 1961, Proposition 7.1.7]. □

Recall the notion of the seminormalisation of a variety (Definition 2.3).

Proposition 6.2. *Let $Y \in \mathsf{Sch}_k^{\mathrm{ft}}$. Then the canonical morphism*

$$\mathcal{O}_{\mathrm{val}}(Y) \cong \mathcal{O}(Y^{\mathrm{sn}})$$

is an isomorphism. The same is true for $\mathcal{O}_{\mathrm{rh}}$, $\mathcal{O}_{\mathrm{cdh}}$ and $\mathcal{O}_{\mathrm{eh}}$ in place of $\mathcal{O}_{\mathrm{val}}$.

Remark 6.3. The proof below works for a general noetherian base scheme S. We (the authors) do not know if the seminormalisation A^{sn} is always noetherian in this setting, but the definition of $\mathcal{O}_{\mathrm{val}}(A^{\mathrm{sn}})$ is, clearly, still valid, and the careful reader will note that the proof below still works, regardless.

Proof. Both $\mathcal{O}_{\mathrm{val}}(-)$ and $\mathcal{O}((-)^{\mathrm{sn}})$ are invariant under $(-)_{\mathrm{red}}$, and are Zariski sheaves, so it suffices to consider the case where Y is reduced and affine. Let $Y = \mathrm{Spec}(A)$. As $\mathrm{Spec}(A^{\mathrm{sn}}) \to \mathrm{Spec}(A)$ is a completely decomposed homeomorphism, $\mathrm{val}(A^{\mathrm{sn}}) \to \mathrm{val}(A)$ is an equivalence of categories, so $\mathcal{O}_{\mathrm{val}}(A) \to \mathcal{O}_{\mathrm{val}}(A^{\mathrm{sn}})$ is an isomorphism. Hence, it suffices to show that if A is a seminormal ring, then $\mathcal{O}(A) = \mathcal{O}_{\mathrm{val}}(A)$. Define $A_{\mathrm{val}} = \mathcal{O}_{\mathrm{val}}(\mathrm{Spec}(A))$. By Lemma 2.6(3) it suffices to show that $A \to A_{\mathrm{val}}$ is subintegral.

First we show that it is integral. By the argument in Lemma 6.1, or by its statement and the comparison of Theorem 4.12, both of the canonical morphisms $A \to A_{\mathrm{val}} \to \prod_{\mathfrak{p}\text{ minimal}} A_{\mathfrak{p}}$ is an embedding into a product of fields [Stacks project, Tag 00EW]. Since $\mathrm{Spec}(A)$ is a noetherian topological space, there are finitely many of them. Now by the definition of A_{val}, the image of A_{val} in each $A_{\mathfrak{p}}$ is contained in any valuation ring of $A_{\mathfrak{p}}$ containing the image of A. Since the normalisation is the intersection of these valuation rings [Stacks project, Tag 090P], it follows that the extension $A \subseteq A_{\mathrm{val}}$ is integral.

To show that $\mathrm{Spec}(A_{\mathrm{val}}) \to \mathrm{Spec}(A)$ is a completely decomposed homeomorphism, it suffices to show that for all fields κ which are residue fields of A or A_{val},

$$\hom_k(A_{\mathrm{val}}, \kappa) \to \hom_k(A, \kappa) \tag{15}$$

is an isomorphism.

Surjectivity: We construct a section of the map of sets (15). For every morphism $\phi : A \to \kappa$ in $\mathrm{val}(A)$ to a residue field κ of A, there is a canonical extension $\phi : A \xrightarrow{\iota} A_{\mathrm{val}} \xrightarrow{\pi_\phi} \kappa$ making the triangle commute:

just take $\pi_\phi : \varprojlim_{\phi' : A \to R' \in \mathrm{val}(A)} R' \to \kappa$ to be the projection to the ϕ-th component of the limit. For κ a general field, take π_ϕ to be the map associated to the residue field corresponding to ϕ.

Injectivity: We show that the section $\phi \mapsto \pi_\phi$ we have just constructed is surjective. That is, for an arbitrary residue field $\psi : A_{\mathrm{val}} \to \kappa$, we claim that $\psi = \pi_{A \to A_{\mathrm{val}} \to \kappa}$. If $\psi \circ \iota$ is the canonical map $A \to A_{\mathfrak{p}}$ to one of the fractions fields of an irreducible component of $\mathrm{Spec}(A)$, then there is a unique lift because $A \subseteq A_{\mathrm{val}} \subseteq \prod_{\mathfrak{p} \text{ minimal}} A_{\mathfrak{p}}$, as we observed above. For a general residue field of A_{val}, there is an A_{val}-valuation ring $R \subseteq A_{\mathfrak{p}}$ of the fraction field $A_{\mathfrak{p}}$ of any irreducible component containing $\mathrm{Spec}(\kappa) \in \mathrm{Spec}(A_{\mathrm{val}})$ whose special point maps to $\mathrm{Spec}(\kappa)$ [EGA II 1961, Proposition 7.1.7] (for the nonnoetherian version, cf. [Stacks project, Tag 00IA]). So we have the commutative diagram

$$\begin{tikzcd} R/\mathfrak{m} & R \arrow[l] \arrow[r] & A_{\mathfrak{p}} \\ \kappa \arrow[u] & A_{\mathrm{val}} \arrow[l, "\psi"'] \arrow[u] & \end{tikzcd}$$

As we have just discussed, the map $A_{\mathrm{val}} \to A_{\mathfrak{p}}$ must be the unique extension $\pi_{A \to A_{\mathrm{val}} \to A_{\mathfrak{p}}}$ of the canonical $A \to A_{\mathfrak{p}}$. As $R \to A_{\mathfrak{p}}$ is injective, the map $A_{\mathrm{val}} \to R$ must also be the canonical $\pi_{A \to A_{\mathrm{val}} \to R}$, and therefore $A_{\mathrm{val}} \to R/\mathfrak{m}$ is $\pi_{A \to A_{\mathrm{val}} \to R \to R/\mathfrak{m}}$, and by injectivity of $\kappa \to R/\mathfrak{m}$, we conclude $\psi = \pi_{A \to A_{\mathrm{val}} \to \kappa}$.

The claim about $\mathcal{O}_{\mathrm{rh}}, \mathcal{O}_{\mathrm{cdh}}, \mathcal{O}_{\mathrm{eh}}$ follows from Theorem 4.12. □

Recall the sdh-topology introduced in [Huber, Kebekus and Kelly 2017, Section 6.2]. It is generated by étale covers and those proper surjective maps that are separably decomposed, i.e., any point has a preimage such that the residue field extension is finite and separable. By de Jong's theorem on alterations [1996], every $X \in \mathrm{Sch}_k^{\mathrm{ft}}$ is sdh-locally smooth. However, sdh-descent fails for differential forms [Huber, Kebekus and Kelly 2017, Proposition 6.6]. The situation is better in degree zero.

Proposition 6.4. *Let k be perfect. Then $\mathcal{O}_{\mathrm{val}} = \mathcal{O}_{\mathrm{sdh}}$.*

Proof. The sdh-topology is stronger than the eh-topology, and we know $\mathcal{O}_{\mathrm{val}} = \mathcal{O}_{\mathrm{eh}}$ (Theorem 4.12). Hence, we have a canonical morphism $\mathcal{O}_{\mathrm{val}} \to \mathcal{O}_{\mathrm{sdh}}$, and an isomorphism $(\mathcal{O}_{\mathrm{val}})_{\mathrm{sdh}} \cong \mathcal{O}_{\mathrm{sdh}}$. If we can show that $\mathcal{O}_{\mathrm{val}}$ is already an sdh sheaf, we are done. The topology is generated by proper separably decomposed morphisms and étale covers. We already know that $\mathcal{O}_{\mathrm{val}}$ is an étale sheaf. Hence it suffices to show that if $Y \to X$ is a proper sdh-cover, which generically is finite and separable, then

$$0 \to \mathcal{O}_{\mathrm{val}}(X) \to \mathcal{O}_{\mathrm{val}}(Y) \to \mathcal{O}_{\mathrm{val}}(Y \times_X Y)$$

is exact.

Recall that since $\mathcal{O}$ is torsion free on valuation rings, $\mathcal{O}_{\mathrm{val}}(X) \to \prod_{x \in X} \kappa(x)$ is injective (Lemma 3.7). For $y \in Y$ with image $x \in X$, the induced map $\kappa(x) \to \kappa(y)$ is injective as a map of fields. As $Y \to X$ is surjective, this implies that $\mathcal{O}_{\mathrm{val}}(X) \to \mathcal{O}_{\mathrm{val}}(Y)$ is injective.

Let $f \in \mathcal{O}_{\mathrm{val}}(Y)$ be in the kernel of the second map. We want to define an element $g \in \mathcal{O}_{\mathrm{val}}(X)$ and start with the component $g_x \in \kappa(x)$ for $x \in X$. Let $y \in Y$ be a preimage of X such that the residue field extension $\kappa(y)/\kappa(x)$ is finite and separable. Let $\lambda/\kappa(x)$ be a finite Galois extension containing $\kappa(y)$. We have a canonical map $l : \mathrm{Spec}(\lambda) \to Y$, and any $\kappa(x)$-automorphism σ of λ gives us a second map $l \circ \sigma$, and this pair of maps define some $(l, l \circ \sigma) : \mathrm{Spec}(\lambda) \to Y \times_X Y$. By the assumption that f is a cocycle, $f_l = (\pi_1^* f)_{(l,l\sigma)} = (\pi_2^* f)_{(l,l\sigma)} = f_{l\sigma}$ in λ. That is, $f_l \in \lambda$ is $\mathrm{Gal}(\lambda/\kappa(x))$-invariant, and therefore, actually lies in $\kappa(x) \subseteq \lambda$. We define $g_{\kappa(x)} := f_\lambda$.

Note that another consequence of the cocycle condition is that $g_{\kappa(x)}$ is independent of the choice of y, even without assuming separability or finiteness. For any other y' over x, we can choose an extension $K/\kappa(x)$ containing both $\kappa(y)$ and $\kappa(y')$, leading to a map $\mathrm{Spec}(K) \to Y \times_X Y$, to which we can apply the cocycle condition to find that $f_y = f_{y'}$ in K via the chosen embeddings, and therefore they also agree in $\kappa(x)$.

It remains to show that the tuple $(g_{\kappa(x)})_{x \in X}$ defines a section of $\mathcal{O}_{\mathrm{val}}(X)$. We continue with the criterion of Lemma 3.7. Let $R \subset \kappa(x)$ be a valuation ring over X. Let $y \in Y$ again be a preimage of x. There is a valuation ring $S \subset \kappa(y)$ such that $R = S \cap \kappa(x)$ [Bourbaki 1964, Chapitre VI, §3.3, Proposition 5]. By the valuative criterion for properness, S is a Y-valuation on $\kappa(y)$. As $f \in \mathcal{O}_{\mathrm{val}}(Y)$, the element $g_{\kappa(x)} = f_{\kappa(y)}$ is in $S \subseteq \kappa(y)$, but it is also in $\kappa(x)$, so $g_{\kappa(x)}$ is in $R \subseteq \kappa(x)$. Therefore, let us write it as g_R. Let x_0 be the image of $\mathrm{Spec}(R/\mathfrak{m}_R) \to X$, and y_0 the image of $\mathrm{Spec}(S/\mathfrak{m}_S) \to Y$). To finish, we must show that g_R agrees with g_{x_0} in $R/\mathfrak{m}$. But $g_R|_{R/\mathfrak{m}_R}|_{S/\mathfrak{m}_S} = g_R|_S|_{S/\mathfrak{m}_S} = f_S|_{S/\mathfrak{m}_S} = f_{S/\mathfrak{m}_S} = f_{y_0}|_{S/\mathfrak{m}_S} = g_{x_0}|_{y_0}|_{S/\mathfrak{m}_S} = g_{x_0}|_{R/\mathfrak{m}_R}|_{S/\mathfrak{m}_S}$ and $R/\mathfrak{m}_R \to S/\mathfrak{m}_S$ is injective, so $g_R|_{R/\mathfrak{m}_R} = g_{x_0}|_{R/\mathfrak{m}_R}$. □

Remark 6.5. Let us point out where the above proof breaks for Ω^n. The argument for injectivity is actually valid because we can choose the preimage y of x to be separable. The construction of each $g_{\kappa(x)}$ is fine, as well as independence of the choice of y used in the construction. However, for y over x which are not separable, we cannot necessarily check that $g_x|_y = f_y$. Choosing y/x separable in the last paragraph, we can show that each $g_{\kappa(x)}$ lifts to any X-valuation ring of $\kappa(x)$, but we cannot ensure that $R/\mathfrak{m}_R \to S/\mathfrak{m}_S$ is separable, nor its image $x_0 \to y_0$, so we cannot check that we have a well-defined section.

In fact, not being able to control this kind of ramification is precisely why the sdh-topology is not suitable for working with differential forms; cf. [Huber, Kebekus and Kelly 2017, Example 6.5].

On the other hand, Proposition 6.4 is valid for any representable presheaf h^Y for any scheme Y. Moreover, using the same proof, we can show that $\Omega^n_{\mathrm{val}}(X) = \Omega^n_{\mathrm{sdh}}(X)$ whenever $\dim X \leq n$.

Proposition 6.6.
$$\mathcal{O}_{\mathrm{sdh}} = \mathcal{O}_{\mathrm{dvr}}.$$

Proof. The same arguments as in the last proof show that $\mathcal{O}_{\mathrm{dvr}}$ has sdh-descent. (In the above notation: if R is a discrete valuation ring, then S can also be chosen as a discrete valuation ring.) As pointed out before, any $X \in \mathrm{Sch}_k^{\mathrm{ft}}$ is smooth locally for the sdh-topology. Hence it suffices to compare the values on smooth varieties. In this case we have on the one hand $\mathcal{O}_{\mathrm{sdh}}(X) = \mathcal{O}_{\mathrm{val}}(X) = \mathcal{O}(X)$, on the other hand $\mathcal{O}_{\mathrm{dvr}}(X) = \mathcal{O}(X)$ by [Huber, Kebekus and Kelly 2017, Remark 4.3.3]. □

Remark 6.7. Hence we have
$$\mathcal{O}_{\mathrm{rh}} = \mathcal{O}_{\mathrm{cdh}} = \mathcal{O}_{\mathrm{eh}} = \mathcal{O}_{\mathrm{val}} = \mathcal{O}_{\mathrm{dvr}} = \mathcal{O}_{\mathrm{sdh}}.$$

However, in positive characteristic $\mathcal{O}_{\mathrm{h}} \neq \mathcal{O}_{\mathrm{val}}$ because $\mathcal{O}_{\mathrm{val}}$ does not have descent for Frobenius covers. See Proposition 6.2 and Remark 6.15(1).

The following property is well-known for the ordinary structure sheaf under the assumption that Y is normal. It will be useful in connection with cohomological descent questions [Huber and Kelly $\geq$ 2018].

Proposition 6.8. *Let $Y' \to Y$ be a* cdp-*morphism in* $\mathrm{Sch}_k^{\mathrm{ft}}$ *with geometrically connected fibres. Then*

$$\mathcal{O}_{\mathrm{eh}}(Y) \to \mathcal{O}_{\mathrm{eh}}(Y')$$

is an isomorphism.

Proof. We use induction on the dimension of Y. Note the hypotheses are preserved by all base changes along Y. Without loss of generality, all schemes are assumed to be reduced. If $\dim Y = -1$ (i.e., $Y = \varnothing$), then $Y' = \varnothing$ and the proposition follows from $\mathcal{O}_{\mathrm{eh}}(\varnothing) = 0$. In dimension ≥ 0, let $\pi : \widetilde{Y} \to Y$ be the normalisation of Y. Let $Z \subset Y$ be the locus where π fails to be an isomorphism and $E = \pi^{-1}Z$ the preimage. Let $\widetilde{Y}'$, Z' and E' be the base changes to Y'. We have a commutative diagram of blowup sequences

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{O}_{\mathrm{eh}}(Y') & \longrightarrow & \mathcal{O}_{\mathrm{eh}}(\widetilde{Y}') \oplus \mathcal{O}_{\mathrm{eh}}(Z') & \longrightarrow & \mathcal{O}_{\mathrm{eh}}(E') \\
 & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & \mathcal{O}_{\mathrm{eh}}(Y) & \longrightarrow & \mathcal{O}_{\mathrm{eh}}(\widetilde{Y}) \oplus \mathcal{O}_{\mathrm{eh}}(Z) & \longrightarrow & \mathcal{O}_{\mathrm{eh}}(E)
\end{array}$$

By the induction hypothesis, it now suffices to prove $\mathcal{O}_{\mathrm{eh}}(\widetilde{Y}) \cong \mathcal{O}_{\mathrm{eh}}(\widetilde{Y}')$. Since $\mathcal{O}_{\mathrm{eh}}((-)^{\mathrm{sn}}) = \mathcal{O}((-)^{\mathrm{sn}})$ by Proposition 6.2, so it suffices, in fact, to show that $\mathcal{O}(\widetilde{Y}'^{\mathrm{sn}}) \to \mathcal{O}(\widetilde{Y})$ is an isomorphism.

Since $\widetilde{Y}'^{\mathrm{sn}} \to \widetilde{Y}'$ is a proper homeomorphism, we are dealing with a proper surjective morphism $\widetilde{Y}'^{\mathrm{sn}} \to \widetilde{Y}$ to a normal scheme. In this situation, Stein factorisation [EGA III_1 1961, Théorème 4.3.1] gives us a factorisation $\widetilde{Y}'^{\mathrm{sn}} \to W \to \widetilde{Y}$ such that $\mathcal{O}(\widetilde{Y}'^{\mathrm{sn}}) \cong \mathcal{O}(W)$ and such that $W \to \widetilde{Y}$ is finite. Since $\widetilde{Y}'^{\mathrm{sn}} \to \widetilde{Y}$ has connected fibres, so does $W \to \widetilde{Y}$ [EGA III_1 1961, Corollaire 4.3.3]. We also deduce that because $\widetilde{Y}'^{\mathrm{sn}}$ is reduced so is W, and because $\widetilde{Y}'$ is completely decomposed, so is $\widetilde{Y}'^{\mathrm{sn}} \to \widetilde{Y}$ and therefore $W \to \widetilde{Y}$ also. In particular, since $W \to \widetilde{Y}$ is completely decomposed and W reduced, the fibre over the generic point of $\widetilde{Y}$ must be an isomorphism. Replacing W with its normalisation, $\widetilde{W} \to \widetilde{Y}$, we have a finite birational morphism between normal schemes. This can only be an isomorphism, so $W \to \widetilde{Y}$ was an isomorphism, and $\mathcal{O}(W) = \mathcal{O}(\widetilde{Y})$.

To summarise, $\mathcal{O}_{\mathrm{eh}}(\widetilde{Y}') \cong \mathcal{O}_{\mathrm{eh}}(\widetilde{Y}'^{\mathrm{sn}}) \cong \mathcal{O}(\widetilde{Y}'^{\mathrm{sn}}) \cong \mathcal{O}(W) \cong \mathcal{O}(\widetilde{Y})$. □

6B. ***Top degree differentials.*** Recall the notion of a birational morphism of schemes in the nonreduced case from Section 2A.

Proposition 6.9. *Let $X/k \in \mathrm{Sch}_k^{\mathrm{ft}}$ be of dimension at most d.*

(1) *$\Omega^d_{\mathrm{cdh}}(X)$ is a birational invariant, i.e., it remains unchanged under proper surjective birational morphisms.*

(2) *We have*

$$\Omega^d_{\mathrm{cdh}}(X) = \varinjlim_{X' \to X} \Omega^d(X'),$$

where the colimit is over proper surjective birational morphisms $X' \to X$.

(3) *Elements of $\Omega^d_{\mathrm{cdh}}(X) = \Omega^d_{\mathrm{val}}(X)$ are determined by their value on the total ring of fractions $Q(X)$, and the integrality condition only needs to be tested on valuations of the function fields. In particular, it is torsion free.*

(4) *More precisely, if X is irreducible of dimension d, then by Lemma 3.7,*

$$\Omega^d_{\mathrm{val}}(X) = \bigcap_{\mathrm{Spec}(k(X)) \to \mathrm{Spec}(R) \to X \in \mathrm{rval}(X)} \Omega^d(R). \tag{16}$$

In general, if $X = X_1 \cup \cdots \cup X_N$ is the decomposition into irreducible component, then

$$\Omega^d_{\mathrm{val}}(X) = \bigoplus_{i=1}^{N} \Omega^d_{\mathrm{val}}(X_i) \quad \textit{and} \quad \Omega^d_{\mathrm{val}}(X) = \Omega^d_{\mathrm{val}}(\tilde{X}).$$

Proof. Note that $\Omega^d_{\mathrm{cdh}} = \Omega^d_{\mathrm{val}}$ vanishes on schemes of dimension less that d. Hence the first statement is immediate from the sequence for abstract blowup squares.

The third statement follows from Lemma 3.7, the fact that $\Omega^n(K) = 0$ for $n > \mathrm{trdeg}(K/k)$ and the valuative criterion for properness. The explicit formula is immediate from this.

For the second statement consider

$$X \mapsto \widetilde{\Omega}^d(X) := \varinjlim_{X' \to X} \Omega^d(X').$$

By definition this is a birational invariant. We claim that $\widetilde{\Omega}^d$ is torsion free. Note that X' can always be refined by the disjoint union of its irreducible components with their reduced structure. Let ω be a torsion element of $\widetilde{\Omega}^d(X)$. It is represented by a differential form on some $X_1 \to X$. After restriction to some further $X_2 \to X_1$ it vanishes on a dense open subset. Then there is a proper birational morphism $X_3 \to X_2$ such that $\omega|_{X_3} = 0$. This was shown in [Huber, Kebekus and Kelly 2017, Theorem A.3] (for a recap see Theorem 4.6 combined with Theorem 2.20). Hence $\omega = 0$ in the direct limit.

By torsion freeness, we have $\widetilde{\Omega}^d(X) = 0$ if the dimension of X is less than d. Hence $\widetilde{\Omega}^d$ is a presheaf on the category of k-schemes of dimension at most d. It is a Zariski sheaf because Ω^d is. It has descent for abstract blowup squares by birational invariance and vanishing in smaller dimensions. Hence it is an rh-sheaf. By the universal property, there is a natural map

$$\Omega^d_{\mathrm{rh}} \to \widetilde{\Omega}^d.$$

The map

$$\widetilde{\Omega}^d(X) = \varinjlim_{X' \to X} \Omega^d(X') \to \varinjlim_{X' \to X} \Omega^d_{\mathrm{rh}}(X') = \Omega^d_{\mathrm{rh}}(X)$$

induces a natural map in the other direction. We check that they are inverse to each other. Both sheaves are torsion free; hence it suffices to consider generic points where it is true. $\square$

Remark 6.10. Note that the description in (16) can also be interpreted as the global sections on the Riemann–Zariski space $\Gamma(\mathsf{RZ}(X), \Omega^d_{RZ(X)/k})$.

In the smooth case, this gives a formula involving only ordinary differential forms.

Corollary 6.11. *Let X be a smooth k-scheme of dimension d. Then*

$$\Omega^d(X) = \varinjlim_{X' \to X} \Omega^d(X'),$$

where the colimit is over proper birational morphisms $X' \to X$.

Proposition 6.12. *On the category of k-schemes of dimension at most d, we have*

$$\Omega^d_{\mathrm{val}} = \Omega^d_{\mathrm{eh}} = \Omega^d_{\mathrm{sdh}} = \Omega^d_{\mathrm{dvr}}.$$

Proof. The first isomorphism is Theorem 4.12. For the other two, the same proofs as for Proposition 6.4 and Proposition 6.6 work. $\square$

6C. *Representable sheaves.* Note that $\Omega^0 = \mathcal{O} = \hom(-, \mathbb{A}^1)$. In this section we extend our results to all representable sheaves over a general noetherian base S. We use the following notation for representable presheaves on $\mathsf{Sch}^{\mathrm{ft}}_S$:

$$h^Y(-) = \hom_{\mathsf{Sch}_S}(-, Y), \qquad Y \in \mathsf{Sch}^{\mathrm{ft}}_S.$$

Note that this presheaf satisfies the properties of Remark 4.1. Notice also that the h^Y are torsion free in the sense of Definition 2.22 — this is exactly the valuative criterion for separatedness.

Lemma 6.13. *Suppose that the noetherian base scheme S is Nagata. Let $X' \to X$ be a finite completely decomposed surjective morphism in Sch_S, and suppose that X' is seminormal. Then the coequalisers*

$$\mathrm{coeq}(X' \times_X X' \rightrightarrows X') = C, \qquad \mathrm{coeq}((X' \times_X X')^{\mathrm{sn}} \rightrightarrows X') = D$$

exist in Sch_S, we have $D = X^{\mathrm{sn}}$ and the canonical morphisms $D \to C \to X$ are finite completely decomposed homeomorphisms.

Proof. Using the description [Ferrand 2003, Scolie 4.3], one easily constructs the coequaliser in the category of locally ringed spaces by taking the coequaliser in the category of sets, equipping it with the quotient topology, and the equaliser of the direct images of the structure sheaves. Using this description, one readily deduces from $X' \to X$ being finite that $D \to C \to X$ are homeomorphisms. Note that $X^{\mathrm{sn}} \to X$ is also a homeomorphism. Now since $X' \to X$ is an rh-cover, it follows from Proposition 6.2 and Remark 6.3 that $\mathcal{O}(X^{\mathrm{sn}}) = \mathrm{eq}(\mathcal{O}(X') \rightrightarrows \mathcal{O}((X' \times_X X')^{\mathrm{sn}})$. The same holds for any open $U \subseteq X$. That is, the canonical morphism $D \to X^{\mathrm{sn}}$ of locally ringed spaces is an isomorphism on topological spaces *and* structure sheaves. In other words, it is an isomorphism. Finally, note that we have a canonical inclusion of sheaves $\mathcal{O}_X \subseteq \mathcal{O}_C \subseteq \mathcal{O}_{X^{\mathrm{sn}}}$. For any open affine $U \subseteq X$ of X, it follows that $\mathrm{Spec}(\mathcal{O}_{X^{\mathrm{sn}}}(U)) \to \mathrm{Spec}(\mathcal{O}_C(U)) \to \mathrm{Spec}(\mathcal{O}_X(U))$ are homeomorphisms on topological spaces. Hence, C is a scheme. $\square$

Proposition 6.14. *Suppose that the noetherian base scheme S is Nagata. Then for every $X, Y \in \mathsf{Sch}_S^{\mathrm{ft}}$ the canonical morphisms*

$$h^Y_{\mathrm{rh}}(X) = h^Y(X^{\mathrm{sn}}) \tag{17}$$

are isomorphisms. The natural maps

$$h^Y_{\mathrm{rh}} \to h^Y_{\mathrm{cdh}} \to h^Y_{\mathrm{eh}} \to h^Y_{\mathrm{sdh}} \to h^Y_{\mathrm{val}} \tag{18}$$

are isomorphisms of presheaves on $\mathsf{Sch}_S^{\mathrm{ft}}$.

Proof. We claim that h^Y_{red} is rh-separated, where

$$h^Y_{\mathrm{red}}(-) = h^Y((-)_{\mathrm{red}}).$$

Let $f, g \in h^Y_{\mathrm{red}}(X)$ with $f = g$ in $h^Y_{\mathrm{rh}}(X)$. In particular, $f_\eta = g_\eta$ at all generic points of X. But as X_{red} is reduced and $Y \to S$ is separated, this implies $f = g$. Since h^Y_{red} is a Zariski sheaf, in light of the factorisation of Remark 2.2(2), we have

$$h^Y_{\mathrm{rh}}(X) = h^Y_{\mathrm{cdp}}(X)$$

when X is reduced (see [Kelly 2017, Proposition 2.1.16]), and hence, in general, as both h^Y_{rh} and h^Y_{cdp} are unchanged by reducing the structure sheaf.

By cdp-separatedness of h^Y_{red} we have

$$h^Y_{\mathrm{cdp}}(X) = \varinjlim_{\mathrm{cdp}} \check{H}^0(X'/X, h^Y_{\mathrm{red}}), \tag{19}$$

where the colimit is over all cdp-covers $p : X' \to X$. We claim that

$$\varinjlim_{\text{comp. dec. homeo.}} \check{H}^0(X'/X, h^Y_{\mathrm{red}}) \to \varinjlim_{\mathrm{cdp}} \check{H}^0(X'/X, h^Y_{\mathrm{red}}) \tag{20}$$

is an isomorphism, where the first colimit is over completely decomposed homeomorphisms. Let $p : X' \to X$ be a cdp-cover. For such a cover, define $X'' = \underline{\mathrm{Spec}}\, p_*\mathcal{O}_{X'}$. Since $q : X' \to X''$ is proper, the topological space of X'' is the quotient of the topological space X' via this morphism. Hence, any morphism $f' \in \mathrm{eq}(h^Y_{\mathrm{red}}(X') \rightrightarrows h^Y_{\mathrm{red}}(X' \times_Y X'))$ factors through X'' as a morphism of *topological spaces*. But we have $q_*\mathcal{O}_{X'} = \mathcal{O}_{X''}$ by construction, and therefore f comes from some $f'' \in h^Y_{\mathrm{red}}(X'')$. Then, since $(X' \times_X X')_{\mathrm{red}} \to (X'' \times_X X'')_{\mathrm{red}}$ is dominant with reduced source, we actually have $f'' \in \check{H}^0(X''/X, h^Y_{\mathrm{red}})$. Replacing X'' by $(X'')^{\mathrm{sn}}$ (this is where we use the assumption that S is Nagata), we are in the situation of Lemma 6.13, and find that $f''|_{(X'')^{\mathrm{sn}}}$ comes from some $g \in h^Y_{\mathrm{red}}(D)$ for some completely decomposed homeomorphism $D \to X$. As $(D \times_X D)_{\mathrm{red}} = D_{\mathrm{red}}$, we have $g \in \check{H}^0(D/X, h^Y_{\mathrm{red}})$, so we have shown that (20) is surjective. It is clearly injective, as any refinement $X'' \to X' \to X$ of a completely decomposed homeomorphism by a cdp-morphism is dominant. Hence, (20) is an isomorphism.

Finally, it follows from Lemma 2.6(3) that $X^{\mathrm{sn}} \to X$ is an initial object in the category of completely decomposed homeomorphisms to X. So

$$h^Y_{\mathrm{cdp}}(X) = \varinjlim_{\text{comp. dec. homeo.}} \check{H}^0(X'/X, h^Y_{\mathrm{red}}) = \check{H}^0(X^{\mathrm{sn}}/X, h^Y_{\mathrm{red}}) = h^Y(X^{\mathrm{sn}}).$$

The isomorphisms (18), except for h^Y_{sdh}, are Theorem 4.12. Injectivity of $h^Y_{\mathrm{sdh}} \to h^Y_{\mathrm{val}}$ has the same proof as injectivity of $h^Y_{\mathrm{eh}} \to h^Y_{\mathrm{val}}$; see the proof of Corollary 4.7. $\square$

Remark 6.15. (1) This is analogous to the comparison $h^Y_{\mathrm{h}}(X) = h^Y(X^{\mathrm{sn}})$ in characteristic zero; see [Huber and Jörder 2014, Proposition 4.5; Voevodsky 1996, Section 3.2].

(2) In fact, in general we have $h^Y_{\mathrm{h}}(X) = h^Y(X^{\mathrm{wn}})$, where Y^{wn} is the absolute weak normalisation [Rydh 2010, Definition B.1]. For noetherian reduced schemes in pure positive characteristic, Y^{wn} is the perfect closure of Y in $\prod_{i=1}^n K_i^a$, the product of the algebraic closures of the function fields of its irreducible components. This holds much more generally: it is true for any algebraic space Y locally of finite presentation [Rydh 2010, Theorem 8.16].

In particular, the categories of representable rh-, cdh-, and eh-sheaves on $\mathsf{Sch}^{\mathrm{ft}}_S$ agree.

Corollary 6.16 (cf. [Voevodsky 1996, Theorem 3.2.9]). *Suppose the noetherian scheme S is Nagata. The category of representable* rh-*sheaves on $\mathsf{Sch}^{\mathrm{ft}}_S$ is a localisation of the category $\mathsf{Sch}^{\mathrm{ft}}_S$ with respect to completely decomposed homeomorphisms. In other words, it is obtained by formally inverting morphisms of the form $X_{\mathrm{red}} \to X$, then formally inverting subintegral extensions.*

Proof. Certainly, the functor $X \mapsto h^X_{\mathrm{rh}}$ factors through the localisation functor $(-)_{\mathrm{red}} : \mathsf{Sch}^{\mathrm{ft}}_S \to (\mathsf{Sch}^{\mathrm{ft}}_S)_{\mathrm{red}}$. Now, it is straightforward to check that the class $\mathcal{S}$ of subintegral extensions of reduced schemes are a multiplicative system:

(1) $\mathcal{S}$ is closed under composition.

(2) For every $t : Z \to Y$ in $\mathcal{S}$ and $g : X \to Y$ in $(\mathsf{Sch}^{\mathrm{ft}}_S)_{\mathrm{red}}$ there is an $s : W \to X$ in $\mathcal{S}$ and $f : W \to Z$ in $(\mathsf{Sch}^{\mathrm{ft}}_S)_{\mathrm{red}}$ with $gs = tf$.

(3) If $f, g : X \rightrightarrows Y$ are parallel morphisms in $(\mathsf{Sch}^{\mathrm{ft}}_S)_{\mathrm{red}}$, then the following are equivalent:
   (a) $sf = sg$ for some $s \in \mathcal{S}$ with source Y.
   (b) $ft = gt$ for some $t \in \mathcal{S}$ with target X.

(In fact, the latter two conditions are equivalent to $f = g$ in this case.)

Since $\mathcal{S}$ is a multiplicative system, the hom sets in the localisation $\mathcal{S}^{-1}(\mathsf{Sch}^{\mathrm{ft}}_S)_{\mathrm{red}}$ are calculated by the formula

$$\hom_{\mathcal{S}^{-1}(\mathsf{Sch}^{\mathrm{ft}}_S)_{\mathrm{red}}}(X, Y) = \varinjlim_{X' \to X \in \mathcal{S}} \hom(X', Y) = \hom(X^{\mathrm{sn}}, Y);$$

see [Gabriel and Zisman 1967; Weibel 1994, Theorem 10.3.7]. But, by Proposition 6.14, this is equal to $\hom_{\mathsf{Shv}_{\mathrm{rh}}(\mathsf{Sch}^{\mathrm{ft}}_S)}(h^X_{\mathrm{rh}}, h^Y_{\mathrm{rh}})$. $\square$

Corollary 6.17 (cf. [Voevodsky 1996, Theorem 3.2.10]). *Suppose our noetherian base scheme S is Nagata. Let $\mathsf{Shv}^{\mathrm{rep}}_{\mathrm{rh}}(\mathsf{Sch}^{\mathrm{ft}}_S) \subseteq \mathsf{Shv}_{\mathrm{rh}}(\mathsf{Sch}^{\mathrm{ft}}_S)$ denote the full subcategory of representable* rh-*sheaves. The Yoneda functor $h^-_{\mathrm{rh}} : \mathsf{Sch}^{\mathrm{ft}}_S \to \mathsf{Shv}^{\mathrm{rep}}_{\mathrm{rh}}(\mathsf{Sch}^{\mathrm{ft}}_S)$ admits a left adjoint. The counit of the adjunction is the seminormalisation $X^{\mathrm{sn}} \to X$. In particular, for any schemes $X, Y \in \mathsf{Sch}^{\mathrm{ft}}_S$ with X seminormal, one has*

$$\hom(h^X_{\mathrm{rh}}, h^Y_{\mathrm{rh}}) = \hom_{\mathsf{Sch}^{\mathrm{ft}}_S}(X, Y).$$

Proof. We have $\hom_{\mathsf{Shv}_{\mathrm{rh}}}(h^X_{\mathrm{rh}}, h^Y_{\mathrm{rh}}) = h^Y_{\mathrm{rh}}(X) = h^Y(X^{\mathrm{sn}}) = \hom(X^{\mathrm{sn}}, Y)$. □

7. Future directions

7A. *Relation to Berkovich spaces.* Berkovich [1990] introduced a generalisation of rigid geometry in terms of seminorms. The sheaves $\Omega^n_{\mathrm{val}\le 1}$ seem to be connected to Berkovich spaces. We review a very small part of the theory.

Recall that a *multiplicative nonarchimedean norm* on a field K is a group homomorphism $|\cdot| : K^* \to \mathbb{R}_{>0}$ (the latter equipped with multiplication) such that $|f+g| \le \max(|f|, |g|)$. This is usually extended to a map of sets $|\cdot| : K \to \mathbb{R}_{\ge 0}$ by setting $|0| = 0$.

Key Lemma 7.1. *The set of multiplicative nonarchimedean norms on a field K is the same as the set of pairs $(R, |\cdot|)$ where R is a valuation ring of K, and $|\cdot| : K^*/R^* \to \mathbb{R}_{>0}$ is an injective group homomorphism. Under this bijection, the ring R corresponds to $|\cdot|^{-1}[0,1]$. Since $\mathbb{R}_{>0}$ has no convex subgroups, such valuation rings R necessarily have rank* 1 (*or rank* 0 *if $R = K$*).

Proof. Obvious. □

Let K be a field equipped with a multiplicative nonarchimedean norm $\|\cdot\| : K^* \to \mathbb{R}_{>0}$. If X is a K-variety, the *Berkovich space* X^{an} of X, as a set, consists of pairs $(x, |\cdot|)$ where x is a point of X, and $|\cdot| : \kappa(x) \to \mathbb{R}_{>0}$ is a multiplicative nonarchimedean norm extending $\|\cdot\|$. This set is equipped with a structure of locally ringed space such that the projection $\pi : X^{\mathrm{an}} \to X; (x, |\cdot|) \mapsto x$ is a morphism of locally ringed spaces.

On the other hand, recall that Lemma 3.7 described $\Omega^n_{\mathrm{val}}(X)$ as

$$\left\{ (s_x) \in \prod_{x\in X} \Omega^n(x) \;\middle|\; \begin{array}{l} \text{for every } k\text{-valuation ring } R \subseteq k(x) \text{ of rank} \le 1 \text{ we have} \\ \qquad s_x \in \Omega^n(R) \text{ and } s_x|_{R/\mathfrak{m}} = s_y|_{R/\mathfrak{m}} \text{ where } y = \mathrm{Image}(\mathrm{Spec}(R/\mathfrak{m})) \end{array} \right\}. \tag{21}$$

In particular, every section $s \in \Omega^n_{\mathrm{val}}(X)$ gives a function $X^{\mathrm{an}} \to \coprod_{(x,|\cdot|)} \Omega^n(\mathscr{H}(x))$ such that the image of $(x, |\cdot|)$ lands in the corresponding component. Here, $\mathscr{H}(x)$ is the completion $\widehat{\kappa(x)}$ of the normed field $\kappa(x)$. Similarly, we could apply Ω^n to the structure sheaf of the locally ringed space X^{an}, and obtain a ring morphism $\Omega^n_{X^{\mathrm{an}}/K}(X^{\mathrm{an}}) \to \coprod_{(x,|\cdot|)} \Omega^n(\mathscr{H}(x))$.

Question 7.2. Is the image of one of

$$\Omega^n_{X^{\mathrm{an}}/K}(X^{\mathrm{an}}) \to \coprod_{(x,|\cdot|)} \Omega^n(\mathscr{H}(x)) \quad \text{and} \quad \Omega^n_{\mathrm{val}}(X) \to \coprod_{(x,|\cdot|)} \Omega^n(\mathscr{H}(x))$$

contained in the image of the other? Does $\Omega^n_{\mathrm{val}}(X) = \Omega^n_{\mathrm{cdh}}(X)$ have an intrinsic description in terms of analytic spaces?

7B. *F-singularities and reflexive differentials.* Recall that reflexive differentials are defined as the double dual $\Omega^{[n]}_X = (\Omega^n_X)^{**}$. One of the results of [Huber and Jörder 2014] is that on a klt base space X, cdh-differentials recover reflexive differential forms:

$$\Omega_X^{[n]}(X) = \Omega_{\mathrm{cdh}}^n(X).$$

One can ask when such a formula is possible in positive characteristic. For example, what is an appropriate replacement for the klt base space hypothesis?

There is an active area of research in positive characteristic birational geometry studying singularities defined via the Frobenius which are analogues of singularities arising in the minimal model program. These former are called F-singularities; log terminal, log canonical, rational and du Bois correspond to F-regular, F-pure/F-split, F-rational and F-injective, respectively; see [Schwede 2010, Remark 17.11]. Under this dictionary, Kawamata log terminal (i.e., klt) corresponds to strongly F-regular [Schwede 2010, Corollary 17.10].

Consequently, we arrive at the following question.

Question 7.3 (Blickle). If a normal scheme X is strongly F-regular, do we have $\Omega_X^{[n]}(X) \cong \Omega_{\mathrm{cdh}}^n(X)$?

In the special case $n = d_X = \dim X$, we have

$$\Omega_{\mathrm{cdh}}^{d_X}(X) = \Omega_{\mathrm{val}}^{d_X}(X) = \varprojlim_{R\in\mathrm{val}(X)} \Omega^{d_X}(R) = \varprojlim_{R\in\mathrm{val}(X), R\subseteq k(X)} \Omega^{d_X}(R) = \Omega_{RZ(X)}^{d_X}(RZ(X)),$$

and so the question becomes:

Question 7.4. If a normal scheme X of dimension d_X is strongly F-regular, do we have $\Omega_X^{[d_X]}(X) \cong \varinjlim_{X'\to X} \Omega_{X'}^{d_X}(X')$? Here the colimit is over proper birational morphisms $X' \to X$.

Remark 7.5. Under the assumption of resolution of singularities, Question 7.4 is true: being strongly F-regular implies being pseudorational, which means (by definition) that for any proper birational morphism $\pi : X' \to X$ the direct image $\pi_*\omega_{X'}$ of the canonical dualizing sheaf $\omega_{X'}$ of X' is ω_X, the canonical dualizing sheaf of X. If X' is smooth over the base, we have $\omega_{X'} = \Omega_{X/k}^{d_X}$. On the other hand, we also have $\Omega_X^{[d_X]} = \omega_X$. So if we restrict the colimit to those X' which are smooth, we have

$$\Omega_X^{[d_X]} = \omega_X = \varinjlim_{\substack{X'\to X\\ X'\text{smooth}}} \omega_{X'}(X') = \varinjlim_{\substack{X'\to X\\ X'\text{smooth}}} \Omega_{X'}^{d_X}(X').$$

Under the assumption of resolution of singularities, the colimit over X' smooth is the same as the colimit over all X'.

We remark that an alternative description of reflexive differentials when X is klt of characteristic zero is given in [Greb et al. 2011] by $\Omega_X^{[p]} \cong \pi_*\Omega_{X'}^p$, where $\pi : X' \to X$ is a log resolution, and that this implies an isomorphism $\Omega_X^{[d_X]}(X) \cong \Omega_{\mathrm{val}}^{d_X}(X)$ (in characteristic zero), where $d_X = \dim X$. That is, the characteristic zero version of Question 7.3 is true.

Let us list some facts about strongly F-regular schemes that allow us to replace the F-regular hypothesis in Question 7.3 with an equivalent more explicit hypothesis, which may help develop a strategy for a proof.

To begin with, a ring R is *strongly F-regular* if and only if its test ideal is equal to R [Schwede 2010, Proposition 16.9]. This implies that R is Cohen–Macaulay, and in particular, that the sheaf ω_R is a dualising object in the derived category.

As for test ideals, it is shown in [Blickle et al. 2015] that in positive characteristic, the test ideal of a normal variety X over a perfect field can be defined as the intersection of the images of certain trace maps, with the intersection taken over all generically finite proper separable maps $\pi : Y \to X$ with Y regular. Here, the trace map comes from the trace morphism $\pi_! \pi^! \omega_X^\bullet \to \omega_X^\bullet$. Moreover, there exists a $Y \to X$ in the indexing set whose image agrees with this intersection.

So in the affine case, with our replacement hypothesis which is equivalent to "strongly F-regular", Question 7.3 becomes:

Question 7.6. Let X be a normal affine variety over a perfect field of positive characteristic, with $\mathbb{Q}$-Cartier canonical divisor. Suppose that for every generically finite proper separable morphism $\pi : Y \to X$ with regular source, the trace morphism

$$\pi_* \mathcal{O}_Y(\lceil K_Y - \pi^* K_X \rceil) \xrightarrow{\text{trace}} \mathcal{O}_X.$$

is surjective; cf. [Blickle et al. 2015, Main Theorem]. Let $\iota : X_{\text{reg}} \to X$ be the inclusion of the regular locus. Is the canonical morphism

$$\Omega^n_{\text{val}}(X) \to \Omega^n_{\text{val}}(X_{\text{reg}})$$

an isomorphism?

In this formulation, we have used that for any normal scheme X with regular locus $\iota : X_{\text{reg}} \to X$ one has $\Omega^{[n]}_X = \iota_* \Omega^n_{X_{\text{reg}}}$. Since we know that $\Omega^n = \Omega^n_{\text{val}}$ on regular schemes, we obtain the description

$$\Omega^{[n]}_X(X) = (\iota_* \Omega^n_{X_{\text{reg}}})(X) = \Omega^n_{X_{\text{reg}}}(X_{\text{reg}}) = \Omega^n_{\text{val}}(X_{\text{reg}}).$$

Acknowledgements

It is a pleasure to thank Manuel Blickle, Fumiharu Kato, Stefan Kebekus, Simon Pepin-Lehalleur, Kay Rülling, David Rydh and Vasudevan Srinivas for inspiring discussions and answers to our mathematical questions. We also greatly appreciate the referee's careful reading.

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Communicated by János Kollár
Received 2017-07-05 Revised 2018-01-10 Accepted 2018-03-10

annette.huber@math.uni-freiburg.de *Mathematisches Institut, Albert-Ludwigs-Universität Freiburg, Freiburg im Breisgau, Germany*

shanekelly@math.titech.ac.jp *Department of Mathematics, Tokyo Institute of Technology, Tokyo, Japan*

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Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

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Algebra & Number Theory

Volume 12 No. 3 2018