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Irreducible components of minuscule affine Deligne–Lusztig varieties

Paul Hamacher and Eva Viehmann

We examine the set of $J_b(F)$-orbits in the set of irreducible components of affine Deligne–Lusztig varieties for a hyperspecial subgroup and minuscule coweight μ. Our description implies in particular that its number of elements is bounded by the dimension of a suitable weight space in the Weyl module associated with μ of the dual group.

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1. Introduction

Let F be a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$ and Γ its absolute Galois group. We denote by $\mathcal{O}_F$ and $k_F \cong \mathbb{F}_q$ its ring of integers and its residue field, and by ϵ a fixed uniformizer. Let L denote the completion of the maximal unramified extension of F, and $\mathcal{O}_L$ its ring of integers. Its residue field is an algebraic closure k of k_F. We denote by σ the Frobenius of L over F and of k over k_F.

Let G be a reductive group scheme over $\mathcal{O}_F$, and denote $K = G(\mathcal{O}_L)$. Then G_F is automatically unramified. We fix $S \subset T \subset B \subset G$, where S is a maximal split torus, T a maximal torus, and B a Borel subgroup of G. Let W be the absolute Weyl group of G. There exist k_F-ind schemes called the loop group LG, the positive loop group L^+G, and the affine Grassmannian $\mathcal{G}r_G := LG/L^+G$ of G whose k-valued points are canonically identified with $G(L)$, $K = G(\mathcal{O}_L)$, and $G(L)/G(\mathcal{O}_L)$, respectively (compare [Pappas and Rapoport 2008; Zhu 2017; Bhatt and Scholze 2017]).

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Let $\mu \in X_*(T)_{\mathrm{dom}}$, and let $b \in G(L)$. Then the affine Deligne–Lusztig variety associated with b and μ is the reduced subscheme $X_\mu(b)$ of $\mathcal{G}r_G$ whose k-valued points are

$$X_\mu(b)(k) = \{g \in G(L)/K \mid g^{-1}b\sigma(g) \in K\mu(\epsilon)K\}.$$

Let $X_{\preceq\mu}(b) = \bigcup_{\mu'\preceq\mu} X_{\mu'}(b)$ where $\mu' \preceq \mu$ if $\mu - \mu'$ is a nonnegative integral linear combination of positive coroots. It is closed in the affine Grassmannian and called the closed affine Deligne–Lusztig variety. For minuscule μ (the case we are mainly interested in for this paper) it agrees with $X_\mu(b)$.

Notice that up to isomorphism, both affine Deligne–Lusztig varieties depend only on the $G(L)$-σ-conjugacy class $[b] \in B(G)$ of b. An affine Deligne–Lusztig variety $X_\mu(b)$ or $X_{\preceq\mu}(b)$ is nonempty if and only if $[b] \in B(G, \mu)$, a finite subset of $B(G)$. The following basic assertion seems to be well known, but we could not find a reference in the literature.

Lemma 1.1. *The scheme $X_\mu(b)$ is locally of finite type in the equal characteristic case and locally of perfectly finite type in the case of unequal characteristic.*

Proof. The proof of this is the same as the corresponding part of the analogous statement for moduli spaces of local G-shtukas; compare the proof of Theorem 6.3 in [Hartl and Viehmann 2011] (where only the first half of page 113 is needed). In that proof, the case of equal characteristic and split G is considered. However, the general statement follows from the same proof. □

Notice that in general $X_\mu(b)$ is not quasicompact since it may have infinitely many irreducible components. It is conjectured to be equidimensional, but this has not been proven in full generality yet. In Section 3 we give an overview of the cases where equidimensionality has been proven. In the case of μ minuscule, which we are primarily interested in here, there are only a few exceptional cases where this is not yet known.

Definition 1.2. For a finite-dimensional k-scheme X we denote by $\Sigma(X)$ the set of irreducible components of X and by $\Sigma^{\mathrm{top}}(X) \subset \Sigma(X)$ the subset of those irreducible components which are top-dimensional.

The affine Deligne–Lusztig varieties $X_\mu(b)$ and $X_{\preceq\mu}(b)$ carry a natural action (by left multiplication) by the group

$$J_b(F) = \{g \in G(L) \mid g^{-1}b\sigma(g) = b\}.$$

This action induces an action of $J_b(F)$ on the set of irreducible components.

A complete description of the set of orbits was previously only known for the groups GL_n and GSp_{2n} and minuscule μ where the action is transitive [Viehmann 2008a; 2008b], and for some other particular cases; see for example [Vollaard and Wedhorn 2011] for a particular family of unitary groups and minuscule μ.

To describe the (conjectured) number of orbits, denote by $\widehat{G}$ the dual group of G in the sense of Deligne and Lusztig. That is, $\widehat{G}$ is the reductive group scheme over $\mathcal{O}_F$ that contains a Borel subgroup $\widehat{B}$ with maximal torus $\widehat{T}$ and maximal split torus $\hat{S}$ such that there exists a Galois equivariant isomorphism

$X^*(\widehat{T}) \cong X_*(T)$ identifying simple coroots of $\widehat{T}$ with simple roots of T. For any $\mu \in X_*(T)_{\mathrm{dom}} = X^*(\widehat{T})_{\mathrm{dom}}$ we denote by V_μ the associated Weyl module of $\widehat{G}_{\mathbb{O}_L}$.

In the following we use an element $\lambda_G(b) \in X^*(\widehat{T}^\Gamma)$ that we define in Section 2. Its restriction λ to $\hat{S}$ can be seen as a "best integral approximation" of the Newton point ν_b of $[b]$, while its precise value in $X^*(\widehat{T}^\Gamma)$ will depend on the Kottwitz point $\kappa_G(b)$. We choose a lift $\tilde{\lambda} \in X_*(T)$.

Conjecture 1.3 (Chen and Zhu). *There exists a canonical bijection between $J_b(F) \setminus \Sigma(X_\mu(b))$ and the basis of $V_\mu(\lambda_G(b))$ constructed by Mirković and Vilonen* [2007], *where $V_\mu(\lambda_G(b))$ denotes the $\lambda_G(b)$-weight space* (*for the action of $\widehat{T}^\Gamma$*) *of V_μ.*

In this paper, we describe the set $J_b(F) \setminus \Sigma^{\mathrm{top}}(X_\mu(b))$ for minuscule μ. Our main result, Theorem 5.12, implies in particular the following theorem.

Theorem 1.4. *Let $\mu \in X_*(T)_{\mathrm{dom}}$ be minuscule, $b \in [b] \in B(G, \mu)$, and $\tilde{\lambda} \in X_*(T)$ be an associated element as in Section 2. There exists a canonical surjective map*

$$\phi : W.\mu \cap [\tilde{\lambda} + (1-\sigma)X_*(T)] \twoheadrightarrow J_b(F) \setminus \Sigma^{\mathrm{top}}(X_\mu(b)).$$

Moreover, this map is a bijection in the following cases:

(1) *G is split and*

(2) $[b] \cap \mathrm{Cent}_G(\nu_b)$ *is a union of superbasic σ-conjugacy classes in* $\mathrm{Cent}_G(\nu_b)$.

Remark 1.5. (a) Let us explain how the theorem is a special case of the conjecture. Since μ is minuscule, we have for any $\tilde{\mu} \in X_*(T)$

$$\dim V_\mu(\tilde{\mu}) = \begin{cases} 1 & \text{if } \tilde{\mu} \in W.\mu, \\ 0 & \text{otherwise,} \end{cases}$$

where now $V_\mu(\tilde{\mu})$ denotes the $\tilde{\mu}$-weight space for the action of $\widehat{T}$. Thus, indeed we obtain a bijection between the Mirković–Vilonen basis of $V_\mu(\lambda)$ and $W.\mu \cap [\tilde{\lambda} + (1-\sigma)X_*(T)]$.

(b) We can replace the weight space $V_\mu(\lambda_G(b))$ by the weight space $V_\mu(\lambda)$ for the action of $\hat{S}$ in Conjecture 1.3. A priori one might expect the second space to be bigger; the equality is a consequence of the relation between λ and the Kottwitz point $\kappa_G(b)$ (see Remark 2.6 for details).

(c) An analogous formula has first been shown by Xiao and Zhu [2017] for $[b]$ such that the F-ranks of J_b and G coincide. In this case one can simply choose $\lambda = \nu_b$, the Newton point of $[b]$. It was then observed by Chen and Zhu (in oral communication) that an expression similar to the above should give $|J_b(F) \setminus \Sigma(X_\mu(b))|$ also for general $[b]$, and all μ.

(d) In particular, Theorems 1.4 and 5.12 apply to all cases that correspond to Newton strata in Shimura varieties of Hodge type.

In the case where b is superbasic, we prove the following stronger result, which was conjectured in [Hamacher 2015a]. For the ordering $\leq$ compare the definition at the top of page 1615.

Proposition 1.6. *Assume $b \in G(L)$ is superbasic. There exists a decomposition into disjoint $J_b(F)$-stable locally closed subschemes*

$$X_\mu(b) = \bigcup_{\substack{\tilde\mu \in W.\mu \\ \tilde\mu|_{\hat S} \preceq \nu_b}} C_{\tilde\mu}$$

such that $C_{\tilde\mu}$ intersected with any connected component of $\mathcal{G}r_G$ is universally homeomorphic to an affine space. These affine spaces are of dimension $d(\tilde\mu) := \sum \lfloor \langle \tilde\mu - \mu_{\mathrm{adom}}, \hat\omega_F \rangle \rfloor$ where we take the sum over all relative fundamental coweights $\hat\omega_F$ of $\widehat{G}$ and where μ_{adom} denotes the antidominant representative in the Weyl group orbit of μ.

Note that varying b within $[b]$ only changes $X_\mu(b)$ by an isomorphism. For suitably chosen $b \in [b]$, the connected components of $C_{\tilde\mu}$ are precisely the intersections of $X_\mu(b)$ with some Iwahori-orbit on $\mathcal{G}r_G$ [Chen and Viehmann 2018, §3]. Since the latter form a stratification on $\mathcal{G}r_G$, we can apply the localization long exact sequence to calculate the cohomology of $X_\mu(b)$. For example for the constant sheaf one obtains the following result.

Corollary 1.7. *Assume $b \in G(L)$ is superbasic, and denote by $J_b(F)^0$ the (unique) parahoric subgroup of $J_b(F)$. Then the $J_b(F)$-equivariant cohomology of $X_\mu(b)$ (for $\ell \neq p$) is given by*

$$H_c^{2i+1}(X_\mu(b), \mathbb{Q}_\ell) = 0,$$
$$H_c^{2i}(X_\mu(b), \mathbb{Q}_\ell) = \text{c-ind}_{J_b(F)^0}^{J_b(F)} V_i,$$

where V_i is a diagonalizable $J_b(F)^0$-representation with coefficients in $\mathbb{Q}_\ell$ and of dimension

$$\#\{\tilde\mu \in W.\mu \mid d(\tilde\mu) = i\}.$$

2. Definition of λ

We associate with every σ-conjugacy class $[b]$ a not necessarily dominant coinvariant $\lambda_G(b) \in X^*(\widehat{T})_\Gamma$ which lifts the Kottwitz point of b and at the same time is a "best approximation" of the Newton point (in a sense to be made precise below). In the split case it is closely connected to the notion of σ-straight elements in the extended affine Weyl group of G.

Invariants of σ-conjugacy classes. By work of Kottwitz [1985], a σ-conjugacy class $[b] \in B(G)$ is uniquely determined by two invariants: the Newton point $\nu_G(b) \in X_*(S)_{\mathbb{Q},\mathrm{dom}}$ and the Kottwitz point $\kappa_G(b) \in \pi_1(G)_\Gamma$. Here $\pi_1(G)$ denotes Borovoi's fundamental group, i.e., the quotient of $X_*(T)$ by its coroot lattice. We also consider the Kottwitz homomorphism w_G as in [Kottwitz 1985]. Let $w : X_*(T) \twoheadrightarrow \pi_1(G)$ denote the canonical projection. By the Cartan decomposition $G(L) = \coprod_{\mu \in X_*(T)_{\mathrm{dom}}} K\mu(\epsilon)K$, and we extend w to a map $w_G : G(L) \to \pi_1(G)$ mapping $K\mu(\epsilon)K$ to $w(\mu)$. Then for every $b \in G(L)$ the projection of $w_G(b)$ to $\pi_1(G)_\Gamma$ coincides with $\kappa_G(b)$.

We define a partial order $\preceq$ on $X^*(\widehat{T})$ such that $\mu' \preceq \mu$ holds if and only if $\mu - \mu'$ is a linear combination of positive roots with nonnegative, integral coefficients. Since the set of positive roots is preserved by the Galois action, this descends to a partial order on $X^*(\widehat{T})_\Gamma$. Similarly, we define its rational analogue $\leq$ on $X^*(T)_{\mathbb{Q}}$ such that $\mu \leq \mu'$ holds if and only if $\mu - \mu'$ is a linear combination of positive roots with nonnegative, rational coefficients. By the same argument as above this order descends to $X^*(\widehat{T})_{\mathbb{Q},\Gamma} = X^*(\hat{S})$.

Lemma/Definition 2.1. *Let $b \in G(L)$. Then the set*

$$\{\tilde{\lambda} \in X^*(\widehat{T})_\Gamma \mid w(\tilde{\lambda}) = \kappa_G(b),\ \tilde{\lambda}|_{\hat{S}} \leq \nu_G(b)\}$$

has a unique maximum $\lambda_G(b)$ characterized by the property that $w(\lambda_G(b)) = \kappa_G(b)$ and that for every relative fundamental coweight $\omega^\vee_{\widehat{G},F}$ of $\widehat{G}$, one has

$$\langle \lambda_G(b) - \nu_G(b), \omega^\vee_{\widehat{G},F}\rangle \in (-1, 0]. \tag{2.2}$$

Proof. Denote by $\widehat{Q} \subset X^*(\widehat{T})$ the root lattice. Then the restriction $X^*(\widehat{T}) \twoheadrightarrow X^*(\hat{S})$ canonically identifies the relative root lattice with $\widehat{Q}_\Gamma$. Note that the preimage $w^{-1}(\kappa_G(b))_\Gamma$ in $X^*(\widehat{T})_\Gamma$ is a $\widehat{Q}_\Gamma$-coset. Thus, one has $\lambda' \succeq \lambda$ for two elements in $w^{-1}(\kappa_G(b))_\Gamma$ if and only if

$$\langle \lambda', \omega^\vee_{\widehat{G},F}\rangle - \langle \lambda, \omega^\vee_{\widehat{G},F}\rangle \geq 0$$

for all relative fundamental coweights $\omega^\vee_{\widehat{G},F}$ of $\widehat{G}$ and moreover the left-hand side always has integral value. Thus, if a $\lambda_G(b)$ as in (2.2) exists, it is the unique maximum. One easily constructs such a $\lambda_G(b)$ by choosing some $\lambda' \in w^{-1}(\kappa_G(b))_\Gamma$ and defining

$$\lambda_G(b) := \lambda' - \sum_{\hat{\beta}} \lceil \langle \lambda' - \nu_G(b), \omega^\vee_{\hat{\beta}}\rangle \rceil \cdot \hat{\beta},$$

where the sum runs over all positive simple roots $\hat{\beta} \in \widehat{Q}_\Gamma$ and $\omega^\vee_{\hat{\beta}}$ denotes the corresponding fundamental coweight. □

Example 2.3. Assume that $G = \mathrm{GL}_n$, B is the upper-triangular Borel subgroup, and that $S = T$ is the diagonal torus. Then $\lambda_G(b)$ has the following geometric interpretation. To an element $\nu \in \mathbb{Q}^n \cong X_*(\hat{S})_{\mathbb{Q}}$, we associate a polygon $P(\nu)$ which is defined over $[0, n]$ with starting point $(0, 0)$ and slope ν_i over $(i-1, i)$. We denote by f_ν the corresponding piecewise linear function. Then $P(\nu_G(b))$ is the (concave) Newton polygon of b and $P(\lambda_G(b))$ is the largest polygon below $P(\nu_G(b))$ with integral slopes and break points. Indeed, the fundamental coweights of GL_n are given by $\omega_i = (\underbrace{1, \dots, 1}_{i \text{ times}}, \underbrace{0, \dots, 0}_{n-i \text{ times}})$; thus,

$$\langle \lambda_G(b) - \nu_G(b), \omega_i\rangle = f_{\lambda_G(b)}(i) - f_{\nu_G(b)}(i),$$

which implies $f_{\lambda_G(b)}(i) = \lfloor f_{\nu_G(b)}(i) \rfloor$ by (2.2). An example is illustrated in Figure 1.

Lemma 2.4. *Let $f : H \to G$ be a morphism of reductive groups over $\mathcal{O}_F$. Then we have $\lambda_G(f(b)) = f(\lambda_H(b))$ in the following cases*:

(1) *f is a central isogeny and*

(2) *f is the embedding of a standard Levi subgroup, such that $\nu_H(b)$ is G-dominant.*

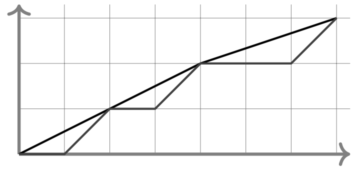

Figure 1. The polygons associated to $\nu_G(b)$ and $\lambda_G(b)$ for $[b] \in B(\mathrm{GL}_7)$ given by $\nu_G(b) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Proof. If f is a central isogeny, we have $X^*(\widehat{T}_H) = X^*(\widehat{T}_G) \times_{\pi_1(G)} \pi_1(H)$ compatibly with the obvious Galois action and partial order on the right-hand side. Thus, f and λ commute.

Now assume that H is a standard Levi subgroup of G and $\nu_H(b)$ is dominant, i.e., $\nu_H(b) = \nu_G(b)$. By (2.2) we have $-1 < \langle f(\lambda_H(b)) - \nu_G(b), \omega^\vee_{\widehat{H},F}\rangle \leq 0$ for every relative fundamental coweight of H. Let $\omega^\vee_{\widehat{G},F}$ be a relative fundamental coweight of G, but not of H. Then $\omega^\vee_{\widehat{G},F}$ factorizes through the center of H; thus, for every quasicharacter $\nu' \in X_*(\widehat{T})_{\mathbb{Q}}$ the value of $\langle \nu', \omega^\vee_{\widehat{G},F}\rangle$ is determined by the image of ν' in $\pi_1(H)_{\Gamma,\mathbb{Q}}$. In $\pi_1(H)_{\Gamma,\mathbb{Q}}$ we have equalities

$$\text{(image of } \nu_H(b)) = \text{(image of } \kappa_H(b)) = \text{(image of } \lambda_H(b));$$

thus, $\langle \nu_H(b) - \lambda_H(b), \omega^\vee_{\widehat{G},F}\rangle = 0$. □

Notation 2.5. For fixed $b \in G(L)$ we denote by $\tilde{\lambda} \in X^*(\widehat{T})$ an arbitrary but fixed lift of $\lambda_G(b)$ and by λ its image in $X^*(\hat{S})$.

Remark 2.6. Since G is quasisplit, the maximal torus of the derived group T^{der} is induced and hence $\widehat{T^{\mathrm{der}}}^{\Gamma} \subseteq \hat{S}$. Thus, any two elements in $X^*(\widehat{T}^{\Gamma})$ with the same image in $X^*(\hat{S})$ differ by a central cocharacter and thus have a different image in $\pi_1(G)_\Gamma$. In particular

$$\{\tilde{\mu} \in X^*(\widehat{T}) \mid \tilde{\mu}|_{\widehat{T}^\Gamma} = \lambda_G(b),\ w_G(\tilde{\mu}) = w_G(\mu)\} = \{\tilde{\mu} \in X^*(\widehat{T}) \mid \tilde{\mu}|_{\hat{S}} = \lambda,\ w_G(\tilde{\mu}) = w_G(\mu)\}.$$

Since $V_\mu(\tilde{\mu}) = 0$ unless $\tilde{\mu} \leq \mu$, this implies $V_\mu(\lambda_G(b)) = V_\mu(\lambda)$.

A group-theoretic definition of λ_G in the split case. We denote by $\widetilde{W} = \widetilde{W}_G := (\mathrm{Norm}_G(T))(L)/T(O_L)$ the extended affine Weyl group of G. Recall that we have canonical isomorphisms $\widetilde{W}_G \cong X_*(T) \rtimes W \cong W_a \rtimes \Omega_G$ where W_a denotes the affine Weyl group of G and $\Omega_G \subset \widetilde{W}_G$ the set of elements stabilizing the base alcove, which we choose as the unique alcove in the dominant Weyl chamber whose closure contains 0. In particular, we can lift the length function ℓ on W_a to $\widetilde{W}_G$.

The embedding $\mathrm{Norm}_G(T) \hookrightarrow G$ induces a natural map $B(\widetilde{W}_G) \to B(G)$, where $B(\widetilde{W}_G)$ denotes the set of $\widetilde{W}_G$-σ-conjugacy classes in $\widetilde{W}_G$. In general the notion of $\widetilde{W}_G$-conjugacy is finer than the notion of $G(L)$-conjugacy. Hence, we consider only a certain subset of $B(\widetilde{W}_G)$.

Definition 2.7. (1) We call $x \in \widetilde{W}_G$ basic if it is contained in Ω_G. A σ-conjugacy class $O \in B(\widetilde{W}_G)$ is called basic if it contains a basic element.

(2) An element $x \in \widetilde{W}_G$ is called σ-straight if it satisfies

$$\ell(x\sigma(x)\cdots\sigma^{n-1}(x)) = \ell(x) + \ell(\sigma(x)) + \cdots + \ell(\sigma^{n-1}(x)).$$

for any nonnegative integer n. Note that the right-hand side might also be written as $n \cdot \ell(x)$. A σ-conjugacy class $O \in B(\widetilde{W}_G)$ is called straight if it contains a σ-straight element.

He and Nie gave a characterization of the set of straight σ-conjugacy classes which is analogous to Kottwitz's description of $B(G)$ [1985, §6].

Proposition 2.8 [He and Nie 2014, Proposition 3.2]. *A σ-conjugacy class $O \in B(\widetilde{W}_G)$ is straight if and only if it contains a basic σ-conjugacy class $O' \in B(\widetilde{W}_M)$ for some standard Levi subgroup $M \subset G$.*

Finally, by [He and Nie 2014, Theorem 3.3] each $[b] \in B(G)$ contains a unique straight $O_{[b]} \in B(\widetilde{W}_G)$. We obtain the following description of λ_G in the split case.

Proposition 2.9. *Let G be a split group over $\mathcal{O}_F$, let $b \in G(L)$, and let $x \in O_{[b]}$ be a σ-straight element. Denote by λ' its image under the canonical projection $\widetilde{W}_G \to X_*(T)$. Then $\lambda'_{\mathrm{dom}} = \lambda_G(b)_{\mathrm{dom}}$.*

Proof. By Proposition 2.8 there exists a standard Levi subgroup $M \subset G$ and an M-basic element $x_M \in \Omega_M$ such that x and x_M are $\widetilde{W}_G$-conjugate. By [He and Nie 2015, Proposition 4.5] any two such elements are even W-conjugate and thus correspond to the same element in $X_*(T)_{\mathrm{dom}}$. Since the same holds true for $\lambda_G(b)_{\mathrm{dom}}$ by Lemma 2.4, it suffices to prove the proposition in the basic case, i.e., when $\nu_G(b)$ is central.

If $[b]$ is basic, then x is basic; thus, λ' is the (unique) dominant minuscule character with $w(\lambda') = \kappa_G(b)$; compare [Bourbaki 1968, §2, Proposition 6]. Hence, it suffices to show that $\lambda_G(b)$ is minuscule. By Lemma 2.4(2) we may assume that G is of adjoint type. This leaves finitely many cases, which can easily be checked using the explicit description of root systems in [Bourbaki 1968]. $\square$

3. Equidimensionality

While it is conjectured that $X_\mu(b)$ is equidimensional [Rapoport 2005, Conjecture 5.10], this has not yet been proven in all cases. We give a partial result after reviewing the necessary geometry of $X_\mu(b)$ first.

Connected components. Let $w_G : G(L) \to \pi_1(G)$ be the Kottwitz homomorphism, as considered in [Kottwitz 1985]; compare the bottom of page 1614. It induces a map $\mathcal{G}r_G(k) \to \pi_1(G)$. After base change to $\operatorname{Spec} k$, this induces isomorphisms $\pi_0(LG_k) \cong \pi_0(\mathcal{G}r_{G,k}) \cong \pi_1(G)$; compare [Pappas and Rapoport 2008, Theorem 0.1] in the equal characteristic case and [Zhu 2017, Proposition 1.21] in the mixed characteristic case. Here we used that as G is unramified, the action of the inertia subgroup of the absolute Galois group of F on $\pi_1(G)$ is trivial.

For $\omega \in \pi_1(G)$, we let LG^ω and $\mathcal{G}r_G^\omega$ be the corresponding connected components. Denote for any subgroup $H \subset LG_k$ and subscheme $X \subset \mathcal{G}r_{G,k}$ the intersection $H^\omega := H \cap LG^\omega$ and $X^\omega := X \cap \mathcal{G}r_G^\omega$.

In particular, $X_\mu(b)^\omega$ is a union of connected components, and the $J_b(F)$-orbit of $X_\mu(b)^\omega$ equals $X_\mu(b)$ by [Nie 2015, Theorem 1.2] (see also [Chen et al. 2015, Theorem 1.2]) whenever $X_\mu(b)^\omega$ is nonempty. One can even show that under some mild condition on the triple $(G, [b], \mu)$ every connected component of $X_\mu(b)$ is of the form $X_\mu(b)^\omega$ (see [Nie 2015, Theorem 1.1] and also [Chen et al. 2015, Theorem 1.1]), but we will not need this result.

The following general result on affine flag varieties is formulated in greater generality than needed in this paper. We will only apply it in the case where $H = G$ is a reductive group scheme. For consistency we denote affine flag varieties by the same symbol $\mathcal{G}r$ as affine Grassmannians.

Proposition 3.1. *Let $f : H' \to H$ be a morphism of parahoric group schemes over $\mathcal{O}_F$ such that the induced homomorphism on their adjoint groups is an isomorphism. Then the induced morphism on connected components of affine flag varieties*

$$f^\omega_{\mathcal{G}r} : \mathcal{G}r^\omega_{H'} \to \mathcal{G}r^{f(\omega)}_H$$

is a universal homeomorphism.

Proof. This is proven in [Pappas and Rapoport 2008, §6] if $\operatorname{char} F = p$ and p does not divide the order of $\pi_1(H'_{\mathrm{der}})$ or $\pi_1(H_{\mathrm{der}})$ (see also [He and Zhou 2016, Proposition 4.3] for the statement if $\operatorname{char} F = 0$). We briefly recall the proof in [Pappas and Rapoport 2008] and explain how to generalize it.

Note that it suffices to show that $f^\omega_{\mathcal{G}r}$ is bijective on geometric points. Indeed, it is a morphism of ind-proper ind-schemes (see [Richarz 2016, Corollary 2.3] if $\operatorname{char} F = p$ and [Zhu 2017, §1.5.2] if $\operatorname{char} F = 0$) and thus universally closed.

By homogeneity under the actions of $H'(L)$ and $H(L)$, respectively, we may assume $\omega = 0$. Denote by H_{der} the derived group of H and by $\widetilde{H}$ the simply connected cover of H_{der}. Since we have a commutative diagram

$$\begin{tikzcd} H' \arrow[rrrr, bend left, "f"] & H'_{\mathrm{der}} \arrow[l, hook'] & \widetilde{H}' = \widetilde{H} \arrow[l, two heads] \arrow[r, two heads] & H_{\mathrm{der}} \arrow[r, hook] & H \end{tikzcd}$$

it suffices to prove the theorem in the following two special cases.

Case 1: $H' = H_{\mathrm{der}}$. One can show that $f^0_{\mathcal{G}r}$ is universally bijective using the argument in [Pappas and Rapoport 2008, p. 144].

Case 2: *H is semisimple and $H' = \widetilde{H}$*. The following argument can be found in [Pappas and Rapoport 2008, p. 140–141]. Fix an algebraically closed field $l \supset k$, and let $M \supset L$ be the corresponding field extension of ramification index 1. We denote by Z the kernel of $\widetilde{H} \to H$ and let T and $\widetilde{T}$ denote the Néron models of fixed maximal tori in H_F and $\widetilde{H}_F$ satisfying $\widetilde{T}_F = f^{-1}(T_F)$. Since $\widetilde{H}_F$ is simply connected, $\widetilde{T}_F$ is an induced torus, i.e., there exist finite field extensions F_i/F such that

$$\widetilde{T}^0 \cong \prod_i \operatorname{Res}_{\mathcal{O}_{F_i}/\mathcal{O}_F} \mathbb{G}_m;$$

thus, there exists an $n \in \mathbb{N}$ such that

$$Z_F \subset \prod_i \operatorname{Res}_{F_i/F} \mu_n.$$

In particular, we have $Z(M) \subset \widetilde{T}^0(\mathbb{O}_M)$. Since $\widetilde{T}^0 \subset \widetilde{H}$, $f^0_{\mathscr{G}r}$ is injective on geometric points. The surjectivity is a direct consequence of [Pappas and Rapoport 2008, Appendix, Lemma 14]. □

Remark 3.2. If char $F = p$ and p does not divide the order of $\pi_1(H'_{\mathrm{der}})$ or $\pi_1(H_{\mathrm{der}})$, it is shown in [Pappas and Rapoport 2008, §6] that $f^\omega_{\mathscr{G}r}$ even induces an isomorphism of the underlying reduced ind-schemes. However, Pappas and Rapoport [2008, Example 6.4] show that this is not necessarily the case when we drop the condition on p. On the other hand $f^\omega_{\mathscr{G}r}$ is always an isomorphism in the case char $F = 0$, since universal homeomorphisms of perfect schemes are isomorphisms by [Bhatt and Scholze 2017, Lemma 3.8].

Let G^{ad} be the adjoint group of G. We denote by a subscript "ad" the image of an element of $G(L)$, $X_*(T)$, or $\pi_1(G)$ in $G^{\mathrm{ad}}(L)$, $X_*(T^{\mathrm{ad}})$, or $\pi_1(G^{\mathrm{ad}})$, respectively. By [Chen et al. 2015, Corollary 2.4.2], the homeomorphism of Proposition 3.1 induces a universal homeomorphism

$$X_\mu(b)^\omega \to X_{\mu_{\mathrm{ad}}}(b_{\mathrm{ad}})^{\omega_{\mathrm{ad}}} \tag{3.3}$$

whenever $X_\mu(b)^\omega$ is nonempty.

Equidimensionality for some affine Deligne–Lusztig varieties. Equidimensionality is known to hold in the following cases.

Theorem 3.4. *Let G, b, and μ be as above.*

(1) *If* char $F = p$, *then $X_\mu(b)$ and $X_{\preceq\mu}(b)$ are equidimensional. Furthermore, $X_{\preceq\mu}(b)$ is the closure of $X_\mu(b)$.*

(2) *Let F be an unramified extension of $\mathbb{Q}_p$, and let G be classical, μ be minuscule, and either $p \neq 2$ or all simple factors of G^{ad} be of type A or C. Then $X_\mu(b)$ is equidimensional.*

Proof. Assume first that char $F = p$. In the case where G is split the assertion is proven in [Hartl and Viehmann 2012, Corollary 6.8] by identifying the formal neighborhood of a closed point in the affine Deligne–Lusztig variety with a certain closed subscheme in the deformation space of a local G-shtuka. We briefly explain how to generalize the arguments in the proof of [Hartl and Viehmann 2012, Corollary 6.8] to arbitrary reductive group schemes over $\mathbb{O}_F$.

The main ingredient is the following result in [Viehmann and Wu 2018], generalizing [Hartl and Viehmann 2012, Theorem 6.6]. Let $x = gK \in X_{\preceq\mu}(b)(k)$, and denote $b' := gb\sigma(g)^{-1}$. Consider the deformation functor

$$\begin{aligned} \mathscr{D}ef_{b',0} &: (\mathrm{Art}/k) \to (\mathrm{Sets}),\\ &A \mapsto \{\tilde{b} \in (\overline{K\mu(\epsilon)K})(A) \mid \tilde{b}_k = b'\}/_{\cong} \end{aligned}$$

where $\tilde{b} \cong \tilde{b}'$ if there is an $h \in G(A[\![\epsilon]\!])$ with $h_k = 1$ and $h^{-1}\tilde{b}\sigma(h) = \tilde{b}'$. By [Viehmann and Wu 2018, Proposition 2.6] this functor is prorepresented by the formal completion of $K \setminus K\mu(\epsilon)K$ at b'. Moreover, the universal object has a unique algebraization by [Viehmann and Wu 2018, Lemma 2.8]. We denote by $D_{b',0}$ the algebraization of $(K \setminus K\mu(\epsilon)K)^\wedge_{b'}$ and by $\tilde{b} \in LG(D_{b',0})$ a lift of the universal object. We denote by $N_{[b],0} \subset D_{b',0}$ the minimal Newton stratum, that is, the set of all geometric points $\bar{s} : \operatorname{Spec} k_{\bar{s}} \to D_{b',0}$ such that $\tilde{b}_{\bar{s}}$ is $G(k_{\bar{s}}((\epsilon)))$-$\sigma$-conjugate to b (or b'). Since $N_{[b],0}$ is closed, we may equip it with the structure of a reduced subscheme. By [Viehmann and Wu 2018, Theorems 2.9 and 2.11] there exists a surjective finite morphism

$$\operatorname{Spec} k[\![x_1, \dots, x_{2\langle \rho_G, \nu_G(b)\rangle}]\!] \widehat{\times} X_{\preceq\mu}(b)_x^\wedge \to N_{[b],0}$$

where ρ_G denotes the half-sum of all absolute positive roots in G and $X_{\preceq\mu}(b)_x^\wedge$ the algebraization of the completion of $X_{\preceq\mu}(b)$ in x. In particular, we get

$$\begin{aligned} \dim N_{[b],0} &= 2\langle \rho_G, \nu_G(b)\rangle + \dim X_{\preceq\mu}(b)_x^\wedge \\ &\leq \langle \rho_G, \mu + \nu_G(b)\rangle - \tfrac{1}{2}\operatorname{def}_G(b). \end{aligned}$$

Here the last inequality follows from the dimension formula of $X_{\preceq\mu}(b)$ in [Hamacher 2015a, Theorem 1.1] and equality holds if and only if $\dim X_{\preceq\mu}(b)_x^\wedge = \dim X_{\preceq\mu}(b)$. The Newton stratification on $D_{b',0}$ satisfies strong purity in the sense of [Viehmann 2015, Definition 5.8]. Indeed, this is shown for $G = \mathrm{GL}_n$ in [Viehmann 2013, Theorem 7] and the general case follows by [Hamacher 2017, Proposition 2.2]. Thus, the conditions of [Viehmann 2015, Lemma 5.12] are satisfied and we get the dimension formula and closure relations of all Newton strata in $D_{b',0}$. In particular,

$$\dim N_{[b],0} = \langle \rho_G, \mu + \nu_G(b)\rangle - \tfrac{1}{2}\operatorname{def}_G(b).$$

Thus, $\dim X_{\preceq\mu}(b)_x^\wedge = \dim X_{\preceq\mu}(b)$ and since x was an arbitrary closed geometric point of $X_{\preceq\mu}(b)$, this implies equidimensionality. Since $\dim X_{\preceq\mu'}(b) < \dim X_{\preceq\mu}(b)$ for every $\mu' \prec \mu$ by [Hamacher 2015a, Theorem 1.1] this also implies the equidimensionality of $X_{\preceq\mu}(b)$ and that $X_\mu(b)$ is dense in $X_{\preceq\mu}(b)$.

Now consider $F = \mathbb{Q}_p$, $p \neq 2$, and assume first that there exists a faithful representation $\rho : G \hookrightarrow \mathrm{GL}_n$ such that the action of $\mathbb{G}_m$ via $\rho(\mu)$ has weights 0 and 1. Then we can associate a Rapoport–Zink space of Hodge type $\mathscr{M}_{G,\mu}(b)$ to the triple (G, μ, b), whose perfection equals $X_\mu(b)$ by [Zhu 2017, Theorem 3.10]. Since $\mathscr{M}_{G,\mu}(b)$ is equidimensional by [Hamacher 2017, Theorem 1.3], so is $X_\mu(b)$.

Now the morphism $X_\mu(b) \to X_{\mu_{\mathrm{ad}}}(b_{\mathrm{ad}})$ induced by the canonical projection $G \twoheadrightarrow G^{\mathrm{ad}}$ is an isomorphism on connected components by (3.3). Thus, all connected components of $X_{\mu_{\mathrm{ad}}}(b_{\mathrm{ad}})$ which are contained in the image of $X_\mu(b)$ are equidimensional. Since all connected components are isomorphic to each other by [Chen et al. 2015, Theorem 1.2], this implies that $X_{\mu_{\mathrm{ad}}}(b_{\mathrm{ad}})$ is equidimensional. Thus, any affine Deligne–Lusztig variety with G classical and adjoint and μ minuscule is equidimensional. Applying (3.3) once more, the claim follows for $p \neq 2$. If $p = 2$, the spaces $\mathscr{M}_{G,\mu}(b)$ are only defined if (G, μ, b) is of PEL type, but in this case the rest of the proof is identical.

If F is an unramified field extension of $\mathbb{Q}_p$, let $G' = \operatorname{Res}_{O_F/\mathbb{Z}_p} G$ and $\mu' = (\mu, 0, \dots, 0)$ and $b' = (b, 1, \dots, 1)$ with respect to the identification $G'_L \cong \prod_{F \hookrightarrow L} G$. By [Zhu 2017, Lemma 3.6] and the Cartesian diagram below it, we have $X_{\mu'}(b') \cong X_\mu(b)$. Thus, $X_\mu(b)$ is equidimensional. □

4. Irreducible components in the superbasic case

In this section we prove Theorem 1.4 for superbasic σ-conjugacy classes. In [Hamacher 2015a, §8] this has been reduced to a purely combinatorial statement, which we prove using the bijectivity of sweep maps on rational Dyck paths.

Superbasic σ-conjugacy classes. An element $b \in G(L)$ or the corresponding σ-conjugacy class $[b] \in B(G)$ is called superbasic if no element of $[b]$ is contained in a proper Levi subgroup of G defined over F.

Remark 4.1 [Chen et al. 2015, §3.1]. (1) If b is superbasic in $G(L)$, then the simple factors of the adjoint group G^{ad} are of the form $\mathrm{Res}_{F_d|F}\,\mathrm{PGL}_n$ for unramified extensions F_d of F (of degree d) and $n \geq 2$. In particular, $X_\mu(b)$ is equidimensional if char $F = p$ or F is an unramified extension of $\mathbb{Q}_p$.

(2) For every $[b] \in B(G)$ there is a standard parabolic subgroup $P \subset G$ defined over F and with the following property. Let T be a fixed maximal torus of G and M the Levi factor of P containing T. Then there is a $b \in [b] \cap M(L)$ which is superbasic in M.

We first consider the special case where $[b]$ is superbasic and where G is of the form $\mathrm{Res}_{F_d|F}\,\mathrm{GL}_n$ for some d and n. In this case we give a proof using EL-charts as in [Hamacher 2015a] (see also [de Jong and Oort 2000] for the split case). We then reduce the general superbasic case to this particular case.

For G as above $L \otimes_F F_d \cong \prod_{\tau: F_d \hookrightarrow L} L$ yields an identification

$$G(L) = \prod_{\tau \in I} \mathrm{GL}_n(L)$$

mapping $g \in G(L)$ to a tuple $(g_\tau)_{\tau \in I}$ where $I := \mathrm{Gal}(F_d, F) \cong \mathbb{Z}/d\mathbb{Z}$. Let $S \subset T \subset B \subset G$ be the split diagonal torus, the diagonal torus, and the upper-triangular Borel, respectively. We have a canonical identification $X_*(T) \cong (\mathbb{Z}^n)^{|I|}$. Then the dominant elements in $X_*(T)$ are precisely the $\mu = (\mu_\tau)_{\tau \in I} \in X_*(T)$ such that the components of μ_τ are weakly decreasing for each τ.

We identify $X_*(S)$ with the invariants $X_*(T)^I = \mathbb{Z}^n$; thus,

$$\mu|_{\hat{S}} = \sum_{\tau \in I} \mu_\tau .$$

Moreover, this identifies the partial order $\leq$ on $X_*(S)_{\mathbb{Q}}$ with the dominance order on $\mathbb{Q}^n$.

A combinatorial identity. An important tool when considering the combinatorics of EL-charts is the sweep map defined by Armstrong, Loehr, and Warrington [Armstrong et al. 2015]. We need a multiple-component version of it, which turns out to be easily realized as a special case of the classical sweep map.

Notation 4.2. By a word $\boldsymbol{w}$ we mean a finite sequence of integers $w_1 \cdots w_r$. For $1 \leq k \leq r$ we define the level of $\boldsymbol{w}$ at k by $l(\boldsymbol{w})_k := \sum_{i=1}^{k} w_i$. We consider the following sets for fixed sequences of integers $a_{\tau,1}, \ldots, a_{\tau,n}$ where $1 \leq \tau \leq d$.

(1) Let $\mathcal{A}_{\mathbb{Z}}^{(d)}$ denote the set of words $\boldsymbol{w} = w_1 \cdots w_{d\cdot n}$ such that the subword $\boldsymbol{w}_{(\tau)} := w_\tau w_{\tau+d} \cdots w_{\tau+(n-1)\cdot d}$ is a rearrangement of $a_{\tau,1}, \ldots, a_{\tau,n}$ for any $\tau \in \{1, \ldots, d\}$.

(2) Denote by $\mathscr{A}_{\mathbb{N}}^{(d)} \subset \mathscr{A}_{\mathbb{Z}}^{(d)}$ the subset of words whose level at multiples of d is nonnegative. Following [Thomas and Williams 2017; Armstrong et al. 2015], we call its elements (d-component) Dyck words.

Definition 4.3. The sweep map $\mathrm{sw}^{(d)} : \mathscr{A}_{\mathbb{Z}}^{(d)} \to \mathscr{A}_{\mathbb{Z}}^{(d)}$ is the map that sorts $\boldsymbol{w}$ according to its level by permuting $\boldsymbol{w}_{(\tau)}$ using the following algorithm. Initialize $\mathrm{sw}^{(d)}(\boldsymbol{w})_{(\tau)} = \varnothing$ for any $1 \leq \tau \leq d$. For each a down from -1 to $-\infty$ and then down from ∞ to 0 read $\boldsymbol{w}_{(\tau)}$ from right to left and append to $\mathrm{sw}^{(d)}(\boldsymbol{w})_{(\tau)}$ all letters w_k such that $l(\boldsymbol{w})_k = a$.

We deduce the bijectivity of $\mathrm{sw}^{(d)}$ from Williams' result for the classical sweep map in [Thomas and Williams 2017].

Proposition 4.4. $\mathrm{sw}^{(d)}$ *is bijective and preserves* $\mathscr{A}_{\mathbb{N}}^{(d)}$.

Proof. If $d = 1$, the map $\mathrm{sw}^{(1)}$ is precisely the sweep map defined in [Thomas and Williams 2017] and the proposition is proven in [Thomas and Williams 2017, Theorems 6.1 and 6.3]. In order to reduce to this case, we need to construct an injection $\mathscr{A}_{\mathbb{Z}}^{(d)} \hookrightarrow \mathscr{A}_{\mathbb{Z}}^{(1)}$ which identifies Dyck words and preserves the sweep map, i.e., such that the diagram

$$\begin{tikzcd} \mathscr{A}_{\mathbb{Z}}^{(d)} \arrow[r, hook] \arrow[d, "\mathrm{sw}^{(d)}"] & \mathscr{A}_{\mathbb{Z}}^{(1)} \arrow[d, "\mathrm{sw}^{(1)}"] \\ \mathscr{A}_{\mathbb{Z}}^{(d)} \arrow[r, hook] & \mathscr{A}_{\mathbb{Z}}^{(1)} \end{tikzcd} \tag{4.5}$$

commutes. Note that part of this construction is also the choice of a sequence $\{a'_1, \ldots, a'_{n\cdot d}\}$ for $\mathscr{A}_{\mathbb{Z}}^{(1)}$.

In preparation, fix an integer N big enough such that for any $\boldsymbol{w} \in \mathscr{A}_{\mathbb{Z}}^{(d)}$ and $1 \leq \tau \leq d$ as above the inequalities

$$\min\{l(\boldsymbol{w})_k + N \mid k \equiv \tau \pmod d\} > \max\{l(\boldsymbol{w})_k \mid k \equiv \tau - 1 \pmod d\}, \tag{4.6}$$

$$\min\{l(\boldsymbol{w})_k + \tau \cdot N \mid k \equiv \tau \pmod d\} \geq 0 \tag{4.7}$$

hold. We now construct a map $\mathscr{A}_{\mathbb{Z}}^{(d)} \to \mathscr{A}_{\mathbb{Z}}^{(1)}$, $\boldsymbol{w} \mapsto \boldsymbol{w}^{+N}$ satisfying the conditions above as follows. For given $\boldsymbol{w}$, let $\boldsymbol{w}^{+N}$ be the word which one obtains by replacing $w_{\tau+(i-1)\cdot d}$ by

$$w'_{\tau+(i-1)\cdot d} := \begin{cases} w_{\tau+(i-1)\cdot d} + N & \text{if } \tau \neq d, \\ w_{\tau+(i-1)\cdot d} - N \cdot (d-1) & \text{if } \tau = d \end{cases}$$

for $1 \leq i \leq n$ and $1 \leq \tau \leq d$. Then $\boldsymbol{w}^{+N} \in \mathscr{A}_{\mathbb{Z}}^{(1)}$ for the choice $\{a'_1, \ldots, a'_{n\cdot d}\}$, where

$$a'_{\tau+(i-1)\cdot d} := \begin{cases} a_{\tau,i} + N & \text{if } \tau \neq d, \\ a_{\tau,i} - N \cdot (d-1) & \text{if } \tau = d. \end{cases}$$

The map $\boldsymbol{w} \mapsto \boldsymbol{w}^{+N}$ is obviously injective. Note that for any k we have $l(\boldsymbol{w}^{+N})_k = l(\boldsymbol{w})_k + \bar{k} \cdot N$ where $0 \leq \bar{k} \leq d-1$ denotes the residue of k modulo d. Thus, $l(\boldsymbol{w}^{+N})_k = l(\boldsymbol{w})_k$ if k is a multiple of d, and $l(\boldsymbol{w}^{+n}) \geq 0$ by (4.7) otherwise. Hence, $\boldsymbol{w}^{+N} \in \mathscr{A}_{\mathbb{N}}^{(1)}$ if any only if $\boldsymbol{w} \in \mathscr{A}_{\mathbb{N}}^{(d)}$.

By (4.6), we have $\min_i l(\boldsymbol{w}^{+N})_{\tau+(i-1)\cdot d} > \max_i l(\boldsymbol{w}^{+N})_{\varsigma+(i-1)\cdot d}$ for all $1 \le \tau < \varsigma \le d-1$ or $\varsigma < \tau = d$. Thus, the permutation of letters of $\boldsymbol{w}^{+N}$ induced by the classical sweep map decomposes into a product of permutations of the subsets $\{\tau, \tau+d, \tau+2d, \ldots\}$. Since moreover $l(\boldsymbol{w}^{+N})_{\tau+(i-1)\cdot d} \ge l(\boldsymbol{w}^{+N})_{\tau+(j-1)\cdot d}$ if and only if $l(\boldsymbol{w})_{\tau+(i-1)\cdot d} \ge l(\boldsymbol{w})_{\tau+(j-1)\cdot d}$, the permutations induced by the classical sweep map $\mathrm{sw}^{(1)}$ applied to $\boldsymbol{w}^{+N}$ and $\mathrm{sw}^{(d)}$ applied to $\boldsymbol{w}$ coincide. In other words, the diagram (4.5) commutes. □

Characterization of EL-charts. Throughout the section, we fix a positive integer m coprime to n and denote $\nu|_{\hat{S}} = (m/n, \ldots, m/n) \in X^*(\hat{S}) = X_*(T)^I$. Let $m_1, \ldots, m_d$ be arbitrary integers such that $m_1 + \cdots + m_d = m$. We shall later make convenient choices of them depending on μ. We recall the notion of EL-charts as they were presented in [Hamacher 2015a, §5].

Let $\mathbb{Z}^{(d)} := \coprod_{\tau\in I} \mathbb{Z}_{(\tau)}$ be the disjoint union of d copies of $\mathbb{Z}$. We impose the notation that for any subset $A \subset \mathbb{Z}^{(d)}$ we write $A_{(\tau)} := A \cap \mathbb{Z}_{(\tau)}$. For $x \in \mathbb{Z}$ we denote by $x_{(\tau)}$ the corresponding element of $\mathbb{Z}_{(\tau)}$ and write $|a_{(\tau)}| := a$. We equip $\mathbb{Z}^{(d)}$ with a partial order $\le$ defined by

$$x_{(\tau)} \le y_{(\varsigma)} \;:\Longleftrightarrow\; \tau = \varsigma \text{ and } x \le y$$

and a $\mathbb{Z}$-action given by

$$x_{(\tau)} + z := (x+z)_{(\tau)}.$$

Furthermore, we consider a $\mathbb{Z}$-equivariant function $f : \mathbb{Z}^{(d)} \to \mathbb{Z}^{(d)}$ with

$$f(a_{(\tau)}) = a_{(\tau+1)} + m_\tau.$$

In particular, $f(\mathbb{Z}_{(\tau)}) = \mathbb{Z}_{(\tau+1)}$ and $f^d(a) = a + m$.

Definition 4.8. (1) An EL-chart is a nonempty subset $A \subset \mathbb{Z}^{(d)}$ which is bounded from below and satisfies $f(A) \subset A$ and $A + n \subset A$.

(2) Two EL-charts A and A' are called equivalent if there exists an integer z such that $A + z = A'$. We write $A \sim A'$.

Let A be an EL-chart and $B = A \setminus (A+n)$. It is easy to see that $\#B_{(\tau)} = n$ for all $\tau \in I$. We define a sequence $b_0, \ldots, b_{d\cdot n}$ as follows. Let $b_0 = b_{n\cdot d} = \min B_{(0)}$, and for given b_i let $b_{i+1} \in B$ be the unique element of the form

$$b_{i+1} = f(b_i) - \mu'_{i+1} \cdot n$$

for a nonnegative integer μ'_i. These elements are indeed distinct: if $b_i = b_j$, then obviously $i \equiv j \pmod d$ and then $b_{i+k\cdot d} \equiv b_i + k\cdot m \pmod n$ implies that $i = j$ as m and n are coprime.

It will later be helpful to distinguish the b_i and μ'_i of different components. For this we change the index set to $I \times \{1, \ldots, n\}$ via

$$b_{\tau,i} := b_{\tau+(i-1)\cdot d},$$

$$\mu'_{\tau,i} := \mu'_{\tau+(i-1)\cdot d}.$$

Here we choose the set of representatives $\{1, \ldots, d\} \subset \mathbb{Z}$ of I.

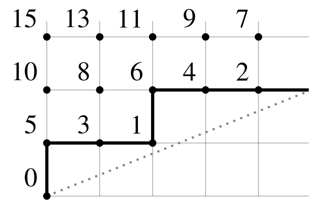

Figure 2. The associated Dyck path and $(5, -2)$-levels for $m = 5$, $n = 7$, and $A = \mathbb{N}_0$.

Definition 4.9. With the notation above, μ' is called the type of A.

Remark 4.10. This definition differs slightly from the definition of the type in [Hamacher 2015a, p. 12822]. In this article we choose the indices such that $\mu'_{\tau,i}$ measures the difference between $b_{\tau,i}$ and $b_{\tau-1,i}$ while in [Hamacher 2015a] it yields the difference between $b_{\tau,i}$ and $b_{\tau+1,i}$. Since one can alternate between those two notions by replacing f by $\bar{f} := f^{-1}$ and μ by $(-\mu)_{\mathrm{dom}}$, we can still use the combinatorial results of [Hamacher 2015a]. Moreover, we consider the Borel of *upper*-triangular matrices instead of lower-triangular matrices in [Hamacher 2015a], thus inverting the order on $X_*(S)$ and $X_*(T)$.

The type characterizes an EL-chart up to equivalence.

Lemma 4.11 [Hamacher 2015a, Lemma 5.3]. *Let*

$$P_{m,n,d} := \{\mu' \in (\mathbb{Z}^n_{\geq 0})^{|I|} \mid \mu'|_{\hat{S}} \leq \nu|_{\hat{S}}\}.$$

Then the type of any EL-chart A lies in $P_{m,n,d}$, and the type defines a bijection

$$\{\textit{EL-charts}\}/_\sim \leftrightarrow P_{m,n,d}.$$

Example 4.12. There are two important special cases of EL-charts.

(a) An EL-chart is called small if $A + n \subset f(A)$, in other words if its type only has entries 0 and 1. They correspond to the affine Deligne–Lusztig varieties with minuscule Hodge point.

(b) A semimodule is an EL-chart $A \subset \mathbb{Z}$. These are the invariants that occur in the split case.

There is a bijection between small semimodules up to equivalence and rational Dyck paths from $(0, 0)$ to $(n - m, m)$, that is, lattice paths allowing only steps in the north and east directions which stay above the diagonal. This gives a purely combinatorial motivation for the definitions below.

The bijection is given as follows (see [Gorsky and Mazin 2013] for more details). With a given equivalence class $[A]$ of small semimodules, we associate the path which goes east at the i-th step if $\mathrm{type}(A)_i = 0$ and north if $\mathrm{type}(A)_i = 1$. By the above lemma, this map is well defined and a bijection. Moreover, if we choose $\min A = 0$, then one can recover A from the Dyck path as the set of $(m, m - n)$-levels in the sense of [Armstrong et al. 2015] of points on or above the path, giving the inverse to the bijection. An example is illustrated in Figure 2.

There is another invariant of EL-charts which is more important for the application of this theory, as it allows us to calculate the dimension of strata inside the affine Deligne–Lusztig variety.

Definition 4.13. Let A be an EL-chart of type μ', and let $b_{\tau,i}$ be defined as above. For each $\tau \in I$ let $\tilde{b}_{\tau,1} > \cdots > \tilde{b}_{\tau,n}$ be the elements of $B_{(\tau)}$ arranged in decreasing order. Define

$$\tilde{\mu}_{\tau,i} = \mu'_{\tau,i'}$$

where i is the unique number such that $\tilde{b}_{\tau,i} = b_{\tau,i'}$. We call $\tilde{\mu}$ the cotype of A.

It is shown in [Hamacher 2015a, p. 12831] that $\operatorname{cotype}(A) \in P_{m,n,d}$. Since the cotype is obviously invariant under equivalence, we obtain a map

$$\zeta : P_{m,n,d} \to P_{m,n,d}, \quad \operatorname{type}(A) \mapsto \operatorname{cotype}(A).$$

We claim that ζ is bijective. For this we note that ζ is the composition of

$$\mu' \mapsto (w_k := m_{k \,(\operatorname{mod} d)} - \mu'_k \cdot n)_{k=1,\dots,n\cdot d} \overset{\operatorname{sw}^{(d)}}{\mapsto} (\widetilde{w}_k) \mapsto \Big(\frac{m_\tau - \widetilde{w}_{\tau+i\cdot d}}{n}\Big)_{\tau,i}. \tag{4.14}$$

Thus, its bijectivity follows from Proposition 4.4.

Example 4.15. For $d=1$, $n=7$, $m=5$, and $A=\mathbb{N}_0$, we can describe (4.14) as follows. In Figure 2 one sees that $\mu' := \operatorname{type}(A) = (0, 1, 1, 0, 1, 1, 1)$. This is mapped to the word $\boldsymbol{w} = (5, -2, -2, 5, -2, -2, -2)$, whose levels $l(\boldsymbol{w}) = (5, 3, 1, 6, 4, 2, 0)$ are the corresponding elements of $B := A \setminus (A+n)$. Thus, applying the sweep map, which sorts the letters of $\boldsymbol{w}$ according to their levels, is nothing else than permuting the letters such that the corresponding elements of B get arranged in decreasing order. Now $\operatorname{sw}(\boldsymbol{w}) = (5, 5, -2, -2, -2, -2, -2)$, which yields $\zeta(\mu') = (0, 0, 1, 1, 1, 1, 1)$.

Altogether, we obtain the following theorem, which generalizes the result of [Thomas and Williams 2017, Corollary 6.4]. It was conjectured in [Hamacher 2015a, Conjecture 8.3] and in the split case by de Jong and Oort [2000, Remark 6.16].

Theorem 4.16. *The cotype induces a bijection*

$$\{\textit{EL-charts}\}/_\sim \leftrightarrow P_{m,n,d}.$$

The superbasic case. Proposition 1.6 is a direct consequence of Theorem 4.16 together with the relation between orbits of irreducible components and EL-charts in [Hamacher 2015a, §8]. We briefly recall this relation for the reader's convenience before proving Proposition 1.6.

When applying the results of the previous subsection to affine Deligne–Lusztig varieties, we consider EL-charts satisfying certain additional criteria.

Definition 4.17. Let A be an EL-chart.

(1) A is called normalized if $\sum_{b\in B_{(0)}} |b| = \binom{n}{2}$ where $B_{(0)} = A_{(0)} \setminus (A_{(0)} + n)$.

(2) The Hodge point of A is defined as $\operatorname{type}(A)_{\mathrm{dom}}$.

Note that every EL-chart is equivalent to a unique normalized EL-chart. Let $P_\mu := \{\mu' \in P_{m,n,d} \mid \mu'_{\mathrm{dom}} = \mu\}$. Then by Lemma 4.11 $A \mapsto \mathrm{type}(A)$ induces a bijection

$$\{\text{normalized EL-charts with Hodge point } \mu\} \leftrightarrow P_\mu.$$

It is easy to see that ζ stabilizes P_μ. Thus, Theorem 4.16 says that $A \mapsto \mathrm{cotype}(A)$ induces a bijection between the set of normalized EL-charts with Hodge point μ and P_μ.

For every minuscule $\mu \in X_*(T)_{\mathrm{dom}}$ there exists a unique basic σ-conjugacy class in $B(G, \mu)$. We choose a representative of this σ-conjugacy class as follows. Let $m_\tau = \mathrm{val}\det\mu(\epsilon)_\tau$, and choose $b = ((b_{\tau,i,j})_{i,j=1}^n)_{\tau\in I}$ with

$$b_{\tau,i,j} = \begin{cases} \epsilon^{\lfloor (i+m_\tau)/n\rfloor} & \text{if } j - i \equiv m_\tau \pmod n, \\ 0 & \text{otherwise.} \end{cases}$$

Then the invariants $\lambda, \nu \in X^*(\hat{S}) = X^*(\widehat{T})_\Gamma \cong \mathbb{Z}^n$ are given by $\nu = (m/(d\cdot n), \ldots, m/(d\cdot n))$ with $m = \sum_{\tau\in I} m_\tau$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_i = \lfloor i\cdot m/n\rfloor - \lfloor (i-1)\cdot m/n\rfloor$. The requirement that b is in fact superbasic corresponds to the assertion that m and n are coprime.

By our choice of b, the variety $X_\mu(b)^0 := X_\mu(b) \cap \mathscr{G}r_G^0$ is nonempty. In [Hamacher 2015a; 2015b] we constructed a $J_b(F)^0$-invariant cellular decomposition

$$X_\mu(b)^0 = \bigcup_A S_A$$

where the union runs over all normalized EL-charts with Hodge-point μ. We denote

$$V_A := \{(i, j) \mid b_i < b_j,\ \mu'_i = \mu'_j + 1\}.$$

In [Hamacher 2015a, Proposition 6.5; 2015b, Proposition 13.9] we show that $\mathbb{A}^{V_A} \xrightarrow{\sim} S_A$ by constructing an element $g_A \in LG(\mathbb{A}^{V_A})$ and a corresponding basis $(v_{\tau,i})$ of the universal G-lattice over S_A. In particular $\dim S_A = \# V_A$.

Following the calculations of the term S_1 in [Hamacher 2015a, p. 12831], one obtains $\# V_A$ from $\tilde{\mu}$ using the formula

$$\# V_A = \sum \lfloor \langle \tilde{\mu}|_{\hat{S}} - \mu_{\mathrm{adom}}, \hat{\omega}_F^\vee\rangle \rfloor,$$

where the sum runs over all relative fundamental coweights $\hat{\omega}_F^\vee$ of $\widehat{G}$ and μ_{adom} denotes the antidominant element in the W-orbit of μ. In particular, S_A is top-dimensional if and only if $\mathrm{cotype}(A)|_{\hat{S}} = \lambda$.

Proof of Proposition 1.6. Let G be arbitrary. We assume without loss of generality that $b \in K\mu(\epsilon)K$; thus, $X_\mu(b)^0 \neq \varnothing$. Since $J_b(F)$ acts transitively on $\pi_0(X_\mu(b))$ by [Chen et al. 2015, Theorem 1.2], it suffices to construct $C_{\tilde{\mu}}^0 := C_{\tilde{\mu}} \cap X_\mu(b)^0$, which have to be $J_b(F)^0$-stable and universally homeomorphic to affine spaces of the correct dimension. In particular, we may take $C^0_{\mathrm{cotype}(A)} = S_A$ if $G = \mathrm{Res}_{F_d/F}\mathrm{GL}_n$.

By Remark 4.1 we have $G^{\mathrm{ad}} \cong \prod_{i=1}^n \mathrm{Res}_{F_{d_i}/F}\mathrm{PGL}_{n_i}$. Let $G' = \prod_{i=1}^n \mathrm{Res}_{F_{d_i}/F}\mathrm{GL}_{n_i}$ and b' and μ' be lifts of b_{ad} and μ_{ad} to G', such that $X_{\mu'}(b')^0 \neq \varnothing$. We identify the underlying topological spaces $X_\mu(b)^0 = X_{\mu_{\mathrm{ad}}}(b_{\mathrm{ad}})^0 = X_{\mu'}(b')^0$ via the homeomorphism (3.3). Thus, we get a cellular decomposition of $X_\mu(b)^0$ per transport of structure from $X_{\mu'}(b')^0$. Since it is $J_{b'}(F)^0$-stable, we consider the canonical projections $J_b(F)^0 \xrightarrow{p} J_{b_{\mathrm{ad}}}(F)^0 \xleftarrow{q} J_{b'}(F)^0$. It suffices to show that q is surjective (implying that the decomposition is $J_{b_{\mathrm{ad}}}(F)^0$-stable) and that the $J_b(F)^0$-action factors through $J_{b_{\mathrm{ad}}}(F)^0$.

To prove the surjectivity, let $j \in J_{b_{\mathrm{ad}}}(F)^0$ and choose a preimage $g \in G(L)^0$ of j. The element g satisfies $g^{-1}b\sigma(g) = zb$ for some $z \in Z'(L) \cap G(L)^0 = Z(\mathbb{O}_L)$, where Z' denotes the center of G'. We choose $z' \in Z'(\mathbb{O}_L)$ with $(z')^{-1}\sigma(z') = z^{-1}$. Then $gz' \in J_{b'}(F)^0$ maps to j, as claimed.

Now an elementary calculation of the kernel shows that we have an exact sequence

$$1 \to Z(\mathbb{O}_F) \to J_b(F)^0 \to J_{b_{\mathrm{ad}}}(F)^0,$$

where Z denotes the center of G. Since $Z(\mathbb{O}_F)$ acts trivially on $\mathcal{G}r_G$, the $J_b(F)^0$-action factors through $J_{b_{\mathrm{ad}}}(F)^0$, as claimed. □

Corollary 4.18. *Conjecture 1.3 is true if b is superbasic and μ minuscule.*

Proof. We have

$$J_b(F) \setminus \Sigma(X_\mu(b)) \cong \{\tilde{\mu} \in P_\mu \mid C_{\tilde{\mu}} \text{ top-dimensional}\} \cong \{\tilde{\mu} \in W.\mu \mid \tilde{\mu}|_{\hat{S}} = \lambda\}. \qquad \square$$

5. Reduction to the superbasic case

In this section we consider the general case of Theorem 1.4; i.e., G is an unramified reductive group over F, μ is minuscule, and b is an arbitrary element of $G(L)$. The goal is to use a reduction method, first introduced in [Görtz et al. 2006], to relate to the superbasic case.

Let $P \subset G$ be a smallest standard parabolic subgroup of G, defined over F and with the following property. Let M be the Levi factor of P containing T. Then we want that $M(L)$ contains a σ-conjugate of b which is superbasic in M. Fix a representative $b \in M(L)$ of $[b]_G = [b]$. Then we furthermore want that the M-dominant Newton point of b is already G-dominant. For existence of such P, M, and b compare Remark 4.1. We write $P = M \cdot N$ where N denotes the unipotent radical of P. Since $b \in M(L)$, this induces a decomposition

$$J_b(F) \cap P(L) = (J_b(F) \cap M(L)) \cdot (J_b(F) \cap N(L)).$$

Throughout the section, we may refer to subschemes of the loop group or Grassmannian by their k-valued points to improve readability, e.g., write K instead of L^+G or $N(L)$ instead of LN. We denote $K_M = M(\mathbb{O}_L)$, $K_N = N(\mathbb{O}_L)$, and $K_P = P(\mathbb{O}_L)$.

We consider the variety

$$X_\mu^{M\subset G}(b) = \{gK_M \in \mathcal{G}r_M \mid g^{-1}b\sigma(g) \in K\mu K\}.$$

Then we have $X_\mu^{M\subset G}(b) = \coprod_{\mu' \in I_{\mu,b}} X_{\mu'}^M(b)$ where $I_{\mu,b}$ is the set of M-conjugacy classes of cocharacters μ' in the G-conjugacy class of μ with $[b]_M \in B(M,\mu')$. As $[b]_M$ is basic in M, this latter condition is equivalent to $\kappa_M(b) = \kappa_M(\mu')$ in $\pi_1(M)_\Gamma$. We identify an element of $I_{\mu,b}$ with its M-dominant representative in $X_*(T)$. Note that $I_{\mu,b}$ is nonempty and finite, but may have more than one element if G is not split.

Notation 5.1. Note that $X_\mu^{M\subset G}(b)$ is in general not equidimensional, although the individual summands are conjectured to be. We define

$$\Sigma'(X_\mu^{M\subset G}(b)) := \bigcup_{\mu'\in I_{\mu,b}} \Sigma^{\mathrm{top}}(X_{\mu'}^M(b)).$$

Using Corollary 4.18 we can show that $X_\mu^{M\subset G}(b)$ has the same number of orbits of irreducible components as given by the right-hand side of Theorem 1.4.

Lemma 5.2. $$(J_b(F)\cap M(L))\setminus \Sigma'(X_\mu^{M\subset G}(b)) = W.\mu\cap[\tilde{\lambda}+(1-\sigma)X_*(T)]$$

Proof. By Corollary 4.18 we have

$$(J_b(F)\cap M(L))\setminus \bigcup_{\mu'\in I_{\mu,b}} \Sigma^{\mathrm{top}}(X_{\mu'}^M(b)) = \bigcup_{\mu'\in I_{\mu,b}} (W_M.\mu'\cap[\tilde{\lambda}_M+(1-\sigma)X_*(T)]).$$

Here the unions on both sides are disjoint, and $\tilde{\lambda}_M=\tilde{\lambda}_M(b)$ denotes the element associated with $[b]\in B(M)$ whereas $\tilde{\lambda}=\tilde{\lambda}_G(b)$. By Lemma 2.4, the above union is equal to $\bigcup_{\mu'\in I_{\mu,b}}(W_M.\mu'\cap[\tilde{\lambda}+(1-\sigma)X_*(T)])$. As $\tilde{\lambda}$ is minuscule, the set $W_M.\mu'\cap[\tilde{\lambda}+(1-\sigma)X_*(T)]$ is nonempty for a given $\mu'\in W.\mu$ if and only if $\kappa_M(\mu')=\kappa_M(\tilde{\lambda})$ $(=\kappa_M(b))$, i.e., if and only if $\mu'\in I_{\mu,b}$. Hence, $\bigcup_{\mu'\in I_{\mu,b}}(W_M.\mu'\cap[\tilde{\lambda}+(1-\sigma)X_*(T)]) = W.\mu\cap[\tilde{\lambda}+(1-\sigma)X_*(T)]$. □

In order to relate the irreducible components of $X_\mu^{M\subset G}(b)$ to those of $X_\mu(b)$, we consider the variety

$$X_\mu^{P\subset G}(b) := \{gK_P\in \mathcal{G}r_P \mid g^{-1}b\sigma(g)\in K\mu(\epsilon)K\}$$

as an intermediate object. The inclusion $P\hookrightarrow G$ induces a natural map $X_\mu^{P\subset G}(b)\to X_\mu(b)$. Using the Iwasawa decomposition $G(L)=P(L)K$ we see that this map is surjective, and in fact $X_\mu^{P\subset G}(b)$ is nothing but a decomposition of $X_\mu^G(b)$ into locally closed subsets (see, e.g., [Hamacher 2015a, Lemma 2.2]). Thus, we obtain a natural bijection

$$\Sigma^{\mathrm{top}}(X_\mu^{P\subset G}(b))\to \Sigma^{\mathrm{top}}(X_\mu^G(b))$$

which induces a surjection

$$\alpha_\Sigma : (J_b(F)\cap P(L))\setminus \Sigma^{\mathrm{top}}(X_\mu^{P\subset G}(b)) \twoheadrightarrow J_b(F)\setminus \Sigma^{\mathrm{top}}(X_\mu^G(b)). \tag{5.3}$$

Furthermore, $\dim X_\mu^{P\subset G}(b)=\dim X_\mu^G(b)$.

On the other hand, the restriction of the canonical projection $\mathcal{G}r_P\twoheadrightarrow\mathcal{G}r_M$ induces a surjective morphism

$$\beta : X_\mu^{P\subset G}(b)\to X_\mu^{M\subset G}(b)$$

by [Hamacher 2015a, Proposition 2.9]. Moreover the fiber dimension for $x\in X_{\mu'}^M(b)$ is given by

$$\dim\beta^{-1}(x)=\dim X_\mu^{P\subset G}(b)-\dim X_{\mu'}^M(b) \tag{5.4}$$

[Hamacher 2015a, Lemma 2.8 and Proposition 2.9(2)], using that for minuscule μ, equality in [Hamacher 2015a, Lemma 2.8] always holds, and using the dimension formula [Hamacher 2015a, Theorem 1.1]. Note that this only depends on μ' (but indeed depends on the choice of $\mu' \in I_{\mu,b}$), but not on the point x.

Lemma 5.5. *β induces a well defined surjective map*

$$\beta_\Sigma : \Sigma^{\mathrm{top}}(X_\mu^{P\subset G}(b)) \to \Sigma'(X_\mu^{M\subset G}(b)).$$

It is $J_b(F)\cap P(L)$-equivariant for the natural action on the left-hand side, and the action through the natural projection $J_b(F)\cap P(L) \twoheadrightarrow J_b(F)\cap M(L)$ on the right-hand side.

Recall that a subset of $G(L)$ is called bounded if it is contained in a finite union of K-double cosets.

Proof. Let $\mathscr{C}$ be a top-dimensional irreducible component of $X_\mu^{P\subset G}(b)$. Then $\beta(\mathscr{C})$ is irreducible and thus contained in one of the open and closed subschemes $X_{\mu'}^M(b)$. By (5.4), its dimension is equal to $\dim(X_{\mu'}^M(b))$; hence, $\beta(\mathscr{C})$ is a dense subscheme of one of the irreducible components of $X_{\mu'}^M(b)$. In this way we obtain the claimed map β_Σ. It is surjective and $J_b(F)\cap P(L)$-equivariant because the same holds for β. □

Proposition 5.6. *Let $Z\subset X_\mu^{M\subset G}(b)$ be an irreducible subscheme. Then $J_b(F)\cap N(L)$ acts transitively on $\Sigma(\beta^{-1}(Z))$.*

In the proof we need the following remark.

Remark 5.7. For $x\in \widetilde{W}$ let IxI be the locally closed subscheme of LG whose k-valued points are $I(k)xI(k)$. Let Y be a scheme and $g\in (IxI)(Y)$. Then we claim that there are elements $i_1, i_2 \in I(Y)$ with $g = i_1 x i_2$. In equal characteristic, this is [Hartl and Viehmann 2012, Lemma 2.4] (whose proof shows the above statement, although the lemma only claims the assertion étale locally on Y). Let us explain how to modify the proof to deduce the above statement in general: we consider the morphism $I/(I\cap xIx^{-1}) \to LG/I$ to the affine flag variety given by $g\mapsto gx$. By writing down the obvious inverse one sees that it is an immersion with image IxI/I.

Let $g\in (IxI)(Y)$ and $\bar g$ be its image in the affine flag variety. Then the above shows that $\bar g$ is the image of some $\bar i \in I/(I\cap xIx^{-1})(Y)$. Note that $I/(I\cap xIx^{-1}) = I_0/(I_0\cap xI_0x^{-1})$ where I_0 is the unipotent radical of I. By [Hartl and Viehmann 2012, Lemma 2.1] we can thus lift $\bar i \in I_0/(I_0\cap xI_0x^{-1})(Y)$ to an element $i_1\in I_0(Y)$ which is as claimed.

Proof of Proposition 5.6. As we have to take an inverse image of an element under σ later in this proof, we replace all occurring ind-schemes by their perfections. Note that this does not change the underlying topological spaces of the schemes. Moreover, since we may check the assertion on an open covering of Z, we may replace Z by an open subscheme $Y\subset Z$ containing one fixed but arbitrary point $z\in Z(k)$.

Étale locally there is a lifting of the inclusion $Z\hookrightarrow X_{\mu'}^M(b)$ to LM [Pappas and Rapoport 2008, Lemma 1.4] (the proof also works for char $F=0$; compare [Zhu 2017, Proposition 1.20]). Thus, there exists $Y'\to Z$ étale with $z\in \mathrm{im}(Y'\to Z)$ such that there exists a lift $\iota : Y'\to LM$. By replacing Y' by an irreducible component if necessary, we may assume that Y' is again irreducible. We denote by Y the image of Y' in Z, and by $y \in Y'$ a point mapping to z.

We denote

$$\Phi = \{(m, n) \in \iota(Y') \times N(L) \mid mnK_P \in X_\mu^{P\subset G}(b)\}$$

and $b_m := m^{-1}b\sigma(m)$ for any $m \in M(L)$. For $g = mn \in P(L)$ we have

$$g^{-1}b\sigma(g) = b_m \cdot [b_m^{-1}n^{-1}b_m\sigma(n)] \tag{5.8}$$

where the bracket is in $N(L)$ and where $b_m \in M(L)$. The condition $gK_P \in \beta^{-1}(Y)$ is then equivalent to the condition that we may choose $m \cdot n \in gK_P$ with $m \in \iota(Y') \subset LM$ and $n \in N(L)$ such that the last bracket is in $N(L) \cap b_m^{-1}K\mu(\epsilon)K$. Thus, we have a morphism

$$\gamma : \Phi \to \mathscr{E} := \{(m, c) \mid m \in \iota(Y'),\ c \in N(L) \cap b_m^{-1}K\mu(\epsilon)K\}, \quad (m, n) \mapsto (m, b_m^{-1}n^{-1}b_m\sigma(n)).$$

In order to get an easier description of $\mathscr{E}$, we show that one can assume $b_m \in K_M \cdot \mu'$ after further shrinking Y and replacing ι if necessary. Let $x \in \widetilde{W}$ such that $I_M x I_M \subset K_M\mu'(\epsilon)K_M$ is the open cell, where I_M denotes the standard Iwahori subgroup of M. Then $K\mu(\epsilon)K = KxK$, and we fix $k_0, k_0' \in K$ such that $b_{\iota(y)} = k_0xk_0'$. We replace Y' (and thus Y) by the open neighborhood of y such that $b_m \in k_0 \cdot I_M x I_M \cdot k_0'$ for all $m \in \iota(Y')$. By Remark 5.7 we have a global decomposition $b_m = k_0i_1xi_2k_0'$ with $i_j \in I_M(\iota(Y'))$. As $Y \subseteq X_{\mu'}^M(b)$ we have $x = w_1\mu'w_2 \in W_M\mu W_M$; thus, $b_m = k_0i_1w_1\mu'(\epsilon)w_2i_2k_0'$. We now replace m by $m\sigma^{-1}(w_2i_2k_0')^{-1} \in mK_M$ and modify ι accordingly. With respect to this new choice we obtain a decomposition of b_m of the form $k_1\mu'(\epsilon)$ with $k_1 = \sigma^{-1}(w_2i_2k_0')k_0i_1w_1 \in L^+M(\iota(Y'))$. Now

$$\begin{aligned} N(L) \cap b_m^{-1}K\mu(\epsilon)K &= N(L) \cap \mu'(\epsilon)^{-1}K\mu(\epsilon)K \\ &= \mu'(\epsilon)^{-1}(N(L) \cdot \mu'(\epsilon) \cap K\mu(\epsilon)K). \end{aligned}$$

Note that this only depends on the constant element μ'. Hence,

$$\mathscr{E} = \iota(Y') \times (N(L) \cap \mu'(\epsilon)^{-1}K\mu(\epsilon)K).$$

Claim 1: *$\mathscr{E}$ is irreducible.* As $\iota(Y')$ is irreducible, we have to show that $N(L) \cap \mu'(\epsilon)^{-1}K\mu(\epsilon)K$ is irreducible. For this we consider the morphism $\mathrm{pr}_{\mu'} : N(L) \to N(L)\mu'(\epsilon)K \subset \mathscr{G}r_G$, $n \mapsto \mu'(\epsilon)n$. Then $N(L) \cap \mu'(\epsilon)^{-1}K\mu(\epsilon)K$ is the preimage of $N(L)\mu'(\epsilon)K \cap K\mu(\epsilon)K$, which is irreducible by [Mirković and Vilonen 2007, Corollary 13.2]. On the other hand $\mathrm{pr}_{\mu'}$ is a K_N-torsor, since it is surjective and factorizes as

$$N(L) \to \mathscr{G}r_N \hookrightarrow \mathscr{G}r_G \xrightarrow{\mu'(\epsilon)\cdot} \mathscr{G}r_G.$$

Here the first map is the projection, a K_N-torsor. The second is the natural closed embedding, and the third the isomorphism obtained by left multiplication by $\mu'(\epsilon)$. As K_N is also irreducible, this completes the proof of Claim 1.

Claim 2: *Let $\mathscr{F} \subseteq \Phi$ be a nonempty open subscheme with $\mathscr{F} = \mathscr{F}K_N$ where K_N acts by right multiplication on the second component. Then its image under γ contains an open subscheme of $\mathscr{E}$. In particular, it is dense by Claim 1.*

Fix an irreducible component C of Φ such that its intersection with $\mathscr{F}$ is nonempty. We may replace $\mathscr{F}$ by an open and dense subscheme of points only contained in the one irreducible component C. As $\mathscr{F}$ is invariant under right multiplication by K_N and m is contained in a bounded subscheme of LM, its image under γ is invariant under right multiplication by some (sufficiently small) open subgroup K'_N of K_N (this follows from the same proof as [Görtz et al. 2006, Proposition 5.3.1], which carries over literally to the unramified case and the case char $F = 0$). Thus, it is enough to show that the image of $\gamma(\mathscr{F})$ in $\iota(Y') \times (N(L) \cap \mu'(\epsilon)^{-1} K\mu K)/K'_N$ is open. Let $g_0 \in \mathscr{F}$, and let $U = \operatorname{Spec} R$ be an affine open neighborhood of $\gamma(g_0)$ in $\mathscr{E}$. After possibly replacing K'_N by a smaller open subgroup we may assume that U is K'_N-invariant. Let (m_U, n_U) be the universal element. Then m_U and n_U are contained in bounded subsets of LM and LN, respectively. By Corollary 5.11 there is an étale covering R' of R and a morphism $\operatorname{Spec} R' \to \Phi$ such that the composite with γ and the quotient modulo K'_N maps $\operatorname{Spec} R'$ surjectively to U/K'_N. Intersecting $\operatorname{Spec} R'$ with the inverse image of the open subscheme $\mathscr{F}$ of Φ and using that $R \to R'$ is finite étale, we obtain an open subscheme of $\operatorname{Spec} R'$, or of $\mathscr{F}$ mapping surjectively to an open neighborhood of $g_0 K'_N$. This implies the claim.

Finally, we show that all irreducible components of $\beta^{-1}(Y)$ are contained in one $J_b(F) \cap N(L)$-orbit of irreducible components of $X_\mu^{P\subset G}(b)$. Let $\mathscr{D}$ and $\mathscr{D}'$ be irreducible components of $\beta^{-1}(Y)$. We have to show that all dense open subsets D and D' of the two components contain points p and p' which are in the same $J_b(F)$-orbit. Consider the K_N-torsor

$$\phi : \Phi \to \beta^{-1}(Y), \quad (m, n) \mapsto mnK_P.$$

Then it is enough to show that for all nonempty open subsets C_1 and C_2 of Φ with $C_i K_N = C_i$ there are points $q_i \in C_i$ and a $j \in J$ with $\phi(q_1) = j\phi(q_2)$. This latter condition follows if we can show that $\gamma(q_1) = \gamma(q_2)$. But by Claim 2, $\gamma(C_1)$ and $\gamma(C_2)$ are both open and dense in $\mathscr{E}$, which implies the existence of such q_1 and q_2. $\square$

Corollary 5.9. *β_Σ induces a bijection*

$$(J_b(F) \cap P(L)) \setminus \Sigma(X_\mu^{P\subset G}(b)) \xrightarrow{1:1} (J_b(F) \cap M(L)) \setminus \Sigma(X_\mu^{M\subset G}(b))$$

which restricts to

$$(J_b(F) \cap P(L)) \setminus \Sigma^{\mathrm{top}}(X_\mu^{P\subset G}(b)) \xrightarrow{1:1} (J_b(F) \cap M(L)) \setminus \Sigma'(X_\mu^{M\subset G}(b)).$$

In particular $X_\mu^{P\subset G}(b)$ is equidimensional if and only if the $X_{\mu'}^M(b)$ are for all $\mu' \in I_{\mu,b}$.

We use the following notation. Let R be an integral k-algebra. In the arithmetic case we assume R to be perfect and let $\mathscr{R} = W_{\mathcal{O}_F}(R)$. In the function field case, let $\mathscr{R} = R[\![\epsilon]\!]$. In both cases let $\mathscr{R}_L = \mathscr{R}[1/\epsilon]$.

For $m \in M(\mathcal{R}_L)$ consider the map

$$f_m : LN_R \to LN_R, \quad n \mapsto (m^{-1}n^{-1}m)\sigma(n).$$

Lemma 5.10 (Chen, Kisin, and Viehmann). *Let $b \in [b] \cap M(L)$ with $b\sigma(b)\cdots\sigma^{l_0-1}(b) = \epsilon^{l_0\nu_b}$ for some $l_0 > 0$ such that $l_0\nu_b \in X_*(T)$. Let R be an integral k-algebra, $\mathcal{R}$ and $\mathcal{R}_L$ as above, and $y \in N(\mathcal{R}_L)$ contained in a bounded subscheme. Further let $x_1 \in \operatorname{Spec} R(k)$ and $z_1 \in N(L)$ with $f_b(z_1) = y(x_1)$. Then for any bounded open subgroup $K' \subset N(L)$ there exists a finite étale covering $R \to R'$ with associated $\mathcal{R} \to \mathcal{R}'$ and $z \in N(\mathcal{R}'_L)$ such that*

(1) *for every k-valued point x of R' we have $f_b(z(x))K' = y(x)K'$ and*

(2) *there exists a point $x'_1 \in \operatorname{Spec} R'(k)$ over x_1 such that $z(x'_1) = z_1$.*

Proof. This is [Chen et al. 2015, Lemma 3.4.4], except for the fact that there R is assumed to be smooth, and only the case of mixed characteristic is considered. But actually, none of these assumptions is needed in the proof given there. □

Corollary 5.11. *Let $b \in [b] \cap M(L)$ and R and $\mathcal{R}$ be as in the previous lemma. Let $m \in M(\mathcal{R}_L)$, and $y \in N(\mathcal{R}_L)$, each contained in a bounded subscheme. Further let $x_1 \in \operatorname{Spec} R(k)$ and $z_1 \in N(L)$ with $f_b(z_1) = y(x_1)$. Let $b_m = m^{-1}b\sigma(m) \in M(\mathcal{R}_L)$. Then for any bounded open subgroup $K' \subset N(L)$ there exists a finite étale covering $R \to R'$ with associated extension $\mathcal{R} \to \mathcal{R}'$ and $z \in N(\mathcal{R}'_L)$ such that*

(1) *for every k-valued point x of R' we have $f_{b_m}(z(x))K' = y(x)K'$ and*

(2) *there exists a point $x'_1 \in \operatorname{Spec} R'(k)$ over x_1 such that $z(x'_1) = z_1$.*

Proof. For $n \in N(L)$ we have

$$\begin{aligned} f_{b_m}(n) &= (\sigma(m)^{-1}b^{-1}m)n^{-1}(m^{-1}b\sigma(m))\sigma(n) \\ &= \sigma(m)^{-1}b^{-1}(mn^{-1}m^{-1})b\sigma(mnm^{-1})\sigma(m) \\ &= \sigma(m)^{-1}f_b(mnm^{-1})\sigma(m). \end{aligned}$$

By the boundedness assumption on m, there is a bounded open subgroup K'' such that

$$\sigma(m(x))^{-1}K''\sigma(m(x)) \in K'$$

for all $\bar{k}$-valued points x of $\operatorname{Spec} R$. Applying Lemma 5.10 to $\sigma(m)y\sigma(m)^{-1}$ and K'', and conjugating the result by m, we obtain the desired lifting with respect to f_{b_m}. □

Theorem 5.12. *Let $\mu \in X_*(T)_{\mathrm{dom}}$ be minuscule, $b \in [b] \in B(G, \mu)$, and $\tilde{\lambda} \in X_*(T)$ be an associated element. Then the map*

$$\phi = \alpha_\Sigma \circ \beta_\Sigma^{-1} : W.\mu \cap [\tilde{\lambda} + (1-\sigma)X_*(T)] \to J_b(F) \setminus \Sigma^{\mathrm{top}}(X_\mu(b))$$

constructed above is surjective and it is bijective if and only if $J_b(F)$ acts trivially on

$$(J_b(F) \cap P(L)) \setminus \Sigma^{\mathrm{top}}(X_\mu(b)).$$

Proof. From Lemma 5.2, Corollary 5.9, and (5.3) we obtain the claimed maps

$$\begin{aligned} W.\mu \cap [\tilde{\lambda} + (1-\sigma)X_*(T)] &= (J_b(F) \cap M(L)) \setminus \Sigma(X_\mu^{M\subset G}(b)) \\ &\overset{\beta_\Sigma^{-1}}{\to} (J_b(F) \cap P(L)) \setminus \Sigma^{\mathrm{top}}(X_\mu^{P\subset G}(b)) \\ &\overset{\alpha_\Sigma}{\twoheadrightarrow} J_b(F) \setminus \Sigma^{\mathrm{top}}(X_\mu(b)). \end{aligned}$$

As $\Sigma^{\mathrm{top}}(X_\mu(b)) \cong \Sigma^{\mathrm{top}}(X_\mu^{P\subset G}(b))$, this description also implies the assertion about bijectivity. □

Proof of Theorem 1.4. The first assertion is a direct consequence of the previous theorem.

If G is split, then $W.\mu \cap [\tilde{\lambda} + (1-\sigma)X_*(T)] = \{\tilde{\lambda}\}$ has only one element; hence, the map is also injective.

If the second condition holds, then $J_b(F) \subset P(L)$; hence, α_Σ and also ϕ are bijective. □

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