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Semistable Chow–Hall algebras of quivers and quantized Donaldson–Thomas invariants

Hans Franzen and Markus Reineke

The semistable ChowHa of a quiver with stability is defined as an analog of the cohomological Hall algebra of Kontsevich and Soibelman via convolution in equivariant Chow groups of semistable loci in representation varieties of quivers. We prove several structural results on the semistable ChowHa, namely isomorphism of the cycle map, a tensor product decomposition, and a tautological presentation. For symmetric quivers, this leads to an identification of their quantized Donaldson–Thomas invariants with the Chow–Betti numbers of moduli spaces.

1. Introduction

The cohomological Hall algebra, or CoHa for short, of a quiver is defined in [Kontsevich and Soibelman 2011] as an analog of the Hall algebra construction of Ringel [1990] in equivariant cohomology of representation varieties. In [Kontsevich and Soibelman 2011] the CoHa serves as a tool for the study of quantized Donaldson–Thomas invariants of quivers, and in particular their integrality properties, since it admits a purely algebraic description as a shuffle algebra "with kernel" on spaces of symmetric polynomials. In [Efimov 2012], the CoHa of a symmetric quiver is shown to be a free super-commutative algebra, proving the positivity of quantized Donaldson–Thomas invariants in this case.

In another direction, the CoHa is used in [Franzen 2016; 2018] to determine the ring structure on the cohomology of noncommutative Hilbert schemes and more general framed moduli spaces of quiver representations, as defined in [Engel and Reineke 2009].

Already in [Franzen 2018] it turns out that a "local" version of the CoHa (the semistable CoHa), constructed via convolution on semistable loci of representation varieties with respect to a stability, is particularly useful, and that it is also convenient to replace equivariant cohomology by equivariant Chow groups.

In the present paper, we study this local version, called the semistable ChowHa, more systematically and demonstrate their utility both for understanding the structure of the CoHa and for the study of quantized Donaldson–Thomas invariants.

We prove the following structural properties of the semistable ChowHa:

The equivariant cycle map between the semistable ChowHa and the semistable CoHa is an isomorphism (Corollary 5.6), which can be viewed as a generalization of a result of [King and Walter 1995] on the cycle

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map for fine moduli spaces of quivers. We exhibit a tensor product decomposition (Theorem 6.2) of the CoHa into all semistable CoHa's for various slopes of the stability, categorifying the Harder–Narasimhan, or wall-crossing, formula of [Reineke 2003]. We give a "tautological" presentation of the semistable ChowHa (Theorem 8.1), in the spirit of [Franzen 2015], which generalizes the algebraic description of [Kontsevich and Soibelman 2011] of the CoHa. Quite surprisingly, such a tautological presentation remains valid for the equivariant Chow groups of stable loci in representation varieties (Theorem 9.1). From this, we conclude that the quantized Donaldson–Thomas invariants of a symmetric quiver are given by the Poincaré polynomials of the Chow groups of moduli spaces of stable quiver representations (Theorem 9.2). This shows that quantized Donaldson–Thomas invariants are of algebro-geometric origin; compare [Meinhardt and Reineke 2014] where quantized Donaldson–Thomas invariants are interpreted via intersection cohomology of moduli spaces of semistable quiver representations.

The proofs of these structural results basically only use the Harder–Narasimhan stratification of representation varieties of [Reineke 2003], properties of equivariant Chow groups, and the result of Efimov [2012]. All structural results are illustrated by examples in Section 10: We first give a complete description of the Hall algebra of a two-cycle quiver, which is the only symmetric quiver with known representation theory apart from the trivial and the one-loop quiver whose CoHa's are already described in [Kontsevich and Soibelman 2011]. Then we consider the only nontrivial (i.e., not isomorphic to the CoHa of a trivial quiver) semistable ChowHa for the Kronecker quiver — we observe that it is not super-commutative, but still has the same Poincaré–Hilbert series as a free super-commutative algebra; for this, the representation theory of the Kronecker quiver is used essentially. Then we illustrate the calculation of Chow–Betti numbers of moduli spaces of stable representations in the context of classical invariant theory, and finally hint at an algebraic derivation of the explicit formula of [Reineke 2012] for quantized Donaldson–Thomas invariants of multiple loop quivers.

The paper is organized as follows:

After reviewing basic facts on quiver representations (Section 2) and the definition of quantized Donaldson–Thomas invariants (Section 3), we define the semistable ChowHa in Section 4. In Section 5 we study the cycle map from ChowHa to CoHa, and prove it to be an isomorphism via induction over Harder–Narasimhan strata and framing techniques. The Harder–Narasimhan stratification is also used in Section 6 to derive the tensor product decomposition of the ChowHa. We recall the algebraic description of the CoHa of [Kontsevich and Soibelman 2011] and Efimov's theorem in Section 7. Again using the Harder–Narasimhan stratification, we obtain the algebraic description of the semistable ChowHa in Section 8. In Section 9, the previous results are combined to identify quantized Donaldson–Thomas invariants and Chow–Betti numbers. Finally, the examples mentioned above are developed in Section 10.

2. A reminder on quiver representations

Let Q be a quiver — i.e., a finite oriented graph — whose set of vertices and arrows we denote by Q_0 and Q_1, respectively. We will often suppress the dependency on Q in the notation. The bilinear form $\chi = \chi_Q$ on $\mathbb{Z}^{Q_0}$ defined by

$$\chi(d,e)=\sum_{i\in Q_0} d_i e_e-\sum_{\alpha:i\to j} d_i e_j=\sum_{i,j}(\delta_{i,j}-a_{i,j})d_i e_j$$

is called the Euler form of Q. Here, $a_{i,j}$ is the number of arrows from i to j in Q. We denote the antisymmetrization $\chi(d,e)-\chi(e,d)$ of the Euler form by $\langle d,e\rangle$. Let $\Gamma=\mathbb{Z}_{\geq 0}^{Q_0}$ be the monoid of dimension vectors of Q.

Let k be a field. A representation M of Q over k is a collection of finite-dimensional vector spaces M_i with $i\in Q_0$ together with linear maps $M_\alpha: M_i\to M_j$ for every arrow $\alpha:i\to j$. See [Assem et al. 2006] for more details. The tuple $\underline{\dim} M=(\dim M_i\mid i\in Q_0)\in\Gamma$ is called the dimension vector of M. For a dimension vector $d\in\Gamma$, we define $R_d(k)$ to be the vector space

$$R_d(k)=\bigoplus_{\alpha:i\to j}\mathrm{Hom}(k^{d_i},k^{d_j})$$

on which we have an action of the group $G_d(k)=\prod_{i\in Q_0}\mathrm{Gl}_{d_i}(k)$ via base change. An element of $R_d(k)$ is a representation of Q on the vector spaces $(k^{d_i})_i$. Being an affine space, $R_d(k)$ admits a $\mathbb{Z}$-model, i.e., there exists a scheme R_d whose set of k-valued points is $R_d(k)$. Likewise, there is a group scheme G_d which is a $\mathbb{Z}$-model for $G_d(k)$.

We introduce a stability condition θ of Q, that is, a linear form $\mathbb{Z}^{Q_0}\to\mathbb{Z}$. For a nonzero dimension vector d, the rational number

$$\frac{\theta(d)}{\sum_i d_i}$$

is called the θ-slope of d. For a rational number μ, let $\Gamma^{\theta,\mu}$ be the submonoid of all $d\in\Gamma$ with $d=0$ or whose θ-slope is μ. If M is a nonzero representation of Q over k, the θ-slope of M is defined as the slope of its dimension vector. A representation M of Q over k is called *θ-semistable* if no nonzero subrepresentation of M has larger θ-slope than M. It is called *θ-stable* if the θ-slope of every nonzero subrepresentation M' is strictly less than the slope of M, unless M' agrees with M. There is a Zariski-open subset $R_d^{\theta-\mathrm{sst}}$ of the scheme R_d whose set of k-valued points is the set of θ-semistable representations of Q. There is also an open subset $R_d^{\theta-\mathrm{st}}$ of $R_d^{\theta-\mathrm{sst}}$ parametrizing absolutely θ-stable representations, that means $R_d^{\theta-\mathrm{st}}(k)$ consists of those $M\in R_d(k)$ such that $M\otimes_k K$ is θ-stable for every finite extension $K\mid k$.

3. Quantum Donaldson–Thomas invariants

Fix a prime power q and let $\mathbb{F}=\mathbb{F}_q$ be the finite field with q elements. Then $R_d(\mathbb{F})$ and $G_d(\mathbb{F})$ are finite sets, and we consider the set of orbits $R_d(\mathbb{F})/G_d(\mathbb{F})$. We define the *completed Hall algebra* of Q as the vector space

$$H_q=H_q((Q))=\Big\{f\mid f:\bigsqcup_{d\in\Gamma}R_d(\mathbb{F})/G_d(\mathbb{F})\to\mathbb{Q}\Big\}$$

equipped with the following convolution-type multiplication: for two functions f and g, we define

$$(f * g)(X) = \sum_{U \subseteq X} f(U) g(X/U),$$

the sum ranging over all subrepresentations of X. Note that this sum is finite. This multiplication turns H_q into an associative algebra. We define yet another algebra. Set $\mathbb{T}_q := \mathbb{Q}(q^{1/2})[\![t_i \mid i \in \mathbb{Q}_0]\!]$ and let the multiplication be given by

$$t^d \circ t^e = (-q^{1/2})^{\langle d, e \rangle} t^{d+e}.$$

It is shown in [Reineke 2003] that the so-called integration map $\int : H_q \to \mathbb{T}_q$ defined by

$$\int f = \sum_{[X]} \frac{(-q^{1/2})^{\chi(\underline{\dim} X, \underline{\dim} X)}}{\sharp \operatorname{Aut}(X)} \cdot f(X) \cdot t^{\underline{\dim} X}$$

is a homomorphism of algebras. Define $\mathbf{1} \in H_q$ to be the function with $\mathbf{1}(X) = 1$ for all $[X]$. An easy computation shows that $A(q, t) := \int \mathbf{1}$ equals

$$A(q, t) = \sum_d (-q^{1/2})^{-\chi(d,d)} \prod_i \prod_{\nu=1}^{d_i} (1 - q^{-\nu})^{-1} t^d.$$

For a stability condition θ of Q and a rational number μ, we define $\mathbf{1}^{\theta,\mu} \in H_q$ as the sum of the characteristic functions on $R_d^{\theta-\mathrm{sst}}(\mathbb{F})/G_d(\mathbb{F})$ over all $d \in \Gamma^{\theta,\mu}$. Set $A^{\theta,\mu}(q, t) = \int \mathbf{1}^{\theta,\mu}$. Using a Harder–Narasimhan type recursion, it is shown in [Reineke 2003] that:

Theorem 3.1. $$\mathbf{1} = \overleftarrow{\prod_{\mu \in \mathbb{Q}}} \mathbf{1}^{\theta,\mu} \quad \textit{in } H_q.$$

This implies that the series A and $A^{\theta,\mu}$ relate in the same way in the twisted power series ring $\mathbb{T}_q$.

Let R be the power series ring $\mathbb{Q}(q^{1/2})[\![t_i \mid i \in Q_0]\!]$ with the usual multiplication. Let R_+ be the set of power series without constant coefficient. There exists a unique continuous bijection $\operatorname{Exp} : R_+ \to 1 + R_+$ such that $\operatorname{Exp}(f + g) = \operatorname{Exp}(f) \operatorname{Exp}(g)$ and

$$\operatorname{Exp}(q^{k/2} t^d) = \frac{1}{1 - q^{k/2} t^d}$$

for every $k \in \mathbb{Z}$ and $d \in \Gamma$. This function is called the plethystic exponential. We call the stability condition θ generic for the slope $\mu \in \mathbb{Q}$ (or μ-generic) if $\langle d, e \rangle = 0$ for all $d, e \in \Gamma^{\theta,\mu}$. Assuming that θ is μ-generic, the series $A^{\theta,\mu}$ can be displayed as a plethystic exponential.

Theorem 3.2 [Kontsevich and Soibelman 2011]. *For a μ-generic stability condition θ, there are polynomials $\tilde{\Omega}_d^\theta(q) = \sum_k \tilde{\Omega}_{d,2k}^\theta q^k$ in $\mathbb{Z}[q]$ for every nonzero dimension vector d of slope μ such that*

$$A^{\theta,\mu}(q^{-1}, t) = \operatorname{Exp}\left(\frac{1}{1-q} \sum_d (-q^{1/2})^{\chi(d,d)} \tilde{\Omega}_d^\theta(q) t^d \right).$$

Definition 3.3. If θ is a μ-generic stability condition and d is a nonzero dimension vector of slope μ then the coefficients of

$$\Omega_d^\theta(q) = \sum_{k\in\mathbb{Z}} \Omega_{d,k}^\theta q^{k/2} := q^{\chi(d,d)/2}\tilde{\Omega}_d^\theta(q) \in \mathbb{Z}[q^{\pm 1/2}]$$

are called the *quantum Donaldson–Thomas invariants* of Q with respect to θ.

When the quiver Q is symmetric, that means the Euler form of Q is a symmetric bilinear form, then every stability condition is μ-generic for every value $\mu \in \mathbb{Q}$. For example, the trivial stability condition is 0-generic and we can define the Donaldson–Thomas invariants $\Omega_{d,k}^0$ for every $d \neq 0$. Note that Theorem 3.1 implies $\Omega_{d,k}^0 = \Omega_{d,k}^\theta$ for every θ. We may therefore write $\Omega_{d,k}$ in this case.

4. The semistable ChowHa

Fix an algebraically closed field k. Abusing the notation from the first section, we will use the symbols R_d, $R_d^{\theta-\mathrm{sst}}$, $R_d^{\theta-\mathrm{st}}$, and G_d for the base extensions of the respective $\mathbb{Z}$-models to $\operatorname{Spec} k$.

Let d be a dimension vector for Q. We define $\mathscr{A}_d^{\theta-\mathrm{sst}}$ to be the G_d-equivariant Chow ring with rational coefficients of the semistable locus

$$\mathscr{A}_d^{\theta-\mathrm{sst}}(Q) = A_{G_d}^*(R_d^{\theta-\mathrm{sst}})_{\mathbb{Q}}.$$

For the definition of equivariant Chow groups and rings, see [Edidin and Graham 1998]. As we will always work with rational coefficients, we will often omit it in the notation. We define $\mathscr{A}^{\theta-\mathrm{sst},\mu}$ as the graded vector space

$$\mathscr{A}^{\theta-\mathrm{sst},\mu}(Q) = \bigoplus_{d\in\Gamma^{\theta,\mu}} \mathscr{A}_d^{\theta-\mathrm{sst}}.$$

We mimic Kontsevich and Soibelman's construction [2011] of the cohomological Hall algebra (CoHa) of a quiver with stability and trivial potential.

For two dimension vectors d and e of the same slope μ, we set $Z_{d,e}$ as the subspace of R_{d+e} of representations M which have a block upper triangular structure as indicated, i.e., for every arrow $\alpha : i \to j$, the linear map M_α sends the first d_i coordinate vectors of $k^{d_i+e_i}$ into the subspace of $k^{d_j+e_j}$ spanned by the first d_j coordinate vectors. We consider

$$R_d \times R_e \leftarrow Z_{d,e} \to R_{d+e},$$

the left-hand map sending a representation $M = \left(\begin{smallmatrix} M' & * \\ & M'' \end{smallmatrix}\right)$ to the pair (M', M''), and the right-hand map being the inclusion. These maps are called Hecke correspondences. For a short exact sequence $0 \to M' \to M \to M'' \to 0$ of representations *of the same slope*, M is θ-semistable if and only if both M'

and M'' are. We thus obtain cartesian squares

$$\begin{array}{ccccc} R_d \times R_e & \longleftarrow & Z_{d,e} & \longrightarrow & R_{d+e} \\ \cup\!| & & \cup\!| & & \cup\!| \\ R_d^{\mathrm{sst}} \times R_e^{\mathrm{sst}} & \longleftarrow & Z_{d,e} \cap R_{d+e}^{\mathrm{sst}} & \longrightarrow & R_{d+e}^{\mathrm{sst}}. \end{array}$$

The action of $G = G_{d+e}$ on R_{d+e} restricts to an action of the parabolic $P = \left(\begin{smallmatrix} G_d & * \\ & G_e \end{smallmatrix}\right)$ on $Z_{d,e}$, and the map $Z_{d,e} \to R_d \times R_e$ is compatible with the action of its Levi $L = G_d \times G_e$ on $R_d \times R_e$. With respect to these actions, the respective semistable loci are invariant. This gives rise to morphisms

$$(R_d^{\mathrm{sst}} \times R_e^{\mathrm{sst}}) \times^L G \leftarrow Z_{d,e}^{\mathrm{sst}} \times^L G \to Z_{d,e}^{\mathrm{sst}} \times^P G \to R_{d+e}^{\mathrm{sst}} \times^P G \to R_{d+e}.$$

We see that:

- $(R_d^{\mathrm{sst}} \times R_e^{\mathrm{sst}}) \times^L G \leftarrow Z_{d,e}^{\mathrm{sst}} \times^L G$ is a G-equivariant (trivial) vector bundle.
- $Z_{d,e}^{\mathrm{sst}} \times^L G \to Z_{d,e}^{\mathrm{sst}} \times^P G$ is a fibration whose fiber P/L is an affine space (as L is the Levi of P in G) and thus induces an isomorphism in G-equivariant intersection theory.
- $Z_{d,e}^{\mathrm{sst}} \times^P G \to R_{d+e}^{\mathrm{sst}} \times^P G$ is a G-equivariant regular embedding of relative dimension $s_1 = \sum_{\alpha: i \to j} d_i e_j$.
- $R_{d+e}^{\mathrm{sst}} \times^P G \to R_{d+e}$ is proper as G/P is complete, and the dimension of G/P is $s_0 = \sum_i d_i e_i$.

The above morphisms give rise to maps in equivariant intersection theory

$$A_L^n(R_d^{\mathrm{sst}} \times R_e^{\mathrm{sst}}) \xrightarrow{\sim} A_L^n(Z_{d,e}^{\mathrm{sst}}) \xleftarrow{\sim} A_P^n(Z_{d,e}^{\mathrm{sst}}) \to A_P^{n+s_1}(R_{d+e}^{\mathrm{sst}}) \to A_G^{n+s_1-s_0}(R_{d+e}^{\mathrm{sst}}).$$

Note that $s_1 - s_0$ equals $-\chi(d, e)$, the negative of the Euler form of d and e. Composing with the equivariant exterior product map $A_{G_d}^*(R_d^{\mathrm{sst}}) \otimes A_{G_e}^*(R_e^{\mathrm{sst}}) \to A_L^*(R_d^{\mathrm{sst}} \times R_e^{\mathrm{sst}})$, we obtain a linear map

$$\mathscr{A}_d^{\mathrm{sst}} \otimes \mathscr{A}_e^{\mathrm{sst}} \to \mathscr{A}_{d+e}^{\mathrm{sst}}.$$

The proof of [Kontsevich and Soibelman 2011, Theorem 1] also shows that we thus obtain an associative $\Gamma^{\theta,\mu}$-graded algebra. In analogy to Kontsevich and Soibelman's terminology, we define:

Definition 4.1. The algebra $\mathscr{A}^{\theta-\mathrm{sst},\mu}(Q)$ is called the θ-semistable *Chow–Hall algebra* (ChowHa) of slope μ of Q.

For the special case that θ is zero (i.e., $R_d^{\mathrm{sst}} = R_d$) and $\mu = 0$, we write $\mathscr{A}$ instead of $\mathscr{A}^{0-\mathrm{sst},0}$ and call it the *ChowHa* of Q.

5. ChowHa vs. CoHa

We discuss the relation between the semistable ChowHa and Kontsevich and Soibelman's semistable CoHa. Let k be the field of complex numbers. There is an equivariant analog (see [Edidin and Graham

1998, 2.8]) of the cycle map from [Fulton 1984, Chapter 19]. Concretely, there is a homomorphism of rings which doubles degrees

$$A^*_{G_d}(R_d^{\theta-\mathrm{sst}}) \to H^*_{G_d}(R_d^{\theta-\mathrm{sst}})$$

for every stability condition θ and every dimension vector d.

Theorem 5.1. *The equivariant cycle map $A^*_{G_d}(R_d^{\theta-\mathrm{sst}}) \to H^*_{G_d}(R_d^{\theta-\mathrm{sst}})$ is an isomorphism. In particular, $R_d^{\theta-\mathrm{sst}}$ has no odd-dimensional G_d-equivariant cohomology.*

This theorem generalizes a result due to King and Walter [1995, Theorem 3(c)]. They show the above assertion for an acyclic quiver, an indivisible dimension vector and a stability condition for which stability and semistability coincide. In this case, there exists a geometric PG_d-quotient $R_d^{\theta-(\mathrm{s})\mathrm{st}} \to M_d^\theta$, and the G_d-equivariant Chow and cohomology groups agree with the tensor product of the ordinary Chow and cohomology groups of the quotient with a polynomial ring $\mathbb{Q}[z] \cong A^*_{\mathbb{G}_m}(\mathrm{pt}) \cong H^*_{\mathbb{G}_m}(\mathrm{pt})$.

We need a general lemma in order to prove Theorem 5.1. Let X be a complex algebraic scheme embedded into a nonsingular variety $\overline{X}$ of complex dimension N; for the rest of this section, a *scheme* will be a complex algebraic scheme (see [Fulton 1984, B.1.1]) which admits such an embedding. The Borel–Moore homology $H_k(X)$ is isomorphic to the singular cohomology $H^{2N-k}(\overline{X}, \overline{X}-X)$. We consider the cycle map $\mathrm{cl}: A_k(X) \to H_{2k}(X)$.

Lemma 5.2. *Let X be a scheme, Y a closed subscheme of X and U the open complement*:

(1) *If both Y and U have no odd-dimensional homology then X does not have odd-dimensional homology.*

(2) *If $A_k(Y) \to H_{2k}(Y)$ and $A_k(U) \to H_{2k}(U)$ are isomorphisms and $H_{2k+1}(U) = 0$ then $A_k(X) \to H_{2k}(X)$ is an isomorphism.*

(3) *Suppose that $A_k(Y) \to H_{2k}(Y)$ and $A_k(X) \to H_{2k}(X)$ are isomorphisms and $H_{2k-1}(Y) = 0$. Then $A_k(U) \to H_{2k}(U)$ is also an isomorphism.*

Proof. The first assertion is clear by the long exact sequence in homology. To prove the second statement, we consider the diagram:

$$\begin{tikzcd}
 & A_k(Y) \arrow[r] \arrow[d] & A_k(X) \arrow[r] \arrow[d] & A_k(U) \arrow[r] \arrow[d] & 0 & \\
\cdots \arrow[r] & H_{2k+1}(U) \arrow[r] & H_{2k}(Y) \arrow[r] & H_{2k}(X) \arrow[r] & H_{2k}(U) \arrow[r] & H_{2k-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

The left and right vertical maps being isomorphisms and the left-most term in the lower row being zero by assumption, the claim follows by applying the snake lemma. The third claim follows by a diagram chase in the same diagram. We give the proof for completeness. Let $u \in H_{2k}(U)$. There exists $x \in H_{2k}(X)$ with $j^*x = u$ where $j: U \to X$ is the open embedding. We find a unique $\xi \in A_k(X)$ with $\mathrm{cl}_X\, \xi = x$, so $u = j^* \mathrm{cl}_X\, \xi = \mathrm{cl}_U\, j^*\xi$ which proves the surjectivity of cl_U. To show that cl_U is injective, let $\upsilon \in A_k(U)$ with $\mathrm{cl}_U\, \upsilon = 0$. For an inverse image $\xi \in A_k(X)$ of υ under j^*, we obtain $j^* \mathrm{cl}_X\, \xi = 0$, whence there exists $y \in H_{2k}(Y)$ such that $i_* y = \mathrm{cl}_X\, \xi$. Here $i: Y \to X$ denotes the closed immersion. Let $\eta \in A_k(Y)$

be the unique cycle with $\mathrm{cl}_Y\,\eta = y$. We get $\mathrm{cl}_X\,i_*\eta = i_*y = \mathrm{cl}_X\,\xi$ and thus $i_*\eta = \xi$ by injectivity of cl_X. This implies $\upsilon = j^*i_*\eta = 0$. □

An immediate consequence of the above lemma is the following:

Lemma 5.3. *Suppose that a scheme X has a filtration $X = X_N \supseteq \cdots \supseteq X_1 \supseteq X_0 = \varnothing$ by closed subschemes such that the cycle map for the successive complements $S_i = X_i - X_{i-1}$ is an isomorphism for all i. Then cl_X is an isomorphism and, moreover, we have noncanonical isomorphisms*

$$A_*(X) \cong \bigoplus_{i=0}^{N} A_*(S_i) \quad \text{and} \quad A^*(X) \cong \bigoplus_{i=0}^{N} A^{*-\mathrm{codim}_X S_i}(S_i)$$

by choosing sections of the surjections $A_*(X_i) \to A_*(S_i)$ for all i.

We now turn to an equivariant setup. Let G be a reductive linear algebraic group acting on a scheme X of complex dimension n. For an index i, we choose a representation V of G and an open subset $E \subseteq V$ such that a principal bundle quotient E/G exists and such that $\mathrm{codim}_V(V - E) > n - i$. Then, the group

$$A_k^G(X) = A_{i+\dim V - \dim G}(X \times^G E)$$

is independent of the choice of E and V. In the same vein, for an index j with $2\,\mathrm{codim}_V(V-E) > 2n - j$, we can define equivariant Borel–Moore homology via ordinary Borel–Moore homology, namely

$$H_j^G(X) = H_{j+2\dim V - 2\dim G}(X \times^G E)$$

(see [Edidin and Graham 1998]). If X is smooth then $H_j^G(X)$ is dual to $H^{2n-j}(X \times^G E)$ which is isomorphic to $H^{2n-j}(X \times^G EG) = H_G^{2n-j}(X)$ (where EG is the classifying space for G).

We consider the equivariant cycle map $\mathrm{cl} : A_k^G(X) \to H_{2k}^G(X)$ which is defined as the ordinary cycle map $\mathrm{cl} : A_{i+\dim V - \dim G}(X \times^G E) \to H_{2k+2\dim V - 2\dim G}(X \times^G E)$ (again independent of $E \subseteq V$). For complementary open and closed subschemes U and Y of X which are G-invariant, we choose $E \subseteq V$ such that the principal bundle quotient E/G exists and $\mathrm{codim}_V(V - E) > n - i$ (note that all the equivariant versions of the groups appearing in the diagram in the proof of Lemma 5.2 can be defined using E) and apply Lemma 5.2 to the complementary open/closed subschemes $U \times^G E$ and $Y \times^G E$. We thus obtain:

Corollary 5.4. *In the above equivariant situation, Lemmas 5.2 and 5.3 hold for equivariant Chow and Borel–Moore homology groups.*

Proof of Theorem 5.1. We prove Theorem 5.1 in two steps:

(1) Prove that the odd-dimensional equivariant cohomology of the θ-semistable locus vanishes by reducing the arbitrary case to a situation where stability and semistability agree. There, the statement is known thanks to Reineke [2003].

(2) Prove that the equivariant cycle map $A_k^{G_d}(R_d^{\theta-\mathrm{sst}}) \to H_{2k}^{G_d}(R_d^{\theta-\mathrm{sst}})$ is an isomorphism by induction over the Harder–Narasimhan strata.

Step 1: First, assume that d is a θ-coprime dimension vector. This means that there is no subdimension vector $0 \neq d' \leq d$ with the same slope as d apart from d itself. In this case, θ-semistability and θ-stability on R_d agree and there exists a smooth geometric PG_d-quotient $R_d^{\theta-(\mathrm{s})\mathrm{st}} \to M_d^\theta$. Here, $\mathrm{PG}_d = G_d/\mathbb{C}^\times$. In [Reineke 2003, Theorem 6.7] it is shown that the odd-dimensional cohomology of M_d^θ vanishes. But as by the existence of a geometric quotient

$$H^*_{G_d}(R_d^{\theta-\mathrm{sst}}) \cong H^*(M_d^\theta) \otimes H^*_{\mathbb{C}^\times}(\mathrm{pt}),$$

it follows that $R_d^{\theta-\mathrm{sst}}$ has no odd-dimensional cohomology.

Now, let d be arbitrary. We show that for a fixed — not necessarily positive — integer k, there exist a quiver $\hat{Q}$, a stability condition $\hat{\theta}$, a $\hat{\theta}$-coprime dimension vector $\hat{d}$, and an integer $s \geq 0$ (all depending on k) for which

$$H_k^{G_d}(R_d^{\theta-\mathrm{sst}}) \cong H_{k+2s}^{\mathrm{PG}_{\hat{d}}}(R_{\hat{d}}^{\hat{\theta}-(\mathrm{s})\mathrm{st}}(\hat{Q})). \tag{1}$$

For a dimension vector n of Q, we consider $\hat{R}_{d,n} = R_d \times F_n$ where $F_n = \bigoplus_i \mathrm{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{d_i})$. The space $\hat{R}_{d,n}$ is the space of representations of the framed quiver $\hat{Q}$ of dimension vector $\hat{d}$ which arise as follows (see [Engel and Reineke 2009, Definition 3.1]): we add an extra vertex ∞ to the vertexes of Q, i.e., $\hat{Q}_0 = Q_0 \sqcup \{\infty\}$ and, in addition to the arrows of Q, we have n_i arrows from ∞ heading to i for all $i \in Q_0$. The dimension vector $\hat{d}$ is defined by $\hat{d}_i = d_i$ for $i \in Q_0$ and $\hat{d}_\infty = 1$ and is indivisible. The structure group $G_{\hat{d}}$ is $\mathbb{C}^\times \times G_d$, whence we can identify $\mathrm{PG}_{\hat{d}}$ with G_d. We define $\hat{\theta}$ in the same way as in [loc. cit., Definition 3.1]. The following are equivalent for a framed representation $(M, f) \in \hat{R}_{d,n}$ (see [loc. cit., Proposition 3.3]):

- (M, f) is $\hat{\theta}$-semistable.
- (M, f) is $\hat{\theta}$-stable.
- M is θ-semistable and the (θ-)slope of every proper subrepresentation M' of M which contains the image of f is strictly less than the slope of M.

We denote the set of $\hat{\theta}$-(semi)stable points of $\hat{R}_{d,n}$ with $\hat{R}_{d,n}^\theta$. It is, by the above characterization, an open subset of $R_d^{\theta-\mathrm{sst}} \times F_n$. Let $\hat{R}_{d,n}^x$ denote the complement of $\hat{R}_{d,n}^\theta$ inside $R_d^{\theta-\mathrm{sst}} \times F_n$. As $R_d^{\theta-\mathrm{sst}} \times F_n$ is a G_d-equivariant vector bundle over $R_d^{\theta-\mathrm{sst}}$, we obtain

$$H_k^{G_d}(R_d^{\theta-\mathrm{sst}}) \cong H_{k+2d\cdot n}^{G_d}(R_d^{\theta-\mathrm{sst}} \times F_n)$$

where $d \cdot n := \sum_i d_i n_i = \dim_{\mathbb{C}} F_n$. We thus obtain a long exact sequence

$$\cdots \to H_{k+2d\cdot n}^{G_d}(\hat{R}_{d,n}^x) \to H_k^{G_d}(R_d^{\theta-\mathrm{sst}}) \to H_{k+2d\cdot n}^{G_d}(\hat{R}_{d,n}^\theta) \to H_{k-1+2d\cdot n}^{G_d}(\hat{R}_{d,n}^x) \to \cdots$$

in equivariant Borel–Moore homology. The equivariant BM homology groups $H_l^{G_d}(\hat{R}_{d,n}^x)$ vanish if l exceeds $2 \dim \hat{R}_{d,n}^x$. So in order to show that (1) is an isomorphism, it suffices to find a framing datum n such that the (complex) dimension of $\hat{R}_{d,n}^x$ is smaller than $(k-1)/2 + d \cdot n$. As shown in the proof of

[Franzen 2018, Theorem 3.2], $\hat{R}^x_{d,n}$ is the union of Harder–Narasimhan strata

$$\hat{R}^x_{d,n} = \bigsqcup \hat{R}^{\mathrm{HN}}_{(\hat{p},q),n}$$

over all proper subdimension vectors p of d which have the same slope (and $q = d - p$). The set $\hat{R}^{\mathrm{HN}}_{(\hat{p},q),n}$ is defined as follows: let $L(M, f)$ be minimal among those representations of the same slope as M which contain im f. We set $\hat{R}^{\mathrm{HN}}_{(\hat{p},q),n}$ as the set of all $(M, f) \in R^{\theta-\mathrm{sst}}_d \times F_n$ with $\underline{\dim} L(M, f) = p$. As

$$\hat{R}^{\mathrm{HN}}_{(\hat{p},q),n} \cong \left(\begin{pmatrix} R^{\mathrm{sst}}_p & * \\ & R^{\mathrm{sst}}_q \end{pmatrix} \times \begin{pmatrix} F_p \\ 0 \end{pmatrix} \right) \times^{P_{p,q}} G_d,$$

the dimension of this stratum — if nonempty — equals

$$\sum_{\alpha: i \to j} (d_i d_j - p_i q_j) + \sum_i p_i n_i - \sum_i (d_i^2 - p_i q_i) + \sum_i d_i^2 = \dim(R_d) + d \cdot n + \chi(p, q) - q \cdot n.$$

Choosing n large enough such that

$$q \cdot n > \dim(R_d) - \tfrac{1}{2}(k-1) + \chi(d - q, q)$$

for all subdimension vectors $0 \neq q \leq d$ of the same slope as d (which is possible as these are finitely many nonzero dimension vectors q), we find that the dimension of $\hat{R}^x_{d,n}$ is smaller than $(k-1)/2 + d \cdot n$, as desired.

Similar arguments were also used by Davison and Meinhardt [2016, Lemma 4.1].

Step 2: Let Q, θ and d be arbitrary. We consider the open and closed complementary subsets R^{sst}_d and R^{unst}_d. As R^{sst}_d is smooth (of dimension $n = \sum_{\alpha: i \to j} d_i d_j$), we have $A^{G_d}_i(R^{\mathrm{sst}}_d) \cong A^{n-i}_{G_d}(R^{\mathrm{sst}}_d)$ and $H^{G_d}_j(R^{\mathrm{sst}}_d) \cong H^{2n-j}_{G_d}(R^{\mathrm{sst}}_d)$. By Corollary 5.4 (concretely, the equivariant analog of part (3) of Lemma 5.2), it suffices to show that

$$\mathrm{cl} : A^{G_d}_*(R^{\mathrm{unst}}_d) \to H^{G_d}_{2*}(R^{\mathrm{unst}}_d)$$

is an isomorphism. If $R^{\mathrm{unst}}_d = \varnothing$, then the assertion is clear. So let us assume that R^{unst}_d is nonempty. The unstable locus admits a stratification into locally closed (irreducible) subsets $R^{\mathrm{HN}}_{d^*}$, the Harder–Narasimhan strata, by [Reineke 2003, Proposition 3.4]. By [loc. cit., Proposition 3.7], they can be ordered in such a way that the union of the first n strata is closed for all n, thus yielding a filtration by G_d-invariant closed subsets like in Lemma 5.3. Thus it suffices to prove that

$$A^{G_d}_*(R^{\mathrm{HN}}_{d^*}) \to H^{G_d}_*(R^{\mathrm{HN}}_{d^*})$$

is an isomorphism for all HN types $d^* = (d^1, \ldots, d^l)$ of d; this includes showing that all HN strata have even cohomology. But by the proof of [loc. cit., Proposition 3.4], we have

$$R^{\mathrm{HN}}_{d^*} \cong Z^{\mathrm{sst}}_{d^*} \times^{P_{d^*}} G_d,$$

where $Z^{\mathrm{sst}}_{d^*}$ is a (trivial) vector bundle over $R^{\mathrm{sst}}_{d^1} \times \cdots \times R^{\mathrm{sst}}_{d^l}$, and P_{d^*} is a parabolic subgroup of G_d with Levi $G_{d^1} \times \cdots \times G_{d^l}$.

In particular, $R_{d^*}^{\mathrm{HN}}$ is smooth, therefore we can again identify Borel–Moore homology with cohomology. Moreover, $A^i_{G_d}(R_{d^*}^{\mathrm{HN}}) \cong A^i_{G_{d^1}\times\cdots\times G_{d^l}}(R_{d^1}^{\mathrm{sst}} \times \cdots \times R_{d^l}^{\mathrm{sst}})$, and similarly for equivariant cohomology. This shows in particular that the equivariant odd-dimensional cohomology of $R_{d^*}^{\mathrm{HN}}$ vanishes by the first step of the proof. Now we argue by induction on the dimension vector d, where the set of dimension vectors is partially ordered by $d \le e$ if $d_i \le e_i$ for all i; with respect to this order, all d^ν's are strictly smaller than d. We thus assume that the equivariant cycle map for each $R_{d^\nu}^{\mathrm{sst}}$ is an isomorphism. Then, [Totaro 1999, Lemma 6.2], which can be generalized to equivariant Chow groups, implies that the equivariant exterior product map

$$A^*_{G_{d^1}}(R_{d^1}^{\mathrm{sst}}) \otimes \cdots \otimes A^*_{G_{d^l}}(R_{d^l}^{\mathrm{sst}}) \to A^*_{G_{d^1}\times\cdots\times G_{d^l}}(R_{d^1}^{\mathrm{sst}} \times \cdots \times R_{d^l}^{\mathrm{sst}})$$

is an isomorphism (even with integral coefficients). As the $R_{d^\nu}^{\mathrm{sst}}$'s have even cohomology, the Künneth map is an isomorphism. We are thus reduced to proving the assertion for minimal dimension vectors d, i.e., $d = 0$. But there the statement is obviously true. □

Remark 5.5. Theorem 5.1 is valid for integer coefficients.

Corollary 5.6. *The cycle map induces an isomorphism $\mathcal{A}^{\mathrm{sst},\mu} \to \mathcal{H}^{\mathrm{sst},\mu}$ of algebras.*

Proof. The multiplication both in the semistable ChowHa and in the semistable CoHa $\mathcal{H}^{\mathrm{sst},\mu}$ are constructed by means of the same Hecke correspondences. Moreover, the cycle map is compatible with push-forward and pull-back. □

6. Tensor product decomposition

We apply Corollary 5.4 to the Harder–Narasimhan filtration, like in the proof of Theorem 5.1. There exists a filtration $R_d = X_N \supseteq \cdots \supseteq X_1 \supseteq X_0 = 0$ by closed subsets such that each of the successive complements $S_i := X_i - X_{i-1}$ equals one of the Harder–Narasimhan strata $R_{d^*}^{\mathrm{HN}}$ (and vice versa). We have argued in the proof of Theorem 5.1 that the push-forward $A_*^{G_d}(X_i) \to A_*^{G_d}(R_d)$ is injective for every i. Any choice of sections $\sigma_i : A_*^{G_d}(S_i) \to A_*^{G_d}(X_i)$ of the surjections $A_*^{G_d}(X_i) \to A_*^{G_d}(S_i)$ gives rise to injections $\tilde{\sigma}_i : A_*^{G_d}(S_i) \to A_*^{G_d}(R_d)$ and yields an isomorphism

$$\bigoplus_{d^*} A_*^{G_d}(R_{d^*}^{\mathrm{HN}}) = \bigoplus_{i=1}^{N} A_*^{G_d}(S_i) \xrightarrow{\sim} A_*^{G_d}(R_d).$$

For a Harder–Narasimhan type d^* we denote the inclusion $A_*^{G_d}(R_{d^*}^{\mathrm{HN}}) \to A_*^{G_d}(R_d)$ with $\tilde{\sigma}_{d^*}$. Using the cohomological grading of the Chow groups, we get

$$\tilde{\sigma}_{d^*} : A_{G_d}^{*-\mathrm{codim}_{R_d}(R_{d^*}^{\mathrm{HN}})}(R_{d^*}^{\mathrm{HN}}) \to A^*_{G_d}(R_d).$$

The codimension of $R_{d^*}^{\mathrm{HN}}$ in R_d can easily be computed as $\chi(d^*) := \sum_{r<s} \chi(d^r, d^s)$. Recall that the Harder–Narasimhan stratum $R_{d^*}^{\mathrm{HN}}$ is isomorphic to $Z_{d^*}^{\mathrm{sst}} \times^{P_{d^*}} G_d$, where $Z_{d^*}^{\mathrm{sst}}$ is the inverse image of

$R^{\mathrm{sst}}_{d^1} \times \cdots \times R^{\mathrm{sst}}_{d^l}$ under the projection map of the P_{d^*}-equivariant vector bundle

$$Z_{d^*} = \begin{pmatrix} R_{d^1} & & * \\ & \ddots & \\ & & R_{d^l} \end{pmatrix} \to R_{d^1} \times \cdots \times R_{d^l}.$$

As the map $\pi : Z_{d^*} \times^{P_{d^*}} G_d \to R_d$ is proper its image is closed (and irreducible) in R_d and contains $R^{\mathrm{HN}}_{d^*}$ as an open subset. This implies that the image of π agrees with the Zariski closure of $R^{\mathrm{HN}}_{d^*}$. Let i be an index such that $R^{\mathrm{HN}}_{d^*} = S_i$ in the above filtration. We summarize the situation in the following diagram:

$$\begin{tikzcd} R^{\mathrm{HN}}_{d^*} \arrow[r, phantom, "\subseteq"] & \overline{R^{\mathrm{HN}}_{d^*}} \arrow[r, phantom, "\subseteq"] & X_i \arrow[r, phantom, "\subseteq"] & R_d \\ Z^{\mathrm{sst}}_{d^*} \times^{P_{d^*}} G_d \arrow[u, "\cong"] \arrow[r, phantom, "\subseteq"] & Z_{d^*} \times^{P_{d^*}} G_d \arrow[u, "\tilde{\pi}"] \arrow[urr, "\pi"] & & \end{tikzcd}$$

The square on the left-hand side is cartesian with open inclusions. The other inclusions are closed embeddings. We pass to Chow groups. This yields the following commutative diagram

$$\begin{tikzcd} A^{G_d}_*(R^{\mathrm{HN}}_{d^*}) & A^{G_d}_*(\overline{R^{\mathrm{HN}}_{d^*}}) \arrow[l] \arrow[r] & A^{G_d}_*(X_i) \arrow[r] & A^{G_d}_*(R_d) \\ A^{G_d}_*(Z^{\mathrm{sst}}_{d^*} \times^{P_{d^*}} G_d) \arrow[u, "\cong"] & A^{G_d}_*(Z_{d^*} \times^{P_{d^*}} G_d) \arrow[l] \arrow[u, "\tilde{\pi}_*"] \arrow[urr, "\pi_*"] & & \end{tikzcd}$$

Now we use the two natural isomorphisms $A^*_{G_d}(Z_{d^*} \times^{P_{d^*}} G_d) \cong A^*_{G_{d^1}}(R_{d^1}) \otimes \cdots \otimes A^*_{G_{d^l}}(R_{d^l})$ and $A^*_{G_d}(Z^{\mathrm{sst}}_{d^*} \times^{P_{d^*}} G_d) \cong A^*_{G_{d^1}}(R^{\mathrm{sst}}_{d^1}) \otimes \cdots \otimes A^*_{G_{d^l}}(R^{\mathrm{sst}}_{d^l})$. We work with the cohomological grading here to avoid having to take degree shifts into account. Let us choose a section of the pull-back of each of the open embeddings $R^{\mathrm{sst}}_{d^\nu} \subseteq R_{d^\nu}$. Let

$$\tau_{d^*} : A^*_{G_{d^1}}(R^{\mathrm{sst}}_{d^1}) \otimes \cdots \otimes A^*_{G_{d^l}}(R^{\mathrm{sst}}_{d^l}) \to A^*_{G_{d^1}}(R_{d^1}) \otimes \cdots \otimes A^*_{G_{d^l}}(R_{d^l})$$

be the resulting section. We get a section $A^*_{G_d}(Z^{\mathrm{sst}}_{d^*} \times^{P_{d^*}} G_d) \to A^*_{G_d}(Z_{d^*} \times^{P_{d^*}} G_d)$ which we, by abuse of notation, also call τ_{d^*}. The composition

$$s_{d^*} : A^*_{G_d}(R^{\mathrm{HN}}_{d^*}) \xrightarrow{\sim} A^*_{G_d}(Z^{\mathrm{sst}}_{d^*} \times^{P_{d^*}} G_d) \xrightarrow{\tau_{d^*}} A^*_{G_d}(Z_{d^*} \times^{P_{d^*}} G_d) \xrightarrow{\tilde{\pi}_*} A^*_{G_d}(\overline{R^{\mathrm{HN}}_{d^*}})$$

is then a section of the pull-back $A^*_{G_d}(\overline{R^{\mathrm{HN}}_{d^*}}) \to A^*_{G_d}(R^{\mathrm{HN}}_{d^*})$ because of the following:

Lemma 6.1. *Let G be an algebraic group and let*

$$\begin{tikzcd} X' \arrow[r, "j'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "j"] & Y \end{tikzcd}$$

be a cartesian square of G-schemes, where j is an open embedding, f is a proper morphism and f' is an isomorphism. Let $s' : A_*^G(Y') \to A_*^G(Y)$ be a section of the open pull-back $j'^* : A_*^G(X) \to A_*^G(X')$. Then the composition $s := f_* s' f_*'^{-1}$ is a section of $j^* : A_*^G(Y) \to A_*^G(Y')$.

Proof. Passing to equivariant Chow groups, we obtain a commutative square

$$\begin{tikzcd} A_*^G(X') \arrow[d, "f_*'"] & A_*^G(X) \arrow[l, "j'^*"'] \arrow[d, "f_*"] \\ A_*^G(Y') & A_*^G(Y) \arrow[l, "j^*"'] \end{tikzcd}$$

in which f_*' is an isomorphism. Let $\alpha \in A_*^G(Y')$. We compute

$$j^* s(\alpha) = j^* f_* s' f_*'^{-1}(\alpha) = f_*' j'^* s' f_*'^{-1}(\alpha) = f_*' f_*'^{-1}(\alpha) = \alpha.$$

This proves the lemma. □

Now we apply the above lemma again, this time to the cartesian diagram

$$\begin{tikzcd} R_{d^*}^{\mathrm{HN}} \arrow[r] \arrow[d, equal] & \overline{R_{d^*}^{\mathrm{HN}}} \arrow[d] \\ S_i \arrow[r] & X_i \end{tikzcd}$$

— the vertical map being the closed immersion, and the horizontal maps the open embeddings — and to $s' = s_{d^*}$. The composition

$$\sigma_i : A_*^{G_d}(S_i) = A_*^{G_d}(R_{d^*}^{\mathrm{HN}}) \xrightarrow{s_{d^*}} A_*^{G_d}(\overline{R_{d^*}^{\mathrm{HN}}}) \to A_*^{G_d}(X_i)$$

is hence a section of $A_*^{G_d}(X_i) \to A_*^{G_d}(S_i)$. If we now form the inclusions $\tilde{\sigma}_{d^*} : A_*^{G_d}(R_{d^*}^{\mathrm{HN}}) \to A_*^{G_d}(R_d)$ as described at the beginning of this section we have ensured that we obtain a commutative diagram

$$\begin{tikzcd} A_{G_{d^1}}^*(R_{d^1}^{\mathrm{sst}}) \otimes \cdots \otimes A_{G_{d^l}}^*(R_{d^l}^{\mathrm{sst}}) \arrow[r, "\cong"] \arrow[d, "\tau_{d^*}"] & A_{G_d}^*(R_d^{\mathrm{HN}}) \arrow[d, "\tilde{\sigma}_{d^*}"] \\ A_{G_{d^1}}^*(R_{d^1}) \otimes \cdots \otimes A_{G_{d^l}}^*(R_{d^l}) \arrow[r] & A_{G_d}^{*+\chi(d^*)}(R_d) \end{tikzcd}$$

in which the lower horizontal map is the ChowHa multiplication. We define the descending tensor product $\bigotimes_{\mu\in\mathbb{Q}}^{\leftarrow} \mathscr{A}^{\mathrm{sst},\mu}$ as the Γ-graded vector space

$$\bigoplus_d \bigoplus_{d^*} A_{G_{d^1}}^*(R_{d^1}^{\mathrm{sst}}) \otimes \cdots \otimes A_{G_{d^l}}^*(R_{d^l}^{\mathrm{sst}})$$

where the inner sum ranges over all Harder–Narasimhan types d^* summing to d (i.e., tuples $d^* = (d^1, \ldots, d^l)$ of dimension vectors of slopes $\mu^1 > \cdots > \mu^l$ such that $d^1 + \cdots + d^l = d$). The above considerations then prove:

Theorem 6.2. *The ChowHa multiplication induces an isomorphism*

$$\overleftarrow{\bigotimes_{\mu\in\mathbb{Q}}}\mathscr{A}^{\theta-\mathrm{sst},\mu}\xrightarrow{\sim}\mathscr{A}$$

of Γ-graded vector spaces between the descending tensor product of the θ-semistable ChowHa's over all possible slopes and the ChowHa.

Remark 6.3. The theorem is valid with integral coefficients, for an arbitrary quiver, and does not require the stability condition to be generic. Theorem 6.2 has been proved with different methods by Rimányi [2013] for the CoHa of a Dynkin quiver which is not an orientation of E_8.

7. Structure of the CoHa of a symmetric quiver

The CoHa and ChowHa of a quiver are described explicitly in [Kontsevich and Soibelman 2011]. Since we will make use of this description, we recall it here: The equivariant Chow ring $A^*_{G_d}(R_d)\cong A^*_{G_d}(\mathrm{pt})$ is isomorphic to

$$\mathbb{Q}[x_{i,r}\mid i\in Q_0,\,1\le r\le d_i]^{W_d},$$

where $W_d=\prod_i S_{d_i}$ is the Weyl group of a maximal torus of G_d. We may regard the variables $x_{i,r}$ (located in degree 1) as a basis for the character group of this torus or as the Chern roots of the G_d-linear vector bundle $R_d\times k^{d_i}\to R_d$ with G_d acting on k^{d_i} by its i-th factor.

Theorem 7.1 [Kontsevich and Soibelman 2011, Theorem 2]. *For $f\in\mathscr{A}_d$ and $g\in\mathscr{A}_e$, the product $f*g$ equals the function*

$$\sum f(x_{i,\sigma_i(r)}\mid i,1\le r\le d_i)\cdot g(x_{i,\sigma_i(d_i+s)}\mid i,1\le s\le e_i)\cdot\prod_{i,j\in Q_0}\prod_{r=1}^{d_i}\prod_{s=1}^{e_j}(x_{j,\sigma_j(d_j+s)}-x_{i,\sigma_i(r)})^{a_{i,j}-\delta_{i,j}}.$$

The sum ranges over all (d,e)-shuffles $\sigma=(\sigma_i\mid i)\in W_{d+e}$, that means each σ_i is a (d_i,e_i)-shuffle permutation.

We assume that the stability condition θ is μ-generic. In this case, we can equip the semistable ChowHa of slope μ with a refined grading: setting

$$\mathscr{A}^{\mathrm{sst}}_{(d,n)}=\begin{cases}A^{(n-\chi(d,d))/2}_{G_d}(R^{\mathrm{sst}}_d) & n\equiv\chi(d,d)\pmod 2,\\ 0 & n\not\equiv\chi(d,d)\pmod 2,\end{cases}$$

it is easy to see that the multiplication map becomes bigraded, thus $\mathscr{A}^{\mathrm{sst}}_{(d,n)}\otimes\mathscr{A}^{\mathrm{sst}}_{(e,m)}\to\mathscr{A}^{\mathrm{sst}}_{(d+e,n+m)}$.

Like in Section 3, we consider again the case of a symmetric quiver and the trivial stability condition. In this situation, it is immediate from the formula in the above theorem that $f*g=(-1)^{\chi(d,e)}g*f$ for $f\in\mathscr{A}_d$ and $g\in\mathscr{A}_e$. One can show (see [Kontsevich and Soibelman 2011, §2.6]) that there exists a bilinear form ψ on the $\mathbb{Z}/2\mathbb{Z}$-vector space $(\mathbb{Z}/2\mathbb{Z})^{Q_0}$ such that $f\star g=(-1)^{\psi(d,e)}f*g$ is a super-commutative multiplication, when defining the parity of an element of bidegree (d, n) to be the parity of n. We see that the generating series $P(q,t) = \sum_d \sum_k (-1)^k \dim \mathcal{A}_{(d,k)} q^{k/2} t^d$ is

$$\sum_d (-q^{1/2})^{\chi(d,d)} \prod_i \prod_{\nu=1}^{d_i} (1-q^\nu)^{-1} t^d.$$

So, $P(q,t) = A(q^{-1}, t)$. By Theorem 3.2, the generating series has a product expansion

$$P(q,t) = \prod_d \prod_k \prod_{n \geq 0} (1 - q^{n+k/2} t^d)^{(-1)^{k-1} \Omega_{d,k}}.$$

As a free super-commutative algebra with a generator in bidegree (d,k) has the generating series $(1-q^{k/2}t^d)^{(-1)^{k-1}} = \operatorname{Exp}((-1)^k q^{k/2} t^d)$, Kontsevich and Soibelman [2011] made a conjecture which was eventually proved by Efimov.

Theorem 7.2 [Efimov 2012, Theorem 1.1]. *For a symmetric quiver Q, the algebra $\mathcal{A}(Q)$, equipped with the super-commutative multiplication $\star$, is isomorphic to a free super-commutative algebra over a $(\Gamma \times \mathbb{Z})$-graded vector space $V = V^{\text{prim}} \otimes \mathbb{Q}[z]$, where z lives in bidegree $(0,2)$, and $\bigoplus_k V^{\text{prim}}_{d,k}$ is finite-dimensional for every d.*

This result implies that the Donaldson–Thomas invariants $\Omega_{d,k}$ must agree with the dimension of $V^{\text{prim}}_{(d,k)}$ and must therefore be nonnegative. We will give another characterization of the primitive part of the CoHa in Theorem 9.2.

8. Tautological presentation of the semistable ChowHa

We investigate the relation between the semistable ChowHa $\mathcal{A}^{\theta-\text{sst},\mu}$ and the ChowHa $\mathcal{A}$ of a quiver Q. For a dimension vector d of slope μ, we consider the open embedding

$$R_d^{\text{sst}} \to R_d$$

which gives rise to a surjective map $A^*_{G_d}(R_d) \to A^*_{G_d}(R_d^{\text{sst}})$. As the Hecke correspondences for the semistable ChowHa are given by restricting the Hecke correspondences of $\mathcal{A}$ to the semistable loci, these open pull-backs are compatible with the multiplication, i.e., they induce a surjective homomorphism of Γ-graded algebras

$$\mathcal{A} \to \mathcal{A}^{\theta-\text{sst},\mu}.$$

Here, we regard $\mathcal{A}^{\theta-\text{sst},\mu}$ as a Γ-graded algebra by extending it trivially to every dimension vector whose slope is not s. We can describe the kernel explicitly.

Theorem 8.1. *The kernel of the natural map $\mathcal{A}_d \to \mathcal{A}_d^{\theta-\text{sst}}$ equals the sum*

$$\sum \mathcal{A}_p * \mathcal{A}_q$$

over all pairs (p,q) of dimension vectors of Q which sum to d and such that $\mu(p) > \mu(q)$.

The key ingredient of the proof of this result is a purely intersection-theoretic lemma. Following [Fulton 1984, B.1.1], we call a k-scheme algebraic if it is separated and of finite type over $\operatorname{Spec} k$. Thus, a variety is an algebraic scheme which is integral.

Lemma 8.2. *Let $f : X \to Y$ be a surjective, proper morphism of algebraic k-schemes. Then the push-forward $f_* : A_*(X)_{\mathbb{Q}} \to A_*(Y)_{\mathbb{Q}}$ is surjective.*

Proof. It is obviously sufficient to prove that, for every dominant morphism $f : X \to Y$ of an algebraic scheme X to a variety Y, there exists a subvariety W of X of dimension $\dim W = \dim Y$ which dominates Y. This is a local statement, so we may assume X and Y to be affine, say $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. The morphism f corresponds to an extension $A \hookrightarrow B$ of rings. We therefore need to show that there exists a prime ideal $\mathfrak{q}$ of B with $\mathfrak{q} \cap A = (0)$ such that the induced extension

$$Q(B/\mathfrak{q}) \mid Q(A)$$

is finite. Let $K = Q(A)$ and $R = B \otimes_A K$. By Noether normalization, there exist $b_1, \dots, b_n \in R$, algebraically independent over K, such that $K[b_1, \dots, b_n] \subseteq R$ is a finite (and hence integral) ring-extension. Without loss of generality, we may assume $b_1, \dots, b_n \in B$. Choose a set of generators $c_1, \dots, c_s$ of R as a $K[b_1, \dots, b_n]$-algebra and polynomials $p_i(T) \in K[b_1, \dots, b_n][T]$ such that $p_i(c_i) = 0$. We find an element $s \in A - \{0\}$ such that the coefficients of all the p_i's lie in $A_s[b_1, \dots, b_n]$ and $p_i(c_i) = 0$ holds in B_s. This implies that B_s is an integral $A_s[b_1, \dots, b_n]$-algebra which yields the surjectivity of the map $\operatorname{Spec} B_s \to \operatorname{Spec} A_s[b_1, \dots, b_n]$. We consider the prime ideal $\mathfrak{p}' = (b_1, \dots, b_n)$ of $A_s[b_1, \dots, b_n]$ and find a prime ideal $\mathfrak{q}'$ of B_s which lies above it. Then $B_s/\mathfrak{q}'$ is an integral extension of $A_s[b_1, \dots, b_n]/\mathfrak{p}' = A_s$ and therefore, setting $\mathfrak{q} = \mathfrak{q}' \cap B$, the extension

$$Q(B/\mathfrak{q}) = Q(B_s/\mathfrak{q}') \mid Q(A_s) = Q(A)$$

is finite. $\square$

Proof of Theorem 8.1. Let R_d^{unst} be the complement of R_d^{sst} in R_d. Then, we have an exact sequence

$$A_m^G(R_d^{\text{unst}}) \to A_m^G(R_d) \to A_m^G(R_d^{\text{sst}}) \to 0,$$

where $G = G_d$. For a decomposition $d = p + q$, let $R_{p,q}$ be the closed subset of R_d of all representations which possess a subrepresentation of dimension vector p. It is the G-saturation of $Z_{p,q}$. The G-action gives a surjective, proper morphism

$$Z_{p,q} \times^{P_{p,q}} G \to R_{p,q},$$

where $P_{p,q}$ is the parabolic $\begin{pmatrix} G_p & * \\ & G_q \end{pmatrix}$. The unstable locus R_d^{unst} equals the union $\bigcup R_{p,q}$ over all decompositions $d = p + q$ where the slope of p is larger than the slope of q. Let us call these decompositions θ-forbidden. We obtain, using Lemma 8.2 and [Fulton 1984, Example 1.3.1(c)], that the sequence

$$\bigoplus A_m^G(Z_{p,q} \times^{P_{p,q}} G) \to A_m^G(R_d) \to A_m^G(R_d^{\text{sst}}) \to 0$$

is exact when passing to rational coefficients — the direct sum being taken over all forbidden decompositions $d = p+q$. Setting $n = \dim R_d - m$, we identify

$$A_m^G(Z_{p,q} \times^{P_{p,q}} G) \cong A_{P_{p,q}}^{n+\chi(p,q)}(Z_{p,q}) \cong A_{G_p \times G_q}^{n+\chi(p,q)}(R_p \times R_q)$$

like in Section 4. As the equivariant product map $A_{G_p}^*(R_p) \otimes A_{G_q}^*(R_q) \to A_{G_p \times G_q}^*(R_p \times R_q)$ is an isomorphism (which is clear from the explicit description given above), we have shown that

$$\bigoplus_{\substack{p+q=d \\ \text{forbidden}}} \bigoplus_{k+l=n+\chi(p,q)} A_{G_p}^k(R_p)_{\mathbb{Q}} \otimes A_{G_q}^l(R_q)_{\mathbb{Q}} \to A_{G_d}^n(R_d)_{\mathbb{Q}} \to A_{G_d}^n(R_d^{\mathrm{sst}})_{\mathbb{Q}} \to 0$$

is an exact sequence. The first map in this sequence is precisely the ChowHa-multiplication. This proves the theorem. □

9. The primitive part of the semistable ChowHa

As a next step, we analyze the kernel of the pull-back $A_G^*(R_d^{\mathrm{sst}}) \to A_G^*(R_d^{\mathrm{st}})$ induced by the open embedding of the stable locus into the semistable locus. A semistable representation $M \in R_d$ is not stable if and only if there exists a proper subrepresentation of the same slope. For a decomposition $d = p+q$ into subdimension vectors of the same slope, we define $R_{p,q}^{\mathrm{sst}}$ as the subset of those $M \in R_d^{\mathrm{sst}}$ which admit a subrepresentation of dimension vector p. Therefore, the set of properly semistable representations is the union

$$R_d^{\mathrm{sst}} - R_d^{\mathrm{st}} = \bigcup R_{p,q}^{\mathrm{sst}}$$

over all decompositions $d = p+q$ such that p and q have the same slope and are both nonzero. We have, yet again, a surjective, proper morphism

$$\begin{pmatrix} R_p^{\mathrm{sst}} & * \\ & R_q^{\mathrm{sst}} \end{pmatrix} \to R_{p,q}^{\mathrm{sst}}.$$

A result of Totaro [1999, Lemma 6.1], which can easily be transferred to equivariant Chow rings, shows that the exterior product $A_{G_p}^*(R_p^{\mathrm{sst}}) \otimes A_{G_q}^*(R_q^{\mathrm{sst}}) \to A_{G_p \times G_q}^*(R_p^{\mathrm{sst}} \times R_q^{\mathrm{sst}})$ is an isomorphism. Following the arguments of the proof of Theorem 8.1, we obtain:

Theorem 9.1. *The kernel of the surjection $\mathscr{A}_d^{\theta-\mathrm{sst}} \to \mathscr{A}_d^{\theta-\mathrm{st}}$ is the sum*

$$\sum \mathscr{A}_p^{\theta-\mathrm{sst}} * \mathscr{A}_q^{\theta-\mathrm{sst}}$$

over all decompositions $d = p+q$ into nonzero subdimension vectors of the same θ-slope.

In other words, the graded vector space $\mathscr{A}^{\theta-\mathrm{st},\mu} = \bigoplus_{d \in \Gamma^{\theta,\mu}} \mathscr{A}_d^{\theta-\mathrm{st}}$ equipped with the trivial multiplication (by which we mean that the product of two homogeneous elements of positive degree is set to be zero) is isomorphic to the quotient $\mathscr{A}^{\theta-\mathrm{sst},\mu}/(\mathscr{A}_+^{\theta-\mathrm{sst},\mu} * \mathscr{A}_+^{\theta-\mathrm{sst},\mu})$ of the semistable ChowHa modulo the square of its augmentation ideal $\mathscr{A}_+^{\theta-\mathrm{sst}}$.

Again, we consider the case of a symmetric quiver Q. We have deduced from Theorem 8.1 that $\mathscr{A}^{\theta-\mathrm{sst},\mu}$ is free super-commutative over $V^{\theta,\mu}=\bigoplus_{d\in\Gamma^{\theta,\mu}}V_d$. The quotient of the augmentation ideal of a free super-commutative algebra by its square is isomorphic to the primitive part of the algebra, i.e., in our case

$$V_d\cong\mathscr{A}_d^{\theta-\mathrm{st}}=A^*_{G_d}(R_d^{\theta-\mathrm{st}})\cong A^*_{\mathrm{PG}_d}(R_d^{\theta-\mathrm{st}})\otimes A^*_{\mathbb{G}_m}(\mathrm{pt})$$

for every $d\neq 0$. As $V_d=V_d^{\mathrm{prim}}\otimes\mathbb{Q}[z]$, we deduce that

$$V_{d,k}^{\mathrm{prim}}=\begin{cases}A_{\mathrm{PG}_d}^{(k-\chi(d,d))/2}(R_d^{\mathrm{st}}) & k\equiv\chi(d,d)\pmod 2,\\ 0 & k\not\equiv\chi(d,d)\pmod 2.\end{cases}$$

Assuming that $R_d^{\theta-\mathrm{st}}$ is nonempty and denoting by $M_d^{\theta-\mathrm{st}}$ the geometric quotient $R_d^{\theta-\mathrm{st}}/\,\mathrm{PG}_d$ (which we call the stable moduli space), we get

$$A^j_{\mathrm{PG}_d}(R_d^{\theta-\mathrm{st}})=A^{\mathrm{PG}_d}_{\dim R_d-j}(R_d^{\theta-\mathrm{st}})=A_{\dim R_d-\dim\mathrm{PG}_d-j}(M_d^{\theta-\mathrm{st}})$$

and $\dim M_d^{\theta-\mathrm{st}}=\dim R_d-\dim\mathrm{PG}_d=1-\chi(d,d)$. This yields that the Donaldson–Thomas invariants of Q are given by the Chow–Betti numbers of the stable moduli spaces, more precisely:

Theorem 9.2. *For a symmetric quiver Q, a stability condition θ and a dimension vector $d\neq 0$, the Donaldson–Thomas invariant $\Omega_{d,k}$ equals*

$$\Omega_{d,k}=\begin{cases}\dim A_{1-(k+\chi(d,d))/2}(M_d^{\mathrm{st}}) & \text{if } k\equiv\chi(d,d)\pmod 2 \text{ and } M_d^{\mathrm{st}}\neq\varnothing,\\ 0 & \text{otherwise.}\end{cases}$$

In particular, $\Omega_{d,k}$ can only be nonzero if $\chi(d,d)\le k\le 2-\chi(d,d)$.

Remark 9.3. The range for the nonvanishing of the Donaldson–Thomas invariants from the above theorem yields that the number $N_d(Q)$ in [Efimov 2012, Corollary 4.1] can be chosen as $1-\chi(d,d)$, i.e., the dimension of $M_d^{\theta-\mathrm{st}}$.

10. Examples

10.1. ***The two-cycle quiver.*** We start by illustrating the tensor product decomposition from Theorem 6.2. There are exactly three connected symmetric quivers which are not wild, that is, for which a classification of their finite-dimensional representations up to isomorphism is known. Namely, these are:

- The quiver L_0 of Dynkin type A_1 with a single vertex and no arrows.
- The quiver L_1 of extended Dynkin type $\tilde{A}_0$ consisting of a single vertex and a single loop.
- The quiver Q of extended Dynkin type $\tilde{A}_1$ with two vertices i and j and single arrows $i\to j$ and $j\to i$, respectively.

For the quivers L_0 and L_1, the structure of the CoHa is determined in [Kontsevich and Soibelman 2011]. Namely, we have

$$\mathscr{A}(L_0)\cong S^*(\mathbb{Q}(1,1)[z])\quad\text{and}\quad\mathscr{A}(L_1)\cong S^*(\mathbb{Q}(1,0)[z]),$$

where S^* denotes the free super-commutative algebra, $\mathbb{Q}(d, i)$ denotes a one-dimensional $\mathbb{Q}$-space placed in bidegree (d, i), and z denotes the element in bidegree $(0, 2)$ as in Theorem 7.2.

The structure of the CoHa of Q is described in [Franzen 2018, Corollary 2.5]; here we give a simplified derivation of this result using the present methods. We consider the stability θ given by $\theta(d_i, d_j) = d_i$ (note that any nontrivial stability is equivalent to θ or $-\theta$ in the sense that the class of (semi)stable representation is the same). Let a representation M of Q of dimension vector d be given by vector spaces V_i and V_j and linear maps $f : V_i \to V_j$ and $g : V_j \to V_i$. We claim that this representation is θ-semistable if and only if $V_i = 0$, or $V_j = 0$, or $\dim V_i = \dim V_j$ and f is an isomorphism; moreover, it is θ-stable if it is θ-semistable and $\dim V_i, \dim V_j \leq 1$.

The case $(\dim V_i)\cdot(\dim V_j) = 0$ being trivial, we assume $\dim V_i, \dim V_j \geq 1$. Suppose M is θ-semistable. If f is not injective, we choose a vector $0 \neq v \in V_i$ in the kernel of f, yielding a subrepresentation U of dimension vector $(1, 0)$. Then we find $1 = \mu(U) \leq \mu(M) = \dim V_i/(\dim V_i + \dim V_j)$, thus $\dim V_j = 0$, a contradiction. Thus f is injective, and $(V_i, f(V_i))$ defines a subrepresentation U' of dimension vector $(\dim V_i, \dim V_i)$ of M. Then we find $\frac{1}{2} = \mu(U') \leq \mu(M)$, thus $\dim V_j \leq \dim V_i$, which already implies $\dim V_i = \dim V_j$ and shows that f is an isomorphism. Conversely every representation M consisting of vector spaces V_i and V_j of the same dimension, an isomorphism $f : V_i \to V_j$, and an arbitrary linear map $g : V_j \to V_i$ is θ-semistable: the subrepresentations of M are of the form $U = (U_i, U_j)$ for some subspaces satisfying $f(U_i) \subseteq U_j$ and $g(U_j) \subseteq U_i$. Injectivity of f implies $\dim U_i \leq \dim U_j$ and thus $\mu(U) \leq \frac{1}{2} = \mu(M)$. To show that stability forces $\dim V_i = 1 = \dim V_j$ we argue as follows: as f is an isomorphism, we may assume without loss of generality that $V_1 = V_2 = V$ and $f = \mathrm{id}_V$. Let $v \in V$ be an eigenvector of g to some eigenvalue λ. The subspaces $U_1 = U_2 = \langle v \rangle$ then provide a subrepresentation of M of dimension vector $(1, 1)$.

This analysis provides identifications

$$A^*_{G_d}(R_d^{\theta-\mathrm{sst}}) \cong A^*_{\mathrm{Gl}_n(k)}(\mathrm{pt}) \quad \text{for } d = (n, 0) \text{ or } d = (0, n),$$
$$A^*_{G_d}(R_d^{\theta-\mathrm{sst}}) \cong A^*_{\mathrm{Gl}_n(k)}(M_{n\times n}(k)) \quad \text{for } d = (n, n),$$

which we recognize as the homogeneous parts of the CoHa of L_0 and L_1, respectively. These identifications obviously being compatible with the respective Hecke correspondences defining the multiplications, we see that

$$\mathcal{A}^{\theta-\mathrm{sst},1}(Q) \cong \mathcal{A}(L_0) \cong \mathcal{A}^{\theta-\mathrm{sst},0}(Q) \quad \text{and} \quad \mathcal{A}^{\theta-\mathrm{sst},1/2}(Q) \cong \mathcal{A}(L_1).$$

By Theorem 6.2, we thus arrive at an isomorphism of graded vector spaces

$$\mathcal{A}(Q) \cong S^*\big((\mathbb{Q}((1,0),1) \oplus \mathbb{Q}((0,1),1) \oplus \mathbb{Q}((1,1),0))[z]\big).$$

10.2. ***The Kronecker quiver.*** Now we consider the Kronecker quiver K_2 with two vertices i and j and two arrows from i to j. As we will use results from Section 5, we work over the field of complex numbers. Again we consider the stability $\theta(d_i, d_j) = d_i$. This is again a case where the representation theory of the quiver is known: up to isomorphism, there exist unique (θ-stable) indecomposable representations

P_n and I_n for each of the dimension vectors $(n, n+1)$ and $(n+1, n)$, respectively, for $n \geq 0$, and there exist one-parametric families $R_n(\lambda)$ of (θ-semistable) indecomposables for each of the dimension vectors (n, n) for $n \geq 0$ and $\lambda \in \mathbb{P}^1(\mathbb{C})$. Arguing as in the first example, we can conclude that

$$\mathscr{A}^{\theta-\mathrm{sst},\mu(d)}(K_2) \cong S^*(\mathbb{Q}(d,1)[z]) \quad \text{for } d = (n, n+1) \text{ or } d = (n+1, n),$$

and

$$\mathscr{A}^{\theta-\mathrm{sst},\mu}(K_2) = 0 \quad \text{if } \mu \notin \left\{0, \tfrac{1}{3}, \tfrac{2}{5}, \tfrac{3}{7}, \ldots, \tfrac{1}{2}, \ldots \tfrac{4}{7}, \tfrac{3}{5}, \tfrac{2}{3}, 1\right\}.$$

It remains to consider $\mathscr{A}^{\theta-\mathrm{sst},1/2}(K_2)$.

We construct a stratification of the θ-semistable locus in $R_{(n,n)}(K_2) \cong M_{n\times n}(\mathbb{C}) \times M_{n\times n}(\mathbb{C})$, on which $G = \mathrm{Gl}_n(\mathbb{C}) \times \mathrm{Gl}_n(\mathbb{C})$ acts via $(g, h) \cdot (A, B) = (hAg^{-1}, hBg^{-1})$. For $0 \leq r \leq n$, we define S_r as the G-saturation of the set of pairs of matrices

$$\left(\begin{pmatrix} E_r & 0 \\ 0 & N \end{pmatrix}, \begin{pmatrix} A & 0 \\ 0 & E_{n-r} \end{pmatrix}\right),$$

where E_i denotes an $i \times i$-identity matrix, A denotes an arbitrary $r \times r$-matrix, and N denotes a nilpotent $(n-r) \times (n-r)$-matrix. We claim that every S_r is locally closed, their union equals the θ-semistable locus, and the closure of S_r equals the union of the $S_{r'}$ for $r' \leq r$.

The representation $R_n(\lambda)$ is given explicitly by the matrices $(E_n, \lambda E_n + J_n)$ for $\lambda \neq \infty$, and by (J_n, E_n) for $\lambda = \infty$, where J_n is the nilpotent $n \times n$-Jordan block. As noted above, a θ-semistable representation of M of dimension vector (n, n) is of the form

$$M = R_{n_1}(\lambda_1) \oplus \cdots \oplus R_{n_k}(\lambda_k)$$

for $n = n_1 + \cdots + n_k$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{P}^1(\mathbb{C})$, uniquely defined up to reordering. Now we reorder the direct sum and assume that $\lambda_1, \ldots, \lambda_j \neq \infty$ and $\lambda_{j+1} = \cdots = \lambda_k = \infty$. Using the above explicit form of the representations $R_n(\lambda)$, we see that M is represented by a pair of block matrices of the form

$$\left(\begin{pmatrix} E_r & 0 \\ 0 & N \end{pmatrix}, \begin{pmatrix} A & 0 \\ 0 & E_{n-r} \end{pmatrix}\right)$$

with N nilpotent and A arbitrary. All claimed properties of the stratification follow.

Now we claim that

$$S_r \cong (\mathrm{Gl}_n(\mathbb{C}) \times \mathrm{Gl}_n(\mathbb{C})) \times^{\mathrm{Gl}_r(\mathbb{C}) \times \mathrm{Gl}_{n-r}(\mathbb{C})} (M_r(\mathbb{C}) \times N_{n-r}(\mathbb{C})),$$

where the group $\mathrm{Gl}_r(\mathbb{C}) \times \mathrm{Gl}_{n-r}(\mathbb{C})$ is considered as a subgroup of $\mathrm{Gl}_n(\mathbb{C}) \times \mathrm{Gl}_n(\mathbb{C})$ by mapping a pair (g_1, h_4) to $\left(\left(\begin{smallmatrix} g_1 & 0 \\ 0 & h_4 \end{smallmatrix}\right), \left(\begin{smallmatrix} g_1 & 0 \\ 0 & h_4 \end{smallmatrix}\right)\right)$. We consider the stabilizer of the set of matrices in the above block form. So we take $g, h \in \mathrm{Gl}_n(\mathbb{C})$, written as block matrices

$$g = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix}, \quad h = \begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix},$$

and assume we are given matrices $A, A' \in M_{r\times r}(\mathbb{C})$ and $N, N' \in N_{n-r}(\mathbb{C})$, the nilpotent cone of $(n-r)\times(n-r)$ matrices, such that

$$\begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix}\begin{pmatrix} E_r & 0 \\ 0 & N \end{pmatrix} = \begin{pmatrix} E_r & 0 \\ 0 & N' \end{pmatrix}\begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & E_{n-r} \end{pmatrix} = \begin{pmatrix} A' & 0 \\ 0 & E_{n-r} \end{pmatrix}\begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix}.$$

From these equations we first conclude $h_1 = g_1$ and $h_4 = g_4$, thus $h_2 = A'g_2$ and $g_2 = h_2N$, which yields $h_2 = A'h_2N$. By induction, this implies $h_2 = (A')^k h_2 N^k$ for all $k \geq 1$. But N is nilpotent, thus $h_2 = 0$, thus $g_2 = 0$. Similarly, we can conclude $h_3 = 0$ and $g_3 = 0$. But then g_1 and g_4 are invertible, and $A' = g_1 A g_1^{-1}$ as well as $N' = g_4 N g_4^{-1}$. This proves the claim.

To obtain information on the Chow groups from this stratification using Lemma 5.3, we first have to analyze the Chow groups of nilpotent cones.

The nilpotent cone $N_d(\mathbb{C})$ is irreducible of dimension $d^2 - d$, and the $\mathrm{Gl}_d(\mathbb{C})$-orbits $\mathbb{O}_\lambda$ in N_d are parametrized by partitions λ in $\mathcal{P}_d$, the set of partitions of d (we denote by $\mathcal{P}$ the union of all $\mathcal{P}_d$'s). The stabilizer G_λ of a point in $\mathbb{O}_\lambda$ has dimension $\langle\lambda,\lambda\rangle = \sum_{i,j}\min(m_i, m_j)m_i m_j$, and its reductive part is isomorphic to $\prod_i \mathrm{Gl}_{m_i}(\mathbb{C})$, where $m_i = m_i(\lambda)$ denotes the multiplicity of i as a part of λ, for $i \geq 1$. We can thus apply Lemma 5.3 and reduce the structure group — note that in characteristic zero, an orbit is isomorphic to the quotient of the group by the stabilizer of a point — to get

$$A^*_{\mathrm{Gl}_d(\mathbb{C})}(N_d(\mathbb{C})) \cong \bigoplus_{\lambda\in\mathcal{P}_d} A^{*+d-\langle\lambda,\lambda\rangle}_{G_\lambda}(\mathrm{pt}),$$

and the equivariant cycle map for $N_d(\mathbb{C})$ is an isomorphism.

This enables us to again apply Lemma 5.3, this time to the stratification $(S_r)_r$. We compute (using $\operatorname{codim} S_r = n - r$):

$$\begin{aligned} A^*_{\mathrm{Gl}_n(\mathbb{C})\times\mathrm{Gl}_n(\mathbb{C})}(R^{\theta-\mathrm{sst}}_{(n,n)}(K_2)) &\cong \bigoplus_{r=0}^{n} A^{*-n+r}_{\mathrm{Gl}_n(\mathbb{C})\times\mathrm{Gl}_n(\mathbb{C})}(S_r) \\ &\cong \bigoplus_{r=0}^{n} A^{*-n+r}_{\mathrm{Gl}_r(\mathbb{C})\times\mathrm{Gl}_{n-r}(\mathbb{C})}(M_r(\mathbb{C})\times N_{n-r}(\mathbb{C})) \\ &\cong \bigoplus_{r=0}^{n} A^*_{\mathrm{Gl}_r(\mathbb{C})}(M_r(\mathbb{C}))\otimes A^{*-n+r}_{\mathrm{Gl}_{n-r}(\mathbb{C})}(N_{n-r}(\mathbb{C})) \\ &\cong \bigoplus_{r=0}^{n} A^*_{\mathrm{Gl}_r(\mathbb{C})}(\mathrm{pt})\otimes \bigoplus_{\lambda\in\mathcal{P}_{n-r}} A^{*-\langle\lambda,\lambda\rangle}_{G_\lambda}(\mathrm{pt}). \end{aligned}$$

Summing over all n, we obtain

$$\mathcal{A}^{\theta-\mathrm{sst},1/2}_{(*,2*)}(K_2) \cong \bigoplus_{n\geq 0} A^*_{\mathrm{Gl}_n(\mathbb{C})\times\mathrm{Gl}_n(\mathbb{C})}(R^{\theta-\mathrm{sst}}_{(n,n)}(K_2)) \cong \Bigl(\bigoplus_{r\geq 0} A^*_{\mathrm{Gl}_r(\mathbb{C})}(\mathrm{pt})\Bigr)\otimes\Bigl(\bigoplus_{\lambda\in\mathcal{P}} A^{*-\langle\lambda,\lambda\rangle}_{G_\lambda}(\mathrm{pt})\Bigr).$$

The generating function of the bigraded space $\mathscr{A}^{\theta-\mathrm{sst},1/2}(K_2)$ therefore equals

$$\left(\sum_{n\geq 0}\frac{t^n}{(1-q)\cdots(1-q^n)}\right)\cdot\left(\sum_{\lambda}\frac{q^{-\langle\lambda,\lambda\rangle}t^{|\lambda|}}{\prod_{i\geq 1}((1-q)\cdots(1-q^{m_i}))}\right)=\prod_{i\geq 0}\frac{1}{1-q^it}\cdot\prod_{i\geq 1}\frac{1}{1-q^it}$$

by standard identities. We thus arrive at an isomorphism of bigraded $\mathbb{Q}$-spaces

$$\mathscr{A}^{\theta-\mathrm{sst},1/2}(K_2)\cong S^*((\mathbb{Q}((1,1),0)\oplus\mathbb{Q}((1,1),2))[z]).$$

However, this is not an isomorphism of algebras, since we will now exhibit an example showing that the algebra $\mathscr{A}^{\theta-\mathrm{sst},1/2}(K_2)$ is not super-commutative.

We use the algebraic description of the CoHa of Section 7 together with Theorem 8.1. We have

$$\mathscr{A}_{(m,n)}(K_2)\cong\mathbb{Q}[x_1,\dots,x_m,y_1,\dots,y_n]^{S_m\times S_n}$$

with multiplication given as in Section 7. By Theorem 8.1, $\mathscr{A}^{\theta-\mathrm{sst},1/2}_{(1,1)}(K_2)$ is the factor of $\mathscr{A}_{(1,1)}(K_2)\cong\mathbb{Q}[x_1,y_1]$ by the image of the multiplication map $\mathscr{A}_{(1,0)}(K_2)\otimes\mathscr{A}_{(0,1)}(K_2)\to\mathscr{A}_{(1,1)}(K_2)$, thus

$$\mathscr{A}^{\theta-\mathrm{sst},1/2}_{(1,1)}(K_2)\cong\mathbb{Q}[x,y]/(x-y)^2.$$

Again by Theorem 8.1, $\mathscr{A}^{\theta-\mathrm{sst},1/2}_{(2,2)}(K_2)$ is the factor of $\mathscr{A}_{(2,2)}(K_2)$ by the image of the multiplication map

$$\mathscr{A}_{(2,1)}(K_2)\otimes\mathscr{A}_{(0,1)}(K_2)\oplus\mathscr{A}_{(1,0)}(K_2)\otimes\mathscr{A}_{(1,2)}(K_2)\to\mathscr{A}_{(2,2)}(K_2).$$

The degree of an element in this image is at least $-\chi((2,1),(0,1))=-\chi((1,0),(1,2))=3$. A direct calculation shows that for the elements $1,x,y\in\mathscr{A}^{\theta-\mathrm{sst},1/2}_{(1,1)}(K_2)$, we have in $\mathscr{A}^{\theta-\mathrm{sst},1/2}_{(2,2)}(K_2)$:

$$\begin{aligned}
1*1&=2,\\
1*x&=y1+y2,\\
x*1&=2(x1+x2)-(y1+y2),\\
1*y&=-(x1+x2)+2(y1+y2),\ \text{and}\\
y*1&=x1+x2.
\end{aligned}$$

In particular, the (anti)commutator of 1 and x does not vanish.

10.3. ***Donaldson–Thomas invariants as Chow–Betti numbers.*** Next, we illustrate Theorem 9.2. We consider the symmetric quiver Q with two vertices i and j and $n\geq 1$ arrows from i to j and from j to i, and the dimension vector $d=(1,r)$ for $r\leq n$. To determine the quantized Donaldson–Thomas invariant $\Omega_{d,k}$, we use the stability $\theta=(r,-1)$, for which d is coprime. Therefore, $\Omega_{d,k}=\Omega^{\theta}_{d,k}$ equals the (suitably shifted) Poincaré polynomial of the cohomology of the moduli space $R^{\theta-\mathrm{sst}}_d(Q)/\mathrm{PG}_d$, which is isomorphic to a vector bundle over the Grassmannian $\mathrm{Gr}_r(k^n)$ [Reineke 2017, §6.1]. By Theorem 9.2, we can also compute $\Omega_{d,k}$ as the (suitably shifted) Poincaré polynomial of the Chow ring of the moduli space $R^{0-\mathrm{st}}_d(Q)/\mathrm{PG}_d$. Again by [Reineke 2017], this moduli space is isomorphic to the space X of $n\times n$-matrices of rank r. Mapping such a matrix to its image defines a $\mathrm{Gl}_n(k)$-equivariant fibration

$X \to \mathrm{Gr}_r(k^n)$, whose fiber is isomorphic to the space of $r \times n$-matrices of highest rank. The latter being open in an affine space, its Chow ring reduces to $\mathbb{Q}$, thus the Chow ring of X is isomorphic to the Chow ring of $\mathrm{Gr}_r(k^n)$ as expected.

10.4. ***Multiple loop quivers.*** Finally, we consider the quiver L_m with a single vertex and $m \geq 2$ loops. The quantized Donaldson–Thomas invariants are computed explicitly in [Reineke 2012].

All stability conditions are equivalent for this quiver. Let M_d^{simp} be the moduli space of simple (equivalently, stable) representations of L_m of dimension d. It is obtained as the geometric quotient $R_d^{\mathrm{simp}}/\,\mathrm{PGl}_d$. The Chow ring $A^*(M_d^{\mathrm{simp}})_{\mathbb{Q}} = A^*_{\mathrm{PGl}_d}(R_d^{\mathrm{simp}})_{\mathbb{Q}}$ (we will always work with rational coefficients in this subsection and therefore neglect it in the notation) is a quotient of the equivariant Chow ring $A^*_{\mathrm{PGl}_d}(R_d) = A^*_{\mathrm{PGl}_d}(\mathrm{pt})$. The group of characters of a maximal torus of Gl_d identifies with the free abelian group in letters $x_1, \dots, x_d$, the natural action of the Weyl group $W = S_d$ being the permutation action. A maximal torus of PGl_d is given by the quotient of the chosen maximal torus of Gl_d by the diagonally embedded multiplicative group. The corresponding Weyl group is also S_d and the character group is then the submodule

$$X_d = \mathrm{Sp}_{(d-1,1)} = \{a_1x_1 + \cdots + a_dx_d \mid a_1 + \cdots + a_d = 0\}.$$

The symmetric algebra $\mathrm{Sym}(X_d)$ over X_d is the subalgebra of $\mathbb{Q}[x_1, \dots, x_d]$ generated by $x_j - x_i$ (with $i < j$) and the equivariant Chow ring $A^*_{\mathrm{PGl}_d}(R_d)$ is therefore $\mathrm{Sym}(X_d)^{S_d}$ which identifies with a subalgebra of $A^*_{G_d}(\mathrm{pt}) = \mathbb{Q}[x_1, \dots, x_d]^{S_d}$. As in the proofs of Theorems 8.1 and 9.1, the kernel of $A^*_{\mathrm{PGl}_d}(R_d) \to A^*_{\mathrm{PGl}_d}(R_d^{\mathrm{simp}})$ is then given by the image of

$$\bigoplus_{\substack{p+q=d\\ p,q>0}} A^*_{\mathrm{PGl}_d}(Z_{p,q} \times^{P_{p,q}} \mathrm{PGl}_d) \to A^*_{\mathrm{PGl}_d}(R_d),$$

where $P_{p,q}$ is the obvious parabolic subgroup of PGl_d — this, by the way, can be done for an arbitrary quiver and for the kernels $A^*_{\mathrm{PG}_d}(R_d) \to A^*_{\mathrm{PG}_d}(R_d^{\theta-\mathrm{sst}})$ and $A^*_{\mathrm{PG}_d}(R_d^{\theta-\mathrm{sst}}) \to A^*_{\mathrm{PG}_d}(R_d^{\theta-\mathrm{st}})$. The ring $A^*_{\mathrm{PGl}_d}(Z_{p,q})$ is isomorphic to $\mathrm{Sym}(X_d)^{S_p \times S_q}$ and the push-forward map

$$m_{p,q} : A^*_{\mathrm{PGl}_d}(Z_{p,q} \times^{P_{p,q}} \mathrm{PGl}_d) \to A^*_{\mathrm{PGl}_d}(R_d)$$

can be described algebraically and looks just like the explicit formula from [Kontsevich and Soibelman 2011, Theorem 2], i.e., given by a shuffle product with kernel $\prod_{i=1}^{p}\prod_{j=1}^{q}(x_{p+j} - x_i)^{m-1}$. The relations in $A^*_{\mathrm{PGl}_d}(R_d)$ which present $A^*(M_d^{\mathrm{simp}})$ thus have at least degree $(m-1)(d-1)$. In other words, for every $0 \leq i < (m-1)(d-1)$, we get

$$A^i(M_d^{\mathrm{simp}}) \cong A^i_{\mathrm{PGl}_d}(R_d) = \mathrm{Sym}^i(X_d)^{S_d}.$$

The generating series of $\mathrm{Sym}(X_d)^{S_d}$ is

$$\frac{1}{(1-q^2)\cdots(1-q^d)} = \sum_{i\geq 0} \sharp\{(k_2, \dots, k_d) \mid 2k_2 + \cdots + dk_d\}q^i$$

and using Theorem 9.2, we obtain a description of the first few Donaldson–Thomas invariants.

Proposition 10.4.1. *For the m-loop quiver, the Donaldson–Thomas invariant $\Omega_{d,k}$ for a nonnegative integer d and an integer k of the same parity as $(1-m)d^2$ satisfying*

$$(1-m)d^2 \le k < (1-m)(d^2-2d+2)$$

computes as

$$\Omega_{d,k} = \sharp\big\{(k_2,\dots,k_d) \mid 2k_2+\cdots+dk_d = \tfrac{1}{2}((m-1)d^2+k)\big\}.$$

We conclude the subsection with a computation of the numbers $\Omega_{2,k}$. The ring $\mathrm{Sym}(X_2)^{S_2}$ is the subalgebra of $\mathbb{Q}[x_1,x_2]^{S_2}$ which is generated by $(x_2-x_1)^2$. Abbreviate $\Delta = x_2 - x_1$. As a $\mathrm{Sym}(X_2)^{S_2}$-module, $\mathrm{Sym}(X_2)$ is generated by 1 and Δ. The push-forward map $m_{1,1} : \mathrm{Sym}(X_2) \to \mathrm{Sym}(X_2)^{S_2}$ sends $f(x_1,x_2)$ to

$$(f(x_1,x_2) + (-1)^{m-1} f(x_2,x_1))\Delta^{m-1}$$

and therefore, the image of $m_{1,1}$ is the ideal of $\mathrm{Sym}(X_2)^{S_2} = \mathbb{Q}[\Delta^2]$ which is generated by $\Delta^{2\lfloor m/2\rfloor}$ (i.e., Δ^m if m is even and Δ^{m-1} if m is odd). We have shown that

$$\Omega_{2,k} = \begin{cases} 1 & \text{if } k \equiv 0 \pmod 4 \text{ and } 4(1-m)-2 \le k \le 4(\lfloor m/2\rfloor - m), \\ 0 & \text{otherwise.} \end{cases}$$

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References

[Assem et al. 2006] I. Assem, D. Simson, and A. Skowroński, *Elements of the representation theory of associative algebras, I*, London Math. Soc. Student Texts **65**, Cambridge Univ. Press, 2006. MR Zbl

[Davison and Meinhardt 2016] B. Davison and S. Meinhardt, "Cohomological Donaldson–Thomas theory of a quiver with potential and quantum enveloping algebras", preprint, 2016. arXiv

[Edidin and Graham 1998] D. Edidin and W. Graham, "Equivariant intersection theory", *Invent. Math.* **131**:3 (1998), 595–634. MR Zbl

[Efimov 2012] A. I. Efimov, "Cohomological Hall algebra of a symmetric quiver", *Compos. Math.* **148**:4 (2012), 1133–1146. MR Zbl

[Engel and Reineke 2009] J. Engel and M. Reineke, "Smooth models of quiver moduli", *Math. Z.* **262**:4 (2009), 817–848. MR Zbl

[Franzen 2015] H. Franzen, "Chow rings of fine quiver moduli are tautologically presented", *Math. Z.* **279**:3-4 (2015), 1197–1223. MR Zbl

[Franzen 2016] H. Franzen, "On cohomology rings of non-commutative Hilbert schemes and CoHa-modules", *Math. Res. Lett.* **23**:3 (2016), 805–840. MR Zbl

[Franzen 2018] H. Franzen, "On the semi-stable CoHa and its modules arising from smooth models", *J. Algebra* **503** (2018), 121–145. MR

[Fulton 1984] W. Fulton, *Intersection theory*, Ergebnisse der Mathematik (3) **2**, Springer, 1984. MR Zbl

[King and Walter 1995] A. D. King and C. H. Walter, “On Chow rings of fine moduli spaces of modules”, *J. Reine Angew. Math.* **461** (1995), 179–187. MR Zbl

[Kontsevich and Soibelman 2011] M. Kontsevich and Y. Soibelman, “Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson–Thomas invariants”, *Commun. Number Theory Phys.* **5**:2 (2011), 231–352. MR Zbl

[Meinhardt and Reineke 2014] S. Meinhardt and M. Reineke, “Donaldson–Thomas invariants versus intersection cohomology of quiver moduli”, preprint, 2014. arXiv

[Reineke 2003] M. Reineke, “The Harder–Narasimhan system in quantum groups and cohomology of quiver moduli”, *Invent. Math.* **152**:2 (2003), 349–368. MR Zbl

[Reineke 2012] M. Reineke, “Degenerate cohomological Hall algebra and quantized Donaldson–Thomas invariants for m-loop quivers”, *Doc. Math.* **17** (2012), 1–22. MR Zbl

[Reineke 2017] M. Reineke, “Quiver moduli and small desingularizations of some GIT quotients”, pp. 613–635 in *Representation theory: current trends and perspectives*, edited by H. Krause et al., Eur. Math. Soc., Zürich, 2017. MR Zbl

[Rimányi 2013] R. Rimányi, “On the cohomological Hall algebra of Dynkin quivers”, preprint, 2013. arXiv

[Ringel 1990] C. M. Ringel, “Hall algebras and quantum groups”, *Invent. Math.* **101**:3 (1990), 583–591. MR Zbl

[Totaro 1999] B. Totaro, “The Chow ring of a classifying space”, pp. 249–281 in *Algebraic K-theory* (Seattle, 1997), edited by W. Raskind and C. Weibel, Proc. Sympos. Pure Math. **67**, Amer. Math. Soc., Providence, RI, 1999. MR Zbl

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