Algebraic de Rham theory for
weakly holomorphic modular forms of level one

Francis Brown and Richard Hain

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We establish an Eichler–Shimura isomorphism for weakly modular forms of level one. We do this by relating weakly modular forms with rational Fourier coefficients to the algebraic de Rham cohomology of the modular curve with twisted coefficients. This leads to formulae for the periods and quasiperiods of modular forms.

1. Introduction

Let M_n denote the $\mathbb{Q}$-vector space of modular forms of weight n and level one with rational Fourier coefficients. Let $S_n \subset M_n$ denote the subspace of cusp forms. The Eichler–Shimura isomorphism [Eichler 1957; Shimura 1959] is usually expressed as a pair of isomorphisms

$$
\begin{aligned}
M_{n+2} \otimes_{\mathbb{Q}} \mathbb{C} &\xrightarrow{\sim} H^1(\mathrm{SL}_2(\mathbb{Z}); V_n^{\mathrm{B}})^+ \otimes_{\mathbb{Q}} \mathbb{C}, \\
S_{n+2} \otimes_{\mathbb{Q}} \mathbb{C} &\xrightarrow{\sim} H^1(\mathrm{SL}_2(\mathbb{Z}); V_n^{\mathrm{B}})^- \otimes_{\mathbb{Q}} \mathbb{C},
\end{aligned}
\tag{1-1}
$$

where the right-hand side denotes group cohomology with coefficients in $V_n^{\mathrm{B}} = \mathrm{Sym}^n V^{\mathrm{B}}$, where V^{B} denotes the standard two-dimensional representation of SL_2 over $\mathbb{Q}$ with basis $\boldsymbol{a}, \boldsymbol{b}$, and $\pm$ denote eigenspaces with respect to the real Frobenius (complex conjugation). One wants to think of this theorem as a special case of the comparison isomorphism between algebraic de Rham cohomology and Betti cohomology, each of which has a natural $\mathbb{Q}$-structure. The rational structures on these groups then enables one to define periods. Each cuspidal Hecke eigenspace, like an elliptic curve, should have four periods (two periods and two quasiperiods) corresponding to the entries of a 2×2 period matrix. However, the isomorphisms (1-1) do not generate enough periods since each one only produces a single period for every modular form. To obtain a full set of periods, one needs to consider "modular forms of the second kind."

In this note, we compute the algebraic de Rham cohomology of the moduli stack $\mathcal{M}_{1,1}$ of elliptic curves and relate it to weakly holomorphic modular forms (modular forms which are holomorphic on the upper half plane but with poles at the cusp). From this, we deduce a $\mathbb{Q}$-de Rham–Eichler–Shimura isomorphism, and a definition of the period matrix of a Hecke eigenspace.

Before stating the main results, it may be instructive to review the familiar case of an elliptic curve E over $\mathbb{Q}$ with equation $y^2 = 4x^3 - ux - v$. The de Rham cohomology $H^1_{dR}(E, \mathbb{Q})$ is a two-dimensional vector space over $\mathbb{Q}$, as is the Betti (singular) cohomology $H^1(E(\mathbb{C}); \mathbb{Q})$. The comparison isomorphism is a canonical isomorphism

$$H^1_{dR}(E; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H^1(E(\mathbb{C}); \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}. \tag{1-2}$$

On the other hand, the space $F^1 H^1_{dR}(E; \mathbb{Q}) := H^0(E; \Omega^1_{E/\mathbb{Q}})$ is one-dimensional and spanned by the holomorphic differential dx/y. The Betti cohomology $H^1(E(\mathbb{C}); \mathbb{Q})$ splits into two eigenspaces under the action of complex conjugation, with eigenvalues ± 1. The analogue of the Eichler–Shimura isomorphisms, in this setting, are exactly the formulae

$$H^0(E; \Omega^1_{E/\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H^1(E(\mathbb{C}); \mathbb{Q})^+ \otimes_{\mathbb{Q}} \mathbb{C},$$

$$H^0(E; \Omega^1_{E/\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H^1(E(\mathbb{C}); \mathbb{Q})^- \otimes_{\mathbb{Q}} \mathbb{C},$$

given by integrating the form dx/y over invariant and antiinvariant cycles in $E(\mathbb{C})$ with respect to complex conjugation, respectively. This is clearly a weaker statement than the comparison isomorphism (1-2). To obtain all periods of the elliptic curve, one needs to consider, in addition, period integrals of the differential xdx/y of the *second kind*, which provides an isomorphism

$$H^1_{dR}(E; \mathbb{Q}) \cong \mathbb{Q}\frac{dx}{y} \oplus \mathbb{Q}\frac{xdx}{y}. \tag{1-3}$$

Remark 1.1. There is also an isomorphism of

$$H^1(E(\mathbb{C}); \mathbb{C})/H^0(E; \Omega^1_{E/\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{C}$$

with the space of antiholomorphic differentials on $E(\mathbb{C})$. Since this isomorphism is only defined over $\mathbb{C}$, one loses the rational structure, and cannot define periods in this manner. The analogue for modular forms is the isomorphism

$$(H^1(\mathrm{SL}_2(\mathbb{Z}); V_n^B) \otimes_{\mathbb{Q}} \mathbb{C})/(M_{n+2} \otimes_{\mathbb{Q}} \mathbb{C})$$

with the space of antiholomorphic cusp forms of weight $n + 2$.

1A. *Statement of the theorem.* Let $\mathcal{E}$ denote the universal elliptic curve over the moduli stack $\mathcal{M}_{1,1}$ (over $\mathbb{Q}$) of elliptic curves. It defines a rank two algebraic vector bundle $\mathcal{V}$, equipped with the Gauss–Manin connection ∇. For all $n \geq 1$, set

$$\mathcal{V}_n = \mathrm{Sym}^n \mathcal{V}$$

and denote the induced connection by ∇ also. Grothendieck defined algebraic de Rham cohomology, which is a finite-dimensional $\mathbb{Q}$-vector space:

$$H^1_{dR}(\mathcal{M}_{1,1}; \mathcal{V}_n).$$

In order to describe this space in terms of modular forms, for each $n \in \mathbb{Z}$, let $M_n^!$ denote the $\mathbb{Q}$-vector space of weakly holomorphic modular forms of weight n that have a Fourier expansion

$$\sum_{k \geq -N} a_k q^k$$

with $a_k \in \mathbb{Q}$. They can have negative weights. Such a form is called a cusp form if $a_0 = 0$. Let $S_n^! \subset M_n^!$ denote the subspace of cusp forms. Now consider the differential operator

$$\mathcal{D} = q \frac{d}{dq}. \tag{1-4}$$

It does not in general preserve modularity, but an identity due to Bol [1949] implies that its powers induce a linear map

$$\mathcal{D}^{n+1} : M_{-n}^! \to S_{n+2}^!$$

for every $n \geq 0$. Our main theorem was inspired by the recent paper [Guerzhoy 2008]. After writing this note, we learnt that similar results for modular curves of higher level were implicitly obtained by Coleman [1996] in the p-adic setting, and independently by Scholl [1985]. As pointed out in the very recent paper [Kazalicki and Scholl 2016], a description of algebraic de Rham cohomology in terms of modular forms of the second kind seems not to have been stated explicitly anywhere in the literature up until that point. An approach using the Cousin resolution was subsequently given in [Candelori 2014]. The following theorem can be indirectly deduced from the results of these papers by viewing a modular form of level one as an invariant form of levels 3 and 4.

Theorem 1.2. *For each $n \geq 0$, there is a canonical isomorphism of $\mathbb{Q}$-vector spaces*

$$\varpi : M_{n+2}^! / \mathcal{D}^{n+1} M_{-n}^! \xrightarrow{\sim} H_{\mathrm{dR}}^1(\mathcal{M}_{1,1}; \mathcal{V}_n).$$

The space on the left contains the space of holomorphic modular forms as a subspace:

$$M_{n+2} \subset M_{n+2}^! / \mathcal{D}^{n+1} M_{-n}^!.$$

More precisely, the group $H_{\mathrm{dR}}^1(\mathcal{M}_{1,1}; \mathcal{V}_n)$ carries a natural Hodge filtration

$$H_{\mathrm{dR}}^1(\mathcal{M}_{1,1}; \mathcal{V}_n) = F^0 \supset F^1 = \cdots = F^{n+1} \supset F^{n+2} = 0$$

and F^{n+1} is the image of M_{n+2} under ϖ. That is,

$$\varpi : M_{n+2} \xrightarrow{\sim} F^{n+1} H_{\mathrm{dR}}^1(\mathcal{M}_{1,1}; \mathcal{V}_n).$$

A splitting of the Hodge filtration is discussed in Section 6.

1B. *Comparison isomorphism.* Grothendieck's algebraic de Rham theorem implies (see Section 3) that there is a canonical isomorphism of complex vector spaces

$$H_{\mathrm{dR}}^1(\mathcal{M}_{1,1}; \mathcal{V}_n) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H^1(\mathcal{M}_{1,1}(\mathbb{C}); \mathbb{V}_n^{\mathrm{B}}),$$

where the right-hand space is the Betti (singular) cohomology of $\mathcal{M}_{1,1}(\mathbb{C})$ with coefficients in the complex local system $\mathbb{V}_n^{\mathrm{B}} = \mathrm{Sym}^{n-1} R^1 \pi_* \mathbb{C}$, where $\pi : \mathcal{E} \to \mathcal{M}_{1,1}$ is the universal elliptic curve. Let V_n^{B} denote its fibre $H^0(\mathfrak{H}, \mathbb{V}_n^{\mathrm{B}})$ at the tangent vector $\partial/\partial q$. Since $\mathcal{M}_{1,1}(\mathbb{C})$ is the orbifold quotient of the upper half plane by $\mathrm{SL}_2(\mathbb{Z})$, its cohomology is computed by group cohomology of $\mathrm{SL}_2(\mathbb{Z})$ and we immediately deduce the following consequence of Grothendieck's theorem:

Corollary 1.3. *There is a canonical isomorphism*

$$\mathrm{comp}_{\mathrm{B},\mathrm{dR}} : H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}; \mathscr{V}_n) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H^1(\mathrm{SL}_2(\mathbb{Z}); V_n^{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}).$$

Combined with the previous theorem, we deduce an algebraic de Rham version of the Eichler–Shimura isomorphism. It is the analogue for modular forms of the isomorphism (1-2).

Corollary 1.4. *There is a canonical isomorphism*

$$M^!_{n+2}/\mathcal{D}^{n+1} M^!_{-n} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H^1(\mathrm{SL}_2(\mathbb{Z}); V_n^{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}). \tag{1-5}$$

The dimension of the space on the left-hand side was computed in [Guerzhoy 2008], the dimension of the right-hand space by Eichler–Shimura: both are $1 + 2 \dim S_n$. Restricting the previous isomorphism to the subspace M_{n+2} of holomorphic modular forms, and projecting onto the positive or negative eigenspaces with respect to complex conjugation on the right-hand space gives back the two isomorphisms (1-1).

For any weakly holomorphic modular form $f \in M^!_{n+2}$, its image under the comparison isomorphism is given explicitly by the cohomology class of the cocycle:

$$\gamma \mapsto (2\pi i)^{n+1} \int_{\gamma^{-1} z_0}^{z_0} f(z)(z\boldsymbol{a} - \boldsymbol{b})^n \, dz, \tag{1-6}$$

where $\boldsymbol{a}, \boldsymbol{b}$ is a basis of V^{B}, which we think of as the first rational Betti cohomology group of the elliptic curve $\mathbb{C}/(\mathbb{Z} \oplus z\mathbb{Z})$. Its cohomology class does not depend on the choice of basepoint $z_0 \in \mathfrak{H}$. A different version of this map (and without the rational structures) was described in [Bringmann et al. 2013]. See also [Bruggeman et al. 2014, Theorem A].

1C. *Periods.* The isomorphism (1-5) is compatible with the action of Hecke operators. The action of Hecke operators on the left-hand side was defined in [Guerzhoy 2008]. The eigenspace of an Eisenstein series is one-dimensional, that corresponding to a cusp form is two-dimensional. Let f be a cusp Hecke eigenform and $K_f \subset \mathbb{R}$ the field generated by its Fourier coefficients. Let

$$V_f^{\mathrm{dR}} \subset \left(M^!_{n+2}/D^{n+1} M^!_{-n}\right) \otimes_{\mathbb{Q}} K_f$$

denote the Hecke eigenspace generated by f. It is a two-dimensional K_f-vector space. Let

$$V_f^{\mathrm{B}} \subset H^1\left(\mathrm{SL}_2(\mathbb{Z}); V_n^{\mathrm{B}} \otimes_{\mathbb{Q}} K_f\right)$$

denote the corresponding Betti eigenspace. It is also a two-dimensional K_f-vector space, and decomposes into invariant and antiinvariant eigenspaces with respect to the real Frobenius. We deduce from (1-5) a

canonical isomorphism

$$\mathrm{comp}_{\mathrm{B,dR}} : V_f^{\mathrm{dR}} \otimes_{K_f} \mathbb{C} \xrightarrow{\sim} V_f^{\mathrm{B}} \otimes_{K_f} \mathbb{C}.$$

Definition 1.5. Define a *period matrix* P_f of f to be the matrix of $\mathrm{comp}_{\mathrm{B,dR}}$ written in a K_f-basis of V_f^{dR} and V_f^{B}. We can assume that the basis of V_f^{B} is compatible with decomposition into eigenspaces for the action of the real Frobenius. It is of the form

$$P_f = \begin{pmatrix} \eta_f^+ & \omega_f^+ \\ i\eta_f^- & i\omega_f^- \end{pmatrix},$$

where $\omega_f^{\pm}, \eta_f^{\pm} \in \mathbb{R}$. It is well-defined up to right multiplication by a lower-triangular matrix with entries in K_f, and entries in $K_f^{\times}$ on the diagonal. The $\omega_f^+, i\omega_f^-$ are the holomorphic periods [Manin 1973].

Remark 1.6. Only the holomorphic periods $\omega_f^+, i\omega_f^-$ can be obtained from the classical Eichler–Shimura isomorphisms (1-1).

Theorem 1.7. *If f has weight $2n$, then* $\det(P_f) \in (2\pi i)^{2n-1} K_f^{\times}$.

2. Definitions and background

2A. *Weakly holomorphic modular forms.*

Definition 2.1. For every $n \in \mathbb{Z}$, let $M_n^!$ denote the $\mathbb{Q}$-algebra of weakly holomorphic modular forms of level 1 and weight n with rational Fourier coefficients. It is the $\mathbb{Q}$-vector space of holomorphic functions $f : \mathfrak{H} \to \mathbb{C}$ on the upper half plane $\mathfrak{H}$ such that

$$f(\gamma z) = (cz + d)^n f(z) \quad \text{for all} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \tag{2-1}$$

which admit a Fourier expansion of the form

$$f = \sum_{k \geq -N} a_k q^k, \quad a_k \in \mathbb{Q}.$$

The space $S_n^! \subset M_n^!$ of cusp forms is the subspace of functions satisfying $a_0 = 0$.

Proposition 2.2 (Bol's identity). *For all $n \geq 0$, there is a linear map*

$$\mathcal{D}^{n+1} : M_{-n}^! \to M_{n+2}^!.$$

Proof. This result follows automatically from our proof of Theorem 1.2. We provide a more direct proof for completeness.

For $f : \mathfrak{H} \to \mathbb{C}$ a real analytic modular form of weight $n \in \mathbb{Z}$ define

$$d_n f = \frac{1}{2\pi i} \left(\frac{\partial f}{\partial z} + \frac{nf}{z - \bar{z}} \right).$$

It is well-known by Maass, and easily verified, that this operator respects the transformation property (2-1), and therefore $d_n f$ is a real analytic modular form of weight $n + 2$.

The proposition follows from the following identity, for all $n \geq 0$:

$$\mathcal{D}^{n+1} = d_n d_{n-2} \cdots d_{2-n} d_{-n}.$$

To verify this, write a real analytic function on $\mathfrak{H}$ as a formal power series in $(z - \bar{z})$ and $\bar{z}$. It suffices to verify the formula for

$$f_{a,b} = (2\pi i (z - \bar{z}))^a (-2\pi i \bar{z})^b,$$

where $a, b \geq 0$. We check that $d_m f_{a,b} = (a + m) f_{a-1,b}$, and hence

$$d_n d_{n-2} \cdots d_{2-n} d_{-n} (f_{a,b}) = a(a - 1) \cdots (a - n) f_{a-n-1,b}.$$

On the other hand, $\log q = 2\pi i z$ and $\log \bar{q} = -2\pi i \bar{z}$ and hence

$$f_{a,b} = (\log q + \log \bar{q})^a (\log \bar{q})^b.$$

Since $\mathcal{D} = q \partial/\partial q = \partial/\partial(\log q)$, the operator $\mathcal{D}^{n+1}$ acts on $f_{a,b}$ in an identical manner. $\square$

2B. *Moduli of elliptic curves.* Let $\Bbbk$ be a commutative ring with $6 \in \Bbbk^\times$. In much of this paper, $\Bbbk$ will be either the universal such ring, $\Bbbk = \mathbb{Z}\left[\frac{1}{6}\right]$, or $\Bbbk = \mathbb{Q}$. The formula

$$\lambda \cdot u = \lambda^4 u, \quad \lambda \cdot v = \lambda^6 v$$

defines a left action

$$\mu : \mathbb{G}_m \times \mathbb{A}^2 \to \mathbb{A}^2 \tag{2-2}$$

of the multiplicative group $\mathbb{G}_m$ on the affine plane $\mathbb{A}^2 := \operatorname{Spec} \Bbbk[u, v]$, and defines a grading on $\Bbbk[u, v]$ called the weight. The discriminant function

$$\Delta = u^3 - 27v^2$$

has weight 12 under this action, so that $\mathbb{G}_m$ also acts on the graded ring $\Bbbk[u, v][\Delta^{-1}]$. Let D be the vanishing locus of the discriminant Δ. Set

$$X = \mathbb{A}^2 - D := \operatorname{Spec} \Bbbk[u, v][\Delta^{-1}].$$

Then X is the moduli scheme of elliptic curves over $\Bbbk$ together with a nonzero abelian differential, [Katz and Mazur 1985, §(2.2.6)]. Its coordinate ring

$$\mathcal{O}(X) = \Bbbk[u, v][\Delta^{-1}] = \bigoplus_{m \text{ even}} \operatorname{gr}_m \mathcal{O}(X)$$

is graded by the $\mathbb{G}_m$ action. The sheaf of regular functions on X will be denoted by $\mathcal{O}_X$.

The universal elliptic curve $\mathcal{E}$ over X is the subvariety of $\mathbb{P}^2 \times_\Bbbk X$ which is the Zariski closure of the affine scheme defined by the equation

$$y^2 = 4x^3 - ux - v \in \mathcal{O}(X)[x, y].$$

The universal abelian differential on $\mathcal{E}$ is dx/y. The multiplicative group $\mathbb{G}_m$ acts on (x, y) by $\lambda \cdot (x, y) = (\lambda^2 x, \lambda^3 y)$ and on the abelian differential by $\lambda \cdot dx/y = \lambda^{-1} dx/y$.

The moduli stack of elliptic curves over $\Bbbk$ is the stack quotient

$$\mathcal{M}_{1,1}/\Bbbk = \mathbb{G}_m \backslash\backslash X$$

of X by $\mathbb{G}_m$. Its Deligne–Mumford compactification is

$$\overline{\mathcal{M}}_{1,1}/\Bbbk = \mathbb{G}_m \backslash\backslash Y,$$

where $Y = \mathbb{A}^2 - \{0\}$. In down to earth terms, to work on $\mathcal{M}_{1,1}/\Bbbk$ is to work $\mathbb{G}_m$-equivariantly on X, and to work on $\overline{\mathcal{M}}_{1,1}/\Bbbk$ is to work $\mathbb{G}_m$-equivariantly on Y.

2C. *Upper half plane description.* When $\Bbbk = \mathbb{C}$, this description of $\mathcal{M}_{1,1}$ relates to the traditional upper half plane model via Eisenstein series. Denote by G_{2n} the normalised Eisenstein series

$$\mathsf{G}_{2n}(q) = -\frac{B_{2n}}{4n} + \sum_{m=1}^{\infty} \sigma_{2n-1}(m) q^m$$

of weight $2n$ where B_k is the k-th Bernoulli number, $q = e^{2\pi i z}$, and $\sigma_k(m) = \sum_{d \mid m} d^k$ the divisor function.

Define a map $\rho : \mathfrak{H} \to X(\mathbb{C})$ by $\rho(z) = (u(z), v(z))$, where

$$u = 20\mathsf{G}_4(z) \quad \text{and} \quad v = \tfrac{7}{3}\mathsf{G}_6(z). \tag{2-3}$$

The map ρ factors through the punctured q-disk and induces a graded ring isomorphism to the space of holomorphic modular forms

$$\rho^* : \mathbb{Q}[u, v] \xrightarrow{\sim} \bigoplus_{n \text{ even}} M_n.$$

The pull-back of the discriminant Δ is the Ramanujan τ-function $\Delta(z)$, which vanishes nowhere on $\mathfrak{H}$. It follows that ρ induces a graded ring isomorphism

$$\rho^* : \mathcal{O}(X) \xrightarrow{\sim} \bigoplus_{n \text{ even}} M_n^!.$$

and that the image of ρ is indeed contained in $X(\mathbb{C})$. The ring of weakly modular forms is therefore nothing other than the affine ring of functions on X.

The elliptic curve $\mathbb{C}/(\mathbb{Z} \oplus z\mathbb{Z})$ is mapped isomorphically to the elliptic curve

$$y^2 = 4x^3 - 20\mathsf{G}_4(z) - \tfrac{7}{3}\mathsf{G}_6(z)$$

by $w \mapsto (\wp_z(w)/(2\pi i)^2, \wp_z'(w)/(2\pi i)^3)$, where $\wp_z(w)$ denotes the Weierstrass $\wp$-function. The abelian differential dx/y pulls back to $2\pi i \, dw$. This implies that the map (2-3) induces an isomorphism

$$\mathrm{SL}_2(\mathbb{Z}) \backslash\backslash \mathfrak{H} \xrightarrow{\sim} (\mathbb{G}_m \backslash\backslash X)(\mathbb{C}) = \mathcal{M}_{1,1}(\mathbb{C})$$

of analytic stacks, since $\rho(\gamma z) = ((cz+d)^4 u(z), (cz+d)^6 v(z))$ for $\gamma \in SL_2(\mathbb{Z})$ of the form (2-1), by modularity of G_4 and G_6.

The following lemma motivates the choice of the basis of one-forms considered later.

Lemma 2.3. *Pulling back along* (2-3) *we have*

$$\frac{2u\,dv - 3v\,du}{\Delta} = \frac{2}{3}\frac{dq}{q}.$$

Proof. Formulae due to Ramanujan imply that

$$\mathcal{D}E_2 = (E_2^2 - E_4)/12, \quad \mathcal{D}E_4 = (E_2 E_4 - E_6)/3, \quad \mathcal{D}E_6 = (E_2 E_6 - E_4^2)/2,$$

where $E_{2n} = -(4n/B_{2n})G_{2n}$ are the Eisenstein series normalised such that their constant Fourier coefficient is 1. These are easily verified by computing the first few terms in their Fourier expansion [Zagier 2008, Proposition 15]. It follows that

$$3E_6\,\mathcal{D}E_4 - 2E_4\,\mathcal{D}E_6 = E_4^3 - E_6^2 = 1728\Delta.$$

Now substitute $E_4 = 240G_4 = 12u$ and $E_6 = -504G_6 = -216v$. $\qquad\square$

Remark 2.4. The functions u, v have the following q-expansions:

$$u = 20\,G_4 = \tfrac{1}{12} + 20q + 180q^2 + 560q^3 + 1460q^4 + \cdots$$
$$v = \tfrac{7}{3}\,G_6 = -\tfrac{1}{216} + \tfrac{7}{3}q + 77q^2 + \tfrac{1708}{3}q^3 + \tfrac{7399}{3}q^4 + \cdots$$

They have coefficients in $\mathbb{Z}\left[\tfrac{1}{6}\right]$. Furthermore,

$$\Delta^{-1} = \frac{1}{q} + 24 + 324q + 3200q^2 + 25650q^3 + 176256q^4 + \cdots$$

has integer coefficients. It follows that

$$\frac{M^!_{n+2}}{\mathcal{D}^{n+1}M^!_{-n}}$$

has a natural $\mathbb{Z}\left[\tfrac{1}{6}\right]$-structure, given by series whose Fourier coefficients lie in $\mathbb{Z}\left[\tfrac{1}{6}\right]$. This is because the operator $\mathcal{D}$ acts on the ring $\mathbb{Z}[\![q]\!]$.

The $\mathbb{Z}\left[\tfrac{1}{6}\right]$ structure on the affine ring of $X/\mathbb{Z}\left[\tfrac{1}{6}\right]$ coincides, by Remark 2.4, with the $\mathbb{Z}\left[\tfrac{1}{6}\right]$ structure on Fourier expansions.

2D. *Differential forms.* The following one-forms play a special role:

$$\psi = \frac{1}{12}\frac{d\Delta}{\Delta}, \quad \omega = \frac{3}{2}\left(\frac{2u\,dv - 3v\,du}{\Delta}\right) \in \Omega^1(X). \tag{2-4}$$

They have weights 0 and -2, respectively. By Lemma 2.3, we have

$$\rho^*\omega = \frac{dq}{q}.$$

Both ω and ψ have logarithmic singularities along the discriminant locus D.

Lemma 2.5. *If $h \in \mathrm{gr}_m \mathcal{O}(X)$, then $\Delta \psi \wedge \omega = \frac{3}{4} du \wedge dv$ and*

$$\frac{dh}{h} \wedge \omega = m\psi \wedge \omega, \quad d(h\omega) = (m-2)h\psi \wedge \omega. \tag{2-5}$$

Proof. Since $d\log(h_1 h_2) = d\log(h_1) + d\log(h_2)$, it suffices to verify the first equation for $h = u$ and $h = \Delta$. For the second equation, use $d\omega = -2\psi \wedge \omega$ to write

$$d(h\omega) = h\frac{dh}{h} \wedge \omega - 2h\psi \wedge \omega$$

and conclude using the first equation. $\qquad\square$

In particular, since ψ and ω are pointwise linearly independent, we have

$$\Omega^1(X) = \mathcal{O}(X)du \oplus \mathcal{O}(X)dv \cong \mathcal{O}(X)\psi \oplus \mathcal{O}(X)\omega.$$

Corollary 2.6. *For every $f \in \mathrm{gr}_m \mathcal{O}(X)$ we have*

$$df = \vartheta(f)\omega + mf\,\psi,$$

where $\vartheta : \mathcal{O}(X) \to \mathcal{O}(X)$ is the derivation of weight two defined by

$$\vartheta = 6v\frac{\partial}{\partial u} + \frac{u^2}{3}\frac{\partial}{\partial v}. \tag{2-6}$$

Proof. There exist unique $f_0, f_1 \in \mathcal{O}(X)$ such that

$$df = f_0\omega + f_1\psi.$$

Use (2-5) to deduce that $df \wedge \omega = mf\psi \wedge \omega$, which yields $f_1 = mf$. The linear map $\vartheta : f \mapsto f_0$ is a derivation which necessarily satisfies $\vartheta(\Delta) = 0$. This is certainly true of the formula (2-6). It therefore suffices to verify that $\vartheta(u) = 6v$. For this, use the fact that $d\Delta = 3u^2\,du - 54v\,dv$ to deduce that $du \wedge \psi = \frac{-9v}{2}(du \wedge dv)/\Delta$. Comparing with $\omega \wedge \psi = -\frac{3}{4}(du \wedge dv)/\Delta$ implies that $\vartheta(u) = 6v$ as required. $\qquad\square$

Remark 2.7. The derivation ϑ is closely related to the Serre derivative [Zagier 2008, (53)].

Consider the pull-back along (2-2) followed by the natural map

$$\Omega^1_X \xrightarrow{\mu^*} \Omega^1_{\mathbb{G}_m \times X} \to \Omega^1_{(\mathbb{G}_m \times X)/X}$$

to relative Kähler differentials. Taking global sections gives a natural $\mathcal{O}(X)$-linear map

$$\pi^* : \Omega^1(X) \to \mathcal{O}(X) \otimes_{\Bbbk} \Omega^1(\mathbb{G}_m) = \mathcal{O}(X)[\lambda^{\pm}]\,d\log\lambda,$$

where λ is the coordinate on $\mathbb{G}_m$.

Say that an element of $\Omega^1(X)$ is *proportional* to ω if it lies in the subspace $\mathcal{O}(X)\omega$ of $\Omega^1(X)$ spanned by ω.

Lemma 2.8. *A form $\eta \in \Omega^1(X)$ is proportional to ω if and only if $\pi^*\eta = 0$. Furthermore, $\pi^*\psi = d\log\lambda$.*

Proof. Via the natural isomorphism of $\mathcal{O}(X)[\lambda^{\pm}]$-modules,

$$\big(\Omega^1(\mathbb{G}_m) \otimes_{\Bbbk} \mathcal{O}(X)\big) \oplus \big(\Omega^1(X) \otimes_{\Bbbk} \Bbbk[\lambda^{\pm}]\big) \to \Omega^1(\mathbb{G}_m \times X),$$

we can compute

$$\mu^*(\omega) = (0, \lambda^{-2}\omega) \quad \text{and} \quad \mu^*(\psi) = (d\log\lambda, \psi). \tag{2-7}$$

This follows from calculating

$$\mu^*(2u\,dv - 3v\,du) = 2\lambda^4 u\,d(\lambda^6 v) - 3\lambda^6 v\,d(\lambda^4 u) = \lambda^{10}\omega$$

and noting that the terms involving $d\lambda$ cancel. More generally, we verify that for any $h \in \mathrm{gr}_m\mathcal{O}(X)$, we have

$$\mu^*\,dh = d(\lambda^m h) = \lambda^m\,dh + m\lambda^{m-1}h\,d\lambda$$

and therefore $\mu^*\,d\log h = (m\,d\log\lambda, d\log h)$. Setting $h = \Delta$ proves (2-7), from which the lemma immediately follows. $\square$

2E. *Alternative description of $\mathcal{M}_{1,1/\mathbb{Q}}$.* A complementary approach to constructing $\mathcal{M}_{1,1}$ as a stack over $\Bbbk$ is as a quotient of the affine subscheme Z of X defined by $\Delta = 1$. Its affine ring $\mathcal{O}(Z)$ is $\Bbbk[u, v]/I$, where I is the graded ideal of $\Bbbk[u, v]$ generated by $\Delta - 1$, where $\Bbbk$ is any commutative ring with $6 \in \Bbbk^{\times}$. Since Δ has weight 12, the affine group scheme

$$\boldsymbol{\mu}_{12} = \mathrm{Spec}\,\Bbbk[\lambda]/(\lambda^{12} - 1),$$

with λ group-like, acts on Z.

Remark 2.9. The affine scheme Z is isomorphic, over $\Bbbk$, to the Fermat cubic minus its identity element. In fact, the closure of the locus $u^3 - 27v^2 = 1/4$ is isomorphic to the Fermat cubic $x^3 + y^3 = 1$, where $u = 1/(x + y)$ and $v = (x - y)/6(x + y)$.

The inclusion $\boldsymbol{\mu}_{12} \hookrightarrow \mathbb{G}_m$ induces an isomorphism of (the constant group scheme) $\mathbb{Z}/12\mathbb{Z}$ with the character group of $\boldsymbol{\mu}_{12}$. Denote the congruence class of $n \bmod 12$ by $[n]$. The $\boldsymbol{\mu}_{12}$ action on Z gives a $\mathbb{Z}/12\mathbb{Z}$-grading of its coordinate ring:

$$\mathcal{O}(Z) = \bigoplus_{n \bmod 12} \mathcal{O}(Z)_{[n]}.$$

If n is odd then $\mathcal{O}(Z)_{[n]} = 0$. Since the inclusion $j : Z \hookrightarrow X$ is $\boldsymbol{\mu}_{12}$ equivariant, the restriction homomorphism induces a ring homomorphism

$$j^* : \mathcal{O}(X)^{\mathbb{G}_m} \to \mathcal{O}(Z)^{\boldsymbol{\mu}_{12}} \tag{2-8}$$

and, for each $m \in \mathbb{Z}$, a homomorphism

$$j^* : \mathrm{gr}_m\mathcal{O}(X) \to \mathcal{O}(Z)_{[m]} \tag{2-9}$$

of modules over (2-8).

Lemma 2.10. *The homomorphism* (2-8) *is an isomorphism, so that*

$$j : \boldsymbol{\mu}_{12} \backslash\backslash Z \xrightarrow{\sim} \mathbb{G}_m \backslash\backslash X$$

is an isomorphism of stacks. Moreover, for each $m \in \mathbb{Z}$, (2-9) *is an isomorphism.*

Proof. Since $\Delta - 1$ is $\boldsymbol{\mu}_{12}$-invariant, it suffices for the first part to show that

$$\mathcal{O}(X)^{\mathbb{G}_m} = \Bbbk[u, v, \Delta^{-1}]^{\mathbb{G}_m} \longrightarrow \Bbbk[\bar{u}, \bar{v}]^{\boldsymbol{\mu}_{12}} / I^{\boldsymbol{\mu}_{12}} = \mathcal{O}(Z)^{\boldsymbol{\mu}_{12}}$$

is an isomorphism, where $\bar{u} = j^*(u)$ and $\bar{v} = j^*(v)$ are the images of u, v. Since $\bar{u}$ has weight 4 and $\bar{v}$ has weight 6, it follows that $\Bbbk[\bar{u}, \bar{v}]^{\boldsymbol{\mu}_{12}} = \Bbbk[\bar{u}^3, \bar{v}^2]$. The inverse is induced by the map

$$\Bbbk[\bar{u}^3, \bar{v}^2] \to \Bbbk[u, v, \Delta^{-1}]^{\mathbb{G}_m}$$

which sends $\bar{u}^3$ to $u^3 \Delta^{-1}$ and $\bar{v}^2$ to $v^2 \Delta^{-1}$. It vanishes on $I^{\boldsymbol{\mu}_{12}}$. This proves that (2-8) is an isomorphism.

For all integers k, multiplication by Δ^k gives an isomorphism

$$\mathrm{gr}_m \mathcal{O}(X) \xrightarrow{\sim} \mathrm{gr}_{m+12k} \mathcal{O}(X)$$

of $\mathrm{gr}_0 \mathcal{O}(X) = \mathcal{O}(X)^{\mathbb{G}_m}$-modules. It therefore suffices to prove that (2-9) is an isomorphism for $m = 0, 4, 6, 8, 10, 14$, which form a complete set of representatives for even numbers modulo 12. But $\mathcal{O}(Z)$ is isomorphic to the free $\mathbb{Z}/12\mathbb{Z}$-graded $\mathcal{O}(Z)^{\boldsymbol{\mu}_{12}} \cong \Bbbk[\bar{u}^3, \bar{v}^2]$-module generated by monomials in $\bar{u}$ and $\bar{v}$ that are of degree < 3 in $\bar{u}$ and degree < 2 in $\bar{v}$, where $\bar{u}^3 - 27\bar{v}^2 = 1$. These are

$$1, \bar{u}, \bar{v}, \bar{u}^2, \bar{u}\bar{v}, \bar{u}^2\bar{v},$$

and have weights 0, 4, 6, 8, 10, 14, respectively. Similarly, $\mathrm{gr}_m \mathcal{O}(X)$ is isomorphic to the free graded $\mathcal{O}(X)^{\mathbb{G}_m} \cong \Bbbk[u^3\Delta^{-1}, v^2\Delta^{-1}]$-module generated by the monomials $1, u, v, u^2, uv, u^2v$. $\square$

2F. *Gauss–Manin connection* [Katz 1973, §A1]. The vector bundle $\mathcal{V}$ over X is defined to be the restriction of the trivial rank 2 vector bundle

$$\overline{\mathcal{V}} := \mathcal{O}_Y \mathsf{S} \oplus \mathcal{O}_Y \mathsf{T}$$

on $Y := \mathbb{A}^2 - \{0\}$ to X. The multiplicative group acts on it by

$$\lambda \cdot \mathsf{S} = \lambda\mathsf{S}, \quad \lambda \cdot \mathsf{T} = \lambda^{-1}\mathsf{T}.$$

So $\overline{\mathcal{V}}$ can be regarded as a vector bundle on the moduli stack $\overline{\mathcal{M}}_{1,1}$ and $\mathcal{V}$ as a vector bundle on $\mathcal{M}_{1,1}$. Set $\mathcal{V}_n = \mathrm{Sym}^n \mathcal{V}$ for all $n \geq 1$.

The connection on $\overline{\mathcal{V}}$, and its symmetric powers

$$\overline{\mathcal{V}}_n := \mathrm{Sym}^n \overline{\mathcal{V}} = \bigoplus_{j+k=n} \mathcal{O}_Y \mathsf{S}^j \mathsf{T}^k$$

is defined by

$$\nabla = d + \begin{pmatrix} \mathsf{S} & \mathsf{T} \end{pmatrix} \begin{pmatrix} \psi & \omega \\ -\dfrac{u}{12}\omega & -\psi \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial \mathsf{S}} \\ \dfrac{\partial}{\partial \mathsf{T}} \end{pmatrix}. \tag{2-10}$$

It is $\mathbb{G}_m$-invariant, and thus defines a rational connection on $\overline{\mathcal{V}}_n$ with regular singularities and nilpotent residue along the discriminant divisor D.

Set $\mathbb{V} = R^1\pi_*\Bbbk_X$ where $\pi : \mathcal{E} \to X$ is the universal elliptic curve. It is proved in [Hain 2013, Proposition 19.6] that when $\Bbbk \subset \mathbb{C}$ there is a natural isomorphism

$$\mathbb{V}^{\mathrm{B}} \otimes_{\Bbbk} \mathcal{O}_X^{\mathrm{an}} \cong \mathcal{V}^{\mathrm{an}} \otimes_{\Bbbk} \mathbb{C}$$

of bundles with connection over $X(\mathbb{C})$, where the left hand bundle is endowed with the Gauss–Manin connection. Under this isomorphism T corresponds to the section dx/y of $(R^1\pi_*\mathbb{C}_X) \otimes_{\mathbb{C}} \mathcal{O}_X$ and S to the section $x\,dx/y$. For later use, we set

$$\mathbb{V}_n^{\mathrm{B}} = \mathrm{Sym}^n \mathbb{V}^{\mathrm{B}}. \tag{2-11}$$

When pulled back to the upper half plane (and q-disk), the connection can also be written down in terms of the frame A and T, where A is the section corresponding to the Poincaré dual of the element $\boldsymbol{a}$ of $H_1(\mathbb{Z}/(\mathbb{Z} + z\mathbb{Z}))$ corresponding to the curve from 0 to 1. The two framings are related by[1]

$$\mathsf{S} = \mathsf{A} + 2\,\mathsf{G}_2(q)\mathsf{T}. \tag{2-12}$$

In this frame, the Gauss–Manin connection is given by

$$\nabla = d + \mathsf{A}\frac{\partial}{\partial \mathsf{T}} \otimes \frac{dq}{q} = 2\pi i\left(\mathcal{D} + \mathsf{A}\frac{\partial}{\partial \mathsf{T}}\right) \otimes dz, \tag{2-13}$$

where, as above, $\mathcal{D}$ is the differential operator $q\partial/\partial q = (2\pi i)^{-1}\partial/\partial z$.

Lemma 2.11. *If* $f = \sum_{j+k=n} f^{j,k}\mathsf{A}^j\mathsf{T}^k$ *is a real analytic section of* $\mathcal{V}_n$, *then*

$$\nabla f = 2\pi i \sum_{j+k=n} \left(\mathcal{D}f^{j,k} + (k+1)f^{j-1,k+1}\right) dz \otimes \mathsf{A}^j\mathsf{T}^k,$$

where we define $f^{n+1,-1} = f^{-1,n+1} = 0$.

Proof. Apply the connection (2-13) to the section f. $\qquad\qquad\square$

Pulling back via the map π^* defines a relative connection on $\mathbb{G}_m$ over $\mathcal{O}_X$, which by (2-7) is of the form

$$\pi^*(\nabla) = d + \begin{pmatrix} \mathsf{S} & \mathsf{T} \end{pmatrix} \begin{pmatrix} \dfrac{d\lambda}{\lambda} & 0 \\ 0 & -\dfrac{d\lambda}{\lambda} \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial \mathsf{S}} \\ \dfrac{\partial}{\partial \mathsf{T}} \end{pmatrix} \tag{2-14}$$

A section of this relative connection is flat if and only if it is $\mathbb{G}_m$-equivariant.

---

[1] This follows from the formulas in [Hain 2013, §19.3]: T and S are obtained from $\hat{T}$ and $\hat{S}$ there by setting $\xi = (2\pi i)^{-1}$ and taking A to be $\boldsymbol{a}$.

3. De Rham cohomology

In this section, we take $\Bbbk = \mathbb{Q}$. Since X is affine, it follows from the version of Grothendieck's algebraic de Rham theorem with coefficients in a connection [Deligne 1970, Corollary 6.3] that for each $n \geq 0$, $H^1_{dR}(X, \mathscr{V}_n)$ is computed by the complex

$$\Omega^\bullet(X, \mathscr{V}_n) := \left[\Gamma(X, \mathscr{V}_n) \xrightarrow{\nabla} \Gamma(X, \Omega^1_X \otimes_{\mathcal{O}_X} \mathscr{V}_n) \xrightarrow{\nabla} \Gamma(X, \Omega^2_X \otimes_{\mathcal{O}_X} \mathscr{V}_n) \right].$$

Since ∇ is equivariant with respect to the action of $\mathbb{G}_m$, we obtain the subcomplex

$$\Omega^\bullet(X, \mathscr{V}_n)^{\mathbb{G}_m} = \left[\Gamma(X, \mathscr{V}_n)^{\mathbb{G}_m} \xrightarrow{\nabla} \Gamma(X, \Omega^1_X \otimes_{\mathcal{O}_X} \mathscr{V}_n)^{\mathbb{G}_m} \xrightarrow{\nabla} \Gamma(X, \Omega^2_X \otimes_{\mathcal{O}_X} \mathscr{V}_n)^{\mathbb{G}_m} \right] \tag{3-1}$$

of invariant forms. Since $\mathbb{G}_m$ is connected, this also computes the de Rham cohomology of X with coefficients in $\mathscr{V}_n$. The Leray spectral sequence for the $\mathbb{G}_m$-bundle $p : X \to \mathcal{M}_{1,1}$

$$H^j(\mathcal{M}_{1,1}, R^k p_* \mathscr{V}_n) \to H^{j+k}(X, \mathscr{V}_n)$$

has only two nonvanishing rows, which implies that there is an exact sequence

$$0 \to H^j_{dR}(\mathcal{M}_{1,1}, \mathscr{V}_n) \xrightarrow{p^*} H^j_{dR}(X, \mathscr{V}_n) \to H^{j-1}_{dR}(\mathcal{M}_{1,1}, R^1 p_* \mathscr{V}_n) \to 0 \tag{3-2}$$

for all $j \geq 0$, where we recall that $\mathscr{V}_n$ denotes both the bundle $p_* \mathscr{V}_n$ on $\mathcal{M}_{1,1}$ and $\mathscr{V}_n$ on X. Since $\mathscr{V}_n$ is trivial on the $\mathbb{G}_m$ orbits on X,

$$H^{j-1}_{dR}(\mathcal{M}_{1,1}, R^1 p_* \mathscr{V}_n) \cong H^1_{dR}(\mathbb{G}_m; \mathbb{Q}) \otimes_{\mathbb{Q}} H^{j-1}_{dR}(\mathcal{M}_{1,1}, \mathscr{V}_n).$$

If $n > 0$, then $H^j(\mathcal{M}_{1,1}, \mathscr{V}_n)$ vanishes when $j \neq 1$. We deduce natural isomorphisms

$$p^* : H^1_{dR}(\mathcal{M}_{1,1}, \mathscr{V}_n) \xrightarrow{\sim} H^1_{dR}(X, \mathscr{V}_n)$$

for all $n > 0$ and

$$\pi^* : H^1_{dR}(X; \mathscr{V}_0) \xrightarrow{\sim} H^1_{dR}(\mathbb{G}_m; \mathbb{Q}).$$

By computations in Section 2D, the left-hand side is generated by $[\psi]$.

Remark 3.1. Since $\mathcal{M}_{1,1} = \boldsymbol{\mu}_{12} \backslash\!\backslash Z$, the homology of the complex $\Omega^\bullet(Z, j^* \mathscr{V}_n)^{\boldsymbol{\mu}_{12}}$ is $H^\bullet_{dR}(\mathcal{M}_{1,1}, \mathscr{V}_n)$. Below (see Proposition 3.8) we show that there is a canonical isomorphism

$$\Omega^\bullet(X, \mathscr{V}_n)^{\mathbb{G}_m} \cong \Omega^\bullet(Z, j^* \mathscr{V}_n)^{\boldsymbol{\mu}_{12}} \oplus \Omega^\bullet(Z, j^* \mathscr{V}_n)^{\boldsymbol{\mu}_{12}}[-1],$$

where the shift is given by multiplication by ψ. This gives a natural splitting of (3-2), where the shift $[-1]$ is given by cup product with $[\psi]$:

$$H^\bullet_{dR}(X, \mathscr{V}_n) \cong H^\bullet_{dR}(\mathcal{M}_{1,1}, \mathscr{V}_n) \oplus H^\bullet_{dR}(\mathcal{M}_{1,1}, \mathscr{V}_n)[-1]. \tag{3-3}$$

3A. *The subcomplex* $\Omega^\bullet(X, \mathscr{V}_n)^\omega$. Recall that we can identify $\mathrm{gr}_n \mathcal{O}(X)$ with $M_n^!$ via the isomorphism

$$\rho^* : \mathrm{gr}_n \mathcal{O}(X) = \mathrm{gr}_n \mathbb{Q}[u, v][\Delta^{-1}] \xrightarrow{\sim} M_n^!$$

since every holomorphic modular form can be uniquely written as a polynomial in the Eisenstein series u and v.

Definition 3.2. For each $n \geq 0$, define

$$\varpi : M_{n+2}^! \to \Gamma(X, \Omega_X^1 \otimes_{\mathcal{O}_X} \mathscr{V}_n)^{\mathbb{G}_m} \tag{3-4}$$

to be the $\mathbb{Q}$-linear map that takes f to $\omega_f := f \omega \, \mathsf{T}^n$, where ω and T were defined in (2-4) and Section 2F.

The first task in proving Theorem 1.2 is to show that ϖ induces a linear map

$$\varpi : M_{n+2}^! / D^{n+1} M_{-n}^! \to H_{\mathrm{dR}}^1(\mathcal{M}_{1,1}; \mathscr{V}_n). \tag{3-5}$$

Here we take the first steps in this direction. The following lemma implies that ω_f is ∇-closed.

Lemma 3.3. *For all $j, k \geq 0$ and $f \in M_{k-j+2}^!$, the one-form*

$$\eta = f \omega \, \mathsf{S}^j \mathsf{T}^k \in \Gamma(X, \omega \otimes \mathscr{V}_{j+k})$$

is $\mathbb{G}_m$-invariant and satisfies $\nabla \eta = 0$. Consequently,

$$\Gamma(X, \omega \otimes \mathscr{V}_{j+k})^{\mathbb{G}_m} \subset \ker \nabla. \tag{3-6}$$

Proof. We have $f \in \mathrm{gr}_{k-j+2} \mathcal{O}(X)$. By the Leibniz rule

$$\nabla \eta = d(f\omega) \mathsf{S}^j \mathsf{T}^k - f\omega \wedge \nabla \mathsf{S}^j \mathsf{T}^k. \tag{3-7}$$

From the connection formula (2-10), we have

$$\nabla \mathsf{S}^j \mathsf{T}^k \equiv (j - k) \psi \, \mathsf{S}^j \mathsf{T}^k \pmod{\omega \mathcal{O}(X)}$$

and therefore

$$f\omega \wedge \nabla \mathsf{S}^j \mathsf{T}^k = (k - j) f \psi \wedge \omega \mathsf{S}^j \mathsf{T}^k.$$

By the second equation of (2-5), we find that

$$d(f\omega) = (k - j + 2 - 2) f \, \psi \wedge \omega = (k - j) f \, \psi \wedge \omega.$$

Substituting the two previous expressions into (3-7) implies that $\nabla \eta = 0$. $\square$

Lemma 3.4. *The image of $\nabla : \Gamma(X, \mathscr{V}_n)^{\mathbb{G}_m} \to \Gamma(X, \Omega_X^1 \otimes \mathscr{V}_n)^{\mathbb{G}_m}$ lies in the subspace $\Gamma(X, \omega \otimes \mathscr{V}_n)^{\mathbb{G}_m}$ of forms proportional to ω.*

Proof. It suffices, by Lemma 2.8, to show that if

$$f = \sum_{j+k=n} f^{j,k} \mathsf{S}^j \mathsf{T}^k$$

is $\mathbb{G}_m$-equivariant, then $\pi^*(\nabla f) = 0$. It follows from (2-14) that

$$\pi^*(\nabla f) = \sum_{j+k=n} \left(\pi^* df^{j,k} + (j-k)\lambda^{k-j} f^{j,k} \frac{d\lambda}{\lambda} \right) \mathsf{S}^j \mathsf{T}^k.$$

Each term in brackets vanishes, since it expresses the fact that $\lambda \cdot f^{j,k} = \lambda^{k-j} f^{j,k}$, i.e., $f^{j,k} \in \mathrm{gr}_{k-j}\mathcal{O}$, which is equivalent to the $\mathbb{G}_m$-equivariance of f since S has weight $+1$ and T has weight -1. □

This lemma implies that ∇ acts on $\Gamma(X, \mathscr{V}_n)^{\mathbb{G}_m}$ via the connection

$$\nabla^\omega = \omega \vartheta + \begin{pmatrix} \mathsf{S} & \mathsf{T} \end{pmatrix} \begin{pmatrix} 0 & \omega \\ -\frac{u}{12}\omega & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial \mathsf{S}} \\ \frac{\partial}{\partial \mathsf{T}} \end{pmatrix}, \tag{3-8}$$

where ϑ is the operator (2-6).

The following is an immediate consequence of Lemmas 3.3 and 3.4.

Corollary 3.5. *For all $n \geq 0$,*

$$\Omega^\bullet(X, \mathscr{V}_n)^\omega := \left[\Gamma(X, \mathscr{V}_n)^{\mathbb{G}_m} \xrightarrow{\nabla^\omega} \Gamma(X, \omega \otimes \mathscr{V}_n)^{\mathbb{G}_m} \longrightarrow 0 \right] \tag{3-9}$$

is a subcomplex of $\Omega^\bullet(X, \mathscr{V}_n)^{\mathbb{G}_m}$.

The kernel of the restriction mapping

$$j^* : \Omega^\bullet(X, \mathscr{V}_n)^{\mathbb{G}_m} \to \Omega^\bullet(Z, j^*\mathscr{V}_n)^{\boldsymbol{\mu}_{12}}$$

contains the ideal generated by ψ. It therefore induces a homomorphism

$$j^* : \Omega^\bullet(X, \mathscr{V}_n)^\omega \to \Omega^\bullet(Z, j^*\mathscr{V}_n)^{\boldsymbol{\mu}_{12}}. \tag{3-10}$$

Lemma 3.6. *For all $n \geq 0$, the restriction map (3-10) is an isomorphism.*

Proof. Both complexes have length 2. That j^* is an isomorphism in degree 0 follows directly from Lemma 2.10. To prove the assertion in degree 1, note that elements of $\Gamma(X, \omega \otimes \mathscr{V}_n)^{\mathbb{G}_m}$ are of the form

$$\sum_{k+l=n} f^{k,l} \omega \mathsf{S}^k \mathsf{T}^l \quad \text{where } f^{k,l} \in \mathrm{gr}_{l-k+2}\mathcal{O}(X)$$

and elements of $\Gamma(Z, j^*\mathscr{V}_n)^{\boldsymbol{\mu}_{12}}$ are of the form

$$\sum_{k+l=n} g^{k,l} j^*(\omega) \mathsf{S}^k \mathsf{T}^l \quad \text{where } g^{k,l} \in \mathrm{gr}_{[l-k+2]}\mathcal{O}(Z),$$

since $j^*(\omega)$ generates $\Omega^1(Z)$, which follows from the fact that $\omega \wedge \psi \neq 0$ and therefore $j^*(\omega) \neq 0$. Injectivity follows from Lemma 2.10: if $j^*(f^{k,l})$ vanishes in $\mathrm{gr}_{[l-k+2]}\mathcal{O}(Z)$, it follows that $f^{k,l} = 0$. For the surjectivity, observe that

$$j^* \left(\sum_{k+l=n} G^{k,l} \omega \mathsf{S}^k \mathsf{T}^l \right) = \sum_{k+l=n} g^{k,l} j^*(\omega) \mathsf{S}^k \mathsf{T}^l,$$

where $G^{k,l} \in \mathrm{gr}_{l-k+2}\mathcal{O}(X)$ is the unique preimage of $g^{k,l} \in \mathrm{gr}_{[l-k+2]}\mathcal{O}(Z)$. □

Since the map $Z \to \boldsymbol{\mu}_{12} \backslash\backslash Z = \mathcal{M}_{1,1}$ is an étale map of stacks over $\mathbb{Q}$, the complex

$$\Gamma(Z, \Omega_Z^\bullet \otimes j^* \mathscr{V}_n)^{\boldsymbol{\mu}_{12}} \tag{3-11}$$

computes $H_{\mathrm{dR}}^\bullet(\mathcal{M}_{1,1}, \mathscr{V}_n)$ for all $n \geq 0$. Thus we have:

Corollary 3.7. *The complex* $\left(\Omega^\bullet(X, \mathscr{V}_n)^\omega, \nabla^\omega\right)$ *computes* $H_{\mathrm{dR}}^\bullet(\mathcal{M}_{1,1}, \mathscr{V}_n)$ *for all* $n \geq 0$.

We conclude this section by showing that the isomorphism (3-3) lifts to the level of de Rham complexes.

Proposition 3.8. *There is a canonical isomorphism*

$$\Omega^\bullet(X, \mathscr{V}_n)^{\mathbb{G}_m} \cong \Omega^\bullet(Z, j^* \mathscr{V}_n)^{\boldsymbol{\mu}_{12}} \oplus \Omega^\bullet(Z, j^* \mathscr{V}_n)^{\boldsymbol{\mu}_{12}}[-1]$$

of complexes.

Proof. The quotient of $\Omega^\bullet(X, \mathscr{V}_n)^{\mathbb{G}_m}$ by $\Omega^\bullet(X, \mathscr{V}_n)^\omega$ is $\psi \otimes \Omega^\bullet(X, \mathscr{V}_n)^\omega$:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma(X, \mathscr{V}_n)^{\mathbb{G}_m} & \xrightarrow{\nabla^\omega} & \Gamma(X, \omega \otimes \mathscr{V}_n)^{\mathbb{G}_m} & & \longrightarrow & 0 \\
& & \| & & \downarrow & & & \downarrow \\
0 & \longrightarrow & \Gamma(X, \mathscr{V}_n)^{\mathbb{G}_m} & \xrightarrow{\nabla} & \Gamma(X, \Omega_X^1 \otimes \mathscr{V}_n)^{\mathbb{G}_m} & \xrightarrow{\nabla} & \Gamma(X, \Omega_X^2 \otimes \mathscr{V}_n)^{\mathbb{G}_m} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| \\
0 & \longrightarrow & 0 & \longrightarrow & \psi \otimes \Gamma(X, \mathscr{V}_n)^{\mathbb{G}_m} & \xrightarrow{\nabla^\omega} & \psi \otimes \Gamma(X, \omega \otimes \mathscr{V}_n)^{\mathbb{G}_m} & \longrightarrow & 0
\end{array}
$$

This exact sequence of complexes is naturally split since the third row defines a subcomplex of the second. The result follows from Lemma 3.6 since $\psi \otimes \Omega^\bullet(X, \mathscr{V}_n)^\omega \cong \Omega^\bullet(X, \mathscr{V}_n)^\omega[-1]$. $\qquad \square$

4. Proof of Theorem 1.2

In this section, we work over $\Bbbk = \mathbb{Q}$.

4A. *Heads and tails.* We show that the "head" of an element f of $\Gamma(X, \omega \otimes \mathscr{V}_n)^{\mathbb{G}_m}$ is related to the "tail" of ∇f by the Bol operator.

Lemma 4.1. *For all $n \geq 0$ there exists a unique $\mathbb{Q}$-linear map*

$$\phi : M_{-n}^! \to \Gamma(X, \mathscr{V}_n)^{\mathbb{G}_m}$$

such that if we write

$$\phi = \sum_{j+k=n} \phi^{j,k} \mathsf{S}^j \mathsf{T}^k,$$

where $\phi^{j,k} \in \mathrm{Hom}(M_{-n}^!, \mathrm{gr}_{k-j} \mathcal{O}(X))$, then we have

$$\phi^{n,0}(f) = f \quad \text{and} \quad \nabla \phi(f) \in \omega \mathcal{O}(X) \mathsf{T}^n. \tag{4-1}$$

In other words, given a weakly holomorphic modular form f of weight $-n$, there is a unique section of $\mathscr{V}_n$ which coincides with f in the first component, and whose image under ∇ vanishes in all components save the last.

Proof. Suppose that $f \in M^!_{-n}$. We shall construct the components $f^{j,k} := \phi^{j,k}(f)$ inductively in k. For $k = 0$, we have $f^{n,0} = f$. The connection acts via (3-8) which we write

$$\nabla^\omega = \omega\left(\vartheta + \mathsf{S}\frac{\partial}{\partial \mathsf{T}} - \frac{u}{12}\,\mathsf{T}\frac{\partial}{\partial \mathsf{S}}\right).$$

Suppose that $f^{a,b}$ is defined for $b \le k < n$. The coefficient of $\mathsf{S}^j\mathsf{T}^k$ in $\nabla^\omega\phi(f)$ is

$$\omega\left(\vartheta(f^{j,k}) + (k+1)\,f^{j-1,k+1} - (j+1)\frac{u}{12}\,f^{j+1,k-1}\right).$$

There is a unique $f^{j-1,k+1} \in \mathrm{gr}_{k-j+2}\mathcal{O}(X)$ that makes this vanish; namely

$$f^{j-1,k+1} = \frac{1}{k+1}\left(\frac{j+1}{12}uf^{j+1,k-1} - \vartheta(f^{j,k})\right).$$

By induction, these equations determine $\phi(f)$ uniquely. □

Note that the inductive definition of ϕ involves dividing by $k+1$ for $1 \le k < n$.

Lemma 4.2 (heads and tails). *The diagram*

$$
\begin{array}{ccc}
M^!_{-n} & \xrightarrow{\ \phi\ } & \Gamma(X, \mathscr{V}_n)^{\mathbb{G}_m} \\[4pt]
{\scriptstyle \mathcal{D}^{n+1}/n!}\Big\downarrow & & \Big\downarrow{\scriptstyle \nabla} \\[4pt]
M^!_{n+2} & \xrightarrow{\ \varpi\ } & \Gamma(X, \omega \otimes \mathscr{V}_n)^{\mathbb{G}_m}
\end{array}
$$

commutes for all $n \ge 0$, where $\mathcal{D}^{n+1}$ is the Bol operator.

Proof. Let $f \in M^!_{-n}$, and let $\phi(f)$ be the unique section constructed in Lemma 4.1, whose coefficient of S^n is f. Perform the change of gauge (2-12). In this gauge, the coefficient of A^n in $\phi(f)$ is f:

$$\phi(f) = \sum_{j+k=n} F^{j,k}\mathsf{A}^j\mathsf{T}^k, \quad \text{where } F^{n,0} = f.$$

The defining property of ϕ is that $\nabla\phi(f)$ is a multiple of $\omega\mathsf{T}^n$. Let $r(f) \in \mathcal{O}(X)$ be the coefficient. That is,

$$\nabla\phi(f) = r(f)\omega\,\mathsf{T}^n.$$

This condition is preserved under the change of gauge (2-12), so that

$$\nabla\phi(f) = r(f)\omega\,\mathsf{T}^n = r(f)\frac{dq}{q}\mathsf{T}^n = 2\pi i r(f)\,dz\mathsf{T}^n.$$

By Lemma 2.11, we obtain the system of equations

$$F^{n,0} = f, \quad \mathcal{D}F^{j,k} + (k+1)F^{j-1,k+1} = 0, \quad \mathcal{D}F^{0,n} = r(f).$$

It follows that $r(f) = (-1)^n\mathcal{D}^{n+1}f/n!$. This proves the result since if f is nonzero n must be even and $(-1)^n = 1$. □

Remark 4.3. The previous two lemmas imply a relation between the Bol operator, multiplication by u, and the Serre derivative ϑ. Compare [Swinnerton-Dyer 1973, (25)].

We have therefore proved the existence of (3-5). It remains to prove that it is an isomorphism.

4B. ***Proof of injectivity of*** **(3-5).** Suppose that $\omega_g \in \Gamma(X, \omega \otimes \mathcal{V}_n)^{\mathbb{G}_m}$ is exact. Write $\omega_g = \nabla f$, where $f \in \Gamma(X, \mathcal{V}_n)^{\mathbb{G}_m}$. Consider the linear map

$$\Gamma(X, \mathcal{V}_n)^{\mathbb{G}_m} \to M^!_{-n}, \qquad \sum_{j+k=n} f^{j,k} \mathsf{S}^j \mathsf{T}^k \mapsto f^{n,0}.$$

Since ω_g has the property that all coefficients $\mathsf{S}^j \mathsf{T}^k$ except for $j = 0$, $k = n$ vanish, it follows from the uniqueness in Lemma 4.1 that $f = \phi(f^{n,0})$. By Lemma 4.2, it follows that ω_g is in the image of $D^{n+1} M^!_{-n}$.

4C. ***Proof of surjectivity*** **(3-5).** To complete the proof, we must show that the map (3-5) is surjective. By the algebraic de Rham Theorem (Section 3), every class in $H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}; \mathcal{V}_n)$ is represented by a section of the form

$$\eta = \sum_{j+k=n} f^{j,k} \omega \, \mathsf{S}^j \mathsf{T}^k,$$

where $f^{j,k} \in \mathrm{gr}_{k-j+2} \mathcal{O}(X)$. We show by induction that such a form is equivalent, modulo the image of ∇, to one in which $f^{j,k}$ vanishes for all $j > 0$. Suppose that $0 \le j \le n$ is largest such that $f^{j,k}$ is nonzero. If j is zero then there is nothing to prove, so assume $j > 0$. It follows from the connection formula (2-10) that

$$\nabla(g \, \mathsf{S}^{j-1} \mathsf{T}^{k+1}) \equiv (k+1) g \, \omega \mathsf{S}^j \mathsf{T}^k \quad \mathrm{mod} \ \mathsf{T}^{k+1}$$

for any $g \in \mathcal{O}(X)$ of weight $k - j + 2$. Therefore, by replacing η by

$$\eta - \frac{1}{(k+1)} \nabla(f^{j,k} \, \mathsf{S}^{j-1} \mathsf{T}^{k+1}),$$

we can assume that $f^{j,k}$ vanishes, since the second term lies in $\Gamma(X, \omega \otimes \mathcal{V}_n)^{\mathbb{G}_m}$ by Lemma 3.4. Proceeding in this manner, we deduce that every cohomology class in $H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}; \mathcal{V}_n)$ is represented by a form in the image of (3-4).

Remark 4.4. The above argument works in characteristic 0 or $> \max\{3, n\}$.

5. Periods of Cusp Forms

Let $f \in S_n$ and $g \in S^!_n$. Write $f = \sum_{m>0} a_m q^m$ and $g = \sum_m b_m q^m$. In [Guerzhoy 2008, Theorem 1], a Hecke-equivariant pairing is defined, up to a sign, by

$$\{f, g\} = \sum_{m \in \mathbb{Z}} \frac{a_m b_{-m}}{m^{n-1}}. \tag{5-1}$$

In this section we show it extends to all $f \in S^!_n$ and that it corresponds to the image of $\omega_f \otimes \omega_g$ under the de Rham incarnation of the cup product

$$H^1_{\mathrm{cusp}}(\mathcal{M}_{1,1}, \mathbb{V}^B_n) \otimes H^1_{\mathrm{cusp}}(\mathcal{M}_{1,1}, \mathbb{V}^B_n) \to H^2_{\mathrm{cusp}}(\mathcal{M}_{1,1}, \mathbb{Q}) \xrightarrow{\sim} \mathbb{Q}$$

induced by the natural pairing $\mathbb{V}_n^B \otimes \mathbb{V}_n^B \to \mathbb{Q}$, where $\mathbb{V}_n^B$ is the local system (2-11).

5A. *Inner products.* Let $V^{dR} = \mathbb{Q}A \oplus \mathbb{Q}T$ be the fibre of the vector bundle $\overline{\mathscr{V}}$ over the cusp. At $q = 0$,

$$S = A + 2G_2(0)T = A - \tfrac{1}{12}T$$

so V^{dR} is also spanned by S and T. Define a skew symmetric inner product

$$\langle \ , \ \rangle_{dR} : V^{dR} \otimes V^{dR} \to \mathbb{Q}$$

by declaring that $\langle T, A \rangle = 1$. This is the natural inner product on V^{dR} and corresponds to the cup product pairing on the first de Rham cohomology group of an elliptic curve (see [Hain 2013, Proposition 19.1]). It extends to a $(-1)^n$ symmetric inner product on $V_n^{dR} := \operatorname{Sym}^n V^{dR}$ by

$$\langle v_1 v_2 \cdots v_n, w_1 w_2 \cdots w_n \rangle = \sum_{\sigma \in \Sigma_n} \prod_{j=1}^{n} \langle v_j, w_{\sigma(j)} \rangle \quad v_j, w_k \in V^{dR}.$$

In particular,

$$\langle A^n, T^n \rangle = \langle S^n, T^n \rangle = (-1)^n n! \tag{5-2}$$

This inner product induces an inner product $\langle \ , \ \rangle_{dR} : \mathscr{V}_n \otimes \mathscr{V}_n \to \mathcal{O}_X$ which is flat with respect to the connection ∇.

There is also a Betti version. As in the introduction, we set

$$V^B = \mathbb{Q}\boldsymbol{a} \oplus \mathbb{Q}\boldsymbol{b}$$

and $V_n^B = \operatorname{Sym}^n V_B$. The unique skew symmetric inner product $\langle \ , \ \rangle_B$ on V_B satisfying $\langle \boldsymbol{a}, \boldsymbol{b} \rangle_B = 1$ induces a $(-1)^n$ symmetric inner product

$$\langle \ , \ \rangle : V_n^B \otimes V_n^B \to \mathbb{Q}$$

satisfying $\langle \boldsymbol{a}^n, \boldsymbol{b}^n \rangle = n!$ as above.

For each tangent vector $\vec{v} = e^\lambda \partial/\partial q$ of the origin of the q-disk there is a comparison isomorphism

$$\operatorname{comp}_{B,dR} : V^{dR} \otimes \mathbb{C} \to V^B \otimes \mathbb{C},$$

which is defined by

$$\operatorname{comp}_{B,dR}(A) = \boldsymbol{a}, \quad \operatorname{comp}_{B,dR}(T) = -2\pi i \boldsymbol{b} + \lambda \boldsymbol{a}.$$

It corresponds to the limit mixed Hodge structure on the first cohomology of the first order smoothing of the nodal cubic in the direction of $\vec{v}$. Observe that, for all λ, two inner products are related by

$$\operatorname{comp}_{B,dR}^* \langle \ , \ \rangle_{dR} = (2\pi i)^{-n} \langle \ , \ \rangle_B \tag{5-3}$$

via the comparison map $\operatorname{comp}_{B,dR} : V_n^{dR} \otimes \mathbb{C} \to V_n^B \otimes \mathbb{C}$.

5B. *Residue maps.* The pullback of $(\mathcal{V}_n, \nabla)$ to a formal neighbourhood of the origin of the q-disk $\mathbb{D}^*$ along the map

$$\rho : \mathbb{D}^* \to X(\mathbb{C})$$

defined in Section 2C is the free $\mathbb{Q}[\![q]\!]$-module with basis $\{\mathsf{A}^n, \mathsf{A}^{n-1}\mathsf{T}, \ldots, \mathsf{T}^n\}$ endowed with the connection (2-13). It can also be expressed in the frame $\{\mathsf{S}^j\mathsf{T}^k\}$ using the formal change of gauge (2-12).

The fraction field of $\mathbb{Q}[\![q]\!]$ is the ring $\mathbb{Q}(\!(q)\!) := \mathbb{Q}[\![q]\!][q^{-1}]$ of Laurent series. Set

$$\Omega^1(\mathbb{D}^*, \mathcal{V}_n) := \mathbb{Q}(\!(q)\!)dq \otimes V_n^{\mathrm{dR}}.$$

Define the local residue map $\Omega^1(\mathbb{D}^*, \mathcal{V}_n) \to \mathbb{Q}$ to be the composite

$$\mathrm{Res} : \Omega^1(\mathbb{D}^*, \mathcal{V}_n) \to V_n^{\mathrm{dR}} \to V_n^{\mathrm{dR}}/\mathsf{A}V_{n-1}^{\mathrm{dR}} \xrightarrow{\sim} \mathbb{Q} \qquad (5\text{-}4)$$

of the usual residue map with the map that sends A to 0 and T to 1. That is,

$$\mathrm{Res}\left(\sum_{j+k=n} \sum_{m \in \mathbb{Z}} a_m^{j,k} q^m \frac{dq}{q} \mathsf{A}^j \mathsf{T}^k \right) = a_0^{0,n}.$$

Observe that if $f \in \mathbb{Q}(\!(q)\!) \otimes V_n$, then $\mathrm{Res}(\nabla f) = 0$, since by Equation (2-13), an exact section satisfies $\nabla f \equiv 0 \bmod \mathsf{A}$.

The residue map

$$\mathrm{Res} : \Gamma(X, \omega \otimes \mathcal{V}_n)^{\mathbb{G}_m} \to \mathbb{Q}$$

is defined to be the composite of the restriction map

$$\rho^* : \Gamma(X, \omega \otimes \mathcal{V}_n)^{\mathbb{G}_m} \to \Omega^1(\mathbb{D}^*, \mathcal{V}_n).$$

followed by the residue map (5-4).

5C. *Cuspidal de Rham cohomology.* Define the local cuspidal de Rham complex $\Omega^\bullet(\mathbb{D}^*, \mathcal{V}_n)$ by

$$\Omega_{\mathrm{cusp}}^0(\mathbb{D}^*, \mathcal{V}_n) := \left\{ \sum_{j+k=n} a_m^{j,k} q^m \mathsf{A}^j \mathsf{T}^k : a_0^{n,0} = 0 \right\}$$

$$\Omega_{\mathrm{cusp}}^1(\mathbb{D}^*, \mathcal{V}_n) := \ker\left(\mathrm{Res} : \mathbb{Q}(\!(q)\!)\frac{dq}{q} \otimes V_n^{\mathrm{dR}} \to \mathbb{Q} \right).$$

It is closed under the differential ∇. Since $H^0(\Omega^\bullet(\mathbb{D}^*, \mathcal{V}_n))$ is spanned by A^n, we see that $H^0(\Omega_{\mathrm{cusp}}^\bullet(\mathbb{D}^*, \mathcal{V}_n))$ vanishes. The following lemma implies that this complex is acyclic. It is a local version of Lemma 4.2. Its proof is similar.

Lemma 5.1 (local heads and tails). *For all $n \geq 0$, every element of $\Omega_{\mathrm{cusp}}^1(\mathbb{D}^*, \mathcal{V}_n)$ is exact in $\Omega_{\mathrm{cusp}}^\bullet(\mathbb{D}^*, \mathcal{V}_n)$. If*

$$f = \sum_{j+k=n} f^{j,k} \mathsf{A}^j \mathsf{T}^k \in \mathbb{Q}(\!(q)\!) V_n^{\mathrm{dR}} \quad \textit{and} \quad \nabla(f) = \sum_{k \neq 0} a_k q^k \frac{dq}{q} \mathsf{T}^n,$$

then

$$(-1)^n n! \left(q \frac{d}{dq} \right)^{n+1} f^{n,0} = \sum_{k \neq 0} a_k q^k.$$

Consequently, the head $f^{n,0}$ of f is

$$f^{n,0} = \frac{(-1)^n}{n!} \sum_{k \neq 0} \frac{a_k q^k}{k^{n+1}}. \qquad \square$$

The restriction mapping $\rho : \Omega^\bullet(X, \mathscr{V}_n)^\omega \to \Omega^\bullet(\mathbb{D}^*, \mathscr{V}_n)$ commutes with ∇. Consequently,

$$\Omega^\bullet_{\mathrm{cusp}}(X, \mathscr{V}_n)^\omega := \rho^{-1}\big(\Omega^\bullet_{\mathrm{cusp}}(\mathbb{D}^*, \mathscr{V}_n) \big)$$

is a subcomplex of $\Omega^\bullet(X, \mathscr{V}_n)^\omega$.

Lemma 5.2. *The complex $\Omega^\bullet_{\mathrm{cusp}}(X, \mathscr{V}_n)^\omega$ computes the cuspidal de Rham cohomology of $\mathcal{M}_{1,1}/\mathbb{Q}$. That is, the comparison isomorphism*

$$\mathrm{comp}_{\mathrm{B,dR}} : H^\bullet_{\mathrm{dR}}(\mathcal{M}_{1,1}, \mathscr{V}_n) \otimes \mathbb{C} \xrightarrow{\sim} H^\bullet(\mathrm{SL}_2(\mathbb{Z}), V_n^{\mathrm{B}}) \otimes \mathbb{C}$$

restricts to an isomorphism

$$\mathrm{comp}_{\mathrm{B,dR}} : H^\bullet_{\mathrm{cusp,dR}}(\mathcal{M}_{1,1}, \mathscr{V}_n) \otimes \mathbb{C} \xrightarrow{\sim} H^\bullet_{\mathrm{cusp}}(\mathrm{SL}_2(\mathbb{Z}), V_n^{\mathrm{B}}) \otimes \mathbb{C}.$$

Proof. This follows directly from the exactness of the sequence

$$0 \to \Omega^\bullet_{\mathrm{cusp}}(X, \mathscr{V}_n)^\omega \to \Omega^\bullet(X, \mathscr{V}_n)^\omega \to \left[\mathbb{Q}\mathsf{A}^n \xrightarrow{0} \mathbb{Q}\frac{dq}{q}\mathsf{T}^n \right] \to 0.$$

Alternatively, it can be deduced directly from [Theorem 1.2.](#) $\qquad \square$

5D. *The residue pairing.* By [Lemma 5.1](#), the first homology of $\Omega^\bullet_{\mathrm{cusp}}(\mathbb{D}^*, \mathscr{V}_n)$ vanishes. Define the local residue pairing

$$\{ \ , \ \} : \Omega^1_{\mathrm{cusp}}(\mathbb{D}^*, \mathscr{V}_n) \otimes \Omega^1(\mathbb{D}^*, \mathscr{V}_n) \to \mathbb{Q}$$

as follows. [Lemma 5.1](#) implies that each $\xi \in \Omega^1_{\mathrm{cusp}}(\mathbb{D}^*, \mathscr{V}_n)$ can be uniquely written $\xi = \nabla F$, where $F \in \Omega^0_{\mathrm{cusp}}(\mathbb{D}^*, \mathscr{V}_n)$. We set

$$\{\xi, \eta\} := \{\nabla F, \eta\} = \mathrm{Res}_{q=0}\langle F, \eta \rangle.$$

Lemma 5.3. *This pairing is well-defined. We have*

$$\left\{ \sum_{j \neq 0} a_j q^j \frac{dq}{q}\mathsf{T}^n, \sum_{k \in \mathbb{Z}} b_k q^k \frac{dq}{q}\mathsf{T}^n \right\} = \sum_{k+l=0} \frac{a_k b_l}{k^{n+1}}.$$

It is $(-1)^{n+1}$ symmetric when restricted to $\Omega^1_{\mathrm{cusp}}(\mathbb{D}^*, \mathscr{V}_n)^{\otimes 2}$.

Proof. That it is well-defined follows from the uniqueness of F. To see that it is $(-1)^{n+1}$ symmetric on $\Omega^1_{\text{cusp}}(\mathbb{D}^*, \mathscr{V}_n)$, write $\xi = \nabla F$ and $\eta = \nabla G$, where $F, G \in \Omega^0_{\text{cusp}}(\mathbb{D}^*, \mathscr{V}_n)$. By flatness of the inner product,

$$d\langle F, G \rangle = \langle \nabla F, G \rangle - (-1)^n \langle \nabla G, F \rangle.$$

The symmetry property follows as the left-hand side has vanishing residue.

The formula is an immediate consequence of Lemma 5.1 as

$$\left\{ \nabla \left(\sum_{j+k=n} f^{j,k} \mathsf{A}^j \mathsf{T}^k \right), g \, dq \, \mathsf{T}^n \right\} = \sum_{j+k=n} \text{Res}_{q=0} \, f^{j,k} g \langle \mathsf{A}^j \mathsf{T}^k, \mathsf{T}^n \rangle$$

$$= (-1)^n n! \, \text{Res}_{q=0} \, f^{n,0} g. \qquad \square$$

By composing with the restriction maps

$$\Omega^1_{\text{cusp}}(X, \mathscr{V}_n)^\omega \to \Omega^1_{\text{cusp}}(\mathbb{D}^*, \mathscr{V}_n) \text{ and } \Omega^1(\mathcal{M}_{1,1}, \mathscr{V}_n) \to \Omega^1(\mathbb{D}^*, \mathscr{V}_n)$$

we obtain a well defined pairing

$$\{ \ , \ \} : \Omega^1_{\text{cusp}}(X, \mathscr{V}_n)^\omega \otimes \Omega^1(\mathcal{M}_{1,1}, \mathscr{V}_n) \to \mathbb{Q}. \qquad (5\text{-}5)$$

Lemma 5.4. *The pairing* (5-5) *has the property that* $\{\nabla f, \xi\} = 0$ *for all* $f \in \Omega^0(X, \mathscr{V}_n)^\omega$ *and* $\xi \in \Omega^1(X, \mathscr{V}_n)^\omega$. *Consequently, it induces a well-defined pairing*

$$\int^{\text{dR}} : H^1_{\text{cusp,dR}}(\mathcal{M}_{1,1}, \mathscr{V}_n) \otimes H^1_{\text{dR}}(\mathcal{M}_{1,1}, \mathscr{V}_n) \to \mathbb{Q}$$

such that the diagram

commutes.

Proof. The one-form $\langle f, \xi \rangle$ is an element of $\Gamma(X, \omega \otimes \mathscr{V}_0)^{\mathbb{G}_m}$, and therefore defines a class in the cohomology of the complex (3-9). We showed that the latter computes $H^1_{\text{dR}}(\mathcal{M}_{1,1}; \mathscr{V}_0) = 0$, and so $\langle f, \xi \rangle$ is exact. Its residue therefore vanishes.

Since the pairing (5-5) is $(-1)^{n+1}$ symmetric on $\Omega^1_{\text{cusp}}(\mathcal{M}_{1,1}, \mathscr{V}_n)$, and since $\nabla f \in \Omega^1_{\text{cusp}}(\mathcal{M}_{1,1}, \mathscr{V}_n)$ for all $f \in \Omega^0(X, \mathscr{V}_n)$, this implies that $\{\xi, \nabla f\} = 0$ for all $\xi \in \Omega^1_{\text{cusp}}(\mathcal{M}_{1,1}, \mathscr{V}_n)$ and that the pairing is well-defined on cohomology.

The formula for the pairing is a direct consequence of Lemma 5.3. $\qquad \square$

5E. *Relation to cup product.* Our final task is to determine how $\int^{\text{dR}}$ is related to the cup product. For this we need to discuss relative cohomology and its relation to cuspidal cohomology. Let Γ_∞ be the subgroup of $\text{SL}_2(\mathbb{Z})$ generated by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right)$. Throughout this section, we assume that $n > 0$.

We have the exact sequence

$$0 \to H^0(\Gamma_\infty; V_n^{\mathrm{B}}) \to H^1(\mathrm{SL}_2(\mathbb{Z}), \Gamma_\infty; V_n^{\mathrm{B}}) \to H^1_{\mathrm{cusp}}(\mathrm{SL}_2(\mathbb{Z}); V_n^{\mathrm{B}}) \to 0, \tag{5-6}$$

which is dual to the exact sequence

$$0 \to H^1_{\mathrm{cusp}}(\mathrm{SL}_2(\mathbb{Z}); V_n^{\mathrm{B}}) \to H^1(\mathrm{SL}_2(\mathbb{Z}); V_n^{\mathrm{B}}) \to H^1(\Gamma_\infty; V_n^{\mathrm{B}}) \to 0$$

under the cup product pairing

$$\int^{\mathrm{B}} : H^1(\mathrm{SL}_2(\mathbb{Z}), \Gamma_\infty; V_n^{\mathrm{B}}) \otimes H^1(\mathrm{SL}_2(\mathbb{Z}); V_n^{\mathrm{B}}) \to H^2(\mathrm{SL}_2(\mathbb{Z}), \Gamma_\infty; \mathbb{Q}) \cong \mathbb{Q}$$

induced by $\langle \ , \ \rangle_{\mathrm{B}}$.

Remark 5.5. It is helpful to note that $H^0(\Gamma_\infty, V_n^{\mathrm{B}}) = \mathbb{Q}\boldsymbol{a}^n$ and $H^1(\Gamma_\infty, V_n^{\mathrm{B}}) = V_n^{\mathrm{B}}/\boldsymbol{a}V_{n-1}^{\mathrm{B}} \cong \mathbb{Q}\boldsymbol{b}^n$.

The algebraic analogue of the relative cohomology group above is the de Rham cohomology

$$H^\bullet_{\mathrm{dR}}(\mathcal{M}_{1,1}, \mathbb{D}^*; \mathscr{V}_n),$$

which is defined to be the cohomology of the complex

$$\Omega^\bullet(X, \mathbb{D}^*; \mathscr{V}_n)^\omega := \mathrm{cone}\big[\Omega^\bullet(X, \mathscr{V}_n)^\omega \to \Omega^\bullet(\mathbb{D}^*, \mathscr{V}_n)\big][-1].$$

There is a comparison isomorphism

$$\mathrm{comp}_{\mathrm{B,dR}} : H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, \mathbb{D}^*; \mathscr{V}_n) \otimes \mathbb{C} \to H^1(\mathrm{SL}_2(\mathbb{Z}), \Gamma_\infty; V_n^{\mathrm{B}}) \otimes \mathbb{C}.$$

There is a short exact sequence

$$0 \to \mathbb{Q}\mathrm{A}^n \to H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, \mathbb{D}^*; \mathscr{V}_n) \to H^1_{\mathrm{cusp,dR}}(\mathcal{M}_{1,1}, \mathscr{V}_n) \to 0 \tag{5-7}$$

which maps to (5-6) after tensoring both sequences with $\mathbb{C}$.

In order to use the cup product to construct a pairing between $H^1_{\mathrm{cusp,dR}}(\mathcal{M}_{1,1}, \mathscr{V}_n)$ and $H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, \mathscr{V}_n)$, we need to choose a splitting of (5-7). Here we use the splitting

$$s : H^1_{\mathrm{cusp,dR}}(\mathcal{M}_{1,1}, \mathscr{V}_n) \to H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, \mathbb{D}^*; \mathscr{V}_n)$$

that is defined by taking the class of $\xi \in \Omega^1_{\mathrm{cusp}}(X, \mathscr{V}_n)^\omega$ to the class of (ξ, F), where F is the unique element of $\Omega^0_{\mathrm{cusp}}(\mathbb{D}^*, \mathscr{V}_n)$ whose derivative is the restriction of ξ to $\mathbb{D}^*$.

Proposition 5.6. *The diagram*

$$
\begin{array}{ccc}
H^1_{\mathrm{cusp,dR}}(\mathcal{M}_{1,1}, \mathscr{V}_n) \otimes H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, \mathscr{V}_n) \otimes \mathbb{C} & \xrightarrow{\ \int^{\mathrm{dR}}\ } & \mathbb{C}\,\boldsymbol{e}^{\mathrm{dR}} \\
\Big\downarrow{\scriptstyle \mathrm{comp}_{\mathrm{B,dR}}^{\otimes 2} \circ (s \otimes 1)} & & \Big\downarrow \\
H^1(\mathrm{SL}_2(\mathbb{Z}), \Gamma_\infty; V_n) \otimes H^1(\mathcal{M}_{1,1}^{\mathrm{an}}; \mathbb{V}_n) \otimes \mathbb{C} & \xrightarrow{\ \int^{\mathrm{B}}\ } & \mathbb{C}\,\boldsymbol{e}^{\mathrm{B}}
\end{array}
$$

commutes, where $e^{dR} = (2\pi i)^{n+1} e^B$ *and where* $\int^B$ *denotes the cup product induced by* $\langle \ , \ \rangle_B$ *evaluated on the fundamental class of* $\overline{\mathcal{M}}_{1,1}(\mathbb{C})$.

Proof. Suppose that $\xi \in \Omega^1_{cusp}(X; \mathcal{V}_n)$ is cuspidal and $\eta \in \Omega^1(X, \mathcal{V}_n)^\omega$. They represent cohomology classes. Let U be the analytic q-disk $\{q \in \mathbb{C} : |q| < e^{-2\pi}\}$ and $U' = U - \{0\}$. Since ξ is cuspidal, its restriction to U' is exact. Choose a meromorphic section $\widetilde{F} \in \Gamma(U', \mathcal{V}_n^{an})$ such that $\nabla \widetilde{F} = \xi$ on U' and the image F of $\widetilde{F}$ in $\Omega^0(\mathbb{D}^*, \mathcal{V}_n)$ is the unique element of $F \in \Omega^0_{cusp}(\mathbb{D}^*, \mathcal{V}_n)$ such that ∇F is the q-expansion of ξ.

Choose $r, R \in \mathbb{R}$ such that $0 < r < R < e^{-2\pi}$. Choose a smooth function $\rho : U \to \mathbb{R}_{\geq 0}$ which vanishes outside the annulus

$$A := \{q \in U : r \leq |q| \leq R\}$$

and is equal to 1 identically when $|q| \leq r$. Then $\widetilde{\xi} := \xi - \nabla(\rho \widetilde{F})$ extends by 0 to a smooth 1-form on the orbifold $\overline{\mathcal{M}}_{1,1}^{an}$ with values in $\mathcal{V}_n \otimes \mathbb{C}$, which vanishes in a neighbourhood of the cusp and equals ξ on $|q| > R$.

Since $\rho \widetilde{F}$ is a smooth section of $\mathcal{V}_n \otimes \mathbb{C}$ over $\mathcal{M}_{1,1}^{an}$, $\widetilde{\xi}$ is a smooth form that represents the same class in $H^1_{cusp}(\mathcal{M}_{1,1}, \mathcal{V}_n)$ as ξ. Since $\nabla(\rho \widetilde{F})$ is supported in the annulus A and since $\xi \wedge \eta = 0$, $\widetilde{\xi} \wedge \eta$ is supported in A. Using (5-3), we have

$$\int^B (\xi \otimes \eta) = \int_{\overline{\mathcal{M}}_{1,1}^{an}} \langle \widetilde{\xi}, \eta \rangle_B = (2\pi i)^n \int_{\overline{\mathcal{M}}_{1,1}^{an}} \langle \widetilde{\xi}, \eta \rangle_{dR} = -(2\pi i)^n \int_A d\langle \rho \widetilde{F}, \eta \rangle_{dR}$$

$$\overset{\text{(Stokes)}}{=} (2\pi i)^n \int_{|q|=r} \langle \widetilde{F}, \eta \rangle_{dR} = (2\pi i)^{n+1} \operatorname{Res}_{q=0} \widetilde{F} \eta$$

$$= (2\pi i)^{n+1} \{\xi, \eta\} = (2\pi i)^{n+1} \int^{dR} (\xi \otimes \eta). \qquad \square$$

Remark 5.7. The Hecke invariance of Guerzhoy's inner product (5-1) on cusp forms follows directly from this description of the inner product using the projection formula.[2]

Remark 5.8. The section s defined above cannot be Hecke invariant. If it were, the Eisenstein series G_{n+2} would be orthogonal to $S^!_{n+2}$ under the inner product $\{ \ , \ \}$. However, this is not the case, since it would contradict the discussion in Section 6: orthogonality with respect to G_{n+2} and cuspidality are two distinct linear conditions on the space of weakly holomorphic modular forms.

Consider, by way of example, the case $n = 10$. There is a unique $\mathbb{Q}$-linear combination $f_{-1} \in S^!_{12}$ of the weakly holomorphic modular forms $G_{24}\Delta^{-1}$, G_{12} and Δ such that $f_{-1} = q^{-1} + O(q^2)$. It is given explicitly by

$$f_{-1} = q^{-1} + 47709536\, q^2 + 39862705122\, q^3 + 7552626810624\, q^4 + \cdots .$$

Its class in $S^!_{12}/\mathcal{D}^{11} M^!_{-10}$ is a Hecke eigenform, with the same eigenvalues as Δ. Since the leading coefficient of Δ is q, we have

$$\{\Delta, f_{-1}\} = 1.$$

---

[2]This states that if $f : X \to Y$ is a smooth proper morphism, then $(f_* a) \cdot b = f_*(a \cdot f^* b)$, where $a \in H^\bullet(X; f^* \mathbb{V})$ and $b \in H^\bullet(Y, \mathbb{V})$.

On the other hand, from the Fourier expansion

$$G_{12} = \tfrac{691}{65520} + q + 2049q^2 + 177148q^3 + \cdots,$$

we find that a naive application of the formula (5-1) to f_{-1} and G_{12} would give $\{G_{12}, f_{-1}\} = 1$, and hence $S_{12}^!$ is not orthogonal to $G_{12}\mathbb{Q}$.

5F. *Proof of Theorem 1.7.* Suppose that f is a Hecke eigen cusp form of weight $2n$. Denote the associated 2-dimensional subspace of $H^1(\mathcal{M}_{1,1}, \mathscr{V}_{2n-2}) \otimes K_f$ by V_f. It has Betti and de Rham realisations V_f^B and V_f^{dR} related by the comparison isomorphism $\mathrm{comp}_{B,dR} : V_f^{dR} \otimes_{K_f} \mathbb{C} \to V_f^B \otimes_{K_f} \mathbb{C}$. The cup product induces a nondegenerate, skew-symmetric pairing

$$\langle\ ,\ \rangle : V_f \otimes V_f \to K_f(-2n+1).$$

It has Betti and de Rham realisations

$$\langle\ ,\ \rangle_B = \int^B \otimes\, e^B \quad \text{and} \quad \langle\ ,\ \rangle_{dR} = \int^{dR} \otimes\, e^{dR}.$$

Let α_f, β_f be a K_f-de Rham basis of V_f^{dR}. There is a K_f basis r_f^+, r_f^- of V_f^B with $\langle r_f^+, r_f^- \rangle_B = e_B$. Then

$$(\alpha_f\ \beta_f) = (r_f^+\ r_f^-) \begin{pmatrix} \eta_f^+ & \omega_f^+ \\ i\eta_f^- & i\omega_f^- \end{pmatrix} = (r_f^+\ r_f^-) P_f,$$

where $\eta_f^\pm$ and $\omega_f^\pm$ are real numbers. By Proposition 5.6,

$$\begin{aligned}
\langle \mathrm{comp}_{B,dR}(\omega_f), \mathrm{comp}_{B,dR}(\eta_f) \rangle_B &= \langle \eta_f^+ r_f^+ + i\eta_f^- r_f^-,\ \omega_f^+ r_f^+ + i\omega_f^- r_f^- \rangle_B \\
&= \det(P_f)\, e_B \\
&= (2\pi i)^{-2n+1} \det(P_f)\, e_{dR}.
\end{aligned}$$

Since $\langle \omega_f, \eta_f \rangle_{dR} \in K_f^\times\, e_{dR}$, this implies that $\det(P_f) \in (2\pi i)^{2n-1} K_f^\times$.

6. A $\mathbb{Q}$-de Rham splitting of the Hodge filtration

Let $\ell = \dim S_n$, so that

$$\dim M_n^! / \mathcal{D}^{n-1} M_{2-n}^! = \dim H_{dR}^1(\mathcal{M}_{1,1}, \mathscr{V}_{n-2}) = 2\ell + 1.$$

Let ord_∞ denote the order of vanishing at the cusp. It follows from the Riemann–Roch formula, as noted in [Duke and Jenkins 2008], that

$$\mathrm{ord}_\infty f \leq \ell$$

for all $f \in M_n^!$. Furthermore, it was shown in [Guerzhoy 2008] that for any $f \in M_n^!$, there exists a unique representative of f modulo $\mathcal{D}^{n-1} M_{2-n}^!$ such that

$$\mathrm{ord}_\infty f \geq -\ell.$$

Since the dimension of $M_n^!/\mathcal{D}^{n-1}M_{2-n}^!$ is exactly $2\ell+1$, it follows that such an f is uniquely determined by its $2\ell+1$ Fourier coefficients $(a_{-\ell}, \dots, a_\ell) \in \mathbb{Q}^{2\ell+1}$, where

$$f = \sum_{n \geq -\ell} a_n q^n,$$

and, conversely, any vector $(a_{-\ell}, \dots, a_\ell) \in \mathbb{Q}^{2\ell+1}$ uniquely determines an element in $M_n^!/D^{n-1}M_{2-n}^!$. It follows that the functions $f \in M_n^!$ of the form

$$f_m = q^m + O(q^{\ell+1})$$

for every $-\ell \leq m \leq \ell$, are a $\mathbb{Q}$-basis for $H_{\mathrm{dR}}^1(\mathcal{M}_{1,1}; \mathscr{V}_n)$, by Theorem 1.2. These functions satisfy some remarkable properties, studied in [Duke and Jenkins 2008].

This basis simultaneously gives a $\mathbb{Q}$-de Rham splitting of the Hodge filtration

$$H_{\mathrm{dR}}^1(\mathcal{M}_{1,1}, \mathscr{V}_{n-2}) = F^0 \supset F^1 = \cdots = F^{n-2} \supset F^{n-1} = M_n \supset F^n = 0$$

and of the weight filtration

$$0 = W_{n-2} \subset H_{\mathrm{cusp,dR}}^1(\mathcal{M}_{1,1}, \mathscr{V}_{n-2}) = W_{n-1} \subset W_n = \cdots = W_{2n-3} \subset W_{2n-2} = H_{\mathrm{dR}}^1(\mathcal{M}_{1,1}, \mathscr{V}_{n-2}).$$

The splitting of

$$0 \to F^{n-1}H_{\mathrm{cusp,dR}}^1(\mathcal{M}_{1,1}, \mathscr{V}_{n-2}) \to H_{\mathrm{cusp,dR}}^1(\mathcal{M}_{1,1}, \mathscr{V}_{n-2}) \to \mathrm{gr}_F^{n-1}H_{\mathrm{cusp,dR}}^1(\mathcal{M}_{1,1}, \mathscr{V}_{n-2}) \to 0$$

is given by lifting $\mathrm{gr}_F^0 H_{\mathrm{cusp,dR}}^1(\mathcal{M}_{1,1}, \mathscr{V}_{n-2})$ to the subspace of $M_n^!/\mathcal{D}^{n-1}M_{2-n}^!$ consisting of those f whose Fourier coefficients a_j vanish when $0 \leq j \leq \ell$. The splitting of the weight filtration is given by the Eisenstein series.

Appendix: Framings

The aim of this appendix is to explain the choice of the power of $2\pi i$ in the cocycle formula (1-6). Here we take $\Bbbk = \mathbb{Q}$.

Recall that $\pi : \mathcal{E} \to X$ denotes the universal elliptic curve and that $\mathbb{V}_n^{\mathrm{B}}$ denotes the n-th symmetric power of $R^1\pi_*\mathbb{Q}$. It underlies a variation of Hodge structure of weight n. To give a framing of $\mathbb{V}_n^{\mathrm{B}}$, it suffices to give a framing of $\mathbb{V}^{\mathrm{B}} := \mathbb{V}_1^{\mathrm{B}}$.

The pullback of $\mathbb{V}^{\mathrm{B}}$ to $\mathfrak{H}$ along $\rho : \mathfrak{H} \to X$ is the trivial local system whose fibre over $z \in \mathfrak{H}$ is $H^1(E_z)$, where $E_z := \mathbb{C}/(\mathbb{Z} \oplus z\mathbb{Z})$. We identify $H^1(E_z)$ with its dual $H_1(E_z) \cong \mathrm{Hom}(H_1(E_z), \mathbb{Z})$ by Poincaré duality

$$\mathrm{P} : H_1(E_z) \to H^1(E_z), \quad \mathrm{P}(c) := \langle c, \ \rangle,$$

where $\langle \ , \ \rangle$ denotes the intersection pairing. On the level of Hodge structures, Poincaré duality is an isomorphism

$$\mathrm{P} : H_1(E_z, \mathbb{Q}) \to H^1(E_z, \mathbb{Q}(1)).$$

Denote the standard basis of $H_1(E_z; \mathbb{Z})$ by $\boldsymbol{a}$, $\boldsymbol{b}$. These classes correspond to the lattice points 1 and z, respectively. Denote the dual basis of $H^1(E_z) \cong \operatorname{Hom}(H_1(E_z), \mathbb{Z})$ by $\check{\boldsymbol{a}}$, $\check{\boldsymbol{b}}$. Then $\check{\boldsymbol{b}} = \mathrm{P}(\boldsymbol{a})$ and $\check{\boldsymbol{a}} = \mathrm{P}(\boldsymbol{b})$. We identify the (Betti) components of $H_1(E)$ and $H^1(E)$ via P. With this identification

$$dw = -\boldsymbol{b} + z\boldsymbol{a},$$

where w is the coordinate in the universal covering $\mathbb{C}$ of E_z. The abelian differential dx/y on the elliptic curve corresponding to $\rho(z)$ is $2\pi i\, dw$. This is the section T of $\mathscr{V}_n$. So T corresponds to the section

$$\mathsf{T} = 2\pi i\, dw = 2\pi i (z\boldsymbol{a} - \boldsymbol{b})$$

of $\mathscr{V}^{\mathrm{an}} := \mathbb{V} \otimes \mathcal{O}(\mathfrak{H})$.

Each $f \in M^!_{n+2}$ corresponds to an element $h(u, v) \in \mathcal{O}(X)$. The corresponding 1-form ω_f is $h\mathsf{T}^n\omega$. So

$$\rho^*\omega_f = (2\pi i)^n f(z)(z\boldsymbol{a} - \boldsymbol{b})^n \frac{dq}{q} = (2\pi i)^{n+1} f(z)(z\boldsymbol{a} - \boldsymbol{b})^n\, dz.$$

Acknowledgments

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